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Title A MATHEMATICAL MODEL FOR OPTIMUM TIMBER ALLOCATION

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Operations research is concerned with the science of decision and uses mathematical models to aid in making these decisions.

In this thesis operations research is used in connection with the lumber industry. The problem of concern has to do with a lumber company that is unable to supply the mill demand with its own supply of timber and must buy from the government to make up the deficiency. The decision to be made is how much company timber and how much government timber should there be allocated to the mill in any given year during the period under study.

The model used in this paper is a linear programming model. Other operations research techniques presented in connection with the allocation problem are post-optimality analysis and regression analysis. The model is such that almost all calculations are done on a computer.

A MATHEMATICAL MODEL FOR OPTIMUM TIMBER ALLOCATION

by

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A MATHEMATICAL MODEL FOR OPTIMUM TIMBER ALLOCATION

INTRODUCTION

This thesis is concerned with formulating a model for use by lumber companies who must integrate two sources of timber in a clear-cut management plan. The purpose of the model is to determine optimum allocation of timber. The two sources of timber referred to are company holdings, and government timber that is obtained through Forest Service timber sales.

A linear programming model is used to solve the problem. The model is built step by step with special emphasis on cost and growth determination. A numerical example is given to demonstrate the use of the model.

It should be stated that the purpose of this paper is not to dictate a management plan, but to present a means by which a lumber company can improve their plan through information obtained by the model under various simulated conditions provided by management personnel.

FOREST MANAGEMENT PROBLEM

General Description

This thesis is concerned with the mathematical representation of the management of lands bearing even-aged stands of West Coast Douglas Fir when clear-cut logging practice is used. Even-aged stands of timber are defined by (3, p. 154) to be

> Stands in which the dominant trees originated at about the same time and, following a period of establishment, and frequently protection as under a shelterwood, developed under essentially full light conditions. However, for a forest to be considered as even-aged from a forest organization standpoint, it must be composed of stands that not only are essentially even-aged but are also of sufficient size and unity that they constitute significant management units.

A clear cutting plan applied to these even-aged stands refers to cutting all merchantable timber on a tract of land until the desired volume is attained. Clear-cut logging is used on West Coast Douglas Fir because it is the most economical and it provides the best means of regeneration. Intermediate cuts can occur in addition to clear cutting. However, they are not considered in the problem studied here.

A problem arises when the timber owned by the company is not sufficient to supply the demand of the mill and other sources of timber must be tapped to maintain production. This situation occurs when company holdings consist of two general classes of timber (Figure I), timber which is merchantable and young growth. The

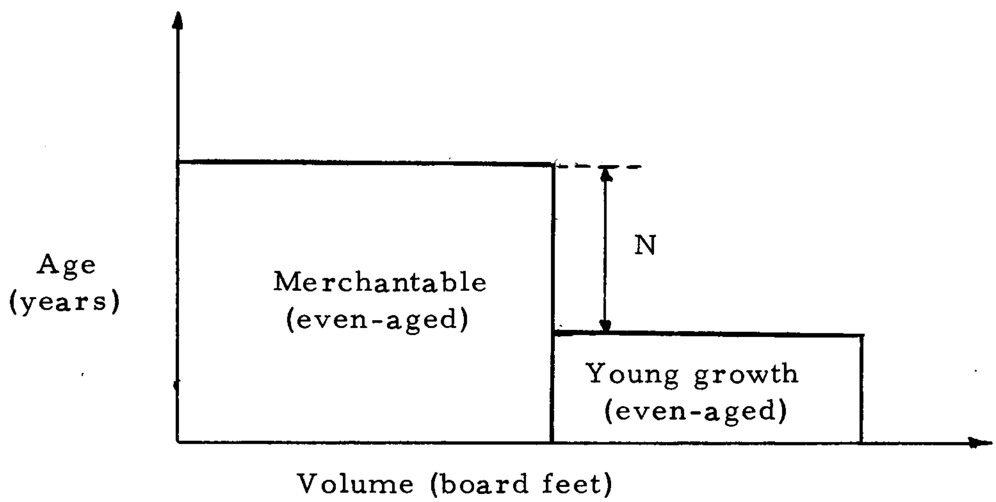

Figure I. Company Timber Available.

timber to be harvested and utilized by the mill must come from the merchantable class for a period lasting N years. At the end of the initial N year period, young growth will have attained merchantable age. If the mill demand exceeds the supply of timber that is available for cutting in the N years, additional timber must be purchased. The company timber is then integrated with that purchased to form a management plan. For purposes of this paper it is assumed that the timber will be purchased from the government (United States Forest Service) and that the supply of government timber will be adequate to fill all demands so created.

This presents to the company management the problem of how much company timber should be used and how much should be bought on a year by year basis over the N year period. In other words, the company would like to find an ideal allocation of its own timber subject to two constraints; firstly, the amount of timber allocated to the mill from company and government sources in any given year must equal the mill demand for that year, and secondly, the total input of company timber over the N years in concern must be less than or equal to original merchantable volume plus new growth. For each year a decision should be made as to the allocation, and this sequence of decisions over the years defines a policy which is minimized with respect to cost.

An important assumption to this problem is that the degree to which sustained yield is practiced is unimportant. Where sustained yield is defined by the Society of American Foresters (1950) as follows: "Management of a forest property for continuing production with the aim of achieving, at the earliest practicable time, an approximate balance between net growth and harvest, either by annual or somewhat longer periods." What is important is the continued stability and prosperity of the forest economy, of which the lumber company is a part, and that the forest lands the company owns or buys timber from are productively managed. This assumption concerning sustained yield is the result of the limited supply of company timber and the desire of the company to obtain the most economical cutting plan.

It also must be postulated that all standing merchantable timber will be capable of putting on new growth. This is because timber which has reached a state of maturity such that growth is no longer in existence must be cut. This would then eliminate the decision of which source of timber to tap.

