This thesis deals with the effect of pressure coupling on the stresses in shells having geometric discontinuities. The problems are, 1) cylinder with an axially symmetric thickness change and 2) cylinder with a hemispherical end closure. Solutions to the above problems are based on differential equations that included the straightening effect of the axial tension load. In each of the problems the only load was internal pressure. Our main objective was to obtain curves giving distribution of dimensionless discontinuity stresses for these problems.

The problem we have solved first was the cylinder with a thickness change. We first hypothetically separated the cylinders at the thickness change and wrote the differential equations for the individual cylinders. The solutions to the differential equations that were thought to be of interest have been retained, and the boundary conditions of
compatibility and force equilibrium used to evaluate the integration constants. Since the thinner cylinder would be the higher stressed, the emphasis of the analysis was placed on the thinner cylinder, from this point on. Non-dimensionalizing of the interface stress resultants, integration constants, solutions to the differential equations, and stress resultants, to put the results in terms of the minimum number of variables, was accomplished. The dimensionless equations were programmed for the IBM 1410 computer and numerical values representing the dimensionless stress curves obtained and these curves were plotted by an IBM 1627 plotter.

For the second problem the cylindrical and hemispherical portions were hypothetically separated and the solution for the sphere given by Cline ("Journal of Applied Mechanics", March 1963) has been used for the hemispherical portion and the solution found in the first problem used for the cylindrical portion. Again the boundary conditions of compatibility and force equilibrium enabled us to determine the integration constants. After non-dimensionalizing the interface stress resultants and the integration constants, to put the equations in terms of the minimum number of dimensionless variables, we wrote the non-dimensional stress equations. Again the dimensionless equations were programmed for the IBM 1410 computer and the numerical values representing the dimensionless discontinuity stresses obtained and the curves plotted by an IBM 1627 plotter.
Dimensionless design curves of the discontinuity stresses, the goal of this thesis, are presented.
EFFECT OF PRESSURE COUPLING ON DISCONTINUITY STRESSES IN SHELLS

by

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NOTATIONS

\[ M \] Bending stress resultant.
\[ N \] Tensile stress resultant.
\[ Q \] Shear stress resultant.
\[ a_0 \] Radius.
\[ x \] Meridional coordinate.
\[ \theta \] Circumferential coordinate.
\[ h_1, h_2 \] Thickness for cylinders one and two respectively.
\[ M_0, N_0, Q_0 \] Stress resultants at discontinuity.
\[ M_x, N_x, Q_x \] Stress resultants at a point where \( x \) equals constant.
\[ N_\theta \] Stress resultant at a point where \( \theta \) equals constant.
\[ w \] Deflection of the middle surface.
\[ E \] Young's Modulus.
\[ \mu \] Poisson's ratio.
\[ \rho \] Internal pressure.

\[ D = \frac{Eh^3}{12(1-\mu^2)} \] Flexural Rigidity.

\[ \beta = \left( \frac{Eh}{4Da_0} \right)^{\frac{1}{4}} = \left[ \frac{3(1-\mu^2)}{2h^2a_0} \right]^{\frac{1}{4}} \] Cylindrical shell parameters.

\[ \eta = \frac{N_\phi^2}{2D\beta^2} = \frac{p(\frac{2}{E})}{\frac{a_0}{h} \left[ 3(1-\mu^2) \right]^\frac{1}{2}} \]
\[ w_p = \frac{p a_0}{E h} (1 - \frac{\alpha}{2}) \]

Deflection of the middle surface due to internal pressure of the cylinder.

\( e \)

Base for natural logarithm.

\( c \)

A complex constant.

\( w_x \)

Total deflection of the middle surface of the cylinder.

\( \gamma_x \)

Rotation of the meridian of the middle surface of the cylinder.

\( C_n \)

Integration constants for the cylinder (\( n = 1, 2, \cdots, 12 \)).

\( w_0 \)

Total deflection of middle surface of cylinders or hemisphere at discontinuity.

\( \gamma_0 \)

Rotation of middle surface meridians for cylinders or hemisphere at discontinuity.

1, 2

Used as subscript to denote cylinder one or two respectively, unless defined otherwise.

\( H_I \)

Constant whose use is obvious from text, \( \eta \) between zero and two, see page 14.

\( C_{71}, C_{81} \)

Integration constants from boundary condition for cylinder one, \( \eta_1 \) equals two, see page 16.

\( H_{II} \)

Constant whose use is obvious from text, \( \eta_2 \) equals two, see page 17.

\( \sigma_x \)

Meridional stress for cylinder.

\( \sigma_{\phi} \)

Meridional stress for hemisphere.

\( \sigma_\theta \)

Circumferential stress.

\( C_{31}, C_{41} \)

Integration constants from boundary condition for cylinder one, \( \eta \) between zero and two.

1

Used as subscript denotes region where \( \eta \) is between zero and two.
\( \Pi \)  
Used as subscript denotes point where \( \eta \) equals two.

( )  
All barred quantities denote they are non-dimensional.

\( h_c, h_s \)  
Thickness for cylinder and hemisphere respectively.

\( \phi \)  
Meridional coordinate for the hemisphere.

\[ \rho = \frac{p a_0}{4 E h} \lambda^2 = \frac{\eta}{2} \]  
Spherical shell parameters.

\[ \lambda = [12(1 - \mu^2)]^{1/2} \left( \frac{a_0}{h} \right) = \sqrt{2} a_0 \beta \]

\[ L = \frac{d^2}{d \phi^2} + \cot \phi \frac{d}{d \phi} (\cot \phi) \]  
Differential operator.

\( \xi \)  
A dependent variable, see page 32.

( )  
Denotes derivative with respect to the argument.

\( m \)  
Denotes integers \( (m = 1, 2, 3 \cdots) \).

\( R, J \)  
Denotes real and imaginary parts of a complex number.

\( A, B, A_0, B_0, C_{01}, C_{02}, D_{01}, D_{02} \)  
Constants of integration applying to hemisphere.

\( z \)  
Argument of the Bessel's function where \( \eta \) is between zero and two.

\( z_1, z_2 \)  
Arguments of the Bessel's function where \( \eta \) is greater than two.

\( I_0(z), K_0(z) \)  
Modified Bessel's function.
U, V  Functions as defined in text, see page 34.

( )'  Denotes derivative with respect to \( \phi \).

\( \alpha \)  Value of the coordinate \( \phi \) at the edge of sphere.

\( G_\phi \)  Shear stress resultant that does not include pressure coupling.

\( G_\phi, M_\phi, N_\phi, Q_\phi \)  Stress resultants for the hemisphere at a point where \( \phi \) equals a constant.

\( w_s, w_c \)  Radial deflection of middle surface due to discontinuity loads.

\( \gamma_s, \gamma_c \)  Rotation of meridian of middle surface due to discontinuity loads.

\( \Delta \)  A constant whose use is obvious from the text, see page 36.

e  Used as subscript denotes edge of sphere.

\( C_{ij} \)  Denotes the hemisphere influence coefficients \((i = 1, 2, \ j = 1, 2)\).

\( f(\alpha), g(\alpha) \)  Functions defined in text, see page 36.

\( S_{ij} \)  Denotes the cylinder influence coefficients \((i = 1, 2, \ j = 1, 2)\).

\( M_{xc}, Q_{xc} \)  Stress resultants for the cylinder--cylinder-to-hemisphere juncture at a point where \( x \) equals a constant.

c, s  Used as subscript denotes cylinder or hemisphere respectively.
EFFECT OF PRESSURE COUPLING ON DISCONTINUITY STRESSES IN SHELLS

CHAPTER I

INTRODUCTION

Changes in thickness of pressurized shells are of interest to industry, since, as engineers are well aware, this will change the biaxial stress field by introducing discontinuity loads, bending, and shear, which increase the stresses in the cylinder. The stress analyst has handled this problem by neglecting the straightening effect of the axial tension load in the differential equations for a cylinder with capped ends and loaded by internal pressure only. For most industrial purposes this type of analysis sufficed but with the coming of the space age the need for more exact analysis has become imperative due to the use of very low safety factors, the high degree of reliability required, and the extreme consciousness of excess weight in missiles.

The problem that is dealt with in this paper is a step towards a more exact solution by the inclusion of the pressurization load in the equations of equilibrium at the geometry change in the cylinder. Two types of geometry changes are considered, 1) thickness change in the cylinder and 2) transition from cylinder to hemisphere at one of the closed ends of the cylinder. The differential equations were based on the original shape of the cylinder and hemisphere, except for inclusion
of the straightening effect of the axial tension load. Small deflection theory was assumed to apply.

The following is a brief explanation of how the problem treated in this analysis will be solved. Initially the differential equations will be found then solved and the solutions will then be non-dimensionalized. Non-dimensionalizing of the solutions is for the purpose of writing the relationships in terms of the minimum number of variables. From these non-dimensional solutions, the non-dimensional stress equations can be written and programed to generate curves of the non-dimensional stresses. These are then the design curves which will be the primary goal of this study.
CHAPTER II

DIFFERENTIAL EQUATIONS FOR A SYMMETRICALLY LOADED CYLINDER WITH AXIAL TENSION

Before the stress distribution in a given cylinder can be found, the manner in which the loads enter the cylinder must be known. When the above is known, we can observe the following three basic relations; 1) force equilibrium (moment and shear), 2) strain-displacement, and 3) stress-strain (Hooke's Law). These, along with the proper assumptions, lead to a mathematical model whose differential equations can be written and solved.

The above outlined method forms the basis for solving the problem of stresses in a cylindrical shell with an axially symmetric meridional thickness change, and loaded by constant internal pressure only.

2:1 Description of the Cylinder

A sketch of the region of thickness change in the cylinder is shown in Figure 2:1. The coordinate system used is also shown in Figure 2:1 with \( x \) designating the coordinate along the axis of revolution.

Thickness change in the cylinder is only in the meridional direction and is axisymmetric with respect to the axis of revolution. The cylinder in both meridional directions from the thickness change is
assumed to have constant thickness and to be long enough to make the region of thickness change free from end effects.

Loading of the cylinder is static and is by internal pressure only.

2:2 Derivation of the Differential Equations

Figure 2:2 shows the region under study with the thick and thin cylinders hypothetically separated with the internal forces that were acting at the discontinuity now shown as edge loads. A typical differential element of the cylinder is shown in Figure 2:3A. But, the condition of axial symmetry of the thickness change and the constant internal pressure indicates that the stress distribution is independent of \( \theta \) and
hence, the simplified differential element in Figure 2:3B describes the problem completely for the derivation of the differential equations. We shall assume that small deflection theory applies (deflection is small compared to the thickness of the cylinder), and that the shell is thin, i.e.,

\[ a_0 > > w \]  \hspace{1cm} (2:1)

\[ \frac{dN_x}{dx} \approx 0 \]  \hspace{1cm} (2:2)
Noting that force and moment equilibrium for the differential element must hold, leads to two of the required equations. By summing the shear and pressure forces in Figure 2:3B and recalling that hoop restraint (4, p. 209) is also present, we have the shear equilibrium equation as,

\[ uN_x + p \frac{dQ}{dx} + \mu N_x \frac{dx}{dx} = 0 \]  \hspace{1cm} (2:3)

where, \( \frac{Eh}{a_0} \) \( wdx \) is the elastic hoop restraint term. Also from Figure 2:3B, we have the moment equilibrium equation as,

\[ dM_x + Q_x dx + \frac{P}{2} (dx)^2 - N_x dw = 0 \]  \hspace{1cm} (2:4)
N \frac{dw}{dx} is the axial tension term and represents the pressure coupling effect in the differential equation. This term is generally neglected by most authors in the derivation of the differential equations for a shell under constant internal pressure. But, this term has been included here because as will be shown it does represent a substantial quantity if the pressure is high. $M_x$, $Q_x$, and $N_x$ are the meridional or axial stress resultants and are moment, shear, and axial load per unit width, respectively. Internal pressure, middle surface radius, radial deflection and Poisson's ratio are denoted by $p$, $a_0$, $w$, and $\mu$ respectively.

When Equation (2:4) is used to eliminate $dQ_x$ in Equation (2:3) and products of differentials are assumed small and neglected, we have as the resulting differential equation,

$$\frac{d^2 M_x}{dx^2} - N_x \frac{d^2 w}{dx^2} + \frac{Eh}{a_0^2} w = p - \frac{\mu N_x}{a_0}$$

(2:5)

Since the cylinder is assumed to carry axial "blow-off" load due to pressure acting on a capped end, and no other axial loads, the axial equilibrium condition requires that

$$N_x = \frac{pa_0}{2}.$$  

(2:6)

The moment-curvature relation,
\[ M_x = D \frac{d^2 w}{dx^2}, \quad (2.7) \]

where

\[ D = \frac{Eh^3}{12(l-u^2)} \quad (2.8) \]

is the flexural rigidity, may be used to eliminate \( M_x \) from Equation (2.5). When the definitions

\[ \beta = \left( \frac{Eh}{4Da_0} \right)^{\frac{1}{4}} = \left[ \frac{3(1-\mu^2)}{h^2 a_0^2} \right]^{\frac{1}{4}} \quad (2.9) \]

\[ \eta = \frac{N_x}{2D\beta^2} = \frac{P}{E} \left( \frac{a_0}{h} \right)^2 \left[ 3(1-\mu^2) \right]^{\frac{1}{2}} \quad (2.10) \]

are introduced, the differential equation (2.5) becomes

\[ \frac{d^4 w}{dx^4} - 2\beta^2 \eta \frac{d^2 w}{dx^2} + 4\beta^4 w = \frac{P}{D} \left( 1 - \frac{\mu^2}{2} \right). \quad (2.11) \]

Since \( \beta \) and \( \eta \) are constants with respect to \( x \), this equation is a standard form of a non-homogeneous fourth order differential equation with constant coefficients which is easily solved (9, p. 101-104).

2:3 Solutions to the Differential Equation

The particular solution to the differential equation,
\[
\frac{d^4 w}{dx^4} - 2\beta^2 \eta \frac{d^2 w}{dx^2} + 4\beta^4 w = \frac{P}{D} \left( 1 - \frac{\mu}{2} \right) \quad (2:11)
\]

can be written immediately as,

\[
w_p = \frac{pa_0^2}{Eh} \left( 1 - \frac{\mu}{2} \right) \quad (2:12)
\]

This solution represents the free radial deflection of a cylinder under internal pressure. The solutions to the homogeneous equation,

\[
\frac{d^4 w}{dx^4} - 2\beta^2 \eta \frac{d^2 w}{dx^2} + 4\beta^4 w = 0 \quad (2:13)
\]

are of the form

\[
w = e^{cx} \quad (2:14)
\]

where \( c \) is a complex constant. When the assumed solution is substituted back into the differential equation, we find that the constant \( c \) may be any of the four values,

\[
c = \pm \beta \sqrt{\eta \pm \sqrt{\eta^2 - 4}} \quad (2:15)
\]

where \( \eta \) can be assumed positive by limiting the internal pressure to positive values.

The real solutions to the homogeneous differential equation are given below for positive values of internal pressure. The form of the
solutions will depend on the value of $\eta$ which is a measure of the amount of "pressure coupling":

$$0 \leq \eta < 2 : \quad w = e^{\pm \sqrt{\frac{n+2}{2}} \beta x} \sin \left( \frac{2-\eta}{2} \beta x \right)$$

(2:16)

$$\eta = 2 : \quad w = e^{\pm \sqrt{2} \beta x} (1, x)$$

(2:17)

$$\eta > 2 : \quad w = e^{\pm \sqrt{\eta \pm \sqrt{\eta^2 - 4}} \beta x}$$

(2:18)

We may write the general solution to the differential equation (2:11), as the sum of the particular solution and an arbitrary linear combination of the solutions to the homogeneous equation given above for each area defined by $\eta$. The resulting equations are,

$$0 \leq \eta < 2 :$$

$$w = \frac{pa}{2} \sqrt{\frac{E}{h}} (1 - \frac{h}{2}) + e^{\left( \frac{n+2}{2} \right) \beta x} \left[ C_1 \cos \left( \frac{2-\eta}{2} \right)^{\frac{1}{2}} \beta x \right] + C_2 \sin \left( \frac{2-\eta}{2} \right)^{\frac{1}{2}} \beta x]$$

$$- e^{\left( \frac{n+2}{2} \right)^{\frac{1}{2}} \beta x} \left[ C_3 \cos \left( \frac{2-\eta}{2} \right)^{\frac{1}{2}} \beta x \right] + C_4 \sin \left( \frac{2-\eta}{2} \right)^{\frac{1}{2}} \beta x]$$

$$+ \frac{pa}{2} \sqrt{\frac{E}{h}} (1 - \frac{h}{2}) + e^{\sqrt{2} \beta x} (C_5 + xC_6) + e^{- \sqrt{2} \beta x} (C_7 + xC_8)$$

(2:19)

$$\eta = 2 :$$

$$w = \frac{pa}{2} \sqrt{\frac{E}{h}} (1 - \frac{h}{2}) + e^{\sqrt{2} \beta x} (C_5 + xC_6) + e^{- \sqrt{2} \beta x} (C_7 + xC_8)$$

(2:20)
\( \eta > 2: \)

\[
\frac{p a_0^2}{E h} (1 - \frac{\mu}{2}) + C_{9e} \sqrt{\eta + \sqrt{\eta^2 - 4 \beta x}} + C_{10e} \sqrt{\eta + \sqrt{\eta^2 - 4 \beta x}}^2
+ C_{11e} \sqrt{\eta - \sqrt{\eta^2 - 4 \beta x}} + C_{12e} \sqrt{\eta - \sqrt{\eta^2 - 4 \beta x}}
\]  

(2.21)

The constants of integration \( C_n \) will be evaluated from known boundary conditions.

