SIAM J. MATH. ANAL.
Vol. 1, No. 2, May 1970

WELL-POSED PROBLEMS FOR A PARTIAL DIFFERENTIAL EQUATION OF ORDER $2m + 1$*

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We are concerned here with well-posed problems for the partial differential equation

$$u_t(x, t) + \gamma M u_t(x, t) + Lu(x, t) = f(x, t)$$

containing the elliptic differential operator M of order $2m$ and the differential operator L of order $\leqq 2m$. Hilbert space methods are used to formulate and solve an abstract form of the problem and to discuss existence, uniqueness, asymptotic behavior and boundary conditions of a solution.

The formulation of a generalized problem is the objective of § 1, and we shall have reason to consider two types of solutions, called weak and strong. Sufficient conditions on the operator M are given for the existence and uniqueness of a weak solution to the generalized problem. These conditions constitute elliptic hypotheses on M and are discussed briefly in § 3. Similar assumptions on L lead to results on the asymptotic behavior of a weak solution. The case in which M and L are equal and self-adjoint is discussed in § 2, and it is here that the role of the coefficient γ of the equation appears first. Special as it is, this is a situation that often arises in applications, and there has been considerable interest in this coefficient γ [4], [25]. The weak and strong solutions are distinguished not only by regularity conditions but also by their associated boundary conditions. It first appears in § 5 that it is possible to prescribe too many (independent) boundary conditions on a strong solution, but in the applications it is seen that the interdependence of these conditions is built into the assumptions on the domains of the operators. Two examples of applications appear in § 6 with a discussion of the types of boundary conditions that are appropriate.

1. The generalized problem. Let G be a nonempty open set in the n-dimensional real Euclidean space, $\mathbb{R}^n$, whose boundary ∂G is an $(n - 1)$-dimensional manifold with G lying on one side of it. $C^\infty(G)$ is the space of infinitely differentiable functions on G, and $C_0^\infty(G)$ is the linear subspace of $C^\infty(G)$ consisting of functions with compact support in G. The Sobolev space $H^m(G) = H^m$ is the Hilbert space of (equivalence classes of) functions in $L^2(G)$, all of whose distributional derivatives through order m belong to $L^2(G)$. The inner product and norm are given, respectively, by

$$(f, g)_m = \sum \left\{ \int_G D^\alpha f \overline{D^\alpha g}\, dx : |\alpha| \leqq m \right\}$$

and $\|f\|_m = \sqrt{(f,f)_m}$, where $\alpha = (\alpha_1, \cdots, \alpha_n)$ denotes an n-tuple of nonnegative integers, and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_2^{\alpha_2}}$$

is a derivative of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

* Received by the editors September 18, 1969.

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$H_0^m(G) = H_0^m$ is the closure of $C_0^\infty(G)$ in H^m; it is known that if ∂G is m times continuously differentiable and ϕ is in $C^{m-1}(\mathrm{cl}\,(G))$ then ϕ is in $H_0^m(G)$ if and only if it is in $H^m(G)$ and vanishes on ∂G together with all derivatives of order $\leqq m - 1$. Hence, $\phi \in H_0^m$ is a weak Dirichlet boundary condition. In order to determine other boundary conditions, we let V be a closed subspace of H^m that contains $C_0^\infty(G)$ and define the norm on V by $\|\phi\|_V = \|\phi\|_m$ for $\phi \in V$.

We shall consider the equation

$$u'(t) + \gamma \mathscr{M} u'(t) + \mathscr{L} u(t) = f(t) \tag{1.1}$$

containing the indicated vector-valued functions and the partial differential operators of order $2m$ in the divergence forms

$$\mathscr{M} = \sum \{(-1)^{|\rho|} D^\rho m^{\rho\sigma}(x) D^\sigma : |\rho|, |\sigma| \leqq m\}, \tag{1.2}$$

$$\mathscr{L} = \sum \{(-1)^{|\rho|} D^\rho l^{\rho\sigma}(x) D^\sigma : |\rho|, |\sigma| \leqq m\}. \tag{1.3}$$

Since we are concerned with *weak* solutions, it suffices to require only that the coefficients in (1.2) and (1.3) be bounded and measurable on G. This implies that the sesquilinear forms

$$m(\phi, \psi) = \sum \{(m^{\rho\sigma} D^\sigma \phi, D^\rho \psi)_0 : |\rho|, |\sigma| \leqq m\}, \tag{1.4}$$

$$l(\phi, \psi) = \sum \{(l^{\rho\sigma} D^\sigma \phi, D^\rho \psi)_0 : |\rho|, |\sigma| \leqq m\} \tag{1.5}$$

are bounded on V; in particular, for all ϕ and ψ in V we have

$$|m(\phi, \psi)| \leqq K_m \|\phi\|_V \|\psi\|_V, \tag{1.6}$$

$$|l(\phi, \psi)| \leqq K_l \|\phi\|_V \|\psi\|_V, \tag{1.7}$$

where $K_m = \sup\{\|m^{\rho\sigma}\|_\infty\}$ and $K_l = \sup\{\|l^{\rho\sigma}\|_\infty\}$. These sesquilinear forms can be used to specify solutions of (1.1) in V, since for any u in V the conjugate linear maps $\phi \mapsto m(u, \phi)$ and $\phi \mapsto l(u, \phi)$ are continuous from $\mathscr{D}(G)$ into $\mathbb{C}$, where $\mathscr{D}(G)$ is the linear space $C_0^\infty(G)$ with the topology of L. Schwartz [11], [19]. These maps determine elements of $\mathscr{D}'(G)$, the space of distributions, and they satisfy

$$m(u, \phi) = \langle \mathscr{M} u, \bar{\phi} \rangle, \tag{1.8}$$

$$l(u, \phi) = \langle \mathscr{L} u, \bar{\phi} \rangle \tag{1.9}$$

for all ϕ in $\mathscr{D}(G)$. The operators $\mathscr{M}$ and $\mathscr{L}$ map V into $\mathscr{D}'(G)$.

Let H be the Hilbert space $L^2(G)$. Define linear subsets of H by $D(M) = \{u \in V : \mathscr{M}(u) \in H$ and (1.8) holds for all $\phi \in V\}$ and $D(L) = \{u \in V : \mathscr{L}(u) \in H$ and (1.9) holds for all ϕ in $V\}$, and let M and L denote the restrictions of $\mathscr{M}$ and $\mathscr{L}$ to $D(M)$ and $D(L)$, respectively. Then M and L are unbounded operators on H whose domains are contained in V [3], [10]. Furthermore, for any u in $D(M)$,

$$m(u, v) = (Mu, v)_H \tag{1.10}$$

for all v in V and

$$l(u, v) = (Lu, v)_H \tag{1.11}$$

for all u in $D(L)$ and v in V.

