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Title: A Study on Student Perceptions of Principles of Logic and Their Application to the Principle of Mathematical Induction

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The purpose of this article is to explore student reasoning with regard to problems in logic, particularly those related to the Principle of Mathematical Induction (PMI). The five case studies presented build off of work done by other researchers, most notably Dubinsky and Harel, who both looked at how students' schemes for logic, proof, and pattern generalization may be applied to PMI. This study examines the schemes students use for tasks involving logical implication both with and without quantifiers, modus ponens, and pattern generalization using data collected from hour-long interviews and then examines the same students' schemes when presented with tasks directly related to PMI in a second hour-long interview. We found that in addition to misrepresentations of logical statements, students also have difficulty representing ideas as mathematical statements and that this is a significant barrier to proof. Additionally, in some cases students had developed schemes which were valuable for tasks on the first interview but did not apply these schemes within the context of induction. Furthermore, it was found that students had formed false beliefs for which they had the tools to determine were false, but for which they had never been challenged or forced into a state of disequilibrium.

A Study on Student Perceptions of Principles of Logic and Their Application to the Principle of Mathematical Induction

by

Allan R. Boone

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I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes the release of my thesis to any reader upon request.

Allan R. Boone, Author

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TABLE OF CONTENTS

Page

LIST OF TABLES

A Study on Student Perceptions of Principles of Logic and Their Application to the Principle of Mathematical Induction

Chapter 1: Introduction

Mathematical induction is a mainstay of proof based courses in mathematics, being taught both in undergraduate courses like discrete math and number theory that introduce proof techniques, and in some honors classes at the high school level. Despite the frequency with which induction is used, it is considered a very difficult concept for many students. Dubinsky (Dubinsky, 1986a) summed this up with his statement:

> "Indeed, if you question students- even those who have had several mathematics courses- although almost all of them will have heard of induction, not many of them will be able to say anything intelligent about what it is, much less actually use it in a problem" (p.305).

A number of studies have been done to examine how students understand and apply induction. Of particular interest were the studies of Dubinsky and Harel, each of which directly contributed to the framing of this study, specifically Dubinsky's genetic decomposition of the Principle of Mathematical Induction (PMI) and Harel's proof schemes and DNR-based model. Although there were several studies detailing how students worked with induction, there was a gap in the literature in connecting students' performance with PMI and their schemes used to work with other areas of logic and reasoning which were related to PMI. The following case studies were an attempt to examine the following questions:

- What schemes do students use for determining the truth of an if-then statement? Are these schemes different when quantifiers are involved?
- How do students understand and apply modus ponens in a general setting?
- What schemes do students use for discovering and asserting the validity of patterns?

- How do all these schemes relate to how students understand and apply induction? Does a lack of understanding PMI stem from an incomplete understanding of prior topics or is there another gap that had not yet been identified?

This was a qualitative study in which four Discrete Math students participated in two one-hour interviews (and a fifth student participated in the first of these interviews), the first of these interviews focused on tasks that involved pre-induction topics, while the second, administered three to four weeks later, focused tasks directly related to PMI. The details of what questions were asked and their rationales for being included are in Chapter 3: Methods.

In Chapter 4: Analysis, the results of the study are presented on a question by question basis, summarizing each student's response to each question. Since a major focus of the study is on what schemes are being employed by students, the order of which students are described varies from question to question as students employing similar schemes may be grouped together.

Finally, in Chapter 5 we present the conclusions. As we discuss the results in light of our questions of interest, we also identify limitations of the study, implications for practitioners and questions for future research.

Chapter 2: Literature Review

A Mathematical analysis of the Principle of Mathematical Induction

Harel and Brown (Harel & Brown, 2008) noted that there are two common ways to state the Principle of Mathematical Induction: The first is to let $S \subseteq N$. Then if $1 \in S\ and\ n \in S \rightarrow n+1 \in S\ for\ any\ n \in N\ it\ follows\ that\ S = N$. Alternatively, you could let $P(1), P(2), \ldots$ be a sequence of statements involving 1,2,… respectively. Then if $P(1)$ is true and $\forall n \in N\ P(n) \rightarrow P(n+1)$ is also true, then we can conclude $\forall n \in N\ P(n)$. The second approach may also be written in the form of propositional functions. In this case P would be a propositional function on the domain of natural numbers, that is for any natural number *n*, *P* outputs a proposition *P*(n). In any case, showing *P*(1) or $1 \in S$ is called the base case and showing that for all natural numbers *n* that $n \in S \rightarrow n+1 \in S\ or\ P(n) \rightarrow P(n+1)$ is called the induction step. The conceptual idea of starting with *P*(1) and concluding *P*(2), which lets us conclude *P*(3), and so on is an application of modus ponens- the logical conclusion of the statement *B* given statements of the form $A\ and\ A \rightarrow B$- as an infinite process.

PMI is strongly tied to recursive processes, as when given a statement that works for some natural number *n*, to show that it also works for *n*+1, one often writes this $(n+1)^{th}$ case in terms of the n^{th} case. One must use a process that transforms a statement involving *n* to one involving *n*+1 or vice versa.

For example, one could use PMI to prove that the formula $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ holds for all natural numbers n in the following way:

Let *P*(n) be the statement “$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.”

Then, $P(1)$ is the statement "$\sum_{i=1}^{1} i = \frac{1(1+1)}{2}$," that is "$1 = \frac{1*2}{2}$" which we know to be true.

Now fix $n \in N$ such that $P(n)$ is true.

Then, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ so $\sum_{i=1}^{n} i + n + 1 = \frac{n(n+1)}{2} + n + 1$, and after a few lines of algebra becomes $\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$ verifying that $P(n+1)$ is true. And so we have that $\forall\, n \in N\ P(n) \rightarrow P(n+1)$. Finally, by the Principle of Mathematical Induction, the statement P(n) must be true for all natural numbers *n*. QED

The big take away in the induction step of this proof was the demonstration that the formula using *n*+1 was a transformation of the one using *n*, notably that $\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + n + 1$. And one could either think of the sum as being built up by adding a new term for each natural number or as a process in which each sum, besides the initial sum of only one term, is recursively defined in terms of its predecessor. Either way suggests a method for the student to transform a statement involving $\sum_{i=1}^{n} i$ into one involving $\sum_{i=1}^{n+1} i$. Modifying the rest of the statement, then, is simply an application of the idea that in order to preserve equality, one must apply the same algebraic procedure to both sides of the equality. This process of transforming one statement into another will be discussed further in the theories of Harel.

Some writers (e.g. Stylianides, Stylianides, & Philippou, 2007) instead begin with a slightly abstracted version in which the base case is to show $P(n_0)$ for some integer n_0, then after proving the induction step, one can conclude that *P* is true on the set $\{n \in \mathbb{Z} | n \geq n_0\}$. This may be because in Stylianides' study, one topic of concern was how students would attempt to evaluate the truth of a propositional function on values that were less than the value of the base case, and so the modified base case was a necessity of the study. Harel (2002) in his studies gave problems that require the modified base case, but did not include this in his definition of PMI.

Beyond adjusting the base case, even further abstractions can be made within PMI. However, aside from Stylianides there does not appear to be anything in the literature about how well students are able to determine truth sets of propositional functions with an abstracted version of PMI, and nothing beyond the modification of the base case. This was a motivation for examining how students respond to looking at more abstract problems which modify PMI but preserve the basic features of using a base case and repeated modus ponens to generate a truth set.

Student Thinking and Learning About PMI

There are two significant perspectives that have been applied to how students learn Principle of Mathematical Induction. Dubinsky applied the idea genetic decompositions to PMI in the late 1980s. Harel followed this with his ideas of proof schemes in the late 1990s and early 21st century, which he applied to PMI among other topics. In this section, I'm going to describe these perspectives followed by a synthesis of the findings of the studies done on PMI.

Dubinsky (1986), in his work, organized the knowledge needed to understand and perform PMI into what he calls a "genetic decomposition." A genetic decomposition is essentially a map of concepts, showing how understanding of one concept develops into another, or in many cases how two or more concepts are required to develop a new one. In Dubinsky's model, there are three basic ideas that students must have first established in order to understand and write proofs using PMI: method of proof, the concept of a function, and logical necessity. The latter two are needed to explain induction and all three are needed in order to apply induction and then solve problems that involve it. In Dubinsky's model, the idea of function is needed in order to convert natural numbers into statements that involve them. For example if $P(n) = "1 + 3 + \cdots + (2n - 1) = n^2"$ to understand that this defines a relationship that includes such pairs as $P(1) = 1=1^2$," P(2)="$1 + 3 = 2^2$, " , etc. This is called generalization. Coordinating the notion of function with the encapsulation of logical necessity is necessary to create propositional functions that involve implications. For example, if we let $Q(n) = (P(n) \rightarrow P(n + 1))$ using the P from above, we can get statements like $Q(1) =$ "$If\ 1 = 1^2\ then\ 1 + 3 = 2^2$." On another branch, logical necessity derives modus ponens. Modus Ponens brings the concept that $P(1) \rightarrow P(2)$ does not require $P(1)$ nor $P(2)$ to be true, just that when both $P(1)$ and $P(1) \rightarrow P(2)$ are true that $P(2)$ must be true as well. Coordinating this with the development of functions that can involve implications allows students to understand PMI. Combining this understanding with method of proof allows for the application of induction and written proof.

