BORDERED TRIANGLES

by

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1. INTRODUCTION

The purpose of this paper is to consider some properties of the triangle. We will consider first the properties of the triangle obtained by constructing triangles on the sides of a given triangle. Next we will consider the triangle formed by corresponding sides of triangles each having a vertex coincident with a vertex of the given triangle. Finally, we will consider some properties of the various triangles formed by bordering a given triangle by squares.

In this paper we will use a system of complex coordinates. The points in the plane will be designated by capital letters and their affixes by the corresponding lower case letters. Numbers known to be real will be designated by Greek letters. Using this system we may write the affix of any point in the plane in terms of the affixes of any other two points of the plane in the following manner:

$$c = r'a + rb,$$

where $r' = 1 - r$ and r is the complex number $ke^{i\theta}$, θ being the angle from AB to AC, and k the ratio AC/AB. By AC we mean the directed length from point A to point C.

2. THE DIRECT PROBLEM

Upon the sides of a given triangle $A_1A_2A_3$ let us construct triangles $B_1A_2A_3$, $B_2A_3A_1$, $B_3A_1A_2$, of fixed shapes. In this section we will consider some properties of triangle $B_1B_2B_3$. The points B_1, B_2, B_3 have affixes given by

$$
\begin{aligned}
b_1 &= r_1'a_2 + r_1a_3, \\
b_2 &= r_2'a_3 + r_2a_1, \qquad (1)\\
b_3 &= r_3'a_1 + r_3a_2.
\end{aligned}
$$

These equations may be solved for the r's, giving

$$
\begin{aligned}
r_1 &= (b_1 - a_2)/(a_3 - a_2),\ r_1' = (b_1 - a_3)/(a_2 - a_3), \\
r_2 &= (b_2 - a_3)/(a_1 - a_3),\ r_2' = (b_2 - a_1)/(a_3 - a_1), \qquad (2)\\
r_3 &= (b_3 - a_1)/(a_2 - a_1),\ r_3' = (b_3 - a_2)/(a_1 - a_2).
\end{aligned}
$$

LEMMA 2.1. <u>A necessary and sufficient condition that triangles $Q_1Q_2Q_3$ and $C_1C_2C_3$ be directly similar is that</u>

$$
\begin{vmatrix} q_1 & c_1 & 1 \\ q_2 & c_2 & 1 \\ q_3 & c_3 & 1 \end{vmatrix} = 0.
$$

It is evident that a necessary and sufficient condition that the triangles be directly similar is that a homology exist which transforms Q_1Q_2 into C_1C_2 and Q_2Q_3 into C_2C_3. Thus, if the triangles are directly similar, there exists a constant k such that

$$q_2 - q_1 = k(c_2 - c_1) \quad \text{and} \quad q_3 - q_2 = k(c_3 - c_2).$$

Then

$$D \equiv \begin{vmatrix} q_1 & c_1 & 1 \\ q_2 & c_2 & 1 \\ q_3 & c_3 & 1 \end{vmatrix} = \begin{vmatrix} q_1 & c_1 & 1 \\ q_2-q_1 & c_2-c_1 & 0 \\ q_3-q_2 & c_3-c_2 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} q_1 & c_1 & 1 \\ k(c_2-c_1) & c_2-c_1 & 0 \\ k(c_3-c_2) & c_3-c_2 & 0 \end{vmatrix} = 0.$$

Conversely, if $D = 0$, there exists a constant k such that

$$q_2 - q_1 = k(c_2 - c_1) \quad \text{and} \quad q_3 - q_2 = k(c_3 - c_2),$$

and the triangles are directly similar.

THEOREM 2.2. If on the sides of an arbitrarily chosen triangle $A_1A_2A_3$ triangles $B_1A_2A_3$, $B_2A_3A_1$, $B_3A_1A_2$, of fixed shapes, be constructed, a necessary and sufficient condition that triangle $B_1B_2B_3$ be directly similar to a given triangle $C_1C_2C_3$, no matter what triangle $A_1A_2A_3$ is chosen, is that a point D exist such that triangles $B_1A_2A_3$, $B_2A_3A_1$, $B_3A_1A_2$ are directly similar respectively to triangles DC_3C_2, DC_1C_3, DC_2C_1. (4, p.532)

Substituting in D of lemma 2.1 the values for the b's given in (1) we have

$$D = \begin{vmatrix} r_1a_3 + r_1'a_2 & c_1 & 1 \\ r_2a_1 + r_2'a_3 & c_2 & 1 \\ r_3a_2 + r_3'a_1 & c_3 & 1 \end{vmatrix} = 0,$$

as the condition that triangle $B_1B_2B_3$ be directly similar to triangle $C_1C_2C_3$. Expanding we have

$$D = c_3(r_2a_1 + r_2'a_3) - c_2(r_3a_2 + r_3'a_1) - c_3(r_1a_3 + r_1'a_2)$$
$$+ c_1(r_3a_2 + r_3'a_1) + c_2(r_1a_3 + r_1'a_2) - c_1(r_2a_1 + r_2'a_3) = 0.$$

Since this must be true for all triangles $A_1A_2A_3$, we may equate the coefficients of the a's to zero, obtaining

$$r_2(c_3 - c_1) + r_3'(c_1 - c_2) = 0,$$
$$r_3(c_1 - c_2) + r_1'(c_2 - c_3) = 0,$$
$$r_1(c_2 - c_3) + r_2'(c_3 - c_1) = 0.$$

Solutions of these for the r's are

$$r_1 = (d - c_3)/(c_2 - c_3),$$
$$r_2 = (d - c_1)/(c_3 - c_1),$$
$$r_3 = (d - c_2)/(c_1 - c_2),$$

where d is an arbitrary complex number. Substituting in these equations the values of the r's found in (2) we obtain,

$$(b_1 - a_2)/(a_3 - a_2) = (d - c_3)/(c_2 - c_3),$$
$$(3) \qquad (b_2 - a_3)/(a_1 - a_3) = (d - c_1)/(c_3 - c_1),$$
$$(b_3 - a_1)/(a_2 - a_1) = (d - c_2)/(c_1 - c_2).$$

These are conditions that triangles $B_1A_2A_3$, $B_2A_3A_1$, $B_3A_1A_2$ be directly similar to triangles DA_3A_2, DA_1A_3, DA_2A_1.

