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This paper is about the computation of the stresses on a rigid body from a knowledge of the far field velocities in exterior Stokes and Oseen flows. The surface of the body is assumed to be bounded and smooth, and the body is assumed to move with constant velocity. We give fundamental solutions and derive boundary integral equations for the stresses. As it turns out, these integral equations are singular, and their null space is spanned by the normal to the body. We then discretize the problem by replacing the body by an approximating polyhedron with triangular faces. Using a collocation method, each integral equation delivers a linear system. Since its matrix approximates a singular integral operator, the matrix is ill-conditioned, and the solution is unstable. However, since we know that the problem is uniquely solvable in the hyperspace orthogonal to the normal, we use regularization methods to get stable solutions and project them in the normal direction onto the hyperspace.

Computation of the Stresses on a Rigid Body in Exterior Stokes and Oseen Flows

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Markus Schuster, Author

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1 Introduction

Design and construction of objects that are supposed to move in viscous fluids require information of what effect the motion has on the object. Therefore, one needs to know the stress distribution that the fluid exerts on the surface of the body. In this paper, we give a method to compute these surface forces in the case of incompressible fluids, i.e. for fluids with constant density. The bodies which we look at are assumed to have a smooth surface and move with constant velocity.

In this section we introduce the notation, assumptions and the equations that we will use. In section 2 the integral equations are derived, and in section 3 we investigate the properties of the corresponding integral operators. Section 4 is devoted to the discretization of the integral equations. One obtains a linear system, and section 5 is about the computation of the matrix of this system. Each entry has to be computed by numerical integration. The integration formula and corresponding functions are explained in section 6. An application is given in section 7 for a sphere. In this case, the creation of a triangulation is easy and we have analytical solutions available to compare with our numerical results. However, the program works for other bodies as well, as long as we are given an appropriate triangulation.

The computational part has been worked out in the mathematical programming language MATLAB [1] and can be obtained by an email to `markus.schuster@orst.edu`.

[1] MATLAB is a product of Mathworks, Inc.

1.1 Notation

$\mathbb{R}$ is the set of real numbers. Points in $\mathbb{R}^3$ will be denoted by $\mathbf{x}, \xi$ etc., and their coordinates by x_i, ξ_i etc. For a subset $U \subset \mathbb{R}^3$ we write $\overline{U}$ and ∂U for the closure in $\mathbb{R}^3$ and the boundary of U, respectively. An open set $D \subset \mathbb{R}^3$ is called a **domain**, if $\partial D = \partial(\mathbb{R}^3 \setminus \overline{D})$, i.e. if all boundary points are accessible from the outside.

$|\cdot|$ is the Euclidean Norm, i.e. $|\mathbf{x}|^2 = \sum_{i=1}^n x_i x_i$, if $\mathbf{x} \in \mathbb{R}^n$. Throughout the whole paper we use the **Einstein summation convention**, that means, we don't write summations over indices that occur twice in a product. For instance, we write $|\mathbf{x}|^2 = x_i x_i$.

For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ we introduce

$$\begin{aligned} \mathbf{d} &= \mathbf{d}_{\mathbf{xy}} = \mathbf{x} - \mathbf{y} \\ r &= r_{\mathbf{xy}} = |\mathbf{d}|. \end{aligned}$$

Derivatives with respect to $\mathbf{y}$ are written with the comma-notation $f_{,i} = \frac{\partial}{\partial y_i} f$. Some examples are $r_{,i} = -d_i/r$, $r_{,i} r_{,i} = 1$ or $d_{i,j} = -\delta_{ij}$, where δ_{ij} is the Kronecker delta. One can express the gradient, divergence and Laplace operators respectively by $\nabla f = f_{,i}$, $\nabla \cdot \mathbf{g} = g_{i,i}$ and $\Delta f = \nabla \cdot \nabla f = f_{,ii}$.

We often need spheres or balls around some points. Therefore, we introduce for $\mathbf{x} \in \mathbb{R}^3$ and $\lambda > 0$

$$\begin{aligned} S_\lambda(\mathbf{x}) &= \{\mathbf{y} \in \mathbb{R}^3 : r_{\mathbf{xy}} = \lambda\} \\ B_\lambda(\mathbf{x}) &= \{\mathbf{y} \in \mathbb{R}^3 : r_{\mathbf{xy}} < \lambda\} \end{aligned}$$

For a subset $D \in \mathbb{R}^3$ we write $C(D)$ for the space of continuous, real-valued functions on D. If D is open and $k > 0$ is an integer, $C^k(D)$ is the space of functions possessing continuous derivatives up to and including order k on D; C^∞ is defined as the intersection of all the spaces $C^k(D)$. $C^k(\overline{D})$ is the space of elements of $C^k(D)$

that extend continuously to the closure $\overline{D}$. If, furthermore, $0 < \chi < 1$, then $C^{k,\chi}$ is the space of k-times Hölder continuously differentiable functions.

We say a function $g(\mathbf{x})$ is of order $O(f(\mathbf{x}))$ as $\mathbf{x} \to \mathbf{a}$, if there is a constant c and an $\varepsilon > 0$, such that for all $\mathbf{x} \in B_\varepsilon(\mathbf{a})$ one has $|g(\mathbf{x})| \leq c \cdot |f(\mathbf{x})|$. If $\mathbf{a}$ is infinite, the estimate should hold for every $\mathbf{x}$ with norm bigger than a certain R. A function $g(\mathbf{x})$ is said to be of order $o(f(\mathbf{x}))$ as $\mathbf{x} \to \mathbf{a}$, if $g(\mathbf{x})/f(\mathbf{x}) \to 0$ as $\mathbf{x} \to \mathbf{a}$. $o(f(\mathbf{x}))$-functions are also $O(f(\mathbf{x}))$-functions.

1.2 Derivation of the Stokes and Oseen Equations

Suppose we have a fixed body given in an inertial system, surrounded by an incompressible fluid. We take the x_1-axis in the direction of the flow. $\overline{G}$ is the set of coordinates of the body, ∂G its boundary, and $G := \overline{G} \setminus \partial G$ the interior of the body, which is assumed to be a domain. The exterior domain, which is filled by the fluid, is denoted by Ω.

The density ρ and viscosity μ of the fluid are constant. The velocity $\mathbf{u}$ and the pressure p are stationary, i.e. they don't depend on time. On ∂G we require $\mathbf{u} = \mathbf{0}$ (no-slip-condition), and at infinity we assume to have $\mathbf{u} = (U,\ 0,\ 0)^\top$. In Ω, the fluid is exhibited to volume forces, and on ∂G, the body exerts surface forces on the fluid. We denote the density of the volume forces by $\mathbf{f}$ and the density of the surface forces by $\mathbf{g}$. $\mathbf{f}$ is assumed to be a potential force, for instance a gravitational force. $\mathbf{g}$ consists of normal stresses (pressures) and tangential stresses, which are due to the deformation of the viscous fluid. For small velocities the following approximation has turned out to agree very well with reality:

$$\mathbf{g} = -p\mathbf{n} + \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^\top)\mathbf{n}$$

where $\mathbf{n}$ is the unit normal on ∂G that points inside the body and thus outside the exterior domain Ω. From the principle of conservation of momentum, one can then

derive (see Oseen [26]) the equations that govern the motion of the fluid. They are called the **Incompressible Steady Navier-Stokes-Equations**:

$$\begin{aligned} \mu\Delta\mathbf{u} - \nabla p &= -\rho\mathbf{f} + \rho(\nabla\mathbf{u})\mathbf{u} \\ \nabla\cdot\mathbf{u} &= 0. \end{aligned}$$

Notice that these equations are linear in p. We replace p by $p + p^*$ and decompose:

$$\begin{aligned} \nabla p^* &= \rho\mathbf{f} \\ \mu\Delta\mathbf{u} - \nabla p &= \rho(\nabla\mathbf{u})\mathbf{u} \\ \nabla\cdot\mathbf{u} &= 0 \end{aligned}$$

Under the above assumption on $\mathbf{f}$, the solution p^* of the first eqation is just the potential of $\rho\mathbf{f}$. All of the following investigations will concentrate on the second and third equation. The boundary condition for the pressure at infinity is then $p = 0$.

It is customary to put the equations in a dimensionless form. Therefore, we set $\mathbf{y} = \mathbf{x}/L$, where L is at the moment an arbitrary length and $\mathbf{y}$ is dimensionless. Then we define $\mathbf{w}(\mathbf{y}) := \mathbf{u}(\mathbf{x})/U$ and $q(\mathbf{y}) := Lp(\mathbf{x})/\mu U$. We thus get the equations

$$\begin{aligned} \Delta\mathbf{w} - \nabla q &= \frac{\rho UL}{\mu}(\nabla\mathbf{w})\mathbf{w} \\ \nabla\cdot\mathbf{w} &= 0, \end{aligned}$$

where now the derivatives are taken with respect to $\mathbf{y}$. If L is chosen as a characteristic length of the body, e.g. its diameter, then the number $\mathcal{R} = \rho UL/\mu$ is called the **Reynolds Number.**

In order to have zero boundary conditions at infinity, we set $\mathbf{v}(\mathbf{y}) = \mathbf{w}(\mathbf{y}) - (1\ 0\ 0)^\top$ and get

$$\begin{aligned} \Delta\mathbf{v} - \nabla q &= \mathcal{R}\tfrac{\partial}{\partial y_1}\mathbf{v} + \mathcal{R}(\nabla\mathbf{v})\mathbf{v} \\ \nabla\cdot\mathbf{v} &= 0. \end{aligned} \tag{1}$$

Boundary conditions are

$$\mathbf{v}(\mathbf{y}) = (-1\ 0\ 0)^\top \text{ for } \mathbf{y} \in \partial G$$

$$\begin{aligned} \mathbf{v}(\mathbf{y}) &\to \mathbf{0} \text{ as } |\mathbf{y}| \to \infty \\ q(\mathbf{y}) &\to 0 \text{ as } |\mathbf{y}| \to \infty. \end{aligned}$$

The **Stokes Equations** are obtained from (1) by neglecting the right side of the equations:

$$\begin{aligned} \Delta \mathbf{v} - \nabla q &= \mathbf{0} \\ \nabla \cdot \mathbf{v} &= 0 \end{aligned} \tag{2}$$

If we neglect only the nonlinear term, we end up with the **Oseen Equations**

$$\begin{aligned} \Delta \mathbf{v} - \mathcal{R} \tfrac{\partial}{\partial y_1} \mathbf{v} - \nabla q &= \mathbf{0} \\ \nabla \cdot \mathbf{v} &= 0. \end{aligned} \tag{3}$$

One can express the stress $\mathbf{g}$ in terms of $\mathbf{v}$ and q:

$$\begin{aligned} \mathbf{g} &= -(p + p^*)\mathbf{n} + \mu(\nabla_{\mathbf{x}} \mathbf{u} + \nabla_{\mathbf{x}} \mathbf{u}^\top)\mathbf{n} \\ &= -p^* \mathbf{n} + \frac{\mu U}{L}(-q + \nabla_{\mathbf{y}} \mathbf{w} + \nabla_{\mathbf{y}} \mathbf{w}^\top)\mathbf{n} \\ &= -p^* \mathbf{n} + \frac{\mu U}{L} \underbrace{\left(-q\mathbf{n} + (\nabla \mathbf{v} + \nabla \mathbf{v}^\top)\mathbf{n}\right)}_{\mathbf{t}} \end{aligned}$$

The expression in brackets, which we denote by $\mathbf{t}$, is the one we solve for in the following. We define the **stress tensor** in index notation

$$T_{ij}(\mathbf{v}, q)(\mathbf{y}) = -q(\mathbf{y})\delta_{ij} + v_{i,j}(\mathbf{y}) + v_{j,i}(\mathbf{y}) \tag{4}$$

and can then write $\mathbf{t}$ in index notation conveniently as

$$t_i = T_{ij}(\mathbf{v}, q) n_j. \tag{5}$$

Remark. The above simplifications of the steady Navier-Stokes equations change the solutions $(\mathbf{v}, q)$ considerably. Near the body one can justify them only for small Reynolds numbers $\mathcal{R}$. At infinity the Stokes solutions have nothing to do with the steady Navier-Stokes solutions, since it can be shown that

$$\mathcal{R} \frac{|\frac{\partial}{\partial y_1} \mathbf{v} + (\nabla \mathbf{v})\mathbf{v}|}{|\Delta \mathbf{v}|} = O(|\mathbf{y}|) \text{ as } |\mathbf{y}| \to \infty$$

However, the assumption that leads to the Oseen equations is reasonable:

$$\mathcal{R}\frac{|(\nabla \mathbf{v})\mathbf{v}|}{|\Delta \mathbf{v}|} = O(|\mathbf{y}|^{-1/2}) \quad \text{as} \quad |\mathbf{y}| \to \infty$$

We refer to the next section for the proof of these results.

1.3 Admissible Bodies

For the derivation of the integral equations, we have to make an assumption on our body. We require that its surface ∂G is bounded and everywhere smooth, in the sense that we can everywhere parametrize it locally with continuously differentiable functions. Therefore, we introduce a technical definition:

Definition 1.1 *Let $k \in N_0$ and $\chi \in [0,1]$. G has $N^{k,\chi}$-**property**, if for all $\mathbf{x} \in \partial G$ there exists a neighborhood U of $\mathbf{x}$, an orthogonal coordinate-transformation $A : \mathbb{R}^3 \to \mathbb{R}^3$, constants $\alpha, \beta > 0$ and a mapping $a : \mathbb{R}^2 \to \mathbb{R}$ with the following properties, where we denote the new coordinates (given by A) by $\mathbf{y} = (y_1, y_2, y_3)$ and $W(\alpha) := \{(y_1, y_2) \mid |y_i| < \alpha \text{ for } i = 1, 2\}$.*
We demand:

$$\begin{aligned} a &\in C^{k,\chi}(W(\alpha)) \\ U \cap \partial G &= \{\mathbf{y} \in \mathbb{R}^3 \,|(y_1, y_2) \in W(\alpha) \text{ and } y_3 = a(y_1, y_2)\} \\ U \cap G &= \{\mathbf{y} \in \mathbb{R}^3 \,|(y_1, y_2) \in W(\alpha) \text{ and } a(y_1, y_2) < y_3 < a(y_1, y_2) + \beta\} \\ U \cap G^C &= \{\mathbf{y} \in \mathbb{R}^3 \,|(y_1, y_2) \in W(\alpha) \text{ and } a(y_1, y_2) - \beta < y_3 < a(y_1, y_2)\} \end{aligned}$$

In this paper, we assume all over the place that G has $N^{1,0}$-**property**. This is equivalent to saying that G has $C^{1,0}$-**property**. Necessary conditions for that are that G has **segment property** and **uniform cone property**. We refer to Wloka [34] for definitions and the proof of these statements.

By $\mathbf{n}$ we mean the unit normal on ∂G, which points into direction of G and thus away from the exterior domain Ω. In the paper we use the following two properties of bounded $C^{1,0}$-surfaces:

- $\int_{\partial G} dS_{\mathbf{y}} < \infty$
- $\mathbf{n}(\mathbf{x}) - \mathbf{n}(\mathbf{y}) = O(r)$

2 Derivation of the Integral Equations

This section is devoted to a derivation of the integral equations for the boundary stresses $\mathbf{t}$. We start with a generalized version of Green's Formula, compute fundamental solutions for the Stokes and Oseen equations and obtain then Fredholm integral equations of the first kind, which we use afterwards to solve the problem numerically.