Harel (Harel & Sowder, 1998) also brought his theories on proof schemes to the table. According to Harel "A person's proof scheme consists of what constitutes ascertaining and persuading for that person" (p. 241). Although this could include writing a formal mathematical proof, there are other forms of proof scheme as well. A common proof scheme is the empirical proof scheme where one becomes convinced of a statement's truth by seeing a number of examples in which it is true. Another one is the appeal to authority, in which a student believes the truth of a statement, simply because an instructor told them it was true. This proof scheme is particularly dangerous in the context of mathematical induction, because such an acceptance allows students to see PMI as merely a formulaic approach in which they write a proof that is validated simply because it follows a certain template.

The proof scheme that is critical to develop for the understanding of PMI is the transformational proof scheme, particularly one that utilizes process pattern generalization. A transformational proof scheme is one in which objects are transformed from one to another with a goal in mind. Process Pattern Generalization is the ability to recognize a pattern and a method that can be applied to the problem repeatedly. Putting these two together, in a problem that motivates proof by induction, a process can be used repeatedly to transform $P(1)$ into $P(2)$ into $P(3)$ … into $P(n)$ into $P(n+1)$, etc.

Although there have been relatively few studies on how students think about mathematical induction, some clear trends have been found, often supporting the theories of Dubinsky and Harel. One of the more universally found misconceptions is the idea that PMI constitutes circular reasoning (Palla, Potari, & Spyrou, 2012; David, Grassl, Hauk, Mendoza-Spencer, & Yestness, 2009; Harel, 2002;Harel & Brown, 2008; Movshovitz-Hadar, 1993; Ernest, 1984). One clear example of this was from Moshovitz-Hadar's observation of a course in which students were looking at an invalid proof by induction and one student said "That's the point. In mathematical induction, we assume what we want to prove and then prove it." (Molshovitz-Hadar 1993 p. 262) Similar observations have been made by Dubinsky, Harel, Ernst, Davis, and Palla. An issue that may cause a belief about circular reasoning is that every proof by mathematical induction will include the line with a statement similar to:

$Suppose\ for\ some\ arbitrary\ n \in N, that\ P(n) is\ true.$

This line looks very similar to the conclusion that for any *n*, *P*(n) is true. Often textbook writers try to differentiate use the variable *k* in the inductive step, although this is a solely a style difference. The purpose of using a different variable is to show that *k* is one particular fixed natural number at this stage of the proof. There has not been, as far as I can tell, any studies done on whether the use of *k* instead of *n* in the inductive step helps students, though I would hypothesize that such a difference has minimal effect.

Ernst (1984) suggests that the problem students have with circular reasoning as it pertains to PMI, is an issue with understanding the logical nature of implication and as well as the nature of quantifiers, concepts that Dubinsky listed as necessary for PMI in his genetic decomposition. Students often do not realize that the hypothesis of the induction step is that $P(n)$ is true for only one particular value and that they go on to show that this means that P is also true for the next value as well. A number of studies (e.g.Palla et al., 2012; Stylianides et al., 2007) have shown that many students believe that the inductive step is proving the statement P(n+1) or even just *n+1*, but when asked what they mean by this are rarely able to give a coherent answer.

What students often fail to understand, then is that the inductive step is proving that $\forall n\ P(n) \rightarrow P(n+1)$, a quantified implication statement, rather than simply P(n+1). David (David et al., 2009) observed in his study of pre-service teachers that students mentioned difficulties trying to connect relationships like P(k) and P(k+1), which could suggest a lack of understanding in either the concepts of functions or of implications. In Stylianides' (2007) study of pre-service teachers, students were given a statement that was always false, specifically "$1 + 3 + \cdots + (2n-1) = n^2 + 3.$" Going through the algebra of the inductive step makes it easy to show that for any value of n that $P(n) \rightarrow P(n+1)$is true [where P(n) is the above statement]. One student was asked if the statement holds for k, does that mean it also holds for k+1. The student responded that it did not because the statement is false when k=1. This, however, suggests that the student lacks an understanding of what it means for a statement to be vacuously true. With sufficiently strong conceptual understanding, students would ideally be able to see that the proposition that follows from induction step would be true when P is always false- and thus see the necessity for the base case. This is a concept, though, that is taught prior to PMI, as it is prerequisite to PMI.

Also in Stylianides , students were given a proof that $n! > 2^n \; for \; n \geq 5$. They were then asked whether "$n! > 2^n$" was true for n=3, n=4, n=6, and n=10. In the study, 44% of the education majors and 36% of the math majors said that this was false for n=4, without checking. In subsequent interviews, some of these students realized that the proof said nothing about the truth values for $n < 5$, and so it needed to be checked. This could suggest a lack of understanding logic (the proof did not show that it was true, so it must be false) or an authoritative proof scheme ("if it was true for n=4, it would have been proven that way as proofs get written to be as encompassing as possible") or possibly something else.

Instructional Interventions in PMI

Based on his theories, of genetic decomposition, Dubinsky created an alternative instructional technique, utilizing the programming languages SETL and ISETL, first working with method of proof (Dubinsky, 1986b). In this study, once students began work with propositional functions, they would use the computer to verify truth values of a proposition at various values, with the goal of developing generalization of the function. After basic functions, they developed functions that were based on implications. After defining a function P, they could define a function Q by $(n) = P(n) \rightarrow P(n+1)$. In doing so, students could see the application in the program. That is, whenever $P(n_0)$ and $Q(n)$ had constant value 'true' for $n \geq n_0$ it could be seen through the process that $P(n)$ had to be true whenever $n \geq n_0$. Getting this concept down, made the application to writing proofs simpler. Dubinsky got very positive results with students able to correctly apply induction in 75% of given problems of notable difficulty, with full proofs 53% of the time. However, according to Brown (2008), Dubinsky's results were irreproducible.

After Dubinsky's work in the late 1980's, there was a period of about ten years, before more work was done on modifying the instruction of proof by mathematical induction. Harel (2002) brought a new theoretical approach to the problem, using the model for DNR-Based instruction. DNR stands for Duality, Necessity, and Repeated Reasoning. Harel's assertion of duality is that both how students think affects their understanding of concepts and their understanding of concepts affects how they think (Brown, 2008). In the context of PMI, how students understand certain content, such as recursion problems changes how they think about what constitutes proof (Harel, 2002). The necessity portion of the DNR model states that students are more likely to learn if they see a need for what they are being taught. And finally, students need to practice reasoning in what they are doing. In Harel's approach, he gave students induction type problems well before explicitly teaching PMI.

Since the use of PMI is strongly tied to recursive processes, Harel gave students problems to solve that would involve creating some sort of recursive process. He called these "quasi-induction" problems. Examples of this were finding the number of steps to the Tower of Hanoi problem or the number of weighings required to find the false coin given 3^n coins. Quasi-induction could either be used by the method of ascent, building up from *k*=1 to *k*=2 and continuing the process until *k*=*n*, a method that shows the very core of PMI. Alternatively, quasi-induction may be implemented by descent, where one starts with the value of *n* and then uses a process that goes backwards until you get back to 1, showing equivalence of a formula working for arbitrary value n to it working for 1, essentially PMI in reverse. One fascinating example was in the Tower of Hanoi problem, one student derived a process showing that $S_{n+1} = 2 * S_n + 1$. Another student, going through a few examples noticed that there was a pattern that $S_n = 2^n - 1$. When prompted by the instructor, they derived that both of these being true implied that $S_{n+1} = 2(2^n - 1) + 1 = 2^{n+1} - 1$, the exact derivation that would be used in an induction proof.

The purpose of these problems was to have students develop what Harel calls "process pattern generalization," in this case finding a process that can be applied repeatedly to generate the desired result, either by building up (ascending) or breaking down (descending). Seeing that a formula holds for a certain set of values is what is called results pattern generalization. Simply doing a brute force of the Tower of Hanoi and recognizing that it can be done in $2^n - 1$ steps each time is an example of results pattern generalization, while finding the recursion is a process pattern, which is stronger as it allows for the opportunity to write a general proof. In another study, Harel and Brown (2008) gave students a false pattern with dividing a circle into regions by drawing lines between n points. Up until *n*=*5*, the circle gets divided into 2^{n-1} regions but this breaks for larger values of *n*. The purpose of this was to make students realize that simply generating results for a finite set without finding a process behind them, does not guarantee that the pattern will hold.

After this initial stage then the second stage was focused on making quasi-induction an interiorized process pattern generalization. That is, making students consciously aware of their internalized scheme. This is the point in time where problems like the sum of finite sums appear. Unfortunately, students did not realize the connection between the problems right away, but were able to see the connection when the formula was rewritten as a sequence of statements.

Harel (2002) also showed that, unfortunately, when PMI was introduced formally, the students did not see it as an abstraction of quasi-induction right away, and intervention was required including individual meetings with the instructors. Students were then given problem sets with problems to be solved by PMI. About 75% of problems were, with most of the rest proved by other means. Students tended to avoid induction on problems such as “prove $2^n < n!\, for\ n \geq 4$,” that they said could be proven more efficiently.

Another study by David (2009), looked at two classes at a university for pre-service teachers, one of which used a traditional approach for induction and another that used a necessity-based approach based on Harel’s work. Unfortunately, they did not give any sort of comparison in the results of the performances of the two classes.