The *generalized problem* is the following: Let V and H be Hilbert spaces for which the injections $\mathscr{D}(G) \hookrightarrow V \hookrightarrow H$ are continuous and $\mathscr{D}(G)$ is dense in H. Let m and l be sesquilinear forms on V which satisfy (1.6) and (1.7). Let u_0 belong to V, and let f be a continuous map of $\mathbb{R}$ into H. Find a continuously differentiable function u of $\mathbb{R}$ into V such that $u(0) = u_0$ and

$$(u'(t), v)_H + \gamma m(u'(t), v) + l(u(t), v) = (f(t), v)_H \tag{1.12}$$

for all v in V and t in $\mathbb{R}$.

A solution of the generalized problem is a *weak solution* of (1.1), since for all ϕ in $C_0^\infty(G)$ it follows from (1.8) and (1.9) that

$$\langle u'(t), \bar{\phi}\rangle + \gamma\langle \mathscr{M}u'(t), \bar{\phi}\rangle + \langle \mathscr{L}u(t), \bar{\phi}\rangle = \langle f(t), \bar{\phi}\rangle,$$

hence (1.1) holds in $\mathscr{D}'(G)$. Furthermore, if $u(t)$ belongs to $D(L)$ and $u'(t)$ to $D(M)$ for all t in $\mathbb{R}$, then

$$u'(t) + \gamma Mu'(t) + Lu(t) = f(t)$$

in H, and $u(t)$ is called a *strong solution* of (1.12).

We shall hereafter assume, with no loss of generality, that

$$\|\phi\|_V \geqq \|\phi\|_H \tag{1.13}$$

for ϕ in V.

2. A special case with $L = M = L^*$. We first use the method of eigenfunction expansions to obtain a rather precise description of solutions of the generalized problem with $m \equiv l$. Assume that

$$l(u, v) = \overline{l(v, u)} \qquad \text{for all } u, v \text{ in } V, \tag{2.1}$$

$$l(u, u) \geqq k_l\|u\|_V^2 \qquad \text{for all } u \text{ in } V, k_l > 0, \tag{2.2}$$

and the injection

$$V \hookrightarrow H \quad \text{is completely continuous.} \tag{2.3}$$

The condition (2.1) implies that L is symmetric, while (2.2) implies that L is a bijection of $D(L)$ onto H [11], [12], [15], [16]. In fact, (2.2) and (1.11) imply that for any ϕ in H,

$$k_l\|L^{-1}\phi\|_V^2 \leqq (\phi, L^{-1}\phi)_H \leqq \|\phi\|_H\|L^{-1}\phi\|_H,$$

so L^{-1} is continuous from H into V and satisfies

$$\|L^{-1}\phi\|_V \leqq k_l^{-1}\|\phi\|_H$$

for all ϕ in H. The condition (2.3) will be satisfied if G is bounded and either $V = H_0^m(G)$ or ∂G is sufficiently smooth [1], [5], [16].

From (1.7), (2.1) and (2.2) it follows that the sesquilinear form $[u, v] \equiv l(u, v)$ is an inner product on V for which the associated norm $\|u\|_l \equiv [u, u]^{1/2}$ is equivalent to the norm $\|u\|_V$. Letting K be the restriction of L^{-1} to V, we see that

$$[Ku, v] = (u, v)_H \tag{2.4}$$

for all u and v in V, and from (2.1) it follows that K is symmetric on V with respect to

$[\cdot,\cdot]$. Also, K is the composition of the continuous operator $L^{-1}: H \to V$ and the completely continuous injection, so K is a completely continuous operator on V.

The spectral resolution of completely continuous and symmetric operators is well known [18]: there is a complete orthonormal sequence $\{\phi_n\}$ of eigenvectors of K in V and associated eigenvalues $\{\rho_n\}$ such that

(2.5a) $$K\phi_n = \rho_n\phi_n \qquad \text{for all } n \geqq 1,$$

(2.5b) $$[\phi_m, \phi_n] = \delta_{m,n} \qquad \text{for all } m, n \geqq 1,$$

(2.5c) $$\rho_1 \geqq \rho_2 \geqq \rho_3 \geqq \cdots \rho_n \to 0 \quad \text{as } n \to \infty,$$

and every v in V can be written as

(2.5d) $$v = \sum_{n \geqq 1} [v, \phi_n]\phi_n.$$

Let $\lambda_n = (\rho_n)^{-1}$; the sequence $\{\lambda_n\}$ is nondecreasing and unbounded by (2.5c), and $L\phi_n = \lambda_n\phi_n$ for $n \geqq 1$. Since $(\phi_m, \phi_n)_H = [K\phi_m, \phi_n] = \rho_m\delta_{m,n}$ for $m, n \geqq 1$, the sequence $\{\lambda_n^{1/2}\phi_n\}$ is orthonormal in H. It is also complete, for if f is in H there is a u in V with $Lu = f$. The sequence $u_n = \sum_{k=1}^{n} [u, \phi_k]\phi_k = \sum_{k=1}^{n} (f, \phi_k)_H\phi_k$ converges in V to u, hence $u_n \to u$ in H. The sequence $Lu_n = \sum_{k=1}^{n} (f, \lambda_k^{1/2}\phi_k)_H\lambda_k^{1/2}\phi_k$ converges in H, since it is the Fourier expansion of f by $\{\lambda_k^{1/2}\phi_k\}$, so L being closed implies $Lu = f = \sum_{k=1}^{\infty} (f, \lambda_k^{1/2}\phi_k)_H\lambda_k^{1/2}\phi_k$.

Let $u(t)$ be a solution of the generalized problem. For each t in $\mathbb{R}$ there is a unique sequence $\{u_n(t)\}$ of complex numbers for which

(2.6) $$u(t) = \sum_{n \geqq 1} u_n(t)\phi_n.$$

These Fourier coefficients are given by $u_n(t) = [u(t), \phi_n]$, so each is a continuously differentiable function which satisfies the initial condition

(2.7) $$u_n(0) = [u_0, \phi_n].$$

If $s_n(t)$ denotes the nth partial sum of the series (2.6), then $s_n(t)$ converges to $u(t)$ in V. The continuity of $u(t)$ implies that this convergence is uniform on compact subsets of $\mathbb{R}$. To verify this, let $g_n(t) = \|u(t) - s_n(t)\|_l^2$. Then each g_n is continuous, the sequence $g_n(t)$ converges to zero for each t, and from

$$g_n(t) = \sum_{k=n+1}^{\infty} |u_k(t)|^2$$

it follows that the sequence is monotone, so the convergence is uniform on each compact subset of $\mathbb{R}$ by a well-known theorem of Dini [12].