COROLLARY 2.3. *If, on the sides of an arbitrarily chosen triangle $A_1A_2A_3$, similar triangles $B_1A_2A_3$, $B_2A_3A_1$, $B_3A_1A_2$ of fixed shapes be constructed, a necessary and sufficient condition that triangle $B_1B_2B_3$ be equilateral for any triangle $A_1A_2A_3$ is that triangles $B_1A_2A_3$, $B_2A_3A_1$, $B_3A_1A_2$ be 120° isosceles with vertex angles at B_1, B_2, B_3.*

If triangle $B_1B_2B_3$ is equilateral and the bordering triangles are similar, then the point D of theorem 2.2 must be the centroid of triangle $C_1C_2C_3$. Consequently, the bordering triangles must be 120° isosceles. Conversely, if the bordering triangles are 120° isosceles, triangle $B_1B_2B_3$ must be equilateral. For now

$$r_1 = r_2 = r_3 = r \equiv e^{i\pi/6}, \; r' = \bar{r},$$

whence, setting $t = e^{i\pi/3}$,

$$\begin{aligned} t(b_3 - b_1) &= (a_1 - a_2)\bar{r}t + (a_2 - a_3)rt \\ &= (a_1 - a_2)r + (a_2 - a_3)(r - \bar{r}) \\ &= \bar{r}(a_3 - a_2) + r(a_1 - a_3) \\ &= b_2 - b_1. \end{aligned}$$

COROLLARY 2.4. *If, on the sides of triangle $A_1A_2A_3$, triangles $B_1A_2A_3$, $B_2A_3A_1$, $B_3A_1A_2$, of fixed shapes be constructed, a necessary and sufficient condition that points B_1, B_2, B_3 be collinear for any triangle $A_1A_2A_3$ is that*

$$\measuredangle B_1A_2A_3 = \measuredangle B_2A_1A_3,$$
$$\measuredangle B_2A_3A_1 = \measuredangle B_3A_2A_1,$$
$$\measuredangle B_3A_1A_2 = \measuredangle B_1A_3A_2. \quad (4, \text{p.532})$$

Here, by ABC, we mean the directed angle from line AB to line BC, that is, the positive angle through which line AB must be rotated to coincide with line BC. It follows that directed angles are equivalent if they differ by any integral multiple of . (2, p.12)

If points B_1, B_2, B_3 are to be collinear, then points C_1, C_2, C_3 must be collinear, and

$$\measuredangle DC_1C_3 = \measuredangle DC_1C_2,$$
$$\measuredangle DC_2C_1 = \measuredangle DC_2C_3,$$
$$\measuredangle DC_3C_2 = \measuredangle DC_3C_1.$$

But, by similarity of triangles, we have

$$\measuredangle B_1A_2A_3 = \measuredangle DC_3C_2 = \measuredangle DC_3C_1 = \measuredangle B_2A_1A_3,$$
$$\measuredangle B_2A_3A_1 = \measuredangle DC_1C_3 = \measuredangle DC_1C_2 = \measuredangle B_3A_2A_1,$$
$$\measuredangle B_3A_1A_2 = \measuredangle DC_2C_1 = \measuredangle DC_2C_3 = \measuredangle B_1A_3A_2.$$

The converse readily follows by reversing the above argument.

If we consider the point D collinear with points C_1, C_2, C_3, we have

COROLLARY 2.5. *Three points* B_1, B_2, B_3 *on the sides of a triangle* $A_1A_2A_3$ *are collinear if and only if*

$$(B_1A_2/B_1A_3)(B_2A_3/B_2A_1)(B_3A_1/B_3A_2) = 1.$$

This is the theorem of Menelaus (2, p.147). The proof follows from the similarity of triangles, for

$$B_1A_2/B_1A_3 = DC_3/DC_2,$$
$$B_2A_3/B_2A_1 = DC_1/DC_3,$$
$$B_3A_1/B_3A_2 = DC_2/DC_1,$$

whence

$$(B_1A_2/B_1A_3)(B_2A_3/B_2A_1)(B_3A_1/B_3A_2) = (DC_3/DC_2)(DC_1/DC_3)(DC_2/DC_1) = 1.$$

The converse is readily established.

Let us now consider the condition that the lines A_1B_1, A_2B_2, A_3B_3 be concurrent. The equations of the lines may be written

$$(\bar{a}_1 - \bar{b}_1)z - (a_1 - b_1)\bar{z} + a_1\bar{b}_1 - \bar{a}_1b_1 = 0,$$
$$(\bar{a}_2 - \bar{b}_2)z - (a_2 - b_2)\bar{z} + a_2\bar{b}_2 - \bar{a}_2b_2 = 0,$$
$$(\bar{a}_3 - \bar{b}_3)z - (a_3 - b_3)\bar{z} + a_3\bar{b}_3 - \bar{a}_3b_3 = 0.$$

Without loss of generality we may take

$$a_1 = a, \quad a_2 = 0, \quad a_3 = 1.$$

Then

$$b_1 = r_1, \; b_2 = r_2' + r_2a, \quad b_3 = r_3'a.$$

The condition that the lines be concurrent is that

$$\begin{vmatrix} \bar{r}_1 - \bar{a} & r_1 - a & \bar{a}r_1 - a\bar{r}_1 \\ \bar{r}_2' + \bar{r}_2'\bar{a} & r_2' + r_2a & 0 \\ 1 - \bar{r}_3'\bar{a} & 1 - r_3'a & \bar{r}_3'\bar{a} - r_3'a \end{vmatrix} = 0.$$