Concluding Statements and Future Research Potential

The literature shows that there are many student difficulties in learning PMI and that many students who learn PMI under the traditional approach have false concepts about PMI. Many students feel like a proof by PMI is just following a prescribed formula. Students are often able to state a definition of PMI word for word from their textbook, but have no knowledge of what it actually means(Palla et al., 2012), the only basis for their belief that PMI works being that their instructor or book told them that it works. Although it would be unrealistic to expect students to come up with PMI on their own, it’s important to make them think about the fundamental concepts behind PMI on their own so that they will believe and understand it when it is presented to them. This is what Moshovitz-Hadar (1993) calls the didactical paradox: “if both the problem and the information about its solution are communicated by the teacher, this deprives the pupil of the conditions necessary for learning and understanding.”

To synthesize some of the above studies that have focused on instruction, it seems that the research has definitely been suggestive that students who learn PMI often lack the perquisite skills cited by Dubinsky- a notion of functions, of logical necessity, and of method of proof. However, the studies done have mostly observed student’s skills with regard to PMI in isolation. It has not been studied whether students who understand all the prerequisite concepts also struggle with PMI. Are there students who have strong understandings of quantifiers, implication and structure of proof, but still struggle with PMI? Are some of these concepts more important than others with respect to understanding PMI? How critical is it for students to develop process pattern generalization in order to construct proofs by PMI? These were the questions that motivated the research of this study.

Chapter 3: Methods

This study primarily utilizes and examines two theories in its framework: Dubinsky's Genetic Decomposition of the Principle of Mathematical Induction and Harel's Proof Schemes, with emphasis on transformational proof schemes. This study was specifically designed to examine student understanding of if-then statements, both with and without quantifiers and generalized propositional functions, as well as students' ability to generalize patterns through recursive processes and to then compare this to their ability to later understand and use induction. This covers the understandings listed by Dubinsky as prerequisite for explaining induction, as well as the transformational schemes that Harel considers important for solving specific induction problems. By examining the relationship between the schemes students develop for analyzing logical statements, utilizing functions and generalizing patterns with their ability to explain and apply induction could give insight into where instruction should be focused. If students are able to perform well in the former set of tasks, yet still struggle with induction, it could suggest that there is another area of knowledge linked with PMI that has not been identified. Alternatively, it is possible that there are multiple schemes that allow for students to successfully answer the more basic questions, but only some of these schemes are able to assimilate the concept of induction. Through the use of interviews, this study not only attempts to examine student competency in the areas of logic and pattern generalization but also looks at what specific schemes students use and their ability to consistently and effectively employ these schemes.

This study involved asking students to answer a sequence of questions in two separate interviews. The first interviews were in either the 4^{th} or 5^{th} week of the term and focused on if-then logic and pattern generalization. The second interviews were given in the 8th week of the term and asked students to explain what induction is as well as write proofs by induction for certain tasks. Additionally, students were asked to answer questions about abstracted versions of induction in which the base case and inductive steps have been altered but in such a way that preserves the nature of the process, in particular generating a truth set through an infinite process of applying modus ponens. The purpose of these later tasks was to examine whether students could apply the ideas of PMI beyond the statements given by their instructor or textbook.

Participants

The students selected for this study were students who were taking one of two versions Discrete Math. One version, MTH 355, is primarily for students majoring in Mathematics The other version, MTH 231, is primarily for students majoring in either Computer Science or Electrical Engineering. Both classes were introductory proof courses and included among their topics: logic, set theory, quantifiers, method of proof and PMI. The students who participated responded to prompts in two one-hour long interviews. The students were selected on a volunteer basis and were paid $20 per interview for their participation in the project. The first interview focused on the preliminary concepts required for understanding PMI:

1) An understanding of the logical if-then both with quantifiers and without
2) modus ponens for both finite and infinite sequences of statements, and
3) process pattern generalization, specifically looking at problems that invoke recursion.

Five students participated in the first interview and four of these students returned for a second interview. The following are the questions asked in the interviews as well as a rationale for why they were included.

Interview 1 Questions

Understanding the logical if-then:

1. Are the following statements true or false? Explain why
   a. If pigs can fly then 2*3=6
   b. If 3+4=7 then 7-4=3
   c. If 2+5=8 then 8-5=7
   d. If -3 < 0 then 3 < 0
   e. If Obama was elected president in 2012 then Obama is the current president.
   f. If Al Gore was elected president in 2000 then Obama was elected president in 2008
   g. If red is a color then a pentagon has five sides.
   h. If Portland is the capital of Oregon then Seattle is the capital of Washington.

2. Can a statement of the form "If *p* then not *p*" ever be true? Explain your answer.

Part of the definition of PMI includes the truth of an if-then statement. The purpose of the question was to see if students interpreted if-then statements using the logical definition or whether they applied other schemes for doing so. The list of statements includes all four combinations of potential truth values for the antecedent and the consequent. Additionally, some statements included a cause and effect relationship between the antecedent and the consequent, while other statements did not. This allows us to examine the question "do students evaluate the truth of an if-then statement solely by the truth values of its components or does the context of the statement affect their interpretation?"

Problem 2 then is a follow up on problem 1 to see how consistently a scheme may be applied. A statement in the form above is designed to be intuitively false, but a strict application of the logical definition would show that it could be true (and in fact, the ability for a statement to imply its own negation allows for proof by contradiction).

Understanding Quantifiers:

3. Are the following true or false? Explain why
   a. $\forall x \in R\ if\ x \in Q\ then\ x^2 \epsilon Q.$
   b. $For\ all\ x\ in\ the\ set\ of\ quadrilaterals, if\ x\ is\ a\ square, then\ x\ is\ a\ rectangle.$
   c. $For\ all\ x\ in\ the\ set\ of\ quadrilaterals, if\ x\ is\ a\ rectangle, then\ x\ is\ a\ square.$

4. On which of the following domains of discourse is the following statement true:
   $\forall x\ if\ x > 2\ then\ x \geq 3$?
   N, Z, Q, R, (-∞,0]

Problem 3 allows for an examination of how the introduction of quantifiers affected how students interpret if-then statements. This is important as the definition of induction includes a quantified if-then statement. Examples of potential misunderstanding of the logical if-then appeared in studies such as that by Stylianides, in which a student was given a particular statement and asked if the statement holds for *k*, does it hold for *k*+1, and answered “no” with the reason that the statement was false when *k*=1. Asking these more basic questions first may give a better indication as to whether such a response comes from a more general misunderstanding of the logical if-then or whether it appears only in the context of more complicated problems.

Problem 4 forces students to consider the properties of different domains upon which they could be doing mathematics. In Palla’s study, 60% of the students said that induction could not be used to prove a statement on the real numbers, but few of them made an appeal to the properties of the natural numbers. This task is given to examine how well students are able to recognize how the properties of the sets they are working with affect the results they obtain. Additionally, the set (-∞,0] is included to once again look at whether students recognize a statement as vacuously true in the additional context of quantified statements.

Modus ponens for finite and infinite sequences of statements:

5. Given a domain of discourse D, what is a propositional function on D?
6. Consider a propositional function P on the set $\{1,2,3,4\}$. What truth values of $P(1)$, $P(2)$, P(3) and P(4) would result in $P(1) \rightarrow P(2), P(2) \rightarrow P(3), and\ P(3) \rightarrow P(4)$ all being true? If we extended the domain of P to N, then what would need to be true to have the statement $\forall n\ P(n) \rightarrow P(n+1)$ be true?

Since the use of propositional functions is not universal in teaching induction and discrete math in general, this question is mainly here to make sure that students are prepared to examine problem 6. Students who are unfamiliar with the term "propositional function" may have it explained to them at this point before looking at problem 6. This problem was not designed for analysis in and of itself.

Problem 6 combines many of the previous ideas together. A student who is able to correctly answer problem 6 should, in theory, have all the skills necessary to understand the nature of why PMI works. The first part of this problem examines whether students are able to both allow a statement to be vacuously true and also to apply modus ponens repeatedly. The second part of the problem examines how students are able to generalize from a finite set to an infinite set. While the solution to the first part of problem 6 may be approached by brute force, the second part requires and therefore examines the students' abilities to apply modus ponens ad infinitum.

Process pattern generalization (involving recursion)

7. Is there a formula for the sum of the first n odd numbers? How would you show that this formula would continue to hold?

11. In a version of takeaway, you and your opponent take turns and may choose to take either 1, 2, or 3 pennies each turn. The player who takes the last penny wins. If the game starts with 6 pennies is there a winning strategy if you pick first? What is it? For an arbitrary number of starting pennies, how can you tell which player has a winning strategy?

Problem 7 was designed to look at two things: first of all, whether students find a pattern in the first place and secondly whether they can justify whether it works. This is similar to a problem given by Palla in which students were asked to find the area of a triangle in which the size grows in a similar pattern. Though 70% of students in Palls's study were able to find a pattern, only 17 of 213 were able to prove that it holds. Similarly, it is hypothesized that most students will be able to identify the sequence of square numbers, but not be able to prove it using a transformational scheme. This problem may also show how comfortable students are with empirical proof schemes.

Problem 11 was intended to examine students' abilities to build recursively off of previously found solutions within the problem. Relatively few steps of logic are required to determine that leaving four pennies is a winning strategy. However, for larger numbers of pennies the number of cases to examine becomes quite large. It is only through using a repeated process, the ability to always leave 4 fewer pennies each time, that one is able to generate a general solution. This problem, then is used to examine how well students are able to use recursive patterns, a valuable skill for figuring out how to generate a proof by induction.