Furthermore, the sequence of formal derivatives $\{s_n'(t)\}$ converges to $u'(t)$ in V. This follows by obtaining the Fourier expansion of $u'(t)$, which converges uniformly on compact subsets of R as above, and integrating this series termwise to obtain $u(t)$. Since $s_n(t) \to u(t)$ and $s_n'(t) \to u'(t)$ in V, we have for any v in V, $l(s_n(t), v) \to l(u(t), v)$ and $\gamma l(s_n'(t), v) + (s_n'(t), v)_H \to \gamma l(u'(t), v) + (u'(t), v)_H$. The sequence

$\{\lambda_n^{1/2}\phi_n\}$ is orthonormal and complete in H, so

$$(f(t), v)_H = (\sum_{n \geqq 1} (f(t), \lambda_n^{1/2}\phi_n)_H \lambda_n^{1/2}\phi_n, v)_H$$

$$= \sum_{n \geqq 1} (f(t), \phi_n)_H[\phi_n, v].$$

Thus, for each t in $\mathbb{R}$ and v in V we have, by (2.4) and (2.5a),

$$\sum_{n \geqq 1} \{(\rho_n + \gamma)u_n'(t) + u_n(t) - (f(t), \phi_n)_H\}[\phi_n, v] = 0,$$

and this yields the necessary condition

$$(\rho_n + \gamma)u_n'(t) + u_n(t) = (f(t), \phi_n)_H, \qquad n \geqq 1, \tag{2.8}$$

for $u(t)$ to be a solution of (1.12).

Let M be the (finite) set of integers m for which $\gamma + \rho_m = 0$, and N the set of integers $n \geqq 1$ for which $\gamma + \rho_n \neq 0$. It follows from (2.7) and (2.8) that for all n in N,

$$u_n(t) = [u_0, \phi_n] \exp(-(\gamma + \rho_n)^{-1}t)$$
$$+ (\gamma + \rho_n)^{-1} \int_0^t \exp((\gamma + \rho_n)^{-1}(\tau - t))(f(\tau), \phi_n)_H \, d\tau,$$

and for m in M we must have $u_m(t) = (f(t), \phi_m)_H$. In particular, the initial function must satisfy the compatibility condition $[u_0, \phi_m] = (f(0), \phi_m)_H$ for all m in M. That is, $\lambda_m u_0 - f(0)$ is orthogonal in H to ϕ_m whenever $\gamma + \rho_m = 0$. These remarks verify the uniqueness and representation statements of the following result.

THEOREM 1. *With the assumptions* (2.1), (2.2) *and* (2.3), *the generalized problem of* § 1 *with* $m \equiv l$ *has at most one solution. A solution exists if and only if for each integer in* $M = \{m : \gamma + \rho_m = 0\}$, *the compatibility condition*

$$l(u_0, \phi_m) = (f(0), \phi_m)_H$$

holds and the function $t \mapsto (f(t), \phi_m)$ *is continuously differentiable. This solution is given by the expansion*

$$\begin{aligned} u(t) = &\sum_{n \in N} [u_0, \phi_n] \exp(-(\gamma + \rho_n)^{-1}t)\phi_n \\ &+ \sum_{n \in N} \left\{ \int_0^t (\gamma + \rho_n)^{-1} \exp((\gamma + \rho_n)^{-1}(\tau - t))(f(\tau), \phi_n)_H \, d\tau \right\} \phi_n \\ &+ \sum_{m \in M} (f(t), \phi_m)_H \phi_m, \end{aligned} \tag{2.9}$$

where N *is the set of integers* $n \geqq 1$ *with* $\gamma + \rho_n \neq 0$.

We need only to verify that the function defined by (2.9) is a solution of the problem. Since the sequence $\{\rho_n\}$ converges, the sequence $\{(\gamma + \rho_n)^{-1}\}$ is uniformly bounded for n in N. If K is a compact subset of $\mathbb{R}$ and $m \geqq n > \sup(M)$, then from the estimate

$$\left\| \sum_{k=n}^{m} [u_0, \phi_k] \exp(-(\gamma + \rho_n)^{-1} t) \phi_k \right\|_l^2 = \sum_{k=n}^{m} |[u_0, \phi_k] \exp(-(\gamma + \rho_n)^{-1} t)|^2$$
$$\leqq \sup\{\exp(-(\gamma + \rho_n)^{-1} t) : n \in N, t \in K\}^2 \cdot \sum_{k=n}^{m} |[u_0, \phi_k]|^2$$

for all t in K and from the convergence of the expansion of u_0 by $\{\phi_k\}$, it follows that the first series in (2.9) converges uniformly on each compact K in $\mathbb{R}$. A similar estimate shows that all derivatives of this series converge uniformly on compact subsets of $\mathbb{R}$, so these can be integrated term-by-term to show that the sum of this series is infinitely differentiable with respect to t in the V-norm (equivalently, the l-norm) and its derivatives are obtained by differentiating the series term-by-term.

In order to discuss the second term in (2.9), let $T > 0$ and $0 \leqq \tau \leqq T$. The continuity of $f:\mathbb{R} \to H$ and of $L^{-1}: H \to V$ imply that the function $L^{-1}f:\mathbb{R} \to V$ is continuous; hence, the series

$$\sum_{n=1}^{\infty} [L^{-1}f(\tau), \phi_n]\phi_n = \sum_{n=1}^{\infty} (f(\tau), \phi_n)_H \phi_n \tag{2.10}$$

converges to $L^{-1}f(\tau)$ for each τ in $\mathbb{R}$, and the convergence is uniform on $[0, T]$ by an argument as above which depends on the theorem of Dini. Letting η denote the supremum of the numbers

$$|(\gamma + \rho_n)^{-1} \exp(\gamma + \rho_n)^{-1}(\tau - t)|$$

over all n in N and τ in $[0, T]$, we obtain the estimate

$$\begin{aligned} &\left\| \sum_{k=n}^{m} (\gamma + \rho_k)^{-1} \exp((\gamma + \rho_k)^{-1}(\tau - t))(f(\tau), \phi_k)_H \phi_k \right\|_l^2 \\ &\qquad = \sum_{k=n}^{m} |(\gamma + \rho_k)^{-1} \exp((\gamma + \rho_k)^{-1}(\tau - t))(f(\tau), \phi_k)_H|^2 \\ &\qquad \leqq \eta^2 \sum_{k=n}^{m} |(f(\tau), \phi_k)_H|^2 \\ &\qquad = \eta^2 \left\| \sum_{k=n}^{m} (f(\tau), \phi_k)_H \phi_k \right\|_l^2 \end{aligned} \tag{2.11}$$

for τ in $[0, T]$ and $m \geqq n > \sup(M)$. But we have shown that the series (2.10) is uniformly convergent on $[0, T]$, hence uniformly Cauchy, so this shows that the series appearing in the first term of (2.11) is uniformly Cauchy on $[0, T]$. We may then integrate this series termwise with respect to τ over the interval $[0, t]$, and this integrated series converges uniformly for all t in $[0, T]$. Thus, the second series in (2.9) converges uniformly on compact subsets of $\mathbb{R}$ to a continuous function from $\mathbb{R}$ into V. Application of a similar argument to the termwise derivative of this series shows that the sum of this series is continuously differentiable in V, and its derivative is the limit in V of the termwise derivative of the series. The convergence is uniform on compact subsets of $\mathbb{R}$.

That the continuously differentiable V-valued function defined by (2.9) is the solution of the generalized problem follows easily by a routine computation similar to that which led to (2.8) above.