2.4 The Edge Stress Resultants

The analysis from this point on in this research will be limited to the areas, \( \eta \) less than or equal to two. \( \eta \) equal to two has been chosen as the upper bound for analysis for two reasons:

1) It is believed that this region will be of most practical interest

2) This will be sufficient to show the designer when pressure coupling may be neglected.

The analysis procedure for \( \eta \) greater than two will require no new analysis method in solving the equations from those used for \( \eta \) less than or equal to two.

\( 0 \leq \eta < 2: \) The basic relationships that are of interest in finding the interface stress resultants for this case are,
Let \( M_0 \) and \( Q_0 \) be the moment and shear stress resultants acting at the interface. They can be found by applying the boundary condition that the deflection and rotation at infinity be finite and evaluating equations (2.19), (2.22), (2.7) and (2.23) at \( x^a \) equal to zero.

We then have as the defining equations at the interface,

\[
w_0 = C_3 + \frac{p a_0}{E h} (1 - \frac{\mu}{2})
\]

\[
\gamma_0 = -\beta \left[ \left( \frac{\eta + 2}{2} \right)^{\frac{1}{2}} C_3 - \left( \frac{2 - \eta}{2} \right)^{\frac{1}{2}} C_4 \right]
\]

\[
M_0 = D \beta^2 \left[ \eta C_3 - (4 - \eta^2) C_4 \right]
\]
\[ Q_0 = -2D\beta^3 \left\{ \left( \frac{\eta + 2}{2} \right)^{\frac{1}{2}} C_3 + \left( \frac{2-\eta}{2} \right)^{\frac{1}{2}} C_4 \right\}. \] (2:27)

\[ C_3 \quad \text{and} \quad C_4 \quad \text{are the integration constants and in terms of the inter-face stress resultants they are,} \]

\[ C_3 = -\frac{\left( \frac{\eta + 2}{2} \right)^{\frac{1}{2}}}{2D\beta^3(\eta + 1)} Q_0 + \frac{1}{2D\beta^2(\eta + 1)} M_0 \] (2:28)

\[ C_4 = -\frac{\eta}{4D\beta^3 \left( \frac{2-\eta}{2} \right)^{\frac{1}{2}}(\eta + 1)} Q_0 - \frac{\left( \frac{\eta + 2}{2} \right)^{\frac{1}{2}}}{2D\beta^2 \left( \frac{2-\eta}{2} \right)^{\frac{1}{2}}(\eta + 1)} M_0 \] (2:29)

From the condition that continuity has to hold, we have that the deflections and rotations at the interface of the hypothetically separated cylinders must be equal, while "Newton's Third Law" requires that the stress resultants acting on the cylinders be equal and opposite. From the above conditions the following relations result,

\[ w_{01} = w_{02} = w_0 \]
\[ \gamma_{01} = \gamma_{02} = \gamma_0 \]
\[ M_{01} = M_{02} = M_0 \] (2:30)
\[ Q_{01} = -Q_{02} = Q_0 \]
\[ N_{01} = N_{02} = N_0 \]
and we write the governing equations for deflection and rotation at the interface as,

\[
\begin{align*}
    w_0 &= w_01 = -\frac{\eta_1 + 2}{2D_2\beta_1^3(\eta_1 + 1)} Q_0 + \frac{1}{2D_2\beta_1^2(\eta_1 + 1)} M_0 + \frac{p\alpha_0}{Eh_1}\left(1 - \frac{h_1}{2}\right) \\
    \gamma_0 &= \gamma_01 = \frac{1}{2D_2\beta_1^2(\eta_1 + 1)} Q_0 - \frac{\eta_1 + 2}{2D_2\beta_1^2(\eta_1 + 1)} M_0 \\
    w_0 &= w_02 = -\frac{\eta_2 + 2}{2D_2\beta_2^3(\eta_2 + 1)} Q_0 + \frac{1}{2D_2\beta_2^2(\eta_2 + 1)} M_0 + \frac{p\alpha_0}{Eh_2}\left(1 - \frac{h_2}{2}\right) \\
    \gamma_0 &= \gamma_02 = \frac{1}{2D_2\beta_2^2(\eta_2 + 1)} Q_0 + \frac{\eta_2 + 2}{2D_2\beta_2^2(\eta_2 + 1)} M_0 .
\end{align*}
\]

This gives four equations in four unknowns which makes the solution for the edge stress resultants \( M_0 \) and \( Q_0 \) in terms of the cylinder parameters \( \eta_1, \beta_1 \) and the known cylinder geometrical quantities possible. First we will let,

\[
H_1 = -\left(\frac{h_1}{h_2}\right)^2 + \frac{h_1}{h_2} - \frac{\eta_1 + 2}{2h_2} + \frac{\eta_1 + 2}{h_2} - \frac{h_1}{h_2} - \frac{h_1}{h_2} (\eta_1 + 2)
\]

\[
- (\eta_1 + 1)(\eta_1 + 2) + \frac{h_1^2}{h_2^2} \left(\frac{h_1^2}{h_2^2} + \frac{h_1^2}{h_2^2} \right)
\]

then,
In the event \( \eta \) for cylinder one or two equals two, the governing equations for that cylinder become,

\[
\begin{align*}
\omega_x &= \frac{p_k a_0}{E h} \left( 1 - \frac{h}{2} \right) + e^\sqrt{2} \beta x (C_5 + xC_6) + e^{-\sqrt{2} \beta x} (C_7 + xC_8) \\
\gamma_x &= \frac{d \omega_x}{dx} \\
M_x &= D \frac{d^2 \omega_x}{dx^2} \\
Q_x &= \frac{p_k a_0}{2} \frac{d \omega_x}{dx} - D \frac{d^3 \omega_x}{dx^3}.
\end{align*}
\]
By using the same boundary conditions and following a similar
development as followed for the case, \( \eta \) between zero and two, we
have the integration constants as,

\[
C_{71} = -\frac{\sqrt{2}}{6D_1^3} Q_0 + \frac{1}{6D_1^2} M_0
\]

\[\text{(2.35)}\]

\[
C_{81} = -\frac{1}{6D_1^2} Q_0 - \frac{\sqrt{2}}{6D_1} M_0
\]

and the deflection and rotation equations for the interface as,

\[
w_0 = w_{01} = -\frac{\sqrt{2}}{6D_1^3} Q_0 + \frac{1}{6D_1^2} M_0 + \dot{w}_{p1}
\]

\[\text{(2.36)}\]

\[
\gamma_0 = \gamma_{01} = \frac{1}{6D_1^2} Q_0 + \frac{\sqrt{2}}{3D_1} M_0
\]

No loss in generality results from our having assumed that the
cylinder with \( \eta \) equal to two, is the cylinder on the right (cylinder
one), therefore the mating cylinder will be cylinder two from the pre-
ceding section whose interface deflection and rotation equations were
found as,
\[
\begin{align*}
\gamma_0 = \gamma_{02} &= \frac{1}{2D_2\beta_2^2(\eta_2 + 1)} Q_0 + \frac{\eta_2 + \frac{1}{2}}{2D_2\beta_2^2(\eta_2 + 1)} M_0
\end{align*}
\]

(2:32)

Again, by equating deflections and rotations and using the equality

\[
\eta_2 = \frac{2}{h_1} \quad \eta_1 = \frac{2}{h_2}
\]

we can find the stress resultants at the cylinder interfaces. First let,

\[
H_{II} = 2 \left[ 3 \left( \frac{1}{h_2} \right)^{\frac{3}{2}} \left( \frac{h_2}{h_1} + 1 \right)^{\frac{1}{2}} + \sqrt{2} \left( 2 \left( \frac{h_2}{h_1} + 1 \right) \right) \right] \left[ 3 \left( \frac{1}{h_2} \right)^{\frac{5}{2}} \left( \frac{h_2}{h_1} + 1 \right)^{\frac{1}{2}} + \sqrt{2} \left( 2 \left( \frac{h_2}{h_1} + 1 \right) \right) \right]
\]

\[
\left( \frac{1}{h_2} - 1 \right)^{\frac{1}{2}}
\]

(2:37)

\[
M_0 = \frac{6D_1\beta_1^2 \left( \frac{h_2^2}{h_1^2} + 1 \right) \left( \frac{1}{h_2^2} - 1 \right)}{H_{II}} (w_{p2} - w_{p1})
\]

(2:38)

\[
Q_0 = \frac{-12D_1\beta_1^3 \left( \frac{h_2^2}{h_1^2} + 1 \right) \left[ 3 \left( \frac{h_2}{h_1} \right)^{\frac{3}{2}} \left( \frac{h_2}{h_1} + 1 \right)^{\frac{1}{2}} + \sqrt{2} \left( 2 \left( \frac{h_2}{h_1} + 1 \right) \right) \right]}{H_{II}} (w_{p2} - w_{p1})
\]
Let us now write the stress equations in terms of the tensile loads, moment and deflection.

The equation giving meridional stress is

\[
\sigma_x = \frac{N_x}{h} \pm \frac{6M_x}{h^2}
\]

and that giving circumferential stress is

\[
\sigma_\theta = \frac{N_\theta}{h} + \frac{Ew}{a_0} \pm \mu \frac{6M_x}{h^2}
\]

Figure 2:4 shows the pressure equilibrating stresses and the axial bending stress. Now shown in the figure is the component represented by the term \(\frac{Ew}{a_0}\), the stress due to the extension of the circumference resulting from the discontinuity moment and shear, and the
component represented by the term $\mu \frac{6M}{h^2}$ is the stress due to the circumferential restraint preventing anticlastic curvature.

2.5 Non-Dimensional Equations

Since the goal of this paper is to develop design curves of practical use to the engineering design analyst, we must find a means of presenting curves such that each curve presented could represent more than a specific case. In order to achieve this goal we will non-dimensionalize the integration constants, deflections, and moments, so that the final stress equations are non-dimensional.

2.5.1 Non-dimensional integration constants: In Section 2.4 the integration constants of interest have been determined as $C_{31}$ and $C_{41}$ for $\eta$ between zero and two, and for $\eta$ equal to two as $C_{71}$ and $C_{81}$:

\begin{align*}
0 \leq \eta < 2: \\
C_{31} &= -\frac{\eta_1 + 2^{\frac{1}{2}}}{2D_1 \beta_1^3 (\eta_1 + 1)} Q_{0I} + \frac{1}{2D_1 \beta_1^2 (\eta_1 + 1)} M_{0I} \\
C_{41} &= -\frac{\eta_1}{4D_1 \beta_1^3 (\frac{2-\eta_1}{2}) (\eta_1 + 1)} Q_{0I} - \frac{\eta_1 + 2^{\frac{1}{2}}}{2D_1 \beta_1^2 (\frac{2-\eta_1}{2}) (\eta_1 + 1)} M_{0I}
\end{align*}
The subscript one is being used since the assumption that cylinder one is the thinner of the two cylinders is being continued.

Before proceeding to non-dimensionalize the integration constants, it will be profitable at this time to non-dimensionalize the interface stress resultants \( M_0 \) and \( Q_0 \) given by Equations (2.34) and (2.38). Let the barred quantities be the non-dimensional stress resultants:

\[
\eta = 2: \\
C_{71} = - \frac{\sqrt{2}}{6D_1 \beta_1^3} Q_{0_{II}} + \frac{1}{6D_1 \beta_1^2} M_{0_{II}} \\
C_{81} = - \frac{1}{6D_1 \beta_1^2} Q_{0_{II}} - \frac{\sqrt{2}}{6D_1 \beta_1} M_{0_{II}}
\]

\[
(2.35)
\]

\[
0 < \eta < 2: \\
M_{0_1} = D_1 \beta_1^2 \bar{M}_{0_1} \\
Q_{0_1} = D_1 \beta_1^3 \bar{Q}_{0_1}
\]

\[
\eta = 2: \\
M_{0_{II}} = D_1 \beta_1^2 \bar{M}_{0_{II}} \\
Q_{0_{II}} = D_1 \beta_1^3 \bar{C}_{0_{II}}
\]

\[
(2.40)
\]
Substituting the non-dimensional form of the stress resultants into the equation for the integration constants, we have

\[ 0 \leq \eta < 2: \]

\[ C_{31} = \frac{\eta_1 + 2}{2(\eta_1 + 1)} \frac{1}{Q_{01}} + \frac{1}{2(\eta_1 + 1)} \frac{1}{M_{01}} \]

\[ = \frac{h_1^2}{(\frac{1}{2} \eta_1 + 1)[(\frac{1}{2} \eta_1 + 1)(\eta_1 + 2)]} \frac{5/2}{\eta_1 + 2} \frac{h_1^2}{h_2^2} \frac{1/2}{(\eta_1 + 2)} \frac{1/2}{(\eta_1 + 2)} \frac{h_1^2}{h_2^2} \]

\[ + \frac{h_1^2}{h_2^2} \frac{2}{(\frac{1}{h_2} - 1)} \frac{w_2 - w_1}{p_2 - p_1} \]

\[ C_{41} = \frac{\eta_1}{4(\frac{1}{2} - \eta_1 \frac{1}{2}) (\eta_1 + 1)} \frac{\frac{\eta_1 + 2}{2}}{2(\frac{1}{2} - \eta_1 \frac{1}{2}) (\eta_1 + 1)} \frac{1}{M_{01}} \]

\[ = \frac{h_1^2}{(\frac{1}{2} \eta_1 + 1)[(\frac{1}{2} \eta_1 + 1)(\eta_1 + 2)]} \frac{5/2}{\eta_1 + 2} \frac{h_1^2}{h_2^2} \frac{1/2}{(\eta_1 + 2)} \frac{1/2}{(\eta_1 + 2)} \frac{h_1^2}{h_2^2} \]

\[ - \frac{\eta_1 + 2}{(\frac{1}{2} - \eta_1 \frac{1}{2})} \frac{w_2 - w_1}{p_2 - p_1} \]
\( \eta = 2 : \)

\[
C_{31} = -\frac{\sqrt{2}}{6} \frac{Q_{0II}}{Q_{0I}} + \frac{1}{6} \frac{M_{0II}}{M_{0I}}

= \frac{h_1^2}{h_2^2} \frac{h_1^{5/2}}{h_2^{5/2}} \frac{h_1^2}{h_2^2} \frac{1}{2} \frac{h_1^2}{h_2^2} \frac{3(2-1/2+1)[2\sqrt{2}(h_{1/2}/h_2^{1/2}) (h_{1/2}+1) + 3h_1^{1/2} + 1]}{h_2^2} (w_{p2}-w_{p1})
\]

\( H_{11} \)

\[ (2: 42) \]

\[ C_{41} = -\frac{\sqrt{2} \beta_1}{6} \frac{Q_{0II}}{Q_{0I}} - \frac{\sqrt{2} \beta_1}{6} \frac{M_{0II}}{M_{0I}} \]

\[
= \frac{h_1^2}{h_2^2} \frac{h_1^{5/2}}{h_2^{5/2}} \frac{h_1^2}{h_2^2} \frac{1}{2} \frac{h_1^2}{h_2^2} \frac{3(2-1/2+1)[2\sqrt{2}(h_{1/2}/h_2^{1/2}) (h_{1/2}+1) + \sqrt{2}(h_1^{1/2} + 1)](w_{p2}-w_{p1})}{h_2^2} \]

\[ H_{11} \]

\[ (2: 43) \]

From the above, the non-dimensional integration constants can be written as,

\[
0 \leq \eta < 2 : \]

\[
C_{31} = \frac{\eta^2}{(w_{p2}-w_{p1})} = \frac{h_1^2}{h_2^2} \frac{h_1^{5/2}}{h_2^{5/2}} \frac{h_1^2}{h_2^2} \frac{1}{2} \frac{h_1^2}{h_2^2} \frac{3(2-1/2+1)[(h_{1/2}/h_2^{1/2}) (h_{1/2}+1) (h_{1/2}+2) (h_{1/2}+2)]}{h_2^2} + \frac{h_1^2}{h_2^2} \frac{h_1^{5/2}}{h_2^{5/2}} \frac{h_1^2}{h_2^2} \frac{1}{2} \frac{h_1^2}{h_2^2} \frac{3(2-1/2+1)(h_{1/2}+1)(h_{1/2}+2) + \eta^2}{h_2^2} \]

\[ H_{11} \]

\[ (2: 43) \]

\[
C_{41} = \frac{\eta^2}{(w_{p2}-w_{p1})} = \frac{h_1^2}{h_2^2} \frac{h_1^{5/2}}{h_2^{5/2}} \frac{h_1^2}{h_2^2} \frac{1}{2} \frac{h_1^2}{h_2^2} \frac{3(2-1/2+1)[(h_{1/2}/h_2^{1/2}) (h_{1/2}+1) (h_{1/2}+2) + \eta^2]}{h_2^2} + \eta_1(h_{1/2}+1)(h_{1/2}+2) - (h_{1/2}+2)(h_{1/2}+2) \]

\[ (2-\eta_1) H_{11} \]

\[ (2: 43) \]
\[ \eta = \frac{2}{h_2} \]

\[ \overline{C_{71}} = \frac{C_{71}}{(w_{p2} - w_{p1})} = \frac{h_1^2}{h_2} \frac{3(2^{1/2} + 1)}{2(\sqrt{2} + 1)} \frac{h_1^{5/2}}{h_2^{3/2}} \frac{1}{h_1^2} + 1 \]

\[ \overline{C_{81}} = \frac{C_{81}}{(w_{p2} - w_{p1})} = \frac{h_1^2}{h_2} \frac{3(2^{1/2} + 1)}{2(\sqrt{2} + 1)} \frac{h_1^{5/2}}{h_2^{3/2}} \frac{1}{h_1^2} + 1 \sqrt{2(\sqrt{2} + 1)} \]

2.5.2 Non-dimensional deflection and moment equations: We have already arbitrarily picked cylinder one to be the thinner cylinder and hence the higher stressed of the two cylinders, therefore the cylinder of interest (6, p. 32).