2.1 Green's Formula and three Lemmas

Proposition 2.1 (Green's Formula) *Let D be a bounded C^1-domain and $\mathbf{b} \in \mathbb{R}^3$ a fixed vector. Assume $(\mathbf{u}, p) \in C^2(\overline{D})^3 \times C^1(\overline{D})$ solves the problem*

$$\left.\begin{array}{c} u_{i,jj} + b_j u_{i,j} - p_{,i} = 0 \\ u_{i,i} = 0 \end{array}\right\} \quad in \quad D$$

and $(\mathbf{v}, q) \in C^2(\overline{D})^3 \times C^1(\overline{D})$ solves the problem

$$\left.\begin{array}{c} v_{i,jj} - b_j v_{i,j} - q_{,i} = 0 \\ v_{i,i} = 0 \end{array}\right\} \quad in \quad D$$

Then we have

$$\int_{\partial D} v_i T_{ij}(\mathbf{u}, p) n_j - u_i T_{ij}(\mathbf{v}, q) n_j + b_j u_i v_i n_j dS_{\mathbf{y}} = 0. \tag{6}$$

Proof: We use the Gauss divergence theorem:

$$\begin{aligned} & \int_{\partial D} v_i T_{ij}(\mathbf{u}, p) n_j - u_i T_{ij}(\mathbf{v}, q) n_j + b_j u_i v_i n_j dS_{\mathbf{y}} \\ = & \int_D \left(v_i T_{ij}(\mathbf{u}, p)\right)_{,j} - \left(u_i T_{ij}(\mathbf{v}, q)\right)_{,j} + \left(b_j u_i v_i\right)_{,j} d\mathbf{y} \\ = & \int_D v_{i,j}(-\delta_{ij} p + u_{i,j} + u_{j,i}) - u_{i,j}(-\delta_{ij} q + v_{i,j} + v_{j,i}) \\ & + v_i(-p_{,i} + u_{i,jj} + u_{j,ij} + b_j u_{i,j}) - u_i(-q_{,i} + v_{i,jj} + v_{j,ij} - b_j v_{i,j}) d\mathbf{y} \\ = & \ 0 \end{aligned}$$

■

Lemma 2.1 *For any $R > 0$ and any $\mathbf{x} \in \mathbb{R}^3$ we have*

$$\int_{S_R(\mathbf{x})} \frac{d_i d_k}{R^4} dS_\mathbf{y} = \frac{4\pi}{3} \cdot \delta_{ik}$$

Proof: Introduce spherical coordinates centered at $\mathbf{x}$:

$$\begin{aligned} y_1 &= x_1 + R\sin\theta\cos\varphi \\ y_2 &= x_2 + R\sin\theta\sin\varphi \\ y_3 &= x_3 + R\cos\theta \\ dS_\mathbf{y} &= R^2\sin\theta d\theta d\varphi \end{aligned}$$

and integrate over $0 \le \varphi \le 2\pi$ and $0 \le \theta \le \pi$. ∎

Lemma 2.2 *Let $\mathbf{x}_0 \in \partial G$. For any $\varepsilon > 0$ there exists an $h > 0$ such that for all $\mathbf{x} \in B_h(\mathbf{x}_0)$ we have*

$$\int_{\partial G \cap B_h(\mathbf{x}_0)} \frac{1}{|\mathbf{x}-\mathbf{z}|} dS_\mathbf{z} < \varepsilon$$

Proof: Let U, A, a and α be like in definition 1.1. We take $b > 0$ so small that $\overline{B_b(\mathbf{x}_0)} \in U$ and define

$$M_b := \left\{ (y_1, y_2) \in W(\alpha) \,\middle|\, (y_1, y_2, a(y_1, y_2)) \in A^{-1}(B_b(\mathbf{x}_0) \cap \partial G) \right\}$$

Since a is continuous, it is clear that

$$\left\{ A(y_1, y_2, a(y_1, y_2)) \,\middle|\, (y_1, y_2) \in \overline{M_b} \right\} \subset U$$

Thus, we can take the maximum $L := \max_{(y_1,y_2)\in\overline{M_b}} |\nabla a(y_1, y_2)|$. Now we choose $h < \min(b, \varepsilon/4\pi\sqrt{1+L^2})$ and estimate for $\mathbf{x} \in B_h(\mathbf{x}_0)$:

$$\begin{aligned} \int_{\partial G \cap B_h(\mathbf{x}_0)} \frac{1}{|\mathbf{x}-\mathbf{z}|} dS_\mathbf{z} &= \int_{A^{-1}(\partial G \cap B_h(\mathbf{x}_0))} \frac{1}{|\mathbf{x} - A\mathbf{y}|} dS_\mathbf{y} \\ &= \int_{A^{-1}(\partial G \cap B_h(\mathbf{x}_0))} \frac{1}{|\xi - \mathbf{y}|} dS_\mathbf{y} \\ &= \int_{M_h} \sqrt{\frac{1 + \nabla a \cdot \nabla a}{(\xi_1 - y_1)^2 + (\xi_2 - y_2)^2 + (\xi_3 - a(y_1, y_2))^2}} d(y_1, y_2) \\ &\le \sqrt{1+L^2} \int_{K_{2h}(\xi)} \frac{1}{\sqrt{(\xi_1 - y_1)^2 + (\xi_2 - y_2)^2}} d(y_1, y_2) \\ &< \varepsilon \end{aligned}$$

Here $\xi = A^{-1}\mathbf{x}$, and $K_{2h}(\xi) \subset \mathbb{R}^2$ is the disk around ξ with radius $2h$. We also used $M_h \subset K_{2h}(\xi)$. ∎

Lemma 2.3 *Assume* $\varphi \in C(\mathbb{R}^3 \setminus \{0\})$, $\varphi(\mathbf{x}) = O(1/|\mathbf{x}|)$ *as* $\mathbf{x} \to \mathbf{0}$ *and* $\psi \in C(\partial G)$. *Then*

$$f(\mathbf{x}) := \int_{\partial G} \varphi(\mathbf{x} - \mathbf{y})\psi(\mathbf{y}) dS_{\mathbf{y}}$$

is continuous in $\mathbb{R}^3$.

Proof: Let $\varepsilon > 0$, $\mathbf{x}_0 \in \mathbb{R}^3$ and R be so large that $G \subset B_R(\mathbf{x}_0)$. For $h < R$ we define $D_h := \overline{B_R(\mathbf{x}_0)} \setminus B_h(\mathbf{x}_0)$. Since $\varphi(\mathbf{x}_0 - \mathbf{y})$ as a function of $\mathbf{y}$ is continuous on D_h and D_h is compact, it's also uniformly continuous, i.e. for all $\varepsilon > 0$ there exists a little $\delta = \delta(h) > 0$ such that for all $\mathbf{y}_1, \mathbf{y}_2 \in D_h$ with $|\mathbf{y}_1 - \mathbf{y}_2| < \delta$ one has $|\varphi(\mathbf{x}_0 - \mathbf{y}_1) - \varphi(\mathbf{x}_0 - \mathbf{y}_2)| < \varepsilon$. Let furthermore M be an upper bound for ψ on ∂G. For $\mathbf{x}_0 \notin \partial G$, we define $d := dist(\{\mathbf{x}_0\}, \partial G)$. Using the above statement for $h := d/2$, we obtain a $\delta > 0$ such that for all $\mathbf{x} \in B_\delta(\mathbf{x}_0)$ one has

$$\begin{aligned} |f(\mathbf{x}) - f(\mathbf{x}_0)| &\leq \int_{\partial G} |\varphi(\mathbf{x}_0 - \mathbf{y}) - \varphi(\mathbf{x} - \mathbf{y})| \cdot |\psi(\mathbf{y})| dS_{\mathbf{y}} \\ &\leq M \int_{\partial G} dS_{\mathbf{y}} \varepsilon \end{aligned}$$

Now assume $\mathbf{x}_0 \in \partial G$. Due to Lemma 2.2, we can find a $d > 0$ such that

$$\int_{\partial G \cap B_d(\mathbf{x}_0)} \frac{1}{r_{\mathbf{xy}}} dS_{\mathbf{y}} < \varepsilon \quad \text{for all } \mathbf{x} \in B_d(\mathbf{x}_0).$$

We additionally require d to be so small that the order estimates for φ can be used. Using the uniform continuity from above for $h := d/2$, we can find a $\delta < h$ such that

$$\int_{\partial G \setminus B_d(\mathbf{x}_0)} |\varphi(\mathbf{x}_0 - \mathbf{y}) - \varphi(\mathbf{x} - \mathbf{y})| \cdot |\psi(\mathbf{y})| dS_{\mathbf{y}} < \varepsilon \quad \text{for all } \mathbf{x} \in B_\delta(\mathbf{x}_0).$$

Then we get for all $\mathbf{x} \in B_\delta(\mathbf{x}_0)$

$$|f(\mathbf{x}) - f(\mathbf{x}_0)| \leq \int_{\partial G} |\varphi(\mathbf{x}_0 - \mathbf{y}) - \varphi(\mathbf{x} - \mathbf{y})| \cdot |\psi(\mathbf{y})| dS_{\mathbf{y}}$$

$$
\begin{aligned}
&\leq \int_{\partial G \setminus B_d(\mathbf{x}_0)} |\varphi(\mathbf{x}_0 - \mathbf{y}) - \varphi(\mathbf{x} - \mathbf{y})| \cdot M dS_{\mathbf{y}} \\
&\quad + \int_{\partial G \cap B_d(\mathbf{x}_0)} \frac{M}{r_{\mathbf{xy}}} dS_{\mathbf{y}} + \int_{\partial G \cap B_d(\mathbf{x}_0)} \frac{M}{r_{\mathbf{x}_0\mathbf{y}}} dS_{\mathbf{y}} \\
&< 3M\varepsilon
\end{aligned}
$$

■

2.2 Stokes Equations

In this section, we derive a Fredholm integral equation of the first kind for the unknown function $t_i = T_{ij}(\mathbf{v}, q)n_j$ on the boundary. We require that the functions $\mathbf{v}(\mathbf{y}) = (v_i(\mathbf{y}))_{i=1,2,3}$ and $q(\mathbf{y})$ are in $(C^2(\overline{\Omega}))^3$ and $C^1(\overline{\Omega})$, respectively, and that they build a solution to the **Stokes Equations**

$$
\left.
\begin{aligned}
\Delta \mathbf{v} - \nabla q &= \mathbf{0} \\
\nabla \cdot \mathbf{v} &= 0
\end{aligned}
\right\} \quad \text{in } \Omega, \tag{7}
$$

$$
\mathbf{v}|_{\partial G} \in C(\partial G) \quad \text{given.} \tag{8}
$$

Furthermore, the following decay-conditions are assumed to hold:

$$
\left.
\begin{aligned}
\mathbf{v} &= O(|\mathbf{y}|^{-1}) \\
\nabla \mathbf{v} &= O(|\mathbf{y}|^{-2}) \\
q &= O(|\mathbf{y}|^{-2})
\end{aligned}
\right\} \quad \text{as } |\mathbf{y}| \to \infty. \tag{9}
$$

Remark. There exists a unique solution to the above problem, as Galdi [9] and Ladyzhenskaya [22] show.

2.2.1 Fundamental Solutions

For fixed $\mathbf{x} \in \mathbb{R}^3$ there are three linearly independent pairs of functions in $(C^\infty(\mathbb{R}^3 \setminus \{\mathbf{x}\}))^3 \times C^\infty(\mathbb{R}^3 \setminus \{\mathbf{x}\})$ which solve the problem

$$
\left.
\begin{aligned}
\Delta \mathbf{E} - \nabla e &= \mathbf{0} \\
\nabla \cdot \mathbf{E} &= 0
\end{aligned}
\right\} \quad \text{in } \mathbb{R}^3 \setminus \{\mathbf{x}\} \tag{10}
$$

and have the lowest possible degree of singularity at $\mathbf{x}$. They are called the **Fundamental Solutions of the Stokes Equations**, denoted by $(\mathbf{E}_k, e_k)$, $k = 1, 2, 3$, and we seek them in the form (see Oseen [26])

$$\begin{aligned} E_{ik} &= \delta_{ik}\Phi_{,ll} - \Phi_{,ik} \\ e_k &= -\Phi_{,llk}, \end{aligned}$$

where $\Phi = \Phi(r)$ is a function in $C^\infty(\mathbb{R}^3 \setminus \{\mathbf{x}\})$, and all derivatives are taken with respect to $\mathbf{y}$. If we require $\Phi_{,lljj} = 0$, then (10) is already satisfied. Therefore, we choose $\Phi(r) := -r/8\pi$, and get the following results:

$$E_{ik} = -\frac{1}{8\pi}\left(\frac{\delta_{ik}}{r_{\mathbf{xy}}} + \frac{d_i d_k}{r_{\mathbf{xy}}^3}\right) \tag{11}$$

$$e_k = \frac{1}{4\pi}\frac{d_k}{r_{\mathbf{xy}}^3} \tag{12}$$

$$E_{ik,j} = -\frac{1}{8\pi}\left(\frac{3d_i d_j d_k}{r_{\mathbf{xy}}^5} + \frac{\delta_{ik}d_j - \delta_{ij}d_k - \delta_{jk}d_i}{r_{\mathbf{xy}}^3}\right) \tag{13}$$

$$\begin{aligned} T_{ij}(\mathbf{E}_k, e_k) &:= -\delta_{ij}e_k + E_{ik,j} + E_{jk,i} \\ &= -\frac{3d_i d_j d_k}{4\pi r_{\mathbf{xy}}^5} \end{aligned} \tag{14}$$

$$T_{ij,j}(\mathbf{E}_k, e_k) = 0 \tag{15}$$

With these results, one can see that the fundamental solutions satisfy the following decay conditions:

$$\left.\begin{aligned} E_{ik}(\mathbf{x},\mathbf{y}) &= O({r_{\mathbf{xy}}}^{-1}) \\ E_{ik,j}(\mathbf{x},\mathbf{y}) &= O({r_{\mathbf{xy}}}^{-2}) \\ e_k(\mathbf{x},\mathbf{y}) &= O({r_{\mathbf{xy}}}^{-2}) \end{aligned}\right\} \quad \text{as } r_{\mathbf{xy}} \to \infty \text{ and as } r_{\mathbf{xy}} \to 0. \tag{16}$$

However, the fundamental solutions have a singularity at the point $\mathbf{x}$, which we now investigate:

Lemma 2.4 *Let* $\mathbf{v}$ *be a function that is continuous in a neighborhood of* $\mathbf{x}$ *and let* $\mathbf{n}$ *be the inward normal to* $S_\varepsilon(\mathbf{x})$, *i.e. the normal that points from the sphere towards its midpoint. Then*

$$\lim_{\varepsilon\to 0}\int_{S_\varepsilon(\mathbf{x})} v_i(\mathbf{y})T_{ij}(\mathbf{E}_k, e_k)(\mathbf{x},\mathbf{y})n_j(\mathbf{y})dS_{\mathbf{y}} = -v_k(\mathbf{x}). \tag{17}$$

Proof:

$$\begin{aligned}
&\int_{S_\varepsilon(\mathbf{x})} v_i T_{ij}(\mathbf{E}_k, e_k) n_j dS_\mathbf{y} \\
&= \int_{S_\varepsilon(\mathbf{x})} v_i(\mathbf{y}) \frac{-3 d_i d_j d_k}{4\pi r^5} \frac{d_j}{r} dS_\mathbf{y} \\
&= -\frac{3}{4\pi} \int_{S_\varepsilon(\mathbf{x})} v_i(\mathbf{y}) \frac{d_i d_k}{4\pi r^4} dS_\mathbf{y} \\
&= -\frac{3}{4\pi} \int_{S_\varepsilon(\mathbf{x})} v_i(\mathbf{x}) \frac{d_i d_k}{4\pi r^4} dS_\mathbf{y} - \frac{3}{4\pi} \int_{S_\varepsilon(\mathbf{x})} (v_i(\mathbf{y}) - v_i(\mathbf{x})) \frac{d_i d_k}{4\pi r^4} dS_\mathbf{y}
\end{aligned}$$

We note that the integrand of the second integral is of order $o(1/\varepsilon^2)$ as $\varepsilon \to 0$. Thus, the second integral vanishes in the limit. Lemma 2.1 concludes the proof. ∎

2.2.2 Integral Equations

To derive the integral equation for the stress, we assume $\mathbf{x} \in \Omega$ and define $D(R, \varepsilon) := B_R(\mathbf{x}) \setminus (\overline{G} \cup \overline{B_\varepsilon(\mathbf{x})})$, where ε is at least so small that $B_{2\varepsilon} \subset \Omega$ and R so big that $G \subset B_R(\mathbf{x})$. It is then clear that $\lim_{\varepsilon \to 0, R \to \infty} D(R, \varepsilon) = \Omega$. By applying Greens Formula (6) on $D(R, \varepsilon)$ and with $(\mathbf{u}, p) = (\mathbf{E}_k, e_k)$, we get

$$0 = \left(\int_{\partial G} + \int_{S_\varepsilon(\mathbf{x})} + \int_{S_R(\mathbf{x})} \right) v_i T_{ij}(\mathbf{E}_k, e_k) n_j - E_{ik} T_{ij}(\mathbf{v}, q) n_j dS_\mathbf{y} \tag{18}$$

Definition 2.2 *For $k = 1, 2, 3$ we introduce the* **Single and Double Layer Potentials**

$$W_k^{(1)}(\mathbf{v}, q)(\mathbf{x}) := \int_{\partial G} E_{ik}(\mathbf{x}, \mathbf{y}) T_{ij}(\mathbf{v}, q)(\mathbf{y}) n_j(\mathbf{y}) dS_\mathbf{y} \tag{19}$$

$$W_k^{(2)}(\mathbf{v})(\mathbf{x}) := \int_{\partial G} v_i(\mathbf{y}) T_{ij}(\mathbf{E}_k, e_k)(\mathbf{x}, \mathbf{y}) n_j(\mathbf{y}) dS_\mathbf{y}, \tag{20}$$

where $\mathbf{n}$ is the normal that points away from the exterior domain.