3. Existence of a solution. The objective in this section is to develop sufficient conditions to guarantee the existence of a solution to the generalized problem of § 1. This development depends on the Lax–Milgram theorem, which gives sufficient conditions on a sesquilinear form in the situation of § 1 for the associated unbounded operator to be onto [11], [15], [16], and the calculus of functions taking values in a Banach space [3], [9]. The major result is Theorem 2, and the two following corollaries give sufficient conditions on the parameter γ in order that the hypothesis of Theorem 2 be fulfilled for the case in which the operator $\mathscr{M}$ is elliptic.

Let the Hilbert spaces H and V and sesquilinear forms m and l be as specified in the generalized problem, and assume further that there is a constant $k > 0$ for which

$$|\gamma m(\phi, \phi) + \|\phi\|_H^2| \geqq k\|\phi\|_V^2 \tag{3.1}$$

for all ϕ in V. This implies that the operator $\gamma M + I$ is a bijection of $D(M)$ onto H and that $D(M)$ is dense in V. It follows then from (1.10), (1.11) and (1.7) that for any ϕ in $D(L)$

$$\begin{aligned} k\|(\gamma M + I)^{-1}L\phi\|_V^2 &\leqq |(L\phi, (\gamma M + I)^{-1}L\phi)_H| \\ &= |l(\phi, (\gamma M + I)^{-1}L\phi)| \\ &\leqq K_l\|\phi\|_V\|(\gamma M + I)^{-1}L\phi\|_V, \end{aligned}$$

and hence the estimate

$$\|(\gamma M + I)^{-1}L\phi\|_V \leqq (K_l/k)\|\phi\|_V \tag{3.2}$$

for all ϕ in $D(L)$. If $D(L)$ is dense in V, it follows from (3.2) that $(\gamma M + I)^{-1}L$ has a unique bounded extension from V into V which we shall hereafter denote by B.

Since B belongs to the Banach algebra $\mathscr{L}(V)$ of continuous linear operators on V, we may define by a power series a one-parameter group of bounded operators on V by

$$\exp(-Bt) = \sum_{n \geqq 0} ((-1)^n/n!)(tB)^n$$

for t in $\mathbb{R}$ [9]. The operator-valued function $t \mapsto \exp(-Bt)$ is differentiable in the uniform operator topology of $\mathscr{L}(V)$ and satisfies

$$\frac{d}{dt}\exp(-Bt) = -B \cdot \exp(-Bt).$$

We now define a V-valued function as follows. Since $(\gamma M + I)^{-1}$ is a bounded map of H into V as a consequence of (3.1), and since $f:\mathbb{R} \to H$ is continuous, it follows that $(\gamma M + I)^{-1}f$ is continuous from $\mathbb{R}$ into V, and so also is the function

$$\tau \mapsto \exp(B(\tau - t))(\gamma M + I)^{-1}f(\tau)$$

for each t in $\mathbb{R}$. Hence we can define for each t in $\mathbb{R}$ an element of V by the formula

$$u(t) = \exp(-Bt) \cdot u_0 + \int_0^t \exp(B(\tau - t)) \cdot (\gamma M + I)^{-1} \cdot f(\tau)\, d\tau, \tag{3.3}$$

where u_0 is the initial condition specified in V. Then $u: \mathbb{R} \to V$ is continuously differentiable and satisfies the equations

$$u'(t) + Bu(t) = (\gamma M + I)^{-1} f(t), \qquad u(0) = u_0 \tag{3.4}$$

in V. From (3.4) we can show that u is a solution of the generalized problem. Let $t \in \mathbb{R}$ and let $\{\phi_n\}$ be a sequence in $D(L)$ for which $\phi_n \to u(t)$ in V. The continuity of B implies by (3.4) that $\{B\phi_n\}$ converges in V to $-u'(t) + (\gamma M + I)^{-1} f(t)$. Since B is an extension of $(\gamma M + I)^{-1} L$ and each ϕ_n is in $D(L)$, it follows that each $B\phi_n$ is in $D(M)$ and

$$(\gamma M + I)(-B\phi_n) + L\phi_n = 0.$$

Thus for each $n \geqq 1$ and each v in V we obtain, by (1.10) and (1.11),

$$\gamma m(-B\phi_n, v) + (-B\phi_n, v)_H + l(\phi_n, v) = 0,$$

and taking the limit in this equation as $n \to \infty$ we obtain (1.12).

The requirement that $D(L)$ be dense in V (which was used twice in the above arguments) is not essential for existence or uniqueness. In particular, $u(t)$ is a solution if and only if $w(t) = e^{-\lambda t} u(t)$ is a solution of the problem with initial data u_0, nonhomogeneous term $F(t) = e^{-\lambda t} f(t)$, and the equations (1.12) with l replaced by

$$\lambda((u, v)_H + \gamma m(u, v)) + l(u, v).$$

By taking λ sufficiently large, say, $\lambda = (K_l + k_l)/k$, it follows from (1.7) and (3.1) that we may assume without loss of generality that

$$|l(\phi, \phi)| \geqq k_l \|\phi\|_V^2 \tag{3.5}$$

for all ϕ in V. From the Lax–Milgram theorem and (3.5) it follows that $D(L)$ is dense in V, and we obtain the following theorem.

THEOREM 2. *If the sesquilinear form m of the generalized problem satisfies* (3.1), *then there exists a solution of this problem, and it is given by the formula* (3.3).

Coercive inequalities like (3.1) and (3.5) are known to hold for the sesquilinear forms associated with strongly elliptic partial differential operators. Garding has verified the following result [8]:

> Let $\mathcal{M}$ be the operator specified by (1.2); if all the coefficients are bounded and measurable, the principal coefficients $\{m^{\rho\sigma} : |\rho| = |\sigma| = m\}$ are uniformly continuous on $\mathrm{cl}\,(G)$, and if
>
> $$\mathrm{Re}\left\{\sum_{|\rho| = |\sigma| = m} \xi^\rho m^{\rho\sigma}(x) \xi^\sigma\right\} \geqq c_0 |\xi|^{2m}, \qquad c_0 > 0,$$
>
> for all real vectors ξ in $\mathbb{R}^n$, then there exist real numbers $c_1 > 0$ and c_2 such that
>
> $$\mathrm{Re}\,\{m(\phi, \phi)\} + c_2 \|\phi\|_0^2 \geqq c_1 \|\phi\|_m^2$$
>
> for all ϕ in $H_0^m(G)$, where G is bounded and open in $\mathbb{R}^n$.

Similar results will be established for some particular examples in § 6 for spaces V other than $H_0^m(G)$. For some other coerciveness results see [13], [16], [17]. We shall make the following assumption: there is a number α such that for every $\varepsilon > 0$ there is a $\beta(\alpha, \varepsilon) > 0$ for which

$$\operatorname{Re} m(\phi, \phi) + (\alpha + \varepsilon)\|\phi\|_H^2 \geqq \beta\|\phi\|_V^2 \tag{3.6}$$

for all ϕ in V.