Since the interviews were a fixed length of time, rather than however long it took to answer all problems, there were extra problems for students who worked quickly, some of which were more difficult. Only one student answered any of these questions. Analysis for these problems, then, was not included, except in the cases in which a response assisted in the analysis of a scheme used in another problem. Here are a list of those problems:

8. Suppose you started stacking pennies. The first stack has one penny, the second stack has two, the third stack has four and so on. Each stack has twice as many pennies as the previous stack. After making n stacks, how many total pennies are there in all stacks? Why will this pattern continue to hold?
9. Construct the following sets:
   a. $A = \{1,3,5\}, B = \{x \in R | \forall y \in A\ x \geq y\}$ what is another way to write B?
   b. $\{x \in R | \exists y \geq 3\ s.t. xy = 6\}$
   c. $\{x \in N | x + 1 \leq x\}$
10. Given a statement of the form "If p then q" what is the converse of this statement? What is the contrapositive? Is either equivalent to the original statement?
12. Consider a $2^n x 2^n$grid with the upper right corner removed. How could you cover this with L shaped blocks of three?

<u>Interview 2 Questions</u>

The purpose of the second interview was to then examine how the same students were understanding and implementing the Principle of Mathematical Induction. They were also given statements similar to induction, but with modifications made to the base case and inductive steps but which preserved the nature of induction, that is the infinite application of modus ponens to determine a truth set. This interview took place 3-4 weeks after the first interview, after students had completed the section on PMI in their class. The questions on this interview and their rationale were as follows:

<u>Explaining the Principle of Mathematical Induction</u>

1. Define the Principle of Mathematical Induction.
2. Why are both the base case and the inductive step necessary? [Or ask them to explain their definition if they do not have these steps]

These questions were used to establish what the students think induction is and what it's used for. The literature suggests some students may give a definition similar to one written in a textbook, while others may describe the process of steps that one goes through in performing a proof by induction. Either way, asking the student to then justify the necessity for the base case and the inductive step is possible. Depending on the response, this may be a place to ask follow up questions, such as "what kind of sets can induction be used on?" or "when you say 'suppose $P(n)$' is that for all values of n or one particular value of n?"

Problem Solving Using PMI

3. Use PMI to prove the following:
   a. In a version of takeaway, you and your opponent take turns and may choose to take either 1, 2, or 3 pennies each turn. Prove that if you can leave a multiple of four pennies, then you have a winning strategy?
   b. Prove that the sum of the first n odd natural numbers is n^2.
   c. Prove that $\log(a_1 * \ldots * a_n) = \log(a_1) + \cdots + \log(a_n)$ for any positive real numbers $a_1, \ldots, a_n$. [You may use the fact that $\log(ab) = \log(a) + \log(b)$ for any positive real numbers a and b]
   d. Consider a $2^n x 2^n$grid with the upper right corner removed. Prove that you can cover this with L-shaped blocks of three for any value of n.
   e. Prove that for $n \geq 4$ that $2^n \leq n!$
4. For problem (e) why is the base case not showing that $P(1)$ is true? How does changing the base case change what is being proven by induction?

The students should be familiar with problems 3a and 3b from the previous interview and may or may not have found the solutions on their own at that time. By having these problems in this interview one can see the relationship between how well students are at generating solutions to problems versus justifying why the answer is what it is. In Ernst's study, he found that students often associate induction with proving finite sum identities, which is what 3b is asking for, while part a is an entirely different type of induction problem. The difference in students' abilities to work with parts a and b may be indicative of the context in which students feel that PMI is appropriate. Due to time concerns, only 3a and 3b were given to all students.

Extension of Inductive Process

5. Suppose P is a propositional function on $\boldsymbol{Z}$ and that;

$$i)\ P(0)\ is\ true\ and\ ii)\ \forall n\ P(n) \rightarrow P(n+3)$$

Then, on what set, if any, can we determine that P is true? On what set, if any, can we determine that P is false?

6. Suppose P is a propositional function on $\boldsymbol{Z}$ and that;

   $i)\ P(4)\ is\ true\ and\ ii)\ \forall n\ P(n) \rightarrow P(n-1) and\ iii) \forall n\ P(n) \rightarrow P(n+1)$
   Then, on what set, if any, can we determine that P is true? On what set, if any, can we determine that P is false?

Problem 5 examines how well students are able to generalize the Principle of Mathematical Induction. Since this particular statement is not one they will have seen in a textbook or been given to them by an instructor, it is up to the student to understand the role that quantifiers play and their ability to apply modus ponens as a repeated process. Additionally, this problem looks at whether students will declare a statement to be false simply because they cannot prove it to be true, as was looked at in Stylianides study. Problem 6 is similar to problem 5. This problem looks at the generalizability of induction. It was expected that students able to answer problem 5 correctly would also be able to answer problem 6 correctly as well.

Chapter 4: Analysis

The participants of the study:

This was a qualitative study in which five students from Discrete Math courses at Oregon State University participated in two hour long interviews. The first interview, held in week 4 or 5 of the term, asked students to answer questions relating to logic, in particular if-then statements, quantifiers, and pattern generalization. Two students, given the pseudonyms Bob and Eugene, were taking MTH 355, a course primarily for math majors. Bob did not have prior experience in proof based courses, while Eugene had previous experience in such courses and was in his last term before graduation. Art, Cal and Dave were all in MTH 231 and had no previous experience in proof based math courses. Cal only attended the first interview; the other four participants took part in both interviews.

Analysis of Student Responses to Problem 1:

Are the following statements true or false? Explain why

a. If pigs can fly then 2*3=6
b. If 3+4=7 then 7-4=3
c. If 2+5=8 then 8-5=7
d. If -3 < 0 then 3 < 0
e. If Obama was elected president in 2012 then Obama is the current president.
f. If Al Gore was elected president in 2000 then Obama was elected president in 2008
g. If red is a color then a pentagon has five sides.
h. If Portland is the capital of Oregon then Seattle is the capital of Washington.

On problem 1, students were given eight if-then statements and asked to evaluate whether they were true or false and to explain why. No hints or additional instructions were given for this statement. It was not specified that they should use “if-then” in the same way in which it is used in their MTH 231 or 355 course, but the students were aware that they were asked to participate in these interview because they were in these classes.

Two of the five students, Art and Eugene, used the definition for the truth of an if-then statement as presented in their discrete math class. Art first wrote down the truth table with the columns for $p, q, p \rightarrow q$. For each statement, he evaluated both the truth of the antecedent and the consequent and wrote down a T or F for each with and arrow in between them. Then he used the truth table to determine whether the statement as a whole was true. Eugene gave identical answers to Art, but instead both statements, he would read the antecedent first. Whenever the antecedent was false, he answered true and went to the next statement. When the antecedent was true, then he'd read the consequent to determine the truth of the statement as a whole.

This suggests that Art and Eugene both interpret the meaning of an "if-then" statement in the same way, and also in the same way it was defined in their classes. However, while Art simply evaluated all truth values and compares to his chart, Eugene had developed a more efficient scheme that allowed him to determine the truth values of if-then statements as soon as he had sufficient information to do so. Bob, Cal, and Dave, on the other hand, used different methods.

Dave appeared to have an internally consistent method for determining the truth of an if-then statement: he simply declared a statement to be true whenever both the antecedent and the consequent were true. He said explicitly "something that's false could not possibly imply something that's true, in my opinion." However, he responded in a less certain manner when both the antecedent and the consequent were false. For part c, he said "if something that's false implies something that's false then that would be true, but it's all false so I'm going to write false." Dave went on to say that he didn't want to say that something was true when parts of it are false. On part h, he has the same dilemma again "If false implies a false in a truth table it would be true, but I don't know, something inside tells me that when they're false I don't want to dictate that it's true so I'm going to write false for h." Dave consistently answered true if and only if both the antecedent and the consequent were true. He acknowledged that when both were false that the implication would also hold from a logical standpoint, but said that he was uncomfortable declaring a statement true when parts of it are false. The use of the phrase "in my opinion" suggests that Dave believed that the meaning of an if-then statement is up to interpretation and that there isn't necessarily one correct way to interpret it.

Bob was similar to Dave in that he only answered true if both the antecedent and the consequent were true. However, on part g, he answered false to "If red is a color then a pentagon has five sides." His reason for this was that "colors and pentagons are mutually disjoint." After all of the statements, when asked if both parts of the statement had to be true, he said that they did. He said that the antecedent needed to be true in order to imply the second part whose name he said he couldn't remember. When asked if the two statements had to be related, he said that was an interesting question and he wasn't sure, but stated that he answered them as if they had to be related. Bob appears to have constructed a scheme for evaluating the truth of if-then statements that involves first verifying that the antecedent is true and then checking that there is some sort of cause and effect relationship that results in the consequent being true. Based on his answers to the follow up questions, he was fully aware of the first step of his scheme and became aware of the second part of the scheme when asked "do the two terms have to be related to each other or do they both just have to be true?" This is consistent with his reasoning throughout the problems, in which he discards all statements with a false antecedent. His response to part e was interesting in that he said that since Obama was elected in 2012, that this was still his term and that nothing happened that would cause him not to be president anymore. So, he did implicitly acknowledge that simply having been elected in 2012 doesn't necessarily mean that he must still be president, but the causal link was still strong enough in his mind to respond true.

