AN ABSTRACT OF THE THESIS OF

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Title: Nonlinear Free Boundary Problems Arising From Melting Processes

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We discuss a mathematical model arising in the melting of a fluid in two spatial dimensions and in time. The model leads to a free boundary value problem for determining the location of the interface as well as the temperature distribution. The movement of the interface depends on the temperatures, the velocities and the material properties at the interface through conditions of dynamical compatibility for energy transfer. In this study, we assume that the density and pressure are constant. A numerical approximation making use of finite differences and the maximum principle is used to present existence and uniqueness theorems and continuous dependence of the solution on the data.

Finally, numerical algorithms for finding approximate solutions and the results of numerical calculations are given.

Nonlinear Free Boundary Problems

Arising From Melting Processes

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TABLE OF CONTENTS

LIST OF FIGURES

Nonlinear Free Boundary Problems Arising From Melting Processes

1. Introduction

In describing change of phase processes (for example, evaporation, freezing, melting, dissolution, etc.), one is frequently faced with a boundary value problem for the heat equation in a domain whose boundary is unknown but which must be determined together with the solution of the differential equations. These phase changes are assumed to take place at some specified temperature and pressure. Such problems are often referred to as Stefan problems.

In this thesis, we shall consider a Stefan problem for parabolic equations in two spatial dimensions and in time. We shall be given an initial condition and an additional condition on the free boundary; namely, the equation for the conservation of energy. We shall think of one phase as occupying a given domain in the xy-plane at the time $t > 0$ with a prescribed initial temperature at time zero and a prescribed flux on the boundary of the domain. We need to determine the temperature of the domain and the location of the free boundary at time t, that is, we shall find the temperature of this phase and the location of the interface between the two phases. The typical problem involves the melting of ice and for simplicity we shall think of that problem in the discussion below. In the melting of ice the interface moves relatively slowly so we can solve the equations on a fixed rectangular grid using finite difference approximations. To track the melting front along grid lines, we will use spline interpolation between grid points. The melting front moves in the direction of the normal to the boundary and depends on the flux condition on the boundary which is occupied at time t. The melting-point of ice means the temperature at which ice is in equilibrium with the adjacent water under the existing pressure. It varies with the pressure and with the purity of the water, both of which are neglected here. The melting temperature differs from point to point, so some portions of a mass of ice melt at a temperature slightly under 0°C, while others require a temperature slightly over 0°C to liquefy them. The consequence is that such a mass will have some of its parts solid and some of its parts liquid. Thus, it is possible that there will not be a sharp boundary between the liquid and the solid phases, but rather a small region where the phases coexist (see Fig. 1.1).

In our case, if we drop hot water onto a single block of ice, then it melts and breaks up into several small ice islands. At that time the ice and water interface (melting front) will not move until all the ice islands have been completely converted into water. There will be a waiting time before melting front moves. We call that stage a "mushy region". In the mushy region, we need to refine our time steps and the mesh points if it is necessary. Physically, the mushy region is a thin region and we shall assume that the ice/water front is sharp so that it is determined locally by a function.

The governing equations of the physical situation are derived in section 2.1 and a complete statement of the problem is given in section 2.2. The literature survey is done in section 2.3.

In the third chapter, we will prove existence and uniqueness theorems for weak solution and discuss variational inequalities. As part of the proof of the existence of a weak solution, we will find some a priori properties of the difference approximation which we will need later in our convergence proofs in Chapter 4.

The fourth chapter deals with a numerical method. A finite difference algorithm is used to get the numerical solution of the temperature of the domain at time t and the normal derivative by a difference quotient is used to find the free boundary. Also, we will give convergence proofs of the difference schemes.

Finally, in Chapter 5, we will solve an example using a numerical procedure which is discussed in Chapter 4 and present the results of the location of the front.

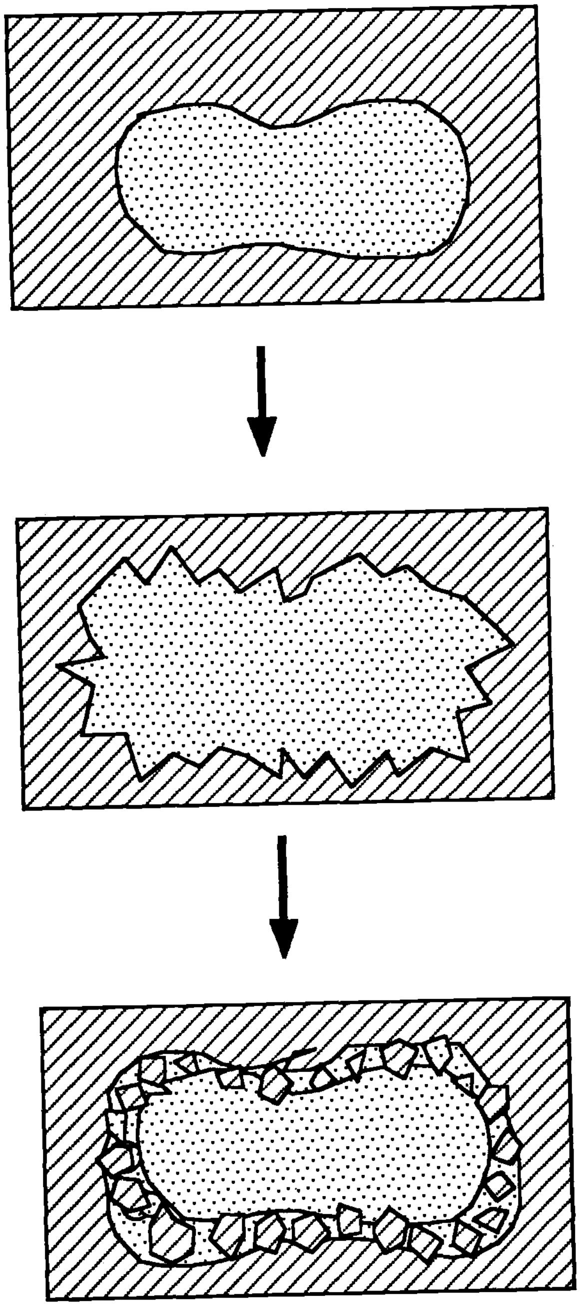


Figure 1.1. Possible Development of a Mushy Region

The Dotted Region is Liquid

The Shaded Region is Solid

2. Physical Description

2.1. Statement of the Physical Problem

The phenomenon of melting is very important in industrial processes such as the casting of metals in foundries , arc welding and the melting of ice.

We shall consider a thin block of ice occupying a domain $-\infty < x < \infty$ and $-\infty < y < \infty$, and drop hot water onto the ice. As the ice starts to melt, the boundary between the water and ice moves. Let us focus our attention on a domain which is occupied by the water and the interface between the water and ice.

The density and pressure of both phases, liquid and solid, are assumed to be the same in the neighborhood of the interface and the liquid remains stationary so that heat is transferred through it only by conduction. We assume that the temperature distribution of the hot water depends on space variables and the temperature distribution of the ice is everywhere 0°C and no internal pressures build up. So we shall deal with the one-phase Stefan problem. The ice will begin to melt and for every time $t > 0$ water will occupy a certain domain. When a body of ice having the shape Fig. 2.1 keeps growing, the interface AB and CD may coincide. Then in the next moment the whole joint boundary will disappear and we will have an ice island. So the free boundary varies in a discontinuous manner and the mushy region starts. This process will continue until the total energy available to the system goes to zero. This energy depends upon the temperature. Our problem is to find the temperature of the domain and the free boundary at any time $t > 0$.

Another problem, which is of great importance in soil science, involves the study of water invading a dry medium. We shall think of a homogeneous, dry, porous medium which is assumed to consist of a large soil slab so that physical properties are determined by two variables. As time passes, the incompressible liquid, in this case, water, will flow toward areas of lower pressure. The unknown boundary, which is called the free boundary, is the wetting front. Let us suppose that the initial distribution of the moisture is known as a function of the space variables. For time $t > 0$, the free boundary is determined by the water flux that is a prescribed function of time. The problem is to find the moisture distribution and the location of the wetting front at any later time $t > 0$.

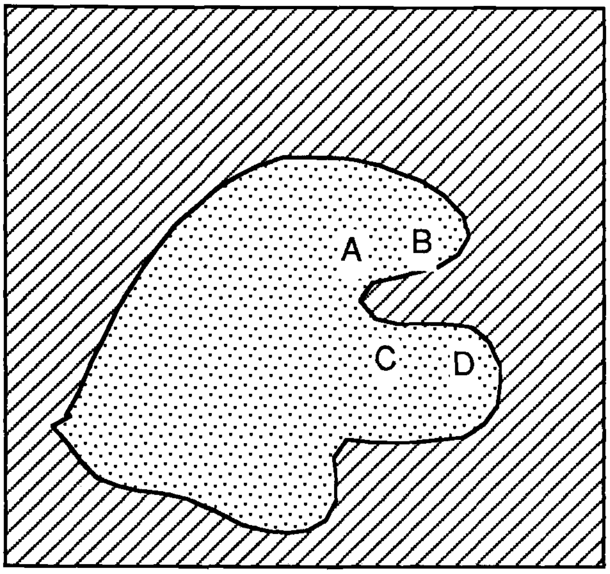


Figure 2.1. A body of Ice

2.2. Statement of the Problem

In the previous section the problem we discussed a change of phase problem in two spatial variables, x and y, and time t. If we think of this problem as the melting ice problem and denote the water temperature which depends on x , y and t by u, then this problem can be formulated as follows.

<u>Problem 1</u>. Find a function $u=u(x,y,t)$ and a domain $\Omega_t = \bigcup_{0 \le \tau \le t} G_\tau \times \{\tau\}$ with boundary $\partial\Omega_t$, the free boundary, such that

$$u_t = a\Delta u \qquad \text{in } \Omega_t, \quad t > 0 \tag{2.1}$$

$$u(x,\ y,\ 0) = \phi(x,y) \qquad \text{on } G_0, \quad t = 0 \tag{2.2}$$

$$u(x,y,t) = f(x,\ y,\ t) = 0 \text{ on } \partial\Omega_t, \quad t \ge 0 \tag{2.3}$$

$$f_t + v\cdot\nabla f = 0 \qquad \text{on } \partial\Omega_t, \quad t \ge 0 \tag{2.4}$$

where a is thermal diffusivity. Without any real loss of generality we will take $a=1$. G_t will denote a bounded domain at time $t > 0$ in two dimensional Euclidean space, $\mathbb{R}^2$, $\overline{G_t}$ denotes its closure and ∂G_t its free boundary. v is the velocity at the position (x,y) at the time t. The operator Δ is called the Laplace operator (i.e. $\Delta u = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$, $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ is the gradient operator and u_t is the partial derivative of u with respect to t. In general, partial derivatives with respect to the time and spatial variables will be denoted by subscripts. Finally, f and ϕ are assumed to be given functions.

2.3. Literature Survey

There have been many developments in the theory and applications of free boundary value problems. The historical development is summarized in L. I. Rubinstein [43] and I. I. Kolodner [33]. Important developments in the study of the free boundary problems have been achieved using by a variational approach which seeks the solution in some "weak" sense. Variational approaches of free boundary value problems are to be found in C. M. Elliott and J. R. Ockendon (1982), A. Friedman (1982), D. Kinderlehrer and G. Stampacchia (1980), D. G. Wilson and Solomon and P. T. Boggs (1977), and J. R. Ockendon and W. R. Hodgkins (1975). Because of the large literature, we will discuss only those papers related to our work. The method mentioned in this thesis is based on the theory introduced by S. Kamin [29] and further developed by A. Friedman [20] for several dimensional Stefan problems.

In [21], Friedman considered the case when the temperature at the boundary is prescribed and water is present at the beginning. His method was a modification of the method of successive approximations. Lazaridis [37] developed a numerical technique with which to treat heat-transfer problems involving a change of phase for the multidimensional problem. But he did not give any proofs for existence and uniqueness and only calculated the location of the interface. In [47], Peckover and Turland considered a wide rectangular block of a solid which is heated on its top horizontal surface and cooled on its horizontal bottom surface. The local stability of the horizontal interface plane (melting front) is studied when the front between the solid phase and the liquid phase is stationary. They considered both semi-infinite media and layers of finite depth.

In [38], G. H. Meyer used an enthalpy transformation for some multidimensional problems and absorption of the phase transition process into the diffusion equation. If this transformation is not applicable, he used a locally one-dimensional Gauss-Seidel type front tracking method coupled with invariant imbedding.

In concluding this survey, we mention the work of Fasano and Primicerio in [12] – [16]. They gave a detailed discussion for change of phase processes in one spatial dimension with the aim of giving an outline of the main features of the mathematical problems related to such phenomena. Moreover, they gave an example and proved the well-posedness of the problem in the classical sense and obtained a better understanding of the typical behaviour of systems with non-uniform melting temperatures.

3. Mathematical Background

3.1 Reduction of the classical problem to a generalized one

Let G_t be a bounded domain in $\mathbb{R}^2$, whose boundary consists of the surface ∂G_t. For any t, $0 \le t \le T$, let $\Omega_t = \bigcup_{0 \le \tau \le t} G_\tau \times \{\tau\}$.

If the ice phase is known to be at temperature 0°C (or very nearly so), then we can assume the temperature of the ice to be 0 and $u(x,y,t)$ to be the temperature of the water. This resulting problem is called the one-phase Stefan problem, which we formulated in the previous section as follows.

Consider the following system of equations for u:

$$u_t = u_{xx} + u_{yy} \quad \text{for } (x, y) \in \Omega_t, \quad 0 < t < T \tag{3.1}$$

$$u(x, y, 0) = \phi(x, y) \quad \text{for } (x, y) \in G_0, \quad t = 0 \tag{3.2}$$

$$u(x,y,t) = f(x, y, t) = 0 \quad \text{on } \partial\Omega_t, \quad t \ge 0 \tag{3.3}$$

$$f_t + v \cdot \nabla f = 0 \quad \text{on } \partial\Omega_t, \quad t \ge 0 \tag{3.4}$$

where v is the velocity at the position (x,y) at the time t, and $f(x,y,t)$ is a C^1 function in $\overline{\Omega_t}$ such that $\partial G_t = \{(x,y,t) \in \Omega_t \mid f(x,y,t) = 0\}$, $\nabla f(x,y,t) \neq 0$ on ∂G_t, $u(x,y,t) > 0$ on Ω_t. $\overline{\Omega_t}$ is closure of Ω_t. The function $\phi(x,y)$ is the initial data for $u(x,y,t)$ and

$$\partial\Omega_t = \bigcup_{0 \le t \le T} \partial G_t, \quad 0 < T \le \infty,$$

is the "free boundary".

The classical, one-phase Stefan problem consists in determining a solution of (3.1) - (3.4).

Definition. A bounded, measurable function, u, in Ω_t is called a generalized solution of (3.1)–(3.4) if

$$\int_0^T \int_{G_\tau} (u\,\Delta\xi + u\,\xi_\tau)\,dx\,dy\,d\tau \quad +$$

$$+ \int_{G_0} \xi(x,y,0)\,u(x,y,0)\,dx\,dy = 0 \qquad (3.5)$$

holds for any function $\xi \in C_0^\infty(\overline{\Omega_t})$. By $C_0^\infty(\overline{\Omega_t})$ we denote the class of functions, ξ, which are infinitely often differentiable on Ω_t and which vanish in a neighborhood of the boundary of $\overline{\Omega_t}$. Such functions are said to have compact support.

Theorem 1. A classical solution of (3.1)–(3.4) in Ω_t is also a generalized solution of (3.1)–(3.4) in Ω_t.

Proof. Let us assume u is a classical solution of (3.1)–(3.4). We multiply both sides of (3.1) by ξ, move the left hand side of (3.1) to right hand side in equation (3.1), and integrate over $\bigcup_{0\le\tau\le T} G_\tau \times \{\tau\}$. Upon applying the Gauss divergence theorem and using the fact that $u=0$ on ∂G_τ, we get after integrating by parts,

$$\int_0^T \int_{G_\tau} (\xi\,\Delta u - \xi\,u_\tau)\,dx\,dy\,d\tau$$

$$=\int_0^T\int_{G_\tau} [\{\mathrm{div}(\xi\ \nabla u)-\ \nabla\xi\cdot\nabla u\}-\xi\ u_\tau\ ]\ dx\ dy\ d\tau$$

$$=-\int_0^T\int_{G_\tau} [\{\mathrm{div}(u\ \nabla\xi)\ -u\ \Delta\xi\ \}+\ \xi u_\tau]\ dxdyd\tau$$

Next an application of the Reynolds' transport theorem yields (see [26] Chapter 1)

$$=\int_0^T\int_{G_\tau} u\ \Delta\xi\ dx\ dy\ d\tau$$

$$-\int_{G_\tau} u\xi(x,y,\tau)dx\ dy\ \Big|_{\tau=0}^{\tau=T}\ +\int_0^T\int_{G_\tau} u\ \xi_\tau\ dx\ dy\ d\tau$$

$$=\int_0^T\int_{G_\tau} (u\Delta\xi\ +\ u\ \xi_\tau\ )dx\ dy\ d\tau\ +$$

$$+\int_{G_0} \xi(x,y,0)u(x,y,0)dx\ dy\ =0.$$

Here ν is the exterior unit normal to ∂G_τ. This equation implies the assertion.

<u>Theorem</u> <u>2</u>. Suppose $u\in C^{2,1}(\Omega_t)$ is a generalized solution of (3.1)−(3.4) in Ω_t. Assume that there exist a continuously differentiable function, Φ in $\overline{\Omega_t}$, satisfying $\Gamma(t)\ =\ \{(x,\ y,\ t)\ \in\Omega_t\ \mid\ \Phi(x,\ y,\ t)\ =0\ \}$, $\nabla\ \Phi\neq 0$ on $\Gamma(t)$, and $\Phi\ >\ 0$ in Ω_t.

Assume that u, $\nabla\ u$ are continuous in $\bigcup_{0\leq\tau\leq T} \overline{G_\tau}\times\{\tau\}$ and Δu, $\frac{\partial u}{\partial t}$ are continuous in $\bigcup_{0<\tau<T} G_\tau\times\{\tau\}$. Then u is a classical solution of (3.1)−(3.4).

Proof. Let u be a weak solution satisfying the assumption of theorem 2. Taking ξ in equation (3.5) with compact support in the neighborhood of a point (x,y,s), where $(x,y,s) \in \Gamma(s)$, in other words, $\xi = 0$ on $\Gamma(t)$ and $\xi(x,y,T) = 0$, we have

$$0=\int_0^T \int_{G_\tau} (u\Delta\xi + u\,\xi_\tau)dx\,dy\,d\tau + $$

$$+\int_{G_0} \xi(x,y,0)u(x,y,0)dx\,dy$$

$$=\int_0^T\int_{G_\tau} (u\,\Delta\xi + u\,\xi_\tau)\,dx\,dy\,d\tau +$$

$$+\int_{G_0} \xi(x,y,0)u(x,y,0)dx\,dy-\int_{G_0} \xi(x,y,T)u(x,y,T)dx\,dy$$

$$=\int_0^T\int_{G_\tau} u\,\Delta\xi\,dx\,dy\,d\tau+\int_0^T\int_{G_\tau} u\,\xi_\tau dx\,dy\,d\tau$$

$$-[\int_{G_0} \xi(x,y,T)u(x,y,T)dx\,dy-\int_{G_0} \xi(x,y,0)u(x,y,0)dx\,dy]$$

$$=\int_0^T\int_{G_\tau} u\,\Delta\xi\,dx\,dy\,d\tau+\int_0^T\int_{G_\tau} u\,\xi_\tau dx\,dy\,d\tau$$

$$-\int_{G_\tau} u\,(x,y,\tau)\xi(x,y,\tau)\,dx\,dy\Big|_{\tau=0}^{\tau=T}$$

$$=\int_0^T\int_{G_\tau} u\,\Delta\xi\,dx\,dy\,d\tau-\int_0^T\int_{G_\tau} u_\tau\,\xi\,dx\,dy\,d\tau$$

$$=\int_0^T\int_{G_\tau} [\mathrm{div}(u\nabla\xi) - \nabla u\cdot\nabla\xi]\,dx\,dy\,d\tau-\int_0^T\int_{G_\tau} u_\tau\,\xi\,dx\,dy\,d\tau$$

$$= - \int_0^T \int_{G_\tau} (\nabla u \cdot \nabla \xi + u_\tau \xi) \, dx\, dy\, d\tau + \int_0^T \int_{G_\tau} \operatorname{div}(\xi \nabla u) dx\, dy\, d\tau$$

$$= \int_0^T \int_{G_\tau} [\operatorname{div}(\xi \nabla u) - \nabla u \cdot \nabla \xi - u_\tau \xi] \, dx\, dy\, d\tau$$

$$= \int_0^T \int_{G_\tau} (\xi \Delta u - \xi u_\tau) \, dx\, dy\, d\tau .$$

We conclude that

$$\int_0^T \int_{G_\tau} (\xi \Delta u - \xi u_\tau) \, dx\, dy\, d\tau = 0 \text{ for arbitrary } \xi,$$

whence, $\Delta u = u_t$.