Since this must be an identity in a we may equate the coefficients of $a^2\bar{a}$, a^2, $a\bar{a}$, and a to zero, obtaining

$$
(4)\quad
\begin{aligned}
&\bar{r}_1(\bar{r}_2 r_3' - r_2\bar{r}_3') + r_3'(r_2 - \bar{r}_2) = 0, \\
&\bar{r}_1(\bar{r}_2' r_3' + r_2) - r_3'(\bar{r}_1 r_2 + \bar{r}_2') = 0, \\
&r_1(\bar{r}_2' r_3' + \bar{r}_2) - \bar{r}_3'(\bar{r}_1 r_2 + \bar{r}_2') + \bar{r}_1(r_2'\bar{r}_3' + \bar{r}_2) - r_3'(r_1\bar{r}_2 + r_2') = 0, \\
&\bar{r}_1(\bar{r}_2' - r_2') + r_3'(\bar{r}_1 r_2' - r_1\bar{r}_2') = 0.
\end{aligned}
$$

Suppose one of the r's is real, say r_1. Then from $(4)_4$ we have

$$r_1(\bar{r}_2' - r_2') + r_3'(r_1 r_2' - r_1\bar{r}_2') = r_1(\bar{r}_2' - r_2')(1 - r_3') = 0.$$

If none of the r's = 0,1, then $\bar{r}_2' = r_2'$ and r_2 is real, and from $(4)_1$

$$r_1(r_2 r_3' - r_2\bar{r}_3') = 0,$$

whence $r_3' = \bar{r}_3'$, and r_3 is real. Equations $(4)_2$ and $(4)_3$ reduce to

$$r_1 r_2 r_3 = r_1' r_2' r_3'.$$

Since all the r's are real, the B's lie on the sides of triangle $A_1A_2A_3$, and the r's are the ratios

$$
\begin{aligned}
r_1 &= A_2B_1/A_2A_3, \quad & r_2 &= A_3B_2/\,A_3A_1, \quad & r_3 &= A_1B_3/A_1A_2, \\
r_1' &= A_3B_1/A_3A_2, \quad & r_2' &= A_1B_2/\,A_1A_3, \quad & r_3' &= A_2B_3/A_2A_1,
\end{aligned}
$$

and the relation $r_1r_2r_3 = r_1'r_2'r_3'$ becomes

$$(B_1A_2/B_1A_3)(B_2A_3/B_2A_1)(B_3A_1/B_3A_2) = -1.$$

Suppose none of the r's is real. Then from $(4)_1$ we have

$$r_3'/\bar{r}_1 = (\bar{r}_2 r_3' - r_2\bar{r}_3')/(\bar{r}_2 - r_2).$$

Since the right member is equal to its own conjugate, it follows that the left member is a real number. That is

$$r_3' = \rho_3\bar{r}_1\ ,\ \rho_3 \text{ real.}$$

Equations $(4)_1$ and $(4)_4$ become

$$\bar{r}_1\bar{r}_2 - r_1 r_2 + r_2 - \bar{r}_2 = 0,$$

and

$$\bar{r}_2' - r_2' + r_3' r_2' - \bar{r}_3'\bar{r}_2' = 0,$$

or

$$r_2 r_1' = \bar{r}_2\bar{r}_1' \qquad \text{and} \qquad r_2' r_3 = \bar{r}_2'\bar{r}_3.$$

Hence the products $\bar{r}_1'\bar{r}_2$ and $\bar{r}_2'\bar{r}_3$ must be real, whence

$$r_1' = (\bar{r}_2\bar{r}_1'/r_2\bar{r}_2)\bar{r}_2 = \rho_1\bar{r}_2\ ,\ \rho_1 \text{ real,}$$

and

$$r_2' = (\bar{r}_2'\bar{r}_3/r_3\bar{r}_3)\bar{r}_3 = \rho_2\bar{r}_3\ ,\ \rho_2 \text{ real.}$$

Now writing the r's in terms of the a's and b's we have

$$(b_1 - a_3)/(a_2 - a_3) = \rho_1(\bar{b}_2 - \bar{a}_3)/(\bar{a}_1 - \bar{a}_3),$$
$$(b_2 - a_1)/(a_3 - a_1) = \rho_2(\bar{b}_3 - \bar{a}_1)/(\bar{a}_2 - \bar{a}_1),$$
$$(b_3 - a_2)/(a_1 - a_2) = \rho_3(\bar{b}_1 - \bar{a}_2)/(\bar{a}_3 - \bar{a}_2).$$

These are conditions that

$$\measuredangle A_2A_3B_1 = -\measuredangle A_1A_3B_2,$$
$$\measuredangle A_3A_1B_2 = -\measuredangle A_2A_1B_3,$$
$$\measuredangle A_1A_2B_3 = -\measuredangle A_3A_2B_1.$$

Thus we have the following theorem. (4, p.534)

THEOREM 2.6. If, on the sides of an arbitrarily chosen triangle $A_1A_2A_3$, triangles $B_1A_2A_3$, $B_2A_3A_1$, $B_3A_1A_2$, of fixed shapes, are constructed, then a necessary and sufficient condition that A_1B_1, A_2B_2, A_3B_3 be concurrent, no matter what triangle $A_1A_2A_3$ is chosen, is

(a) if B_1, B_2, B_3 lie on the sides A_2A_3, A_3A_1, A_1A_2 of triangle $A_1A_2A_3$,

$$(B_1A_2/B_1A_3)(B_2A_3/B_2A_1)(B_3A_1/B_3A_2) = -1;$$

(b) otherwise,

$$\measuredangle A_2A_3B_1 + \measuredangle A_1A_3B_2 = 0,$$
$$\measuredangle A_3A_1B_2 + \measuredangle A_2A_1B_3 = 0,$$
$$\measuredangle A_1A_2B_3 + \measuredangle A_3A_2B_1 = 0.$$

Note that part (a) is the well known theorem of Ceva (2,p.147).