Consider first the generalized problem with $\gamma > 0$. From (3.6) it follows that

$$\operatorname{Re} \gamma m(\phi, \phi) + \|\phi\|_H^2 \geqq \gamma\beta\|\phi\|_V^2 \tag{3.7}$$

for all ϕ in V if $\gamma(\alpha + \varepsilon) \leqq 1$ for some $\varepsilon > 0$ in the case $\alpha \geqq 0$ and for any γ if $\alpha < 0$. Since

$$\operatorname{Re} \gamma m(\phi, \phi) + \|\phi\|_H^2 \leqq |\gamma m(\phi, \phi) + \|\phi\|_H^2|,$$

we obtain from Theorem 2 the following corollary.

COROLLARY 1. *Assume that the sesquilinear form m satisfies* (3.6). *Then the generalized problem has a solution if $\alpha > 0$ and $0 < \gamma < \alpha^{-1}$ or if $\alpha \leqq 0$ and $0 < \gamma$.*

For the case of $\gamma < 0$ the above method is applicable only if (3.6) holds for some $\alpha < 0$, for if

$$-\gamma \operatorname{Re} m(\phi, \phi) - \|\phi\|_H^2 \geqq k\|\phi\|_V^2$$

for all ϕ in V, then

$$\operatorname{Re} m(\phi, \phi) - (-\gamma)^{-1}\|\phi\|_H^2 \geqq (-\gamma)^{-1}k\|\phi\|_V^2,$$

so (3.6) holds with $\alpha = \gamma^{-1} < 0$. Conversely, if (3.6) holds for some $\alpha < 0$, then

$$\begin{aligned} \operatorname{Re}\{-\gamma m(\phi, \phi) - \|\phi\|_H^2\} &\geqq (-\gamma)\beta\|\phi\|_V^2 - [(\alpha + \varepsilon)(-\gamma) + 1]\|\phi\|_H^2 \\ &\geqq (-\gamma)\beta\|\phi\|_V^2 \end{aligned} \tag{3.8}$$

for all ϕ in V if, for some $\varepsilon > 0$, $(\alpha + \varepsilon)(-\gamma) + 1 \leqq 0$, and this is true if $-\gamma > (-\alpha)^{-1}$.

COROLLARY 2. *Assume that the sesquilinear form m satisfies* (3.6) *with* $\alpha < 0$. *Then the generalized problem has a solution if $\gamma < \alpha^{-1}$.*

4. Uniqueness and boundedness. The solution of the generalized problem constructed in § 3 is the only solution. In particular, we shall show that (3.1) yields estimates on the growth of a solution and dependence on the initial data and nonhomogeneous term of (1.12). Estimates of the type (3.6) for m and l and symmetry of m imply that the solution of the homogeneous equation is asymptotically stable, since all such solutions decay exponentially to zero.

Consider the sesquilinear forms m and l introduced above on $V \times V$. For each ϕ in V, the conjugate linear functional $\psi \mapsto m(\phi, \psi)$ on V is bounded by (1.6), so the Riesz–Fréchet theorem [11] implies the existence of a unique $m_0(\phi)$ in V for which

$$m(\phi, \psi) = (m_0(\phi), \psi)_V \tag{4.1}$$

for all ψ in V. This determines a bounded operator $m_0 : V \to V$ whose norm in $\mathscr{L}(V)$ satisfies $\|m_0\| \leqq K_m$ by (1.6). Similarly, there is a unique operator l_0 in $\mathscr{L}(V)$ for which

$$l(\phi, \psi) = (l_0(\phi), \psi)_V \tag{4.2}$$

for all ϕ and ψ in V with $\mathscr{L}(V)$-norm $\|l_0\| \leqq K_l$ by (1.7). The continuity of the injection $V \hookrightarrow H$ suggests the construction of an operator $J: H \to V$ as follows. For each ϕ in H, the conjugate linear form $\psi \mapsto (\phi, \psi)_H$ is continuous on V, so there is a unique $J(\phi)$ in V for which

$$(\phi, \psi)_H = (J\phi, \psi)_V \tag{4.3}$$

for all ψ in V. This operator J maps H into V, and it follows from (1.13) that the $\mathscr{L}(H, V)$-norm of J satisfies

$$\|J\| \leqq 1.$$

Let $v(t)$ be any solution of the generalized problem. It follows from (1.12), (4.1), (4.2) and (4.3) that

$$(J + \gamma m_0)v'(t) + l_0 v(t) = Jf(t) \tag{4.4}$$

in V. That is, $v(t)$ satisfies (4.4) in V with bounded operator coefficients. From the estimate (3.1) and the Lax-Milgram theorem it follows that the bounded operator $\gamma m_0 + J$ on V associated with the V-coercive sesquilinear form $\gamma m(\phi, \psi) + (\phi, \psi)_H$ is a topological isomorphism of V onto V for which the $\mathscr{L}(V)$-norm of the inverse satisfies $\|(\gamma m_0 + J)^{-1}\| \leqq k^{-1}$. Hence the function $v(t)$ satisfies the equation

$$v'(t) + (J + \gamma m_0)^{-1} l_0 \cdot v(t) = (J + \gamma m_0)^{-1} Jf(t). \tag{4.5}$$

Since $v: \mathbb{R} \to V$ is continuously differentiable, the real-valued function

$$\sigma(t) \equiv \|v(t)\|_V^2$$

is continuously differentiable and by (4.5) satisfies

$$\begin{aligned} \sigma'(t) &= 2\,\mathrm{Re}\,(v'(t), v(t))_V \\ &= 2\,\mathrm{Re}\,\{-((J + \gamma m_0)^{-1} l_0 v(t), v(t))_V + ((J + \gamma m_0)^{-1} J \cdot f(t), v(t))_V\} \end{aligned}$$

and this in turn implies

$$\begin{aligned} |\sigma'(t)| &\leqq 2k^{-1} K_l \|v(t)\|_V^2 + 2k^{-1} \|f(t)\|_H \|v(t)\|_V \\ &\leqq k^{-1}(2K_l + 1)\sigma(t) + k^{-1} \|f(t)\|_H^2 \end{aligned} \tag{4.6}$$

for all t in $\mathbb{R}$. From (4.6) we obtain the estimates

$$\begin{aligned} \sigma(t) \leqq{}& \sigma(0) \exp(k^{-1}(2K_l + 1)|t|) \\ &+ k^{-1} \left| \int_0^t \exp(k^{-1}(2K_l + 1)|t - \tau|) \cdot \|f(\tau)\|_H^2 \, d\tau \right| \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} \sigma(t) \geqq{}& \sigma(0) \exp(-k^{-1}(2K_l + 1)|t|) \\ &- k^{-1} \left| \int_0^t \exp(-k^{-1}(2K_l + 1)|t - \tau|) \|f(\tau)\|_H^2 \, d\tau \right| \end{aligned} \tag{4.8}$$

for all t in $\mathbb{R}$.

The linearity of the problem and the preceding remarks yield the following result.

