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Discovering real lie subalgebras of $\mathfrak{e}_6$ using Cartan decompositions

Aaron Wangberg[1,a)] and Tevian Dray[2,b)]
[1]*Department of Mathematics and Statistics, Winona State University, Winona, Minnesota 55987, USA*
[2]*Department of Mathematics, Oregon State University, Corvallis, Oregon 97331, USA*

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The process of complexification is used to classify Lie algebras and identify their Cartan subalgebras. However, this method does not distinguish between real forms of a complex Lie algebra, which can differ in signature. In this paper, we show how Cartan decompositions of a complexified Lie algebra can be combined with information from the Killing form to identify real forms of a given Lie algebra. We apply this technique to $\mathfrak{sl}(3, \mathbb{O})$, a real form of $\mathfrak{e}_6$ with signature (52, 26), thereby identifying chains of real subalgebras and their corresponding Cartan subalgebras within $\mathfrak{e}_6$. Motivated by an explicit construction of $\mathfrak{sl}(3, \mathbb{O})$, we then construct an Abelian group of order 8 which acts on the real forms of $\mathfrak{e}_6$, leading to the identification of 8 particular copies of the 5 real forms of $\mathfrak{e}_6$, which can be distinguished by their relationship to the original copy of $\mathfrak{sl}(3, \mathbb{O})$. *© 2013 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4818503]

I. INTRODUCTION

The group E_6 has a long history of applications in physics, beginning with the original discovery of the Albert algebra and its relationship to exceptional quantum mechanics.[1–3] As a candidate gauge group for a Grand Unified Theory, E_6 is the natural next step in the progression $SU(5)$, $SO(10)$, both of which are known to lead to interesting, albeit ultimately unphysical, models of fundamental particles (see, e.g., Ref. 4), and is closely related to the Standard Model gauge group $SU(3) \times SU(2) \times U(1)$.

A description of the group $E_{6(-26)}$ as $SL(3, \mathbb{O})$ was given in Ref. 5, generalizing the interpretation of $SL(2, \mathbb{O})$ as (the double cover of) $SO(9, 1)$ discussed in Ref. 6. An interpretation combining spinor and vector representations of the Lorentz group in 10 spacetime dimensions was described in Ref. 7, and in Ref. 8 we obtained nested chains of subgroups of $SL(3, \mathbb{O})$ that respect this Lorentzian structure.

The resulting action of E_6 appears to permit an interpretation in terms of electroweak interactions on leptons, suggesting that this approach may lead to models of fundamental particles with physically relevant properties. In particular, the asymmetric nature of the octonionic multiplication table appears to lead naturally[9] to precisely three generations of particles, with single-helicity neutrinos, observed properties of nature which as yet have no theoretical foundation.

In the present work, we use various Cartan decompositions of the Lie algebra $\mathfrak{e}_6$ to further extend this construction in two distinct ways. First, we identify additional real subalgebras of $\mathfrak{sl}(3, \mathbb{O})$, thus completing the explicit construction of nested chains of subgroups of $SL(3, \mathbb{O})$ begun in Ref. 8, building on the Lorentzian structure of $SL(2, \mathbb{O})$. In particular, we locate the "missing" C_4 subgroups of $SL(3, \mathbb{O})$ referred to in Ref. 8, in the form $SU(3, 1, \mathbb{H})$. We then reinterpret the Cartan decompositions used in our construction as (vector space) isomorphisms of the complexified Lie algebra $\mathfrak{e}_6^{\mathbb{C}}$, yielding an Abelian group of such isomorphisms that acts on the 5 real forms of

a)Electronic mail: awangberg@winona.edu
b)Electronic mail: tevian@math.oregonstate.edu

$\mathfrak{e}_6$, allowing us to identify 8 particular copies of these real forms, based on their relationship to the original copy of $\mathfrak{sl}(3, \mathbb{O})$.