Since one of the constraints on the allocation process requires an estimate of growth, a method of estimating growth of existing merchantable timber is needed. Yield tables collectively furnish the largest volume of available data which are usable in estimating growth. A yield table is a tabular presentation giving, for a given age, site, and stocking, the total volume of timber that can be expected per acre. Growth estimates employing yield tables are always net, since an average allowance for mortality due to such variables as insect, fire and storm damage is built into normal yield tables. Davis (3, p. 79) says the following concerning yield tables,

> Good growth estimates can be made with yield tables, particularly for large areas, and for stands more or less uniform in general structure and density. Where available, they are frequently the most convenient method to use. They must, however, be applied with judgement and with full understanding both of the yield tables and the actual stand data to which they are applied.

Growth rate can be illustrated by a normal yield-table curve as in Figure II. Since growth rate enters into this management problem, it is assumed that the stands are more or less uniform in general structure and density so that good estimates of growth rate can be obtained.

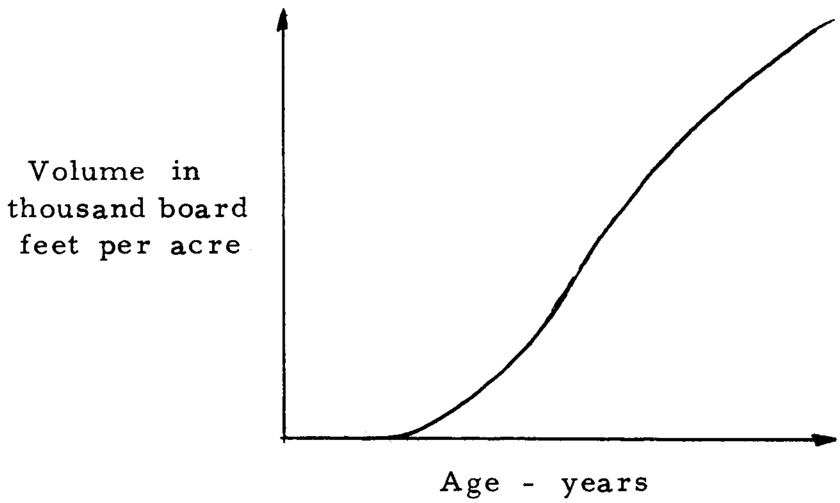

Figure II. Normal Yield-Table Curve.

Because our objective is to provide a method of allocation of timber with minimum cost, costs involved should be carefully stipulated. Here it is assumed that no other costs than those stated are involved. The cost of company timber is composed of logging costs, tax and investment. The tax is that which is paid each year on standing timber owned by the company. Costs of government timber include logging costs and the price at which the timber must be purchased.

An important point concerning yields and costs is the reliability of the available information on their future values. These values must in most cases be labeled uncertain predictions. This paper is concerned only with developing a system which selects the optimum management plan under the set of predictions specified.

LINEAR PROGRAMMING MODEL

Description

The practice of mathematical modelling or model building makes use of mathematical principles to form a model, which corresponds to a system under study in the real world. This correspondence or fit could hardly be expected to be perfect except in a few trivially simple cases. For this reason, a model must be considered only as valid as its stated assumptions. A mathematical model will give one an impression of how a system may operate under the postulated conditions. For example in the problem presented herein, a model will indicate the effect of alternative management plans, for given costs, demands, supplies, and the stated assumptions, and will provide a basis for selection of a specific plan.

The model used in this thesis to solve the forest management problem discussed earlier is called linear programming.

Linear programming is concerned with maximizing or minimizing a linear expression (called the objective function (3)), subject to linear constraints (1). Usually it will be required that the variables have non-negative values (2). The constraints may be inequalities or equations.

The standard form of a linear program is:

Constraints

(1) $$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N = b_2$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$a_{M1}x_1 + a_{M2}x_2 + \cdots + a_{MN}x_N = b_M$$

(2) $$x_j \geq 0 \qquad j = (1, 2, \cdots N)$$

Objective Function

(3) $$C_1x_1 + C_2x_2 + \cdots + C_Nx_N = Z(\min)$$

Any set of values x_j satisfying (1) and (2) is called a feasible solution. The set of all feasible solutions is called the feasible set. The purpose of linear programming is to find a member of this set which optimizes the objective function. In the case of (3), the function is minimized.

A method for solving a linear program is called the simplex method, due to George B. Dantzig, who described it first in 1951.

In looking for optimal solutions, we need only to look at a subset of the feasible set, namely, basic feasible solutions. This is proven by the following theorem from (6, p. 12).

> Unless the feasible region is unbounded, every feasible optimizing solution is a linear combination of basic feasible optimizing solutions.

A basic feasible solution, provided the $\bar{b}_i$ so determined are non-negative is one obtained by putting (1) and (3) into canonical form through the use of row operations. The equations (1) and (3) are in canonical form if the coefficient of x_i is one in one of the equations and zero in the others for $i = 1, 2, 3, \cdots M$

$$
\begin{array}{lllllll}
x_1 & & + \bar{a}_{1, M+1} x_{N+1} & + \cdots & + \bar{a}_{1N} x_N & = & \bar{b}_1 \\
 & x_2 & + \bar{a}_{2, M+1} x_{M+1} & + \cdots & + \bar{a}_{2N} x_N & = & \bar{b}_2 \\
 & & \ddots & & & & \\
 & & x_M + \bar{a}_{M, M+1} x_{M+1} & + \cdots & + \bar{a}_{MN} x_N & = & \bar{b}_M \\
 & & (-Z) + \bar{C}_{M+1} x_{M+1} & + \cdots & + \bar{C}_N x_N & = & -\bar{Z}_0
\end{array}
$$

Building the Model

Dantzig (2, p. 32-35) gives five basic steps to be taken in building a linear programming model.

Step 1: Define the Activity Set. Decompose the entire system under study into all of its elementary functions, the activities. An activity is thought of as a kind of "black box" into which flow tangible inputs and out of which flow the products of the activity. A unit must be chosen for each activity in terms of which its quantity, or level, can be measured.