THEOREM 2.7. If the bordering triangles are directly similar to one another, then the centroids of triangles $A_1A_2A_3$ and $B_1B_2B_3$ coincide.

For

$$\begin{aligned} 3g_b &= b_1 + b_2 + b_3 \\ &= (r'a_2 + ra_3) + (r'a_3 + ra_1) + (r'a_1 + ra_2) \\ &= (r + r')(a_1 + a_2 + a_3) = 3g_a. \end{aligned}$$

The special case where the bordering triangles are flat (the B's falling on the sides of triangle $A_1A_2A_3$), is very old, appearing in Book VIII of Pappus's Collection. The general case was considered by Brocard, and numerous treatments of it have been given.

THEOREM 2.8. *If the bordering triangles are directly similar to one another, then a necessary and sufficient condition for the centroids of the bordering triangles to form an equilateral triangle, no matter what triangle $A_1A_2A_3$ is chosen, is that the bordering triangles themselves be equilateral.*

The affixes g_1, g_2, g_3 of the centroids of the triangles $B_1A_2A_3$, $B_2A_3A_1$, $B_3A_1A_2$ are given by

$$3g_1 = a_2(r' + 1) + a_3(r + 1),$$
$$3g_2 = a_3(r' + 1) + a_1(r + 1),$$
$$3g_3 = a_1(r' + 1) + a_2(r + 1).$$

Therefore

$$3(g_2 - g_1) = a_1(r + 1) - a_2(r' + 1) + a_3(r' - r),$$
$$3(g_3 - g_1) = a_1(r' + 1) - a_2(r' - r) - a_3(r + 1).$$

In order that triangle $G_1G_2G_3$ be equilateral it is necessary and sufficient that

$$g_2 - g_1 = e^{i\pi/3}\,(g_3 - g_1),$$

or, equating coefficients of a_1, a_2, a_3 on the two sides of the equation, that

$$r + 1 = e^{i\pi/3}\,(r' + 1),$$
$$r' + 1 = e^{i\pi/3}\,(r' - r),$$
$$r - r' = e^{i\pi/3}\,(r + 1).$$

Solving any of these three equations for r gives $r = e^{i\pi/3}$.

This theorem is well known and is intimately associated with the Fermat-Torricelli problem and the geometry of the isogonic centers of a triangle (2, pp.218-222).

3. THE INVERSE PROBLEM

The question now arises: Being given triangle $B_1B_2B_3$, under what conditions does there exist a triangle $A_1A_2A_3$ which is bordered by triangles $B_1A_2A_3$, $B_2A_3A_1$, $B_3A_1A_2$ of fixed shapes? The special case where triangle $B_1B_2B_3$ is equilateral was proposed as early as 1869 by Lemoine, and a number of solutions and discussions of this case have been given (1, p.109). In this section we will develop some theorems related to the general inverse problem.

From (1) we may obtain the following expressions for the affixes of the A's:

$$
\begin{aligned}
a_1 &= (-r_2'r_3b_1 + r_3r_1b_2 + r_1'r_2'b_3)/(r_1r_2r_3 + r_1'r_2'r_3'),\\
a_2 &= (r_2'r_3'b_1 - r_3'r_1b_2 + r_1r_2b_3)/(r_1r_2r_3 + r_1'r_2'r_3'),\\
a_3 &= (r_2r_3b_1 + r_3'r_1'b_2 - r_1'r_2b_3)/(r_1r_2r_3 + r_1'r_2'r_3').
\end{aligned} \tag{5}
$$

THEOREM 3.1. <u>Suppose triangle</u> $B_1B_2B_3$ <u>is such that there exists a solution triangle</u> $A_1A_2A_3$ <u>which is bordered by triangles</u> $B_1A_2A_3$, $B_2A_3A_1$, $B_3A_1A_2$ <u>of fixed shapes. If on the sides of triangle</u> $B_1B_2B_3$ <u>triangles</u> $C_1B_3B_2$, $C_2B_1B_3$, $C_3B_2B_1$ <u>are constructed directly similar to triangles</u> $B_1A_2A_3$, $B_2A_3A_1$, $B_3A_1A_2$, <u>then triangle</u> $C_1C_2C_3$ <u>will be directly similar to triangle</u> $A_1A_2A_3$.

The affix of C_1 is given by

$$
\begin{aligned}
c_1 &= r_1'b_3 + r_1b_2\\
&= r_1'(r_3'a_1 + r_3a_2) + r_1(r_2'a_3 + r_2a_1)\\
&= a_1(r_1'r_3' + r_1r_2) + a_2r_1'r_3 + a_3r_1r_2'.
\end{aligned}
$$

Similarly,

$$c_2 = a_1r_2r_3' + a_2(r_2'r_1' + r_2r_3) + a_3r_2'r_1$$

and

$$c_3 = a_1r_3'r_2 + a_2r_3r_1' + a_3(r_3'r_2' + r_3r_1).$$

Therefore

$$\begin{aligned} c_1 - c_2 &= a_1(r_1'r_3' + r_1r_2 - r_2r_3') - a_2(r_2'r_1' + r_2r_3 - r_1'r_3) \\ &= (a_1 - a_2)(r_1r_2r_3 + r_1'r_2'r_3'). \end{aligned}$$

Similarly,

$$c_3 - c_1 = (a_3 - a_1)(r_1r_2r_3 + r_1'r_2'r_3').$$

Thus triangle $C_1C_2C_3$ is directly similar to triangle $A_1A_2A_3$.