Now we can establish two interesting results. The second one is particularly important for the numerical computation in the latter sections.

Theorem 2.1 *For* $\mathbf{x} \in \Omega$ *we have*

$$W_k^{(1)}(\mathbf{v}, q)(\mathbf{x}) = -v_k(\mathbf{x}) + W_k^{(2)}(\mathbf{v})(\mathbf{x}). \tag{21}$$

Proof: We notice by looking at the decay-conditions (9) and (16) that the integral over $S_R(\mathbf{x})$ in (18) vanishs as $R \to \infty$. Since $E_{ik} = O(1/r)$ as $r \to 0$ and $T_{ij}n_j$ is continuous, we get

$$\lim_{\varepsilon \to 0} \int_{S_\epsilon(\mathbf{x})} E_{ik} T_{ij}(\mathbf{v}, q) n_j dS_\mathbf{y} = 0.$$

By a short look at (17), we now get the assertion. ∎

Theorem 2.2 *For* $\mathbf{x} \in \partial G$ *we have*

$$W_k^{(1)}(\mathbf{v}, q)(\mathbf{x}) = -\frac{1}{2} v_k(\mathbf{x}) + W_k^{(2)}(\mathbf{v})(\mathbf{x}). \tag{22}$$

If $\mathbf{v}$ *is constant on the boundary, then*

$$W_k^{(1)}(\mathbf{v}, q)(\mathbf{x}) = -v_k. \tag{23}$$

Proof: Let $(\mathbf{x}_n) \subset \Omega$ be a sequence approaching $\mathbf{x}$. Since $\mathbf{v}$ is continuous, we get from (21)

$$\lim_{\mathbf{x}_n \to \mathbf{x}} W_k^{(1)}(\mathbf{v}, q)(\mathbf{x}_n) = -v_k(\mathbf{x}) + \lim_{\mathbf{x}_n \to \mathbf{x}} W_k^{(2)}(\mathbf{v})(\mathbf{x}_n)$$

It is a result from Odqvist [25] that the double layer potentials $W_k^{(2)}(\mathbf{v})$ are discontinuous across the boundary. In particular we have:

$$\lim_{\Omega \ni \mathbf{x}_n \to \mathbf{x} \in \partial G} W_k^{(2)}(\mathbf{v})(\mathbf{x}_n) = \frac{1}{2} v_k(\mathbf{x}) + W_k^{(2)}(\mathbf{v})(\mathbf{x}) \tag{24}$$

$$\lim_{G \ni \mathbf{x}_n \to \mathbf{x} \in \partial G} W_k^{(2)}(\mathbf{v})(\mathbf{x}_n) = -\frac{1}{2} v_k(\mathbf{x}) + W_k^{(2)}(\mathbf{v})(\mathbf{x}). \tag{25}$$

For the single layer potentials we notice that $E_{ik} = O(1/r)$ as $r \to 0$. Since $T_{ij}n_j$ is continuous on ∂G, we can apply lemma 2.3 to see the continuity of $W_k^{(1)}(\mathbf{v}, q)$:

$$\lim_{\mathbf{x}_n \to \mathbf{x}} W_k^{(1)}(\mathbf{v}, q)(\mathbf{x}_n) = W_k^{(1)}(\mathbf{v}, q)(\mathbf{x})$$

For the second assertion we refer again to Odqvist [25], who showed that for any $\mathbf{v}$ that is constant on the boundary ∂G

$$W_k^{(2)}(\mathbf{v})(\mathbf{x}) = -\frac{1}{2}v_k. \tag{26}$$

holds. This result is actually used to prove the above jump conditions. ∎

2.3 Oseen Equations

Now we require that the functions $\mathbf{v}(\mathbf{y}) = (v_i(\mathbf{y}))_{i=1,2,3}$ and $q(\mathbf{y})$ are in $(C^2(\overline{\Omega}))^3$ and $C^1(\overline{\Omega})$, respectively, and that they build a solution to the **Oseen Equations**

$$\left.\begin{array}{c}(\Delta - 2a\frac{\partial}{\partial y_1})\mathbf{v} - \nabla q = \mathbf{0} \\ \nabla\cdot\mathbf{v} = 0\end{array}\right\} \text{ in } \Omega, \tag{27}$$

$$\mathbf{v}|_{\partial G} \in C(\partial G) \text{ given.} \tag{28}$$

where $a > 0$ is a constant. Furthermore, the following decay-conditions are assumed to hold:

$$\left.\begin{array}{c}\mathbf{v} = O(|\mathbf{y}|^{-1}) \\ \nabla\mathbf{v} = O(|\mathbf{y}|^{-3/2}) \\ q = O(|\mathbf{y}|^{-2})\end{array}\right\} \text{ as } |\mathbf{y}| \to \infty. \tag{29}$$

Remark. There exists a unique solution to the above problem, as Galdi [9] and Ladyzhenskaya [22] show.

2.3.1 Fundamental Solutions

For fixed $\mathbf{x} \in \mathbb{R}^3$ there are three linearly independent pairs of functions in $(C^\infty(\mathbb{R}^3 \setminus \{\mathbf{x}\}))^3 \times C^\infty(\mathbb{R}^3 \setminus \{\mathbf{x}\})$ which solve the problem

$$\left.\begin{array}{c}(\Delta + 2a\frac{\partial}{\partial y_1})\mathbf{E} - \nabla e = \mathbf{0} \\ \nabla\cdot\mathbf{E} = 0\end{array}\right\} \text{ in } \mathbb{R}^3 \setminus \{\mathbf{x}\} \tag{30}$$

and have the lowest possible degree of singularity at $\mathbf{x}$. They are called the **Fundamental Solutions of the Oseen Equations**, denoted by $(\mathbf{E}_k^{Os}, e_k^{Os})$, $k = 1, 2, 3$, and we seek them in the form (see Oseen [26])

$$\begin{aligned} E_{ik}^{Os} &= \delta_{ik}\Phi_{,ll} - \Phi_{,ik} \\ e_k^{Os} &= -\Phi_{,llk} - 2a\Phi_{,1k} \end{aligned}$$

where $\Phi = \Phi(\mathbf{d})$ is a function in $C^\infty(\mathbb{R}^3 \setminus \{\mathbf{x}\})$ and all derivatives are taken with respect to $\mathbf{y}$. If we require $\Phi_{,lljj} + 2a\Phi_{,ll1} = 0$, then (30) is already satisfied. We refer to Oseen [26] for getting

$$\Phi(s) := -\frac{1}{8\pi a}\int_0^{as} \frac{1 - e^{-\tau}}{\tau} d\tau, \tag{31}$$

where $s := r - d_1 \geq 0$. We can then compute the following auxiliary results, where the orders have to be understood as $r \to 0$:

$$\begin{aligned} s_{,i} &= -\frac{d_i}{r} + \delta_{1i} \\ s_{,ij} &= \frac{\delta_{ij}}{r} - \frac{d_i d_j}{r^3} \\ s_{,ijk} &= \frac{\delta_{ij}d_k + \delta_{ik}d_j + \delta_{jk}d_i}{r^3} - \frac{3d_i d_j d_k}{r^5} \\ \Phi' &:= \frac{\partial}{\partial s}\Phi = -\frac{1 - e^{-as}}{8\pi as} = -\frac{1}{8\pi} + O(r) \qquad (32) \\ \Phi'' &:= \frac{\partial^2}{\partial s^2}\Phi = -\frac{ase^{-as} - 1 + e^{-as}}{8\pi as^2} \qquad (33) \\ \Phi''' &:= \frac{\partial^3}{\partial s^3}\Phi = -\frac{-a^2s^2e^{-as} - 2ase^{-as} + 2 - 2e^{-as}}{8\pi as^3} \\ \Phi_{,i} &= s_{,i}\Phi' \\ \Phi_{,ij} &= s_{,ij}\Phi' + s_{,i}s_{,j}\Phi'' \\ \Phi_{,ijk} &= s_{,ijk}\Phi' + (s_{,ij}s_{,k} + s_{,jk}s_{,i} + s_{,ki}s_{,j})\Phi'' + s_{,i}s_{,j}s_{,k}\Phi''' \end{aligned}$$

Now we can compute the fundamental solutions and their derivatives (The orders are again as $r \to 0$):

$$\begin{aligned} E_{ik}^{Os} &= \left(\frac{\delta_{ik}}{r} + \frac{d_i d_k}{r^3}\right)\Phi' + \\ &\quad \left(2\delta_{ik} - \delta_{1i}\delta_{1k} + \frac{\delta_{1i}d_k + \delta_{1k}d_i - 2\delta_{ik}d_1}{r} - \frac{d_i d_k}{r^2}\right)\Phi'' \end{aligned} \tag{34}$$

$$= E_{ik} + O(1) \tag{35}$$

$$e_k^{Os} = e_k \tag{36}$$

$$E_{ik,j}^{Os} = E_{ik,j} + O(\frac{1}{r}) \tag{37}$$

$$T_{ij}(\mathbf{E}_k^{Os}, e_k) := -\delta_{ij} e_k + E_{ik,j}^{Os} + E_{jk,i}^{Os}$$

$$= T_{ij}(\mathbf{E}_k, e_k) + O(\frac{1}{r}) \tag{38}$$

$$T_{ij,j}(\mathbf{E}_k^{Os}, e_k) = -2a E_{ik,1}^{Os} \tag{39}$$

The behavior at infinity is not as obvious as in the Stokes case:

Lemma 2.5 *The fundamental solutions satisfy the following decay conditions:*

$$\left.\begin{array}{c} E_{ik}^{Os}(\mathbf{x},\mathbf{y}) = O(r^{-1}) \\ E_{ik,j}^{Os}(\mathbf{x},\mathbf{y}) = O(r^{-3/2}) \\ e_k(\mathbf{x},\mathbf{y}) = O(r^{-2}) \end{array}\right\} \quad as \quad r \to \infty. \tag{40}$$

Proof: We look at (34). Since Φ' is bounded in $\mathbb{R}^3$, we show the first assertion only for the second term. It's easy to check that Φ'' is of order $O(s^{-1})$ as $r \to \infty$. We can also see the following estimate ($k = 2, 3$):

$$d_k^2 \leq d_2^2 + d_3^2 = r^2 - d_1^2 = (r - d_1)(r + d_1) = s(r + d_1) \leq 2sr$$

Let's denote the second bracket in (34) by F_{ik}. For $k = 2, 3$ we write

$$|F_{11}| = |1 - \frac{d_1^2}{r^2}| \leq 2|\frac{s}{r}|$$

$$|F_{kk}| = |2 - 2\frac{d_1}{r} - \frac{d_k^2}{r^2}| = |2\frac{s}{r} - \frac{d_k^2}{r^2}| \leq 4|\frac{s}{r}|$$

$$|F_{23}| = |\frac{d_2 d_3}{r^2}| \leq 2|\frac{s}{r}|$$

$$|F_{1k}| = |\frac{d_k}{r} - \frac{d_1 d_k}{r^2}| = |\frac{d_k s}{r^2}| \leq |\frac{s}{r}|$$

Thus, F_{ik} is of order $O(s/r)$, and we have shown the first assertion. The second assertion can be shown by a similar reasoning, we refer to Galdi [9]. The third assertion is quite obvious. ∎

Even better estimates can be obtained, if we restrict ourselves on a region outside a wake behind the motion of the sphere:

Lemma 2.6 *For $0 < \lambda < 1$ we define the* **Wake** $W_R^\lambda(\mathbf{x}) := \{\mathbf{y} \in S_R(\mathbf{x}) : d_1 \geq r\lambda\}$. *It's area is given by*

$$\int_{W_R^\lambda(\mathbf{x})} dS_\mathbf{y} = 2\pi R^2(1-\lambda), \tag{41}$$

and outside this wake we have

$$E_{ik}^{Os}(\mathbf{x},\mathbf{y}) = O(\frac{1}{r^2(1-\lambda)^2}) \quad as \quad r \to \infty. \tag{42}$$

Proof: The wake is the intersection of a sphere with radius R around $\mathbf{x}$ and a cone with vertex $\mathbf{x}$, axis in direction of the negative 1-axis and angle φ, where $\cos\varphi = \lambda$. It's then only a matter of computing a surface integral to check that (41) is true. Outside the wake we have $d_1 < r\lambda$, which is equivalent to $s > r(1-\lambda)$. Since the numerators in (32) and (33) are bounded for $s > 0$, we get

$$\begin{aligned} |\Phi'| &\leq \frac{c'}{r(1-\lambda)} < \frac{c'}{r(1-\lambda)^2} \\ |\Phi''| &\leq \frac{c''}{r^2(1-\lambda)^2}, \end{aligned}$$

where c' and c'' are constants that depend only on a. By looking at (34), we can now verify the assertion easily. ∎

Remark. The fundamental solution E_{ik}^{Os} as a function of $\mathbf{y}$ exhibits obviously a "wake" region in direction opposite to what one would expect for a flow past a body in direction of the positive x_1-axis. This is because E_{ik}^{Os} as a function of $\mathbf{y}$ satisfies the adjoint system (30) of the Oseen equations. We refer to Galdi [9].

However, the fundamental solutions have a singularity at the point $\mathbf{x}$, which we now investigate:

Lemma 2.7 *Let* $\mathbf{v}$ *be a function that is continuous in a neighborhood of* $\mathbf{x}$ *and let* $\mathbf{n}$ *be the inward normal to* $S_\varepsilon(\mathbf{x})$*, i.e. the normal that points from the sphere towards its midpoint. Then*

$$\lim_{\varepsilon\to 0}\int_{S_\varepsilon(\mathbf{x})} v_i(\mathbf{y})T_{ij}(\mathbf{E}_k^{Os},e_k)(\mathbf{x},\mathbf{y})n_j(\mathbf{y})dS_\mathbf{y} = -v_k(\mathbf{x}). \tag{43}$$

Proof:

$$\begin{aligned}&\int_{S_\varepsilon(\mathbf{x})} v_iT_{ij}(\mathbf{E}_k^{Os},e_k)n_jdS_\mathbf{y}\\ =\ &\int_{S_\varepsilon(\mathbf{x})} v_iT_{ij}(\mathbf{E}_k,e_k)n_jdS_\mathbf{y} + \int_{S_\varepsilon(\mathbf{x})} v_i\left(T_{ij}(\mathbf{E}_k^{Os},e_k)-T_{ij}(\mathbf{E}_k,e_k)\right)n_jdS_\mathbf{y}\end{aligned}$$

We notice that the integrand of the second integral is of order $O(1/\varepsilon)$ as $\varepsilon\to 0$. Thus, the second integral vanishes in the limit. Lemma 2.4 concludes the proof. ∎

2.3.2 Integral Equations

To derive the integral equation for the surface stress, we assume $\mathbf{x}\in\Omega$ and define $D(R,\varepsilon)$ like in section 2.2.2. By applying Greens Formula (6) on $D(R,\varepsilon)$ and with $(\mathbf{u},p)=(\mathbf{E}_k^{Os},e_k)$, we get

$$0=\left(\int_{\partial G}+\int_{S_\epsilon(\mathbf{x})}+\int_{S_R(\mathbf{x})}\right)v_iT_{ij}(\mathbf{E}_k^{Os},e_k)n_j - E_{ik}^{Os}T_{ij}(\mathbf{v},q)n_j + 2aE_{ik}^{Os}v_in_1dS_\mathbf{y} \tag{44}$$

Definition 2.3 *For* $k=1,2,3$ *we introduce the* **Single and Double Layer Potentials**

$$W_k^{(1,Os)}(\mathbf{v},q)(\mathbf{x}) := \int_{\partial G} E_{ik}^{Os}(\mathbf{x},\mathbf{y})T_{ij}(\mathbf{v},q)(\mathbf{y})n_j(\mathbf{y})dS_\mathbf{y} \tag{45}$$

$$W_k^{(2,Os)}(\mathbf{v})(\mathbf{x}) := \int_{\partial G} v_i(\mathbf{y})T_{ij}(\mathbf{E}_k^{Os},e_k)(\mathbf{x},\mathbf{y})n_j(\mathbf{y})dS_\mathbf{y} \tag{46}$$

$$W_k^{(3,Os)}(\mathbf{v})(\mathbf{x}) := \int_{\partial G} 2av_i(\mathbf{y})E_{ik}^{Os}(\mathbf{x},\mathbf{y})n_1(\mathbf{y})dS_\mathbf{y}, \tag{47}$$

where $\mathbf{n}$ *is the normal that points away from the exterior domain.*

We can now establish two results similar to the Stokes case. We use the second one in the next section for the computation of the stress.