While Bob and Dave appeared to apply a scheme consistently, albeit one that does not correspond to the logical definition of if-then, Cal appeared to interpret the meaning of if-then differently depending on the type of statement. The statements "If pigs can fly then 2*3=6" and "If Al Gore was elected president in 2000, then Obama was elected president in 2008" both have a false antecedent and a true consequent. As a result, Art and Eugene both declared both of these statements true using the logical definition. Bob and Dave both had schemes that declared all statements with false antecedents false and so answered false to both of these questions. Cal, however, answered false to part a and true to part f. He reasoned that if Gore had been elected in 2000 that we wouldn't have had Bush, but there would be no reason that this would prevent Obama from being elected in 2008. However, on part a, he responded that the statement was false since pigs cannot fly. It appeared, then, that Cal's interpretation of the truth of an if-then statement depended on the context of the statement beyond the truth of the antecedent, consequent and the existence of a cause and effect relationship.

In common language, if-then statements typically imply a causal relationship between the antecedent and the consequent. This use in natural language may have been a source of interference for both Bob and Cal. For Bob, his scheme required a causal relationship between the two terms in order to establish truth. Cal, on the other hand, appeared to interpret statements differently depending on their content, allowing an if-then statement to either show a causal relationship or to apply to hypothetical situation. Dave appeared to debate whether an if-then statement is true whenever the antecedent and consequent had the same truth value or whether both had to be true at the same time.

One might expect to see four different interpretations of what it means for an if-then statement of the form “if p then q” to be true:

The “logical” definition, which is going by the definition that $p \rightarrow q$ is equivalent to $\neg p \text{ or } q$.

The “causal” definition, which requires p to be true and a causal link between p and q.

The “conjunctive,” definition which simply requires both p and q to be true.

Finally, the “equivalence” definition, which requires p and q to have the same truth value.

The following chart shows how students in each of these categories would respond:

Question	Logical	Causal	Conjunctive	Equivalence	inconsistent
A	True	False	False	False	
B	True	True	True	True	
C	True	False	False	True	
D	False	False	False	False	
E	True	Either	True	True	
F	True	False	False	False	
G	True	False	True	True	
H	True	False	False	True	
Students	Art and Eugene	Bob	Dave	none	Cal

<u>Analysis of Student Responses to Problem 2</u>:

2. Can a statement of the form “If p then not p” ever be true? Explain your answer.

Problem 2 asked students whether a statement of the form “If p then not p” could ever be true. This problem was chosen because such statements would be considered absurd in natural language and yet could be true using the logical definition. Predictably, students who used schemes other than directly applying the logical definition answered that such statements would have to be false, using terms such as “contradiction” and “paradox” to describe them.

Art, however, (the same student who wrote a truth table before answering problem 1) wrote out the following truth table to describe the situation:

P	$\neg p$	$p \to \neg p$
T	F	F
T	F	F
F	T	T
F	T	T

Using the table, he determined that it can be true and even described it as being true "half the time." When asked if four rows were necessary or if two would have been sufficient, he answered that two would have been sufficient but he's used to writing four rows. Art is again showing the use of a scheme for determining the truth of an implication by first drawing a truth table and then filling it in and using it.

Eugene, on the other hand, thought about this problem for a significant amount of time. His initial reaction was that the answer was no because a statement cannot be simultaneously true and false. He gave the example "if it's Tuesday then it's not Tuesday." Then he considered "but what if it's Friday?" Ultimately, Eugene decided to write false and decided that a statement in such a form would be a contradiction since it would assume the truth of *p* and then imply the falsity of p. Art and Eugene both answered problem 1 identically, but used different schemes to get there. Art was able to transfer his scheme to problem 2, as the answer could be found through utilizing a truth table. However, Eugene's scheme, while working very efficiently for analyzing specific statements, was not able to accommodate a more general statement as well.

<u>Analysis of Student Responses to Problem 3</u>:

3. Are the following true or false? Explain why
   a. $\forall x \in R \; if \; x \in Q \; then \; x^2 \epsilon Q.$
   b. $For \; all \; x \; in \; the \; set \; of \; quadrilaterals, if \; x \; is \; a \; square, then \; x \; is \; a \; rectangle$
   c. $For \; all \; x \; in \; the \; set \; of \; quadrilaterals, if \; x \; is \; a \; rectangle, then \; x \; is \; a \; square$
   d. $\forall x \in R \; \exists \; y \in R \; s.t. xy = 1.$
   e. $\exists x \in R \; \forall \; y \in R \; s.t. xy = 1.$

When quantifiers were involved, students appeared to determine the truth of if-then statements in a much more consistent manner with each other. With one exception, all students answered that 3a and 3b were true and that 3c was false. The exception was that Cal answered false to 3b, but even this was consistent with what he believed to be the definition of a rectangle, as he specifically stated that rectangles have two pairs of equal length sides which are different lengths from each other.

No student verbally expressed any concern for the ambient set. For 3a, no student mentioned what would happen if they were working with an irrational number; they simply stated or showed that the square of a rational number is a rational number. Likewise, for 3b and 3c, no student mentioned what would happen if they looked at a quadrilateral that wasn't a square or a rectangle. Multiple students stated for parts 3b and 3c that all squares are rectangles but not all rectangles are squares.

One possibility is that students have developed a scheme that implicitly converts a statement of the form $\forall x \in A\ if\ x \in B\ then\ P(x)$ to the statement $\forall x \in B, P(x)$ or $\forall x \in A \cap B, P(x)$. Given the particular problems asked, these two forms actually result in the same responses, so additional questions which do not use strict subset containment could be used to explore that possibility. This is a case in which natural language helps rather than hinders the meaning of the if-then. In a quantified if-then statement, there becomes an assumption that the antecedent is true for the variable and so the possibility of the antecedent being false is no longer a worry. Although this allows students to determine the truth of quantified if-then statements in most situations it may cause trouble when the antecedent is always false.

<u>Analysis of Student Responses to Problem 4</u>:

4. On which of the following domains of discourse is the following statement true:
$\forall x \textit{ if } x > 2 \textit{ then } x \geq 3$?
N, Z, Q, R, (-∞,0]

Student answers to problem 4 appeared to be consistent with their answers to problems 1 and 3 and helped support the evidence of the described schemes used in the previous tasks. Art and Eugene both answered $\boldsymbol{N}, \boldsymbol{Z}$ and $(-\infty, 0]$. Art at first wanted to say that all of them were false because he could pick a number that was greater than 2 but less than 3. When asked what natural number is greater than 2 but less than 3, he immediately realized that it's true on the natural numbers and likewise the integers but says it's not true on $\boldsymbol{R}$ or $\boldsymbol{Q}$ because they could be decimals. For the last one, he first said false and explained that he couldn't even pick something above 2; after stating this, he noted that false implies false is true and so it's true. Eugene accepted $\boldsymbol{N}$ and $\boldsymbol{Z}$ quickly and just as quickly rejected $\boldsymbol{Q}$ and $\boldsymbol{R}$, noting 5/2 as an example that breaks it. Eugene called the last one tricky, but then noted that for the given set "x>2" is always false and so the statement as a whole is vacuously true.

Bob, like Eugene, also used the example of 2.5 to declare that the statement is false on both $\boldsymbol{Q}$ and $\boldsymbol{R}$, but affirmed that the statement was true on $\boldsymbol{N}$ and $\boldsymbol{Z}$ since the first number greater than 2 is 3. He says it will not hold on the last set because the base is always false. This shows a consistency with his answer to problem 3 in that he does not worry about the cases in which the antecedent is false in a quantified statement, while being consistent with problem 1 in that he requires an antecedent to be true.

Dave at first started writing false for his answers, using x=1 as a case where x is not greater than 2 (consistent with his conjunctive scheme applied in problem 1). Then he reread the statement and decided that he would have to start with using numbers greater than 2 (consistent with his scheme in problem 3, which first reduces the set to where the antecedent is true). He declared that the statement was false on $(-\infty, 0]$ because that set contained no number greater than 2 or greater than or equal to 3 and discarded ***R*** using the example 2 ½ as a real number greater than 2 but not greater than or equal to 3. His final answer was ***N***, ***Z*** and ***Q***, though in the case of ***Q*** it was unclear whether he actually understood what the set of rational numbers was. So, his scheme was similar to Bob's which should not be surprising given their similarities in answering problems 1 and 3.

Cal showed inconsistent reasoning. Like other students, he used 2 ½ as an example as to why the statement was false on ***Q***. However, he did not apply this same reasoning on ***R*** and said that it was true on ***R***. He said that it was false on the integers because the integers contain negative numbers, while declaring the statement true on the natural numbers.