Step 2: Define the Item Set. Determine the classes of objects, the items which are consumed or produced by the activities, and choose a unit for measuring each item. Select one item such that the net quantity of it produced by the system as a whole measures the "cost" of the entire system.

Step 3: Determine the Input-Output Coefficients. Determine the quantity of each item consumed or produced by the operation of each activity at its unit level. These numbers, the input-output coefficients, are the factors of proportionality between activity levels and item flows.

Step 4: Determine the Exogenous Flows. Determine the net inputs or outputs of the items between the system, taken as a whole, and the outside.

Step 5: Determine the Material Balance Equations. Assign unknown nonnegative activity levels $x_1, x_2, \cdots$, to all the activities: then for each item, write the material balance equation which asserts that the algebraic sum of the flows of that item into each activity (given as the product of the activity level by the appropriate input-output coefficient) is equal to the exogenous flow of the item.

The result of the model-building is thus the collection of mathematical relationships characterizing all the feasible programs of the system. This collection is the linear programming model.

Using these steps it will be possible to build a model for the timber allocation problem.

Post-Optimality Analysis

Sensitivity analysis on the linear programming model is a method of determining the effect on the optimal solution of changes in the input-output coefficients, cost coefficients, and requirements. For example, a sensitivity analysis of the cost coefficients would involve finding the interval of values of one cost coefficient C_j such that the solution still remained optimal for any value of C_j in the interval, while retaining the other cost coefficients at fixed values.

Parametric programming is an extension of sensitivity analysis in which some or all of the cost coefficients are related by a one-parameter linear form $C_j = f_j + g_j t$. Values of t, for which a basis leads to a minimum, form an interval, e.g. $t' < t < t''$. This interval may be just a point and all such intervals form a connected set, e.g. $t_L < t_1 < t_2 < t_3 \cdots < t_u$, where t_L and t_u represent the finite or infinite range over which t can vary. Each interval in the set results in a different optimum solution.

APPLICATION OF LINEAR PROGRAMMING TO THE MANAGEMENT OF EVEN-AGED FORESTS

Building the Model

Using the steps outlined by Dantzig's method, a model can be built for the forest management problem.

1. Define Activity List:

Table I. Activity List.

Activity Number	Activity
1	Cutting company timber in year 1
2	Cutting company timber in year 2
. . .	. . . .
N	Cutting company timber in year N
N+1	Cutting government timber in year 1
N+2	Cutting government timber in year 2
. . .	. . . .
2N	Cutting government timber in year N

It was stated earlier that each activity could be thought of as a "black box" into which flow tangible inputs and out flow the products of the activity. This flow is shown in Figure III. The level of each activity is represented by the unknown quantities;

x_j = amount of company timber cut in year j

x_{N+j} = amount of government timber cut in year j

$$j = 1, 2, 3, \cdots N$$

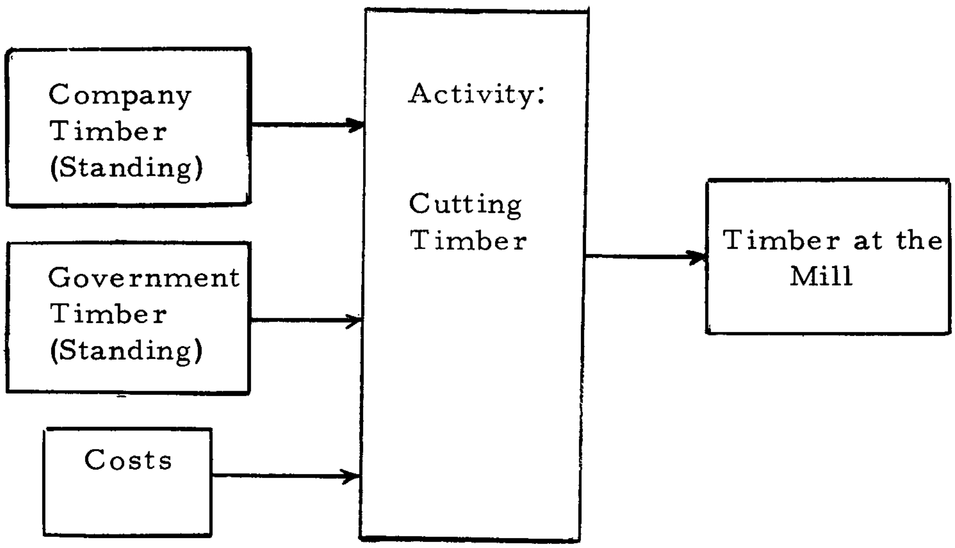

Figure III. Flow Diagram of an Activity.

2. Define Item Set

Except for cost it might seem that there is only one item namely timber. However, economists regard similar items at different locations or different times as different items. Therefore the item list is:

1.	Timber from company	Input
2.	Timber from government	
3.	Timber at mill, year 1	Output
.	. . . . . .	
.		
N+2	Timber at mill, year N	
N+3	Costs	

3. Determine the Input-Output Coefficients

The input-output coefficients determine the quantity of each item consumed by the operation of each activity at its unit level. The coefficients are the factors of proportionality or weights between activity levels and supply or demand. The weights are either zero or one for all items except item one and the cost item. In item one these coefficients are greater than one because for every unit cut, the quantity of timber consumed is composed of the actual unit cut plus the potential growth on that unit.