Theorem 2.3 *For* $\mathbf{x} \in \Omega$ *we have*

$$W_k^{(1,Os)}(\mathbf{v},q)(\mathbf{x}) = -v_k(\mathbf{x}) + W_k^{(2,Os)}(\mathbf{v})(\mathbf{x}) + W_k^{(3,Os)}(\mathbf{v})(\mathbf{x}). \tag{48}$$

Proof: We notice by looking at the decay-conditions (9) and (40) that

$$\lim_{R\to\infty} \int_{S_R(\mathbf{x})} v_i T_{ij}(\mathbf{E}_k^{Os}, e_k) n_j - E_{ik}^{Os} T_{ij}(\mathbf{v},q) n_j dS_\mathbf{y} = 0$$

The third term in (44) makes a little bit more trouble at infinity. We assume R big enough to use the order-estimates and look at Lemma 2.6 to get

$$\begin{aligned}
\left| \int_{S_R(\mathbf{x})} 2aE_{ik}^{Os} v_i n_1 dS_\mathbf{y} \right| &\leq \left(\int_{S_R(\mathbf{x})\setminus W_R^\lambda(\mathbf{x})} + \int_{W_R^\lambda(\mathbf{x})} \right) 2a|E_{ik}^{Os} v_i n_1| dS_\mathbf{y} \\
&\leq \int_{S_R(\mathbf{x})\setminus W_R^\lambda(\mathbf{x})} \frac{c'_k}{R^3(1-\lambda)^2} dS_\mathbf{y} + \int_{W_R^\lambda(\mathbf{x})} \frac{c''_k}{R^2} dS_\mathbf{y} \\
&\leq \frac{4\pi c'_k}{R(1-\lambda)^2} + 2\pi c''_k (1-\lambda)
\end{aligned}$$

with positive constants c'_k and c''_k. Thus, we have for every $0 < \lambda < 1$

$$\lim_{R\to\infty} \left| \int_{S_R(\mathbf{x})} 2aE_{ik}^{Os} v_i n_1 dS_\mathbf{y} \right| \leq 2\pi c''_k (1-\lambda)$$

It's then clear that the integral has to vanish at infinity.

Since $E_{ik}^{Os} = O(1/r)$ as $r \to 0$ and $T_{ij} n_j$ is continuous, we get

$$\lim_{\varepsilon\to 0} \int_{S_\varepsilon(\mathbf{x})} E_{ik}^{Os} T_{ij}(\mathbf{v},q) n_j + 2av_i E_{ik}^{Os} n_1 dS_\mathbf{y} = 0.$$

By a short look at (43), we now get the assertion. ∎

Theorem 2.4 *For* $\mathbf{x} \in \partial G$ *we have*

$$W_k^{(1,Os)}(\mathbf{v},q)(\mathbf{x}) = -\frac{1}{2} v_k(\mathbf{x}) + W_k^{(2,Os)}(\mathbf{v})(\mathbf{x}) + W_k^{(3,Os)}(\mathbf{v})(\mathbf{x}). \tag{49}$$

If $\mathbf{v}$ *is constant on the boundary, then*

$$W_k^{(1,Os)}(\mathbf{v},q)(\mathbf{x}) = -v_k. \tag{50}$$

Proof: Let $(\mathbf{x}_n) \subset \Omega$ be a sequence approaching $\mathbf{x}$. Since $\mathbf{v}$ is continuous, we get from (48)

$$\lim_{\mathbf{x}_n \to \mathbf{x}} W_k^{(1,Os)}(\mathbf{v},q)(\mathbf{x}_n) = -v_k(\mathbf{x}) + \lim_{\mathbf{x}_n \to \mathbf{x}} W_k^{(2,Os)}(\mathbf{v})(\mathbf{x}_n) + \lim_{\mathbf{x}_n \to \mathbf{x}} W_k^{(3,Os)}(\mathbf{v})(\mathbf{x}_n)$$

Let us look at the double layer potentials:

$$W_k^{(2,Os)}(\mathbf{v})(\mathbf{x}_n) \;=\; W_k^{(2)}(\mathbf{v})(\mathbf{x}_n) + \int_{\partial G} v_i \left(T_{ij}(\mathbf{E}_k^{Os}, e_k) - T_{ij}(\mathbf{E}_k, e_k)\right) n_j dS_{\mathbf{y}}.$$

We note that by looking at (38) that the integrand of the second integral is of order $O(1/\varepsilon)$ as $\varepsilon \to 0$, and Lemma 2.3 tells us that this integral is continuous. Using the jump conditions in the Stokes case (24) and (25) we get the same jump conditions in the Oseen case:

$$\lim_{\Omega \ni \mathbf{x}_n \to \mathbf{x} \in \partial G} W_k^{(2,Os)}(\mathbf{v})(\mathbf{x}_n) \;=\; \frac{1}{2} v_k(\mathbf{x}) + W_k^{(2,Os)}(\mathbf{v})(\mathbf{x}) \tag{51}$$

$$\lim_{G \ni \mathbf{x}_n \to \mathbf{x} \in \partial G} W_k^{(2,Os)}(\mathbf{v})(\mathbf{x}_n) \;=\; -\frac{1}{2} v_k(\mathbf{x}) + W_k^{(2,Os)}(\mathbf{v})(\mathbf{x}). \tag{52}$$

For the single layer potentials, we notice that $E_{ik}^{Os} = O(1/\varepsilon)$ as $\varepsilon \to 0$. Since $T_{ij} n_j$ is continuous on ∂G, we again apply Lemma 2.3 to see the continuity of $W_k^{(1,Os)}(\mathbf{v},q)$ and $W_k^{(3,Os)}(\mathbf{v})$:

$$\begin{aligned} \lim_{\mathbf{x}_n \to \mathbf{x}} W_k^{(1,Os)}(\mathbf{v},q)(\mathbf{x}_n) &= W_k^{(1,Os)}(\mathbf{v},q)(\mathbf{x}) \\ \lim_{\mathbf{x}_n \to \mathbf{x}} W_k^{(3,Os)}(\mathbf{v})(\mathbf{x}_n) &= W_k^{(3,Os)}(\mathbf{v})(\mathbf{x}) \end{aligned}$$

To show the second assertion, we use the corresponding result of the Stokes case, namely (26):

$$\begin{aligned} & W_k^{(2,Os)}(\mathbf{v})(\mathbf{x}) + W_k^{(3,Os)}(\mathbf{v})(\mathbf{x}) \\ = \;& W_k^{(2)}(\mathbf{v})(\mathbf{x}) + \int_{\partial G} v_i \left(T_{ij}(\mathbf{E}_k^{Os}, e_k) - T_{ij}(\mathbf{E}_k, e_k)\right) n_j + 2a v_i E_{ik}^{Os}(\mathbf{x},\mathbf{y}) n_1 dS_{\mathbf{y}} \\ = \;& -\frac{1}{2} v_k + \int_{\partial G} v_i \left(T_{ij}(\mathbf{E}_k^{Os}, e_k) - T_{ij}(\mathbf{E}_k, e_k)\right) n_j + 2a v_i E_{ik}^{Os}(\mathbf{x},\mathbf{y}) n_1 dS_{\mathbf{y}} \end{aligned}$$

It remains to show that the integral is zero:

$$\int_{\partial G} v_i \left(T_{ij}(\mathbf{E}_k^{Os}, e_k) - T_{ij}(\mathbf{E}_k, e_k)\right) n_j + 2a v_i E_{ik}^{Os}(\mathbf{x},\mathbf{y}) n_1 dS_{\mathbf{y}}$$

$$
\begin{aligned}
&= \left(\int_{\partial G \setminus B_\lambda(\mathbf{x})} + \int_{G \cap S_\lambda(\mathbf{x})} - \int_{G \cap S_\lambda(\mathbf{x})} + \int_{\partial G \cap B_\lambda(\mathbf{x})} \right) \ \ldots \, dS_{\mathbf{y}} \\
&= \int_{G \setminus B_\lambda(\mathbf{x})} v_i \left(T_{ij,j}(\mathbf{E}_k^{Os}, e_k) + 2a E_{ik,1}^{Os}(\mathbf{x}, \mathbf{y}) \right) - v_i T_{ij,j}(\mathbf{E}_k, e_k) d\mathbf{y} \\
&\quad - \int_{G \cap S_\lambda(\mathbf{x})} \ldots \, dS_{\mathbf{y}} + \int_{\partial G \cap B_\lambda(\mathbf{x})} \ldots \, dS_{\mathbf{y}} \\
&= - \int_{G \cap S_\lambda(\mathbf{x})} \ldots \, dS_{\mathbf{y}} + \int_{\partial G \cap B_\lambda(\mathbf{x})} \ldots \, dS_{\mathbf{y}}
\end{aligned}
$$

In the last equation we used (15) and (39). This holds for every $\lambda > 0$, so let's take $\lambda \to 0$. Since the orders of the integrands are $O(1/r)$ as $r \to 0$, the first integral vanishs obviously, and the second with the help of Lemma 2.2. ∎

3 Investigation of the Integral Equations

In this section, we investigate the integral equations (23) and (50). If we denote the components of the stress by $t_i(\mathbf{y}) := T_{ij}(v,q)n_j(\mathbf{y})$, we can write them like

$$\left.\begin{aligned} \int_{\partial G} E_{ik}(\mathbf{x},\mathbf{y})t_i(\mathbf{y})dS_{\mathbf{y}} &= -v_k \\ \int_{\partial G} E^{Os}_{ik}(\mathbf{x},\mathbf{y})t^{Os}_i(\mathbf{y})dS_{\mathbf{y}} &= -v_k \end{aligned}\right\} \quad \text{for } \mathbf{x} \in \partial G. \tag{53}$$

where v_k is the constant boundary velocity.

As we have seen in the last section, these integral equations arise, if we want to solve the exterior Stokes and Oseen problems, where the boundary velocity is given by a constant vector and the decay at infinity is appropriate. They also arise, if we want to solve the corresponding interior problems, which is easier to see, since we don't have to worry about decay at infinity. All of these problems are uniquely solvable, as long as one prescribes the value of the pressure at one point. We refer to Galdi [9] and Ladyzhenskaya [22] for these statements.

Equations (53) are Fredholm integral equations of the first kind. Their kernels are both weakly singular and symmetric, i.e. the integral equations are self-adjoint. Both of them are singular, as it turns out, and their one-dimensional kernels are spanned by the normal $\mathbf{n}$ on the boundary ∂G:

Theorem 3.1 *The continuous solutions of the homogenous integral equations*

$$\left.\begin{aligned} \int_{\partial G} E_{ik}(\mathbf{x},\mathbf{y})\varphi_i(\mathbf{y})dS_{\mathbf{y}} &= 0 \\ \int_{\partial G} E^{Os}_{ik}(\mathbf{x},\mathbf{y})\varphi^{Os}_i(\mathbf{y})dS_{\mathbf{y}} &= 0 \end{aligned}\right\} \quad \textit{for } \mathbf{x} \in \partial G \tag{54}$$

are multiples of the function $\mathbf{n}$*, i.e. the normal on the boundary* ∂G*.*

Proof: The fact that E_{ik} and E^{Os}_{ik} are both divergence free allows us to write for $\mathbf{x} \in \Omega$

$$\int_{\partial G} E_{ik}(\mathbf{x},\mathbf{y})n_i(\mathbf{y})dS_{\mathbf{y}} = \int_G E_{ik,i}(\mathbf{x},\mathbf{y})dy = 0,$$

and the same in the Oseen case. The continuity of the single layer potentials (lemma 2.3) shows that $\mathbf{n}$ is in fact a solution to the integral equations. We now show that each solution has to be a multiple of $\mathbf{n}$.

Let us at first consider the Stokes case. Suppose we have a continuous solution φ_i to the first equation in (54). By looking at the Stokes equations, one can easily see, that the solution to the exterior problem has to look like $(\mathbf{0}, c^{\text{ext}})$, where c^{ext} is a constant. On the other hand, potential theory tells us how to recover the solution using φ_i. Thus, we have for $\mathbf{x} \in \Omega$

$$\begin{aligned} v_k^{\text{ext}}(\mathbf{x}) &:= -\int_{\partial G} E_{ik}(\mathbf{x},\mathbf{y})\varphi_i(\mathbf{y})dS_{\mathbf{y}} = 0 \\ q^{\text{ext}}(\mathbf{x}) &:= c^{\text{ext}} + \int_{\partial G} e_i(\mathbf{x},\mathbf{y})\varphi_i(\mathbf{y})dS_{\mathbf{y}} = c^{\text{ext}}. \end{aligned}$$

Notice that the integral that occurs in the pressure q^{ext} is a constant, which has to be zero, since it is obviously zero at infinity. Similarly, we get the solution of the interior problem ($\mathbf{x} \in G$):

$$\begin{aligned} v_k^{\text{int}}(\mathbf{x}) &:= -\int_{\partial G} E_{ik}(\mathbf{x},\mathbf{y})\varphi_i(\mathbf{y})dS_{\mathbf{y}} = 0 \\ q^{\text{int}}(\mathbf{x}) &:= c + \int_{\partial G} e_i(\mathbf{x},\mathbf{y})\varphi_i(\mathbf{y})dS_{\mathbf{y}} = c^{\text{int}}, \end{aligned}$$

where now c and c^{int} have to be distinguished, according to the constant value of the integral. We summarize these results by defining on $G \cup \Omega$

$$(\mathbf{v}, q) := \begin{cases} (\mathbf{v}^{\text{ext}}, q^{\text{ext}}) & \text{in } \Omega \\ (\mathbf{v}^{\text{int}}, q^{\text{int}}) & \text{in } G. \end{cases}$$

Next, we define on $G \cup \Omega$ the stress tensor in the usual way:

$$T_{kj}(\mathbf{v}, q)(\mathbf{x}) := -q(\mathbf{x})\delta_{kj} + \frac{\partial}{\partial x_j} v_k(\mathbf{x}) + \frac{\partial}{\partial x_k} v_j(\mathbf{x})$$

Plugging in our above results, we get

$$T_{kj}(\mathbf{v}, q) = \begin{cases} -c^{\text{ext}}\delta_{kj} + \int_{\partial G} T_{kj}(\mathbf{E}_i, e_i)\varphi_i dS_{\mathbf{y}} = -c^{\text{ext}}\delta_{kj} & \text{in } \Omega \\ -c\delta_{kj} + \int_{\partial G} T_{kj}(\mathbf{E}_i, e_i)\varphi_i dS_{\mathbf{y}} = -c^{\text{int}}\delta_{kj} & \text{in } G \end{cases}$$

Now fix $\mathbf{x} \in \partial G$ and let $(\mathbf{x}_n) \subset \Omega$ be a sequence approaching $\mathbf{x}$. In the following equations we use the symmetry in all three indices of the Stokes stress tensor (14), jump condition (24), lemma 2.3 and the fact that (for our surfaces) the functions $n_j(\mathbf{x}) - n_j(\mathbf{y})$ are of order $O(r)$.

$$\begin{aligned} 0 &= \lim_{\mathbf{x}_n \to \mathbf{x}} \int_{\partial G} T_{kj}(\mathbf{E}_i, e_i)(\mathbf{x}_n, \mathbf{y}) \varphi_i(\mathbf{y}) n_j(\mathbf{x}) dS_{\mathbf{y}} \\ &= \lim_{\mathbf{x}_n \to \mathbf{x}} W_k^{(2)}(\varphi)(\mathbf{x}_n) + \lim_{\mathbf{x}_n \to \mathbf{x}} \int_{\partial G} T_{kj}(\mathbf{E}_i, e_i)(\mathbf{x}_n, \mathbf{y}) \varphi_i(\mathbf{y}) \left(n_j(\mathbf{x}) - n_j(\mathbf{y})\right) dS_{\mathbf{y}} \\ &= \frac{1}{2}\varphi_k(\mathbf{x}) + W_k^{(2)}(\varphi)(\mathbf{x}) + \int_{\partial G} T_{kj}(\mathbf{E}_i, e_i)(\mathbf{x}, \mathbf{y}) \varphi_i(\mathbf{y}) \left(n_j(\mathbf{x}) - n_j(\mathbf{y})\right) dS_{\mathbf{y}} \end{aligned}$$

If the sequence $(\mathbf{x}_n) \subset G$ approaches $\mathbf{x}$ from the interior, we use jump condition (25) and get

$$\begin{aligned} (c &- c^{\text{int}}) n_k(\mathbf{x}) \\ &= -\frac{1}{2}\varphi_k(\mathbf{x}) + W_k^{(2)}(\varphi)(\mathbf{x}) + \int_{\partial G} T_{kj}(\mathbf{E}_i, e_i)(\mathbf{x}, \mathbf{y}) \varphi_i(\mathbf{y}) \left(n_j(\mathbf{x}) - n_j(\mathbf{y})\right) dS_{\mathbf{y}} \end{aligned}$$

Subtracting the two equations yields

$$(c^{\text{int}} - c) n_k(\mathbf{x}) = \varphi_k(\mathbf{x}),$$

which proves our theorem in the Stokes case.