This is local, so choose $\xi \in C_0^\infty(\overline{\Omega_t})$. Then all boundary terms vanish. So $f(x,y,t) = 0$ on $\partial\Omega_t$ and we have $u_t = \Delta u$ in Ω_t.

Since $u(x,y,t) = 0 = f(x,y,t)$ on $\partial\Omega_t$, $t \geq 0$. Taking derivative with respect to t, we get $f_t = 0$ and $\nabla f = 0$ on $\partial\Omega_t$.

Hence,

$$f_t + \nabla u \cdot \nabla f = 0 \text{ on } \partial\Omega_t,$$

and we see that equation (3.4) holds.

3.2 Existence and uniqueness theorems.

Let $\Delta x = \Delta y = h$, $\Delta t = k$. We cover the region $\{(x,y) \in \mathbb{R}^2,\ 0 \le t \le T\}$ by a rectangular network by means of lines $t_n = nk$, $x_i = ih$, $y_j = jh$ where i, j run through all the integer and n is an integer lying in the interval $[0, \frac{T}{k}]$. The lattice points of the net have coordinates that are multiples of h and we take T to be a multiple of k. Then

$R_t(h) = \{ (x,y) \in \mathbb{R}^2 \mid (x,y) = (ih, jh),\ i,\ j\text{: integers}\}$

denotes the rectangular network of mesh points.

Let

$$\overline{G_t}(h) = R_t(h) \bigcap \overline{G_t}.$$

If $(x,y) \in R_t(h)$, the points $(x,y) \pm he_1 \pm he_2$ will be called the neighbors of (x,y), where $e_1 = (1,0)$ and $e_2 = (0,1)$. The set of points in $\overline{G_t}(h)$ which have at least one neighbor lying outside $\overline{G_t}(h)$, that is the boundary of $G_t(h)$, will be denoted by $\partial G_t(h)$. Set $G_t(h) = \overline{G_t}(h) - \partial G_t(h)$. Now let $t_n = nk$ and define the sets

$$\Omega_t(h) = G_t(h) \times \{t = t_n \mid n = 1, \ldots, N\},$$

$$S_t(h) = \partial G_t(h) \times \{ t = t_n \mid n = 1, \ldots, N\},$$

and

$$\overline{\Omega_t}(h) = \Omega_t(h) \cup S_t(h) \cup \{\overline{\Omega_t}(h) \times (t = 0)\}.$$

where N will be a fixed integer and $k = \frac{T}{N}$.

We replace the problem (3.1) – (3.4) by a difference scheme. To do that, we define the difference operators.

$$D_x^+ U(x,y,t) = \frac{U(x+h,y,t) - U(x,y,t)}{h} \equiv D_x^+ U_{i,j}^n ,$$

$$D_x^- U(x,y,t) = \frac{U(x,y,t) - U(x-h,y,t)}{h},$$

$$D_y^+ U(x,y,t) = \frac{U(x,y+h,t) - U(x,y,t)}{h},$$

$$D_y^- U(x,y,t) = \frac{U(x,y,t) - U(x,y-h,t)}{h},$$

$$D_t^+ U(x,y,t) = \frac{U(x,y,t+k) - U(x,y,t)}{k},$$

$$D_x^+ \; D_x^- U(x,y,t) = \frac{U(x+h,y,t) - 2\,U(x,y,t) + U(x-h,y,t)}{h^2},$$

$$D_y^+ D_y^- U(x,y,t) = \frac{U(x,y+h,t) - 2\,U(x,y,t) + U(x,y-h,t)}{h^2} ,$$

$$U(x+h,y,t) = U_{i+1,j}^n.$$

We now seek a function, $U(x,y,t)$, defined on $\overline{\Omega_t}(h)$ satisfying

$$D_t^+ a(U(x,y,t)) = D_x^+ \; D_x^- U(x,y,t) + D_y^+ \; D_y^- U(x,y,t)$$

$$\text{for } (x,y,t) \in \Omega_t(h) \qquad (3.6)$$

$$U(x,y,t) = \psi(x,y,t) = 0, \qquad (x,y,t) \in S_t(h) \quad (3.7)$$

$$U(x,y,0) = \phi(x,y), \qquad (x,y) \in G_0(h) \quad (3.8)$$

where we take $U(x,y,0)$ to be zero for $(x,y) \in \partial G_0(h)$ and set $\phi(x,y)$ equal to zero for $(x,y) \in \partial G_0(h)$ when we deal with the difference equations. In general, $a(U(x,y,t))$ is a function of $U(x,y,t)$, in our case, $a(U(x,y,t))$ is equal to $U(x,y,t)$.

From now on we assume that h has been chosen to be very small and certainly less than or equal to the diameter d_t of G_t so that $G_t(h)$ is not empty. If Λ is the number of mesh points in $G_t(h)$, (3.6) represents a nonlinear system of equations in Λ unknowns for each t_n. We shall be dealing with functions which vanish on $\partial G_t(h)$ or with the product of such a function and another function defined on $\overline{G_t}(h)$. It will be convenient to think of these functions as being defined on all of $R_t(h)$ by simply assigning them the value zero in $R_t(h) - G_t(h)$.

__Lemma 1__. At every mesh point lying in $\Omega_t(h)$, there exists a solution $U^n_{i,j}$ to equation (3.6) where $\frac{k}{h^2} < \mu$, and μ depends on

$$C_0 = \max_{\substack{(x,y)\in G_0 \\ (x,y,t)\in S_t(h)}} \{ |\phi(x,y)|, |\psi(x,y,t)|\} \text{ and } a(u), \text{ but}$$

is independent of h, k, x_i, y_j, and t_n.

In this case, $|U^n_{i,j}| \le C_h = \max_{\substack{(x,y)\in G_0 \\ (x,y,t)\in S_t(h)}} \{|\phi(x,y)|, |\tilde{\psi}(x,y,t)|\}$,

where $\tilde{\psi}(x,y,t)$ is a continuous function which will take on the value $\psi(x,y,t)$ over $S_t(h)$ and has 3 continuous derivatives in $\Omega'_t \subset \Omega_t$.

<u>Remark</u>. It is clear that $C_h \to C_0$ as $h \to 0$. Let us consider h to be so small that $C_h \le C_0 + 1$ is true.

<u>Proof</u>. Let us prove lemma 1 using mathematical induction. We assume lemma 1 is true for t_n and prove that it is also true for t_{n+1}. Rewrite equation (3.6). Then

$$\frac{a(U_{i,j}^{n+1}) - a(U_{i,j}^{n})}{k} = \frac{U_{i+1,j}^{n} - 2\, U_{i,j}^{n} + U_{i-1,j}^{n}}{h^2}$$

$$+ \frac{U_{i,j+1}^{n} - 2\, U_{i,j}^{n} + U_{i,j-1}^{n}}{h^2} \qquad (3.9)$$

Multiply both sides of equation (3.9) by k and add $a(U_{i,j}^{n})$ to obtain

$$a(U_{i,j}^{n+1}) = a(U_{i,j}^{n}) + \frac{k}{h^2}(U_{i+1,j}^{n} + U_{i-1,j}^{n} + U_{i,j+1}^{n}$$

$$+ U_{i,j-1}^{n} - 4\, U_{i,j}^{n}) \equiv g_{i,j}^{n} \qquad (3.10)$$

$$|a(U_{i,j}^{n+1})| = |g_{i,j}^{n}| \le |a(U_{i,j}^{n})| + \frac{8\,k}{h^2}\,|U_{i,j}^{n}|$$

$$\le a(C_0 + 1) + \frac{8\,k}{h^2}\, C_h$$

That is,

$$a(-\,C_0 - 1) - 8\, C_h \frac{k}{h^2} \le g_{i,j}^{n} \le a(\,C_0 + 1) + 8\, C_h \frac{k}{h^2}$$

If μ is sufficiently small and $\frac{k}{h^2} < \mu$, then

$$a(-\,C_0 - 1) - 8\, C_h \frac{k}{h^2} > a(\,-\,C_0 - 2) \text{ and}$$

$$a(C_0 + 1) + 8\,C_h \frac{k}{h^2} < a(C_0 + 2).$$

Therefore, $a(-C_0 - 2) < g^n_{i,j} < a(C_0+2)$.

Consequently, $a(U^{n+1}_{i,j}) = g^n_{i,j}$ has a solution $U^{n+1}_{i,j}$ and $|U^{n+1}_{i,j}| < C_0 + 2$. In this case μ depends on C_0 and $a(U)$ where $U = C_0+1,\ C_0+2,\ -(C_0+1),\ -(C_0+2)$.

Now let us prove $|U^{n+1}_{i,j}| \le C_h$ for sufficiently small μ. We prove the contrapositive. Assume, to obtain a contradiction, that we have found $(x_i,\ y_j)$ for which $|U^{n+1}_{i,j}| > C_h$ is true.

Since $a(U^{n+1}_{i,j}) - a(U^n_{i,j}) \ge \eta(C_0 + 2)(U^{n+1}_{i,j} - U^n_{i,j})$ (see [29]),

$$\eta(C_0 + 2)(U^{n+1}_{i,j} - U^n_{i,j}) \le \frac{k}{h^2}(U^n_{i+1,j} + U^n_{i-1,j} + U^n_{i,j+1} + U^n_{i,j-1} - 4\,U^n_{i,j}).$$

Moreover,

$$\eta(C_0 + 2)U^{n+1}_{i,j} \le \frac{k}{h^2}(U^n_{i+1,j} + U^n_{i-1,j} + U^n_{i,j+1} + U^n_{i,j-1}) + U^n_{i,j}\left[\eta(C_0 + 2) - \frac{4k}{h^2}\right] \quad (3.11)$$

Choose $\beta > 0$ such that $\mu \le \dfrac{\eta(C_0 + 2)(1 - \beta)}{4} < \dfrac{\eta(C_0+2)}{4}$. (3.12)

Substituting equation (3.12) into (3.11) and using the induction assumption, we get

$$\eta(C_0 + 2)U_{i,j}^{n+1} < \eta(C_0 + 2)\ C_h .$$

Therefore,

$$U_{i,j}^{n+1} < C_h$$

because of $\eta(C_0 + 2) > 0$.

This is a contradiction, so

$$U_{i,j}^{n+1} \le C_h ,$$

and the proof of the lemma is complete. An immediate consequence of the proof is

<u>Corollary 1</u>. At every mesh point of $\Omega_t(h)$

$$|U_{i,j}^n| \le C_0 + 1.$$

<u>Lemma 2</u>. For $\frac{k}{h^2} < \mu$, the following estimates hold:

$$k \cdot h^2 \sum_{\Omega_t(h)} \left(\frac{U_{i,j}^{n+1} - U_{i,j}^n}{k}\right)^2 \le K_0 \tag{3.13}$$

$$k \cdot h^2 \sum_{\Omega_t(h)} \left(\frac{U_{i+1,j}^n - U_{i,j}^n}{h}\right)^2 + k \cdot h^2 \sum_{\Omega_t(h)} \left(\frac{U_{i,j+1}^n - U_{i,j}^n}{h}\right)^2 \le K_1 \tag{3.14}$$

where in the summation $\sum_{\Omega_t(h)}$ is taken over values of the

$U_{i,j}$ such that at least one of the mesh points (x_i, y_j), (x_{i+1}, y_j), and (x_i, y_{j+1}) lies in $\Omega_t(h)$. K_0 and K_1 are constants independent of h and k.

<u>Proof</u>. Let Ω_t $(0 \le x \le x_0;\ 0 \le y \le y_0;\ 0 \le t \le T)$ denote the interior

mesh points and let $\frac{x_0}{h} = M_1$, $\frac{y_0}{h} = M_2$, where M_1, M_2 are integers. Assume $\gamma(x,y,t) = \frac{\partial \tilde{\psi}}{\partial t}$. We determine mesh points belonging to $\Omega_t(h) = \overline{\Omega_t}$ as a function $\gamma^n_{i,j}$ as follows. On $S_t(h)$, set

$\gamma^n_{i,j} = \dfrac{\psi^{n+1}_{i,j} - \psi^n_{i,j}}{k}$ where $n \le N-1$ and $\gamma^N_{i,j} = \gamma(ih, jh, Nk)$ $(i = 0, \ldots, M_1,\ j = 0, \ldots, M_2)$. Inside Ω_t, we will set $\gamma^n_{i,j} = \gamma(ih, jh, nk)$. The following estimates are justified making use of the fact that $\gamma(x,y,t)$ is twice continuously differentiable with respect to x, y, and t.

$$\left|\frac{\gamma^n_{i,j} - \gamma^{n-1}_{i,j}}{k}\right| = \left|\frac{\gamma(ih,jh,nk) - \gamma(ih,jh,(n-1)k)}{k}\right| = \left|\frac{\partial \gamma}{\partial t}\right| \le K_2$$

$$(1 \le i \le M_1 - 1,\ 1 \le j \le M_2 - 1, 1 \le n \le N) \tag{3.15}$$

Similarly,

$$\left|\frac{\gamma^n_{i+1,j} - \gamma^n_{i,j}}{h}\right| \le K_2 \qquad (0 \le n \le N,\ i = 0,\ i = M_1 - 1) \tag{3.16}$$

$$\left|\frac{\gamma^n_{i,j+1} - \gamma^n_{i,j}}{h}\right| \le K_2 \qquad (0 \le n \le N,\ j = 0,\ M_2 - 1) \tag{3.17}$$

$$\left|\frac{\gamma^n_{i+1,j} - 2\gamma^n_{i,j} + \gamma^n_{i-1,j}}{h^2}\right| \le K_2$$

$$(0 \le n \le N,\ 1 \le i \le M_1 - 1) \quad (3.18)$$

$$\left|\frac{\gamma^n_{i,j+1} - 2\gamma^n_{i,j} + \gamma^n_{i,j-1}}{h^2}\right| \le K_2 \ (0 \le n \le N,\ 1 \le j \le M_2 - 1) \quad (3.19)$$

where K_2 is a positive constant.

Multiply both sides of equation (3.9) by

$h^2 k \left(\dfrac{U^{n+1}_{i,j} - U^n_{i,j}}{k} - \gamma^n_{i,j}\right)$ and sum over i, j, and n from 1 to M_1-1, from 1 to M_2-1, and from 0 to $N-1$, respectively, we obtain

$$h^2 k \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \frac{a(U^{n+1}_{i,j}) - a(U^n_{i,j})}{k} \frac{U^{n+1}_{i,j} - U^n_{i,j}}{k}$$

$$- h^2 k \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \frac{a(U^{n+1}_{i,j}) - a(U^n_{i,j})}{k} (\gamma^n_{i,j})$$

$$- h^2 k \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \left(\frac{U^{n+1}_{i,j} - U^n_{i,j}}{k} - \gamma^n_{i,j}\right)$$

$$\left[\frac{U^n_{i+1,j} - 2U^n_{i,j} + U^n_{i-1,j}}{h^2} + \frac{U^n_{i,j+1} - 2U^n_{i,j} + U^n_{i,j-1}}{h^2}\right]$$

$$= S_1 + S_2 + S_3 + S_4 = 0 \quad (3.20)$$

Using the fact that $\dfrac{a(U) - a(V)}{U - V} \ge \eta(C)$ where

$C = \max\{|U|, |V|\}$ and $|U^n_{i,j}| \le C_0 + 1$, the first summation in the expression will be estimated by

$$S_1 = h^2k \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \frac{a(U^{n+1}_{i,j}) - a(U^n_{i,j})}{k} \frac{U^{n+1}_{i,j} - U^n_{i,j}}{k}$$

$$\ge \eta(C_0 + 1)(h^2k) \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \frac{U^{n+1}_{i,j} - U^n_{i,j}}{k} \frac{U^{n+1}_{i,j} - U^n_{i,j}}{k}$$

$$= \eta(C_0 + 1)(h^2k) \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \left[\frac{U^{n+1}_{i,j} - U^n_{i,j}}{k} \right]^2. \qquad (3.21)$$

For the 2nd sum in equation (3.20), we sum by parts and get

$$S_2 = - h^2k \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \frac{a(U^{n+1}_{i,j}) - a(U^n_{i,j})}{k} (\gamma^n_{i,j})$$

$$= h^2k \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \frac{-\gamma^n_{i,j}\, a(U^{n+1}_{i,j}) + \gamma^{n+1}_{i,j}\, a(U^{n+1}_{i,j})}{k}$$

$$- h^2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} [\, a(U^N_{i,j})(\gamma^N_{i,j}) - a(U^0_{i,j})(\gamma^0_{i,j})]$$

$$= h^2k \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=1}^{N-1} \frac{\gamma^n_{i,j} - \gamma^{n-1}_{i,j}}{k} a(U^n_{i,j})$$

$$- h^2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} [\, a(U^N_{i,j})(\gamma^{N-1}_{i,j}) - a(U^0_{i,j})(\gamma^0_{i,j})]. \qquad (3.22)$$

Hence, $|S_2| \le K_3$ by equation (3.15).

If we apply summation by parts to S_3, we will have

$$S_3 = -h^2k \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \left[\left(\frac{U_{i,j}^{n+1} - U_{i,j}^{n}}{k} - \gamma_{i,j}^{n}\right)\left(\frac{U_{i+1,j}^{n} - 2U_{i,j}^{n} + U_{i-1,j}^{n}}{h^2}\right)\right]$$

$$= h^2k \sum_{i=0}^{M_1-2} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \left(\frac{U_{i+1,j}^{n} - U_{i,j}^{n}}{h^2}\right)\left(\frac{U_{i+1,j}^{n+1} - U_{i+1,j}^{n}}{k} - \gamma_{i+1,j}^{n}\right)$$

$$-h^2k \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \left[\left(\frac{U_{i,j}^{n+1} - U_{i,j}^{n}}{k} - \gamma_{i,j}^{n}\right)\left(\frac{U_{i+1,j}^{n} - U_{i,j}^{n}}{h^2}\right)\right]$$

$$= h^2k \sum_{i=0}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \left(\frac{U_{i+1,j}^{n} - U_{i,j}^{n}}{h^2}\right)\left(\frac{U_{i+1,j}^{n+1} - U_{i+1,j}^{n}}{k} - \gamma_{i+1,j}^{n}\right)$$

$$- h^2k \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \left(\frac{U_{M_1,j}^{n} - U_{M_1-1,j}^{n}}{h^2}\right)\left(\frac{U_{M_1,j}^{n+1} - U_{M_1,j}^{n}}{k} - \gamma_{M_1,j}^{n}\right)$$

$$- h^2k \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \left[\left(\frac{U_{i,j}^{n+1} - U_{i,j}^{n}}{k} - \gamma_{i,j}^{n}\right)\left(\frac{U_{i+1,j}^{n} - U_{i,j}^{n}}{h^2}\right)\right]$$

$$=h^2k \sum_{i=0}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \left(\frac{U_{i+1,j}^n - U_{i,j}^n}{h^2}\right)\left(\frac{U_{i+1,j}^{n+1} - U_{i+1,j}^n}{k} - \gamma_{i+1,j}^n - \frac{U_{i,j}^{n+1} - U_{i,j}^n}{k} + \gamma_{i,j}^n\right)$$

$$- k \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \left[(U_{M_1,j}^n - U_{M_1-1,j}^n)\left(\frac{U_{M_1,j}^{n+1} - U_{M_1,j}^n}{k} - \gamma_{M_1,j}^n\right) - \left(\frac{U_{0,j}^{n+1} - U_{0,j}^n}{k} - \gamma_{0,j}^n\right)(U_{1,j}^n - U_{0,j}^n)\right].$$

Hence,

$$S_3 = h^2k \sum_{i=0}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \left(\frac{U_{i+1,j}^n - U_{i,j}^n}{h^2}\right)\left(\frac{U_{i+1,j}^{n+1} - U_{i+1,j}^n}{k} - \gamma_{i+1,j}^n + \gamma_{i,j}^n - \frac{U_{i,j}^{n+1} - U_{i,j}^n}{k}\right) \qquad (3.23)$$

because of the boundary conditions

$$\gamma_{M_1,j}^n = \frac{U_{M_1,j}^{n+1} - U_{M_1,j}^n}{k} \quad \text{and} \quad \gamma_{0,j}^n = \frac{U_{0,j}^{n+1} - U_{0,j}^n}{k},$$

where $n \le N-1$.