The coefficients or weights for item one are obtained as follows:

It is a given constraint that,

$$\sum_{j=1}^{N} x_j \leq V_C + V_{NG}$$

where;

x_j = amount of company timber cut in year j (M board feet)

V_C = original volume of company merchantable timber (M board feet)

V_{NG} = total new growth (M board feet)

N = number of years

V_{NG} is obtained as follows: Let

K = initial average M board feet volume per acre containing merchantable timber,

β_j = growth rate for year j (average M board feet increase per acre),

V_{NG_j} = total M board feet increase in volume due to new growth in the j^{th} year,

and

$$V_{NG} = \sum_{j=1}^{N} V_{NG_j} .$$

Then

$$V_1 = V_C - x_1$$

$$V_j = V_{j-1} - x_j + V_{j-1} \frac{\beta_{j-1}}{K + \sum_{j=1}^{N-2} \beta_j}$$

$$= V_{j-1} \left(1 + \frac{\beta_{j-1}}{K + \sum_{j=1}^{N-2} \beta_j} \right) - x_j$$

for $j = 2, 3, \ldots, N-1$

$$V_{NG_1} = V_1 \frac{\beta_1}{K}$$

$$V_{NG_2} = \left[\frac{V_1}{K} - \frac{x_2}{K + \beta_1} \right] \beta_2$$

$$V_{NG_3} = \left[\frac{V_2}{K + \beta_1} - \frac{x_3}{K + \beta_1 + \beta_2} \right] \beta_3$$

$$\vdots$$

$$V_{NG_N} = \left[\frac{V_{N-1}}{K + \sum_{j=1}^{N-2} \beta_j} - \frac{X_N}{K + \sum_{j=1}^{N-1} \beta_j} \right] \beta_N$$

which gives

$$\sum_{j=1}^{N} x_j \leq V_C + V_{NG}$$

$$\leq V_C + \sum_{j=1}^{N} V_{NG_j}$$

$$\leq V_1 \frac{\beta_1}{K} + \left[\frac{V_1}{K} - \frac{x_2}{K+\beta_1} \right] \beta_2 + \left[\frac{V_2}{K+\beta_1} - \frac{x_3}{K+\beta_1+\beta_2} \right] \beta_3$$

$$+ \ldots + \left[\frac{V_{N-1}}{K + \sum_{j=1}^{N-2} \beta_j} - \frac{x_N}{K + \sum_{j=1}^{N-1} \beta_j} \right] \beta_N$$

Combining similar terms gives an equation of the form

$$\sum_{j=1}^{N} a_j x_j \leq V_C' .$$

The a_j $j = 1, 2, \ldots N$ are the input coefficients for item one,

e.g. $a_N = 1 + \beta_N / \left(K + \sum_{j=1}^{N-1} \beta_j\right)$.

The β_j used in obtaining the coefficients are found by means of a sigmoid curve representing normal yield for an average site and stocking. This is done in Figure IV by letting $\beta = (B-A)/I$.

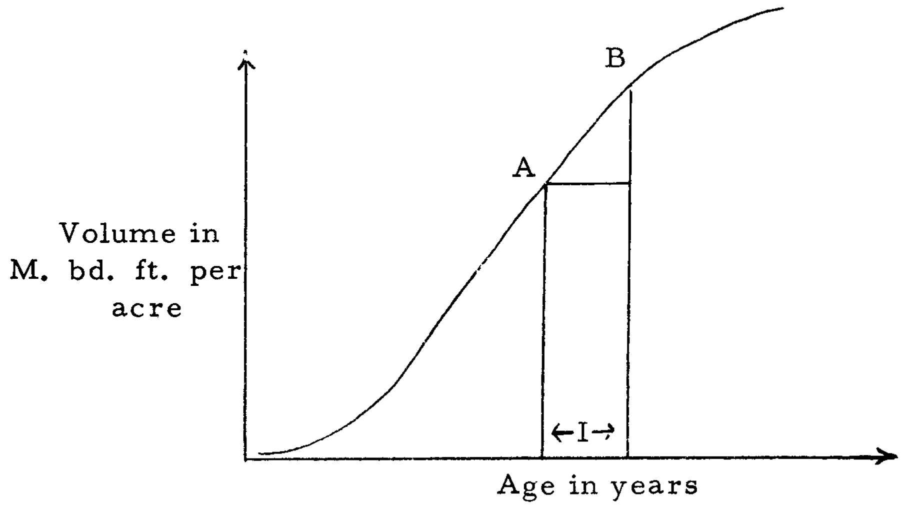

Figure IV. Approximation of Annual Growth Rate from Normal Yield Table Curve.

β is usually approximated over ten year periods therefore

$\beta_1 = \beta_2 = \beta_3 \ldots = \beta_{10} = {}_1\beta$.

The cost coefficients are actually combinations of costs due to several sourcesa

Cost of company timber

1. logging costs
2. taxes
3. investment

Cost of government timber

1. logging costs
2. sale price (see appendix),

where logging costs include

1. felling and bucking
2. yarding - loading
3. hauling
4. road construction

The C_j and C_{N+j} are the cumulative cost to the company of one thousand board feet of timber up until the time it was cut in year j.

$$C_1 = c_{01}$$

$$C_2 = c_{02} + c_{11} + c_{21}$$

$$C_3 = c_{03} + c_{11} + c_{12} + c_{22}$$

$$\vdots \qquad \vdots$$

$$C_N = c_{0N} + \sum_{j=1}^{N-1} c_{1j} + c_{2,N-1}$$

$$C_{N+1} = c_{3,N+1} + c_{4,N+1}$$

$$C_{N+2} = c_{3,N+2} + c_{4,N+2}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$C_{N+N} = c_{3,N+N} + c_{4,N+N}$$

where for $j = 1, 2, \cdots N$

c_{0j} = logging costs of company timber per M board feet in year j

c_{1j} = tax per M board feet for year j

c_{2j} = cost due to investment on M board feet up until it is cut in year j

$c_{3,N+j}$ = logging costs of government timber per M board feet in year j

$c_{4,N+j}$ = sale price of government timber in year j (M board feet)

Table II shows the coefficients in a systematic form.

Table II. Input-Output Coefficients.