The deflection and moment equations from section 2.3 with the boundary conditions in section 2.4 are,

\[ 0 \leq \eta < 2 ; \]

\[ w_{x1} = e^{-(\frac{\eta}{2}) \beta_1 x} \left\{ \frac{C_{31}}{2-\eta_1} \frac{1}{\beta_1} x \right\} + C_{41} \sin \left\{ \frac{2-\eta_1}{2} \beta_1 x \right\} \]

\[ M_{x1} = D_1 \beta_1^2 e^{-(\frac{\eta}{2}) \beta_1 x} \left\{ \frac{\eta_1}{2} \frac{1}{\beta_1} x \right\} \cos \left\{ \frac{2-\eta_1}{2} \beta_1 x \right\} \]

\[ + \left\{ \frac{4-\eta_1}{2} C_{31} + \eta_1 C_{41} \right\} \sin \left\{ \frac{2-\eta_1}{2} \beta_1 x \right\} \]
\[ n = 2: \]

\[ w_{xI} = e^{-\sqrt{2} \beta_1 x} (C_{71} + xC_{81}) \]  

(2.46)

\[ M_{xI} = D_{1} \beta_{1} e^{-\sqrt{2} \beta_1 x} [2\beta_1 C_{71} + (2\beta_1 x - 2\sqrt{2})C_{81}] . \]

Where \( w_{xI} \) and \( w_{xII} \) are the deflections due to the discontinuity loads only.

In order to non-dimensionalize the above equations we substitute the non-dimensional form of the integration constants into the equations and then multiply the resulting deflection equations by \( \frac{Eh_{1}}{p_{a}0} \) and the moment equation by \( \frac{1}{p_{a}0 h_{1}} \). When the preceding is done we have,

\[ 0 \leq \eta < 2 : \]

\[ \frac{w_{xI}}{p_{a}0} = e^{-(\frac{1}{2}) \beta_{1} x} \left[ \frac{\eta_{1} + 2^{\frac{1}{2}}}{C_{31}} \cos \left( \frac{\eta_{1}^{2}}{2} \right) \beta_{1} x \right] + C_{41} \sin \left( \frac{\eta_{1}^{2}}{2} \right) \beta_{1} x \]

(2.47)

\[ \frac{M_{xI}}{p_{a}0 h_{1}} = e^{-(\frac{1}{2}) \beta_{1} x} \left[ \frac{\eta_{1} + 2^{\frac{1}{2}}}{C_{31}} \cos \left( \frac{\eta_{1}^{2}}{2} \right) \beta_{1} x \right] + \left( 4 - \eta_{1}^{2} \right) C_{31} + \frac{2^{\frac{1}{2}}}{C_{41}} \sin \left( \frac{\eta_{1}^{2}}{2} \right) \beta_{1} x \]

\[ \left[ \frac{4(1-\mu) h_{1}^{2}}{3(1-\mu) \frac{h_{1}}{2}^{2}} \right] \]
These are the non-dimensional deflection and moment equations we have sought.

2.6 Non-Dimensional Stress Equations

All the work to this point has been with the thought of obtaining the non-dimensional equations giving the stress distribution in the cylinder.

Non-dimensionalizing the stress equations given in section 2.4 can probably be done in several ways but we choose to non-dimensionalize them by multiplying through by \( \frac{h}{\rho a_0} \). This will then give a direct comparison with nominal hoop stress.

By so doing we have:

\[
\eta = 2:
\]

\[
\frac{w_{xII}}{\rho a_0^2} = \frac{E \frac{h}{\rho a_0}}{\frac{1}{h_2} - 1} \left( \frac{\beta_1}{\sqrt{\rho a_0}} \left[ \frac{1}{C_7} + \beta_1 x \frac{C_8}{h_2} \right] \left( 1 - \frac{\mu}{2} \right) \left( \frac{h_1}{h_2} - 1 \right) \right)
\]

\[
\frac{M_{xII}}{\rho a_0 h_1} = \frac{M_{xII}}{\rho a_0 h_1} = e^{-\sqrt{2}\beta_1 x} \left[ \frac{C_7}{1 + x / \sqrt{2}C_8} \right] \left[ 1 - \frac{\mu}{2} \right] \left( \frac{h_1}{h_2} - 1 \right).
\]

(2:48)
Non-dimensional meridional stress:

\[
\sigma_x = \frac{h \sigma_x}{\rho a_o} = \frac{N x}{\rho a_o} + \frac{6 M_x}{\rho a_o h}
\]

(2:49)

Non-dimensional circumferential stress:

\[
\sigma_\theta = \frac{h \sigma_\theta}{\rho a_o} = \frac{N \theta}{\rho a_o} + \frac{E h w}{\rho a_o^2} \pm \mu \frac{6 M_x}{\rho a_o h}
\]

Let us continue to choose cylinder one the thinner of the two cylinders and therefore the higher stressed of the two. By substituting the non-dimensional deflection and moment equations from section 2:5:2 we have the final non-dimensional stress equations that we have sought:

\[
0 < \eta < 2:
\]

Non-dimensional meridional stress:

\[
\sigma_{xI} = \frac{h_1 \sigma_{xI}}{\rho a_o} = \frac{1}{2} \pm \frac{6 M_{xI}}{\rho a_o h}
\]

(2:50)

Non-dimensional circumferential stress:

\[
\sigma_{\theta I} = \frac{h_1 \sigma_{\theta I}}{\rho a_o} = 1 + \frac{w_{xI}}{\rho a_o} \pm \mu \frac{6 M_{xI}}{\rho a_o h}
\]
$\eta = 2$:

Non-dimensional meridional stress:

$$\frac{-\bar{\sigma}_{xII}}{pa_0} = \frac{h_1}{h_2} \frac{\sigma_{xII}}{pa_0} = \frac{1}{2} \pm \frac{6M}{h_2}$$

(2.51)

Non-dimensional circumferential stress:

$$\frac{-\bar{\sigma}_{\theta II}}{pa_0} = \frac{h_1}{h_2} \frac{\sigma_{\theta II}}{pa_0} = 1 + \frac{w_{xII}}{6M} \pm \mu$$

Note that the stress equations are functions of only three non-dimensional variables: $\frac{h_1}{h_2}$, $\eta_1$ and $\beta_1 x$:

$0 \leq \eta < 2$:

$$\frac{-\bar{\sigma}_{xI}}{h_2} = F_{xI} \left( \frac{h_1}{h_2}, \eta_1, \beta_1 x \right)$$

(2.52)

$$\frac{-\bar{\sigma}_{\theta I}}{h_2} = F_{\theta I} \left( \frac{h_1}{h_2}, \eta_1, \beta_1 x \right)$$

$\eta = 2$:

$$\frac{-\bar{\sigma}_{xII}}{h_2} = F_{xII} \left( \frac{h_1}{h_2}, \eta_1, \beta_1 x \right)$$

(2.53)

$$\frac{-\bar{\sigma}_{\theta II}}{h_2} = F_{\theta II} \left( \frac{h_1}{h_2}, \eta_1, \beta_1 x \right)$$
The stress equations can be solved for the practical problem since the geometry of the cylinders will be known. But, since we wish to present to the designer a quick and accurate means of checking his designs, the computer offered the best means of obtaining numerical results from the stress equations for the design curves. Details on how the computer was used, and the non-dimensional design curves are presented in a later chapter.
CHAPTER III

DIFFERENTIAL EQUATIONS FOR THE HEMISPHERE, AND
STRESSES NEAR ITS JUNCTURE WITH A CYLINDER

Solutions for stresses in the immediate areas of geometrical discontinuity of the type found in end closures of cylindrical pressure vessels will be treated in this chapter. End closures for cylindrical pressure vessels may vary in extremes from flat plates to inverted cones, but probably the most commonly used form is the hemisphere. Analysis will be limited to a hemispherical closure for a cylindrical shell that is loaded by a constant internal pressure.

The method of analysis to be followed has been outlined in the introduction of the previous chapter.

3.1 Description of the Cylinder-Hemisphere Juncture

Figure 3.1 shows a sketch of a cylinder-hemisphere juncture. Spherical coordinates will be used for the hemisphere while the use of the coordinates introduced in Chapter II will be continued for the cylindrical portion. No difficulties from the use of two different coordinate systems will result if their use is limited to their respective regions. For the hemisphere, \( \phi \) will be the meridional variable while \( x \) will denote the meridional variable for the cylinder. The radius is a constant for both shells, and identical. Small deflection
theory is assumed to hold and both shells are assumed to be thin. Thickness for the hemisphere and cylinder are each constant but not necessarily equal to each other. We will require their middle surfaces at the juncture to coincide for all cases. The cylindrical section of the pressure vessel will be assumed to be in the long cylinder range so that interaction between the ends does not exist.

![Diagram](image)

Figure 3: 1

3:2 Governing Equations for the Hemisphere

Equations in this section have been derived in detail by Cline (3); therefore only a brief outline of how the final equations are obtained is presented.
Cline's study includes the pressure coupling effects in spherical shells using assumptions comparable to those used in the preceding chapter for the cylinder. Therefore, the exactness of the equations may be expected to be comparable.

Changes in the notations used by Cline (3) will be made as follows,

<table>
<thead>
<tr>
<th>Notation used here</th>
<th>Notation used by Cline</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_\phi$</td>
<td>$Q$</td>
</tr>
<tr>
<td>$Q_\phi$</td>
<td>$H_2 \sin \phi$</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>$\eta$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$\beta$</td>
</tr>
<tr>
<td>$\mu$</td>
<td>$\nu$</td>
</tr>
</tbody>
</table>

in order to keep the notations from overlapping those already used in Chapter II. Their definitions are listed under notations at the start of this paper.

The governing pair of differential equations for the sphere that Cline has derived is,

$$\left\{ L - \lambda^2 \left[ \rho - \left( \rho^2 - 1 \right)^{\frac{1}{2}} \right] \right\} \frac{G_\phi}{E_h} = 0$$

(3:2)

$$\left\{ L - \lambda^2 \left[ \rho + \left( \rho^2 - 1 \right)^{\frac{1}{2}} \right] \right\} \frac{G_\phi}{E_h} = 0 .$$

Where $G_\phi$ is the transverse shear stress resultant and the
differential operator $L$ and the parameters $\rho$ and $\lambda$ are defined as,

$$L = \frac{d^2}{d\phi^2} + \cot \phi \frac{d}{d\phi} - \cot^2 \phi$$

$$\rho = \frac{p\alpha_0}{4Eh} \lambda^2 = \frac{n}{2}$$

$$\lambda = [12(1-\mu^2)]^{\frac{1}{4}} \left(\frac{\mu}{h}\right)^{\frac{1}{2}} = \sqrt{2} a_0 \beta$$

When the substitution,

$$\frac{G}{Eh} = \left(\frac{\phi}{\sin \phi}\right)^{\frac{1}{2}} \xi(\phi)$$

is made the equations (3.2) become

$$\frac{d^2 \xi}{d\phi^2} + \frac{1}{\phi} \frac{d\xi}{d\phi} - \left\{ \frac{1}{\phi^2} + \frac{2}{\phi^2} + \lambda^2 [\rho - (\rho^2 - 2 \frac{1}{2})] \right\} \xi = 0$$

is a form of Bessel's equation for which the solutions are,
\( \rho < 1 \) \((\eta < 2)\):

\[
\zeta = A_0 \left\{ R[I_0(z)] + iJ[I_0(z)] \right\} + B_0 \left\{ R[K_0(z)] + iJ[K_0(z)] \right\} + \frac{1}{2} \tan^{-1} \left[ \left( 1 - \rho^2 \right)^{\frac{1}{2}} / \rho \right] \quad z = \lambda \phi \tag{3.5}
\]

\( \rho \geq 1 \) \((\eta \geq 2)\):

\[
\zeta = C_{01} I_0(z_1) + C_{02} I_0(z_2) + D_{01} K_0(z_1) + D_{02} K_0(z_2) + \frac{1}{2} \frac{1}{2} \lambda \phi \quad z_{1,2} = \left[ \rho \pm (\rho^2 - 1)^{\frac{1}{2}} \right] \lambda \phi \tag{3.6}
\]

Where \( A_0, B_0, C_{01}, C_{02}, D_{01}, D_{02} \) are constants of integrations and \( R, J \), are standard complex number notations signifying real and imaginary parts. \( I_0, K_0 \) are the modified Bessel's functions \((5, p. 61-64)\). The dot (\( \dot{\quad} \)) denotes derivatives with respect to the arguments.

The singularity of \( K_0(z) \) at \( \phi = 0 \) requires that the constants \( B_0, D_{01}, D_{02} \) be set equal to zero.

From this point we shall retain only the region covered by \( \rho \) less than one \((\eta < \text{less than two})\), for the reasons similar to those given in Chapter II. Also, only the solutions pertaining to the area \( \phi \) between zero and \( \alpha \) as given by Cline will be retained since our problem of the hemisphere is within this area.