Rewrite (3.23) as

$$S_3 = h^2k \sum_{i=0}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \left(\frac{U_{i+1,j}^n - U_{i,j}^n}{h^2}\right)\left(\frac{U_{i+1,j}^{n+1} - U_{i+1,j}^n}{k} - \frac{U_{i,j}^{n+1} - U_{i,j}^n}{k}\right)$$

$$- h^2k \sum_{i=0}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \left(\frac{U_{i+1,j}^n - U_{i,j}^n}{h^2}\right)(\gamma_{i+1,j}^n - \gamma_{i,j}^n)$$

$$= S_5 + S_6, \qquad (3.24)$$

where

$$S_5=h^2k \sum_{i=0}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \left(\frac{U^n_{i+1,j} - U^n_{i,j}}{h^2}\right)\left(\frac{U^{n+1}_{i+1,j} - U^n_{i+1,j}}{k} - \frac{U^{n+1}_{i,j} - U^n_{i,j}}{k}\right)$$

and

$$S_6 = - h^2k \sum_{i=0}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \left(\frac{U^n_{i+1,j} - U^n_{i,j}}{h^2}\right)(\gamma^n_{i+1,j} - \gamma^n_{i,j}).$$

We simplify S_5.

Let $V^n_{i,j} = \dfrac{U^n_{i+1,j} - U^n_{i,j}}{h}$.

$$S_5 = h^2 \sum_{i=0}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} V^n_{i,j}\,(V^{n+1}_{i,j} - V^n_{i,j})$$

$$= \frac{h^2}{2} \sum_{i=0}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} [(V^{n+1}_{i,j})^2 - (V^n_{i,j})^2 - (V^{n+1}_{i,j} - V^n_{i,j})^2]$$

$$= \frac{h^2}{2} \sum_{i=0}^{M_1-1} \sum_{j=1}^{M_2-1} (V^N_{i,j})^2 - \frac{h^2}{2} \sum_{i=0}^{M_1-1} \sum_{j=1}^{M_2-1} (V^0_{i,j})^2 - \frac{h^2}{2} \sum_{i=0}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} (V^{n+1}_{i,j} - V^n_{i,j})^2$$

$$= \frac{h^2}{2} \sum_{i=0}^{M_1-1} \sum_{j=1}^{M_2-1} \left(\frac{U^N_{i+1,j} - U^N_{i,j}}{h}\right)^2 - S_7 - S_8 , \qquad (3.25)$$

with $S_7 = \dfrac{h^2}{2} \displaystyle\sum_{i=0}^{M_1-1} \sum_{j=1}^{M_2-1} \left(\frac{U^0_{i+1,j} - U^0_{i,j}}{h}\right)^2$

and

$$S_8 = \frac{h^2}{2} \sum_{i=0}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} (V^{n+1}_{i,j} - V^n_{i,j})^2.$$

Since the initial function $a_0(x,y) = a(u_0(x,y))$ is smooth,

$$|S_7| \le K_4 .$$

We estimate the summation S_8.

$$S_8 = \frac{1}{2} \sum_{i=0}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} [(U_{i+1,j}^{n+1} - U_{i,j}^{n+1}) - (U_{i+1,j}^{n} - U_{i,j}^{n})]^2$$

$$= \frac{1}{2} \sum_{i=0}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} [(U_{i+1,j}^{n+1} - U_{i+1,j}^{n}) - (U_{i,j}^{n+1} - U_{i,j}^{n})]^2$$

$$\le \sum_{i=0}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} [(U_{i+1,j}^{n+1} - U_{i+1,j}^{n})^2 + (U_{i,j}^{n+1} - U_{i,j}^{n})^2]$$

$$= k^2 \sum_{i=0}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} [(\frac{U_{i+1,j}^{n+1} - U_{i+1,j}^{n}}{k})^2 + (\frac{U_{i,j}^{n+1} - U_{i,j}^{n}}{k})^2]$$

$$= k^2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} (\frac{U_{i,j}^{n+1} - U_{i,j}^{n}}{k})^2 +$$

$$+ k^2 \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} (\frac{U_{M_1,j}^{n+1} - U_{M_1,j}^{n}}{k})^2$$

$$+ k^2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} (\frac{U_{i,j}^{n+1} - U_{i,j}^{n}}{k})^2$$

$$+ k^2 \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} (\frac{U_{0,j}^{n+1} - U_{0,j}^{n}}{k})^2 .$$

This implies that

$$S_8 = 2k^2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \left(\frac{U_{i,j}^{n+1} - U_{i,j}^{n}}{k}\right)^2 +$$

$$+ k^2 \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \left[\left(\frac{U_{0,j}^{n+1} - U_{0,j}^{n}}{k}\right)^2 + \left(\frac{U_{M_1,j}^{n+1} - U_{M_1,j}^{n}}{k}\right)^2 \right]. \quad (3.26)$$

Using the smoothness of $\psi(P)$ and $\frac{k}{h^2} < \mu$, the latter two summations of (3.26) tend to zero as h, k $\to$ 0.

In fact,

$$k^2 \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \left[\left(\frac{U_{0,j}^{n+1} - U_{0,j}^{n}}{k}\right)^2 + \left(\frac{U_{M_1,j}^{n+1} - U_{M_1,j}^{n}}{k}\right)^2 \right.$$

$$= h^2 \left(\frac{k}{h^2}\right) k \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \left\{ \left(\frac{\psi(0,j,(n+1)k) - \psi(0,j,nk)}{k}\right)^2 + \right.$$

$$\left. + \left(\frac{\psi(M_1,j,(n+1)k) - \psi(M_1,j,nk)}{k}\right)^2 \right\} \le K_5 h^2.$$

Therefore,

$$S_8 \le K_5 h^2 + 2k^2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \left(\frac{U_{i,j}^{n+1} - U_{i,j}^{n}}{k}\right)^2$$

$$= K_5 h^2 + \frac{2k}{h^2 \eta (C_0 + 1)} h^2 k \eta (C_0 + 1) \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \left(\frac{U_{i,j}^{n+1} - U_{i,j}^{n}}{k}\right)^2$$

$$\le K_5 h^2 + \frac{2\mu}{\eta (C_0 + 1)} S_1$$

However,

$$\frac{2\mu}{\eta(C_0 + 1)} \le \frac{1-\beta}{2}, \text{ where } \beta > 0.$$

Hence, we will get

$$S_8 \le K_5 h^2 + \frac{S_1(1-\beta)}{2} \tag{3.27}$$

Next,

$$S_6 = -h^2 k \sum_{i=0}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \left(\frac{U_{i+1,j}^n - U_{i,j}^n}{h^2}\right)(\gamma_{i+1,j}^n - \gamma_{i,j}^n)$$

$$= h^2 k \sum_{i=0}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \left[\left(- \frac{\gamma_{i+1,j}^n - \gamma_{i,j}^n}{h^2} (U_{i+1,j}^n)\right.\right.$$

$$\left.\left. + \frac{\gamma_{i+1,j}^n - \gamma_{i,j}^n}{h^2} (U_{i,j}^n)\right)\right]$$

$$= h^2 k \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \frac{-\gamma_{i,j}^n + \gamma_{i-1,j}^n}{h^2} (U_{i,j}^n) +$$

$$+ h^2 k \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \frac{-\gamma_{M_1,j}^n + \gamma_{M_1-1,j}^n}{h^2} (U_{M_1,j}^n) +$$

$$+ h^2 k \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \left[\frac{\gamma_{i+1,j}^n - \gamma_{i,j}^n}{h^2} (U_{i,j}^n) + \right.$$

$$\left. + \frac{\gamma_{1,j}^n - \gamma_{0,j}^n}{h^2} (U_{0,j}^n)\right].$$

$$S_6 = h^2 k \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \frac{\gamma_{i+1,j}^n - 2\gamma_{i,j}^n + \gamma_{i-1,j}^n}{h^2} (U_{i,j}^n)$$

$$-hk \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \left[\frac{\gamma_{M_1,j}^n - \gamma_{M_1-1,j}^n}{h} (U_{M_1,j}^n) \right.$$

$$\left. - \frac{\gamma_{1,j}^n - \gamma_{0,j}^n}{h} (U_{0,j}^n) \right]. \tag{3.28}$$

S_6 is uniformly bounded for all h and k by equations (3.16) and equation (3.18). In other words,

$$|S_6| \le K_6.$$

Let us estimate the fourth sum in equation (3.20). Then

$$S_4 = - \quad h^2 k \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \left[\left(\frac{U_{i,j}^{n+1} - U_{i,j}^{n}}{k} - \gamma_{i,j}^{n} \right) \left(\frac{U_{i,j+1}{}^{n} - 2\, U_{i,j}{}^{n} + U_{i,j-1}{}^{n}}{h^2} \right) \right]$$

$$= h^2 k \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \left[\left(\frac{U_{i,j+1}^{n} - U_{i,j}^{n}}{h^2} \right) \left(\frac{U_{i,j+1}^{n+1} - U_{i,j+1}^{n}}{k} - \gamma_{i,j+1}^{n} + \gamma_{i,j}^{n} - \frac{U_{i,j}^{n+1} - U_{i,j}^{n}}{k} \right) \right]$$

$$- k \sum_{i=1}^{M_1-1} \sum_{n=0}^{N-1} \left[\left(\frac{U_{i,M_2}^{n+1} - U_{i,M_2}^{n}}{k} - \gamma_{i,M_2}^{n} \right) \left(U_{i,M_2}^{n} - U_{i,M_2-1}^{n} \right) - \left(\frac{U_{i,0}^{n+1} - U_{i,0}^{n}}{k} - \gamma_{i,0}^{n} \right) \left(U_{i,1}^{n} - U_{i,0}^{n} \right) \right]. \qquad (3.29)$$

The estimates for the y-coordinate are obtained in a similar manner using $W_{i,j}^{n} = \dfrac{U_{i,j+1}^{n} - U_{i,j}^{n}}{h}$, equation (3.17), and equation (3.19). We obtain

$$S_4 = S_9 + S_{10}$$

$$= \frac{h^2}{2} \sum_{i=1}^{M_1-1} \sum_{j=0}^{M_2-1} \left(\frac{U_{i,j+1}^{N} - U_{i,j}^{N}}{h} \right)^2 - S_{11} - S_{12} + S_{10},$$

$$|S_{11}| \leq K_7, \quad |S_{12}| \leq K_8 h^2 + \frac{S_1(1-\beta)}{2}, \text{ and } |S_{10}| \leq K_9.$$

If we take all the estimates which we have obtained and substitute them into equation (3.20), we will get

$$S_1 + \frac{h^2}{2}\left[\sum_{i=0}^{M_1-1}\sum_{j=1}^{M_2-1}\left(\frac{U^N_{i+1,j} - U^N_{i,j}}{h}\right)^2\right.$$

$$\left. + \sum_{i=1}^{M_1-1}\sum_{j=0}^{M_2-1}\left(\frac{U^N_{i,j+1} - U^N_{i,j}}{h}\right)^2\right]$$

$$\leq K_3 + K_4 + K_7 + (K_5 + K_8)h^2 + S_1(1-\beta) + K_6 + K_9$$

Hence,

$$\beta S_1 + \frac{h^2}{2}\left[\sum_{i=0}^{M_1-1}\sum_{j=1}^{M_2-1}\left(\frac{U^N_{i+1,j} - U^N_{i,j}}{h}\right)^2\right.$$

$$\left. + \sum_{i=1}^{M_1-1}\sum_{j=0}^{M_2-1}\left(\frac{U^N_{i,j+1} - U^N_{i,j}}{h}\right)^2\right]$$

$$\leq K_3 + K_4 + K_7 + (K_5 + K_8)h^2 + K_6 + K_9 \equiv K \qquad (3.30)$$

Equation (3.13) will follow from last inequality and equation (3.21). Let us note equation (3.30) will be justified for any whole number $N_1 \leq N$, with the constant K which may be chosen as one for all $N_1 \leq N$.

Therefore,

$$h^2 \left[\sum_{i=0}^{M_1-1} \sum_{j=1}^{M_2-1} \left(\frac{U^{N_1}_{i+1,j} - U^{N_1}_{i,j}}{h} \right)^2 + \sum_{i=1}^{M_1-1} \sum_{j=0}^{M_2-1} \left(\frac{U^{N_1}_{i,j+1} - U^{N_1}_{i,j}}{h} \right)^2 \right]$$

$$\leq 2\ K.$$

If we multiply this inequality by k and sum over N_1 from 0 to N, we will get equation (3.14), which completes the proof of lemma 2.

<u>Theorem 3</u>. Suppose the following conditions hold:

1) $a_0(x,y) = a(\phi(x,y))$ is continuously differentiable on $\overline{G_t}$, $\psi(x,y,t)$ is three times continuously differentiable on the $\partial\Omega_t$.
2) at $t=0$, $\psi(x,y,t) = C_1 =$ constant
3) $\phi(x,y) = C_1$ near the surface $\partial\Omega_t$.

Then, there exists a generalized solution $u(x,y,t) \in W_2^1(\Omega_t)$ of equation (3.6) on Ω_t which satisfies conditions (3.7) and (3.8).

<u>Remark</u>. In our case, C_1 is zero.

<u>Proof</u>. Let us introduce a $\tilde{u}_h$ such that within the parallelepiped, $\{ ih<x<(i+1)h,\ jh<y<(j+1)h,\ nk<t<(n+1)k\}$, it is given by $\tilde{u}_h(x,y,t) = U^n_{i,j}$. $u_h{}^*$ denotes the function which is linear over x, y, and t in $ih \leq x \leq (i+1)h$; $jh \leq y \leq (j+1)h$; $nk \leq t \leq (n+1)k$ and which coincides with $U^n_{i,j}$. It is clear that $u_h{}^*$ is continuous and has a first generalized derivative. From inequalities (3.13), (3.14), corollary 1, and smoothness of function $\psi(x,y,t)$,

$$\|u_h{}^*\|_{W_2^1\ (\Omega_t)} < K_{11} \text{ holds.}$$

Let $\{h_l\}$ and $\{k_l\}$ denote subsequences of $\{h\}$ and $\{k\}$ which tend to zero as $l \to \infty$. In this case $\frac{k_l}{h_l^2} < \mu$. where $\frac{M_1}{h_l}$, $\frac{M_2}{h_l}$, and $\frac{N}{k_l}$ are integers. From the inequality above, a subsequence $u_{h_s}{}^*$ exists with the following properties:

A) $\{u_{h_s}{}^*\}$ converges weakly in the norm of $W_2^1(\Omega_t)$ to a function $u(x,y,t) \in W_2^1(\Omega_t)$.

B) $\{u_{h_s}{}^*\}$ converges strongly to $u(x,y,t)$ in the norm of $L_2\ (\Omega_t)$.

Hence it follows that the limit function $u(x,y,t)$ satisfies the boundary conditions in $x = 0$, $x = M_1$, $y = 0$, and $y = M_2$ in the sense that

$$\int_0^N [u(h,0,t) - \psi(0,0,t)]^2\, dt \to 0 \text{ as } h \to 0,$$

$$\int_0^N [u(0,h,t) - \psi(0,0,t)]^2\, dt \to 0 \text{ as } h \to 0,$$

$$\int_0^N [u(x_0-h,y_0,t) - \psi(x_0,y_0,t)]^2\, dt \to 0 \text{ as } h \to 0,$$

$$\int_0^N [u(x_0,y_0-h,t) - \psi(x_0,y_0,t)]^2\, dt \to 0 \text{ as } h \to 0.$$

The sequence $\{\tilde{u}_{h_s}\}$ converges to the limit function $u(x,y,t)$ in $L_2(\Omega_t)$. Consequently, there exists a sequence which converges to $u(x,y,t)$ almost everywhere. There is a subsequence of this sequence, denoted again by $\{\tilde{u}_{h_s}\}$, which converges to $u(x,y,t)$ almost everywhere. From the uniform boundness of the $\tilde{u}_{h_s}$, we obtain the uniform boundness of $a(\tilde{u}_{h_s})$ and from this we get the weak compactness of $\{a(\tilde{u}_{h_s})\}$ in $L_2(\Omega_t)$. Therefore, there exists a sequence, $\{a(\tilde{u}_{h_l})\}$, which converges weakly to a function $b(x,y,t) \in L_2(\Omega_t)$. We prove that

$$b(x,y,t) = a(u(x,y,t)). \tag{3.31}$$

Let E_i be the set of those points of Ω_t in which $u_i < u < u_{i+1}$ is true, where u_i is a constant. Over the interval (u_i, u_{i+1}), $a(u)$ will be continuously differentiable. Hence, $a(\tilde{u}_{h_l})$ converges to $a(u)$ almost everywhere on E_i because $\{\tilde{u}_{h_l}\}$ converges to $u(x,y,t)$ almost everywhere on E_i. Therefore, equation (3.31) will be true almost everywhere on E_i. We next prove equation (3.31) to be true for the set D_i which consists of points of Ω_t in which u equals to u_i. We must prove that $a(u_i + 0) \geq b(x,y,t) \geq a(u_i - 0)$ for D_i. We prove the contrapositive. Assume, to obtain a contradiction, we have found an $\epsilon > 0$, $F_i \subset D_i$, and $\text{meas}(F_i) > 0$ such that $b(x,y,t) < a(u_i - 0) - \epsilon$ is true for an $\epsilon > 0$ on F_i. At almost all points $P \in F_i$ $\tilde{u}_{h_s}(P)$ converges to u_i. Consequently, at each point P where K_{13} is sufficiently large (K_{13} depends on P), $a(\tilde{u}_{h_l}(P)) > a(u_i - 0) - \epsilon$ for $l > K_{13}$. For any w we will be able to find a number K_0 such that

$\mathrm{meas}(R[a(\tilde{u}_{h_l}) < a(u_0 - 0) - \epsilon]) < w$ where $R \subset F_i$ and $l > K_0$.

Hence,

$$\iint_{F_i} [\, a(\tilde{u}_{h_l}) - a(u_i - 0) + \epsilon] \, dx \, dy > - Aw \text{ where } l > K_0 \text{ and}$$

A is a positive constant.

If we let l tend to ∞, we find that

$$\iint_{F_i} [\, b(x,y,t) - a(u_i - 0) + \epsilon] \, dx \, dy \geq - Aw.$$

Since w is arbitrary, we have

$$\iint_{F_i} [\, b(x,y,t) - a(u_i - 0) + \epsilon] \, dx \, dy \geq 0,$$

which contradicts the fact that $b(x,y,t) < a(u_i - 0) - \epsilon$ on F_i. Hence $b(x,y,t) \geq a(u_i - 0)$. Similarly, we can prove $b(x,y,t) \leq a(u_i + 0)$ for D_i.

We now prove that $u(x,y,t)$ is our desired generalized solution. Since $|u_h{}^*| \leq C_0 + 1$, $|u(x,y,t)| \leq C_0 + 1$. Therefore, all we need to do is show that $u(x,y,t)$ satisfies the integral identity (3.5) for any twice continuously differentiable function $\xi(x,y,t)$ equal to zero on $\partial\Omega_t$. Let us multiply both sides of equations (3.9) by $h^2 k \xi^n_{i,j}$, where $\xi^n_{i,j} = \xi(ih,jh,nk)$, and then sum over i, j, and n from 1 to $M_1 - 1$, from 1 to $M_2 - 1$, and from 0 to $N-1$, respectively. We find

$$h^2k \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \frac{a(U_{i,j}^{n+1}) - a(U_{i,j}^{n})}{k} (\xi_{i,j}^{n}) -$$

$$-h^2k \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=0}^{N-1} \Big[\Big(\frac{U_{i+1,j}^{n} -2\, U_{i,j}^{n} + U_{i-1,j}^{n}}{h^2} \Big)(\xi_{i,j}^{n})$$

$$+\Big(\frac{U_{i,j+1}^{n} -2\, U_{i,j}^{n} + U_{i,j-1}^{n}}{h^2} \Big)(\xi_{i,j}^{n})\Big] = 0.$$

We now transform this identity making use of summation by parts and the fact that $\xi(x,y,t)$ equals to zero on $\partial\Omega_t$ to obtain

$$-h^2k \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=1}^{N} \Big(\frac{\xi_{i,j}^{n}-\xi_{i,j}^{n-1}}{k}\Big)(a(U_{i,j}^{n}))-$$

$$- h^2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} a(U_{i,j}^{0})(\xi_{i,j}^{0}) +$$

$$+h^2k \sum_{i=0}^{M_1-1} \sum_{j=0}^{M_2-1} \sum_{n=0}^{N-1} \frac{(\xi_{i+1,j}^{n} - \xi_{i,j}^{n})(U_{i+1,j}^{n} - U_{i,j}^{n})}{h^2} +$$

$$+h^2k \sum_{i=0}^{M_1-1} \sum_{j=0}^{M_2-1} \sum_{n=0}^{N-1} \frac{(\xi_{i,j+1}^{n} - \xi_{i,j}^{n})(U_{i,j+1}^{n} - U_{i,j}^{n})}{h^2} = 0.$$

If we make use of the notation and definitions which we introduced above, we can rewrite this last equality as follows:

$$-\int_{k}^{T+k}\int_{h}^{M_2}\int_{h}^{M_1} a(\tilde{u}_h)\Big(\frac{\Delta\tilde{\xi}^{n}}{k}\Big)\, dx\, dy\, dt - h^2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} a(\phi_{i,j})(\xi_{i,j}^{0}) +$$

$$+\int_{0}^{T}\int_{0}^{M_2}\int_{0}^{M_1} \Big[\Big(\frac{\Delta\tilde{u}_i}{h}\Big)\Big(\frac{\Delta\tilde{\xi}_i}{h}\Big) + \Big(\frac{\Delta\tilde{u}_j}{h}\Big)\Big(\frac{\Delta\tilde{\xi}_j}{h}\Big)\Big]\, dx\, dy\, dt = 0. \qquad (3.32)$$

In this equality we pass to the limits: $h \to 0$, $k \to 0$ where h and k run through subsequences $\{h_l\}$ and $\{k_l\}$.