Activities / Item	x_1	x_2	· · · ·	x_N	x_{N+1}	x_{N+2}	· · ·	x_{N+N}
Timber from company	a_1	a_2	· · · ·	a_N				
Timber from government					1	1	· · ·	1
Timber at mill, year 1	1				1			
Timber at mill, year 2		1				1		
· · · ·			⋱				⋱	
Timber at mill, year N				1				1
Costs	C_1	C_2	· · · ·	C_N	C_{N+1}	C_{N+2}	· · ·	C_{N+N}

4. Determine the Exogenous Flows

The next step in building the model is to determine the net inputs or outputs available for each item. See Figure V for "black box" form.

Item

Item		
1.	V_C'	available input
2.	$\sum_{j=1}^{N} D_j - \sum_{j=1}^{N} x_j$	
3.	D_1	available output
.	D_2	
.	.	
.	.	
N+2.	D_N	
N+3.	Z(min)	

V_C' = total supply of company timber

D_j = mill demand for year j

x_j = amount of company timber cut (year j)

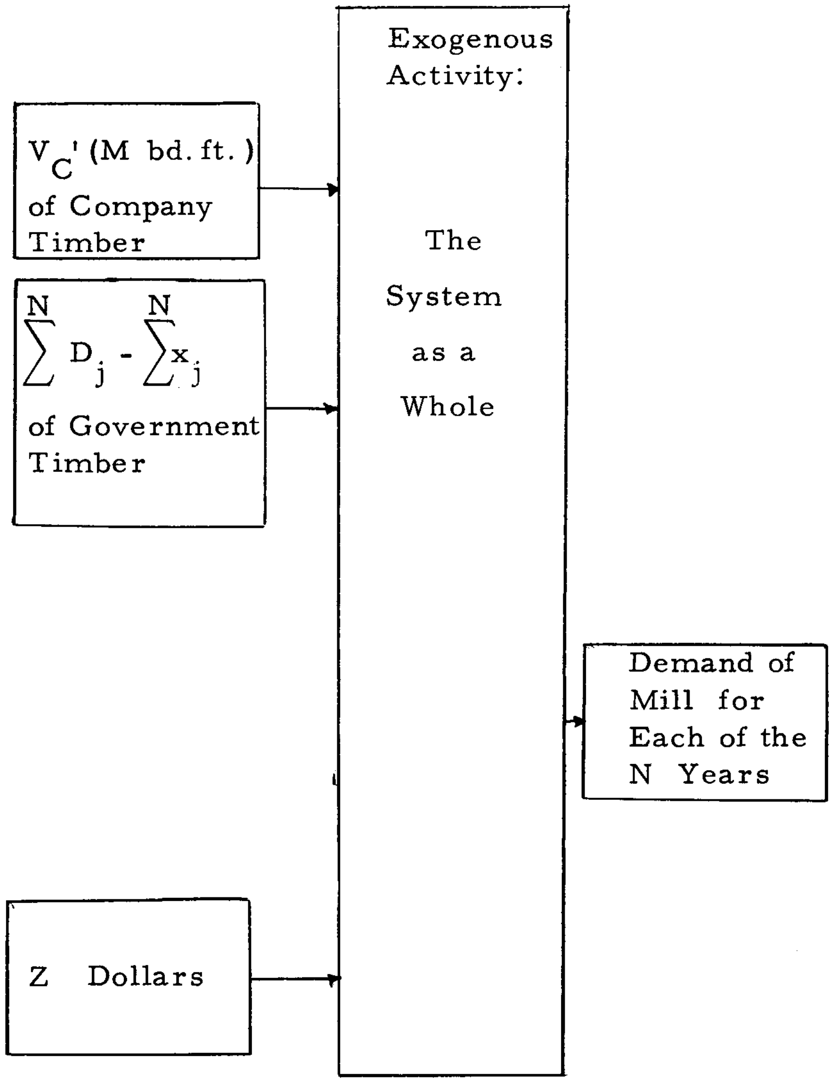

Figure V. Flow Diagram of Exogenous Activity.

5. Determine the Material Balance Equations

The material balance equations assert that the algebraic sum of activity should be equal to the total available input or output:

$$a_1x_1 + a_2x_2 + \cdots + a_Nx_N \leq V_C'$$

$$x_{N+1} + x_{N+2} + \cdots + x_{N+N} = \sum_{j=1}^{N} D_j - \sum_{j=1}^{N} x_j$$

$$x_1 + x_{N+1} = D_1$$

$$x_2 + x_{N+2} = D_2$$

$$\vdots$$

$$x_N + x_{N+N} = D_N$$

$$C_1x_1 + C_2x_2 + \cdots + C_Nx_N + C_{N+1}x_{N+1} + \cdots + C_{N+N}x_{N+N} = Z\,(\min)$$

It can be seen that in the material balance equations, the second equation is equal to the sum of the next N equations following. Therefore it is possible to eliminate the second equation to give an independent system of linear equations.

$$a_1x_1 + a_2x_2 + \cdots + a_Nx_N \leq V_C'$$

$$x_1 + x_{N+1} = D_1$$

$$x_2 + x_{N+2} = D_2$$

$$\vdots$$

$$x_N + x_{N+N} = D_N$$

$$C_1x_1 + \cdots + C_Nx_N + C_{N+1}x_{N+1} + \cdots + C_{N+N}x_{N+N} = Z(\min)$$

This is the model that will be used to determine optimum allocation of timber.

To put this linear program into standard form, we must introduce a slack variable x_{2N+1} into the first constraint. This gives us the equations,

$$a_1x_1 + a_2x_2 + \dots + a_Nx_N + a_{2N+1}\,x_{2N+1} = V_C{}'$$

$$x_1 + x_{N+1} = D_1$$

$$\vdots$$

$$x_N + x_{N+N} = D_N$$

$$C_1x_1 + \dots + C_Nx_N + C_{N+1}x_{N+1} + \dots + C_{2N}x_{2N} + C_{2N+1}x_{2N+1} = Z(\min).$$

C_{2N+1} represents the recovery value of company timber left after the N year period has expired. This value is equal to sale price of the timber minus the non-recoverable capital due to investment interest and tax.