THEOREM 3. *Let the sesquilinear form m of the generalized problem satisfy* (3.1) *for all ϕ in V. If $u_i(t)$, $i = 1, 2$, are solutions of the generalized problem with initial* *data* $u_1(0)$ *and* $u_2(0)$ *and nonhomogeneous terms* $f_1(t)$ *and* $f_2(t)$, *respectively, then* $\sigma(t) = \|u_1(t) - u_2(t)\|_V^2$ *satisfies the growth and decay estimates* (4.7) *and* (4.8) *with* $f = f_1 - f_2$. *In particular, the generalized problem has at most one solution.*

Stronger estimates on the solution can be obtained when the sesquilinear form m is symmetric and satisfies estimates of the form obtained for the corollaries of § 3. Let us assume then that

$$m(\phi, \psi) = \overline{m(\psi, \phi)} \qquad \text{for all } \phi, \psi \text{ in } V, \tag{4.9}$$

and that

$$|\gamma| m(\phi, \phi) + \operatorname{sgn}(\gamma)\|\phi\|_H^2 \geqq |\gamma|\beta\|\phi\|_V^2 \tag{4.10}$$

for all ϕ in V. The condition (4.9) implies that $m(\phi, \phi) = \overline{m(\phi, \phi)}$ is real for each ϕ in V, and (4.10) is equivalent to (3.7) when $\gamma > 0$ and to (3.8) when $\gamma < 0$. Thus (4.10) follows from the coercive estimate (3.6) for certain values of γ.

Let $u(t)$ be a solution of the generalized problem. Then the real-valued function

$$\Sigma(t) \equiv |\gamma| m(u(t), u(t)) + \operatorname{sgn}(\gamma)\|u(t)\|_H^2$$

is continuously differentiable, and from (4.9) we obtain

$$\begin{aligned}\Sigma'(t) &= 2 \operatorname{Re}\{|\gamma| m(u'(t), u(t)) + \operatorname{sgn}(\gamma)(u'(t), u(t))_H\} \\ &= 2 \operatorname{sgn}(\gamma) \operatorname{Re}\{\gamma m(u'(t), u(t)) + (u'(t), u(t))_H\}.\end{aligned}$$

If (1.12) is homogeneous, then

$$\Sigma'(t) = 2 \operatorname{sgn}(\gamma) \operatorname{Re}\{-l(u(t), u(t))\},$$

and if l satisfies the coercive estimate

$$\operatorname{sgn}(\gamma) \operatorname{Re} l(\phi, \phi) \geqq k_l\|\phi\|_V^2 \tag{4.11}$$

for some $k_l > 0$ and all ϕ in V, then we have from this and (1.6) the estimate

$$\Sigma'(t) \leqq -2k_l\|u(t)\|_V^2 \leqq -2k_l(|\gamma|K_m + 1)^{-1}\Sigma(t).$$

But this implies that for all $t \geqq 0$,

$$\Sigma(t) \leqq \exp(-2k_l(|\gamma|K_m + 1)^{-1}t)\Sigma(0).$$

We summarize these results in the following theorem.

THEOREM 4. *Assume that the sesquilinear forms of the generalized problem satisfy* (4.9), (4.10) *and* (4.11). *Then there exists a unique solution to the generalized problem, and if* $f \equiv 0$ *in* (1.12), *then this solution satisfies the estimate*

$$\|u(t)\|_V \leqq (\beta^{-1}K_m + |\gamma|^{-1})\|u_0\|_V \exp(-k_l(|\gamma|K_m + 1)^{-1}t) \tag{4.12}$$

for all $t \geqq 0$.

This last inequality follows from the estimate on $\Sigma(t)$ together with (1.6) and (4.10). Also, (4.10) implies (3.1). By the usual linearity arguments, one may obtain estimates for the solution of the nonhomogeneous equation (1.12) by adding (4.8) with $\sigma(0) = \|u_0\|_V^2 = 0$ and (4.12). The same argument shows that if (4.11) is replaced by the estimate

$$(4.13) \qquad -\operatorname{sgn}(\gamma)\operatorname{Re} l(\phi, \phi) \geqq k_l \|\phi\|_V^2,$$

then one obtains an estimate like (4.12) with the inequality reversed, so the solution *grows* at least exponentially in norm. Finally, we remark that with all the hypotheses above except $f \equiv 0$, the difference of two solutions with different initial data satisfies (4.12), so the effect of initial data is "transient".

5. Weak and strong solutions. The objective of this section is to show that if M and L satisfy elliptic hypotheses and if M is "stronger" than L, then the weak solution of the problem is a strong solution if and only if the initial function u_0 is in the domain of L.

THEOREM 5. *Assume that the sesquilinear forms m and l of the generalized problem satisfy the estimates* (3.1) *and* (3.5), *and that $D(M) \subseteq D(L)$. If u_0 belongs to $D(L)$, then the weak solution* (3.3) *of the generalized problem is a strong solution* (§ 1).

Proof. From the estimate (3.5) it follows that L^{-1} is a continuous injection of H into V. Hence we can define by

$$(5.1) \qquad \|\phi\|_L = \|L\phi\|_H$$

a norm on $D(L)$ for which the injection $D(L) \hookrightarrow V$ is continuous. The completeness of H shows that $D(L)$ is complete in the norm (5.1). The bounded extension B of $(\gamma M + I)^{-1}L$ maps $D(L)$ into $D(M)$, and the assumption above that $D(M) \subseteq D(L)$ implies that B maps $D(L)$ into $D(L)$. Thus B is a continuous linear operator from V into V, and the space $D(L)$ is invariant under B. This implies by the closed graph theorem that B is continuous from $D(L)$ into itself with the norm (5.1). To see this, let $\{\phi_n\}$ be a sequence in $D(L)$ for which $\|\phi_n - x_0\|_L \to 0$ and $\|B\phi_n - y_0\|_L \to 0$ as $n \to \infty$, where y_0 and x_0 are in $D(L)$. Then

$$\begin{aligned}\|y_0 - Bx_0\|_V &\leqq \|y_0 - B\phi_n\|_V + \|B(\phi_n - x_0)\|_V \\ &\leqq \|y_0 - B\phi_n\|_V + \|B\|_{\mathscr{L}(V)}\|\phi_n - x_0\|_V,\end{aligned}$$

and the continuity of the injection $D(L) \hookrightarrow V$ implies that each of these terms converges to zero, so $y_0 = Bx_0$. Thus B is a closed and everywhere-defined linear operator and is hence continuous on $D(L)$ [9], [18].

The significance of the continuity of B on $D(L)$ is that the restrictions of the operators

$$\{\exp(-Bt) : t \text{ in } \mathbb{R}\}$$

are bounded on $D(L)$, and hence the function $t \mapsto \exp(-Bt)u_0$ is in $C^1(D(L))$. Finally, each $(\gamma M + I)^{-1}f(t)$ belongs to $D(M)$, hence also $D(L)$, and an argument like that above shows that $(\gamma M + I)^{-1}$ is continuous from H into $D(L)$, so $f : \mathbb{R} \to H$ being continuous implies that the function

$$t \mapsto \int_0^t \exp(B(\tau - t))(\gamma M + I)^{-1} f(\tau)\, d\tau$$

is in $C^1(D(L))$. Hence the (weak) solution of the generalized problem given by (3.3) is in $C^1(D(L))$, and differentiating this function shows that $u'(t)$ belongs to $D(M)$ for each t in $\mathbb{R}$, so $u(t)$ is a strong solution of the problem.