The piecewise constant sequences of functions $\{\frac{\Delta\tilde{\xi}^n}{k}\}$, $\{\frac{\Delta\tilde{\xi}_i}{h}\}$, and $\{\frac{\Delta\tilde{\xi}_j}{h}\}$ converge strongly to $\frac{\partial\xi}{\partial t}$, $\frac{\partial\xi}{\partial x}$, and $\frac{\partial\xi}{\partial y}$ in $L_2(\Omega_t)$. $a(\tilde{u}_h)$, $\frac{\Delta\tilde{u}_i}{h}$, and $\frac{\Delta\tilde{u}_j}{h}$ converge weakly to $a(u)$, $\frac{\partial u}{\partial x}$, and $\frac{\partial u}{\partial y}$, respectively. (see [29]) Because of the smoothness of the initial function $a_0(x,y) = a(\phi(x,y))$, we have

$$h^2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} a(\phi_{i,j})(\xi^0_{i,j}) \to \int_0^{M_2} \int_0^{M_1} \xi\ a(\phi(x,y))\ dx\ dy$$

$$= \int_0^{M_2} \int_0^{M_1} \xi a_0(x,y)\ dx\ dy.$$

Therefore, we get

$$\iiint_{\Omega_t} a(u)\frac{\partial\xi}{\partial t}\ dx\ dy\ dt - \iiint_{\Omega_t} \left[\frac{\partial u}{\partial x}\frac{\partial\xi}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial\xi}{\partial y} \right] dx\ dy\ dt$$

$$+\int_0^{M_2} \int_0^{M_1} \xi(x,y,0)\ a_0(x,y)\ dx\ dy = 0. \tag{3.33}$$

By the smoothness of ξ and the fact that $u \in W^1_2(\Omega_t)$,

$$\iiint_{\Omega_t} \left[\frac{\partial u}{\partial x}\frac{\partial\xi}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial\xi}{\partial y} \right] dx\ dy\ dt$$

$$= \int_0^T \int_0^{M_2} u\ \frac{\partial\xi}{\partial x}\Big|_{x=M_1} dydt - \int_0^T \int_0^{M_2} u\ \frac{\partial\xi}{\partial x}\Big|_{x=0} dydt +$$

$$+\int_0^T \int_0^{M_1} u \frac{\partial \xi}{\partial y} \Big|_{y=M_2} dxdt - \int_0^T \int_0^{M_1} u \frac{\partial \xi}{\partial y} \Big|_{y=0} dxdt -$$

$$- \iiint_{\Omega_t} u \left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} \right) dx\, dy\, dt$$

$$= - \int_0^T \int_{S_t} \xi \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) dS\, dt - \iiint_{\Omega_t} u \left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} \right) dxdydt$$

$$= - \iiint_{\Omega_t} u \left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} \right) dxdydt. \tag{3.34}$$

If we substitute equation (3.34) into equation (3.33), we obtain

$$\iiint_{\Omega_t} a(u) \frac{\partial \xi}{\partial t} dx\, dy\, dt + \iiint_{\Omega_t} u \left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} \right) dx\, dy\, dt$$

$$+ \int_0^{M_2} \int_0^{M_1} \xi(x,y,0)\; a_0(x,y)\; dx\, dy = 0.$$

This completes the proof of theorem 3.

<u>Theorem</u> <u>4</u>. Suppose the assumptions of theorem 3 hold. Then there exists a unique, generalized solution of equation (3.6) which satisfies conditions (3.7) and (3.8).

<u>Proof</u>. Assume that u, v are two generalized solutions of equation (3.6). Substitute both solutions into the equation (3.5) (definition of a generalized solution) and subtract the resulting equation to get

$$\iiint_{\Omega_t} [(u-v)\; \Delta\xi + (a(u) - a(v)) \frac{\partial \xi}{\partial t}]\; dx\, dy\, dt$$

$$= \iiint_{\Omega_t} (a(u)-a(v)) \left(\frac{\partial \xi}{\partial t} + \frac{u - v}{a(u) - a(v)} \Delta\xi \right) dx\, dy\, dt = 0, \tag{3.35}$$

for $\xi \in C_0^2(\overline{\Omega_t})$ and $\xi = 0$ on $\partial\Omega_t$ and $t = T$.

Let

$$e(x,y,t) = \frac{u(x,y,t) - v(x,y,t)}{a(u(x,y,t)) - a(v(x,y,t))} \quad \text{if } u(x,y,t) \neq v(x,y,t)$$

$$e(x,y,t) = 0 \quad \text{if } u(x,y,t) = v(x,y,t), \tag{3.36}$$

then $e(x,y,t)$ is a measurable function on Ω_t.

Using the fact that $\frac{a(u) - a(v)}{u-v} \geq \eta(M_0)$

where $M_0 = \sup\{|u|, |v|\}$, we obtain

$$0 < e(x,y,t) \leq \frac{1}{\eta(M_0)} \quad \text{if } u \neq v$$

$$e(x,y,t) = 0 \quad \text{if } u = v. \tag{3.37}$$

Hence, $e(x,y,t)$ is a bounded measurable function on Ω_t.

We now approximate e by a sequence of smooth functions $\{\overline{e_n}(x,y,t)\}$ which converges to $e(x,y,t)$ in measure on $L_2(\Omega_t)$ (see 29), that is, choose a sequence such that

$$0 \leq \overline{e_n}(x,y,t) \leq \frac{1}{\eta(M_0)}$$

and

$$\| e - \overline{e_n} \|_{L^2(\Omega_t)} \leq \frac{1}{n} .$$

Set $$e_n(x,y,t) = \overline{e_n}(x,y,t) + \frac{1}{n} .$$

This implies that

$$\frac{1}{n} \leq e_n \leq 1 + \frac{1}{\eta(M_0)} , \tag{3.38}$$

and $$\| e_n - e \|_{L^2(\Omega_t)} \leq \| e_n - \overline{e_n} \|_{L^2(\Omega_t)} + \| \overline{e_n} - e \|_{L^2(\Omega_t)}$$

$$\leq \frac{1}{n} + \frac{1}{n} (\text{meas}(\Omega_t))^{\frac{1}{2}} \to 0 \text{ as } n \to \infty. \tag{3.39}$$

Furthermore,

$$\left\| \frac{e_n - e}{e_n} \right\|_{L^2(\Omega_t)} \leq \frac{1}{\min(e_n)} \|e_n - e\|_{L^2(\Omega_t)}$$

$$\leq n\left(\frac{1}{n} + \frac{1}{n} (\text{meas}(\Omega_t))^{\frac{1}{2}} \right) = 1 + (\text{meas}(\Omega_t))^{\frac{1}{2}}.$$

From the last inequality and the triangle inequality, we have

$$\left\| \frac{e}{e_n} \right\|_{L^2(\Omega_t)} = \left\| \frac{e - e_n}{e_n} + 1 \right\|_{L^2(\Omega_t)}$$

$$\leq \left(1 + (\text{meas}(\Omega_t))^{\frac{1}{2}}\right) + (\text{meas}(\Omega_t))^{\frac{1}{2}} = K,$$

where K is a constant and does not depend on n.

Let us consider the sequence of problems

$$\frac{\partial \xi_n}{\partial t} + e_n \Delta \xi_n = \Phi \quad \text{on } \Omega_t \tag{3.40}$$

$$\xi_n = 0 \quad \text{on } \partial\Omega_t \tag{3.41}$$

$$\xi_n|_{t=T} = 0 \quad \text{on } G_T \tag{3.42}$$

for any $\Phi \in C_0^\infty(\overline{\Omega_t})$.

It is not difficult to show that there exists a $\xi_n(x,y,t) \in C^2(\overline{\Omega_t})$, which satisfies the equations (3.40) -(3.42) (see [29]).

We will show that $|\xi_n| < K_1$ for a constant K_1 which does not depend on n.

Let

$$\xi_n = \overline{\xi_n}\, e^{-\epsilon t} \quad (\epsilon > 0, \text{ any constant}). \tag{3.43}$$

Putting (3.43) into (3.40) and multiplying both sides of (3.40) by $e^{\epsilon t}$, we get

$$\frac{\partial \overline{\xi_n}}{\partial t} + e_n \, \Delta \overline{\xi_n} - \epsilon \overline{\xi_n} = \Phi \, e^{\epsilon t}.$$

If $\overline{\xi_n}$ has a positive maximum at an arbitrary point $P \in \Omega_t$, then

$$\frac{\partial \overline{\xi_n}}{\partial t} \le 0, \; \Delta \overline{\xi_n} \le 0, \; \overline{\xi_n} > 0$$

where

$$0 \le t \le T.$$

This implies that

$$| \; \epsilon \; \overline{\xi_n}(P) \; | \le | \; \Phi \; e^{\epsilon t} \; |.$$

Furthermore,

$$|\max \; \overline{\xi_n}(P) \; | \le \frac{1}{\epsilon} \; |\Phi \; e^{\epsilon t} \; |.$$

Let us multiply both sides of equation (3.40) by $\Delta \xi_n$ and integrate on Ω_t, we get

$$\iiint_{\Omega_t} e_n (\Delta \xi_n)^2 dx\,dy\,dt + \iiint_{\Omega_t} \frac{\partial \xi_n}{\partial t} \Delta \xi_n \, dx\,dy\,dt$$

$$= \iiint_{\Omega_t} e_n (\Delta \xi_n)^2 dx\,dy\,dt - \frac{1}{2} \iiint_{\Omega_t} \frac{\partial}{\partial t} (\nabla \xi_n)^2 dx\,dy\,dt$$

$$= \iiint_{\Omega_t} e_n (\Delta \xi_n)^2 dx\,dy\,dt + \frac{1}{2} \iint_{G_t} (\nabla \xi_n)^2 \, dx\,dy$$

$$= \iiint_{\Omega_t} \Delta \xi_n \, \Phi \, dx\,dy\,dt = \iiint_{\Omega_t} \xi_n \, \Delta \Phi \, dx\,dy\,dt,$$

since $\Phi = 0$ on $\partial \Omega_t$.

Therefore,

$$\iiint_{\Omega_t} e_n (\Delta \xi_n)^2 dx\,dy\,dt \le \iiint_{\Omega_t} \xi_n \, \Delta \Phi \, dx\,dy\,dt \le K_2 \qquad (3.44)$$

where K_2 is a constant which does not depend on n.

Substitute $\xi = \xi_n$ into equation (3.35) and use equation (3.40). We get

$$0 = \iiint_{\Omega_t} (a(u) - a(v))(\frac{\partial \xi_n}{\partial t} + e\, \Delta\xi_n)\, dx\, dy\, dt$$

$$= \iiint_{\Omega_t} (a(u) - a(v))(\frac{\partial \xi_n}{\partial t} + e_n\Delta\xi_n + (e - e_n)\, \Delta\xi_n)\, dx\, dy\, dt$$

$$= \iiint_{\Omega_t} (a(u) - a(v))\, \Phi\, dx\, dy\, dt +$$

$$+ \iiint_{\Omega_t} (a(u) - a(v))(e - e_n)\, \Delta\xi_n\, dx\, dy\, dt$$

$$= \iiint_{\Omega_t} (a(u) - a(v))\, \Phi\, dx\, dy\, dt + J_0 ,$$

where

$$J_0 = \iiint_{\Omega_t} (a(u) - a(v))(e - e_n)\, \Delta\xi_n\, dx\, dy\, dt.$$

If we prove that $J_0 \to 0$ as $n \to \infty$, then

$$\iiint_{\Omega_t} (a(u)-a(v))\, \Phi\, dx\, dy\, dt = 0.$$

J_0 is estimated as follows.

Let

$$J_1 = \iiint_{\Omega_t} |e - e_n||\Delta\xi_n|\, dx\, dy\, dt.$$

This implies that

$$J_1 = \iiint_{\Omega_t} (\sqrt{e} + \sqrt{\overline{e_n}})\,|\sqrt{e} - \sqrt{\overline{e_n}}|\,|\Delta\xi_n|\ dx\ dy\ dt.$$

$$= \iiint_{\Omega_t} \sqrt{\overline{e_n}}\ |\sqrt{e} - \sqrt{\overline{e_n}}|\,|\Delta\xi_n|\ dx\ dy\ dt\ +$$

$$+ \iiint_{\Omega_t} \sqrt{e}\ |\sqrt{e} - \sqrt{\overline{e_n}}|\,|\Delta\xi_n|\ dx\ dy\ dt \tag{3.45}$$

$$= J_2 + J_3\ ,$$

where

$$J_2 = \iiint_{\Omega_t} \sqrt{\overline{e_n}}\ |\sqrt{e} - \sqrt{\overline{e_n}}|\,|\Delta\xi_n|\ dx\ dy\ dt\ ,$$

and

$$J_3 = \iiint_{\Omega_t} \sqrt{e}\ |\sqrt{e} - \sqrt{\overline{e_n}}|\,|\Delta\xi_n|\ dx\ dy\ dt.$$

We first estimate J_2.

Using (3.44) and the Schwarz inequality, we obtain

$$J_2 \le \|\sqrt{e} - \sqrt{\overline{e_n}}\|_{L^2(\Omega_t)}\ \|\sqrt{\overline{e_n}}\ \Delta\ \xi_n\|_{L^2(\Omega_t)}$$

$$\le \|\sqrt{e} - \sqrt{\overline{e_n}}\|_{L^2(\Omega_t)}\ K_2^{\frac{1}{2}} \ \to 0 \text{ as } n \to \infty. \tag{3.46}$$

Since both $\{\overline{e_n}\}$ and $\{e_n\}$ converge to e in measure, $\sqrt{\overline{e_n}}$ converges to $\sqrt{e}$ in measure.

It remains to estimate J_3.

Given an $\epsilon > 0$, let $E_t = \{(x,y,t) \in \Omega_t \,|\ |\sqrt{\overline{e_n}} - \sqrt{e}| > \epsilon\}$.

Because $e_n \to e$ in measure, for any $\delta > 0$ there exists an

$M = M(\epsilon,\delta)$ such that $\text{meas}(E_t) < \delta$ if $n \geq M$.

Setting $F_t = \Omega_t - E_t$ and using (3.37) and (3.38)

$$J_3 = \iiint_{E_t} \sqrt{e}\ |\sqrt{e} - \sqrt{e_n}|\,|\Delta\xi_n|\ dx\ dy\ dt\ +$$

$$+ \iiint_{F_t} \sqrt{e}\ |\sqrt{e} - \sqrt{e_n}|\,|\Delta\xi_n|\ dx\ dy\ dt$$

$$\leq \left(\frac{1}{\eta(M_0)} + 1\right)^{\frac{1}{2}} \iiint_{E_t} \sqrt{e}\ |\Delta\xi_n|\ dx\ dy\ dt$$

$$+ \epsilon \iiint_{F_t} \sqrt{e}\ |\Delta\xi_n|\ dx\ dy\ dt$$

$$\leq \left(\frac{1}{\eta(M_0)} + 1\right)^{\frac{1}{2}} \iiint_{E_t} \frac{\sqrt{e}}{\sqrt{e_n}} \sqrt{e_n}\ |\Delta\xi_n|\ dx\ dy\ dt$$

$$+ \epsilon \iiint_{F_t} \frac{\sqrt{e}}{\sqrt{e_n}} \sqrt{e_n}\ |\Delta\xi_n|\ dx\ dy\ dt$$

Next an application of the Schwarz inequality yields

$$\leq \left(\frac{1}{\eta(M_0)} + 1\right)^{\frac{1}{2}} \|\sqrt{e_n}\ \Delta\ \xi_n\|_{L^2(E_t)} \left\| \frac{\sqrt{e}}{\sqrt{e_n}} \right\|_{L^2(E_t)} +$$

$$+ \epsilon \left\| \frac{\sqrt{e}}{\sqrt{e_n}} \right\|_{L^2(F_t)} \|\sqrt{e_n}\ \Delta\ \xi_n\|_{L^2(F_t)}$$

$$\leq K_3 \left\| \frac{\sqrt{e}}{\sqrt{e_n}} \right\|_{L^2(E_t)} + K_4\ \epsilon$$

$$\leq K_3 \left\{ \iiint_{E_t} \frac{e}{e_n}\ dx\ dy\ dt \right\}^{\frac{1}{2}} + K_4\ \epsilon$$

$$\leq K_3 \left\| \frac{e}{e_n} \right\|^{\frac{1}{2}}_{L^2(E_t)} (\text{meas}(E_t))^{\frac{1}{4}} + K_4 \, \epsilon .$$

Therefore,

$$J_3 \leq K_5 \, \delta^{\frac{1}{4}} + K_4 \, \epsilon \to 0 \text{ as } n \to \infty , \qquad (3.47)$$

where

$$K_3 = \left(\frac{1}{\eta(M_0)} + 1\right)^{\frac{1}{2}} \|\sqrt{e_n} \, \Delta \, \xi_n\|_{L^2(E_t)} ,$$

$$K_4 = \left\| \frac{\sqrt{e}}{\sqrt{e_n}} \right\|_{L^2(F_t)} \|\sqrt{e_n} \, \Delta \, \xi_n\|_{L^2(F_t)} ,$$

and

$$K_5 = K_3 \left\| \frac{e}{e_n} \right\|^{\frac{1}{2}}_{L^2(E_t)} .$$

K_3, K_4 and K_5 are constants which are independent of n. Substituting (3.46) and (3.47) into (3.45) leads to

$$J_1 \to 0 \text{ as } n \to \infty .$$

Consequently,

$$\iiint_{\Omega_t} (a(u)-a(v)) \, \Phi \, dx \, dy \, dt = 0 . \qquad (3.48)$$

for any $\Phi \in C_0^\infty(\overline{\Omega_t})$.

We have proved (3.48) for any $\Phi \in C_0^\infty(\overline{\Omega_t})$. Now let Φ be any function in $L^2(\overline{\Omega_t})$ and construct a sequence of functions Φ_i in $C_0^\infty(\overline{\Omega_t})$ such that $\|\Phi - \Phi_i\|_{L^2(\Omega_t)} \to 0$.

Then (3.48) holds for $\Phi_i \in L^2(\overline{\Omega_t})$. Hence the equality holds also for $\Phi \in L^2(\overline{\Omega_t})$.

From equation (3.48), we get

$$a(u) = a(v).$$

This implies that $u = v$.

Thus we have proven theorem 4.

The proof of the regularity of the solutions of parabolic systems of partial differential equations is found in [24] and [36].

4. Numerical Method

4.1 Finite difference algorithms

In this chapter we shall give a numerical scheme and a proof for the convergence of the algorithm approximating solutions to the our problems which are given in 2.2. Let $\nu(x,y) = (\nu_1(x,y), \nu_2(x,y))$ be the interior unit normal to ∂G_t at the point (x,y). This is a change in notation from Chapter 3, but it is more convenient here to use the interior normal. Then the exterior unit normal is the negative of the interior unit normal to ∂G_t at a certain point. If $h \in \mathbb{R}^2$, let $|h| = (h_1^2 + h_2^2)^{\frac{1}{2}}$. Then under suitable conditions on f, ϕ, and G_t, we will show that the difference schemes constructed below converge in the maximum norm to the solution (2.1) – (2.4) like $O(|h| + k)$ as $|h|$, $k \to 0$, where h_1, h_2, and k are the grid increments for the variables x, y, and t respectively.

The convergence proofs are simple but need the following assumption on the domain G_t, which we shall refer to from now as the assumptions (A) and (B).