Computer Solution

Due to the complexity of the model all calculations are done on a computer. Figure VI illustrates the steps involved.

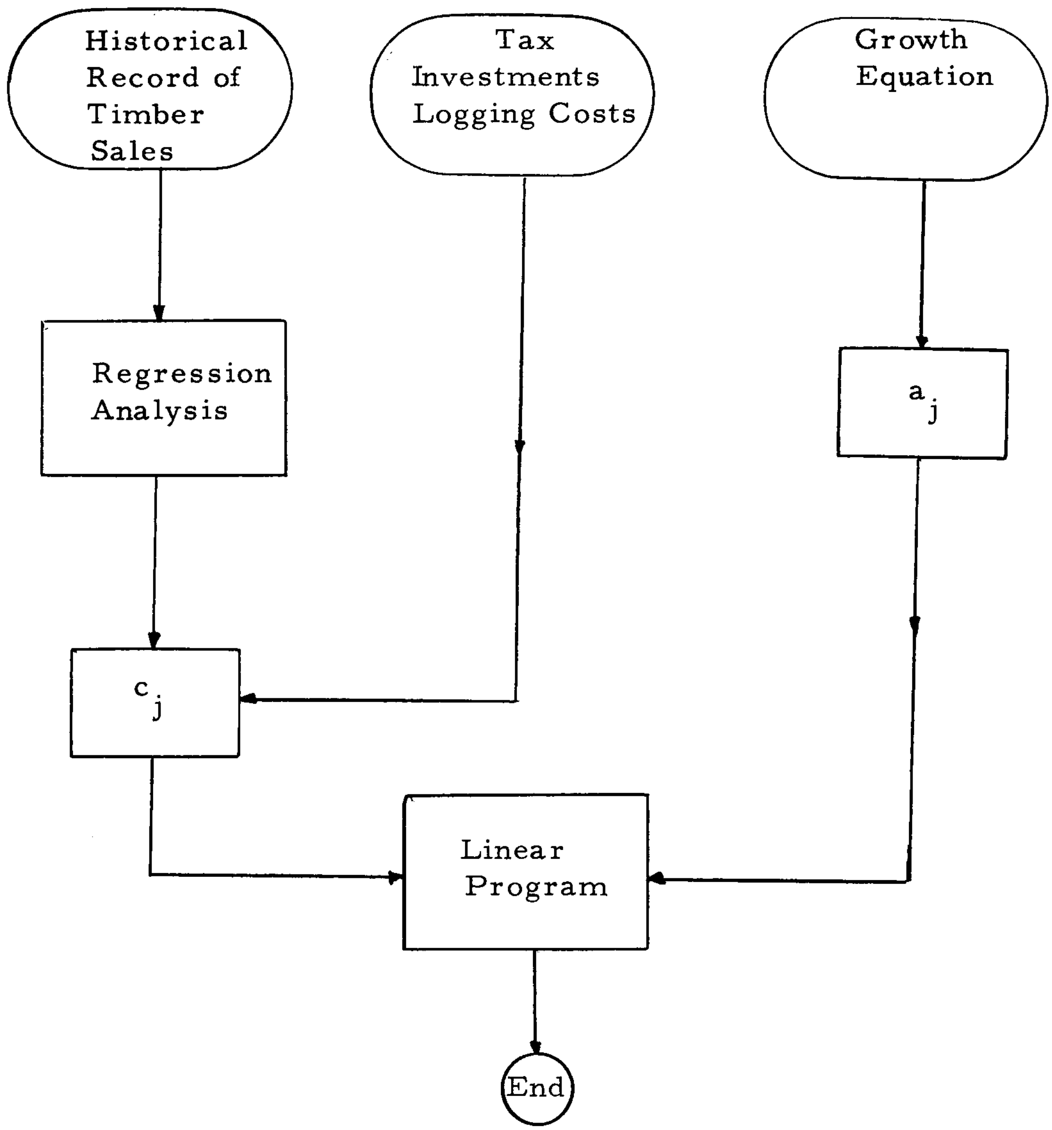

Figure VI. Computer Flow Chart.

Decomposition of the Solution

This particular linear program decomposes rather simply into two specific types of solution, as shown in the following proposition.

<u>Proposition:</u> A set of M equations in a linear programming model of the form,

$$
\begin{array}{lll}
(1) & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N + x_{2N+1} & = b_1 \\
(2) & x_1 \qquad\qquad\qquad + x_{N+1} & = b_2 \\
(3) & \qquad x_2 \qquad\qquad\qquad + x_{N+2} & = b_3 \\
\vdots & \qquad\qquad \ddots \qquad\qquad\qquad \ddots & \\
(M) & \qquad\qquad\qquad x_N \qquad\qquad + x_{N+N} & = b_M
\end{array}
$$

$$x_j \geq 0; \quad x_{N+j} \geq 0; \quad x_{2N+1} \geq 0; \quad a_{ij} > 0; \quad b_i > 0$$

for $i = 1, 2, \cdots M$ and $j = 1, 2, \cdots N$, $M = N + 1$,

has a basic feasible solution which is one of two specific types.

Type 1: The slack variable (x_{2N+1}) equals zero.

The solution for any given value j, excluding at most one value j', is of the form;

$$x_j = 0 \quad \text{and} \quad x_{N+j} = b_i$$

or

$$x_j = b_i \quad \text{and} \quad x_{N+j} = 0,$$

for $j = j'$,

$$x_{j'} = b_{i'} - x_{N+j'} \quad \text{and} \quad x_{N+j'} = b_{i'} - x_{j'}$$

$$x_{j'} > 0, \quad x_{N+j'} > 0, \quad j' = 1, 2, \cdots N.$$

Type 2: The slack variable (x_{2N+1}) is not equal to zero but equal to $b_1 - \Sigma a_{1j} x_j$.