Redefining some of the terms as follows will help in substituting into the equations for the stress resultants;
$$j^1 [l_0 (z)] = \sum_{m=0}^{\infty} \frac{1}{m!} \left( \frac{\lambda \phi}{2} \right)^{2m} \sin \left[ m \tan^{-1} \left( \frac{1-\rho}{\rho} \right)^{1/2} \right]$$

$$I (z) = j^1 [l_0 (z)] = \sum_{m=1}^{\infty} \frac{m}{(m-1)!} \left( \frac{\lambda \phi}{2} \right)^{2m-1} \sin \left[ (2m-1) \tan^{-1} \left( \frac{1-\rho}{\rho} \right)^{1/2} \right]$$

$$R (z) = j^1 [l_0 (z)] = \sum_{m=1}^{\infty} \frac{m}{(m-1)!} \left( \frac{\lambda \phi}{2} \right)^{2m-1} \cos \left[ (2m-1) \tan^{-1} \left( \frac{1-\rho}{\rho} \right)^{1/2} \right]$$

$$U = \left( \frac{\phi}{\sin \phi} \right)^{1/2} \cdot R (z)$$

$$V = \left( \frac{\phi}{\sin \phi} \right)^{1/2} \cdot I (z)$$

Also let,

$$G = \frac{\phi \theta}{Eh} + \left( \frac{\phi}{\sin \phi} \right)^{1/2} \zeta (\phi) = AU + BV$$

Here A and B are the redefined constants of integration and we have inserted the equivalent series representation (2, p.5,79) for the modified Bessel's functions. Now, by substituting the solution given by Equation (3:4) back into the equations as found in (3) by Cline for the meridional stress resultants, rotation of the meridian, and
deflection, and neglecting small quantities the following equations are obtained,

\[
\frac{Q_\phi}{Eh} = - \left( \frac{G + N_\phi \gamma_s}{Eh} \right) = - \left[ (1-\rho^2^2)(AU+BV) + 2\rho(1-\rho^2^2)^{\frac{1}{2}}(BU-AV) \right] \quad (3:8)
\]

\[
M_\phi = \frac{a_0 Eh}{\lambda^2} \left\{ (U' + \mu U \cot \phi) \left[ \rho A - (1-\rho^2^2)^{\frac{1}{2}} B \right] + (V' + \mu V \cot \phi) \left[ \rho B + (1-\rho^2^2)^{\frac{1}{2}} A \right] \right\} \quad (3:9)
\]

\[
\gamma_s = \lambda^2 \left[ (1-\rho^2^2) (BU-AV) - \rho (AU+BV) \right] \quad (3:10)
\]

\[
w_\phi = -a_0 \sin \phi \left[ (1-2\rho^2^2)(AU'+BV') + 2\rho(1-\rho^2^2)^{\frac{1}{2}}(BU'-AV') \right. \\
- \mu(\cot \phi)(1-2\rho^2^2)(AU+BV) - 2\mu(\cot \phi)(1-\rho^2^2)^{\frac{1}{2}}(BU-AV) \right] \quad (3:11)
\]

Where \((\ ))'\) denotes derivative with respect to \(\phi\), and \(M_\phi\), \(Q_\phi\), \(N_\phi\) are the stress resultants with \(\gamma_s\), \(w_\phi\) denoting the rotation of the meridian and transverse deflection of the middle surfaces respectively. \(A\) and \(B\), the integration constants, are evaluated from Equations (3:8) and (3:9) at the edge of the shell \((\phi = \alpha)\).

\[
A = \frac{1}{\Delta} \left\{ (1-2\rho^2^2) I + 2\rho(1-\rho^2^2) R \right\} \frac{\lambda}{a_0} \frac{M_\phi}{Eh(\frac{\alpha}{\sin \alpha})^2} \left[ \frac{1}{\lambda} \frac{H_2 e \sin \alpha}{Eh(\frac{\alpha}{\sin \alpha})^\frac{1}{2}} \right] \quad (3:12)
\]

\[
B = \frac{1}{\Delta} \left\{ 2\rho(1-\rho^2^2) I - (1-2\rho^2^2) R \right\} \frac{\lambda}{a_0} \frac{M_\phi}{Eh(\frac{\alpha}{\sin \alpha})^2} \left[ \frac{1}{\lambda} \frac{H_2 e \sin \alpha}{Eh(\frac{\alpha}{\sin \alpha})^\frac{1}{2}} \right] \quad (3:13)
\]
Where,

$$\Delta = (1 - 2\rho)(\frac{1 + \rho}{2}) (R\dot{I} - IR) + (1 + 2\rho)(\frac{1 - \rho}{2}) (RR + II) + (1 - \rho)^2 \frac{f(\alpha)}{\lambda\alpha} (R^2 + I^2)$$  \hfill (3.14)

and

$$f(\alpha) = (\mu - \frac{1}{2})\alpha \cot \alpha - \frac{1}{2}$$ \hfill (3.15)

$$g(\alpha) = (\mu + \frac{1}{2})\alpha \cot \alpha + \frac{1}{2}$$

$I, I, R, R$ appearing without an argument should be interpreted to mean they are evaluated at $\phi$ equal to $\alpha$; also the subscript "e" denotes the edge where $\phi$ is equal to $\alpha$.

Cline evaluates the equations for the rotation of the meridian and radial deflection at $\phi$ equal to $\alpha$ to obtain the edge rotation of meridian and radial deflection in terms of the edge stress resultants thus defining the influence coefficients $C_{ij}$:

$$w_{0e} = C_{11}Q_{\phi e} + C_{12}M_{\phi e}$$ \hfill (3.16)

$$\gamma_{0e} = C_{21}Q_{\phi e} + C_{22}M_{\phi e}$$ \hfill (3.17)

The influence coefficients for the shell, determined by substituting Equations (3.12) and (3.13) into Equations (3.10) and (3.11),
The stress equations will be written in terms of the tensile loads, moment, and deflection.

The equation giving meridional stress is

\[ \sigma_\phi = \frac{N_\phi}{h} \pm \frac{6 M_\phi}{h^2} \] (3:21)

and that giving circumferential stress is

\[ \sigma_\theta = \frac{N_\theta}{h} + \frac{Ew}{a_0} \pm \mu \frac{6 M_\phi}{h^2} \]

Figure 3:2 shows the pressure equilibrating stresses and the moment stress. Not shown in the figure is the component represented by the term \( \frac{Ew}{a_0} \), the stress due to the extension of the circumference resulting from the discontinuity moment and shear, and the component represented by the term \( \mu \frac{6 M_\phi}{h^2} \) is the stress.
due to the circumferential restraint preventing anticlastic curvature.

In a later section we shall use the above solutions as found by Cline (3) together with the equations for the cylinder in the next section to solve the boundary value problem involving a cylinder-hemisphere juncture. Non-dimensional stress distribution equations for the hemisphere will then be found and solved.

3.3 Governing Equations for the Cylinder

The complete set of equations for the cylinder have already been found in Chapter II. By noting the positive convention for the stress resultants, deflections and rotations that was used in their derivation and changing them as a convenience for this problem, we will
be able to use those equations here.

Figure 3:3 shows the positive convention for the stress resultants to be used for this chapter and matches those used by Cline for the hemisphere.

From the comparison of Figures 3.3 and 2.2 we see that the only change in sign required before using the equations from Chapter II is in the shear terms. This change in the equations derived for cylinder two in Figure 2.2 makes them the required equations for the cylinder in Figure 3:3. The equations are,

\[ w_c = e^{-\left(\frac{\pi + \frac{2}{2}}{2}\right)^{\frac{1}{2}} \beta x} \left[ C_3 \cos\left(\frac{2-\pi}{2}\right)^{\frac{1}{2}} \beta x + C_4 \sin\left(\frac{2-\pi}{2}\right)^{\frac{1}{2}} \beta x \right] \]  

(3.22)
\[ Y_c = \frac{dw}{c dx} = e^{-\left(\frac{\eta + 2}{2}\right)^{\frac{1}{2}} \beta x} \beta \left\{ \left(\frac{\eta + 2}{2}\right) C_3 - \left(\frac{2-\eta}{2}\right)^{\frac{1}{2}} C_4 \right\} \cos\left(\frac{2-\eta}{2}\right)^{\frac{1}{2}} \beta x \]

\[ + \left\{ \left(\frac{2-\eta}{2}\right)^{\frac{1}{2}} C_3 + \left(\frac{\eta + 2}{2}\right)^{\frac{1}{2}} C_4 \right\} \sin\left(\frac{2-\eta}{2}\right)^{\frac{1}{2}} \beta x \] (3:23)

\[ M_{xc} = \frac{d^2 w}{c dx^2} = e^{-\left(\frac{\eta + 2}{2}\right)^{\frac{1}{2}} \beta x} D\beta^2 \left\{ \eta C_3 - (4-\eta) C_4 \right\} \cos\left(\frac{2-\eta}{2}\right)^{\frac{1}{2}} \beta x \]

\[ + \left\{ (4-\eta) C_3 + \eta C_4 \right\} \sin\left(\frac{2-\eta}{2}\right)^{\frac{1}{2}} \beta x \] (3:24)

\[ Q_{xc} = \left[ N_x Y + D\frac{c}{dx} \right] = e^{-\left(\frac{\eta + 2}{2}\right)^{\frac{1}{2}} \beta x} \left\{ -\left(\frac{\eta + 2}{2}\right)^{\frac{1}{2}} \beta x \right\} \right\} + \left\{ -\left(\frac{\eta + 2}{2}\right)^{\frac{1}{2}} \beta C_3 \right\} \cos\left(\frac{2-\eta}{2}\right)^{\frac{1}{2}} \beta x \]

\[ -\left(\frac{\eta + 2}{2}\right)^{\frac{1}{2}} \beta C_4 \right\} \sin\left(\frac{2-\eta}{2}\right)^{\frac{1}{2}} \beta x \] (3:25)

Evaluating Equations (3:23) and (3:24) at the juncture where

\[ x \text{ is equal to zero}, \]

enables the determining of the integration constants \( C_3 \) and \( C_4 \)

\[ C_3 = -\frac{\eta + 2}{2} \left(\frac{\eta + 2}{2}\right)^{\frac{1}{2}} \right\} Q_0 + \frac{1}{2\beta^2 (\eta + 1)} M_0 \] (3:26)

\[ C_4 = -\frac{\eta}{4\beta^2 (\eta + 1)^{\frac{1}{2}}} \left(\frac{2-\eta}{2}\right)^{\frac{1}{2}} \right\} Q_0 - \frac{\left(\frac{\eta + 2}{2}\right)^{\frac{1}{2}}}{2\beta^2 (\eta + 1)^{\frac{1}{2}}} M_0 \] (3:27)
Substituting $C_3$ and $C_4$ in Equations (3:22) and (3:23) and evaluating the resulting equations for $x$ equal to zero, gives the juncture deflection and rotation as,

$$w_{0c} = -\frac{(\eta + 2)^{\frac{1}{2}}}{2D\beta^3(\eta+1)} Q_0 + \frac{1}{2D\beta^2(\eta+1)} M_0 + w_p$$  \hspace{1cm} (3:28)$$

$$\gamma_{0c} = -\frac{1}{2D\beta^2(\eta+1)} Q_0 + \frac{(\eta + 2)^{\frac{1}{2}}}{D\beta(\eta+1)} M_0$$  \hspace{1cm} (3:29)$$

or,

$$w_{0c} = S_{11} Q_0 + S_{12} M_0 + w_p$$  \hspace{1cm} (3:30)$$

$$\gamma_{0c} = S_{21} Q_0 + S_{22} M_0$$  \hspace{1cm} (3:31)$$

The $S_{ij}$'s are the influence coefficients.

$$S_{11} = -\frac{(\eta + 2)^{\frac{1}{2}}}{2D\beta^3(\eta+1)}$$  \hspace{1cm} (3:32)$$

$$S_{12} = -S_{21} = \frac{1}{2D\beta^2(\eta+1)}$$  \hspace{1cm} (3:33)$$

$$S_{22} = \frac{(\eta + 2)^{\frac{1}{2}}}{D\beta(\eta+1)}$$  \hspace{1cm} (3:34)$$

The applicable stress equations for the cylinder has been described in section 2:4 as,
meridional stresses

\[
\sigma_x = \frac{N}{h} \pm \frac{6M}{h^2}
\]

(2.39)

and circumferential stresses

\[
\sigma_\theta = \frac{N_\theta}{h} + \frac{Ew}{a_0} \pm \frac{6M_\theta}{h^2}.
\]

Now that all the pertinent equations for the cylinder have also been defined, the hypothetically separated cylinder and hemisphere can be properly joined by the equations of compatibility at the juncture. This allows the juncture stress resultants to be determined.

3.4 Cylinder-Hemisphere Juncture Stress Resultants

The deflection and rotation at the juncture of the cylinder and hemisphere have been determined in terms of the juncture stress resultants and influence coefficients in the previous two sections.

Hemisphere:

\[
w_{0s} = C_{11}O_{0s} + C_{12}M_{0s} + w_{ps}
\]

(3.16a)

\[
\gamma_{0s} = C_{21}O_{0s} + C_{22}M_{0s}
\]

(3.17)
Cylinder:

\[ w_{0c} = S_{11} Q_{0c} + S_{12} M_{0c} + w_{pc} \]  \hspace{1cm} (3:30)

\[ \gamma_{0c} = S_{21} Q_{0c} + S_{22} M_{0c} \]  \hspace{1cm} (3:31)

where use of the identities:

\[ Q_{0s} = Q_{\phi e} \]
\[ M_{0s} = M_{\phi e} \]  \hspace{1cm} (3:35)
\[ w_{0s} = w_{\phi e} + w_{ps} \]

has been made, and \( w_{ps} \) is the pressure deflection of the sphere under internal pressurization.

From the conditions of compatibility, and Newton's Third Law, the following relations must hold,

\[ w_{0s} = w_{0c} = w_0 \]
\[ \gamma_{0s} = \gamma_{0c} = \gamma_0 \]  \hspace{1cm} (3:36)
\[ Q_{0s} = -Q_{0c} = Q_0 \]
\[ M_{0s} = M_{0c} = M_0 \]

Therefore, by equating the cylinder-hemisphere juncture deflections and rotations, we have that,
Here again, as in Chapter II, the main goal is to obtain non-dimensional stress curves which will allow the design analyst a means of quickly checking designs with a high degree of accuracy. Dimensionless stress equations will make this possible. But, before these equations are obtained, it will be convenient to non-dimensionalize other terms and equations.

3:5:1 Non-dimensional influence coefficients: The influence coefficients have been found in sections 3:2 and 3:3 for the hemisphere and cylinder respectively. As the first step in non-dimensionalizing the influence coefficients, let us substitute the following into their defining equations in sections 3:2 and 3:3,
\[ \alpha = \frac{\pi}{2} \]

\[ \lambda = \lambda_s \]

\[ \eta = \eta_c = \left(\frac{h}{h_c}\right)^2 \eta_s \]

\[ D = \frac{E h^3 c}{12(1-\mu^2)} \]

\[ \beta = \frac{\lambda_c}{\sqrt{2a_0}} = \left(\frac{h}{2h_c}\right)^{\frac{1}{2}} \frac{\lambda_s}{a_0} \]

This will give the following relations,

**Hemisphere:**

\[
C_{11} = \left[ \frac{2}{\lambda_s \pi} \left( \frac{2-\eta_s}{2} \right) (IR - RI) \left( \frac{4}{4} \right) - \frac{2+\eta_s}{2} \left( \frac{2}{2} \right) (II + RR) \left( \frac{4}{4} \right) \right]
\]

\[
+ \frac{1}{\lambda_s \pi^2} \left( \frac{4-\eta_s}{4} \right) \left( I^2 + R^2 \right) + \left( \frac{4-\eta_s}{4} \right) \left( I^2 + R^2 \right) \frac{\lambda_s a_0}{E h_s \Delta} \]

\[
C_{12} = -C_{21} = \left[ \frac{2+\eta_s}{2} \left( \frac{4}{4} \right) (IR - RI) + \left( \frac{4}{4} \right) (II + RR) \right] \frac{\lambda_s^2}{E h_s \Delta} \]

\[
C_{22} = - \left[ \frac{4-\eta_s}{4} \left( I^2 + R^2 \right) \right] \frac{\lambda_s^3}{E h_s a_0 \Delta} \]
Cyliner:

\[
S_{11} = - \frac{\lambda_c a_0}{\text{Eh}_c} \left( \frac{h_s}{h_c} \right)^{\frac{1}{2}} \left( \frac{-\eta_s + 2}{\eta_s + 1} \right) \left( \frac{h_s}{h_c} \right)^{\frac{1}{2}}
\]

\[
S_{12} = - S_{21} = \frac{1}{\text{Eh}_c} \left( \frac{h_s}{h_c} \right)^{\frac{1}{2}} \left( \frac{-\eta_s + 2}{\eta_s + 1} \right) \left( \frac{h_s}{h_c} \right)^{\frac{1}{2}}
\]

\[
S_{22} = \frac{\lambda_c^3}{\text{Eh}_c a_0} \left( \frac{h_s}{h_c} \right)^{\frac{1}{2}} \left( \frac{-\eta_s + 2}{\eta_s + 1} \right) \left( \frac{h_s}{h_c} \right)^{\frac{1}{2}}
\]

We now define the non-dimensional influence coefficients as,

Hemisphere:

\[
\bar{C}_{11} = \left\{ \frac{1}{\lambda_s \pi} (2 - \eta_s) (\text{IR-RI}) \left( \frac{1}{4} \right) - \frac{1}{\lambda_s \pi} (2 + \eta_s) (\text{II+RR}) \left( \frac{1}{4} \right) \right\} \frac{1}{\Delta}
\]

\[
\bar{C}_{12} = - \bar{C}_{21} = \left\{ \frac{2 + \eta_s}{4} (\text{IR-RI}) + \frac{2 - \eta_s}{4} (\text{II+RR}) \right\} \frac{1}{\Delta}
\]

\[
\bar{C}_{22} = - \left\{ \frac{2 - \eta_s}{4} (\text{II+RR}) \right\} \frac{1}{\Delta}
\]
3.5.2 Non-dimensional juncture stress resultants and integration

constants: With the influence coefficients now dimensionless, the non-dimensionalizing of the juncture stress resultants can be accomplished.