THEOREM 3.2. <u>A necessary and sufficient condition that triangle $A_1A_2A_3$ be directly similar to a given triangle $D_1D_2D_3$, no matter what triangle $B_1B_2B_3$ is chosen, is that a point F exist such that triangles $B_1A_2A_3$, $B_2A_3A_1$, $B_3A_1A_2$ are directly similar to triangles FD_2D_3, FD_3D_1, FD_1D_2.</u>

From theorem 3.1 we have triangle $C_1C_2C_3$ directly similar to triangle $A_1A_2A_3$, and from theorem 2.2, a necessary and sufficient condition that triangle $C_1C_2C_3$ be directly similar to triangle $D_1D_2D_3$, no matter what triangle $B_1B_2B_3$ is chosen, is that a point F exist such that triangles $C_1B_3B_2$, $C_2B_1B_3$, $C_3B_2B_1$ are directly similar to triangles FD_2D_3, FD_3D_1, FD_1D_2. But triangles $C_1B_3B_2$, $C_2B_1B_3$, $C_3B_2B_1$ are directly similar to triangles $B_1A_2A_3$, $B_2A_3A_1$, $B_3A_1A_2$. Thus triangles $B_1A_2A_3$,

$B_2A_3A_1$, $B_3A_1A_2$ are directly similar to triangles FD_2D_3, FD_3D_1, FD_1D_2.

COROLLARY 3.3. A necessary and sufficient condition that points A_1, A_2, A_3 be collinear is that $\measuredangle B_1A_3A_2 = \measuredangle B_2A_3A_1$, $\measuredangle B_2A_1A_3 = \measuredangle B_3A_1A_2$, and $\measuredangle B_3A_2A_1 = \measuredangle B_1A_2A_3$.

Since triangles $B_1A_2A_3$, $B_2A_3A_1$, $B_3A_1A_2$ are directly similar to triangles FD_2D_3, FD_3D_1, FD_1D_2, corresponding angles must be equal. But, for D_1, D_2, D_3 to be collinear, it is necessary and sufficient that $\measuredangle FD_3D_2 = \measuredangle FD_3D_1$, $\measuredangle FD_1D_3 = \measuredangle FD_1D_2$, and $\measuredangle FD_2D_1 = \measuredangle FD_2D_3$. This proves the corollary.

If triangles $B_1A_2A_3$, $B_2A_3A_1$, $B_3A_1A_2$ are similar isosceles triangles, of fixed shape, with vertices at B_1, B_2, B_3, then the affixes of A's are given by

$$(6)\qquad \begin{aligned} a_1 &= (-r\bar{r}b_1 + r^2b_2 + \bar{r}^2b_3)/(r^3 + \bar{r}^3), \\ a_2 &= (\bar{r}^2b_1 - r\bar{r}b_2 + r^2b_3)/(r^3 + \bar{r}^3), \\ a_3 &= (r^2b_1 + \bar{r}^2b_2 - r\bar{r}b_3)/(r^3 + \bar{r}^3), \end{aligned}$$

where $r = (1 + i\tan\beta)/2$, β being the base angle of the bordering isosceles triangles.

If $\beta = \pm 30^\circ$, then $r = (\sqrt{3}/3)e^{\pm i\pi/6}$, and

$$r^3 + \bar{r}^3 = (\sqrt{3}/9)(e^{\pm i\pi/2} + e^{\mp i\pi/2}) = 0,$$

and the denominators in (6) vanish. The numerators will also vanish if and only if (considering the numerator of $(6)_1$, for example)

$$\begin{aligned} 0 &= 3(-r\bar{r}b_1 + r^2b_2 + \bar{r}^2b_3) \\ &= -b_1 + e^{\pm i\pi/3}\, b_2 + e^{\mp i\pi/3}\, b_3 \\ &= e^{\pm i\pi/3}(b_2 - b_3) - (b_1 - b_3), \end{aligned}$$

that is, if and only if triangle $B_1B_2B_3$ is equilateral. Thus we have the following theorem.

THEOREM 3.4. <u>If triangles $B_1A_2A_3$, $B_2A_3A_1$, $B_3A_1A_2$ are isosceles triangles with base angles equal to 30°, then there exists a solution triangle $A_1A_2A_3$ if and only if triangle $B_1B_2B_3$ is equilateral, in which case there exists an infinite number of solutions.</u> (3, p.9)

In the rest of this section the bordered triangles will be considered as exteriorly constructed, unless explicit mention is made to the contrary. If the bordering triangles are similar isosceles and $r^3 + \bar{r}^3 \neq 0$, a necessary condition that a solution exist is that the area of $A_1A_2A_3$, in this order be negative, where we take $B_1B_2B_3$ negative. This can be written

$$-A' \equiv \frac{i}{4}\begin{vmatrix} a_1 & \bar{a}_1 & 1 \\ a_2 & \bar{a}_2 & 1 \\ a_3 & \bar{a}_3 & 1 \end{vmatrix} < 0.$$