Assumption (A). Let G_t be such that the solution to (2.1) – (2.4) exists. With the exception of at most a finite number of points, ∂G_t is continuously differentiable. Denote by $\partial G_t'$ the set of points in ∂G_t where ∂G_t is continuously differentiable. For $(x,y) \in \partial G_t'$, construct the straight line originating in (x,y) in the direction of the interior normal $\nu(x,y)$ and denote this line by $L_\nu(x,y)$. Let $S_t(x,y) = \{(x',y') \in \mathbb{R}^2 | L_\nu(x,y) \cap (G_t \setminus (x,y))\}$. We now assume , for each $(x,y) \in \partial G_t^2$, there exists an $(x'',y'') \in S_t(x,y)$ such that

$$\delta(x,y) = \inf_{(x',y') \in S_t(x,y)} |(x,y) - (x',y')|$$

$= |(x,y) - (x'',y'')|$ and further there exists a number $\delta > 0$ such that $\inf_{(x,y) \in \partial G_t^2} \delta(x,y) = \delta > 0$. (see Fig. 4.1)

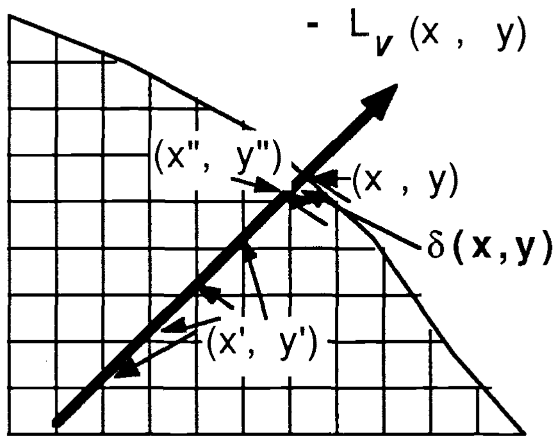


Figure 4.1. The Sketch of $\delta(x,y)$.

One can easily verify that if ∂G_t is twice continuously differentiable, then G_t satisfies assumption (A). Furthermore the "usual" domains arising in practice satisfy assumption (A) and so do our domains.

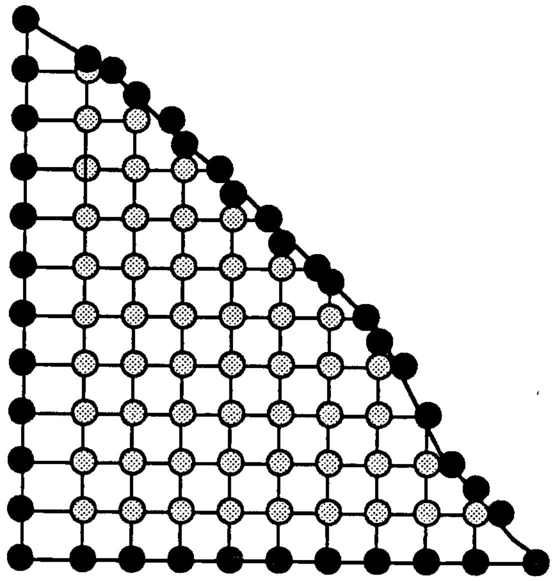
Assumption (B): Let $T > 0$ be a real number. We denote the closed interval $0 \le t \le T$ by $[0,T]$. We shall assume that the known functions $f(x,y,t)$ and $\phi(x,y)$ are defined and continuous on the closed sets $\partial G_t \times [0,T]$ and $\overline{G_0}$ respectively, where $\partial G_t \times [0,T]$ is the topological product of ∂G_t with $[0,T]$, etc.. In addition to requiring that G_t satisfy assumption (A), we assume that ∂G_t is everywhere at least once continuously differentiable. Finally, we assume that f, ϕ, and ∂G_t are constructed so that the solutions $u(x,y,t)$ to (2.1) – (2.4), exist, are uniquely determined, and are three times continuously differentiable with respect to the space variables and twice with respect to the t variable. More precise statements on the assumptions on f, ϕ, and G_t guaranteeing the existence and uniqueness of solutions to (2.1) – (2.4), having the required differentiability properties may be found in Friedman [22] and Il'in, Kalashnikov, and Oleinik [28]. As usual, place a rectangular grid on the three dimensional space (x, y, t) with grid spacing h_1 in the x variable, h_2 in the y variable and k in the t variable. Let $G_t(h)$ be the set of grid points in the open set G_t corresponding to the cross-section of time $t = t_n = nk$. Denote by $\partial G_t(h)$ the set of points where either an x, or a y, or both an x and a y grid line intersects ∂G_t (see Fig. 4.2).

Let $\overline{G_t}(h) = G_t(h) \cup \partial G_t(h)$.

Assume $R_{h,k} = \{(x, y, t) \mid (x, y) \in G_t(h),\ t=t_n,\ n=1, \ \ldots \ ,N\}$

and

$$S_{h,k} = \{(x, y, t) \mid (x, y) \in \partial G_t(h),\ t=t_n,\ n=1, \ \ldots \ ,N\}.$$



$G_t(h)$ = { ◍ | ◍ : Grid points in G_t }

$\partial G_t(h)$ = { ● | ● : Grid line intersects either an x, or a y, or both an x and y}

Figure 4.2. The Sketch of $G_t(h)$ and $\partial G_t(h)$.

Now let $(x,y) \in G_t(h)$. Let $0 < \mu, \lambda, \rho, \sigma \leq 1$ be four numbers depending on (x, y) such that the points $(x+\mu h_1, y)$, $(x-\lambda h_1, y)$, $(x, y+\rho h_2)$, $(x, y-\sigma h_2)$ are in $\overline{G_t}(h)$. We assume $W(x, y,t)$ to be a function defined on $\overline{R}_{h,k}$. Then for $(x,y,t) \in R_{h,k}$ we define the difference operators (see Fig. 4.3).

$$\Delta_h W(x, y, t) = \alpha W(x+\mu h_1, y, t) + \beta W(x-\lambda h_1, y, t)$$
$$+ \gamma W(x, y+\rho h_2, t) + \theta W(x, y-\sigma h_2, t) + \omega W(x, y, t), \quad (4.1)$$

where $\alpha = 2 / h_1{}^2(\mu\lambda + \mu^2)$, $\beta = 2 / h_1{}^2(\mu\lambda + \lambda^2)$,

$\gamma = 2 / h_2{}^2(\rho\sigma + \rho^2)$, $\theta = 2 / h_2{}^2(\rho\sigma + \sigma^2)$, and

$\omega = -2 / h_1{}^2\mu\lambda - 2 / h_2{}^2\rho\sigma$, and

$$D_t\, W(x,y,t) = [W(x,y,t) - W(x,y,t-k)] / k \tag{4.2}$$

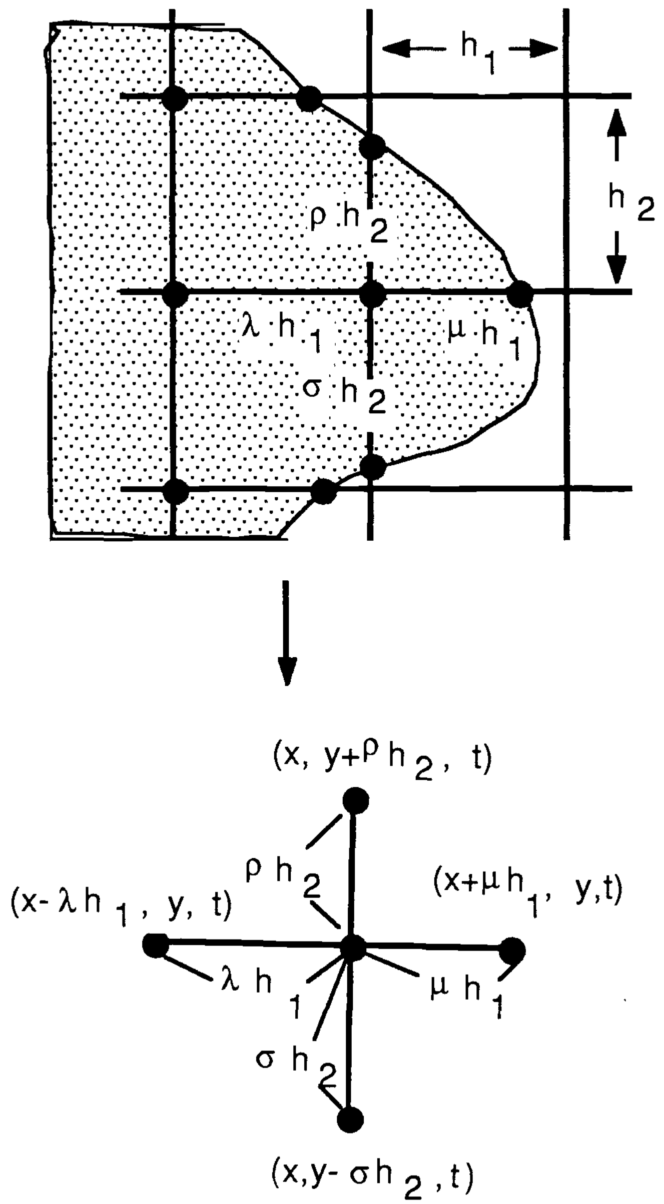


Figure 4.3. The Grid on $G_t(h)$.

If $W(x,y,t)$ is three times continuously differentiable with respect to x, y and twice with respect to t in $G_t \times [0,T]$, then $\Delta_h W - \Delta W = O(|h|)$, $|h| \to 0$, and $D_t W - W_t = O(k)$, $k \to 0$ for $(x,y,t) \in R_{h,k}$.

We now turn to the problem of replacing the normal derivative by a difference quotient. Let $L_\nu(x,y)$ denote the line lying in $G_t \cup \{(x,y)\}$, which originates in $(x,y) \in \partial G_t$ and proceeds along the interior normal ν. There are several cases to consider.

Case 1. Let $(x,y) \in \partial G_t(h)$ and suppose $L_\nu(x,y)$ lies on a grid line, either an x or a y grid line (see Figure 4.4). Denote the grid point on $L_\nu(x,y)$ closest to (x,y) by (x',y') and set

$$D_\nu W(x,y,t) = [W(x',y',t) - W(x,y,t)] \,/\, |(x',y') - (x,y)|. \quad (4.3)$$

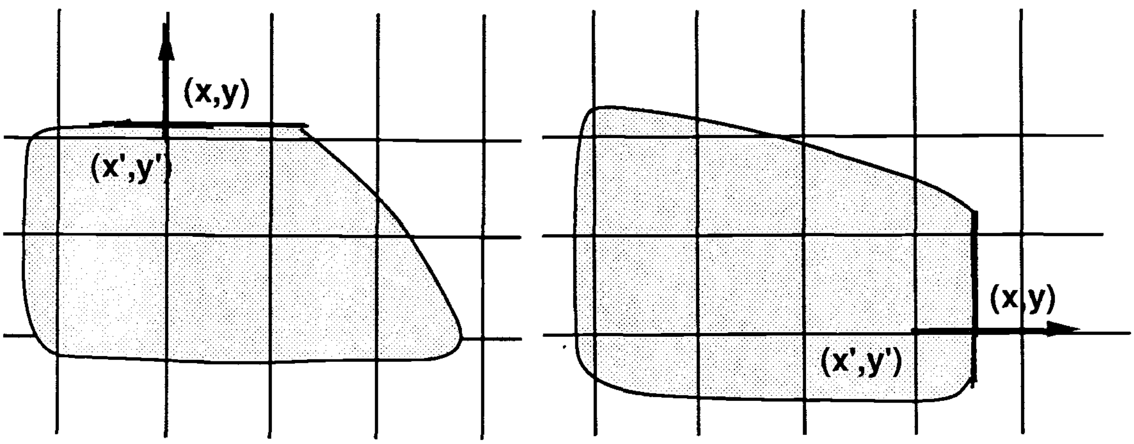


Figure 4.4. The Point (x',y') of Case 1.

Case 2. Let $(x, y) \in \partial G_t(h)$ and suppose $L_\nu(x, y)$ intersects a grid line, either an x or a y grid line (see Fig. 4.5) at a point (ζ, η) where $|(x,y)-(\zeta,\eta)|<|h|$, and where (ζ, η) lies between a point $(x', y') \in G_t(h)$ and a point $(x'', y'') \in \overline{G_t}(h)$ on the x or y grid line as the case may be. Define $W(\zeta,\eta,t)$ by interpolating linearly between $W(x',y',t)$ and $W(x'',y'',t)$ and then set

$$D_\nu W(x,y,t) = [\,W(\zeta,\eta,t) - W(x,y,t)] \,/\, |(\zeta,\eta) - (x,y)|. \qquad (4.4)$$

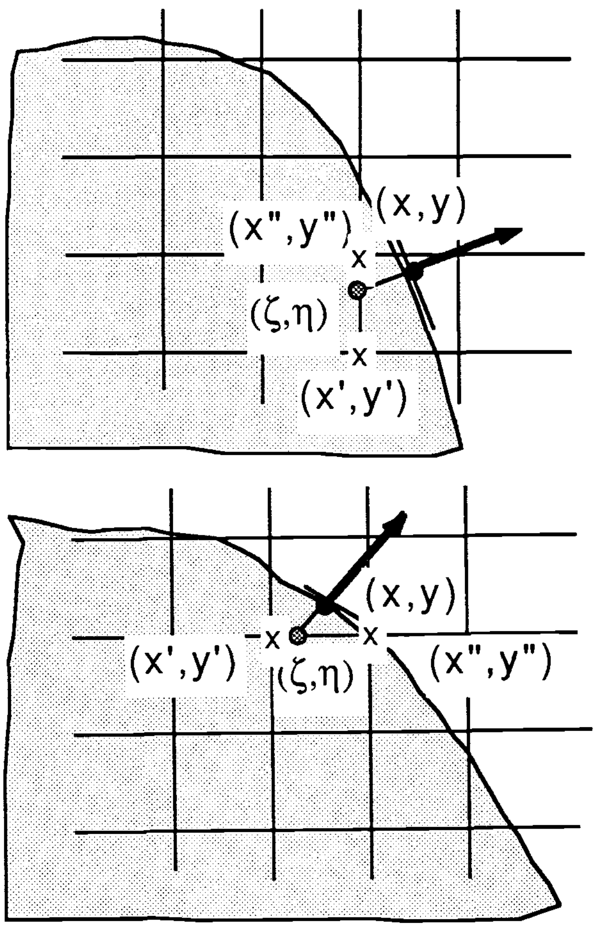


Figure 4.5. The Point (ζ,η) of Case 2.

Case 3. Let $x=(x,y) \in \partial G_t(h)$ and suppose $L_\nu(x,y)$ intersects a point $(x',y') \in G_t(h)$, (see Figure 4.6), $|(x,y) - (x',y')| < |h|$. Then set

$D_\nu W(x,y,t)$

$$= [\, W(x',y',t) - W(x,y,t)] \,/\, |(x',y') - (x,y)|. \quad (4.5)$$

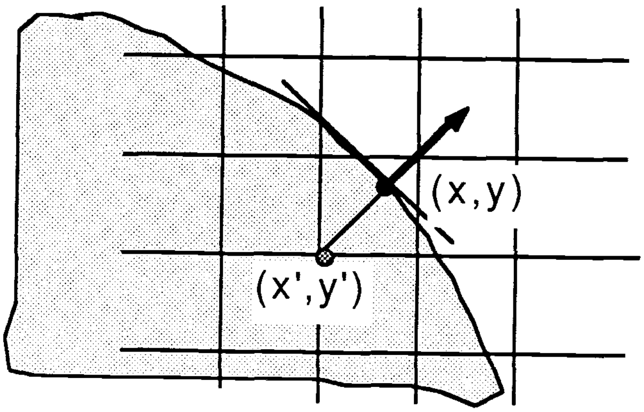


Figure 4.6. The Point (x',y') of Case 3.

It is a consequence of assumption (A) that for $|h|$ chosen sufficiently small, only these cases need to be considered in replacing the normal derivative by a difference quotient and at least one of these cases does occur. From now on we assume that $|h|$ has been chosen sufficiently small so that $D_\nu W(x,y,t)$ is well-defined. If $W(x,y,t)$ is a solution to (2.1) – (2.4) having the required differentiability properties, then for $(x,y,t) \in S_{h,k}$,

$$D_\nu W(x,y,t) - \frac{\partial W(x,y,t)}{\partial \nu} = O(|h|), \quad |h| \to 0.$$

The finite difference schemes for approximating solutions to (2.1) – (2.4) may be formulated as follows. We seek a function, $U(x,y,t)$ for each t, defined for $(x,y,t) \in \bar{R}_{h,k}$ such that

$$D_t U(x,y,t) = \Delta_h U(x,y,t), \quad (x,y,t) \in R_{h,k}, \tag{4.6}$$

$$U(x,y,0) = \phi(x,y), \quad (x,y) \in \bar{G}_0(h), \tag{4.7}$$

$$f(x,y,t) = 0 \text{ for } (x,y) \in S_{h,k} \tag{4.8}$$

$$f_t(x,t) + D_\nu U(x,t)\cdot \nu = 0 \text{ on } (x,y) \in S_{h,k}. \tag{4.9}$$

We shall show that for each h, k the solution $U(x,y,t)$ to the problem (4.6) – (4.9) exists and is uniquely defined. Then we show that the family $\{U\}$ converges to u as $|h|$, $k \to 0$, where u is the solution to (2.1) – (2.4).

<u>Theorem</u> <u>5</u>. The solutions to the difference scheme (4.6) – (4.9) converge weakly to the solution of problem (2.1) – (2.4).

<u>Proof</u>. Let (x,y,t) be an arbitrary point where $(x,y) \in \mathbb{R}^2$ and $0 \leq t \leq T$. Let (x',y',t') and (x'',y'',t'') be mesh points satisfying $x' < x < x''$, $y' < y < y''$, let t be such that $t' < t \leq t''$, and define the function $U(x,y,t;h)$ by interpolating linearly between $U(x',y',t')$ and $U(x'',y'',t'')$. Finally, let $r = \frac{k}{h^2}$ be fixed. Rewrite (3.14) using $D_x^+ U(x,y,t)$ and $D_y^+ U(x,y,t)$ (the notation is given on page 18) and replace $U(x,y,t)$ by $U(x,y,t;h)$. Then the summations over $\Omega_t(h)$ on the left hand side in (3.14) become integrals of $U(x,y,t;h)$ taken over $\mathbb{R}^2 \times (0,t'')$. (In fact, the integrals are over $G_{t''}$, since the $U(x,y,t;h)$ vanish outside $G_{t''}$).

From this point on, we proceed in a manner similar to that given in [28]. The functions $D_x^+ U(x,y,t;h)$ and $D_y^+ U(x,y,t;h)$ are uniformly bounded in $L^2(G_t)$ and for any t, in $L^2(\Omega_t)$ since $D_x^+ U(x,y,t)$ and $D_y^+ U(x,y,t)$ are uniformly bounded (see lemma 2). Consequently, we can choose subsequences $\{U(x,y,t;h_l)\}_{l=1}^{\infty}$, $\{D_x^+ U(x,y,t;h_l)\}_{l=1}^{\infty}$ and $\{D_y^+ U(x,y,t;h_l)\}_{l=1}^{\infty}$, $h_l > h_{l+1} \to 0$ as $l \to \infty$, r constant, which converge weakly to functions $u(x,y,t) \in L^2(\Omega_t)$, $u_x(x,y,t) \in L^2(\Omega_t)$, and $u_y(x,y,t) \in L^2(\Omega_t)$.

As in [28], we can conclude that $u_x(x,y,t)$ and $u_y(x,y,t)$ are the generalized derivatives of $u(x,y,t)$, i.e. $u_x = \frac{\partial u}{\partial x}$ and $u_y = \frac{\partial u}{\partial y}$. The limit function $u \in \overset{0}{W}{}^{1,1}(\Omega_t)$ satisfies the integral identity (3.5), and using the fact that $\phi(x,y) \geq 0$, $u(x,y,0) = \phi(x,y)$. The assertion that u satisfies (3.5) is proven by multiplying (3.6) by a function $\xi \in C_0^{\infty}(\Omega_t)$, $\xi(x,y,T) = 0$ which is continuously differentiable, summing the resulting equality by parts, multiplying by $h_l^2 k$ and then letting $h \to 0$ with r held constant to obtain (3.5). Therefore, we conclude that the limit function, $u(x,y,t)$, of $U(x,y,t;h_l)$ satisfies the problem (2.1) – (2.4) in the sense of (3.5). Since from any subfamily of the family $U(x,y,t;h_l)$, we can find a weakly convergent subsequence converging to a solution to (2.1) –(2.4) and since solutions to (2.1) –(2.4) are unique, we can conclude that the whole family $U(x,y,t;h)$

converges weakly to the unique solution to (2.1) –(2.4) like $O(|h| + k)$. This completes the proof of theorem 5.