From section 3.4 we have,

\[
M_0 = -\frac{p a^2}{2 E h} \frac{h}{c_s} \left[ (2-\mu) - \frac{h}{c_s} (1-\mu) \right] \left( C_{21} - S_{21} \right)
\]

\[
Q_0 = -\frac{p a^2}{2 E h} \frac{h}{c_s} \left[ (2-\mu) - \frac{h}{c_s} (1-\mu) \right] \left( C_{22} - S_{22} \right)
\]

By substituting the non-dimensional influence coefficients the above equations become,
Then the non-dimensional juncture stress resultants are,

\[
\overline{M}_0 = \frac{1}{12(1-\mu^2)^{\frac{1}{2}}} \frac{h^2}{\left[\left(\frac{h}{h_s}\right)^2 - \left(\frac{c}{h_s}\right)^2\right]} \left[\left(\frac{c}{h_s}\right)^2 \overline{S}_{22} - \overline{S}_{21}\right]
\]

\[
\overline{Q}_0 = \frac{h^2}{\lambda_c} \left[\left(\frac{h}{h_s}\right)^2 - \left(\frac{c}{h_s}\right)^2\right] \left[\left(\frac{h}{h_s}\right)^2 \overline{S}_{22} - \overline{S}_{21}\right]
\]

The integration constants can now be non-dimensionalized by substituting the non-dimensional juncture stress resultants in
Equations (3:12) and (3:13) for the hemisphere and Equations (3:26) and (3:27) for the cylinder. By so doing we have,

**Hemisphere:**

\[
A = \frac{p a_0}{E h_s} \left( \frac{2h c}{\pi h_s} \right)^{\frac{1}{2}} \frac{1}{\lambda_s} \frac{2-\eta_s^2}{\eta_s} \left( \frac{4-\eta_s^2}{2} \right) R \left\{ \frac{h c}{h_s} \left( 1-\mu^2 \right) \right\}^{\frac{1}{2}} \frac{M_0}{Q_0}
\]

\[
B = \frac{p a_0}{E h_s} \left( \frac{2h c}{\pi h_s} \right)^{\frac{1}{2}} \frac{1}{\lambda_s} \frac{4-\eta_s^2}{\eta_s} \left( \frac{2-\eta_s^2}{2} \right) R \left\{ \frac{h c}{h_s} \left( 1-\mu^2 \right) \right\}^{\frac{1}{2}} \frac{M_0}{Q_0}
\]

**Cylinder:**

\[
C_3 = -\frac{h_c^2}{h_s^2} \left[ \frac{(s)^2}{(s+2)} \right]^{\frac{1}{2}} \frac{p a_0}{E h_c} \frac{2}{Q_0} + \frac{h_c^2}{h_s^2} \left[ \frac{(s)^2}{(s+1)} \right] \frac{12(1-\mu^2)}{2} \frac{p a_0}{E h_c} \frac{2}{M_0} = \frac{p a_0}{E h_c} C_3 (3:26a)
\]
From the above we see that the non-dimensional integration constants that we have sought are,

**Hemisphere:**

\[
\bar{A} = \frac{2h}{\pi h_s} \left( \frac{1}{\lambda_s \Delta} \left[ \left( \frac{2-\eta_s}{2} \right) I + \eta_s \left( \frac{4-\eta_s}{4} \right) R \right] \right) \frac{h_s}{\lambda_s} \frac{h_s}{\Delta} \frac{1}{\Delta} \bar{M}_0
\]

\[
+ \left\{ \left( \frac{2-\eta_s}{4} \right) I - \left( \frac{4-\eta_s}{4} \right) R - \frac{1}{\lambda_s \pi} \left[ \frac{\eta_s}{2} - \left( \frac{4-\eta_s}{4} \right) R \right] \right\} \bar{Q}_0
\]

\[
(3: 43)
\]

\[
\bar{B} = \frac{2h}{\pi h_s} \left( \frac{1}{\lambda_s \Delta} \left[ \left( \frac{4-\eta_s}{4} \right) I - \left( \frac{2-\eta_s}{2} \right) R \right] \right) \frac{h_s}{\lambda_s} \frac{h_s}{\Delta} \frac{1}{\Delta} \bar{M}_0
\]

\[
- \left\{ \left( \frac{2-\eta_s}{4} \right) I + \left( \frac{4-\eta_s}{4} \right) R - \frac{1}{\lambda_s \pi} \left[ \frac{\eta_s}{2} - \left( \frac{4-\eta_s}{4} \right) I \right] \right\} \bar{Q}_0
\]

\[
(3: 43)
\]
Non-dimensional moment and deflection equations: We are now at a point in the non-dimensionalizing procedure where non-dimensionalizing the moment and deflection equations presented in section 3:2 and 3:3 for the hemisphere and cylinder respectively is possible.

For the hemisphere, by substituting the non-dimensional integration constants the equations become,

Moment:

\[
M_{\phi} = \frac{p a_0 h_s}{[12(1-\mu^2)]^{\frac{1}{2}}} \left\{ (U' + \mu U \cot \phi) \left[ \frac{\eta_s A}{2} - \left( -\frac{4-\eta_s^2}{4} \right)^{\frac{1}{2}} B \right] + (V' + \mu V \cot \phi) \left[ \frac{\eta_s B}{2} + \left( -\frac{4-\eta_s^2}{4} \right)^{\frac{1}{2}} A \right] \right\} = p a_0 h_s M_{\phi}
\]
Deflection:

\[ w_s = \frac{pa_0^2 \sin \phi}{Eh_s} \left\{ \frac{2-\eta_s^2}{2} (AU' + BV') + \eta_s \left( \frac{4-\eta_s^2}{4} \right) (BU' - AV') \right\} \]

\[ -\mu \cot \phi \left( \frac{2-\eta_s^2}{2} \right) (AU + BV) - \mu \eta_s \cot \phi \left( \frac{4-\eta_s^2}{4} \right) (BU - AV) \]  

(3:11a)

\[ \frac{pa_0^2}{Eh_s} \sin \phi w_s. \]

And doing similarly for the cylinder, the equations become,

**Moment:**

\[ M_{xc} = \frac{pa_0 h_c^2}{2(1-\mu)^{\frac{3}{2}}} \left[ \left( \frac{h_s}{h_c} \right) \eta_s + 2 \right] ^{\frac{1}{2}} \left( \frac{h_s^2}{2h_c} \right) \left( \lambda_s \right) \left( \frac{x}{a_0} \right) \frac{h_c^2}{h_s^2} \]

\[ -\left( \frac{4-\left( \frac{h_s}{h_c} \right) \eta_s C_4}{h_c} \right) \cos \left[ \left( \frac{h_s^2}{2h_c} \right) \left( \lambda_s \right) \left( \frac{x}{a_0} \right) \right] \]

\[ + \left( \frac{4-\left( \frac{h_s}{h_c} \right) \eta_s C_3}{h_c} + \left( \frac{h_s}{h_c} \right) \eta_s C_4 \right) \sin \left[ \left( \frac{h_s^2}{2h_c} \right) \left( \lambda_s \right) \left( \frac{x}{a_0} \right) \right] \]

\[ = \frac{pa_0 h_c^2 M_{xc}}{Eh_s} \]
Deflection:

\[
\begin{align*}
\text{Deflection: } w &= \frac{h^2}{2} \left( \frac{s}{h} \right)^{1/2} \\
&= \frac{p a_0}{E h_c} \left[ -\frac{h}{2h_c} \eta_s + 2 - \frac{h}{2h_c} \right] (\lambda_s) \left( \frac{x}{a_0} \right) \left( \frac{\lambda_s}{a_0} \right) \left( \frac{x}{a_0} \right) \\
&= \frac{2 - \frac{h}{h_c} \eta_s}{h_c} \left( \frac{s}{h} \right)^{1/2} \left( \frac{s}{2h_c} \right) (\lambda_s) \left( \frac{x}{a_0} \right) \left( \frac{s}{2h_c} \right) (\lambda_s) \left( \frac{x}{a_0} \right)
\end{align*}
\]

(3.22a)

where the equivalent sphere parameter has replaced the normal cylinder parameter \( \beta \).

We have as the non-dimensional moment and deflection equations the following,

**Hemisphere:**

\[
\begin{align*}
M &= \frac{1}{12(1-\mu)^{1/2}} \left\{ \left( U' + \mu U \cot \phi \right) - \left( \frac{4-\eta_s}{4} \right)^{1/2} \right\} \\
&\quad \times \left[ \frac{\eta_s}{2} + \left( \frac{4-\eta_s}{4} \right) \left( \frac{\eta_s}{2} \right) \right] \\
&\quad + \left( V' + \mu V \cot \phi \right) \\
&\quad \left( \frac{4-\eta_s}{4} \right) \left( \frac{\eta_s}{2} \right) \left( \frac{\eta_s}{2} \right)
\end{align*}
\]

(3.45)

\[
\begin{align*}
w_s &= -\left( \frac{2-\eta_s}{2} \right) \left( U' + BV' \right) + \left( \frac{4-\eta_s}{4} \right) \left( \frac{\eta_s}{2} \right) \left( \frac{\eta_s}{2} \right)
\end{align*}
\]

(3.46)

\[
\begin{align*}
- (\mu \cot \phi) \left( \frac{2-\eta_s}{2} \right) (A U + B V) - (\mu \eta_s \cot \phi) \left( \frac{4-\eta_s}{4} \right) (B U - A V)
\end{align*}
\]
The preliminary work is now completed and in the next section, the final goal of this chapter, the non-dimensional stress equations will be realized.

3: 6 Non-Dimensional Stress Equations

All the work to this point has been with the singular thought of obtaining non-dimensional stress equations. These equations will be
used in obtaining the non-dimensional stress distribution curves for the hemisphere and cylinder in the vicinity of their juncture. These curves will be useful to design analysts and others doing work on shells.

By observing the stress equations in sections 3:2 and 3:3 it becomes evident that non-dimensionalizing of the equations can be accomplished by multiplying each equation through by \( \frac{h_s c}{pa_0} \). Doing this the equations become,

**Hemisphere:**

**meridional**

\[
\frac{h_s \sigma_{\theta s}}{pa_0} = \frac{1}{2} \pm \frac{6M_\theta}{pa_0 h_s}
\]

(3:21a)

**circumferential**

\[
\frac{h_s \sigma_{\phi s}}{pa_0} = \frac{1}{2} + \frac{w_s E h_s}{pa_0 \sin \phi} \pm \frac{6M_\phi}{pa_0 h_s}
\]
Cylinder:

meridional

\[
\frac{h_c \sigma_{xc}}{\rho a_0} = \frac{1}{2} \pm \frac{6M_{xc}}{\rho a_0 h_c}
\]

circumferential

\[
\frac{h_c \sigma_{xc}}{\rho a_0} = 1 + \frac{w E h_c}{\rho c^2} \pm \mu \frac{6M_{xc}}{\rho a_0 h_c}.
\]

The terms of the equations involving the moment and deflection are observed to be in the non-dimensional form. Therefore, in final form the non-dimensional stress equations are,

Hemisphere:

meridional

\[
\bar{\sigma}_{\phi s} = \frac{1}{2} \pm 6M_{\phi}
\]

circumferential

\[
\bar{\sigma}_{0 s} = \frac{1}{2} + \bar{w}_s \pm \mu \bar{6M}_{\phi}
\]
Cylinder:

meridional

\[ \frac{\sigma}{\sigma_{xc}} = \frac{1}{2} \pm 6M_{xc} \]  \hspace{1cm} (3:50)

circumferential

\[ \frac{\sigma}{\theta_c} = 1 + \frac{\mu}{\omega} \pm \mu \Delta M_{xc} \]

Examining the equations for the non-dimensional moments and deflections we see that each of the non-dimensional stresses is a function of four non-dimensional parameters, therefore the following relations can be written;

Hemisphere:

meridional

\[ \frac{\sigma}{\phi_s} = f_{\phi_s} \left( \frac{h}{h_s}, \lambda_s, \eta_s, \phi \right) \]  \hspace{1cm} (3:51)

circumferential

\[ \frac{\sigma}{\theta_s} = f_{\theta_s} \left( \frac{h}{h_s}, \lambda_s, \eta_s, \phi \right) \]
Cylinder:

meridional

\[ \bar{\sigma}_{xc} = f_{xc} \left( \frac{h_c}{h_s}, \lambda_s, \eta_s, \frac{x}{a_0} \right) \] (3:52)

circumferential

\[ \bar{\sigma}_{\theta c} = f_{\theta c} \left( \frac{h_c}{h_s}, \lambda_s, \eta_s, \frac{x}{a_0} \right) \]

These non-dimensional stress equations form the basis for the design curves presented in the next chapter.
CHAPTER IV

THE DESIGN CURVES

The non-dimensional meridional and circumferential stress distribution curves will be obtained from the equations found in Chapters II and III. But, our primary interest is in the discontinuity stresses, therefore we will subtract the membrane stresses (those components not related to discontinuity moment and shear). These equations will be programmed to be solved on an IBM 1410 computer for various values of the variables. Output will be printed and punched out, with the punch cards being used to plot the curves on an IBM 1627 plotter. Both the computer and the plotter have been chosen to do the calculating and plotting because the large amount of work required in obtaining the finished curves made doing the work by hand almost impossible.

Curves for the cylinder with a thickness change have been limited to the thinner cylinder because this is the higher stressed of the two cylinders (6, p. 32). For the case of the hemispherical closure on a cylinder we will have curves for both the hemisphere and the cylinder in order to show overall stress distribution in the area of the juncture. It is of interest to see the manner in which the discontinuity stresses die out along both the hemisphere and cylinder.
For the case of the cylinder-to-hemisphere juncture the infinite series representations for the modified Bessel's functions were used in the calculations for the non-dimensional stress distribution. The problem of where to terminate the series was encountered and this problem was handled by taking the largest value of the argument in the modified Bessel's functions to be computed and using 35 terms of the series, then 50 terms of the series and comparing the results with those using 100 terms of the series.

It was found that the results using 50 and 100 terms differed only in the sixth decimal place, but those for 35 and 50 terms differed in the first decimal place. Therefore, 50 terms was chosen as the number of terms that will adequately represent the series.

The resultant curves for the three areas cited previously will be presented separately in the order,

1) Design curves for the thinner cylinder -- cylinder with a thickness change
2) Design curves for the hemisphere -- hemisphere-to-cylinder juncture
3) Design curves for the cylinder -- hemisphere-to-cylinder juncture.

Equations used in the computer program will also be listed for each
of the three areas given above in the appendices.

4:1 Design Curves for the Thinner Cylinder--Cylinder with a Thickness Change

The equations required for this computer program to calculate the non-dimensional stress distributions and a listing of the computer program are in Appendix I.

Bounds on the variables $\frac{h_1}{h_2}$, $\eta_1$, and $\beta_1x$ will be chosen for the thickness ratio as,

$$0.4 \leq \frac{h_1}{h_2} \leq 1.00$$

for the cylinder parameter,

$$0.0 \leq \eta_1 \leq 2.00$$

and the meridional cylinder parameter,

$$0.0 \leq \beta_1x \leq 4.0$$

The limits on $\eta_1$ have been set in Chapter II for the reasons given on page 11, while the thickness ratio limits have been set from trying to cover the majority of cases in thickness change that normally occur. The limits on $\beta_1x$ have been chosen after looking at preliminary computer output to determine the range in which
discontinuity loads are significant.

Three different checks on the computer results were made. One of these consisted of looking at the results for $\frac{h_1}{h_2}$ equals one for which the discontinuity stresses were zero as they should be since this value for $\frac{h_1}{h_2}$ represents a zero thickness change. Another check was made by comparing a specific curve against the curves given in (6, p. 32). The third check was to see that the stresses went to zero as coordinate values advanced away from the thickness change.

An IBM 1627 plotter was used to translate the punch cards into the non-dimensional stress curves that follow. Only the higher value for each of the meridional and circumferential stress distributions were plotted. The meridional stresses are the curves on the left side of the page and the curves on the right side of the page are the circumferential stresses. Each figure has a set of three curves which represent the three values of $\eta_1$ for each value of thickness ratio.

The design curves follow.
Figure 4.1a \( \frac{h_1}{h_2} = 0.4 \)

Outside Surface

Figure 4.1b \( \frac{h_1}{h_2} = 0.4 \)

Outside Surface
Figure 4:2a \( \frac{h_1}{h_2} = 0.5 \)
Outside Surface

Figure 4:2b \( \frac{h_1}{h_2} = 0.5 \)
Outside Surface
Figure 4.3a \( \frac{h_1}{h_2} = 0.6 \)
Outside Surface

Figure 4.3b \( \frac{h_1}{h_2} = 0.6 \)
Outside Surface
Figure 4:5a \[ \frac{h_1}{h_2} = 0.8 \]

Outside Surface

Figure 4:5b \[ \frac{h_1}{h_2} = 0.8 \]

Outside Surface
Figure 4.6a \( \frac{h_1}{h_2} = 0.9 \)

Outside Surface

Figure 4.6b \( \frac{h_1}{h_2} = 0.9 \)

Outside Surface
4:2 Design Curves for the Hemisphere--Hemisphere-to-Cylinder Juncture

We have developed the required equations for finding the non-dimensional stress distribution for the hemisphere in Chapter III. A listing of the equations required in order to write a computer program to calculate the non-dimensional stress distributions and the computer program are shown in Appendix II.

The variables are \( \frac{h_c}{h_s}, \lambda_s, \eta_s, \phi \) and the bounds that will be used are, for the sphere parameter,

\[ 6 \leq \lambda_s \leq 30 \]

while those for the cylinder parameter are,

\[ 0.001 \leq \eta_s \leq 1.951 \]

and for the meridional variable are,

\[ \frac{\pi}{4} \leq \phi \leq \frac{\pi}{2} \quad \text{or} \quad \frac{3\pi}{8} \leq \phi \leq \frac{\pi}{2}. \]

For the variable \( \frac{h_c}{h_s} \) we will limit the values to one and two in order to keep the calculating time for the computer at a reasonable length. Limits for the variable \( \lambda_s \) have been chosen to cover a specific range of values. The limits on \( \eta_s \) were set in Chapter III for the reasons given on page 11, where the upper value was less than two.
and the lower value greater than or equal to zero. In the actual calculation the lower limit had to be chosen greater than zero due to the limitations of the subroutine for the arctangent that was used. The upper limit on $\phi$ was limited by the juncture point and the lower limit was chosen after some check runs showed where the discontinuities had died out.