Substituting the values for the a's in terms of the b's, we have

$$-4(\bar{r}^3 + r^3)^2A' \equiv i\begin{vmatrix} -b_1r\bar{r} + b_2r^2 + b_3\bar{r}^2 & -\bar{b}_1r\bar{r} + \bar{b}_2\bar{r}^2 + \bar{b}_3r^2 & 1 \\ b_1\bar{r}^2 - r\bar{r}b_2 + r^2b_3 & \bar{b}_1r^2 - \bar{b}_2r\bar{r} + \bar{b}_3\bar{r}^2 & 1 \\ b_1r^2 + b_2\bar{r}^2 - b_3r\bar{r} & \bar{b}_1\bar{r}^2 + \bar{b}_2r^2 - \bar{b}_3r\bar{r} & 1 \end{vmatrix}$$

$$= i\begin{vmatrix} -b_1(r^2+r\bar{r}) + b_2(r^2-\bar{r}^2) + b_3(\bar{r}^2+r\bar{r}) & -\bar{b}_1(\bar{r}^2+r\bar{r}) + \bar{b}_2(\bar{r}^2-r^2) + \bar{b}_3(r^2+r\bar{r}) \\ b_1(\bar{r}^2-r^2) - b_2(r\bar{r}+\bar{r}^2) + b_3(r^2+r\bar{r}) & \bar{b}_1(r^2-\bar{r}^2) - \bar{b}_2(r\bar{r}+r^2) + \bar{b}_3(\bar{r}^2+r\bar{r}) \end{vmatrix}$$

$$= i\Big\{r^2\big[(b_1-b_2)(\bar{b}_2-\bar{b}_1)+(b_1-b_3)(\bar{b}_3-\bar{b}_2)\big] + r\bar{r}\big[(b_2-b_1)(\bar{b}_3-\bar{b}_1)-(b_3-b_1)(\bar{b}_2-\bar{b}_1)$$
$$+ (b_2-b_1)(\bar{b}_3-\bar{b}_2)-(b_3-b_2)(\bar{b}_2-\bar{b}_1)\big] + \bar{r}^2\big[(b_3-b_2)(\bar{b}_3-\bar{b}_1)+(b_2-b_1)(\bar{b}_2-\bar{b}_1)\big]\Big\}.$$

If we denote by B the area of triangle $B_1B_3B_2$ and by k_1, k_2 and k_3 the lengths of the sides,

$$-4A'(r^3 + \bar{r}^3)^2 = i\left[-12Br^2/i + r(12B/i - \sum k_i^2) + (-2B/i + \sum k_i^2/2)\right]$$
$$= -12Br^2 + 12Br - 2B + i(\tfrac{1}{2} - r)\sum k_i^2$$
$$= B(3t^2 + 1) + t\sum k_i^2/2 \quad \text{where } t = \tan\beta .$$

THEOREM 3.5. <u>If the bordering triangles are similar isosceles triangles, a necessary condition that a solution triangle</u> $A_1A_2A_3$ <u>exist is that</u> $B(3t^2 + 1) + (t/2)\sum k_i^2 < 0$.

COROLLARY 3.6. <u>If triangle</u> $B_1B_2B_3$ <u>is a right triangle and the bordering triangles are right isosceles triangles, there exists no solution triangle</u> $A_1A_2A_3$.

For $B = ab/2$, $t = 1$ and our condition is $4ab/2 + (a^2 + b^2 + c^2)/2 < 0$, $4ab + c^2 < 0$, which is impossible.

THEOREM 3.7. <u>If the bordering triangles are isosceles, two of which are directly and one of which is inversely similar to a given triangle, then quadrilateral</u> $A_iB_jB_iB_k$, B_i <u>being the vertex of the inversely similar triangle, will be a parallelogram.</u> (3, p.4)

Suppose B_1 is the vertex of the inversely similar triangle. Then $a_1 - b_2 = b_3 - b_1$, $a_1 - b_3 = b_2 - b_1$ is a condition that $A_1B_2B_1B_3$ be a parallelogram. The affixes of the B's may be written

$$b_1 = ra_2 + \bar{r}a_3$$
$$b_2 = \bar{r}a_3 + ra_1$$
$$b_3 = \bar{r}a_1 + ra_2$$

$$a_1 - b_2 = a_1(1 - r) - \bar{r}a_3 = \bar{r}(a_1 - a_3)$$

$$b_3 - b_1 = \bar{r}(a_1 - a_3) = a_1 - b_2$$

$$a_1 - b_3 = r(a_1 - a_2) = b_2 - b_1$$

thus the theorem.

THEOREM 3.8. <u>A necessary condition that a solution triangle</u> $A_1A_2A_3$ <u>exist for any triangle</u> $B_1B_2B_3$ <u>is that</u>

$$\begin{aligned}\{&r_1\bar{r}'_3[(k_2^2 - k_3^2 - k_1^2)i/2 + 2B] - \bar{r}_1r'_3[(k_2^2 - k_3^2 - k_1^2)i/2 - 2B]\\ &+ r_2\bar{r}'_1[(k_3^2 - k_1^2 - k_2^2)i/2 + 2B] - \bar{r}_2r'_1[(k_3^2 - k_1^2 - k_2^2)i/2 - 2B]\\ &+ \bar{r}'_2r_3[(k_1^2 - k_2^2 - k_3^2)i/2 + 2B] - r'_2\bar{r}_3[(k_1^2 - k_2^2 - k_3^2)i/2 - 2B]\\ &- 4B\}/(r_1r_2r_3 + r'_1r'_2r'_3)(\bar{r}_1\bar{r}_2\bar{r}_3 + \bar{r}'_1\bar{r}'_2\bar{r}'_3) > 0.\end{aligned}$$

Here B is the area of triangle $B_1B_3B_2$ and k_1, k_2, and k_3 are the lengths of the sides. In order that a solution triangle $A_1A_2A_3$ exist, A, the area of triangle $A_1A_3A_2$ must be positive. We may write

$$-A \equiv i/4 \begin{vmatrix} a_1 & \bar{a}_1 & 1 \\ a_2 & \bar{a}_2 & 1 \\ a_3 & \bar{a}_3 & 1 \end{vmatrix} < 0$$

$$-4A \equiv i \begin{vmatrix} a_1 - a_3 & \bar{a}_1 - \bar{a}_3 \\ a_2 - a_3 & \bar{a}_2 - \bar{a}_3 \end{vmatrix}$$

$$= i \begin{vmatrix} -b_1r_3 + b_2(r_1 - r'_3) + b_3r'_1 & -\bar{b}_1\bar{r}_3 + \bar{b}_2(\bar{r}_1 - \bar{r}'_3) + \bar{b}_3\bar{r}'_1 \\ b_1(r'_2 - r_3) - b_2r'_3 + b_3r_2 & \bar{b}_1(\bar{r}'_2 - \bar{r}_3) - \bar{b}_2\bar{r}'_3 + \bar{b}_3\bar{r}_2 \end{vmatrix}$$

By a straightforward expansion of this determinant we arrive at theorem 3.8.