The remaining variables are then

$$x_j = 0 \quad \text{and} \quad x_{N+j} = b_i$$

or

$$x_j = b_i \quad \text{and} \quad x_{N+j} = 0 .$$

Proof:

The equations (1) through (M) are in canonical form therefore we have $(2N+1) - (N+1) = N$ non-basic variables. In the case of a Type 1 solution the slack variable (x_{2N+1}) is a non-basic variable (equal to zero). It follows from the definition of a non-basic variable, that $N-1$ of the equations (2) through (M) must be solved with one basic and one non-basic variable. This would leave one equation of the form,

$$x_{j'} + x_{N+j'} = b_{i'}, \quad j' = 1, 2, \cdots N,$$

without a non-basic variable. However, the N-1 non-basic variables substituted into the (2) through (M) equations,

$$x_j + x_{N+j} = b_i, \quad j \neq j'; \quad j = 1, 2, \cdots N,$$

gives us values of all x_j except j'. Therefore substitute the N-1 known values of x_j into (1) and we get,

$$a_{ij'} x_{j'} = b'$$

$$x_{j'} = b' / a_{ij'},$$

where $b' = b_1 - \sum_{\text{all } j \neq j'} a_{ij} x_j$.

This enables us to solve

$$x_{j'} + x_{N+j'} = b_{i'} ,$$

for the basic variables $x_{j'}$ and $x_{N+j'}$.

If the slack variable is not one of the N non-basic variables, a solution of Type 2 form is attained. This is easily seen as equations (2) through (M) must now contain N non-basic variables, hence a solution of the form,

$$x_{2N+1} = b_1 - \sum a_{ij} x_j$$

$$x_j = 0 \quad \text{and} \quad x_{N+j} = b_i$$

or

$$x_j = b_i \quad \text{and} \quad x_{N+j} = 0 \, .$$

The physical interpretation of the slack variable is that it represents the amount of company timber remaining after N years. If there is no timber left after the N year period, we can say by means of the foregoing proposition, that we will either use all company or all government timber in any given year except for at most one. The exception is the year in which depletion of company timber starts or ends. This may cause a split in the source of timber.

If the slack variable is not zero, all company or all government timber is used in any given year, with the excess or slack company timber remaining at the end of the N years. However, this will not occur very often, due to cost differences.

Numerical Example

The example used here could apply to a local lumber company. Simulated data were obtained through the School of Forestry at Oregon State University.

Table III. Data.

Number of years	N	40
Original volume	V_C	40,000 M
Mill demand for each of the N years	D_j	4,200 M
Average M bd. ft. per acre volume	K	64.5
Constant growth rate		
Years 1966-1975	${}_1\beta$	.7
1976-1985	${}_2\beta$	.6
1986-1995	${}_3\beta$	.525
1996-2005	${}_4\beta$	.475

Since sale price represents a component in the total cost of government timber, the prediction of sale price was done by the method proposed in the appendix of this paper.

The historical record of timber sales on forest districts from which the company might purchase timber was obtained at the Siuslaw National Forest Headquarters. Using the IBM 1410 computer stepwise regression analysis on a sample of 139 observations of appraisal price and sale price over the years 1964 and 1965, the correlation coefficient and regression line were computed. The correlation between appraisal and sale prices was $r = .557$ and the regression of sale prices on appraisal was

$$y_j = 11.40 + 1.03\, x_j \,.$$

A regression analysis was also run on a sample of sales over the past eight years. The value of r, b_0, and b_1 did not differ significantly from the two year result. Most of the difference that did occur would be due to sales that were a result of the Columbus Day storm in 1962. Hence, the two year result was used.

Costs from forestry personnel plus sale price predicted by use of the regression line determined the C_j. The a_{ij} also were calculated with the aid of the IBM 1410 computer.

j	C_j	C_{40+j}	a_j
1	29.31	66.65	1.3612
2	30.54	67.67	1.3468
3	31.63	68.43	1.3326
4	32.95	69.44	1.3188
5	34.12	70.20	1.3052
6	35.36	70.90	1.2919
7	36.64	71.73	1.2788
8	37.99	72.49	1.2611
9	39.40	73.26	1.2536
10	40.78	73.90	1.2413
11	42.40	74.79	1.2293
12	43.92	75.43	1.2190
13	45.52	76.07	1.2088
14	47.18	77.70	1.1989
15	48.93	77.35	1.1891
16	50.76	77.98	1.1795
17	52.71	78.63	1.1700
18	54.73	79.26	1.1607
19	56.86	79.90	1.1515
20	59.09	80.54	1.1425
21	61.46	81.17	1.1336
22	63.93	81.80	1.1258
23	66.52	82.43	1.1180
24	69.25	83.06	1.1104
25	72.08	83.67	1.1029
26	75.15	84.29	1.0955
27	78.34	84.88	1.0882
28	81.70	85.54	1.0810
29	85.24	86.29	1.0739
30	88.97	86.93	1.0668
31	92.89	87.57	1.0599
32	97.03	88.21	1.0536
33	101.40	88.85	1.0473
34	106.00	89.48	1.0412
35	110.98	90.13	1.0351
36	115.97	90.76	1.0291
37	121.39	91.41	1.0231
38	127.09	92.04	1.0171
39	133.12	92.67	1.0114
40	136.91	93.32	1.0056

The solution arrived at using the foregoing data was:

Year	Company	Government
1	4200 M bd. ft.	0 M bd. ft.
2	4200	0
3	4200	0
4	4200	0
5	4200	0
6	4200	0
7	4200	0
8	4200	0
9	4200	0
10	4090	110
11	0	4200
12	0	↓
.	↓	↓
.	↓	↓
.	↓	↓
40	0	4200

Z(min) = \$ 12,057,093.00

As stated earlier in this paper, sensitivity analysis of the cost coefficients refers to finding an interval of values of one cost coefficient, while retaining the other cost coefficients at fixed values, such that the solution still remained optimal for any value of C_j in the interval. The computer program used to solve the preceding problem printed out the sensitivity or indifference range for each C_j and C_{40+j}. For an example, let's look at the interval around C_{41} and C_{47} .