Theorem 5 is really a regularity result, for the domain of an elliptic operator consists of functions which are "smooth". In particular, the global regularity results for elliptic operators can be used to show that B leaves invariant the subspaces $V \cap H^p(G)$, and an argument like that above shows that $u(t)$ belongs to $V \cap H^p(G)$, where the integer p depends on the coefficients in M and L and the boundary of G. The details for the case $V = H_0^1(G)$ for Dirichlet boundary conditions on an equation of order 3 appear in [22].

The interesting distinction between weak and strong solutions is the type of boundary conditions they carry. If $u(t)$ is a strong solution of the generalized problem, then $u(t)$ and $u'(t)$ belong to $D(L)$ and $D(M)$, respectively, and from (1.8) and (1.9) it follows that

$$m(u'(t), v) = (Mu'(t), v)_H \tag{5.2}$$

and

$$l(u(t), v) = (Lu(t), v)_H \tag{5.3}$$

for all v in V. These constitute independent boundary conditions on $u'(t)$ and $u(t)$, respectively, if V *properly* contains $H_0^m(G)$. Also, the conditions that $u(t)$ and $u'(t)$ belong to V constitute boundary conditions if V is properly contained in $H^m(G)$. The conditions (5.2) and (5.3) will be called *strong boundary conditions.*

Suppose $u(t)$ is a weak solution of the generalized problem. Then the identities (1.8), (1.9) and (1.12) imply that

$$u'(t) + \gamma\mathscr{M}u'(t) + \mathscr{L}u(t) = f(t) \tag{5.4}$$

in $\mathscr{D}'(G)$. From (1.12) and (5.4), we obtain the identity

$$(\gamma\mathscr{M}u'(t) + \mathscr{L}u(t), v)_H = \gamma m(u'(t), v) + l(u(t), v) \tag{5.5}$$

for all v in V. This will be called a *weak boundary condition,* since it is certainly implied by the strong boundary conditions.

6. Applications. We shall discuss the implications of our above results in two examples. The first originates in the flow of second order fluids as discussed in [4] and [25], and our results contain most of those in these references. The second example includes the above as well as problems in consolidation of clay [24] and homogeneous fluid flow in fissured rocks [2]. Our results are adequate to discuss all of the boundary value problems associated with these theories as well as many for which no physical applications are known to this writer.

For the first example, let G be the interval $(0, T)$, $T > 0$, and define

$$l(u, v) = \int_0^T u_x \overline{v_x}\, dx$$

for u and v in $H^1(G)$. For functions in $H^1(G)$ we have, for x, y in G,

$$|u(x) - u(y)| = \left| \int_x^y u'(s)\, ds \right| \leqq |x - y|^{1/2} \|u\|_1, \tag{6.1}$$

so $H^1(G)$ contains only continuous functions. Suppose V is a closed subspace of

$H^1(G)$ which contains only functions which vanish at $x = T$. Then for such ϕ in V,

(6.2)
$$\sup \{|\phi(x)| : 0 \leqq x \leqq T\} \leqq (T)^{1/2} \|\phi\|_1.$$

From (6.1) and (6.2) it follows that any sequence $\{\phi_n\}$ of elements of V for which $\|\phi_n\| \leqq 1$ for all n is a sequence of equicontinuous and uniformly bounded functions. By the Ascoli–Arzelà theorem [11], [18], such a sequence has a uniformly convergent subsequence which then converges in the mean-square norm. That is, the injection of V into $H \equiv L^2(G)$ is completely continuous.

For any ϕ in V, we have

$$\int_0^T \left\{|\phi(x)|^2 + x\frac{d}{dx}|\phi(x)|^2\right\} dx = x|\phi(x)|^2\Big|_0^T = 0$$

since $\phi(T) = 0$, so

$$\int_0^T |\phi^2(x)|\, dx \leqq 2\int_0^T x \cdot |\phi(x)| \cdot |\phi'(x)|\, dx.$$

From the inequality $2\alpha\beta \leqq 2\alpha^2 + \beta^2/2$, we obtain

$$\int_0^T |\phi(x)|^2\, dx \leqq \frac{1}{2}\int_0^T |\phi(x)|^2\, dx + 2T^2 \int_0^T |\phi'(x)|^2\, dx,$$

and hence the inequality

(6.3)
$$\|\phi\|_0 \leqq 2T\|\phi_x\|_0$$

for ϕ in V. From (6.3) we have for all u in V,

$$\begin{aligned} l(u, u) &= \int_0^T |u_x|^2\, dx \\ &\geqq \frac{1}{2}\int_0^T |u_x|^2\, dx + \frac{1}{8T^2}\int_0^T |u|^2\, dx \geqq k_l\|u\|_1^2, \end{aligned}$$

where $k_l = \min[1/2, 1/(8T^2)] > 0$. Thus the conditions (2.1), (2.2) and (2.3) are satisfied. By Theorem 1 there is a unique solution of the generalized problem of § 1 for certain values of γ which is then a solution of the equation $u'(t) + \gamma\mathscr{L}u'(t) + \mathscr{L}u(t) = f(t)$, where $\mathscr{L}$ is the distributional derivative $-d^2/dx^2$. Furthermore, the inequality (6.2) shows that for each $x \in (0, T)$, the "evaluation" functional $e_x : u \to u(x)$ from V into $\mathbb{C}$ is continuous, so $u'(t)(x) = \partial[u(t)(x)]/\partial t$ in the equation. If γ is not equal to any of the eigenvalues $\{\rho_n\}$, then the initial data and nonhomogeneous term are prescribed arbitrarily. For the exceptional values, a compatibility condition is necessary and sufficient for the existence of the solution which is given by (2.9). We shall discuss two choices for V and the associated problem.

If $V = \{\phi \text{ in } H^1(G) : \phi(T) = 0\}$ then the sequence of eigenvalues is given by $\rho_n = (2T/((2n-1)\pi))^2$, $n \geqq 1$, and the eigenfunctions are $\cos(\rho_n^{-1/2} x)$.

For u and v in V, we obtain by integrating by parts

(6.4)
$$l(u, v) = (\mathscr{L}u, v)_0 - u_x \cdot v|_0^T,$$

so u is in $D(L)$ if and only if $u_x \cdot v|_0^T = 0$ for all v in V. That is, $u_x(0) = 0$. The condition that u belongs to V implies $u(T) = 0$. Thus the solution $u(t)$ of the generalized problem satisfies the weak boundary conditions

$$(\gamma u_x'(t) + u_x(t))|_{x=0} = 0,$$
$$u(t)|_{x=T} = 0$$

from (5.5) and $u(t) \in V$, respectively. The condition $u'(t)|_{x=T} = 0$ follows from $u'(t) \in V$, but is redundant since it can be obtained from the second condition above by differentiation. If γ is chosen such that (3.1) holds and if u_0 is in $D(L)$ (see above), then the solution satisfies the strong boundary condition

$$u_x(t)|_{x=0} = 0, \qquad u(t)|_{x=T} = 0$$

from (5.3) and $u(t) \in V$. (Note that (5.2) leads to a redundant condition $u_x'(t)|_{x=0} = 0$.)