All students determined that the statement was false on some set by using a counterexample. Four of five students used the same counterexample, whether expressed as $\frac{5}{2}, 2.5 \; or \; 2\frac{1}{2}$, while the other student said that it's false on ***Q*** and ***R*** because they contain decimals. This suggests that these students did have a scheme that tells them a statement of the form $\forall x \; P(x) \rightarrow Q(x)$ can be negated by $\exists x \; s.t. P(x) and \; \neg Q(x)$, whether or not this had been internalized. However, while all students determined that finding such a counterexample was a sufficient condition for negating the statement, it was not a necessary condition for the students who declared the statement false on $(-\infty, 0]$. Since these students' stated justification for answering that the proposition was false on $(-\infty, 0]$ was because the condition of "$x > 2$" could never be met, this could be seen as a consistent pattern with how they answered question 1. Under the scheme used by Bob, Dave and sometimes Cal, a simple if-then statement was negated by determining that the antecedent was false or by determining that the causal relationship did not hold. A quantified if-then statement could then be negated either by determining that the antecedent is always false or that there is a case in which the implication fails. In logical terms, this would mean that "$\forall x\ P(x) \rightarrow Q(x)$" was interpreted as $\exists x\ s.t. P(x)$ and $\forall x\ P(x) \rightarrow Q(x)$ which could rightfully be negated by $\forall x, \neg P(x)$ or $\exists x\ s.t. P(x) and\ \neg Q(x)$, that is by finding a counter-example or by determining that the antecedent is always false.

The existence of such an interpretation would also explain a finding by Stylianides. In his study, Stylianides found that some students believed that: $[\forall k \in N, P(k) \rightarrow P(k+1)] \rightarrow \forall n \in S, P(n)$ where *S* is a proper subset of *N* of the form $\{n | n \geq m > 1, m, n \in N\}$. Such a statement would be implied by the use of the scheme attributed to Bob and Dave for evaluation quantified if-then statements.

<u>Analysis of Student Responses to Problem 6</u>:

5. Given a domain of discourse *D*, what is a propositional function on *D*?

6. Consider a propositional function P on the set {1,2,3,4}. What truth values of P(1), P(2), P(3) and P(4) would result in $P(1) \rightarrow P(2), P(2) \rightarrow P(3), and\ P(3) \rightarrow P(4)$ all being true? If we extended the domain of P to N, then what would need to be true to have the statement $\forall n\ P(n) \rightarrow P(n+1)$ be true?

On problem 6, students who understood the meaning of implication appeared to be able to determine whether the sequence of implications must be true and were able to find examples that satisfied the condition but were not able to generate all the different possibilities in either the finite or the infinite case. Problem 5 was included primarily because the term "propositional function" is not always used. In fact, none of the students responded that they were familiar with the term and had it explained to them.

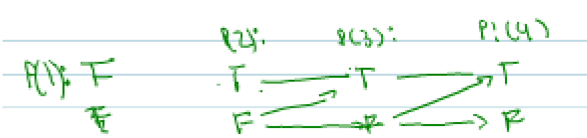
Art first noted the possibility that all statements were true. When asked if that was the only way, he said that all false would work because the implication gets evaluated before the "and." Under further analysis, he determined that making $P(1)$ be false would not restrict the truth values of any of the other statements, but that assigning false values elsewhere could create the problem of having a false implies true. After determining that $P(1)$ could be false and the others true, he made no visible effort to extend this to having both $P(1)$ and $P(2)$ being false. For both the finite and infinite cases of problem 6, he gave the same answer: they could all be true, all false or $P(1)$ false with all the remaining ones being true.

Eugene appeared to use similar reasoning to Art. His first expressed thought, like Art was that P had to be true at all values. Then he considered if statements were false, noting that if *P(1)* is false then $P(1) \to P(2)$ must be true. He then chained together falses and until he got to the last place and decided that it could be anything. After he was satisfied that TTTT, FFFF and FFFT (he had first grouped these as FFFX) worked I asked if he could come up with any others. He said "we have four things so there are usually four things." He then came up with FTTT as his last case. This could be a case in which Eugene's mathematical experience led him astray, deciding that this problem likely fit a pattern in which the size of the domain was equal to the size of the solution set and so once he found four solutions with the given property he assumed he was done.

In generalizing to $\boldsymbol{N}$, Eugene's first two answers were that they could either all be false or all be true, then that the first one could be false but the rest all be true. He then said if they're all false but "the last one." As he struggles with what this could mean, I asked if he could have a case in which $P(1000000)$ is false and $P(1000001)$ is true and what that would mean for $P(1000002)$. He said it would have to be true but that he couldn't know what the next value was.

He then said that he realized what is problem was: there is no last natural number, so the case in which the last one is false and the rest are true does not carry over when the domain is $\boldsymbol{N}$ but the other three cases do. Eugene showed an ability to transfer his solutions to a small problem to a larger version of the problem, noting that three solutions carried over directly and that the fourth could not since it relied on the fact that $\{1,2,3,4\}$ contains a last number. Unfortunately, he missed one case in the smaller version, FFTT, possibly due to an empirical proof scheme in which he decided the size of his solution set was equal to the size of the domain of P, and the case that showed the full generality, that one could have any number of false statements followed only by true statements.

Interestingly, the student who employed the most effective scheme was Cal, who was able to generate all possible truth values in the finite case. Although Cal had employed inconsistent methods for evaluating the truth of if-then statements in previous problems, here he noted that $P(1) \rightarrow P(2)$ is false only when $P(1)$ is true and $P(2)$ is false. It was clear from this problem that Cal was aware of the logical definition of implication even though he had not used it in previous problems. He noted that if $P(1)$ were true then $P(2)$ would have to be true as well, but if $P(1)$ was false that $P(2)$ could be true or false. He then noted that $P(3)$ could be true or false as well, but would have to be true if $P(2)$ was true. I asked if he could draw a tree diagram that shows all the possibilities since the truth values had a dependence relation. He drew the following diagram which showed quite nicely the possible truth values and noted that any set of truth values would work as long as a false value never succeeded a true value.



I then asked if $P(1)$ could be true since that case was not listed and he decided that it could, but only if $P(2)$, $P(3)$ and $P(4)$ were all true as well. When generalizing to $\boldsymbol{N}$ he states that he could have them all be false or he could branch off to true but if he did they would all have to be true after that. When asked what possible truth sets he could have though, he appeared to get confused. The more time that was spent on this problem, the less coherent his responses seemed to get. Like Eugene, he doubted what he can say about specific truth values because subsequent truth values were unknown. As Harel notes, students often have trouble when working with infinite processes and so setting the truth values of infinitely many things at once may be difficult.

Dave said that he likes to use examples and does not like to think generally. He thought about how he would try to plug in variables and writes statements like $P(n) < P(n+1)$. When asked if two propositions can be compared with a less than sign he responds "not logically, no." In the end he decides that we want P to be true everywhere. This is consistent with his established pattern of evaluating $p \rightarrow q$ and p and q. However, the functional notation seems to be a barrier for him.

Bob mentioned induction on this problem and said that we are showing something is an inductive set. When asked what he means by an inductive set he responds "I think in abstract algebra terms it would have to be one to one." When asked about the truth values of P, he notes that if $P(n+1)$ is true then $P(n+2)$ is true and so on, showing that he may have an idea that once a statement is true, all subsequent statements must be true as well, but it's difficult to make any conclusions about Bob's response to this question. He appeared to be dealing with interference, referencing multiple mathematical terms, some of which were not related to the task at hand.

<u>Analysis of Student Responses on Problem 7</u>:

7. Is there a formula for the sum of the first n odd numbers? How would you show that this formula would continue to hold?

On this problem, the two MTH 355 students were able to determine that the outputs were square numbers and were able to show this using an induction like process. Eugene focused on the differences of subsequent terms and wrote $n^2 - (n-1)^2 = 2n - 1$, then noted that $2n - 1$ gave us the odd number that would be added to get to the n^{th} term. Bob, on the other hand, when asked how he would show that the sum of the first *n* odd numbers is n^2wrote $(x+1)^2 = x^2 + 2x + 1$ then noted that this is equal to the previous square added to the next odd number. Bob mentioned the word induction when he did this, suggesting along with his response to problem 6 that he does have a sense that induction is used when there is some sort of sequential process involved.

Art first identified the pattern, stating that "each number is the previous number plus the next odd number." It's not until he computed up the fifth sum that he noted that these are square numbers. Although he was able to construct both a recursive process for calculating sums and found an explicit formula for what they would be, when he is asked how he would show that this formula would continue to hold, he says that he would pick a number down the line and check it. Art was also able to do problem 8 and again was able to find both a recursive process and an explicit formula and again when asked how he could know this would continue to hold, he said to pick a large value of *n* and see if it works. Art expressed a clear empirical proof scheme when analyzing this problem. Although he does not appear to be completely satisfied with that answer, it seems he felt that's the best that could be done.

Cal and Dave were unable to determine that the sum of the first *n* odd numbers were square numbers. Both of these students needed a further explanation of what the question was asking. Dave, after quite a bit of work, was able to express it as a sum $\sum_{i=1}^{n}(2i-1)$ but kept trying to work with this sum, rather than look at specific numbers. This is despite him saying that he doesn't work well with general statements. Cal did, with help in explaining what the question was asking, compute the values up through 36, but did not identify these numbers as square numbers, only noting that they alternate between odd and even. Dave, as well, when asked to compute values, computed up to 9 and noted that it was an odd number. Perhaps the problem statement asking to have them add odd numbers played a role in their thinking that a critical component of the answer would be based on the numbers being odd or even.

The range of results for this question aligns with expectations from the literature. Two students used a transformational scheme to get from one square number to the next, one focusing on addition and the other on subtraction. A third student was convinced by an empirical scheme that the sums would result in square numbers, while two students were unable to find solutions at all.