The Oseen case can be treated by the same idea. The only essential difference occurs at the point where we use the symmetry property of the Stokes stress tensor. However, using (38) leads also to the desired result. ∎

Remark. This implies that the integral equations are not uniquely solvable. We note that in the paper of Young and Acrivos [35] it is stated that the first integral equation is uniquely solvable. However, the proof contains a mistake, and the above theorem shows that in fact the opposite is true.

The next question is: Are the integral equations solvable, and, if so, what do the solutions look like? The answer is given in the next theorem:

Theorem 3.2 *The integral equations (53) have continuous solutions, and they look like*

$$t_i = t_i^p + c \cdot n_i.$$

If we require $\mathbf{t}^p$ *to be orthogonal to* $\mathbf{n}$*, it is uniquely determined in both cases.*

Proof: Both cases can be treated simultaneously. The selfadjoint integral operators map $C(\partial G)$ into itself. Existence of solutions for the equations (53) is by virtue of the Fredholm alternative equivalent to the orthogonality of $\mathbf{v}$ to the kernels of the integral operators. This is easy to verify, if we regard $\mathbf{v}$ as a constant vector on $\overline{G}$:

$$\int_{\partial G} v_i n_i(\mathbf{y}) dS_{\mathbf{y}} = \int_G v_{i,i} d\mathbf{y} = 0$$

The uniqueness of $\mathbf{t}^p$ is immediate. ∎

4 Discretization of the Integral Equations

Our goal is to solve the integral equations (23) and (50). If we denote the components of the stress by $t_i(\mathbf{y}) := T_{ij}(v,q)n_j(\mathbf{y})$, we can write them like

$$-v_k = \int_{\partial G} E_{ik}(\mathbf{x},\mathbf{y})t_i(\mathbf{y})dS_{\mathbf{y}} \quad \text{for all } \mathbf{x} \in \partial G$$

where $\mathbf{E}$ is the tensor of either the Stokes or the Oseen fundamental solutions; the latter one for $a = \mathcal{R}/2$, the half of the Reynolds-Number. Remember that the boundary condition on $\mathbf{v}$ was

$$\mathbf{v} = (-1\ 0\ 0)^\top \quad \text{on} \quad \partial G$$

We make now the following assumptions:

- The body G is replaced by an approximating polyhedron P, and the surface of P consists of triangles $P_1, P_2, ..., P_N$. The corners of the triangles should lie on the boundary ∂G.
- The stress $\mathbf{t}$ we want to solve for is constant on each triangle P_l, i.e. $t_i = \sum_{j=1}^{N} \tau_{il}\varphi_l$, where φ_l is the characteristic function of the triangle P_l.
- To compute the $3N$ unknowns τ_{il}, we require the integral equation not to hold for every $\mathbf{x} \in \partial G$ anymore, but only for the centroids $\mathbf{x}^{(l)}$ of the triangles.

It's then clear that instead of the above integral equation we have the $3N$ equations

$$\delta_{1k} = \sum_{l=1}^{N} \tau_{il} \int_{P_l} E_{ik}(\mathbf{x}^{(n)},\mathbf{y})dS_{\mathbf{y}} \quad (k = 1,2,3, \quad n = 1,2,...,N)$$

We define for $1 \leq n,l \leq N$ the 3×3-matrices

$$F_{nl} = \left(\int_{P_l} E_{ik}(\mathbf{x}^{(n)},\mathbf{y})dS_{\mathbf{y}} \right)_{ik}$$

and set

$$\begin{aligned} \mathbf{M} &= (F_{nl})_{nl} \\ \mathbf{t} &= (\tau_{11}, \tau_{21}, \tau_{31}, \tau_{12}, \tau_{22}, \tau_{32}, \tau_{13}, ..., \tau_{3N})^\top \\ \mathbf{w} &= (1\,0\,0\,1\,0\,0\,1\,0\,0\,...\,1\,0\,0)^\top. \end{aligned}$$

Then we obtain the matrix equation

$$\mathbf{w} = \mathbf{M}\mathbf{t}. \tag{55}$$

The $3N \times 3N$-matrix $\mathbf{M}$ depends on wether we plug in Stokes or Oseen fundamental solutions, and in the latter case additionally on the Reynolds number.

Remark. This kind of discretization is called **Collocation Method**. Other approaches are the **Galerkin Method** with finite elements as trial functions (see Hsiao and Wendland [18]) or the **Galerkin Collocation Method** (see Hsiao, Kopp and Wendland [19, 20]).

4.1 Triangulation

The first part of our discretization requires a uniform triangulation of our body. For latter purposes we also need to compute the inner normals. It is a nontrivial task to do that for arbitrary bodies, and we won't go into details at this point. If the surface of the body is given implicitly, i.e. $\partial G = \{\mathbf{x} \in \mathbb{R}^3 : H(\mathbf{x}) = 0\}$, where H is a certain function, we refer to the work of Widmann [32, 33].

For simple bodies like a sphere it's rather easy to obtain a triangulation and all its connected data. We explain the developed functions in the following. Except for `triangulation.m` and `trinormal.m`, all functions can be used for arbitrary bodies as well, as long as we are given the triangles and the inner normals.

The triangulation of the sphere is produced by the function `creategrid.m`.

```
function creategrid(r,delta);

% input
%   r           ...radius of sphere
%   delta       ...maximal edge-size of triangulation
% output on file tri1.mat
%   a1,a2,a3,b1,b2,b3,c1,c2,c3
%               ...see function triangulation
%   n1,n2,n3    ...see function trinormal
%   m1,m2,m3    ...see function trimidpoints
%   vol         ...see function trivolumes
%   r,delta     ...see input
% output on file tri2.mat
%   simil,rep     ...see function tricongruent
%   simor       ...see function tricongruent2
%   p1,p2,q1,q2,r1,r2
%               ...see function trimap
%   s11,s12,s13,s21,s22,s23,s31,s32,s33
%               ...see function trimap
%   r,delta     ...see input
```

The data stored in `tri1.mat` are the corners, inner normals, midpoints and volumes of the triangles. `tri2.mat` contains rather data that is needed for the computation of the diagonal blocks of the matrix M.

4.1.1 Geometric Data

triangulation.m Computes the corners of our triangulation

```
function [a1,a2,a3,b1,b2,b3,c1,c2,c3,or,oc]=triangluation(delta, r);

% creates triangulation for a sphere with center in the origin.
%
% input
%   delta     ...maximal length of edges
%   r         ...radius of sphere
% output
%   a1        ...x-coordinate of A (m-vector if we have m triangles)
%   a2,a3     ...y,z-coordinate of A
%   b1,b2,b3  ...see a1,a2,a3
%   c1,c2,c3  ...see a1,a2,a3
%   or        ...orientation of triangle, when looking from the inside of the
```

```
%               sphere
%               orientation 1:          orientation -1:
%                    C                       C
%                  /   \                   /   \
%                 A-----B                 B-----A
%   oc       ...octant in which triangle is.
%               here means 1: x+ y+ z+
%                          2: x+ y+ z-
%                          3: x+ y- z+
%                          4: x+ y- z-
%                          5: x- y+ z+
%                          6: x- y+ z-
%                          7: x- y- z+
%                          8: x- y- z-
```

The function starts out with the hexagon $(\pm r, 0, 0), (0, \pm r, 0), (0, 0, \pm r)$. If the edges are longer than `delta`, each triangle is replaced by four other triangles, the corners of which are either corners or projections of edge-midpoints of the original triangle onto the sphere. After some iterations of this process all edges are smaller than `delta`.

trinormal.m The function computes via crossproduct the normals that point inside the body, i.e. outside the exterior domain.

```
function [n1,n2,n3]=trinormal(a1,a2,a3,b1,b2,b3,c1,c2,c3,or);

% creates inner normals of a triangulation
%
% input
%   see function triangulation
%   (but can use this program also for other triangulations!)
% output
%   n1   ...x-coordinate of normal, which points outside the interior domain.
%   n2,n3...y,z-coordinates
```

trirename.m

```
function [a1,a2,a3,b1,b2,b3,c1,c2,c3]=trirename(a1,a2,a3,b1,b2,b3,c1,c2,c3);

% Renames triangulation such that for every triangle the longest side is AB
```

```
% and the orientation is preserved
%
% input, output
%   see function triangulation.m
```

trimidpoints.m

```
function [m1,m2,m3]=trimidpoints(a1,a2,a3,b1,b2,b3,c1,c2,c3);

% computes centroids of elements
%
% input
%   see function triangulation
% output
%   m1    ...1. coordinate of midpoints
%   m2,m3 ...2./3. coordinate of midpoints
```

trivolumes.m

```
function [vol]= trivolumes(a1,a2,a3,b1,b2,b3,c1,c2,c3);

% computes volumes of elements
%
% input
%   see function triangulation
% output
%   vol   ...volumes of triangles
```

4.1.2 Transformation Data

tricongruent.m This function checks which triangles are congruent. The set of triangles is split up into **congruency classes**. From each class we pick arbitrarily one representant.

```
function [sim,rep]=tricongruent(a1,a2,a3,b1,b2,b3,c1,c2,c3);

% Checks triangulation for congruent triangles
%
% input
%   see function triangulation
```

```
% output
%   sim  ...gives congruency relation.
%          sim[k]=m  --> triangle k is congruent to triangle m
%   rep  ...gives a representant of each congruency class
```

trimap.m For any triangle ABC we define its **Local Coordinate-System** by the following properties:

- The origin lies in the centroid M of the triangle;
- The x_1-axis is parallel to $\overline{AB}$ and A lies in the positive x_1-region;
- The x_2-axis lies in the triangle, and C lies in the negative x_2-region;
- All axes are orthogonal to each other, and the system is positively oriented.

Suppose we have given A, B, C and M both in local coordinates x_1, x_2, x_3 and in global coordinates y_1, y_2, y_3. Let P be the affine transformation between them, i.e. $P(\mathbf{x}) := \mathbf{S}\mathbf{x} + \mathbf{t} = \mathbf{y}$. It's easy to check that we have to have $\mathbf{t} = \mathbf{y}_M$. We introduce

$$\begin{aligned} \mathbf{e}_1 &= \mu(\mathbf{y}_A - \mathbf{y}_B) \\ \mathbf{e}_2 &= \lambda(\mathbf{y}_A - \mathbf{y}_C - ((\mathbf{y}_A - \mathbf{y}_C) \cdot \mathbf{e}_1)\mathbf{e}_1), \\ \mathbf{e}_3 &= \mathbf{e}_1 \times \mathbf{e}_2, \end{aligned}$$

where $\mu, \lambda > 0$ are chosen such that $|\mathbf{e}_1| = |\mathbf{e}_2| = 1$. Then one can show that we have to have $S_{ij} := \mathbf{e}_{ji}$. Notice that S is a rotation matrix. Altogether we have

$$P(\mathbf{x}) = \mathbf{S}\mathbf{x} + \mathbf{y}_M \tag{56}$$

$$\det(\frac{\partial P(\mathbf{x})}{\partial \mathbf{x}}) = \det(S) = 1 \tag{57}$$

```
function [p1,p2,q1,q2,r1,r2,s11,s12,s13,s21,s22,s23,s31,s32,s33]=...
   trimap(a1,a2,a3,b1,b2,b3,c1,c2,c3,m1,m2,m3);

% Given the global coordinates for the triangle-corners and -centroids,
% the function computes the corresponding local coordinates and the
```

```
% matrix S of the affine transformation from local to global coordinates.
%
% input:
%   a., b., c.  ...see function triangulation
%   m.          ...see function trimidpoints
% output:
%   p1,p2  ...x-coordinates of A (x3-coordinate =0 due to construction)
%   q1,q2  ...x-coordinates of B (x3-coordinate =0 due to construction)
%   r1,r2  ...x-coordinates of C (x3-coordinate =0 due to construction)
%   s..    ...entries of the orthogonal matrix S
```

tricongruent2.m It is possible that congruent triangles have different local coordinates, namely reflected at the x_2-axis. In `tricongruent.m` we have chosen for each congruency class one represenant. We have to check now for each class, which of its members have the same local coordinates and which of them have reflected ones:

```
function [simor]= tricongruent2(sim,rep,p1,p2);

% checks, which congruent triangles have the same local coordinates and
% which have reflected ones, compared with the representant of
% the congruency class
%
% input
%   see functions tricongruent, trimap
% output
%   simor: simor(k)= +1 if equal
%                  = -1 if reflected
```

5 Computation of the Matrices

In this section, we describe the computation of the matrix **M** in (55). We first give the formulas for the nondiagonal blocks of **M**. The diagonal parts, however, require a closer look. We do that in the Stokes case by using some analytical help. These results can also be used in the Oseen case.

5.1 Nondiagonal Blocks

For the nondiagonal blocks of **M**, the singularity of the integrands does not lie in the corresponding triangles we integrate over.

5.1.1 Stokes

We have to compute for each triangle and each centroid the integral of the Stokes fundamental solutions (11) which looked like

$$\begin{aligned} i = k: \qquad & E_{ii}(\mathbf{x},\mathbf{y}) = -\frac{1}{8\pi}\left(\frac{1}{r} + \frac{d_i^2}{r^3}\right) \\ i \neq k: \qquad & E_{ik}(\mathbf{x},\mathbf{y}) = -\frac{1}{8\pi}\frac{d_i d_k}{r^3} \end{aligned}$$

If the centroid lies in the triangle, we refer to 5.2.1. For the other cases we use the functions `fun_rgd3.m` and `fun_rgnd3.m`, which contain the diagonal and nondiagonal parts of **E**, respectively.