The indifference range for C_{41} which has a value of \$66.65 is,

$$65.60 \leq C_{41} \leq \infty .$$

The optimum solution will not change if C_{41} lies in the stated interval. However, if C_{41} were equal to \$65.00, this would cause x_1 to become non-basic (equal to zero) and x_{41} basic (equal to 4200). This would mean cutting government timber in the first year followed by a depletion of company timber.

The indifference range for C_{47} which has a value of \$71.73 is;

$$70.76 \leq C_{47} \leq \infty .$$

If the value of C_{47} was lowered to \$70.00, x_{47} would become basic (equal to 4200) and x_7 would be equal to zero. In terms of the allocation problem, company timber would be used in the first six years, government timber used the seventh year, with a depletion of company timber to follow.

Assuming that the cost of company and government timber increases at an approximate constant rate as in the example presented, what is the value of looking at the effect of significant changes in isolated cost coefficients or in a neighboring series of cost coefficients. The main value is that it may give some information as to the effect of inflation or other factors that might cause abnormal costs.

On examining the C_j and C_{40+j} for the example, it might be concluded that the reason for cutting company timber first in the example is that initially C_j is quite a bit less than C_{40+j}, plus the fact that C_j and C_{40+j} steadily increase by an approximate constant rate. Instead, the reason seems to be due to the rate of increase by C_j and C_{40+j} with C_j increasing at a faster rate than C_{40+j}. This was shown by a hypothetical example in which C_j was less than C_{N+j}, both increased by a constant amount, and C_j increased at a slower rate than C_{N+j}.

Hypothetical Example

Year	C_j	C_{32+j}
1	22.00	72.50
2	22.45	73.15
3	22.90	73.80
.	.	.
.	.	.
.	.	.
.	.	.
32	36.40	93.30

$V_C' = 14{,}448$ $\qquad$ $D_j = 4200$

The solution to the example was:

Year	Company	Government
1	0	4200
2	0	4200
.	↓	↓
.		
.		
28	1585	2615
29	4200	0
30	4200	0
31	4200	0
32	0	4200

This shows that when the rate of increase of C_{N+j} is greater than C_j government timber is cut first. The year in which depletion of company timber begins (here the 28th year) is dependent on the a_j and the magnitude of the increase rate.

To summarize the sensitivity of numerical examples such as the ones given where C_j and C_{N+j} each increase steadily at an approximate constant rate with $C_j < C_{N+j}$ (in the case of C_j approaching C_{N+j}, C_j may become greater than C_{N+j} in the later stages), the following seems to hold true:

(1) If C_j increases faster than C_{N+j} company timber is cut first.

(2) If C_j increases slower than C_{N+j} government timber is cut first.

(3) The a_j also effect the year in which exhaustion of company timber begins.

(4) The x_j which are equal to a zero (non-basic variables) and the x_j which are equal to a constant (basic variables) are effected by changes which are outside the indifference range. This type of change in the cost coefficient causes a new variable to enter the basis. Therefore, a change to a value outside the indifference range of x_{30} in the hypothetical example will cause x_{60} to become basic (equal to 4200) while x_{30} becomes non-basic (equal to zero).

Sensitivity analysis on the input-output coefficients as well as parametric programming would be useful in an analysis of this example. However, due to the unavailability of these programs for the computer this was not done.

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APPENDIX

APPENDIX

The data are dictated by timber management to give an impression of what results to expect under the conditions postulated. However, management may find the sale price of government timber difficult to estimate. Since this difficulty tends to increase in time, a method must be formulated to predict these prices.

Assuming that the government appraisal price increases on the average by a constant amount each year, it is possible to assign a definite value to appraisal price for any given year. This appraisal price will then enable us to predict a sale price. This is done by obtaining past records of sales on government forest districts where the company in concern would purchase timber. Using a sample of recorded sales, the appraised price is plotted against sale price as in Figure VII.

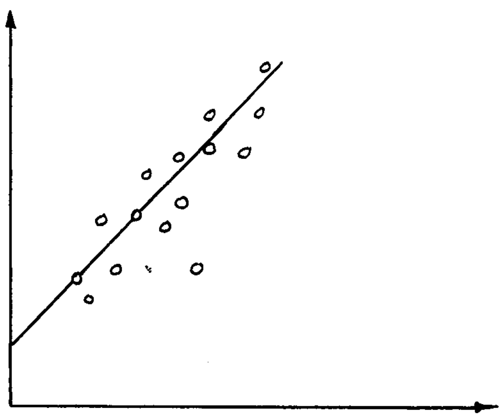

Figure VII. Plotting of Historical Timber Sale Record.

The points are then fitted by a straight line using the least squares procedure. Since the equation for the straight line is the standard simple linear regression model, it can be used as a prediction equation.

General model,

$$y_j = \beta_0 + \beta_i x_j + \varepsilon_j$$

where,

y_j = observation of sale price (dependent variable) for the j^{th} element in population

β_0 = intercept

β_1 = slope of relationship between x and y

x_j = observation of appraisal price (independent variable) for the j^{th} element in the population

ε_j = random error NID$(0, \sigma^2)$

To predict a sale price y for given appraised price x^*, let

$$y = b_0 + b_1 x^*, \qquad b_0 = \hat{\beta}_0, \quad b_1 = \hat{\beta}_1 ,$$

y can now be used as an estimate of some future y, say y^+, to be observed at x^*.

A prediction interval around y^+ would be:

$$b_0 + b_1x^* - A \leq y^+ \leq b_0 + b_1x^* + A$$

where

$$A = t_{\frac{\alpha}{2}} s \sqrt{1 + \frac{1}{n} + \frac{(x^* - \bar{x})^2}{\Sigma(x_i - \bar{x})^2}} \quad .$$