<u>Analysis of Student Responses to Problem 11</u>:

11. In a version of takeaway, you and your opponent take turns and may choose to take either 1, 2, or 3 pennies each turn. The player who takes the last penny wins. If the game starts with 6 pennies is there a winning strategy if you pick first? What is it? For an arbitrary number of starting pennies, how can you tell which player has a winning strategy?

Four of five students had enough time to discuss the "Takeaway" problem (Dave ran out of time before this). Problems 8, 9 and 10 were skipped for all students, except for Art. All four students were able to determine that leaving four pennies was a winning strategy. However, only Art was able to generalize this and determine that leaving a multiple of four pennies is also a winning strategy for larger versions of the game. Art's key insight was stated as "I don't want my opponent to be able to leave four." He determined that the first number for which this was possible was 8, noting these were multiples of 4, he then considered if he left 12 and decided that he'd be able to leave 8, deciding when he left a multiple of 4, he could keep leaving smaller multiples of 4, he was satisfied.

The other students, however, kept trying to look at various cases of starting values and consider all the scenarios that could play out from there and did not use the fact that they could stop once they'd reached a point that was previously solved for. Because of this, the cases became too complicated for them to go through even for small values of n, though Eugene noted that if he started with 8 pennies he would lose and may have had solutions up to $n=12$.

With Bob, I suggested working backwards and he got the "solution" to leave $3n+1$ pennies. His justification was that he knew leaving 4 pennies was a winning strategy and also that no matter how many pennies were left on his turn that leaving $3n+1$ pennies would always be possible. The fact that the possibilities of numbers of pennies left to him by his opponent could also be of the form $3n+1$ was not verbally acknowledged. This was an interesting response in that inductive ideas were present but were misapplied in answering the specific question.

Students in the first interview showed a diverse range of knowledge and ability with respect to concepts which Dubinsky, Harel and others have written are important for students to be able to understand induction. Students Art and Eugene showed that they understood the meaning of the logical implication which is required for the definition of induction. Cal also displayed an understanding of logical implication, though highly contextualized, and was able to apply it to an induction-like process on a finite set, but not an infinite one.

Art also showed the process pattern generalization skills required for a proof by mathematical induction problem, while students Bob and Eugene, perhaps due to their richer mathematical background, were able to do pseudo-induction to show the sum of the first n odd numbers is n^2.

Based on the data collected in the first interview, I hypothesized that Art and Eugene would have the knowledge base necessary to understand the Principle of Mathematical Induction and develop schemes for implementing it. I was less optimistic about Bob and Dave as they did not display an understanding of the logical basis of induction. Cal was difficult to make any predictions about. The inconsistencies shown throughout his work, as well as his struggles to understand certain problem statements was disconcerting, but on problem 6, which was the problem most closely related to the definition of induction, he showed an understanding of the nature of the problem, perhaps more strongly than any other student. Unfortunately, Cal did not participate in a second interview and so we have no further data on him. The following chart summarizes student responses to problems 1, 2, 4, 6, 7 and 11. The responses to problem 3 were all similar enough to not be distinguished from one another.

Analysis of Student Responses on Problems 1 and 2 of Interview 2:

1. Define the Principle of Mathematical Induction.
2. Why are both the base case and the inductive step necessary?

There were some patterns that emerged in the second interview. In the first problem, three of the four students said that PMI was used to show that a formula or equation holds, a limited view of induction noted by Ernst, Harel and others. Only Eugene answered that PMI was used to show that a proposition holds for any value from a set.

Art and Bob both showed that they have an understanding of induction being associated with finding relationships between certain terms. Bob had a notion of an “inductive set,” a term he used in the previous interview but had difficulty explaining, saying that an inductive set is one in which each term follows another. Art used a metaphor of fire escapes saying that a base floor must exist and that a relationship must exist between floors. He notes that in a proof by induction that the base case shows a formula holds for the first one and that the induction step shows “how you get from one thing to the next.” When asked if there was a logical term for the relationship he said it was implication. Dave showed that he has some idea that induction involves a relationship between consecutive terms, as he talked about replacing a variable with its predecessor, though he admitted to being unsure what he meant by this.

Analysis of Student Responses on Problem 3a:

> 3a. In a version of takeaway, you and your opponent take turns and may choose to take either 1, 2, or 3 pennies each turn. Prove that if you can leave a multiple of four pennies, then you have a winning strategy?

Given that students had developed a notion that PMI was used to prove that a formula, and usually a summation or product formula, would hold, it was not surprising that students struggled applying PMI to a problem such as proving that leaving a multiple of four pennies was a winning strategy in Takeaway. Art said that he doesn’t even know how he’d set it up, noting there was no right or left side of an equation to look at. Bob was able to note that given a multiple of 4, that he can leave the previous multiple of 4. He noted that after leaving $4n$ pennies that on his next turn, he’d be able to $4(n-1)$ pennies and gave an explanation why, but said he felt like he wasn’t using induction. Interestingly, this student was able to reason why a solution works, in this case why leaving multiples of 4 is a winning strategy, using a process that is the core principle of induction and yet feel like he isn't using induction. He was eventually able to write a full proof, but not without significant scaffolding.

Eugene was able to write a proof for this problem, but struggled with what the induction means. For the base case, he noted that it would be the situation in which it's his opponents turn to pick and there are four pennies left. For the inductive step, he noted an issue that if $n=4q$ then the next multiple of 4 would be $4q+4$, though noted this is also $4(q+1)$. In doing induction he decides that he can simply add 4 instead of 1 to the value of pennies that he knows is a winning strategy and is able to write a proof for this. It should be noted that he had some difficulty with this and first proved that if a value n is a multiple of 4 that $n+4$ is also multiple of 4 as well and had to be reminded that he was trying to prove that he had a winning strategy by leaving $n+4$ pennies.

Dave was able to understand, though needed an explanation, as to why somebody leaving a multiple of four would then be able to leave a smaller multiple of four the next time. However when asked if this related to induction, he was unsure in his response. When asked to state what the base case would be, he said that it is tough to think generally when the base case is one statement and there are so many cases for what people could choose.

Analysis of Student Responses on Problem 3b:

3b. Prove that the sum of the first n odd natural numbers is n^2.

Problem 3b represented a more classical induction problem in which one has to prove that a summation formula holds for any natural number. Bob and Eugene were able to write mostly complete proofs for this one, just as they had been able to justify this statement with pseudo-induction on the previous interview. Both Bob and Eugene struggled with getting indices correct on their summations at first (they both used sigma notation for the sum) but were able to write the solution correctly. Bob never wrote an inductive hypothesis even though he used one in the proof.

Art and Dave both struggled with this problem and there were similarities. Both students used general subscripts for writing the first n odd numbers e.g. $2x_1 - 1 + 2x_2 - 1 + \cdots 2x_n - 1$ even though when asked, they stated that they had specific values and that $x_1 = 1, x_2 = 2, etc.$ Also, both students truncated their sum in their attempted proofs, reducing to the last one or two terms on the left hand side of the equation, writing statements such as $2n - 1 + 2n + 1 = (n+1)^2$. As far as I know, none of the literature notes students doing this. Given that Art and Dave were in the same Discrete Math class, it could be a unique experience in this class led to this.

<u>Analysis of Student Responses on Problem 5</u>:

> 5. Suppose P is a propositional function on Z and that:
> $$i)\ P(0) is\ true\ and\ ii)\ \forall n P(n) \rightarrow P(n+3)$$
> Then, on what set, if any, can we determine that P is true? On what set, if any, can we determine that P is false?

Art began by being unsure what the statement is asking by “what set,” and notes that P(0) would be the base case and $P(n) \rightarrow P(n+3)$ would be the inductive step but with an n+3 instead of an n+1. He said he didn’t know what to do because he didn’t know what P was, other than he thought it was a function. I had him compare the statement of the problem to problem 6 in the previous interview but he did not see a connection. I asked him to construct a truth table since that is what he did during the previous interview whenever implication came up. He was able to write a truth table with $(0), P(3) and\ P(0) \rightarrow P(3)$. After this, he was able to determine that P(3) had to be true and was able to generalize this for all multiples of 3. He said that the truth value could not be determined elsewhere and noted that even if we knew a value for which it was false, it could still be true for a value 3 greater. He was then able to answer problem 6 without trouble, noting that since the implications let him go both up and down by 1 that he can cover all integers.

Being reminded of the truth table appeared to be a significant scaffold suggestion for Art during the second interview. After this, he was able to give an explanation for why a suppose statement is necessary in an induction proof, as an implication is being proven. He also explained that the base case is necessary since establishing an implication is true does not require any of the statements to be true and so the first statement would have to be proven on its own. Although Art had been successfully using a scheme that linked implication to a truth table during the time of interview 1, at some point in time before interview 2 he stopped using such schemes. He still was a bit shaky on what P would actually represent as seen by his response to problem 3e afterward when he asked if $P(n)$ would be $2^n or\ n!$. I asked him if 2^n could be a true statement and at this point he noted that $P(n)$ would have to be the entire statement "$2^n < n!$." This suggests that difficulties for Art may have stemmed from his notion of a function.

Bob was exposed to problem 5 earlier in the interview, after problem 1, and was asked if the statements were related to induction at all. He said that the only difference was an n+3 instead of an n+1 and it was at this time that Bob talked about inductive sets being those in which one element comes after another. He said the base case gives a starting place and the inductive step gives a relationship for how elements come one after another. Due to time constraints, Bob did not give an answer to this problem.