5.1.2 Oseen

Now we have to determine the integrals of the Oseen fundamental solutions for each triangle and centroid. If the triangle contains the centroid, we refer to 5.2.2, else we

use the functions (34)

$$
\begin{aligned}
& & E_{11}^{Os} &= \left(\frac{1}{r}+\frac{d_1^2}{r^3}\right)\Phi' + \left(1-\frac{d_1^2}{r^2}\right)\Phi'' \\
i=k\neq 1: & & E_{ii}^{Os} &= \left(\frac{1}{r}+\frac{d_i^2}{r^3}\right)\Phi' + \left(2-2\frac{d_1}{r}-\frac{d_i^2}{r^2}\right)\Phi'' \\
i\neq k=1: & & E_{i1}^{Os} &= \frac{d_i d_1}{r^3}\Phi' + \left(\frac{d_i}{r}-\frac{d_i d_1}{r^2}\right)\Phi'' \\
& & E_{23}^{Os} &= \frac{d_2 d_3}{r^3}\Phi' + \left(-\frac{d_2 d_3}{r^2}\right)\Phi'',
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi' &= -\frac{1-e^{-as}}{8\pi as} = -\frac{1}{8\pi}\sum_{i=0}^{\infty}\frac{(-as)^i}{(i+1)!} \\
&= -\frac{1}{8\pi}\sum_{i=0}^{n}\frac{(-as)^i}{(i+1)!} + R'_{n+1} \\
\Phi'' &= -\frac{ase^{-as}-1+e^{-as}}{8\pi as^2} = \frac{a}{8\pi}\sum_{i=0}^{\infty}\frac{i+1}{(i+2)!}(-as)^i \\
&= \frac{a}{8\pi}\sum_{i=0}^{n}\frac{i+1}{(i+2)!}(-as)^i + R''_{n+1}.
\end{aligned}
$$

The diagonal part is implemented in `oseenfun_rgd3.m`, the nondiagonal part in `oseenfun_rgnd3.m`. If as is close to zero, we use Taylor approximations for Φ' and Φ''. To justify this, we assume $as < 1$ and estimate the above remainders:

$$
\begin{aligned}
|R'_{n+1}| &= \frac{1}{8\pi}\sum_{i=0}^{\infty}\frac{(as)^{i+n+1}}{(i+n+2)!} \leq \frac{(as)^{n+1}}{8\pi(n+2)!}\sum_{i=0}^{\infty}\frac{(n+2)!}{(i+n+2)!} \\
&\leq \frac{(as)^{n+1}(n+3)}{8\pi(n+2)(n+2)!} \\
|R''_{n+1}| &= \frac{a}{8\pi}\sum_{i=0}^{\infty}\frac{i+n+2}{(i+n+3)!}(as)^{i+n+1} \leq \frac{a(as)^{n+1}}{8\pi(n+2)!}\sum_{i=0}^{\infty}\frac{(n+2)!}{(i+n+2)!} \\
&\leq \frac{a(as)^{n+1}(n+3)}{8\pi(n+2)(n+2)!}
\end{aligned}
$$

Suppose we want to compute the integrals with a tolerance ε, and the longest edge of all triangles is smaller than δ. If we take the first order Taylor approximation for $as < \varepsilon/10a\delta^2$ (i.e. $n=1$), we can estimate the introduced error by

$$
\int_{\triangle}\frac{2}{r}|R'_2| + 5|R''_2|dS_{\mathbf{y}} \leq \frac{a^2 s\cdot 4}{8\pi\cdot 3\cdot 3!}\int_{\triangle}\frac{2s}{r} + 5asdS_{\mathbf{y}}
$$

$$\leq \frac{\varepsilon}{100}.$$

The Taylor approximations for $n = 1$ look like

$$\begin{aligned} \Phi' &\approx -\frac{1}{8\pi} + \frac{as}{16\pi} \\ \Phi'' &\approx \frac{a}{16\pi} - \frac{a^2 s}{24\pi}, \end{aligned}$$

and with the above reasoning we can neglect the introduced error.

5.2 Diagonal Blocks

5.2.1 Stokes

To avoid numerical integration over weak singularities, we at first compute the integrals analytically for one special triangle PQR, which has nice properties ("basic integrals"). Then we take from each congruency class of triangles one representant, map it into the plane of $\triangle PQR$ and compute the integrals over these images ("representant integrals"). We transfer the obtained values to the other members of the congruency classes ("standard integrals"). Finally, we transform the integrals back, such that they're taken over the actual triangles ("actual integrals").

Basic Integrals Let us consider the triangle in the plane $x_3 = 0$ with the vertices $P = (-r\sqrt{3}, -r)$, $Q = (r\sqrt{3}, r)$, $R = (0, 2r)$, where r is at the moment an arbitrary number. $\triangle PQR$ is equilateral with centroid in $(0, 0)$. We define the functions

$$\begin{aligned} f(\mathbf{x}) &= -\frac{1}{8\pi\sqrt{x_1^2 + x_2^2}} \\ h_{11}(\mathbf{x}) &= -\frac{x_1^2}{8\pi\sqrt{x_1^2 + x_2^2}^3} \\ h_{12}(\mathbf{x}) = h_{21}(\mathbf{x}) &= -\frac{x_1 x_2}{8\pi\sqrt{x_1^2 + x_2^2}^3} \\ h_{22}(\mathbf{x}) &= -\frac{x_2^2}{8\pi\sqrt{x_1^2 + x_2^2}^3} \end{aligned}$$

One can then compute that

$$\begin{aligned}&\int_{\triangle PQR} h_{11}(\mathbf{x})dS_{\mathbf{x}}\\ =\ &\int_{\triangle PQR} h_{22}(\mathbf{x})dS_{\mathbf{x}} = \int_{\triangle PQR} f(\mathbf{x})dS_{\mathbf{x}} + \frac{3r}{8\pi}\ln(\sqrt{3}+2)\end{aligned}$$

Since $f = h_{11} + h_{22}$ and h_{12} is symmetric to the x-axis, we get

$$\begin{aligned}\int_{\triangle PQR} f(\mathbf{x})dS_{\mathbf{x}} &= -\frac{6r}{8\pi}\ln(\sqrt{3}+2)\\ \int_{\triangle PQR} h_{11}(\mathbf{x})dS_{\mathbf{x}} &= -\frac{3r}{8\pi}\ln(\sqrt{3}+2)\\ \int_{\triangle PQR} h_{22}(\mathbf{x})dS_{\mathbf{x}} &= -\frac{3r}{8\pi}\ln(\sqrt{3}+2)\\ \int_{\triangle PQR} h_{12}(\mathbf{x})dS_{\mathbf{x}} &= 0\end{aligned}$$

For latter purposes we choose r so big that $\triangle PQR$ contains the image under P^{-1} of the representant of every congruency class (see section 4.1.2). Such a value is computed in the function `trimaxdist.m`.

```
function [ou1,ou2,ov1,ov2,ow1,ow2,Bf,Bh11,Bh22,Bh12]= intbasic(r);

% Gives exact integral values over the particular triangle PQR
%
% input
%   r       ...see function trimaxdist
% output
%   ou1,ou2,ov1,ov2,ow1,ow2
%           ...corners of triangle PQR
%   Bf,Bh11,Bh22,Bh12
%           ...basic values for big triangle
```

Representant Integrals For each congruency class we take now one representant triangle ABC and compute the integrals of f, h_{11}, h_{12} and h_{22} over the local coordinates of that particular representant. For that purpose, we write

$$\begin{aligned}&\int_{\triangle ABC} f(\mathbf{x})dS_{\mathbf{x}} = \int_{\triangle PQR} f(\mathbf{x})dS_{\mathbf{x}} -\\ &\left(\int_{\triangle PCB} + \int_{\triangle PQC} + \int_{\triangle QAC} + \int_{\triangle QRA} + \int_{\triangle RBA} + \int_{\triangle BRP}\right) f(\mathbf{x})dS_{\mathbf{x}},\end{aligned}$$

and the same for the functions h_{11}, h_{12} and h_{22}. This decomposition of PQR has the advantage that we don't have to integrate numerically over the weak singularity, which would lead to a huge amount of subdivisions in our adaptive integration function. Instead, we only have to integrate over six triangles, where the integrands are smooth. The set-up of the triangulation guarantees that this kind of decomposition is well defined. However, if the triangles become very small, the six triangles come closer and closer to the singularity.

```
function [Rf,Rh11,Rh22,Rh12]= ...
    intsimclass(rep,p1,p2,q1,q2,r1,r2, ou1, ou2, ov1,ov2,ow1,ow2, ...
                Bf,Bh11,Bh22,Bh12,tol,dep,Wf,Kf,Wg,iKg,maxmatsize,tab);

% Computes integral values for representants of similarity classes
%
% input
%   rep              ...see function trisimilar.m
%   p1,p2,q1,q2,r1,r2...corners of trianglesof similarity classes
%   ou1,ou2,ov1,
%   ov2,ow1,ow2      ...corners of big triangle, see function intbasic.m
%   Bf,Bh11,Bh22,Bh12...see function intbasic.m
%   tol,dep          ...
%   Wf,Kf,Wg,iKg     ...
%   maxmatsize       ...
%   tab              ...helpstring for nicer output
% output
%   Rf,Rh11,Rh22,Rh12
%          ...basic values for each similarity class
```

Standard Integrals Suppose now we want to compute the integrals of f, h_{11}, h_{12} and h_{22} over the local coordinates of any other triangle UVW in that class. Due to the construction the local coordinates of the two triangles might either be the same, or they are reflected at the x_2-axis. The function `tricongruent2` (see section 4.1) gives us that information. However, there is not a big problem with this. It's easy to check that

$$\begin{aligned}\int_{\triangle UVW} f(\mathbf{x})dS_{\mathbf{x}} &= \int_{\triangle ABC} f(\mathbf{x})dS_{\mathbf{x}} \\ \int_{\triangle UVW} h_{11}(\mathbf{x})dS_{\mathbf{x}} &= \int_{\triangle ABC} h_{11}(\mathbf{x})dS_{\mathbf{x}}\end{aligned}$$

$$\int_{\triangle UVW} h_{22}(\mathbf{x})dS_{\mathbf{x}} = \int_{\triangle ABC} h_{22}(\mathbf{x})dS_{\mathbf{x}}$$
$$\int_{\triangle UVW} h_{12}(\mathbf{x})dS_{\mathbf{x}} = \pm\int_{\triangle ABC} h_{12}(\mathbf{x})dS_{\mathbf{x}},$$

where $+$ holds, if the coordinates are the same, and $-$, if they are reflected. For this part of the computation we have got the function `intstandard.m`.

Actual Integrals Having all this, we can now compute the integrals that we actually need. Let $\triangle UVW$ be one of our triangles, M its centroid, and the integration takes place over the global ($\mathbf{y}$-) coordinates of that triangle. P is the affine mapping (56), and we have seen that its functional determinant is 1.

$$\begin{aligned}
-\frac{1}{8\pi}\int_{\triangle UVW}\frac{1}{|\mathbf{y}_M-\mathbf{y}|}dS_{\mathbf{y}} &= -\frac{1}{8\pi}\int_{\triangle UVW}\frac{1}{|\mathbf{y}_M-P(\mathbf{x})|}dS_{\mathbf{x}}\\
&= -\frac{1}{8\pi}\int_{\triangle UVW}\frac{1}{|-\mathbf{Sx}|}dS_{\mathbf{x}}\\
&= -\frac{1}{8\pi}\int_{\triangle UVW}\frac{1}{|\mathbf{x}|}dS_{\mathbf{x}}\\
&= \int_{\triangle UVW} f(\mathbf{x})dS_{\mathbf{x}}\\
-\frac{1}{8\pi}\int_{\triangle UVW}\frac{(\mathbf{y}_M-\mathbf{y})_i(\mathbf{y}_M-\mathbf{y})_k}{|\mathbf{y}_M-\mathbf{y}|^3}dS_{\mathbf{y}} &= -\frac{1}{8\pi}\int_{\triangle UVW}\frac{(\mathbf{y}_M-P(\mathbf{x}))_i(\mathbf{y}_M-P(\mathbf{x}))_k}{|\mathbf{y}_M-P(\mathbf{x})|^3}dS_{\mathbf{x}}\\
&= -\frac{1}{8\pi}\sum_{l,m=1}^{3} S_{il}S_{km}\int_{\triangle UVW}\frac{x_l x_m}{\sqrt{x_1^2+x_2^2}^3}dS_{\mathbf{x}}\\
&= \sum_{l,m=1}^{2} S_{il}S_{km}\int_{\triangle UVW} h_{lm}(\mathbf{x})dS_{\mathbf{x}}
\end{aligned}$$

This little computation is implemented in the function `intsingular.m`:

```
function [f,h11,h22,h33,h12,h13,h23]= ...
  intsingular(s11,s12,s13,s21,s22,s23,s31,s32,s33, ...
              Sf,Sh11,Sh22,Sh12);

% Computes integrals over the functions
%                  -1
%   fun_f(y) = ----------
%               8 Pi |m-y|
```

```
%              -1        T          1        3      3x3
%   fun_h(y) = ---- (m-y) * grad(-----) : R --> R
%              8 Pi                |m-y|
% where m are the midpoints of the triangles
%
% input
%   s11,s12,s13,s21,s22,s23,s31,s32,s33
%                       ...see function trimap
%   Sf,Sh11,Sh22,Sh12 ...see function intstandard
% output
%   f                   ...integral of fun_f
%   h11,h22,h33         ...diagonal components of integral of fun_h
%   h12,h13,h23         ...nondiagonal components of integral of fun_h
```

5.2.2 Oseen

We remove the singularity by subtracting the integral over the Stokes fundamental solution, which we have already treated. The integrand of that difference remains bounded, and in particular we get:

$$E_{11}^{Os} - E_{11} = \left(1 + \frac{d_1^2}{r^2}\right) \frac{\Phi' + \frac{1}{8\pi}}{r} + \left(1 - \frac{d_1^2}{r^2}\right) \Phi''$$

$$i = k \neq 1: \qquad E_{ii}^{Os} - E_{ii} = \left(1 + \frac{d_i^2}{r^2}\right) \frac{\Phi' + \frac{1}{8\pi}}{r} + \left(2 - 2\frac{d_1}{r} - \frac{d_i^2}{r^2}\right) \Phi''$$

$$i \neq k = 1: \qquad E_{i1}^{Os} - E_{i1} = \frac{d_i d_1}{r^2} \frac{\Phi' + \frac{1}{8\pi}}{r} + \left(\frac{d_i}{r} - \frac{d_i d_1}{r^2}\right) \Phi''$$

$$E_{23}^{Os} - E_{23} = \frac{d_2 d_3}{r^2} \frac{\Phi' + \frac{1}{8\pi}}{r} + \left(-\frac{d_2 d_3}{r^2}\right) \Phi''$$

where

$$\begin{aligned} \frac{\Phi' + \frac{1}{8\pi}}{r} &= -\frac{1}{8\pi r} \sum_{i=1}^{n} \frac{(-as)^i}{(i+1)!} + \frac{1}{r} R'_{n+1} \\ &= \frac{as}{8\pi r} \sum_{i=0}^{n-1} \frac{(-as)^i}{(i+2)!} + \frac{1}{r} R'_{n+1} \end{aligned}$$

and Φ'' is like in section 5.1.2. The diagonal part is implemented in `oseenfun_sgd3.m`, the nondiagonal part in `oseenfun_sgnd3.m`. If as is close to zero, we use Taylor approximations. To justify this, we assume $as < 1$ and get:

$$|\frac{1}{r} R'_{n+1}| \leq \frac{(as)^{n+1}(n+3)}{8\pi r(n+2)(n+2)!}$$

$$\leq \frac{a(as)^n(n+3)}{4\pi(n+2)(n+2)!}$$

Suppose we want to compute the integrals with a tolerance ε, and the longest edge of all triangles is smaller than δ. Like in the nonsingular case (see 5.1.2) we take the first order Taylor approximation for $as < \varepsilon/10a\delta^2$, and can estimate the introduced error by

$$\begin{aligned}\int_{\triangle} \frac{2}{r}|R'_2| + 5|R''_2|dS_{\mathbf{y}} &\leq \frac{a^2 s \cdot 4}{8\pi \cdot 3 \cdot 3!} \int_{\triangle} \frac{2s}{r} + 5asdS_{\mathbf{y}} \\ &\leq \frac{\varepsilon}{100}\end{aligned}$$

The Taylor approximation for $n = 1$ looks like

$$\frac{\Phi' + \frac{1}{8\pi}}{r} \approx \frac{as}{16\pi r}.$$

Thus, we can neglect the influence of the introduced error.

6 Integration over Triangles

In the last section, we have seen that for each entry of the matrix $\mathbf{M}$ in (55) we have to compute some integrals over triangles. This section gives an idea how we do this. Our previous work allows us to assume that all integrands are continuous on their integration domains.

The general idea is that we compute two approximations of different orders, and if they differ by less than a given tolerance, the higher order approximation is considered to be the value of the integral. The integration formula that we use is explained in 6.1. If the two approximations are not as close together as desired, we subdivide the triangle and do then the same procedure for each of the smaller triangles. One can think of it as a recursion, although it's certainly not a good idea to implement it recursively. See 6.2 for details.

Unfortunately, we don't give an error estimate for the whole method. Furthermore, there are several other methods for integration over triangles, we refer to [24].