If $V = H_0^1(G) = \{\phi \in H^1(G) : \phi(0) = \phi(T) = 0\}$, then the sequence of eigenvalues and functions is given by $\rho_n = (T/(n\pi))^2$ and $\sin(\rho_n^{-1/2}s)$, $n \geqq 1$. For u, v in V, all boundary terms are zero in (6.4), so the identities (5.2), (5.3) and (5.5) do not determine boundary conditions. However $u(t) \in V$ implies the boundary conditions

$$u(t)|_{x=0} = u(t)|_{x=T} = 0.$$

Similar applications hold in spaces of higher dimension. Estimates like (6.3) hold for smooth domains and functions which vanish on a sufficiently large portion of the boundary, and the injection of V into $H = L^2(G)$ is completely continuous if G is bounded and either $V = H_0^m(G)$ or the boundary is m times continuously differentiable [5], [17]. Nonhomogeneous boundary data may be introduced by superposition [22]. The relation between $u'(t)(x)$ and $\partial[u(t)(x)]/\partial t$ is not always so clear as above; see [9, pp. 68–71] for results in this direction.

For a second example, which exhibits more of the "flavor" of these problems, we define the forms

$$m(u, v) = \sum_{i=1}^{n} (u_{x_i}, v_{x_i})_0 + \int_{\partial G} \alpha(s)u(s)\bar{v}(s)\, ds,$$

$$l(u, v) = \sum_{i=1}^{n} (u_{x_i}, v_{x_i})_0 + \int_{\partial G} \beta(s)u(s)\bar{v}(s)\, ds,$$

where G is a bounded open set in $\mathbb{R}^n$ with smooth boundary ∂G, and ds denotes Lebesgue measure on ∂G. The functions α, β are in $L^\infty(\partial G)$ and $\alpha(s) \geqq 0$. By elementary results on "traces" [17], m and l are bounded on $H^1(G)$. Since $\alpha(s) \geqq 0$, it follows that for each $\varepsilon > 0$

$$m(\phi, \phi) + \varepsilon\|\phi\|_0^2 \geqq \min(1, \varepsilon)\|\phi\|_1^2,$$

so (3.6) is satisfied with $V \leqq H^1(G)$, $H = L^2(G)$ and $\alpha = 0$. Hence the generalized problem has a unique solution for each $\gamma > 0$.

If the elements of V satisfy the estimate

$$\|\phi\|_0^2 \leqq K \sum_{i=1}^{n} \|\phi_{x_i}\|_0^2, \tag{6.5}$$

then (3.6) holds for α small but negative, so the generalized problem has a unique solution for all $\gamma < (\alpha)^{-1}$. Furthermore, if (6.5) holds and $\beta(s) \geqq 0$, then the form $l(u, v)$ satisfies an estimate of the form (4.11) with $\gamma > 0$, so the solution is asymptotically stable if $f \equiv 0$ in (1.12). Similarly, if $\gamma < (\alpha)^{-1} < 0$ then (6.5) implies (4.13), and the solution grows exponentially as $t \to \infty$ by the remarks at the end of § 4.

In any case, there exists a unique solution of the generalized problem for γ satisfying either of the two corollaries, and the V-valued function $u(t)$ satisfies in $\mathscr{D}'(G)$ the equation

$$u'(t) - \gamma\Delta_n u'(t) - \Delta_n u(t) = f(t),$$

where $\mathscr{M} = \mathscr{L} = -\Delta_n$ is the Laplace operator in n variables, and the initial condition $u(0) = u_0$. If u and v are sufficiently regular and if ∂G is sufficiently smooth for the divergence theorem to apply [13], then we have (formally)

$$m(u, v) = (-\Delta_n u, v)_0 + \int_{\partial G} \left(\frac{\partial u}{\partial n} + \alpha u\right)\bar{v}\, ds \tag{6.6}$$

and

$$l(u, v) = (-\Delta_n u, v)_0 + \int_{\partial G} \left(\frac{\partial u}{\partial n} + \beta u\right)\bar{v}\, ds \tag{6.7}$$

for all u and v in V, where $\partial/\partial n$ denotes the normal derivative. Thus, the weak boundary condition is

$$\int_{\partial G} \left\{\gamma\left(\frac{\partial u'}{\partial n} + \alpha u'\right) + \left(\frac{\partial u}{\partial n} + \beta u\right)\right\}\bar{v}\, ds = 0 \tag{6.8}$$

for all v in V, while the strong boundary conditions are

$$\int_{\partial G} \left(\frac{\partial u'}{\partial n} + \alpha u'\right)\bar{v}\, ds = 0, \qquad \int_{\partial G} \left(\frac{\partial u}{\partial n} + \beta u\right)\bar{v}\, ds = 0 \tag{6.9}$$

for all v in V. The condition

$$u(t) \in V \tag{6.10}$$

also holds true.

Let the boundary ∂G be equal to the disjoint union of Γ_1 and Γ_2. Let V be the closure in $H^1(G)$ of the space of restrictions to G of those functions in $C_0^\infty(\mathbb{R}^n)$ whose support is disjoint from Γ_1. The condition (6.10) means that each $u(t)$ vanishes on Γ_1 while the condition (6.8) implies that

$$\gamma\left(\frac{\partial u'(t)}{\partial n} + \alpha u'(t)\right) + \frac{\partial u(t)}{\partial n} + \beta u = 0 \tag{6.11}$$

on Γ_2.

If that portion Γ_1 of ∂G on which the Dirichlet condition is prescribed is sufficiently large, then the estimate (6.5) holds for some K and all ϕ in V, and if $\beta \geqq 0$, then one can obtain the estimate (3.5) for l. Finally, the identities (6.6) and

(6.7) show that $D(M) \subseteq D(L)$ if $\alpha \equiv \beta$. From Theorem 5 it follows that if u_0 belongs to $D(L)$, that is, if

$$\frac{\partial u_0}{\partial n} + \alpha u_0 = 0$$

on Γ_2, then the solution of the generalized problem, $u(t)$, satisfies the strong boundary condition

$$\frac{\partial u(t)}{\partial n} + \alpha u(t) = 0 \tag{6.12}$$

on Γ_2 for all t in $\mathbb{R}$. Note that in order to obtain the condition $D(M) \subseteq D(L)$, we had to choose $\alpha \equiv \beta$, and this makes the two conditions in (6.9) dependent, for the first can be obtained from the second by differentiation. Although the boundary conditions obtained by combining (6.10) with (6.11) or (6.12) for various choices of Γ_1, α and β include all cases of physical interest, many other types can be introduced by adding more boundary integrals to the sesquilinear forms.

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