Dave noted that the second statement would mean that he'd have the statements $P(1) \rightarrow P(4), P(2) \rightarrow P(5), etc.$ And so P would have to be true everywhere. This is again consistent with his view in the first interview that an implication is true when both its antecedent and consequent are true. However, he doubts this. Next, he also notes that he knows both $P(0)\ and\ P(0) \rightarrow P(3)$ are true. I asked him if this implies anything about $P(3)$. He responds no and claims that it is so general that he doesn't know what it means. At this point, Dave appears to be overwhelmed by mathematical notation. He has conflicting ideas for what things could mean and is struggles to make sense of it.

Finally, Eugene noted that he had $P(0) and\ P(0) \rightarrow P(3)$ and concluded that P(3) had to be true. His final answer was that $P(0)$ and $P(3)$ had to be true and that at other values P could potentially be true or false. Eugene did not spend much time on this problem, being satisfied with his answer. As he was only given $P(0)$, only one implication was not able to be immediately rejected as vacuously true. Eugene showed in previous problems that he has the schemes necessary to continue to build off of statements that must be sequentially set to true as he showed on problem 6 of the previous interview and may have simply been satisfied with the first answer he came up with.

For future studies, I would expect for there to be three somewhat common responses to this question: the correct response of non-negative multiples of 3, the response of all integers for those who either interpret implication as an 'and' statement or focus on the symbols $\forall n\ P(n)$, and finally the set {0,3} for those who notice the first implication, but do not realize that an infinite sequence of truth values builds off of it.

Chapter 5: Conclusions

Comparisons to Other Studies

A number of observations found in this study support previously made observations. Previous studies have suggested that students think about the structure of a proof by PMI as being a circular argument in nature. When asked whether the inductive hypothesis assumes that a statement is true for all values of n or for one specific value of n, Art and Bob both responded that the assumption was made for all values of n. When asked if this constituted circular reasoning, Art first responded in the affirmative but then answered that he wasn't sure what circular reasoning meant. Bob responded that he hadn't thought about it before but reasoned that it would have to constitute circular reasoning. He expressed dissatisfaction with such a conclusion, but maintained "that is the definition of induction, though."

Bob's response suggests that he had sufficiently developed logical schemes to determine that supposing a propositional function was true for all input values would constitute circular reasoning and that this was problematic. However, he had formed a belief that such a hypothesis was used in a proof by PMI and never had this belief challenged. This is not the only instance of such a phenomenon taking place. Eugene was asked upon which sets could PMI be used. His initial response was that PMI could be used on any set for which adding 1 made sense including the integers, the reals and the complex numbers. When justifying why it could work on the real numbers he said "if you have that it's true for pi, then you have that it's true for pi plus one … but you don't have that it's true for pi plus the square root of two, so I'm going to say the integers."

The Difficulty of Functions

One observation throughout the interviews was that students had difficulties writing down statements in a mathematical form and using propositional functions. While Bob and Eugene were both able to write a proof for problem 3b, Art and Dave were unable to convert this into a mathematical statement which they could prove and so didn't even have the opportunity to apply induction. Problem 3a for Art was particularly telling, because Art was able to find the solution and a transformational process to generate the given solution but responded that he could not think of how this problem could relate to induction. It would appear that the schemes that Art, Bob and Dave had developed for identifying when induction can be used, limits them to identifying problems which involve verifying a formula, rather than any problem which involves applying a repeated process. Because of this, their schemes for induction did not appear to include the translation of a statement into a mathematical form upon which induction could be applied. Even when Bob was able to do this under heavy scaffolding, he said it didn't feel like he was using induction, suggesting that the instruction has not sufficiently shown the broadness of the application of PMI.

Another observed phenomenon was that students may possess knowledge and schemes sufficient to inform their understanding of PMI, but not apply such knowledge. This was especially apparent with Art on Problem 5. On this problem, he was able to derive the truth set of P only after being reminded of his use of the truth tables. After this, Art was able to explain what the inductive step proves and the necessity of the base case in a much more succinct manner than at the beginning of the interview. Interpreting functions was still a difficulty for Art even after this as evidenced by his question on a subsequent problem "Would P(n) be 2^n or $n!$?" to which I asked if either of those could be a true statement and he then determined that the entire statement "$2^n < n!$" would have to be the statement supposed in the hypothesis. Unfortunately, there was insufficient time in the interview to examine whether the scaffolding process resulted in an improved ability of Art for writing proofs by PMI, though he appeared to show greater understanding once he was reminded of his previously used schemes.

This was particularly interesting because Art had previously used truth tables very effectively and had written down truth tables whenever implication appeared during the first interview. From his responses, it was clear that Art never forgot how to use truth tables, but rather had either stopped using them or had not learned to use them in new contexts.

Limitations

The participants in this study were volunteers and so the results might not be as generalizable as they would be with randomly selected students. All four case studies were done at the same university and students were in one of two sections, so some experiences may also have been unique to the school or instructor. Additionally, this study only examined how students responded to the given tasks. No data was collected with respect to how the course was taught.

Consequences for Future Studies

The least examined branch of Dubinsky's Genetic Decomposition for PMI was the notion of a function. In the first interview, students were simply told what propositional functions were for the sake of examining whether they could generate truth set via modus ponens in problem 6. Students were not asked to interpret propositional functions or to write statements in common language in a mathematical form. Although a simple description of propositional functions seemed to be sufficient for students to work with problem 6, and three of five students were able to obtain partial solutions, this problem did not give much insight into students' notion of a function, nor was it designed to do so.

The case study of Art, in particular, suggests that this is a topic that deserves more attention. My hypothesis after the first interview was that Art would be able to effectively apply the Principle of Mathematical Induction, given his use of the logical if-then, his ability to apply modus ponens and to generalize patterns. However, there was little data collected on Art's ability to interpret or translate mathematical statements.

These case studies suggest that the notion of a function and the ability to interpret mathematical statements or translate an idea into a mathematical statement is a barrier for students working in higher mathematics. Additionally, some students develop a scheme that interprets a statement of the form $\forall x\ P(x) \rightarrow Q(x)$ as one of the form $\exists x, P(x)\ and\ \forall x\ P(x) - Q(x)$. In many cases, this interpretation does not cause problems, as in mathematics we rarely prove vacuous statements, but it may cause problems with interpreting induction, particularly with regards to the necessity of the base case. There were also other cases in which students were found to have false beliefs, such as the belief that one supposes the induction hypothesis for all natural numbers or that PMI can be used to prove that a statement is true for all real numbers, and showed the ability to determine that these beliefs were false but had never had these beliefs challenged and thus never went through a stage of disequilibrium to change these beliefs.

Finally, it was found that students often do not relate induction to recursive processes, but rather associate induction with equations which typically involve a finite sum or product. This suggests that Harel's instructional method of introducing recursive problems as a precursor to induction may be more effective in portraying to students when and why PMI is used. At least one of these case studies also suggests that it may be helpful for students to see PMI as a single case of using modus ponens repeatedly to generate a truth. Thus, PMI rather than being its own technique may be assimilated into the schemes of applying modus ponens, but further research on this is still required.

Implications for Instruction

These case studies showed that students, even in higher level courses, have difficulty interpreting and using mathematical notation. Although direct use of propositional functions may not be necessary, and these students had not used such terminology, understanding how to write and interpret statements using variables is crucial to proof writing in general and PMI specifically. As Harel noted in his work, there is often a conceptual gap between seeing a statement such as $\sum_{j=1}^{n} 2j - 1 = n^2$and perceiving a sequence of statements 1 = 1, 1+3 = 4, 1+3+5 = 9, etc. Seeing our results in light of Harel's work suggests that a greater emphasis needs to be placed on understanding variables, what they represent, and how to write and interpret statements that use them. An example task to motivate this understanding might be something like "write an example of a particular statement proven by the induction step in the proof that $\sum_{j=1}^{n} 2j - 1 = n^2$." One such example would be "$1 + 3 = 4 \rightarrow 1 + 3 + 5 = 9$," which shows what the variable *n* represents, as well as motivates the procedure used within the induction step to get from one statement to the next. Similarly, in the "takeaway" problem such an example may be "if leaving 16 pennies is a winning strategy then leaving 20 pennies is a winning strategy," again motivating a procedure for such a strategy. Additionally, in the proofs, it should be expressed as clearly as possible what the variable represent. The statement "Suppose $\sum_{j=1}^{n} 2j - 1 = n^2$" does not show that *n* represents a fixed number, while "Fix $n \in N$ with the property that $\sum_{j=1}^{n} 2j - 1 = n^2$" expresses this more directly.

Additionally, to help students assimilate PMI into their schemes for modus ponens, it may be useful to give students tasks, such as task 5 from interview 2, followed by the inverse of such problems like “How can you modify PMI to prove that a statement is true for all even natural numbers? All integers? All even numbers that are not multiples of four?” Such tasks necessitate students to consider the variable with which they are working and the relationships which they need to have hold. Following the DNR model, how students think about logic and how to express their ideas mathematically affects how they approach tasks such as induction, such tasks bring the necessity of interpreting and expressing mathematical statements and they must follow consistent patterns of reasoning between tasks.

Finally, this study identified a number of different schemes that students employ for interpreting “if-then” statements. The results of employing such schemes are identifiable as long as they are used consistently. This may be used to help identify when students are misusing logic and what is causing them to do so, making such mistakes easier to address.

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