6.1 Integration Formula

To integrate a function f over a triangle ABC, we use the formula

$$\int_{\triangle ABC} f(\mathbf{x}) dS_{\mathbf{x}} \approx \text{area}(\triangle ABC) \cdot \sum_{i+j+k=n} \omega_{ij} f(\frac{i}{n}\mathbf{x}_A + \frac{j}{n}\mathbf{x}_B + \frac{k}{n}\mathbf{x}_C),$$

where ω_{ij} are the weights. The formula is exact for polynomials up to order n and needs $(n+1)(n+2)/2$ knots. We denote them and their weights by n-knots and n-weights, respectively, and the approximation by n-approximation.

We require n to be even. It's easy to see that every $n/2$-knot is also a n-knot. That means, after having computed an n-approximation, we can compute an $n/2$-approximation without needing any additional function evaluations. We take advan-

tage of that to check, if an approximation is accurate or not.

The computation of the weights leads to solving a linear system. We should mention that the matrix of this system is badly scaled if one takes n too big, for instance $n > 14$.

```
function [Wf,Kf,Wg,iKg]=wABC(n);

% computes weights for integration over triangle.
% formula has order n+1, i.e. for polynomials up to order n it's exact.
%
% input
%   n      ...high order (has to be even)
% output
%   Wf     ...weights for order n
%   Kf     ...knots for order n
%   Wg     ...weights for order n/2
%   iKg    ...Kf(iKg) are the knots for order n/2
```

6.2 Adaptive Scheme

Suppose we have N triangles $\triangle_j$ $(j = 1, ..., N)$, a scalar function $f : \mathbb{R}^3 \to \mathbb{R}$ and a vector function $\mathbf{g} : \mathbb{R}^3 \to \mathbb{R}^3$. We write $D := \bigcup_{j=1}^{N} \triangle_j$ for the union of the N triangles, which we assume to be mutually disjoint, except for some sets of measure zero. Our goal is to compute the integrals

$$\int_D f(\mathbf{x}) dS_{\mathbf{x}}$$
$$\int_D \mathbf{g}(\mathbf{x}) dS_{\mathbf{x}}$$
$$\int_{\triangle_j} \mathbf{g}(\mathbf{x}) dS_{\mathbf{x}} \quad \text{for} \quad j = 1, ..., N$$

as fast and as accurately as possible. We can increase the accuracy by taking low tolerances. In addition, we have to increase the order for our integration formula and the number of allowed subdivisions. The more accuracy we want, the more time is needed. On the other hand, the computation time is essentially influenced by the

efficiency of our code. A recursion would be the worst method. Therefore, we tried to minimize the amount of loops, to treat many triangles simultaneously and to make use of the vector computing facilities that MATLAB has to offer.

For each of the above cases we have an own function, which takes advantage of the particular situation. They all use the same idea, have most of the code in common, and their inputs and outputs look like the following:

```
% input:
%   fname          ...matlab-filename of integrand
%   a1,a2,a3       ...N 1./2./3. coordinates triangle-corners A
%   b1,b2,b3       ...N 1./2./3. coordinates triangle-corners B
%   c1,c2,c3       ...N 1./2./3. coordinates triangle-corners C
%   tol            ...desired accuracy of integrals
%   addpar         ...additional parameters for integrand (constants)
%   dep            ...maximal depth of subdivisions
%   Wf             ...higher order weights
%   Kf             ...higher order knots
%   Wg             ...lower order weights
%   iKg            ...lower order knots (as indices of Kf)
%   maxmatrixsize ...maximal allowed size of occuring matrices.
%                     hardware-dependent. choose as high as possible
% output
%   int            ...computed value of integral
%   err            ...flag (=0 if accuracy achieved else =1)
%   echterr        ...max(0,echttol-tol)
%   echttol        ...real accuracy
%   subdiv         ...number of occured subdivisions
```

The input argument `fname` and the output arguments `int`, `echterr` and `echttol` have slightly different meanings, and we explain them for each function seperately. All the other arguments have the same meaning. In `a1` we store the first coordinates of the triangle-corners A. `a2,...,c3` have to be understood similarly, and the values have to be in the same order than `a1`. `tol` is the tolerance for the difference of the two approximations (see above). In `addpar` we can hand over additional arguments for the function, which are constant during the integration. If we don't need any, we hand over an empty vector. `dep` is the number of allowed subdivisions. For the next four arguments, we refer to 6.1. Since we treat as many triangles at once as possible, the number of function evaluation might get very big. The last input argument,

maxmatrixsize, allows us to keep that number under a desired bound. If we don't need to worry about other users on the workstation, we take this value as high as possible. The output argument err is zero, if all integrals could be computed with the desired accuracy, else one. In subdiv we count the number of occurred subdivions.

6.2.1 intABC.m

This function is supposed for the first of the above three cases. Therefore, the input parameter fname has to be the name of the scalar function we want to integrate. This function has to look like the following:

```
function f=anyname(y1,y2,y3,addpar);

% input
%   y1,y2,y3  ...1./2./3. coordinates
%   addpar    ...constant parameters (might be empty)
% output
%   f         ...computed function values
```

The function has to be evaluable at several points at once, and the result f should have the same size as the received arguments y1,y2,y3. It has to expect an additional vector addpar, where parameters can be received that are constant during the integration.

The triangle data is stored in a list named REG, and the elements of that list look like

```
[a1 a2 a3 b1 b2 b3 c1 c2 c3 tritol vol dep].
```

At the beginning each row of REG corresponds to one of the given triangles. tritol is the desired accuracy for the integration over that particular triangle, vol its area, and dep is a number that keeps track of the occured subdivisions. If dep=0, the triangle wouldn't be subdivided anymore, although we might not have achieved the desired accuracy yet.

We have a loop, which in every step takes several triangles from the top of our list, computes the approximated integral values, and if they seem to be good enough or if no more subdivision is allowed, the values are added to a variable `int` that in the end contains the approximated integral value over the whole integration domain. If for some triangles the approximation is not good enough, they are subdivided and put to the end of the `REG`-list. The loop stops, if the list is empty.

To decide, if the approximation is good enough, we actually compute two approxations, one of them with weigths `Wf` and knots `Kf`, and one with weights `Wg` and knots `Kg`. The f-approximation is the better one, because it's based on more knots; as a matter of fact, every g-knot is also an f-knot. Therefore, we have the g-knots given via `Kg=Kf[ikg]`, where `ikg` is just an index vector. We can thus save function evaluations. If the two approximations differ by less than `tritol`, then the f-Approximation is considered to be the value of the particular integral.

If a subdivision is necessary for a triangle, it is divided up in four similar but smaller triangles that are put at the end of the `REG`-list, each of them provided with one fourth of tolerance and volume, and with `dep` reduced by one.

There is a serious bookkeeping of the tolerances. If a value was accepted, because the two approximations differed by let's say $d < tritol$, then $tritol - d > 0$ is added to a variable `tolguthaben`. If a value had to be accepted, although it was't good enough, because `dep` was zero, then $d - tritol \geq 0$ is added to a variable `echterr`. At the end of each loop-cylce, we subtract `echterr` from `tolguthaben`, and if there is something left on `tolguthaben`, we distribute that onto the subdivided triangles and thus allow them to have bigger tolerances.

In the very end, we return the computed integral value `int`, a variable `echttol (= tol + echterr - tolguthaben)`, and `echterr (= max(0,echterr-tolguthaben))`. If `echterr=0`, we have computed the integral value successfully with the desired accuracy.

Special emphasis is put on saving computing time; therefore, everything is vectorized, and in each loop-cylce several hundred triangles can be treated simultaneously. Since the REG-list can blow up at the beginning, we can control via `maxmatrixsize`, how big we allow the function evaluation matrix to be, the size of which is given by the number of triangle times the number of knots of the f-approximation.

6.2.2 intABC3.m

The function works like `intABC.m`, but now for 3-dimensional integrands. Therefore, the integrand specified by `fname` has to have the form

```
function [f1,f2,f3]= fun_rgd3(y1,y2,y3,addpar);
```

We now have 3-vectors of f- and g-approximations, and we thus take the maximum-norm instead of the absolute value of their difference to decide wether the approximation is good enough or not.

6.2.3 intABC3sep.m

The input of this function is exactly the same as in `intABC3.m`. We get for each triangle an own integral value, not like in the previous case, where we got only the sum of them. Thus, the output arguments `int`, `echterr` and `echttol` are not scalars anymore but vectors.

The structure of the code had to be changed slightly. The difference is that we cannot check anymore after each loop-cycle, if we have wasted anything of the tolerances and distribute that on the remaining triangles. This is a disadvantage, if we have to do a lot of subdivisions. However, if most of the triangles don't need subdivisions, we can treat the few exceptions afterwards with `intABC3`.

7 Application on a Sphere

In this section we apply the theory and the program developed to the simple case of a sphere, where we have an analytical solution available. We first investigate the matrices themselves. Since the matrices approximate singular integral operators, we expect them to be ill-conditioned. However, the linear systems which arise should be uniquely solvable in the hyperspace that is orthogonal to the normal direction.

One way of solving a system in that sense is to seek the solution as a linear combination of basis vectors that span this particular hyperspace. However, the computation of such a basis is too expensive unless there exists a clever way of doing it. We do not investigate this further. Instead, we seek a solution in the whole space and afterwards project it onto the hyperspace.

Although the right hand sides of the linear systems are known exactly, we have to take into account an error in the matrix-computation. In order to get stability of the solution against little changes in the matrix, we can't solve directly via Gauss elimination. Instead, we have to make use of regularization techniques like truncated singular value decomposition or **Tychonov regularization**. These and other methods are described in Groetsch [11]. Since we have to deal with big matrices, we can't afford to compute a singular value decomposition, thus, we concentrate on the methods of Tychonov, i.e. instead of the system $\mathbf{Mt} = \mathbf{w}$, we solve the minimization problem

$$\begin{pmatrix} \mathbf{M} \\ \lambda\mathbf{I} \end{pmatrix} \mathbf{t} = \begin{pmatrix} \mathbf{w} \\ \mathbf{0} \end{pmatrix} \quad \text{or}$$
$$(\mathbf{M}^*\mathbf{M} - \lambda^2\mathbf{I})\mathbf{t} = \mathbf{M}^*\mathbf{w},$$

where $\mathbf{I}$ is the identity matrix of the same size as $\mathbf{M}$, $\mathbf{M}^*$ is the adjoint of $\mathbf{M}$ and $\lambda \in \mathbb{R}$ is a parameter we can adjust. If $\lambda = 0$, we get the exact solution, which is unstable, and for large λ we get almost the zero vector. Thus, λ has to be choosen small, and has then an effect of stabilizing the solution by pulling it to zero. An

optimal choice of λ is yet unknown. We tried the values 1e-10, 1e-6 and 1e-2, and noticed that in most cases the second value delivered reasonable results, in the sense that we ended up near the hyperspace.

One eigenvalue of the matrix is almost zero. Therefore, one has to expect that the solution picks up a large amount in direction of the corresponding eigenvector. Numerically, it turns out that the above stabilization has the effect of removing that undesired contribution. It also turns out that this eigenvector is almost parallel with the discretized kernel of the integral equation, given by the normal vector. Therefore, we always projected in direction of the discretized normal.

Our investigation will focus on 80 different matrices. We have triangulations with $8, 32, 128$ and 512 elements, Stokes matrices (we refer to them by $a = 0$) and Oseen matrices for $a = 5e-05, 0.005, 0.5$ and 50. Each of them is computed with different tolerances of the integration routine, namely $1e-3, 1e-5, 1e-7$ and $1e-9$. A few other matrices are available for comparisons.

In the Stokes case and the Oseen case for small a, we compare the computed stresses with the available Stokes solution for a sphere (see Batchelor [1])

$$\mathbf{t} = (-1.5,\ 0,\ 0)^{\top}.$$

As a check for the net force on the body we take the formulas of Oseen [26], which are exact for $a = 0$ (i.e. in the Stokes case) and are themselves approximations in the Oseen case:

$$\mathbf{F} = (-6\pi r L(1 + \frac{3ar}{4L}),\ 0,\ 0)^{\top}$$

where r is the radius and L a characteristic length of the body, which we have both

Table 1: Accuracy of matrices

Comparison of matrices that have been computed with different integration tolerances and the corresponding matrices of tolerance 1e-13. The values are the infinity norms of the matrix differences.

#△	a	1e-03	1e-05	1e-07	1e-09	1e-11	1e-13
8	0	3.1448e-04	4.0984e-05	8.0956e-05	1.6936e-09	3.9920e-16	0
32	0	1.5702e-03	2.8982e-05	5.7244e-05	1.1975e-09	6.1729e-15	0
128	0	1.2867e-03	8.5657e-05	3.3052e-05	6.9581e-10	7.1125e-14	0

chosen to be 1 unit of length. Thus, we expect in the first component the values

$$F_1 \approx \begin{cases} -18.8496 & \text{for } a = 0 \\ -18.8503 & \text{for } a = 5\text{e-}5 \\ -18.9202 & \text{for } a = 5\text{e-}3 \\ -25.9181 & \text{for } a = 5\text{e-}1 \\ -725.7079 & \text{for } a = 5\text{e+}1 \end{cases}$$

7.1 Accuracy and properties of the computed matrices

Since presently an error estimate for the integration routine is not available, we demonstrate in table 1, how accurate the computed matrices actually are. Therefore, we compare each matrix with the most accurately computed matrix of the same type, i.e. same number of elements and same a.

The Oseen equations contain the parameter a, and if $a = 0$, we have actually the Stokes equations. Although we had to do a lot more analytical work for the Oseen equations, we expect the arising matrices for small a to be almost equal to the corresponding Stokes matrices. This is shown in table 2.

In table 3 we give the condition numbers of the matrices. They are computed in the 2−norm, and thus give the ratio of the biggest and smallest singular value. Table 4 gives the 2−norms, and table 5 the smallest singular value of each matrix.

Table 2: Comparison of Stokes and Oseen matrices

The values are the infinity norms of the differences between some Oseen and their corresponding Stokes matrices.

#△	tol	5e-05	5e-03	5e-01	5e+1
8	1e-05	4.1524e-05	3.0048e-03	2.1544e-01	6.3777e-01
	1e-09	3.0194e-05	3.0083e-03	2.1544e-01	6.3773e-01
32	1e-05	7.6739e-05	5.4161e-03	3.3209e-01	7.9265e-01
	1e-09	5.4312e-05	5.3999e-03	3.3209e-01	7.9264e-01
128	1e-05	9.1623e-05	6.4037e-03	3.7716e-01	8.5568e-01
	1e-09	6.4416e-05	6.4001e-03	3.7717e-01	
512	1e-05	9.6000e-05	6.6962e-03	3.9007e-01	8.7771e-01
	1e-09	6.7459e-05	6.6924e-03	3.9007e-01	

For big matrices these computations are very expensive, actually more expensive than solving the systems. One can see that the condition numbers increase and the smallest singular values decrease with decreasing tolerance of integration. The norms, i.e. the biggest singular values are not affected by that.

7.2 Investigation of the solutions

7.2.1 Stokes

Since we have seen in table 3 that we have to deal with big condition numbers, we expect our solutions to be unstable, i.e. that small errors in the matrix computations lead us to wrong results. We investigate the solutions in table 6, which measures the difference of two solutions in the infinity norm. Column 4 and 5 show the resulting net force. Values less than 1e-08 have been replaced by zero.