After the computer printed and punched out the non-dimensional stress distribution values for this case, checks on the computer output were made, examining the intermediate calculations for influence coefficients and the interface stress resultants $M_0$ and $Q_0$. Also a check, that the discontinuity effects approached zero as $\phi$ advanced away from the juncture, was made.

Curves plotted by an IBM 1627 plotter are shown in the following pages. Curves for the meridional and circumferential stresses are plotted but only for the higher value of the inside or outside stress for each set of values for the variables. The method of presenting the curves developed in section 4:1, of showing the meridional stress curves on the left and the circumferential stress curves on the right for each page, was followed. Each figure has four curves where each curve represents a value of $\eta_s$ while $\lambda_s$ and $\frac{h_c}{h_s}$ have been held constant.

Design curves for this section are shown on the following pages.
Figure 4:8a \[
\frac{h_c}{h_s} = 1.0
\]
\[
\lambda_s = 14
\]
Inside surface
Figure 4:9a  \[ \frac{h_s}{h_0} = 1.0 \]

Inside Surface

\[ \frac{h_c}{h_s} = 1.0 \]

\[ \lambda_s = 22 \]

Figure 4:9b  \[ \frac{h_s}{h_0} = 1.0 \]

Inside Surface

\[ \frac{h_c}{h_s} = 1.0 \]

\[ \lambda_s = 22 \]
Figure 4:10a  \( \frac{h_s}{h_s} = 1.0 \)
\( \lambda_s = 30 \)
Inside Surface

Figure 4:10b  \( \frac{h_s}{h_s} = 1.0 \)
\( \lambda_s = 30 \)
Inside Surface
Figure 4:11a \( \frac{h_c}{h_s} = 2 \)
\( \lambda_s = 6 \)
Outside Surface

Figure 4:11b \( \frac{h_c}{h_s} = 2 \)
\( \lambda_s = 6 \)
Outside Surface
Figure 4:12a
\[ \frac{h_s \cdot \eta_s}{\eta_s - 2} = 2.0 \]
\[ \lambda_s = 14.0 \]
Outside Surface

Figure 4:12b
\[ \frac{h_c}{h_s} = 2.0 \]
\[ \lambda_s = 14.0 \]
Outside Surface
Figure 4:13a \[ \frac{h_s}{h} = 2.0 \]
\[ \lambda_s = 22 \]

Outside Surface

Figure 4:13b \[ \frac{h}{h_s} = 2.0 \]
\[ \eta_s = 0.001 \]
\[ \eta_s = 0.651 \]
\[ \eta_s = 1.301 \]
\[ \eta_s = 1.951 \]

Outside Surface
Figure 4:14a
\[ \frac{h_c}{h_s} = 2.0 \]
\[ \lambda_s = 30 \]

Outside Surface

Figure 4:14b
\[ \frac{h_c}{h_s} = 2.0 \]
\[ \lambda_s = 30 \]

Outside Surface
4: 3 Design Curves for the Cylinder--Hemisphere-to-Cylinder Juncture

The equations required for finding the non-dimensional stress distribution for the cylinder have been found in Chapter III. Equations required for writing the computer program that will calculate the stress distribution and a listing of the computer program are shown in Appendix III.

The variable \( \frac{x}{a_0} \) replaces the variable \( \phi \) in section 4: 2 while the other three variables are the same as given in section 4: 2 and they will have the same bounds. Bounds on the variable \( \frac{x}{a_0} \) for the case \( \eta_s \) equals 1.951 and \( \frac{c}{h_s} \) equals one are,

\[
0 \leq \frac{x}{a_0} \leq 1.00
\]

or

\[
0 \leq \frac{x}{a_0} \leq 1.50
\]

For the other cases the bounds are,

\[
0 \leq \frac{x}{a_0} \leq 0.500
\]

or

\[
0 \leq \frac{x}{a_0} \leq 0.250
\]
Upper bounds for this variable were chosen such that all the discontinuity stresses have become insignificant before they were reached.

The form of the computer output was the same as for the case of the hemisphere in section 4.2. Therefore the checks on the output values were the same as those followed in that section.

Curves were again drawn by an IBM 1627 plotter and the choice of values used in the final curves and the method of presenting the curves are the same as in section 4.2 with the exception of the curves where $\eta_s$ equals 0.001 or 1.951 and $\frac{h_c}{h_s}$ equals one. These curves are presented in four separate figures because of the greater number of data points used in obtaining them.

The design curves for the cylinder follow.
Figure 4:15a \( \frac{h_c}{h_s} = 1.0 \)
\( \eta_s = 0.001 \)
Outside Surface

Figure 4:15b \( \frac{h_c}{h_s} = 1.0 \)
\( \eta_s = 0.001 \)
Outside Surface
Figure 4:16a  \( \frac{h_c}{h_s} = 1 \)
\( \eta_s = 1.951 \)
Outside Surface

Figure 4:16b  \( \frac{h_c}{h_s} = 1 \)
\( \eta_s = 1.951 \)
Outside Surface
Figure 4:17a \( \frac{h_c}{h_s} = 1.0 \)
\( \eta_s = 0.001 \)

Outside Surface

Figure 4:17b \( \frac{h_c}{h_s} = 1.0 \)
\( \eta_s = 0.001 \)

Outside Surface
Figure 4:19a  \( \frac{h_c}{h_s} = 2.0 \)
\( \lambda_s = 6 \)
Outside Surface

Figure 4:19b  \( \frac{h_c}{h_s} = 2.0 \)
\( \lambda_s = 6 \)
Outside Surface
Figure 4:20a \( \frac{h_c}{h_s} = 2.0 \)
\( \lambda_s = 14 \)

Outside Surface

Figure 4:20b \( \frac{h_c}{h_s} = 2.0 \)
\( \lambda_s = 14 \)

Outside Surface
Figure 4:21a \[ \frac{h_{c}}{h_{s}} = 2.0 \]
\[ \lambda_{s} = 22 \]

Outside Surface

Figure 4:21b \[ \frac{h_{c}}{h_{s}} = 2.0 \]
\[ \lambda_{s} = 22 \]

Outside Surface
Figure 4:22a
\[ \frac{h_c x}{pa_0} = 2.0 \]
\[ \lambda_s = 30 \]
Outside Surface

Figure 4:22b
\[ \frac{h_c \phi}{pa_0} = 1 \]
\[ \eta_s = 0.001 \]
\[ \eta_s = 0.651 \]
\[ \eta_s = 1.301 \]
\[ \eta_s = 1.951 \]
\[ \lambda_s = 30 \]
Outside Surface
BIBLIOGRAPHY


APPENDICES
The equations for the distribution of discontinuity stress in the cylinder, thickness change in a cylinder, used in the computer program with a listing of the computer program follow. The cases where $\eta$ is less than two and where $\eta$ equals two, are presented separately.

**Equations Used in Computer Programs:**

1. \[ H_1 = -\left(\frac{1}{2} - 1\right)^2 \left(\frac{h_1}{h_2}\right)^2 \left(\frac{\eta_1 + 1}{2}\right)^2 \left(\frac{\eta_1 + 2}{2}\right)^2 - \left(\frac{1}{2} + 1\right) \left(\frac{h_1}{h_2}\right)^2 \left(\frac{\eta_1 + 1}{2}\right)^2 \left(\frac{\eta_1 + 2}{2}\right)^2 \] 

2. \[ C_{31} = -\frac{h_1^2}{h_2^2} \left(\frac{h_1}{h_2}\right)^2 \left(\frac{\eta_1 + 1}{2}\right)^2 \left(\frac{\eta_1 + 2}{2}\right)^2 \left(\frac{1}{2} + \frac{1}{2}\right) + \frac{h_1^2}{h_2^2} \left(\frac{\eta_1 + 1}{2}\right)^2 \left(\frac{\eta_1 + 2}{2}\right)^2 \]
\[ \eta = 2 \]

1. \( H_{II} = 2 \left[ 3 \left( \frac{h_1}{h_2} \right)^{\frac{3}{2}} \left( \frac{h_2}{h_2} + 1 \right) + \sqrt{2} \left( \frac{h_1}{h_2} + 1 \right) \right] \left[ 3 \left( \frac{h_1}{h_2} \right)^{\frac{1}{2}} \left( \frac{h_2}{h_2} + 1 \right) + \sqrt{2} \left( \frac{h_1}{h_2} + 1 \right) \right] - \left( \frac{1}{2} \right) \]

   \[ (2:37) \]

2. \( \overline{C_{71}} = \frac{h_2^{\frac{1}{2}} h_2^{\frac{1}{2}} h_2^{\frac{1}{2}} h_2^{\frac{1}{2}}}{H_{II}} \)

   \[ (2:44) \]
\[
3. \quad \bar{C}_{81} = \frac{h_1^2}{3(2\frac{1}{2} + 1)} \left[ \frac{h_1^2}{h_2^2} \left( \frac{h_1^2}{h_2^2} \right)^{\frac{1}{2}} + \frac{h_1^2}{h_2^2} \right]
\]

4. \[
\bar{w}_{\text{vII}} = e^{-\sqrt{2}\beta_1 x} \left[ \bar{C}_{71} + \beta_1 x \bar{C}_{81} \right] \left[ (1 - \frac{h_1}{2}(\frac{1}{h_2^2} - 1)) \right]
\]

5. \[
\bar{M}_{\text{vII}} = e^{-\sqrt{2}\beta_1 x} \left[ \bar{C}_{71} \right] \left( \beta_1 x - \sqrt{2}\bar{C}_{81} \right) \left[ \frac{(h_1/2)}{2(3(1-\mu^2))} \right] \left( \frac{h_1}{h_2^2} - 1 \right)
\]

6. \[
\bar{\sigma}_{\text{vII}} - \frac{1}{2} = 6 \bar{M}_{\text{vII}}
\]

7. \[
\bar{\sigma}_{\text{vII}} - 1 = \bar{w}_{\text{vII}} \pm \mu 6 \bar{M}_{\text{vII}}
\]
COMBO = (SQRT (HI)**5) + (SQRT (HI)**3)
DEN = 2.0 - ETA
DEM = 2.0 - ETHI
HIET = ETHI + 2.0
ETLO = ETHI + 1.00
HE = SQRT (HIET)
HO = SQRT (ETA2)
HEHO = HE*HO
HUM = ETA + 1.0
SEE = (2.0 - ETA)/ 2.0
SAW = SQRT (SEE)
XE = (ETA + 2.0)/ 2.0
XMU = .333
QT = 4.0 - ETA*ETA
QZ = SQRT (QT)
QUE = 3.0*(1.0 - XMU*XMU)
ADA = ETHI + 1.0
BOO = SQRT (HI)**5
XK7 = 1.0 - (ADA*ADA*ETA2) - (2.0*(HI*HI)) - (COMBO*HUM*ADA*HEHO) + ((HI**4)*(1.0 - HUM*HUM*HIET))
 XC31 = ADA*((ETHI*ETA) + (2.0*ETHI) + ETA + 1.0 + (BOO*HUM*HEHO) + (HI*HI))/ XK7
 XC41 = -(ETA2*ADA*ADA*HO) + (BOO*ETA*ADA*HUM*HE) + (ADA*HO*(1.0 - HI*HI)))/ (SQRT (DEN)*XK7)

DO 15 I = 1, 21
00020 BETA(I) = .00
IF (I-1)20, 20, 21
00021 BETA(I) = BETA(I-1) + .20
GOTO 22
00022 WOW(I) = -SQRT (XE)*BETA(I)
WAM = WOW(I)
WOM = EXP(WAM)
CSAW = COS((SAW)*BETA(I))
SSAW = SIN((SAW)*BETA(I))
WD1(I) = WOM*((XC31*CSAW + XC41*SSAW)*((1.0 - (XMU/2.0)))
XM1(I) = (WOM/4.0)*(((XC31*ETA-QZ*XC41)*CSAW) + ((XC41*ETA + QZ*XC31)*SSAW))*(1.0 - (XMU/2.0))
POSX(I) = 6.0*XM1(I)
XNEG(I) = -6.0*XM1(I)
POST(I) = WD1(I) + (XMU*6.0*XM1(I))
TNEG(I) = WD1(I) - (XMU*6.0*XM1(I))
00015 CONTINUE
WRITE (3, 40) HI, ETA
WRITE (2,42)HI, ETA
WRITE (3,41)(POSX(I), XNEG(I), POST(I), TNEG(I), I=1, 21)
WRITE (2,43)(POSX(I), XNEG(I), POST(I), TNEG(I), I=1, 21)
ETA=ETA+.40
GOTO5
00025 HI=HI+.10
GOTO3
00030 CONTINUE
END

\eta = 2

FORTRAN LISTING 1410-FO-970

DIMENSION BETA(100), POSX(100), POST(100), XNEG(100),
        WD1(100), XM1(100), WOW(100), TNEG(100)
00040 FORMAT (1HK, 5X, 3HHI= , F7.2, 5X, 4HETA= , F7.2)
00041 FORMAT (1H, 4E20.8)
00042 FORMAT (5X, 3HHI= , F7.2, 5X, 4HETA= , F7.2)
00043 FORMAT (4E20.8)
C COMPUTER PROGRAM FOR DETERMINING STRESS
HI=.40
00003 ETA=2.0
1F(HI-1.0)5, 5, 30
00005 IF(ETA-2.00)10, 10, 25
00010 ETIHI=HI*HI*ETA
ETA2=ETA+.2.00
COMBO=(SRT(HI)**5) + (SRT(HI)**3)
DEN=2.0-ETA
DEM=2.0-ETHI
HIET=ETHI+2.0
ETLO=ETHI+1.00
HE=SRT(HIET)
HO=SRT(ETA2)
HEHO=HE*HO
HUM=ETA+1.0
SEE=(2.0-ETA)/ 2.0
SAW=SRT(SEE)
XE=(ETA+.2.0)/ 2.0
XMU=.333
QT=4.0-ETA*ETA
QZ=SRT(QT)
QUE=3.0*(1.0-XMU*XMU)
ADA=ETHI+1.0
BOO=SQRT (HI)**5

XK7 = 1.0 - (ADA*ADA*ETA2) - (2.0*(HI*HI)) - (COMBO*HUM*ADA*HEHO)/(HI**4)*(1.0-HUM*HUM*HIET))

XC31 = -(ADA*((ETHI*ETA) + (2.0*ETHI) + ETA + 1.0 + (BOO*HUM*HEHO)+(HI*HI))/XK7)

XC41 = -(ETA*ADA*ADA*HO) + (BOO*ETA*ADA*HUM*HE) + (ADA*HO*(1.0-HI*HI))/ (SQRT(2.0)*(XK7))

DO151 = 1, 21
IF(I-1)20, 20, 21
00020  BETA(I) = .00
        GOTO 22
00021  BETA(I) = BETA(I-1) + .20
00022  WOW(I) = -SQRT (XE)*BETA(I)
        WAM = WOW(I)
        WOM = EXP(WAM)
        CSAW = COS((SAW) * BETA(I))
        SSAW = SIN((SAW) * BETA(I))
        WD1(I) = WOM*(XC31 + XC41*BETA(I))*((1.0-(XMU/2.0))*HI-1.0)
        XM1(I) = (WOM/2.0)*(XC31 + (BETA(I) - SQRT(2.0))*XC41)*((1.0-(XMU/2.0))*HI-1.0)/ (SQRT(QUE))
        POSX(I) = 6.0*XM1(I)
        XNEG(I) = -6.0*XM1(I)
        POST(I) = WD1(I) + (XMU*6.0*XM1(I))
        TNEG(I) = WD1(I) - (XMU*6.0*XM1(I))
00015  CONTINUE
        WRITE (3, 40) HI, ETA
        WRITE (2, 42) HI, ETA
        WRITE (3, 41) (POSX(I), XNEG(I), POST(I), TNEG(I), I = 1, 21)
        WRITE (2, 43) (POSX(I), XNEG(I), POST(I), TNEG(I), I = 1, 21)
        ETA = ETA + .40
        GOTO 5
00025  HI = HI + .10
        GOTO 3
00030  CONTINUE
END
APPENDIX II

COMPUTER PROGRAM FOR THE NON-DIMENSIONAL STRESS CURVES OF THE HEMISPHERE, HEMISPHERE TO CYLINDER JUNCTURE

Equations used in the computer program for determining the distribution of discontinuity stress in the hemisphere, hemisphere-to-cylinder juncture, and a listing of the computer program follow.