5. Example

In this chapter, we solve a given problem using the finite difference method on uniform spatial grid points for all time steps. A complete set of flow charts illustrating the sequence of calculation by finite difference methods can be found in Appendix. As a model problem let us consider the following free boundary value problem. (see Fig. 5.1)

$$u_t = \Delta u \text{ in } G_t,\ t > 0. \tag{5.1}$$

$$u(x, y, 0) = e^{4-x^2-y^2} - e^3 \text{ on } G_0 \tag{5.2}$$

$$u(x,y,t) = f(x, y, t) = 0 \text{ on } \partial\Omega_t\ ,\ t \geq 0 \tag{5.3}$$

$$f_t + \nabla u \cdot \nabla f = 0 \text{ on } \partial\Omega_t\ ,\ t \geq 0 \tag{5.4}$$

$$\frac{\partial u}{\partial x}(0,y,t) = 0\ ,\ 0 \leq y \leq f(0,y,t) \tag{5.5}$$

$$\frac{\partial u}{\partial y}(x,0,t) = 0\ ,\ 0 \leq x \leq f(x,0,t)\,. \tag{5.6}$$

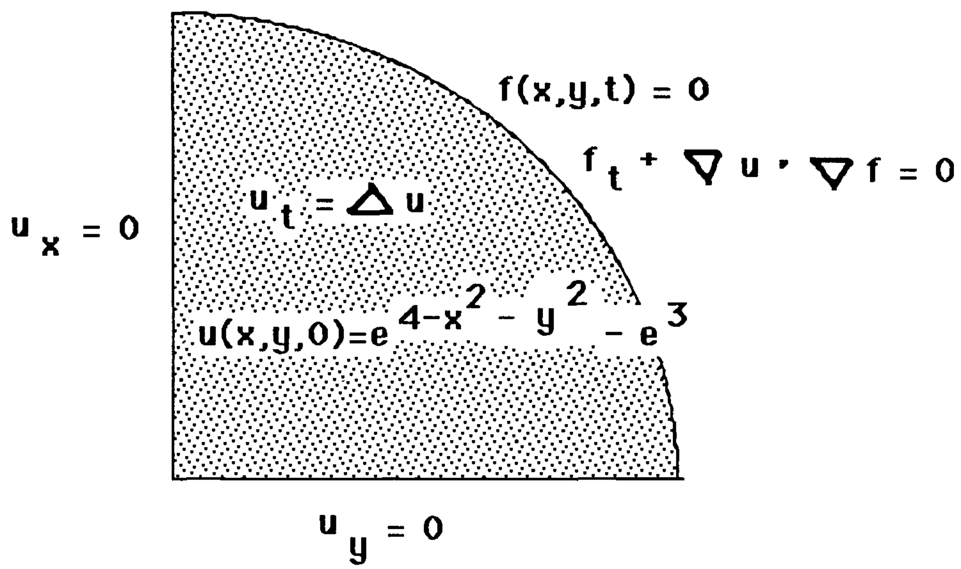


Figure 5.1. Test Problem.

Let us set

$$h_1 = \frac{1}{nh1+1} \quad , \text{ for a fixed integer } nh1 \geq 1$$

$$h_2 = \frac{1}{nh2+1} \quad , \text{ for a fixed integer } nh2 \geq 1$$

$$k = \frac{T}{N} \text{ , for a fixed integer } N \geq 1$$

$$x_i = ih_1 \text{ , } i = 1, \ldots , nh1 + 2$$

$$y_j = jh_2 \text{ , } j = 1, \ldots , nh2 + 2$$

$$t_n = nk \text{ , } n = 0, \ldots , N.$$

$$U^n_{i,j} = U(x_i, y_j, t_n).$$

where k stands for the time increment and h_1 and h_2 are increments of x-axis and y-axis, respectively.

Choose several points on $x^2 + y^2 = 1$ obtained by uniformly incrementing the angle (see Figure 5.2). Then solve the differential equation (5.4) by approximating the normal derivative with a difference quotient at chosen points and using the Euler method. In this way, we can find the location of the front at the new time step. Once we find the boundary, we will approximate the domain temperature using the appropriate difference replacement for the heat equation at the new time step. Calculate the midpoints of the front at the new time step and solve the equation (5.4) on those midpoints again (see Fig 5.3). These steps will give us a new free boundary. Repeat this procedure. We will get the free boundary and the domain temperature at each time step. As we identify the boundary, the points of intersection of the boundary with the grid lines or mesh points lie between the numerically computed front (see Fig 5.4), we will interpolate the front on those points using a 1-dimensional linear spline (see Fig. 5.5). We use two methods to calculate the domain temperature.

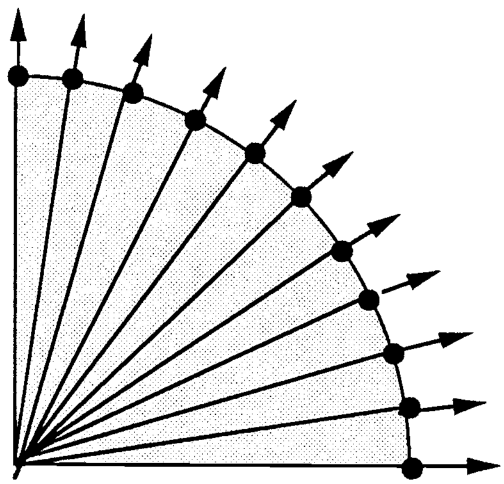


Figure 5.2. The Graph of Starting Points on $x^2 + y^2 = 1$.

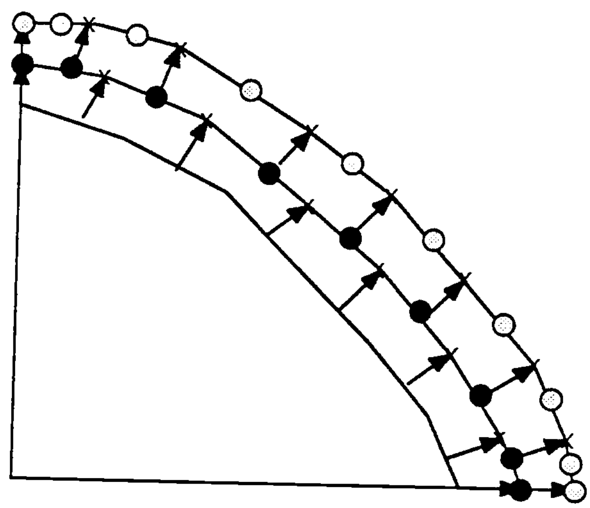


Figure 5.3. The Graph of the Midpoints of the boundaries.

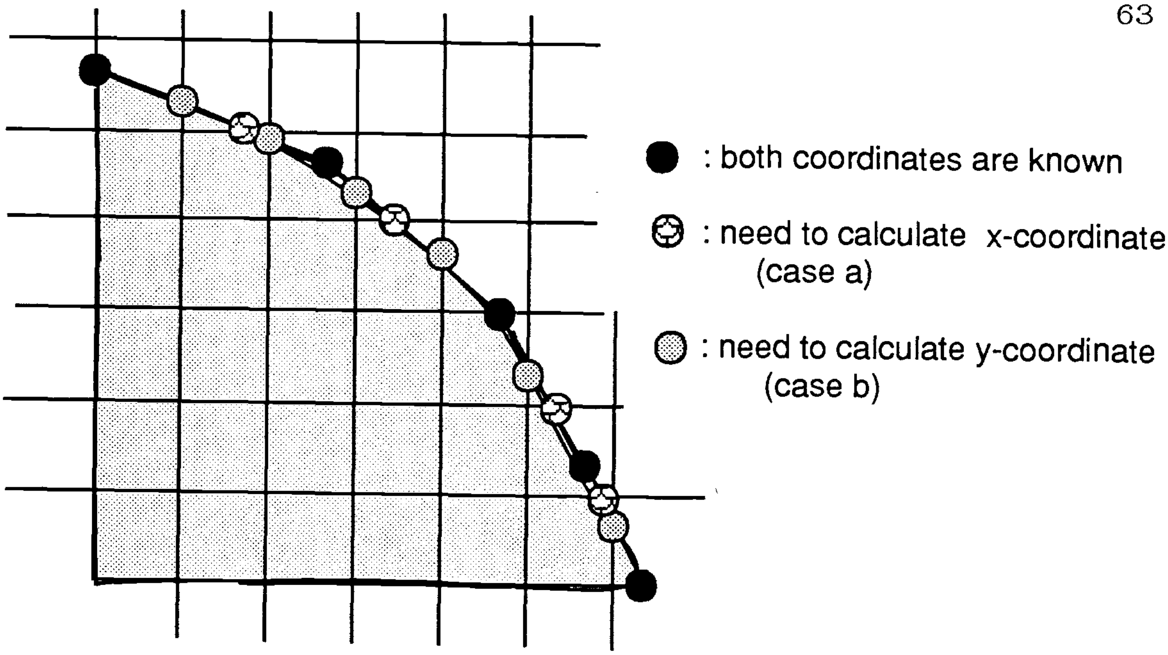
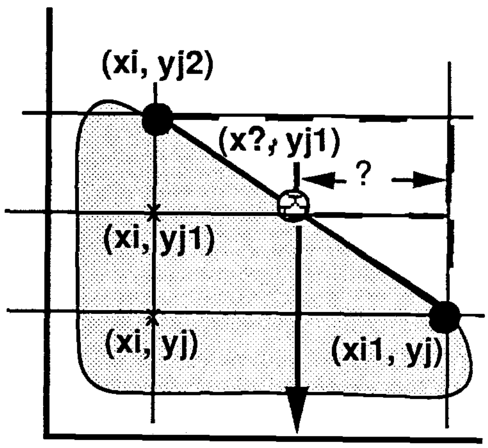
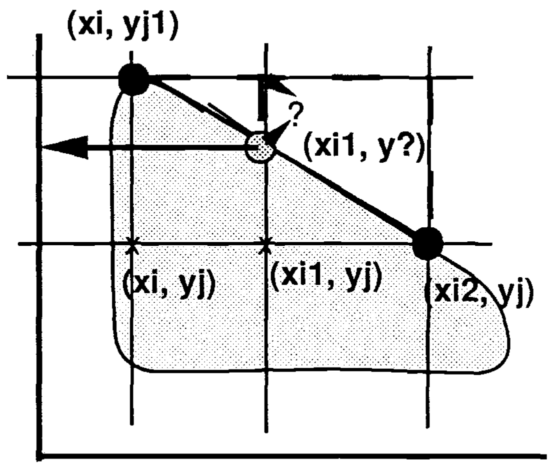


Figure 5.4 The Graph of the Points of the Intersection of the Boundary with the Grid Lines.



case a (find x-coordinate)



case b (find y-coordinate)

Figure 5.5. A 1-dimensional Spline

5.1. Forward difference in time

In this section, we use the forward difference for the time derivative u_t at the fixed time $t = t_n$ as

$$u_t = \frac{U^{n+1}_{i,j} - U^n_{i,j}}{k}. \tag{5.7}$$

Rewrite the right hand side of (5.1) as

$$\Delta u = \alpha U^n_{i+1,j} + \beta U^n_{i-1,j} + \gamma U^n_{i,j+1} + \theta U^n_{i,j-1} + \omega U^n_{i,j}, \tag{5.8}$$

where $\alpha = 2 / h_1{}^2(\mu\lambda + \mu^2)$, $\beta = 2 / h_1{}^2(\mu\lambda + \lambda^2)$,
$\gamma = 2 / h_2{}^2(\rho\sigma + \rho^2)$, $\theta = 2 / h_2{}^2(\rho\sigma + \sigma^2)$, and
$\omega = -2 / h_1{}^2\mu\lambda - 2 / h_2{}^2\rho\sigma$.
(for more details, see Figure 4.3)
Multiply both sides of (5.7) and (5.8) by k and equate the left hand sides to get after some rearranging

$$U^{n+1}_{i,j} = U^n_{i,j} + A\, U^n_{i+1,j} + F\, U^n_{i-1,j} + \\ + C\, U^n_{i,j+1} + D\, U^n_{i,j-1} + E\, U^n_{i,j}, \tag{5.9}$$

where $A = k\alpha$, $F = k\beta$, $C = k\gamma$, $D = k\theta$, $E = k\omega$,
$0 < i < kf1(j)$, $0 < j < ifnp2$.
Here, i, j, kf1(j), and ifnp2 are integers.
The equation (5.9) gives us the temperature distribution in the domain at the new time step. Using these temperature values and repeating the procedure, which we discussed on page 61, we get the free boundary at each

time step. Using this method with the initial temperature of the domain given in equation (5.2) and the boundary conditions (5.3) - (5.6), we calculate the free boundaries as well as the temperature of the domain. The graphs of the computer(CRAY X-MP/48) output of free boundaries are listed in Figure 5.7 – 5.9.

If the exterior normal derivatives, $-\nu$, become negative (see Fig. 5.6), using a 2-dimensional spline to interpolate between the mesh points gives more accurate approximations.