It seems that both the direct solutions (DI) and the Tychonov-regularized solutions (TREG) agree well with the analytical solution (ANA). However, as one can see in the first column, the more accurately we compute our matrices, the more the

Table 3: Condition numbers

a	tol	8	32	128	512
0	1e-3	4.8106e+04	2.7034e+04	2.5986e+04	4.3679e+04
	1e-5	2.2141e+04	1.7138e+05	5.1147e+06	3.0243e+05
	1e-7	2.6505e+11	1.9278e+11	6.7055e+09	3.3726e+08
	1e-9	2.3391e+14	2.5826e+14	2.0139e+12	1.0961e+13
5e-5	1e-3	7.0971e+04	3.4628e+04	3.3619e+04	7.1362e+04
	1e-5	2.5996e+04	4.4093e+05	1.1194e+05	8.2042e+04
	1e-7	2.3071e+05	1.2341e+05	1.1444e+05	1.1262e+05
	1e-9	2.9976e+11	1.1191e+09	6.6305e+07	1.0503e+07
5e-3	1e-3	2.2747e+03	1.2780e+03	1.1825e+03	1.1411e+03
	1e-5	2.2074e+04	1.6284e+05	4.2555e+05	1.5999e+05
	1e-7	1.3087e+11	1.2998e+09	1.0139e+09	3.2310e+08
	1e-9	5.1427e+12	5.8541e+12	1.3101e+12	5.0479e+11
5e-1	1e-3	3.0549e+04	1.0567e+04	1.3946e+04	3.0019e+04
	1e-5	1.7689e+04	1.2887e+05	1.2593e+05	2.2215e+05
	1e-7	2.5709e+10	2.8696e+09	8.4757e+07	2.5410e+08
	1e-9	3.4887e+12	5.2289e+11	1.8351e+12	1.0737e+13
5e+1	1e-3	2.1884e+02	5.7532e+02	5.8022e+02	1.2585e+03
	1e-5	8.9994e+02	5.1318e+03	2.8183e+04	8.5471e+04
	1e-7	1.4133e+08	6.3699e+06	2.4832e+08	1.2448e+07
	1e-9	6.0831e+11	3.4361e+09		

Table 4: Matrix norms

a	tol	8	32	128	512
0	1e-7	5.4441e-01	6.3769e-01	6.7121e-01	6.8019e-01
5e-5	1e-7	5.4438e-01	6.3764e-01	6.7117e-01	6.8014e-01
5e-3	1e-7	5.4299e-01	6.3558e-01	6.6878e-01	6.7766e-01
5e-1	1e-7	4.3473e-01	4.8141e-01	4.9534e-01	4.9864e-01
5e+1	1e-7	2.7599e-02	3.4363e-02	3.7535e-02	3.8665e-02

Table 5: Smallest singular values

a	tol	8	32	128	512
0	1e-3	1.1317e-05	2.3591e-05	2.5830e-05	1.5572e-05
	1e-5	2.4586e-05	3.7208e-06	1.3123e-07	2.2491e-06
	1e-7	2.0540e-12	3.3079e-12	1.0010e-10	2.0168e-09
	1e-9	2.3274e-15	2.4692e-15	3.3329e-13	6.2054e-14
5e-5	1e-3	7.6705e-06	1.8417e-05	1.9964e-05	9.5308e-06
	1e-5	2.0940e-05	1.4461e-06	5.9958e-06	8.2902e-06
	1e-7	2.3596e-06	5.1669e-06	5.8647e-06	6.0391e-06
	1e-9	1.8161e-12	5.6979e-10	1.0123e-08	6.4757e-08
5e-3	1e-3	2.3837e-04	4.9631e-04	5.6421e-04	5.9238e-04
	1e-5	2.4599e-05	3.9031e-06	1.5716e-06	4.2354e-06
	1e-7	4.1489e-12	4.8898e-10	6.5964e-10	2.0974e-09
	1e-9	1.0559e-13	1.0857e-13	5.1048e-13	1.3425e-12
5e-1	1e-3	1.4232e-05	4.5569e-05	3.5518e-05	1.6611e-05
	1e-5	2.4577e-05	3.7356e-06	3.9333e-06	2.2446e-06
	1e-7	1.6909e-11	1.6776e-10	5.8443e-09	1.9624e-09
	1e-9	1.2462e-13	9.2069e-13	2.6992e-13	4.6441e-14
5e+1	1e-3	1.2598e-04	5.9673e-05	6.4964e-05	3.0824e-05
	1e-5	3.0637e-05	6.6958e-06	1.3318e-06	4.5228e-07
	1e-7	1.9528e-10	5.3946e-09	1.5115e-10	3.1060e-09
	1e-9	4.5400e-14	1.0001e-11		

Table 6: Stokes solutions

DI: Solution by Gauss elimination, TREG: Solution by Tychonov regularization (λ = 1e-6), ANA: Analytical Stokes solution for a sphere, Force DI: Force by integrating DI, Force TREG: Force by integrating TREG. Values of first three columns are infinity norms of vector differences. Values less than 1e-08 have been replaced by zero.

#△	tol	DI-TREG	DI-ANA	TREG-ANA	Force DI	Force TREG
8	1e-3	0	3.3077e-01	3.3077e-01	-1.2684e+01	-1.2684e+01
	1e-5	0	3.3101e-01	3.3101e-01	-1.2686e+01	-1.2686e+01
	1e-7	4.7968e-06	3.3103e-01	3.3103e-01	-1.2686e+01	-1.2686e+01
	1e-9	5.0308e-03	3.3594e-01	3.3091e-01	-1.2685e+01	-1.2685e+01
	1e-11	1.4567e-01	4.7658e-01	3.3091e-01	-1.2685e+01	-1.2685e+01
32	1e-3	0	2.1773e-01	2.1773e-01	-1.6607e+01	-1.6607e+01
	1e-5	0	2.1940e-01	2.1940e-01	-1.6607e+01	-1.6607e+01
	1e-7	1.6745e-06	2.1937e-01	2.1937e-01	-1.6607e+01	-1.6607e+01
	1e-9	2.3343e-03	2.2196e-01	2.1963e-01	-1.6607e+01	-1.6607e+01
	1e-11	9.9732e-03	2.2960e-01	2.1963e-01	-1.6607e+01	-1.6607e+01
128	1e-3	0	1.5523e-01	1.5523e-01	-1.8212e+01	-1.8212e+01
	1e-5	0	1.5358e-01	1.5358e-01	-1.8211e+01	-1.8211e+01
	1e-7	1.5583e-07	1.5387e-01	1.5387e-01	-1.8211e+01	-1.8211e+01
	1e-9	7.8022e-05	1.5395e-01	1.5387e-01	-1.8211e+01	-1.8211e+01
	1e-11	7.3702e-02	2.2613e-01	1.5387e-01	-1.8211e+01	-1.8211e+01
512	1e-3	0	6.9694e-02	6.9694e-02	-1.8683e+01	-1.8683e+01
	1e-5	0	6.9486e-02	6.9486e-02	-1.8682e+01	-1.8682e+01
	1e-7	0	6.9193e-02	6.9193e-02	-1.8682e+01	-1.8682e+01
	1e-9	2.8330e-05	6.9207e-02	6.9193e-02	-1.8682e+01	-1.8682e+01

regularized and the direct solutions differ. We can expect this behavior, since we have seen earlier, that the condition numbers grow. Apparently, the difference between the two solutions is of order $O(\text{1e-17} \cdot \text{cond}(\mathbf{M}))$.

The forces turn out not to be affected by the way we solve the equation or the accuracy of our matrices, and for fine triangulations, they seem to approach the value $-6\pi \approx -18.8496$, as expected.

7.2.2 Oseen for small a

In the Oseen case we lose the stability of our solutions, as shown in table 7. For $a =$1e-5 and $a = 0.005$ we still compare the solutions with the analytical solution of the Stokes case, since for small a we don't expect a big change of the stress. We have replaced values less than 0.1 by zero.

The distances between the regularized and the direct solutions get very big, although the regularized solutions remain near the analytical ones. If the value in the first column is very small, then either both the regularized and the direct solution are good, or they are bad. In the last case, a bigger parameter for the Tychonov regularization would lead to the same result as already mentioned.

However, the forces don't exhibit any instability, and they seem to approach the expected values 18.8503 and 18.9202. This behavior seems to show that whatever the solution picks up by the instability, it doesn't contribute to the net force.

7.3 Investigation of the projected solutions

Now we project all the solutions of the last subsection in direction of the discretized normal on the body.

Table 7: Oseen solutions for small a

DI: Solution by Gauss elimination, TREG: Solution by Tychonov regularizaation (λ = 1e-6), ANA: Analytical Stokes solution for a sphere, Force DI: Force by integrating DI, Force TREG: Force by integrating TREG. Values of first three columns are infinity norms of vector differences. Values less than 0.1 have been replaced by zero.

#△	a	tol	DI-TREG	DI-ANA	TREG-ANA	Force DI	Force TREG
8	5e-5	1e-3	0	7.0923e-01	7.0290e-01	-1.2684e+01	-1.2684e+01
		1e-5	0	4.6964e-01	4.6933e-01	-1.2686e+01	-1.2686e+01
		1e-7	0	8.5516e-01	7.7538e-01	-1.2686e+01	-1.2686e+01
		1e-9	2.6464e+04	2.6464e+04	3.3100e-01	-1.2685e+01	-1.2685e+01
	5e-3	1e-3	0	1.3071e+00	1.3071e+00	-1.2730e+01	-1.2730e+01
		1e-5	0	5.2110e-01	5.2078e-01	-1.2718e+01	-1.2718e+01
		1e-7	1.1797e+06	1.1797e+06	3.4051e-01	-1.2698e+01	-1.2718e+01
		1e-9	4.6431e+07	4.6431e+07	3.4040e-01	-1.2717e+01	-1.2717e+01
32	5e-5	1e-3	0	5.4955e-01	5.4857e-01	-1.6608e+01	-1.6608e+01
		1e-5	1.3599e+00	4.4233e+00	3.0635e+00	-1.6608e+01	-1.6608e+01
		1e-7	0	1.3962e+00	1.3537e+00	-1.6608e+01	-1.6608e+01
		1e-9	5.7683e+00	5.9879e+00	2.1978e-01	-1.6608e+01	-1.6608e+01
	5e-3	1e-3	0	1.4878e+00	1.4878e+00	-1.6687e+01	-1.6687e+01
		1e-5	0	2.9447e-01	2.8960e-01	-1.6662e+01	-1.6662e+01
		1e-7	1.6328e+03	1.6330e+03	2.3495e-01	-1.6662e+01	-1.6662e+01
		1e-9	7.3558e+06	7.3558e+06	2.3482e-01	-1.6658e+01	-1.6662e+01
128	5e-5	1e-3	0	5.2690e-01	5.2597e-01	-1.8213e+01	-1.8213e+01
		1e-5	0	1.3916e+00	1.3581e+00	-1.8212e+01	-1.8212e+01
		1e-7	0	1.4197e+00	1.3839e+00	-1.8212e+01	-1.8212e+01
		1e-9	2.3934e+00	2.5009e+00	1.5427e-01	-1.8212e+01	-1.8212e+01
	5e-3	1e-3	0	1.5117e+00	1.5117e+00	-1.8308e+01	-1.8308e+01
		1e-5	1.0561e+05	1.0561e+05	1.6973e-01	-1.8277e+01	-1.8277e+01
		1e-7	4.2481e-01	1.6138e+00	1.1975e+00	-1.8277e+01	-1.8277e+01
		1e-9	8.8513e+01	8.8604e+01	1.6969e-01	-1.8277e+01	-1.8277e+01
512	5e-5	1e-3	0	8.8623e-01	8.7709e-01	-1.8684e+01	-1.8684e+01
		1e-5		1.0093e+00		-1.8683e+01	
		1e-7	0	1.3680e+00	1.3328e+00	-1.8683e+01	-1.8683e+01
		1e-9	1.3978e+00	1.4488e+00	0	-1.8683e+01	-1.8683e+01
	5e-3	1e-3	0	1.4392e+00	1.4392e+00	-1.8784e+01	-1.8784e+01
		1e-5	1.0963e-01	2.1343e+00	2.0249e+00	-1.8752e+01	-1.8752e+01
		1e-7	1.0370e+00	1.0879e+00	0	-1.8752e+01	-1.8752e+01
		1e-9	3.8644e+03	3.8644e+03	0	-1.8752e+01	-1.8752e+01

Table 8: Projected Stokes solutions

DI: Projected solution by Gauss elimination, TREG: Projected solution by Tychonov regularization (λ = 1e-6), ANA: Analytical Stokes solution for a sphere, Normal Force: Force by integrating a norm-1-vector of the discretized kernel of the integral equation. Values of first three columns are infinity norms of vector differences.

#△	tol	DI-TREG	DI-ANA	TREG-ANA	Normal Force
8	1e-3	8.3531e-11	3.3077e-01	3.3077e-01	-2.2204e-16
	1e-5	8.3166e-11	3.3101e-01	3.3101e-01	
	1e-7	8.2987e-11	3.3103e-01	3.3103e-01	
	1e-9	8.3078e-11	3.3091e-01	3.3091e-01	
	1e-11	8.3080e-11	3.3091e-01	3.3091e-01	
32	1e-3	1.7380e-10	2.1773e-01	2.1773e-01	0
	1e-5	1.7201e-10	2.1940e-01	2.1940e-01	
	1e-7	1.7205e-10	2.1937e-01	2.1937e-01	
	1e-9	1.7206e-10	2.1963e-01	2.1963e-01	
	1e-11	1.7206e-10	2.1963e-01	2.1963e-01	
128	1e-3	3.5733e-10	1.5523e-01	1.5523e-01	-2.2204e-16
	1e-5	3.5910e-10	1.5358e-01	1.5358e-01	
	1e-7	3.5910e-10	1.5387e-01	1.5387e-01	
	1e-9	3.5909e-10	1.5387e-01	1.5387e-01	
	1e-11	3.5910e-10	1.5387e-01	1.5387e-01	
512	1e-3	1.5474e-09	6.9694e-02	6.9694e-02	8.8818e-16
	1e-5	1.5433e-09	6.9486e-02	6.9486e-02	
	1e-7	1.5377e-09	6.9193e-02	6.9193e-02	
	1e-9	1.5377e-09	6.9193e-02	6.9193e-02	

7.3.1 Stokes

As one can see in table 8, the projections of the direct and regularized solutions agree very well and are stable, i.e. they don't depend significantly on the tolerance of the matrix computation. Already with tolerance 1e-5 we seem to get right answers. This shows once more, that apparently the effect of the instability is to pick up a vector of arbitrary length in direction of the discretized kernel.

7.3.2 Oseen

Basically, the same statements than in the Stokes case are still true. However, the values of the first column in table 9 do not look as nice as in table 8. It is not quite clear, if this is caused only by numerical peculiarities or if there is any kind of rule behind that, which we haven't discovered yet.

Table 9: Projected Oseen solutions for small a

DI: Projected solution by Gauss elimination, TREG: Projected solution by Tychonov regularization (λ = 1e-6), ANA: Analytical Stokes solution for a sphere. Values are infinity norms of vector differences.

#△	a	tol	DI-TREG	DI-ANA	TREG-ANA
8	5e-5	1e-3	5.6040e-08	3.3093e-01	3.3093e-01
		1e-5	2.8005e-09	3.3117e-01	3.3117e-01
		1e-7	9.5545e-07	3.3116e-01	3.3116e-01
		1e-9	5.2124e-04	3.3123e-01	3.3100e-01
	5e-3	1e-3	3.1116e-08	3.4546e-01	3.4546e-01
		1e-5	1.7787e-09	3.4053e-01	3.4053e-01
		1e-7	1.6582e-02	3.4801e-01	3.4053e-01
		1e-9	1.3423e-04	3.4039e-01	3.4040e-01
32	5e-5	1e-3	6.6860e-06	2.1923e-01	2.1923e-01
		1e-5	2.9335e-04	2.1958e-01	2.1951e-01
		1e-7	1.2302e-06	2.1963e-01	2.1963e-01
		1e-9	4.5009e-07	2.1978e-01	2.1978e-01
	5e-3	1e-3	3.7012e-08	2.4937e-01	2.4937e-01
		1e-5	1.3914e-06	2.3473e-01	2.3473e-01
		1e-7	1.0541e-03	2.3421e-01	2.3456e-01
		1e-9	3.6112e-02	1.9870e-01	2.3482e-01
128	5e-5	1e-3	2.9790e-06	1.5572e-01	1.5572e-01
		1e-5	4.9701e-06	1.5373e-01	1.5373e-01
		1e-7	1.5554e-06	1.5413e-01	1.5413e-01
		1e-9	1.3643e-06	1.5403e-01	1.5403e-01
	5e-3	1e-3	1.7126e-08	1.8053e-01	1.8053e-01
		1e-5	1.0861e-03	1.6921e-01	1.6973e-01
		1e-7	1.1981e-04	1.6893e-01	1.6903e-01
		1e-9	1.2879e-04	1.6969e-01	1.6973e-01
512	5e-5	1e-3	1.3642e-05	7.0496e-02	7.0489e-02
		1e-5		6.9721e-02	
		1e-7	2.4774e-06	6.9354e-02	6.9355e-02
		1e-9	4.8160e-06	6.9301e-02	6.9300e-02
	5e-3	1e-3	2.5961e-08	9.0883e-02	9.0883e-02
		1e-5	1.3706e-04	8.3984e-02	8.3976e-02
		1e-7	3.6366e-06	8.3513e-02	8.3514e-02
		1e-9	2.6034e-04	8.3287e-02	8.3515e-02

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