Equations Used in Computer Program:

1. \[ I(z) = \sum_{m=1}^{\infty} \frac{m}{(m!)} \left( \frac{s}{4} \right)^{2m-1} \left\{ \sin\left[ \left( \frac{2m-1}{2} \right) \tan^{-1} \left( \frac{4 - \eta_s^2}{\eta_s^2} \right) \right] \right\} (m=1,2,3,\ldots) \]

2. \[ I(z) = \sum_{m=1}^{\infty} \frac{1}{m!} \left( \frac{s}{4} \right)^{2m} \left\{ \sin\left[ m \tan^{-1} \left( \frac{4 - \eta_s^2}{\eta_s^2} \right) \right] \right\} (m=1,2,3,\ldots) \]

3. \[ R(z) = \sum_{m=1}^{\infty} \frac{m}{(m!)} \left( \frac{s}{4} \right)^{2m-1} \left\{ \cos\left[ \left( \frac{2m-1}{2} \right) \tan^{-1} \left( \frac{4 - \eta_s^2}{\eta_s^2} \right) \right] \right\} (m=1,2,3,\ldots) \]

4. \[ R(z) = 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \left( \frac{s}{4} \right)^{2m} \left\{ \cos\left[ m \tan^{-1} \left( \frac{4 - \eta_s^2}{\eta_s^2} \right) \right] \right\} (m=1,2,3,\ldots) \]

The above equations are from Equations (3:6) where \( \phi = \frac{\pi}{2} \).
5. \[ \Delta = (1 - \eta_s) \left( \frac{1}{4} \right)^{\frac{1}{2}} \cdot \cdot \cdot \left( \frac{2 + \eta_s}{4} \right)^{\frac{1}{2}} \cdot \cdot \cdot (\text{From Equations (3:14)(3:15)(3:38)}) \]

\[ \Delta = \frac{1}{\lambda_s \pi} \left( \frac{4 - \eta_s^2}{4} \right)^{\frac{1}{2}} (R^2 + \frac{1}{4}) \]

where, \( \alpha = \frac{\pi}{2} \)

6. \[ C_{11} = \frac{1}{\Delta} \left\{ \frac{1}{\lambda_s \pi} (2 - \eta_s) (\text{IR - RI}) \left( \frac{1}{4} \right) - \frac{1}{\lambda_s \pi} (2 + \eta_s) (\text{II} + \text{RR}) \left( \frac{1}{4} \right) \right\} \]

\[ C_{11} = \frac{4 - \eta_s^2}{4} (1 + R^2) \frac{1}{\lambda_s \pi} \left( \frac{2 + \eta_s}{4} \right)^{\frac{1}{2}} \]

\[ + \left( \frac{4 - \eta_s^2}{4} \right)^{\frac{1}{2}} \left( \frac{1}{\lambda_s \pi} \right) \left( \frac{2 + \eta_s}{4} \right)^{\frac{1}{2}} (1^2 + R^2) \] \hspace{1cm} (3:39)

7. \[ C_{12} = C_{21} = \frac{1}{\Delta} \left\{ \left( \frac{2 + \eta_s}{4} \right)^{\frac{1}{2}} \cdot \cdot \cdot \left( \frac{2 - \eta_s}{4} \right)^{\frac{1}{2}} \cdot \cdot \cdot \right\} \]

\[ C_{12} = C_{21} = \frac{1}{\Delta} \left\{ \left( \frac{2 + \eta_s}{4} \right)^{\frac{1}{2}} \cdot \cdot \cdot \left( \frac{2 - \eta_s}{4} \right)^{\frac{1}{2}} \cdot \cdot \cdot \right\} \] \hspace{1cm} (3:39)

8. \[ C_{22} = \frac{1}{\Delta} \left\{ \left( \frac{2 + \eta_s}{4} \right)^{\frac{1}{2}} \cdot \cdot \cdot \left( \frac{2 - \eta_s}{4} \right)^{\frac{1}{2}} \cdot \cdot \cdot \right\} \]

\[ C_{22} = \frac{1}{\Delta} \left\{ \left( \frac{2 + \eta_s}{4} \right)^{\frac{1}{2}} \cdot \cdot \cdot \left( \frac{2 - \eta_s}{4} \right)^{\frac{1}{2}} \cdot \cdot \cdot \right\} \] \hspace{1cm} (3:39)

9. \[ S_{11} = \frac{h^2}{\eta_s + 2} \]

\[ S_{11} = \frac{h^2}{\eta_s + 2} \]

\[ S_{22} = \frac{h^2}{\eta_s + 1} \]

\[ S_{22} = \frac{h^2}{\eta_s + 1} \] \hspace{1cm} (3:40)

10. \[ S_{12} = \frac{h^2}{S_{21} = \frac{1}{\Delta} \left\{ \left( \frac{2 + \eta_s}{4} \right)^{\frac{1}{2}} \cdot \cdot \cdot \left( \frac{2 - \eta_s}{4} \right)^{\frac{1}{2}} \cdot \cdot \cdot \right\} \]

\[ S_{12} = \frac{h^2}{S_{21} = \frac{1}{\Delta} \left\{ \left( \frac{2 + \eta_s}{4} \right)^{\frac{1}{2}} \cdot \cdot \cdot \left( \frac{2 - \eta_s}{4} \right)^{\frac{1}{2}} \cdot \cdot \cdot \right\} \] \hspace{1cm} (3:40)

11. \[ S_{22} = -S_{11} \]

\[ S_{22} = -S_{11} \] \hspace{1cm} (3:40)
\begin{align*}
12. \quad \overline{M}_0 &= \frac{1}{\left[12(1-\mu^2)\right]^{\frac{1}{2}}} \frac{h_c^2}{h_s^2} \frac{\left(\frac{c}{h_s}这不是一个有效的数学表达式。\right)}{C_{21}^{-2} - S_{21}^{-2}} \left[\left(1-\frac{\mu^2}{2}\right) - \frac{\left(\frac{c}{h_s}\right)(1-\mu^2)}{2}\right] \\
13. \quad \overline{Q}_0 &= \frac{h_c^{\frac{5}{2}}}{h_s^{\frac{5}{2}}} \frac{\left(\frac{c}{h_s} \text{不是有效的数学表达式。}\right)}{C_{22}^{-2} - S_{22}^{-2}} \left[\left(1-\frac{\mu^2}{2}\right) - \frac{\left(\frac{c}{h_s}\right)(1-\mu^2)}{2}\right] \\
14. \quad \overline{A} &= \frac{2h_c^{\frac{1}{2}}}{\pi h_s} \frac{1}{\frac{h_s}{\Delta}} \left\{ \frac{2-\eta_s}{2} - \frac{4-\eta_s}{4} \right\} R \left\{ 1 + \frac{\left(\frac{c}{h_s}\right)}{(1-\mu^2)} \right\} \overline{M}_0 \\
&\quad + \left\{ \left(\frac{c}{h_s}\right) + \frac{4-\eta_s}{4} \right\} R \left( -\frac{1}{\lambda_s} \right) \left(\frac{\eta_s}{2} - \frac{4-\eta_s}{4} \right) \overline{Q}_0
\end{align*}
15. \( \overline{B} = \left( \frac{h c}{\pi \hbar s} \right) \left( \frac{1}{\lambda s} \right) \left[ \{ \eta s \left( \frac{-s^2}{4} \right) \} \left( \frac{2 - \eta_s}{2} \right) \cdot \frac{h c}{\hbar s} \left( 1 - \frac{\eta_s^2}{4} \right) \right] \left[ 12 \frac{h c}{\hbar s} (1 - \mu^2) \right] \left( \frac{1}{\lambda s} \right) \overline{M}_0 \)

\[
\begin{align*}
2 - \eta_s & \frac{1}{2} \\
\{ \left( \frac{-s^2}{4} \right) \left( 1 + \left( \frac{2 + \eta_s}{s} \right) \right) R - \frac{1}{\lambda s} & \left[ \frac{\eta_s R}{4} + \left( \frac{4 - \eta_s^2}{4} \right) \right] \} \overline{Q}_0 \end{align*}
\]  

(3.42)

16. \( U = \left( \frac{\phi}{\sin \phi} \right)^{-1} \sum_{m=1}^{\infty} \left( \frac{m}{(m!)^2} \left( \frac{s}{2} \right) \right)^{2m-1} \left( \frac{\phi}{\sin \phi} \right)^{-2} \left( \frac{\phi \cos \phi}{\sin \phi} \right) \left( \frac{\lambda s}{2} \right)^{2m-1} \{ \cos \left( \frac{2m-1}{2} \right) \tan^{-1} \left( \frac{4 - \eta_s^2}{2} \right) \} \) \( \text{for} \ m = 1, 2, 3, \ldots \)  

(3.6)

From Equation (3.6)

17. \( U' = \frac{1}{2} \left( \frac{\sin \phi}{\phi} \right)^{-1} \sum_{m=1}^{\infty} \left( \frac{m}{(m!)^2} \right) \left( \frac{\phi \cos \phi}{\sin \phi} \right) \left( \frac{\lambda s}{2} \right)^{2m-1} \{ \cos \left( \frac{2m-1}{2} \right) \tan^{-1} \left( \frac{4 - \eta_s^2}{2} \right) \} \)  

\( \text{for} \ m = 1, 2, 3, \ldots \)

18. \( V = \left( \frac{\phi}{\sin \phi} \right)^{-1} \sum_{m=1}^{\infty} \left( \frac{m}{(m!)^2} \right) \left( \frac{\lambda s}{2} \right)^{2m-1} \{ \sin \left( \frac{2m-1}{2} \right) \tan^{-1} \left( \frac{4 - \eta_s^2}{2} \right) \} \)  

\( \text{for} \ m = 1, 2, 3, \ldots \) (3.6)
19. \[ V' = \frac{1}{2} \left( \sin \phi \right)^{\frac{1}{2}} \left( \frac{1}{\sin \phi} - \frac{\phi \sin \phi}{\sin \phi} \right) \sum_{m=1}^{\infty} \left( \frac{m}{(m!)^2} \right) \frac{\lambda \phi}{2m-1} \left\{ \sin \left( \frac{2m-1}{2} \right) \tan^{-1} \left( \frac{4-\eta^2}{2} \right) \right\} \right] \]

From Equation (3.6)

\[ + \left( \frac{\phi}{\sin \phi} \right)^{\frac{1}{2}} \sum_{m=1}^{\infty} \left( \frac{m}{(m!)^2} \right) \frac{\lambda \phi}{2m-1} \left\{ \sin \left( \frac{2m-1}{2} \right) \tan^{-1} \left( \frac{4-\eta^2}{2} \right) \right\} \}

(m=1, 2, 3, \cdots)

20. \[ \bar{M}_\phi = \frac{1}{12(1-\mu^2)^{\frac{1}{2}}} \left\{ (U' + \mu U \cot \phi) \left[ \frac{\eta_s A}{2} - \left( \frac{4-\eta^2}{4} \right) \bar{B} \right] + (V' + \mu V \cot \phi) \left[ \frac{\eta_s B}{2} + \left( \frac{4-\eta^2}{4} \right) \bar{A} \right] \right\} \] (3.44)

21. \[ \bar{w}_s = -\left[ \left( \frac{2-\eta^2}{4} \right) (A U' + B V') - \eta_s \left( \frac{2-\eta^2}{4} \right) (B U' - A V') \right] - (\mu \cot \phi) \left( \frac{2-\eta^2}{4} \right) (A U + B V) - (\mu \eta_s \cot \phi) \left( \frac{2-\eta^2}{4} \right) (B U - A V) \] (3.45)

22. \[ \bar{\sigma}_{\phi_s} - \frac{1}{2} = \pm 6 \bar{M}_\phi \]

23. \[ \bar{\sigma}_{\theta s} - \frac{1}{2} = \bar{w}_s \pm \mu 6 \bar{M}_\phi \]
Listing of Computer Program:

FORTRAN LISTING 1410-FO-970

DIMENSIONA(4), CDT(50), CS(50), SDT(50), S(50)
00101 FORMAT (14, E16.8)
00102 FORMAT (///, 8H LAMBDAA=, F6.0, 10X, 4H ETA=, F12.6, /)
00103 FORMAT (1H, 4E20.8)
00104 FORMAT (6H1WHEW.)
00105 FORMAT (10X, 5H C11=, E12.6, 5H C12=, E12.6, 5H C21=,
          E12.6, 5H C22=, E12.6, 10X, 5H S11=, E12.6, 5H S12=,
          E12.6, 5H S21=, E12.6, 5H S22=, E12.6, /10X, 5H
          PQQ0=, E12.6, 5H P00=, E12.6, /)

READ (1, 101) LOOP, UM
HCS=1.0
00012 XLAM=6.0
00010 ETA=0.001
00011 TANI=ATAN(SQRT((2.0/ETA)**2-1.0))

DO 1M=1, LOOP
FM=M
P=FM*TANI
Q=P-.5*TANI
SDT(M)=SIN(Q)
S(M)=SIN(P)
CDT(M)=COS(Q)
CS(M)=COS(P)
PI=3.14159
FLAM=XLAM*PI/4.0
FLAM2=FLAM*FLAM
DOTI=0.0
FREI=DOTI
DOTR=DOTI
FRER=1.0
C1=FLAM
C2=FLAM2
WRITE (3, 102) XLAM, ETA
WRITE (2, 102) XLAM, ETA
DO 2M=1, LOOP
DOTI=DOTI+C1*SDT(M)
FREI=FREI+C2*S(M)
DOTR=DOTR+C1*CDT(M)
FRER=FRER+C2*CS(M)
FM=M
$C_1 = C_1 \cdot \frac{F L A M_2}{(F M \cdot (F M + 1.0))}$

$C_2 = C_2 \cdot \frac{F L A M_2}{((F M + 1.0) \cdot (F M + 2.0))}$

$P I L = X L A M \cdot \frac{1}{\pi}$

$E T A_2 = E T A \cdot \frac{1}{\eta}$

$R I D_T = F R E_R \cdot D O T_R + F R E_I \cdot D O T_I$

$D I F = F R E_R \cdot D O T_I - F R E_I \cdot D O T_R$

$D R 2 I 2 = D O T_R \cdot D O T_R + D O T_I \cdot D O T_I$

$R E I 2 = F R E_I \cdot F R E_I + F R E_R \cdot F R E_R$

$C_1 = \sqrt{2.0 + \eta}$

$C_2 = \sqrt{2.0 - \eta}$

$C_3 = \frac{\sqrt{4.0 - \eta^2}}{2.0}$

$C_4 = \frac{2.0 - \eta^2}{2.0}$

$D E L T_A = (1.0 - \eta) \cdot C_1 \cdot \frac{D I F}{2.0} + (1.0 + \eta) \cdot C_2 \cdot \frac{R I D_T - C_3 \cdot D R 2 I 2}{P I L}$

$C_{11} = \frac{((1.0 - \eta) \cdot C_1 \cdot D I F + (1.0 + \eta) \cdot C_2 \cdot R I D_T - C_3 \cdot D R 2 I 2) \cdot P I L + C_3 \cdot (R 2 I 2 + D R 2 I 2 \cdot P I L^2)}{D E L T_A}$

$C_{12} = \frac{(C_2 \cdot R I D_T - C_1 \cdot D I F) \cdot D E L T_A}{D E L T_A}$

$C_{21} = 0.0 - C_{12}$

$C_{22} = 0.0 - D R 2 I 2 \cdot C_3 \cdot D E L T_A$

$D E N O M = E T A \cdot (H C S \cdot H C S) + 1.0$

$S_{11} = 0.0 - \sqrt{\eta \cdot (E T A) \div (H C S \cdot H C S) + 2.0)} \div D E N O M$

$S_{12} = 1.0 \div D E N O M$

$S_{21} = 0.0 - S_{12}$

$S_{22} = 0.0 - S_{11}$

$D_1 = C_{22} \cdot \sqrt{H C S \cdot H C S} - S_{22}$

$D_2 = C_{11} \cdot \sqrt{H C S \cdot H C S} - S_{11}$

$D_3 = C_{12} \cdot H C S \cdot H C S - S_{12}$

$D_4 = 0.0 - D_3$

$D_6 = \sqrt{(12.0 \cdot (1.0 - U M \cdot U M))}$

$U = 0.0 - (1.0 - U M) \div 2.0 - ((1.0 - U M) \div 2.0) \cdot H C S$

$D E N O M = D_2 \cdot D_1 - D_3 \cdot D_4$

$P Q_0 = 0.0 - U \cdot D_1 \div D E N O M$

$P M_0 = D_4 \cdot U \div (D E N O M \cdot D_6)$

$W R I T E (3, 105) C_{11}, C_{12}, C_1, C_2, S_{11}, S_{12}, S_{21}, S_{22}, P Q_0, P M_0$

$C_5 = \sqrt{(2.0 \cdot H C S)} \div (S Q R T (P I) \cdot D E L T_A \cdot X L A M)$

$C_6 = P M_0 \cdot S Q R T (H C S) \cdot D_6$

$A P = C_5 \cdot (C_6 \cdot C_4 \cdot D O T_I + E T A \cdot C_3 \cdot D O T_R) + P Q_0 \cdot (C_1 \cdot F R E_I - C_2 \cdot F R E_R \cdot (E T A \cdot D O T_I) \div (2.0 \cdot C_3 \cdot D O T_R) \div P I L)$