COROLLARY 3.9. *If the bordering triangles are isosceles triangles of fixed shapes, a necessary condition that a solution triangle* $A_1A_2A_3$ *exist for any triangle* $B_1B_2B_3$ *is that*

$$\left\{r_1(4B + ik_1^2) + r_3(4B + ik_2^2) + r_3(4B + ik_3^2) - 4B(r_1r_2 + r_1r_3 + r_2r_3) - \tfrac{1}{2}\left[i\sum k_i^2 + 4B\right]\right\}/(r_1r_2r_3 + \bar{r}_1\bar{r}_2\bar{r}_3)^2 < 0.$$

For, if the triangles are isosceles, $\bar{r}_1 = r_1'$, $\bar{r}_2 = r_2'$, $\bar{r}_3 = r_3'$ and from theorem 3.9,

$$+4A = i\Big\{\left[r_1+r_3-1\right]\left[(k_2^2-k_3^2-k_1^2)/2-2B/i\right] + \left[r_1+r_2-1\right]\left[(k_3^2-k_2^2-k_1^2)/2 -2B/i\right] + \left[r_2+r_3-1\right]\left[(k_1^2-k_2^2-k_3^2)/2-2B/i\right] + 4B\left[r_1r_2 + r_1r_3 + r_2r_3\right]/i -4B/i\Big\}/(r_1r_2r_3 + \bar{r}_1\bar{r}_2\bar{r}_3)^2$$

$$+4A = \left[-r_1(4B + ik_1^2) - r_2(4B + ik_2^2) - r_3(4B + ik_3^2) +4B(r_1r_2 + r_2r_3 + r_1r_3) + i(k_1^2 + k_2^2 + k_3^2 + 4B/i)\right]/2\ (r_1r_2r_3 + \bar{r}_1\bar{r}_2\bar{r}_3)^2.$$

4. TRIANGLES BORDERED BY SQUARES

Suppose on the sides of a given triangle ABC we construct squares ABA_2A_1, BCB_2B_1, and CAC_2C_1 externally and squares ABA_4A_3, BCB_4B_3, and CAC_4C_3 internally. The centers of the externally described squares will be designated by A', B', and C', and those of the internally described squares by A", B", and C". The affixes of these points are given by

(1) $a_1 = (1 - i)a + ib$ $\qquad$ $a_3 = (1 + i)a - ib$

$a_2 = -ia + (1 + i)b$ $\qquad$ $a_4 = ia + (1 - i)b$

$a' = (1-i)a + (1+i)b /2$ $\qquad$ $a'' = (1+i)a + (1-i)b /2$

$b_1 = (1 - i)b + ic$ $\qquad$ $b_3 = (1 + i)b - ic$

$b_2 = -ib + (1 + i)c$ $\qquad$ $b_4 = ib + (1 - i)c$

$b' = (1-i)b + (1+i)c /2$ $\qquad$ $b'' = (1+i)b + (1-i)c /2$

$c_1 = (1 - i)c + ia$ $\qquad$ $c_3 = (1 + i)c - ia$

$c_2 = -ic + (1 + i)a$ $\qquad$ $c_4 = ic + (1 - i)a$

$c' = (1-i)c + (1+i)a /2$ $\qquad$ $c'' = (1+i)c + (1-i)a /2$

THEOREM 4.1. The centroids of triangles ABC, $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$, $A_4B_4C_4$, A'B'C', and A"B"C" coincide.

$$3g = a + b + c,$$

$$3g_1 = (1 - i)(a + b + c) + i(a + b + c) - 3g,$$

$$3g_2 = -i(a + b + c) + (1 + i)(a + b + c) = 3g,$$

$$3g_3 = (1 + i)(a + b + c) - i(a + b + c) = 3g,$$

$$3g_4 = i(a + b + c) + (1 - i)(a + b + c) = 3g,$$

$$3g' = (1 - i)(a + b + c) + (1 + i)(a + b + c) /2 = 3g,$$

$$3g'' = (1 + i)(a + b + c) + (1 - i)(a + b + c) /2 = 3g.$$

THEOREM 4.2. The areas of the triangles are given by $\Delta_1 = \Delta_2 = 4\Delta + \sum k_i^2/4$, $\Delta_3 = \Delta_4 = 4\Delta - \sum k_i^2/4$, $\Delta' = \Delta + \sum k_i^2/8$, $\Delta'' = \Delta - \sum k_i^2/8$, where k_1, k_2, and k_3 are the lengths of the sides and Δ, Δ_1, Δ_2, Δ', Δ'' are the areas of triangles ACB, $A_1C_1B_1$, $A_2C_2B_2$, A'C'B', A"C"B".

$$-4\Delta_1/i \equiv -4\Delta_2/i \equiv \begin{vmatrix} (1-i)a + ib & (1+i)\bar{a} - i\bar{b} & 1 \\ (1-i)b + ic & (1+i)\bar{b} - i\bar{c} & 1 \\ (1-i)c + ia & (1+i)\bar{c} - i\bar{a} & 1 \end{vmatrix}$$