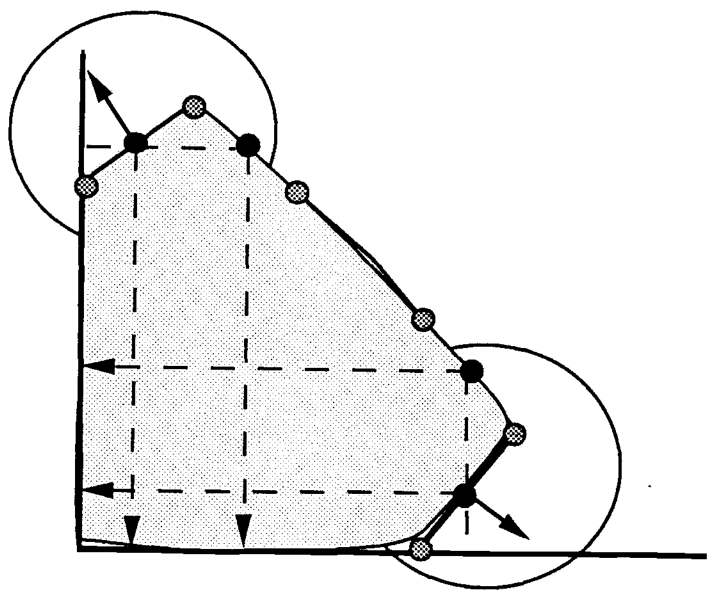


Figure 5.6. The Graph of the Possible Free Boundary

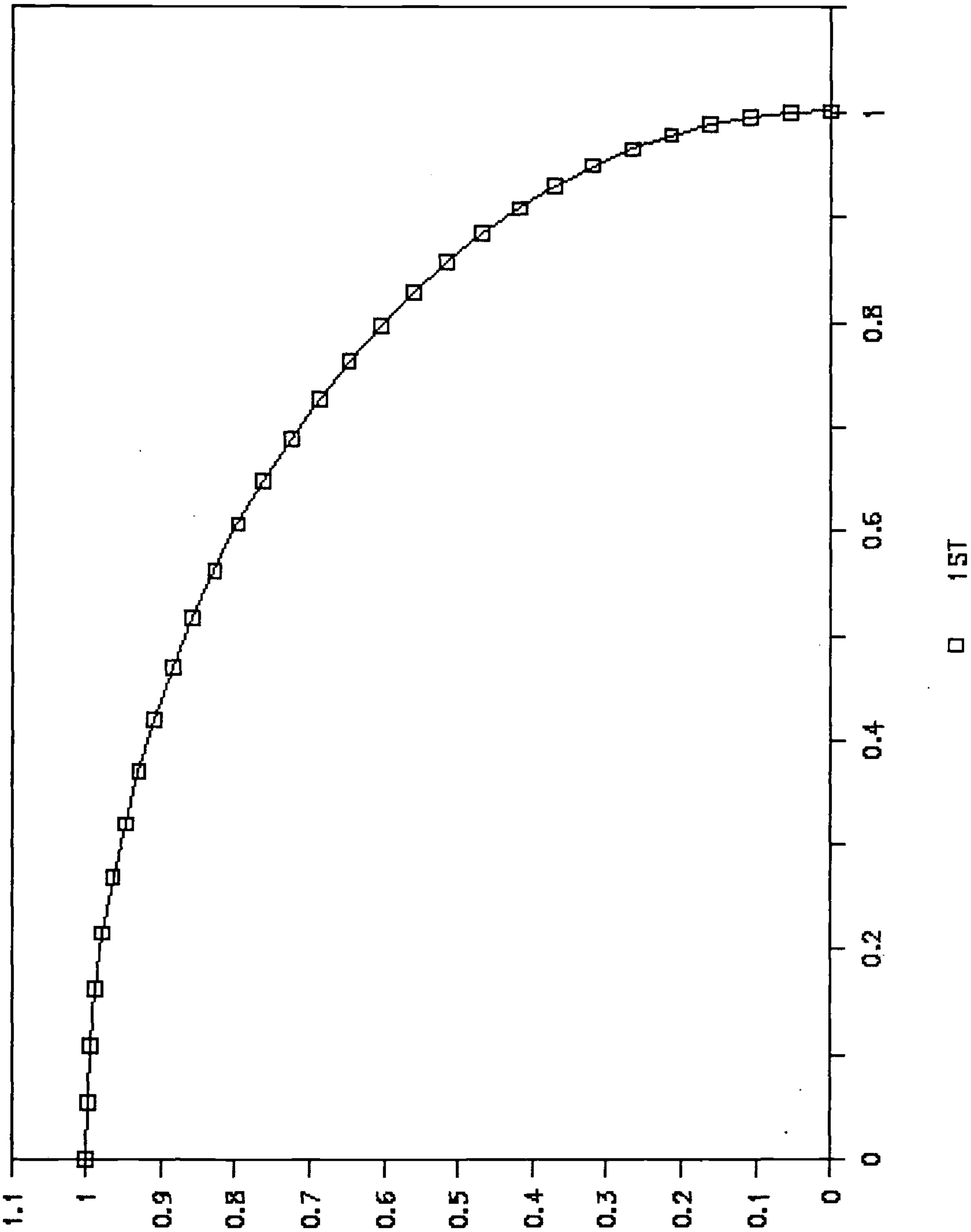


Figure 5.7. 1st Free Boundary for $h_1 = h_2 = 1.3e-2$, $k = 2.4e-5$

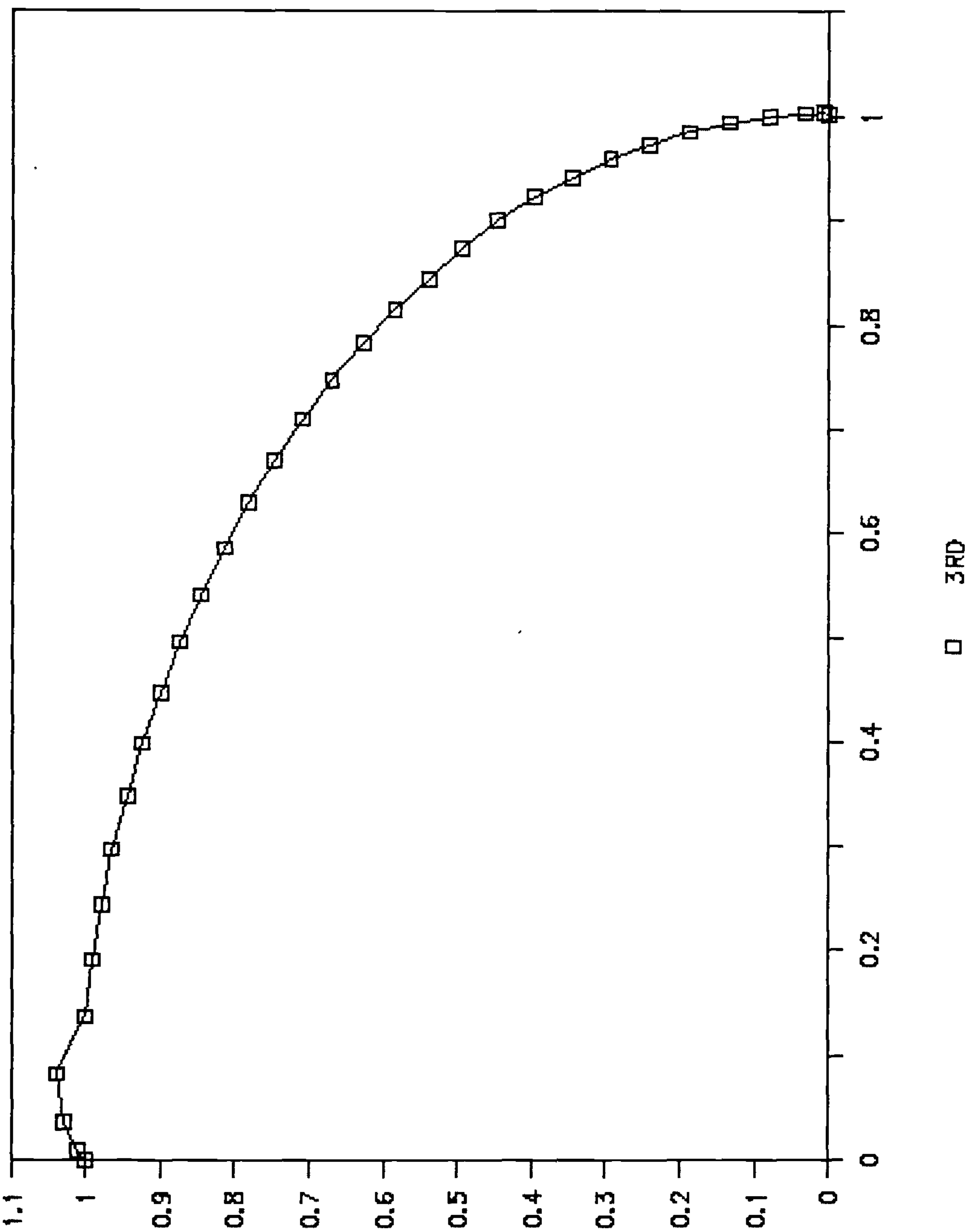


Figure 5.8. 3rd Free Boundary for $h_1 = h_2 = 1.3e-2$, $k = 2.4e-5$

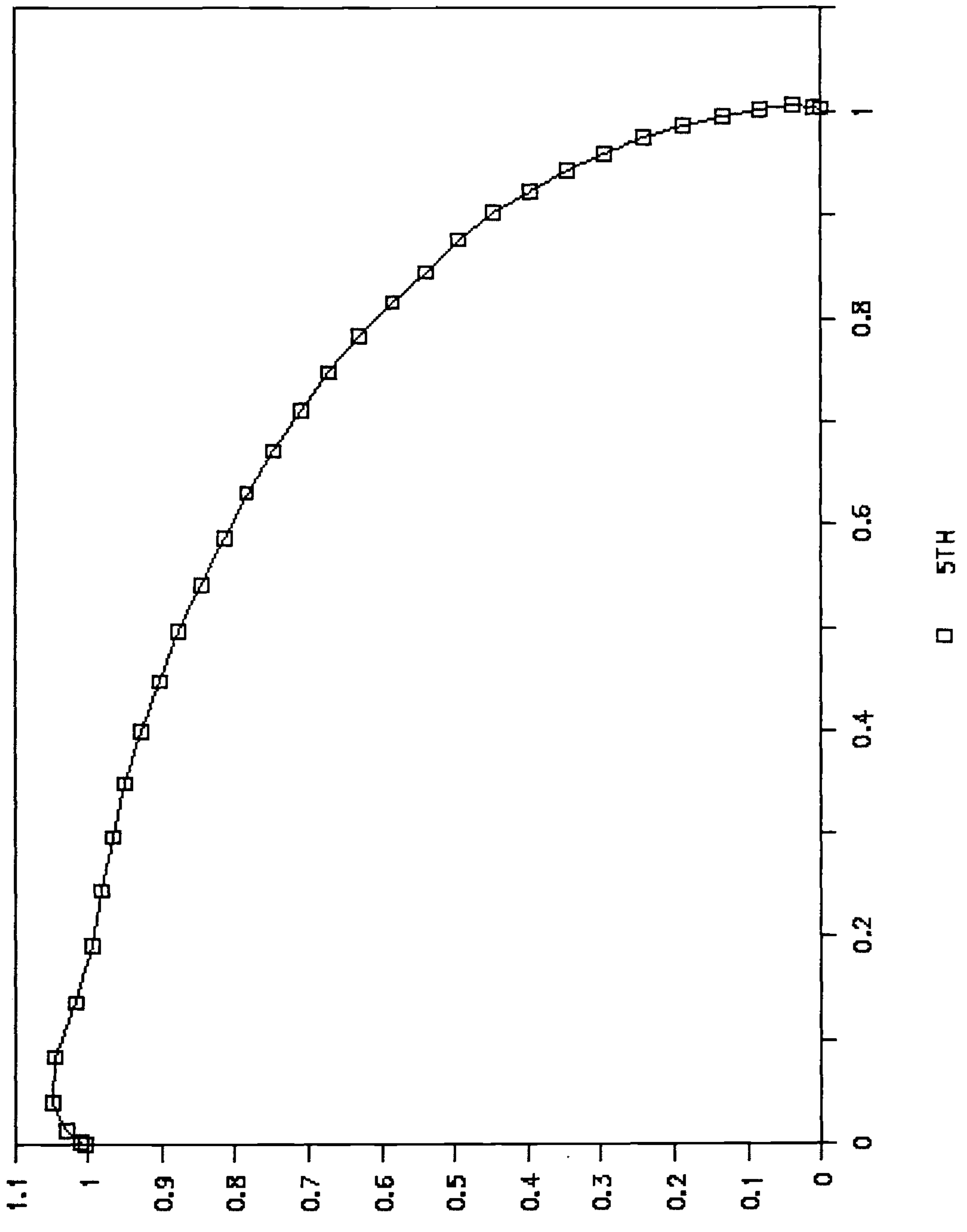


Figure 5.9. 5th Free Boundary for $h_1 = h_2 = 1.3e-2$, $k = 2.4e-5$

5.2. Backward difference in time

Let us use the backward difference for the time derivative u_t at the fixed time $t = t_{n+1}$ and rewrite equation (5.1) as follows:

$$\frac{U^{n+1}_{i,j} - U^{n}_{i,j}}{k} = \alpha\, U^{n+1}_{i+1,j} + \beta\, U^{n+1}_{i-1,j} + \gamma\, U^{n+1}_{i,j+1} + \theta\, U^{n+1}_{i,j-1} + \\ + \omega\, U^{n+1}_{i,j}\ , \qquad (5.10)$$

where $\alpha = 2 / h_1{}^2(\mu\lambda + \mu^2)$, $\beta = 2 / h_1{}^2(\mu\lambda + \lambda^2)$, $\gamma = 2 / h_2{}^2(\rho\sigma + \rho^2)$, $\theta = 2 / h_2{}^2(\rho\sigma + \sigma^2)$, and $\omega = -2 / h_1{}^2\mu\lambda - 2 / h_2{}^2\rho\sigma$.

Multiply both sides of equation (5.10) by k and rearrange the resulting equation. We find

$$(1-E)\ U^{n+1}_{i,j} - A\ U^{n+1}_{i+1,j} - F\ U^{n+1}_{i-1,j} - \\ C\ U^{n+1}_{i,j+1} - D\ U^{n+1}_{i,j-1} = U^{n}_{i,j}\ , \qquad (5.11)$$

where $A = k\alpha$, $F = k\beta$, $C = k\gamma$, $D = k\theta$, $E = k\omega$, $0 < i < ks1(j)$, $0 < j < isnp2$.

Here, i, j, ks1(j), and isnp2 are integers.

The equation (5.11) represents a system of $\sum_{j=1}^{isnp2} ks1(j)$ linear equations for $\sum_{j=1}^{isnp2} ks1(j)$ unknowns of the form

$$M\ x = B,$$

where M is an ($\sum_{j=1}^{isnp2}$ ks1(j)) x ($\sum_{j=1}^{isnp2}$ ks1(j)) tridiagonal matrix with fringes and B is the temperature vector known from previous time step. By solving the system M x = B, we can get the values of $U_{i,j}^{n+1}$, which are the domain temperatures, at each point $(ih_1, jh_2, (n+1)k)$. Repeat the same procedures as in the forward difference case to calculate the free boundaries. To store all the information about M requires large memory spaces which is not available at this time. In the future, CRAY-2 or SCS-40 might be available to handle this problem, but it is very expensive. Further research needs to be done using sparse matrix techniques with new computer structure designs.

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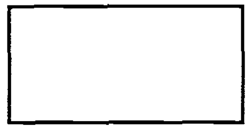
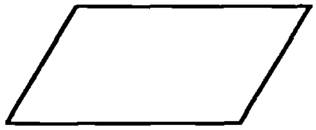
APPENDICES

1. Flowchart Symbols

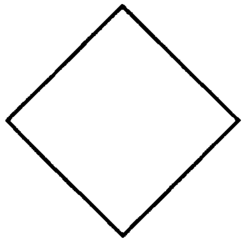
assignments or computations



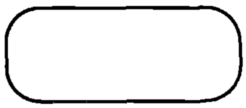
input or output



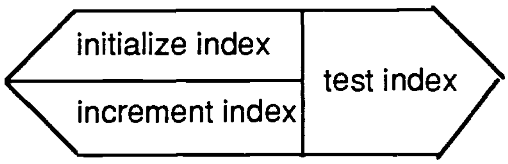
conditions



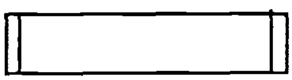
start or stop



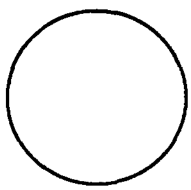
do loop



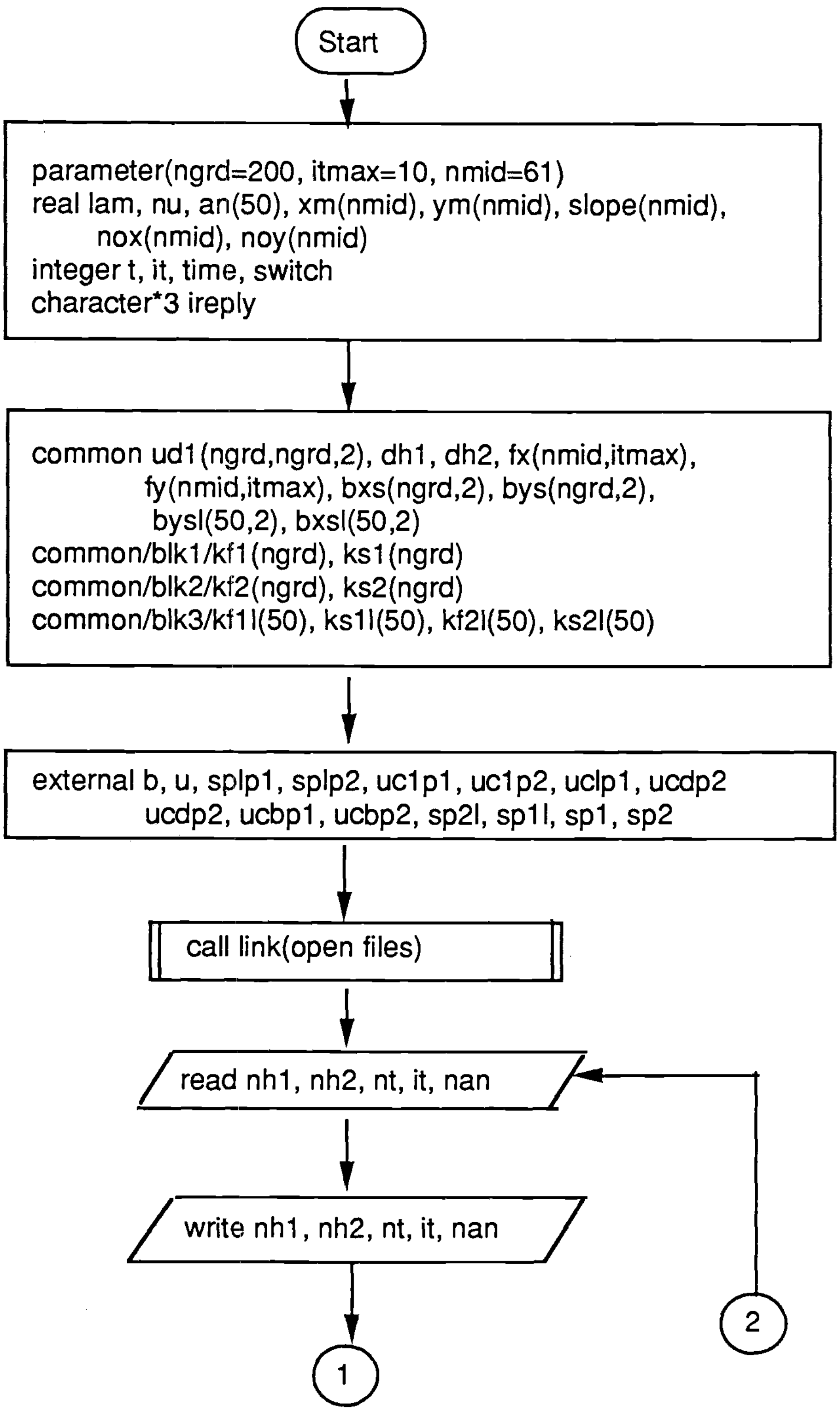
subprogram
(subroutine or function
statement)