$B P = C_5 \cdot (C_6 \cdot (E T A \cdot C_3 \cdot D O T_I - C_4 \cdot D O T_R) - P Q_0 \cdot (C_2 \cdot F R E_I + C_1 \cdot F R E_R \cdot (E T A \cdot D O T_R) \div (2.0 + C_3 \cdot D O T_I) \div P I L)$

$F I N K = P I \div 80.0$

$P H I = P I \div 2.0$

$D_1 = \sin (P H I)$

$D_2 = \sqrt{P H I} \div D_1$

$D_3 = \frac{(1.0 - P H I \cdot C O S (P H I) \div D_1)}{2.0 \cdot D_2 \cdot D_1}$

$U = 0.0$
PU=U
V=U
PV=U
PLAM=XLAM*PHI/2.0
D4=PLAM
D5=0.5
PLAM=D4*D4
DO5 M=1, LOOP
FM=M
U=U+D4*CDT(M)
P U=PU+D5*CDT(M)
V=V+D4*SDT(M)
PV=PV+D5*SDT(M)
D4=D4*PLAM/(FM*(FM+1.0))
D5=D5*PLAM*(FM+0.5)/(FM*(FM+1.0)*(FM-0.5))
UP=D3*U+D2*PU*XLAM
U=D2*U
VP=D3*V+D2*PV*XLAM
V=D2*V
COT=COS(PHI)/D1
UR=UM*COT*(ETA*C3*(BP*U-AP*V)+C4*(AP*U+BP*V))-ETA*C3*(BP*UP-AP*VP)-C4*(AP*UP+BP*VP)
PHIM=((UP+UM*COT*U)*(ETA*AP/2.0-C3*BPH*VP+UM*COT*V)*(ETA(BP/2.0+C3*AP))/SQRT(12.0*(1.0-UM*UM))
A(1)=6.0*PHIM
A(2)=A(1)-12.0*PHIM
A(3)=UR+6.0*UM*PHIM
A(4)=A(3)-12.0*UM*PHIM
WRITE (3, 103)A
WRITE (2, 103)A
PHI=PHI+FINK
IF((Pl/4.0)-PHI)4, 4, 8
ETA=ETA+.6500
IF(ETA-1.951)11, 11, 6
XLAM=XLAM+.5
IF(XLAM-30.0)10, 10, 9
HCS=HCS+1.0
IF(HCS-2.0)12, 12, 7
WRITE(3, 104)
CALLEXIT
END
APPENDIX III

COMPUTER PROGRAM FOR THE NON-DIMENSIONAL STRESS CURVES OF THE CYLINDER, HEMISPHERE TO CYLINDER JUNCTURE

Equations that were programmed for finding the distribution of discontinuity stress in the cylinder, hemisphere-to-cylinder juncture, and a listing of the computer program follow.

Equations Used in Computer Program:

1. \[ I(z) = \sum_{m=1}^{\infty} \frac{1}{(m!)^2} \left( \frac{s}{4} \right)^m \frac{\lambda \pi}{2m-1} \sin\left(\frac{2m-1}{2}\right)\tan^{-1}\left(\frac{\eta_s}{\eta_s^2}\right) \] (m=1,2,3\ldots)

2. \[ I(z) = \sum_{m=1}^{\infty} \frac{1}{(m!)^2} \left( \frac{s}{4} \right)^m \sin\left(\frac{m\pi}{2}\tan^{-1}\left(\frac{\eta_s}{\eta_s^2}\right) \right) \] (m=1,2,3\ldots)

3. \[ R(z) = \sum_{m=1}^{\infty} \frac{1}{(m!)^2} \left( \frac{s}{4} \right)^m \frac{\lambda \pi}{2m-1} \cos\left(\frac{2m-1}{2}\right)\tan^{-1}\left(\frac{\eta_s}{\eta_s^2}\right) \] (m=1,2,3\ldots)

4. \[ R(z) = 1 + \sum_{m=1}^{\infty} \frac{1}{(m!)^2} \left( \frac{s}{4} \right)^m \cos\left(\frac{m\pi}{2}\tan^{-1}\left(\frac{\eta_s}{\eta_s^2}\right) \right) \] (m=1,2,3\ldots)

The above equations are from Equations (3.6) where \( \phi = \frac{\pi}{2} \).
5. $\Delta = (1 - \eta_s) (\frac{2 + \eta_s}{4}) (RI - IR) + (1 + \eta_s) (\frac{2 - \eta_s}{4}) (RR + II)

\begin{align*}
&= \frac{1}{\lambda s \pi} \left( \frac{4 - \eta_s^2}{4} \right) \left( R^2 + I^2 \right) From Equations (3:14)(3:15) \\
&\text{Where, } \alpha = \frac{\pi}{2}
\end{align*}

6. $C_{11} = \frac{1}{\Delta} \left\{ \frac{1}{\lambda s \pi} (2 - \eta_s) (IR - RI) (\frac{s}{4}) - \frac{1}{\lambda s \pi} (2 + \eta_s) (II + RR) (\frac{s}{4}) \right\}

\begin{align*}
&+ \frac{4 - \eta_s^2}{4} \left( I^2 + R^2 \right) + \frac{2 - \eta_s^2}{4} \left( I^2 + R^2 \right) \right\} (3:39)
\end{align*}

7. $C_{12} = -C_{21} = \frac{1}{\Delta} \left\{ \frac{2 + \eta_s}{4} (IR - RI) + (\frac{s}{4}) (II + RR) \right\}$ (3:39)

8. $C_{22} = -\frac{1}{\Delta} \left( \frac{4 - \eta_s^2}{4} \right) \left( I^2 + R^2 \right)$ (3:39)

9. $S_{11} = -\frac{2}{h_s} \left( \frac{-\eta_s + 2}{h_c} \right)$ (3:40)

10. $S_{12} = -S_{21} = \frac{1}{h_s} \left( \frac{\eta_s + 1}{h_c} \right)$ (3:40)

11. $S_{22} = -S_{11}$ (3:40)
12. \[ \overline{M}_0 = \frac{1}{[12(1-\mu^2)]^{\frac{1}{2}}} \left( \frac{h_s}{h_c} \right)^{\frac{2}{2}} \frac{h_s^{\frac{3}{2}}}{h_c^{\frac{3}{2}}} \left[ \left( \frac{c_c}{h_c} \right) C_{22} - S_{22} \right] \left\{ \left( 1 - \frac{1}{2} \right) - \left( \frac{c_c}{h_c} \right) \left( \frac{1}{2} \right) \right\} \] (3: 41)

13. \[ \overline{Q}_0 = \frac{h_s^{\frac{5}{2}}}{h_c^{\frac{5}{2}}} \left[ \left( \frac{c_c}{h_c} \right) C_{22} - S_{22} \right] \left\{ (1 - \mu^2) - \left( \frac{c_c}{h_c} \right) \left( \frac{1}{2} \right) \right\} \] (3: 41)

14. \[ \overline{C}_3 = \frac{\frac{h_s^2}{h_c^2}}{\left( \frac{c_c}{h_c} \eta_s + 1 \right)} \] (3: 43)

15. \[ \overline{C}_4 = -\frac{h_s^2}{h_c^2} \frac{1}{\sqrt{2}} \frac{2 - \left( \frac{c_c}{h_c} \right) \eta_s}{\left( \frac{c_c}{h_c} \right) \eta_s + 1} \] (3: 43)
16. \[ \overline{M}_{xc} = e^{\frac{h}{2c} \eta_s + 2 \left( \frac{1}{2} \right)} \left( \frac{h}{2h_c} \right) \left( \frac{x}{a_0} \right) \sin \left[ \left( \frac{2h}{h_c} \right) \eta_s \left( \frac{1}{2} \right) \left( \frac{x}{a_0} \right) \right] \]

\[ + \left( \frac{4}{h_c} \right)^2 \eta_s \left( \frac{1}{2} \right) \left( \frac{x}{a_0} \right) \sin \left[ \left( \frac{2h}{h_c} \right) \eta_s \left( \frac{1}{2} \right) \left( \frac{x}{a_0} \right) \right] \]

17. \[ \overline{w}_c = e^{\frac{h}{2c} \eta_s + 2 \left( \frac{1}{2} \right)} \left( \frac{h}{2h_c} \right) \left( \frac{x}{a_0} \right) \sin \left[ \left( \frac{2h}{h_c} \right) \eta_s \left( \frac{1}{2} \right) \left( \frac{x}{a_0} \right) \right] \]

\[ \overline{C}_3 \cos \left[ \left( \frac{2h}{h_c} \right) \eta_s \left( \frac{1}{2} \right) \left( \frac{x}{a_0} \right) \right] \]

18. \[ \overline{\sigma}_{xc} = \frac{1}{2} = \pm 6 \overline{M}_{xc} \]

19. \[ \overline{\sigma}_{0c} = \overline{w}_c = \pm \mu \overline{6 M}_{xc} \]
Listing of Computer Program:

FORTRAN LISTING    1410-FO-970

DIMENSION A(4), CDT(50), CS(50), SDT(50), S(50)
00101 FORMAT (14, E16. 8)
00102 FORMAT ( //, 8H LAMBDA=, F6. 0, 10X, 4HETA=, F12.6./)
00103 FORMAT (1H, 4E20. 8)
00104 FORMAT (6H1 WHEW. )
00105 FORMAT (10X, 5H C11= , E12.6, 5H C12=, E12.6, 5H C21=,
          E12.6, 5H C22=, E12.6, / 10X, 5H S11=, E12.6, 5H
          S12=, E12.6, 5H S21 =, E12.6, 5H S22=, E12.6, / 10X, 5H
          P00=, E12.6, 5H PM0=, E12.6.,/ )
00106 FORMAT (3E16. 8)
READ (1, 101) LOOP, UM
READ (1, 106) ETAB, EINK, EMAX, XLAMB, XINK, XMAX,
             XAOB, XAINK, XAMAX
HCS=1.0
00012 XLAM=XLAMB
00010 ETA=ETAB
00111 TANI=ATAN(SQRT((2.0/ ETA)**2-1.0))
DO1M=1, LOOP
FM=M
P=FM*TANI
Q=P-.5*TANI
SDT (M)=SIN(Q)
S(M)=SIN(P)
CDT(M)=COS(Q)
00001 CS(M)=COS(P)
PI=3.14159
FLAM=XLAM*PI/ 4.0
FLAM2=FLAM*FLAM
DOTI=0.0
FREI=DOTI
DOTR=DOTI
FRER=1.0
C1=FLAM
C2=FLAM2
WRITE (3, 102) XLAM, ETA
WRITE (2, 102) XLAM, ETA
DO2M=1, LOOP
DOTI=DOTI+ C1*SDT(M)
FREI=FREI+ C2*S(M)
DOTR=DOTR+ C1*CDT(M)
\[ FRER = FRER + C2 \cdot CS(M) \]
\[ FM = M \]
\[ C1 = C1 \cdot FLAM2 / (FM \cdot (FM + 1.0)) \]
\[ C2 = C2 \cdot FLAM2 / ((FM + 1.0) ** 2) \]
\[ PIL = XLAM \cdot PI \]
\[ ETA2 = ETA \cdot ETA \]
\[ RIDT = FRER \cdot DOTR + FREI \cdot DOTI \]
\[ DIF = FRER \cdot DOT1 - FREI \cdot DOTR \]
\[ DR212 = DOTR \cdot DOTR + DOTI \cdot DOTI \]
\[ R212 = FREI \cdot FREI + FRER \cdot FRER \]
\[ C1 = \sqrt{2.0 + ETA} / 2.0 \]
\[ C2 = \sqrt{2.0 - ETA} / 2.0 \]
\[ C3 = \sqrt{4.0 - ETA2} / 2.0 \]
\[ C4 = (2.0 - ETA2) / 2.0 \]
\[ DELTA = (1.0 - ETA) \cdot C1 \cdot DIF + (1.0 + ETA) \cdot C2 \cdot RIDT - C3 \cdot DR212 / PIL \]
\[ C11 = (((ETA - 2.0) \cdot DIF \cdot C1 - (2.0 + ETA) \cdot RIDT \cdot C2) / PIL + C3 \cdot (R212 + DR212 / PIL ** 2)) / DELTA \]
\[ C12 = (C2 \cdot RIDT - C1 \cdot DIF) / DELTA \]
\[ C21 = 0.0 - C12 \]
\[ C22 = 0.0 - DR212 \cdot C3 / DELTA \]
\[ DENOM = ETA / (HCS * HCS) + 1.0 \]
\[ S11 = 0.0 - (\sqrt{(ETA) / (HCS * HCS) + 2.0}) / DENOM \]
\[ S12 = 1.0 / DENOM \]
\[ S21 = 0.0 - S12 \]
\[ S22 = 0.0 - S11 \]
\[ D1 = C22 \cdot SQRT(HCS ** 5) - S22 \]
\[ D2 = C11 \cdot SQRT(HCS ** 3) - S11 \]
\[ D3 = C12 \cdot HCS \cdot HCS - S12 \]
\[ D4 = 0.0 - D3 \]
\[ D6 = SQRT(12.0 * (1.0 - UM * UM)) \]
\[ U = 0.0 - (1.0 - UM / 2.0 - ((1.0 - UM / 2.0) ** HCS)) \]
\[ DENOM = D2 * D1 - D3 * D4 \]
\[ PQ0 = 0.0 - U * D1 / DENOM \]
\[ PM0 = D4 * U / (DENOM * D6) \]
\[ WRITE (3, 105) C11, C12, C21, C22, S11, S12, S21, S22, PQ0, PM0 \]
\[ XA0 = XA0B \]
\[ HSC2 = 1.0 / (HCS * HCS) \]
\[ DE1 = ETA * HSC2 \]
\[ DE2 = SQRT((2.0 - DE1) / 2.0) \]
\[ DE3 = SQRT((2.0 + DE1) / 2.0) \]
\[ DE4 = DE1 + 1.0 \]
\[ CP3 = (-DE3 * PQ0 * SQRT(2.0)) / PM0 * D6 \]
\[ CP4 = -(DE1 * PQ0 + (SQRT(2.0) * DE3 * PM0 * D6)) / (DE4 * DE2 * SQRT(2.0)) \]
DE4 = XA0 * XLAM / SQRT(2.0 * HCS)
E = EXP(-DE3 * DE4)
DE5 = DE2 * DE4
DS = SIN(DE5)
DC = COS(DE5)
DE6 = SQRT(4.0 - DE1 * DE1)
PU DC = E * (CP3 * DC + CP4 * DS)
DE3 = SQRT(4.0 - DE1 * DE1)
PMXC = E * ((CP3 * DE1 - DE6 * CP4) * DC + (DE6 * CP3 + CP4 * DE1) * DS) / SQRT(48.0 * (1.0 - UM * UM))
A(1) = 6.0 * PMXC
A(2) = A(1) - 12.0 * PMXC
A(3) = PUDC + 6.0 * UM * PMXC
A(4) = A(3) - 12.0 * UM * PMXC
WRITE (3, 103) A
WRITE (2, 103) A
XA0 = XA0 + XAINK
IF(XA0 - XAMAX)4, 4, 8
ETA = ETA + EINK
IF(ETA - EMAX)11, 11, 6
XLAM = XLAM + XINK
IF(XLAM - XMAX)10, 10, 9
HCS = HCS + 1.0
IF(HCS - 2.0)12, 12, 7
WRITE (3, 104)
CALL EXIT
END
APPENDIX IV

COMPUTER PROGRAM FOR CURVE PLOTTING

A listing of the computer program used for curve plotting on an IBM 1627 plotter coupled to an IBM 1620 computer follows.

PLOTTING ROUTINE
DIMENSION Y(4)
1 IF (SENSE SWITCH 1 ) 4, 2
2 IF (SENSE SWITCH 2) 10, 3
3 PAUSE
   GO TO 1
4 ACCEPT 101, I
101 FORMAT (II)
   PAUSE
   IF (SENSE SWITCH 4) 4, 5
5 CALL PLOT (99)
   X = 0.0
   CALL PLOT (90, 0.0, 0.0)
8 READ 100, Y
100 FORMAT (4E20.8)
   CALL PLOT (90, X, Y(I))
   X = X+1.0
   IF (X-21.0) 8, 8, 9
9 TYPE 102
102 FORMAT (9HPLOT DONE,// )
   GO TO 3
10 ACCEPT 103, XMIN, XMAX, XL, XD, YMIN, YMAX, YL, YD
103 FORMAT (E20.0)
   PAUSE
   IF (SENSE SWITCH 4) 10, 11
11 CALL PLOT (99)
   CALL PLOT (201, XMIN, XMAX, XL, XD, YMIN, YMAX, YL, YD)
   CALL PLOT (90, XMIN, YMIN)
   CALL PLOT (90, XMIN, YMAX)
   CALL PLOT (99)
   CALL PLOT (90, XMAX, 0.0)
   CALL PLOT (90, XMIN, 0.0)
   CALL PLOT (99)
TYPE 104

104 FORMAT (12HSCALING DONE)
GO TO 3
END