By the laws of addition of determinants,

$$-4\Delta_1/i \equiv (1+i)(1-i)\begin{vmatrix} a & \bar{a} & 1 \\ b & \bar{b} & 1 \\ c & \bar{c} & 1 \end{vmatrix} - i(1-i)\begin{vmatrix} a & \bar{b} & 1 \\ b & \bar{c} & 1 \\ c & \bar{a} & 1 \end{vmatrix}$$

$$+ \; i \begin{vmatrix} b & \bar{a} & 1 \\ c & \bar{b} & 1 \\ a & \bar{c} & 1 \end{vmatrix} + \begin{vmatrix} b & \bar{b} & 1 \\ c & \bar{c} & 1 \\ a & \bar{a} & 1 \end{vmatrix} .$$

$$-4\Delta_1/i = -16\Delta/i + i\sum k_i^2, \quad \Delta_1 = 4\Delta + \sum k_i^2/4.$$

$$-4\Delta_3/i \equiv -4\Delta_4/i \equiv \begin{vmatrix} (1+i)a - ib & (1-i)\bar{a} + i\bar{b} & 1 \\ (1+i)b - ic & (1-i)\bar{b} + i\bar{c} & 1 \\ (1+i)c - ia & (1-i)\bar{c} + i\bar{a} & 1 \end{vmatrix}$$

$$-4\Delta_3/i \equiv (1+i)(1-i)\begin{vmatrix} a & \bar{a} & 1 \\ b & \bar{b} & 1 \\ c & \bar{c} & 1 \end{vmatrix} + i(1+i)\begin{vmatrix} a & \bar{b} & 1 \\ b & \bar{c} & 1 \\ c & \bar{a} & 1 \end{vmatrix}$$

$$- i(1-i)\begin{vmatrix} b & \bar{a} & 1 \\ c & \bar{b} & 1 \\ a & \bar{c} & 1 \end{vmatrix} + \begin{vmatrix} b & \bar{b} & 1 \\ c & \bar{c} & 1 \\ a & \bar{a} & 1 \end{vmatrix}$$

$$-4\Delta_3/i = -8\Delta/i - i\sum k_i^2 - 4\Delta/i - 4\Delta/i$$

$$\Delta_3 = \Delta_4 = 4\Delta - \sum k_i^2/4$$

$$-4\Delta'/i = 1/4\begin{vmatrix} (1-i)a + (1+i)b & (1+i)\bar{a} + (1-i)\bar{b} & 1 \\ (1-i)b + (1+i)c & (1+i)\bar{b} + (1-i)\bar{c} & 1 \\ (1-i)c + (1+i)a & (1+i)\bar{c} + (1-i)\bar{a} & 1 \end{vmatrix}$$

$$-16\Delta'/i = (1+i)(1-i)\begin{vmatrix} a & \bar{a} & 1 \\ b & \bar{b} & 1 \\ c & \bar{c} & 1 \end{vmatrix} + (1-i)^2\begin{vmatrix} a & \bar{b} & 1 \\ b & \bar{c} & 1 \\ c & \bar{a} & 1 \end{vmatrix}$$

$$+ (1+i)^2\begin{vmatrix} b & \bar{a} & 1 \\ c & \bar{b} & 1 \\ a & \bar{c} & 1 \end{vmatrix} + (1+i)(1-i)\begin{vmatrix} b & \bar{b} & 1 \\ c & \bar{c} & 1 \\ a & \bar{a} & 1 \end{vmatrix}$$

$$\Delta' = \Delta/2 + \sum k_i^2/8 + \Delta/2 = \Delta + \sum k_i^2/8.$$

$$-4\Delta''/i = 1/4\begin{vmatrix} (1+i)a + (1-i)b & (1-i)\bar{a} + (1+i)\bar{b} & 1 \\ (1+i)b + (1-i)c & (1-i)\bar{b} + (1+i)\bar{c} & 1 \\ (1+i)c + (1-i)a & (1-i)\bar{c} + (1+i)\bar{a} & 1 \end{vmatrix}$$

$$-16\Delta''/i = (1+i)(1-i)\begin{vmatrix} a & \bar{a} & 1 \\ b & \bar{b} & 1 \\ c & \bar{c} & 1 \end{vmatrix} + (1+i)^2\begin{vmatrix} a & \bar{b} & 1 \\ b & \bar{c} & 1 \\ c & \bar{a} & 1 \end{vmatrix}$$

$$+ (1-i)^2\begin{vmatrix} b & \bar{a} & 1 \\ c & \bar{b} & 1 \\ a & \bar{c} & 1 \end{vmatrix} + (1+i)(1-i)\begin{vmatrix} b & \bar{b} & 1 \\ c & \bar{c} & 1 \\ a & \bar{a} & 1 \end{vmatrix}$$

THEOREM 4.3. <u>The sums of the squares of the lengths of the principal diagonals</u> A'B", B'C", C'A", <u>of the hexagon</u> A'A"B'B"C'C" <u>formed by the centers of the squares is equal to twice the sum of the squares of the lengths of the sides of triangle</u> ABC.

$$a' - b'' = (1-i)(a-c) \qquad \bar{a}' - \bar{b}'' = (1+i)(\bar{a}-\bar{c})$$

$$b' - c'' = (1-i)(b-a) \qquad \bar{b}' - \bar{c}'' = (1+i)(\bar{b}-\bar{a})$$

$$c' - a'' = (1-i)(c-b) \qquad \bar{c}' - \bar{a}'' = (1+i)(\bar{c}-\bar{b})$$

$$(a' - b'')(\bar{a}' - \bar{b}'') + (b' - c'')(\bar{b}' - \bar{c}'') + (c' - a'')(\bar{c}' - \bar{a}'')$$

$$= (1-i)(1+i)\ (a-c)(\bar{a}-\bar{c}) + (b-a)(\bar{b}-\bar{a}) + (c-b)(\bar{c}-\bar{b})$$

$$= 2\ \Sigma k_1^2.$$

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