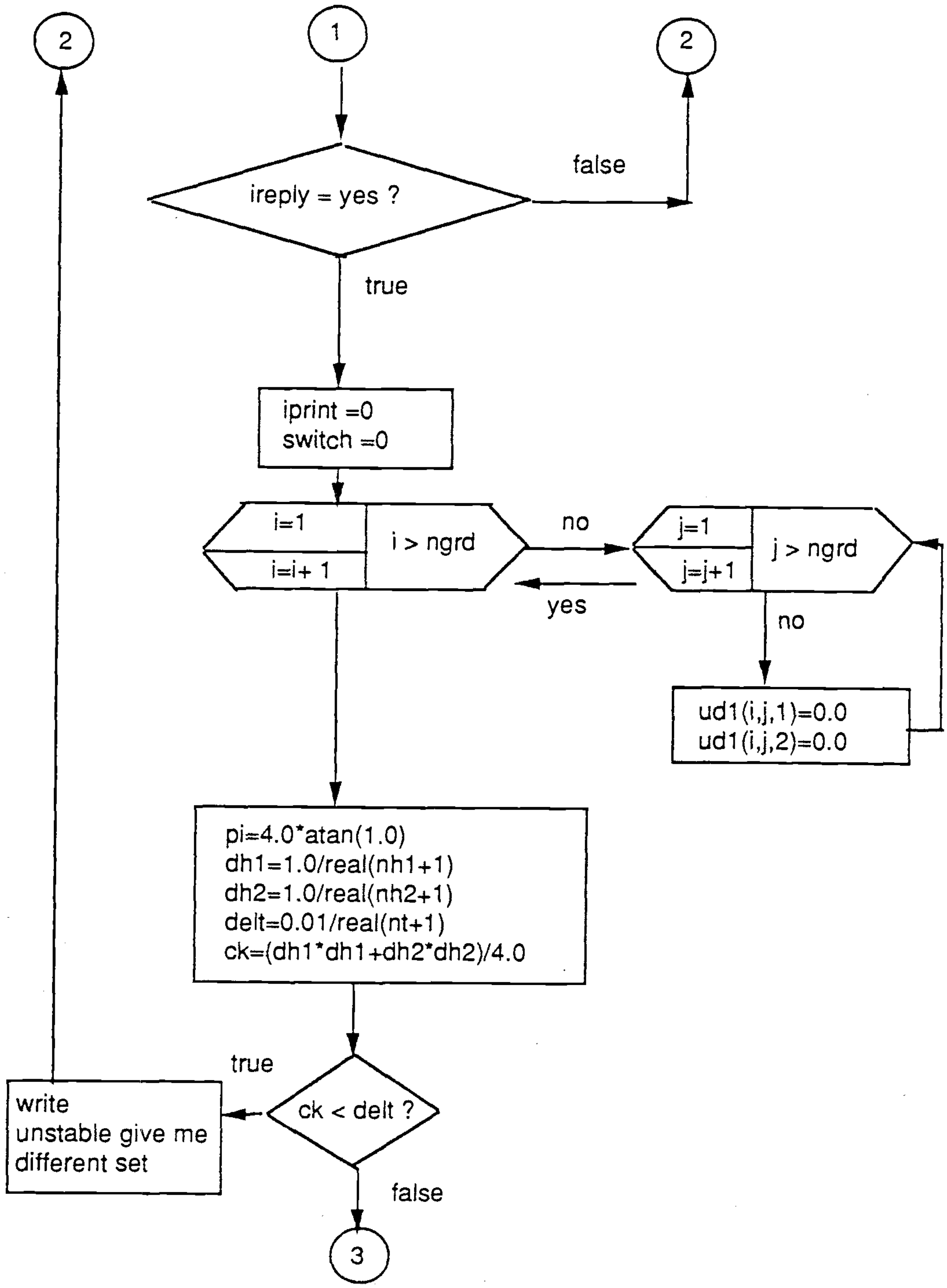


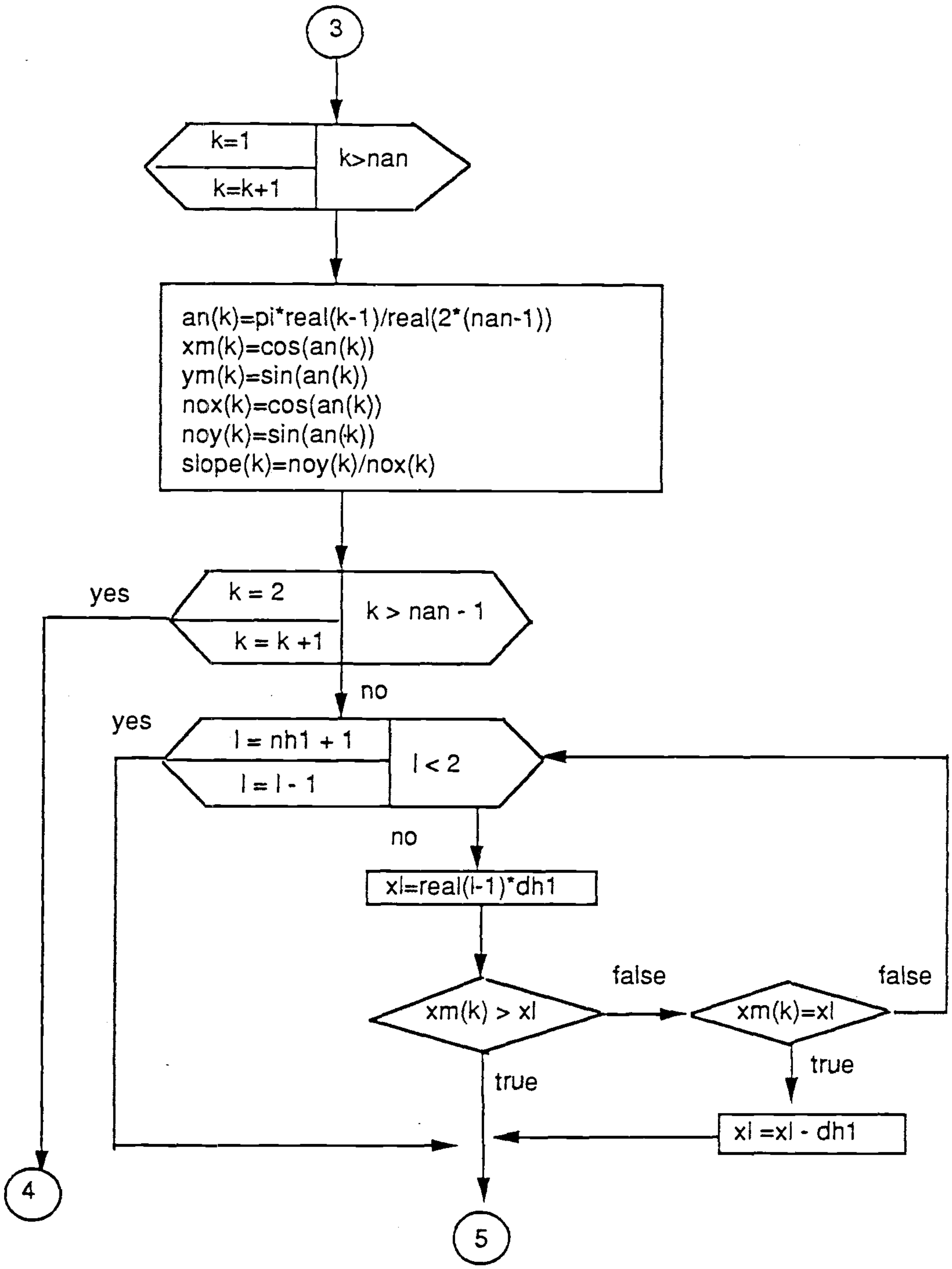
connector

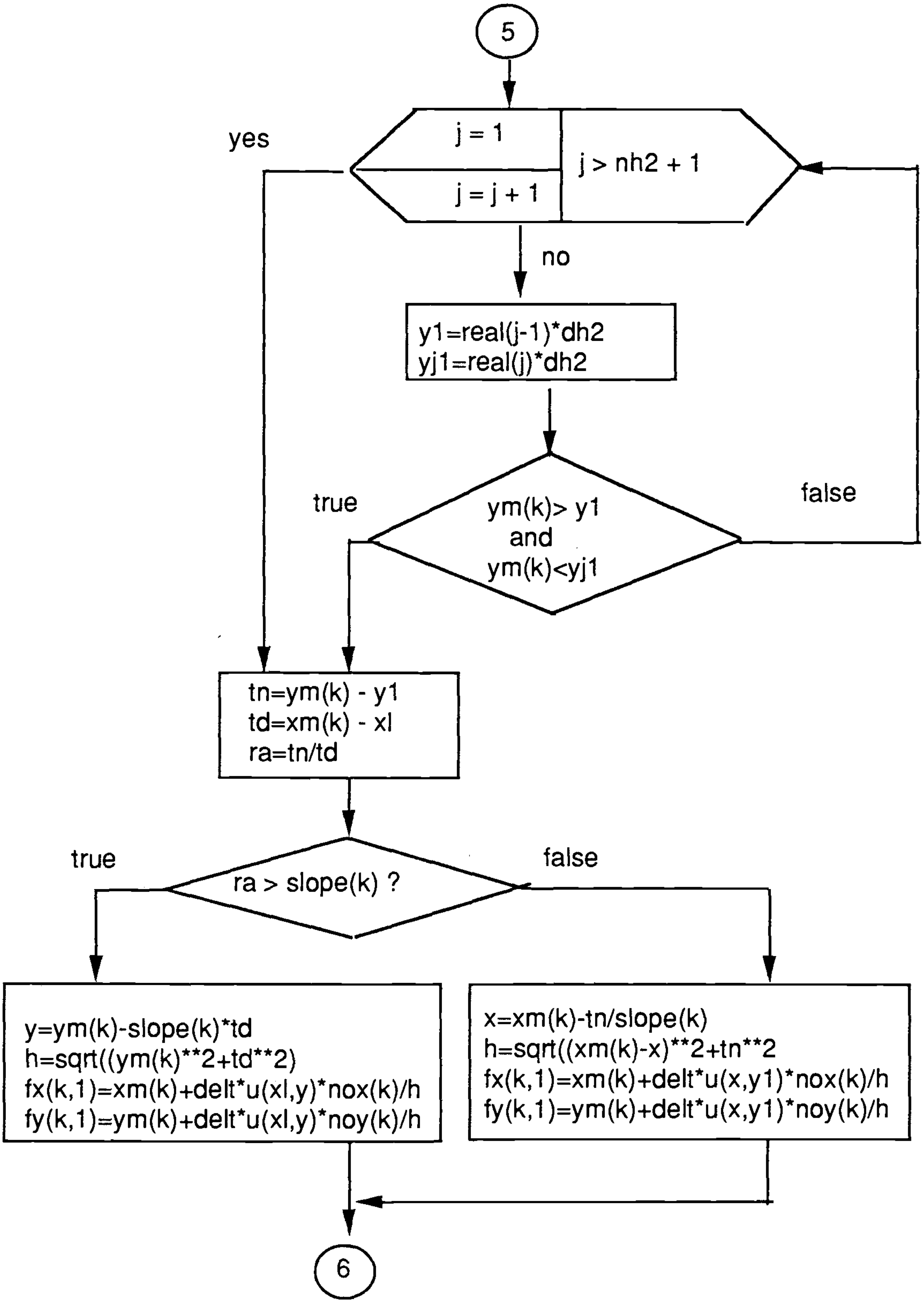


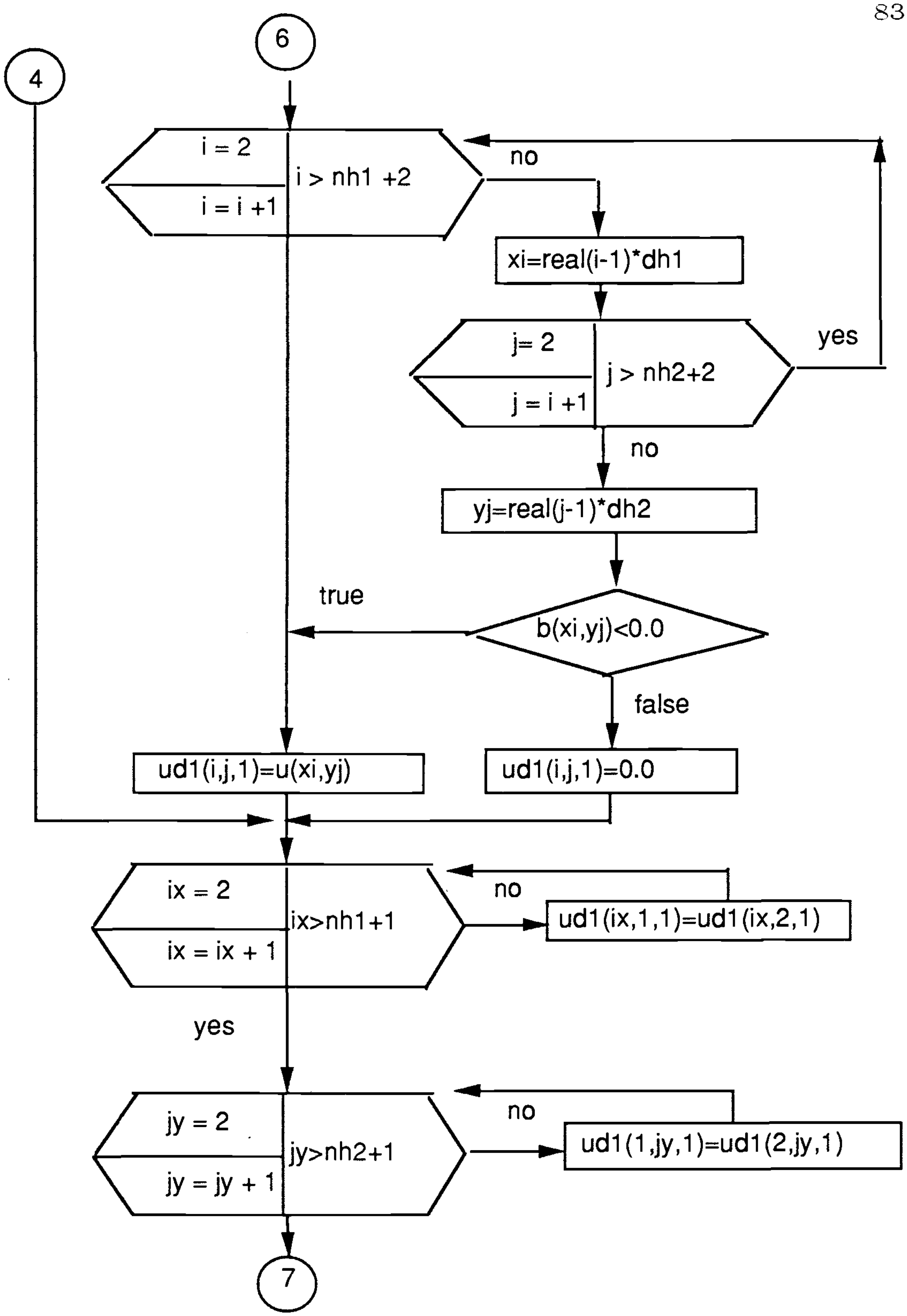
2. Flowchart of the Computer Program for the Finite Difference Scheme(forward in time)

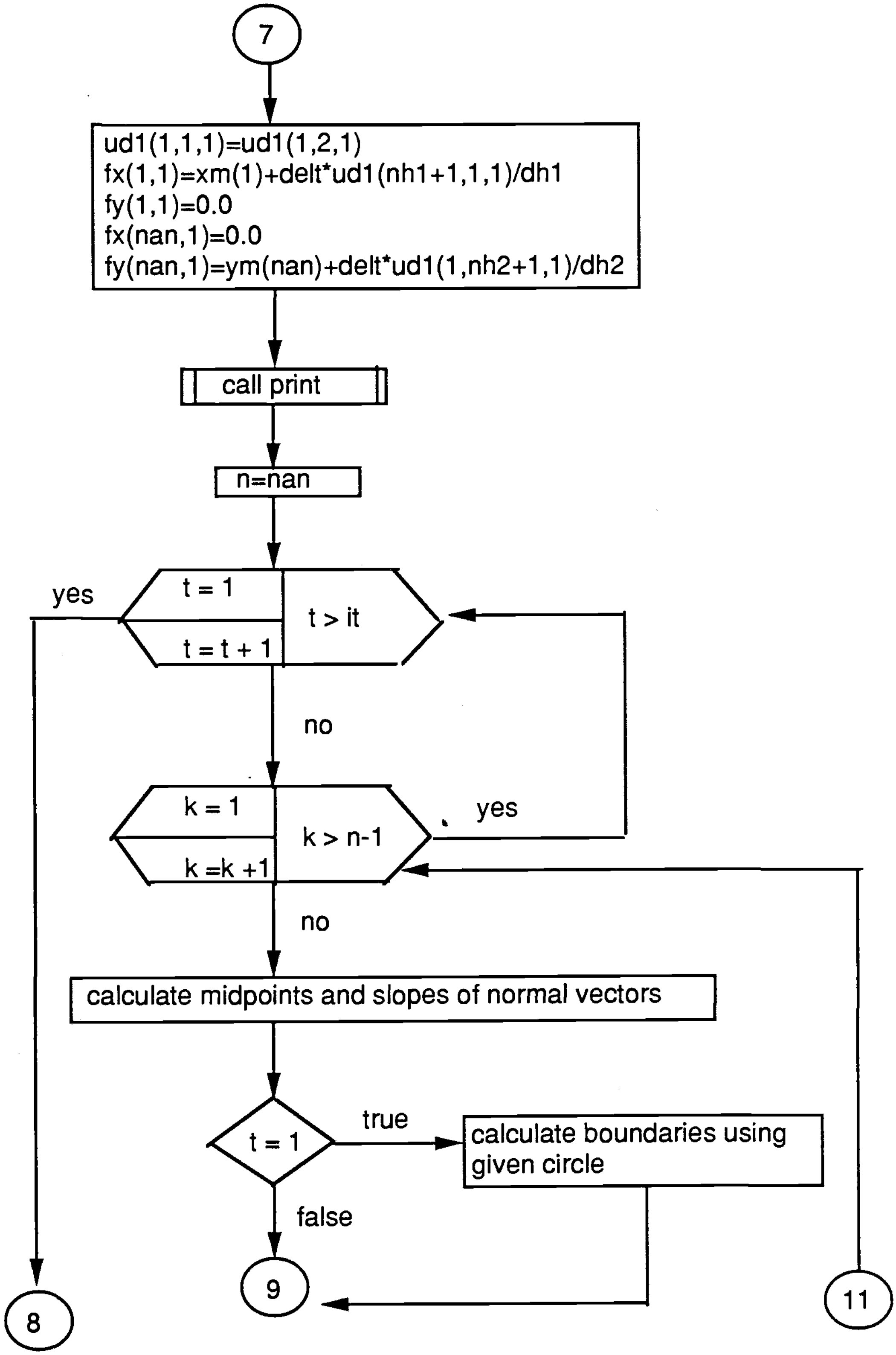


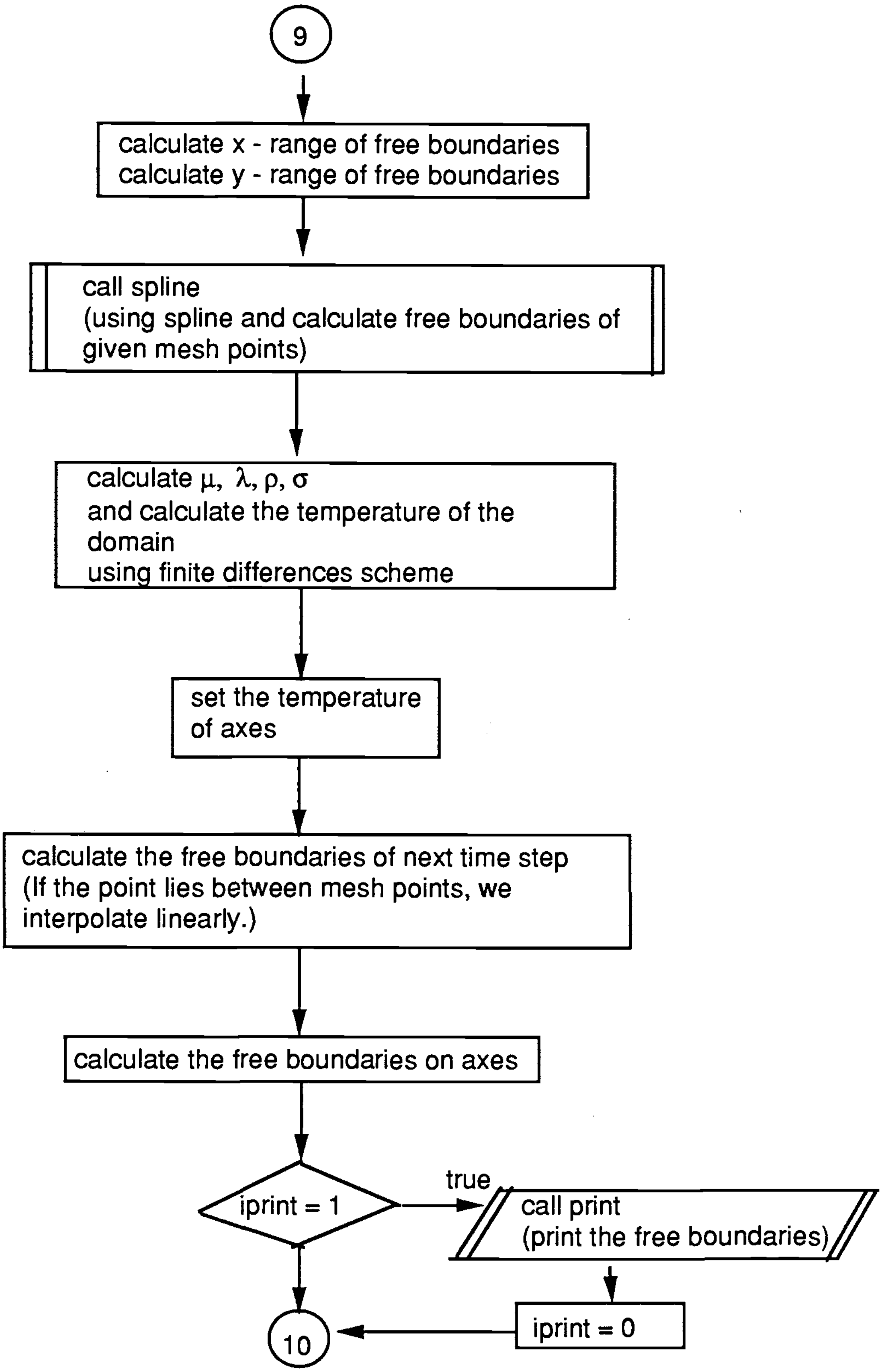

2
1
2
ireply = yes ?
false
true
iprint =0
switch =0
i=1
i=i+ 1
i > ngrd
no
yes
j=1
j=j+1
j > ngrd
no
ud1(i,j,1)=0.0
ud1(i,j,2)=0.0
pi=4.0*atan(1.0)
dh1=1.0/real(nh1+1)
dh2=1.0/real(nh2+1)
delt=0.01/real(nt+1)
ck=(dh1*dh1+dh2*dh2)/4.0
true
write
unstable give me
different set
ck < delt ?
false
3

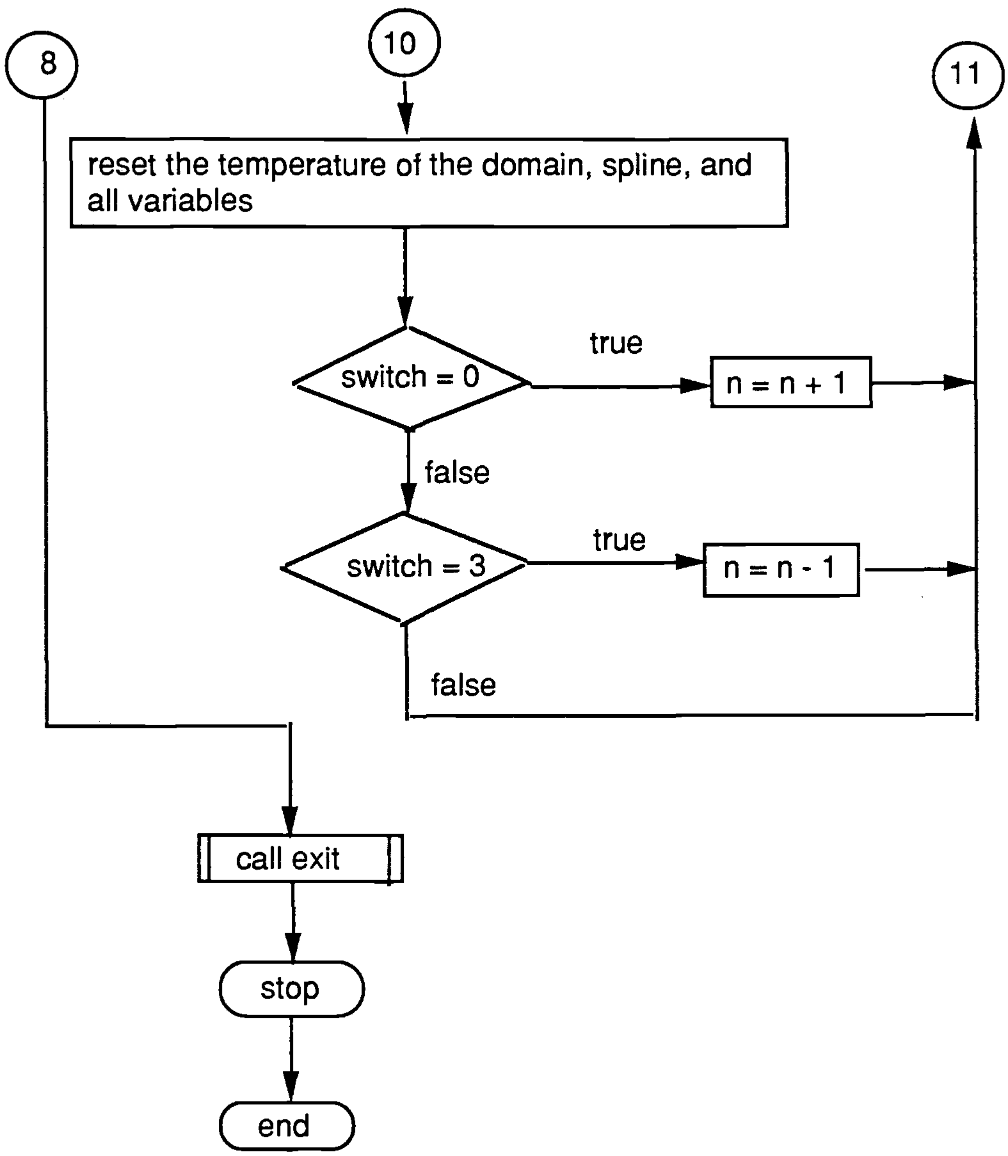

3
k=1
k=k+1
k>nan
an(k)=pi*real(k-1)/real(2*(nan-1))
xm(k)=cos(an(k))
ym(k)=sin(an(k))
nox(k)=cos(an(k))
noy(k)=sin(an(k))
slope(k)=noy(k)/nox(k)
yes
k = 2
k = k +1
k > nan - 1
no
yes
l = nh1 + 1
l = l - 1
l < 2
no
xl=real(l-1)*dh1
false
false
xm(k) > xl
xm(k)=xl
true
true
xl =xl - dh1
4
5


5
yes
j = 1
j = j + 1
j > nh2 + 1
no
y1=real(j-1)*dh2
yj1=real(j)*dh2
true
false
ym(k)> y1
and
ym(k)<yj1
tn=ym(k) - y1
td=xm(k) - xl
ra=tn/td
true
false
ra > slope(k) ?
y=ym(k)-slope(k)*td
h=sqrt((ym(k)**2+td**2)
fx(k,1)=xm(k)+delt*u(xl,y)*nox(k)/h
fy(k,1)=ym(k)+delt*u(xl,y)*noy(k)/h
x=xm(k)-tn/slope(k)
h=sqrt((xm(k)-x)**2+tn**2
fx(k,1)=xm(k)+delt*u(x,y1)*nox(k)/h
fy(k,1)=ym(k)+delt*u(x,y1)*noy(k)/h
6


6
4
i = 2
i > nh1 +2
i = i +1
no
xi=real(i-1)*dh1
j= 2
j > nh2+2
j = i +1
yes
no
yj=real(j-1)*dh2
true
b(xi,yj)<0.0
false
ud1(i,j,1)=u(xi,yj)
ud1(i,j,1)=0.0
ix = 2
ix>nh1+1
ix = ix + 1
no
ud1(ix,1,1)=ud1(ix,2,1)
yes
jy = 2
jy>nh2+1
jy = jy + 1
no
ud1(1,jy,1)=ud1(2,jy,1)
7


7
ud1(1,1,1)=ud1(1,2,1)
fx(1,1)=xm(1)+delt*ud1(nh1+1,1,1)/dh1
fy(1,1)=0.0
fx(nan,1)=0.0
fy(nan,1)=ym(nan)+delt*ud1(1,nh2+1,1)/dh2
call print
n=nan
t = 1
t = t + 1
t > it
yes
no
k = 1
k =k +1
k > n-1
yes
no
calculate midpoints and slopes of normal vectors
t = 1
true
calculate boundaries using given circle
false
8
9
11


9
calculate x - range of free boundaries
calculate y - range of free boundaries
call spline
(using spline and calculate free boundaries of given mesh points)
calculate μ, λ, ρ, σ
and calculate the temperature of the domain
using finite differences scheme
set the temperature of axes
calculate the free boundaries of next time step
(If the point lies between mesh points, we interpolate linearly.)
calculate the free boundaries on axes
iprint = 1
true
call print
(print the free boundaries)
iprint = 0
10


8
10
11
reset the temperature of the domain, spline, and all variables
switch = 0
true
n = n + 1
false
switch = 3
true
n = n - 1
false
call exit
stop
end