We briefly review the basic structure of $SL(3, \mathbb{O})$ in Sec. II A, and the basic properties of Cartan decompositions in Sec. II B. We construct three important Cartan decompositions of $\mathfrak{e}_6$ in Sec. III, and use them to construct nested chains of subalgebras of $\mathfrak{sl}(3, \mathbb{O})$ in Sec. IV, which are used in Sec. IV B to construct a tower of subalgebras in $\mathfrak{sl}(3, \mathbb{O})$ based upon our preferred basis of the Cartan subalgebra, completing the program begun in Ref. 8. Finally, we explore the relationships between the real forms of $\mathfrak{e}_6$ in Sec. V.

II. THE BASICS

A. $SL(3, \mathbb{O})$

We summarize here the description of the Lie group $SL(3, \mathbb{O}) = E_{6(-26)}$, as given in Refs. 5,7,8, and 10. The *Albert algebra* $\mathbf{H}_3(\mathbb{O})$ consists of the 3×3 octonionic Hermitian matrices. A *complex* matrix $\mathcal{M}$ is one whose components lie in some complex subalgebra of $\mathbb{O}$; such matrices act on $\mathcal{X} \in \mathbf{H}_3(\mathbb{O})$ as

$$\mathcal{X} \longmapsto \mathcal{M}\mathcal{X}\mathcal{M}^\dagger \tag{1}$$

and $SL(3, \mathbb{O})$ can be defined as the (composition of) such transformations that preserve the determinant of $\mathcal{X}$. Each *Jordan matrix* $\mathcal{X}$ can be decomposed as

$$\mathcal{X} = \left(\begin{array}{c|c} \mathbf{X} & \theta \\ \hline \theta^\dagger & \cdot \end{array}\right) \tag{2}$$

and $\mathbf{M} \in SL(2, \mathbb{O}) \subset SL(3, \mathbb{O})$ acts on *vectors* $\mathbf{X}$ and *spinors* θ via the embedding

$$\mathbf{M} \longmapsto \mathcal{M} = \left(\begin{array}{c|c} \mathbf{M} & 0 \\ \hline 0 & 1 \end{array}\right). \tag{3}$$

An explicit identification of elements of $SL(2, \mathbb{O})$ with the (double cover of the) Lorentz group $SO(9, 1)$ was given in Ref. 6, naturally generalizing the identification of $SL(2, \mathbb{C})$ with (the double cover of) $SO(3, 1)$. We refer to elements of $SL(3, \mathbb{O})$ of the form (3) as being of *type* 1. Cyclically permuting rows and columns in (3) results in analogous embeddings of type 2 (1 in the upper left) and type 3 (1 in the middle). There are thus 3 natural embeddings of $SL(2, \mathbb{O})$ sitting inside $SL(3, \mathbb{O})$, whose elements we label according to their type and character as a Lorentz transformation. Thus, the 45 elements of type 1 $SL(2, \mathbb{O})$ consist of 9 *boosts* B^1_{tq} and 36 independent *rotations* R^1_{qr}, where the spatial labels q, r run over x, z, and the imaginary octonionic units $\{i, j, k, k\ell, j\ell, i\ell, \ell\}$. Some of these 45 *generators* of type 1 $SL(2, \mathbb{O})$ require the use of more than one transformation of the form (1), a phenomenon we call *nesting*.

The generators of the 3 natural copies of $SL(2, \mathbb{O})$ do indeed span $SL(3, \mathbb{O})$, but are not independent, so we introduce a preferred set of generators. First of all, although there are 3 natural chains of subgroups of the form $G_2 \subset SO(7) \subset SO(9, 1)$, the 3 copies of G_2 are in fact the same, so we dispense with the superscript labeling type, and call the 14 generators $\{A_q, G_q\}$, with q now ranging over the imaginary octonionic units. We denote the remaining 7 generators of (type 1) $SO(7)$ by S^1_q. Second, by triality there is only one copy of $SO(8)$, so we add the 7 generators R^1_{xq}, for a total of 28 generators so far. The remaining 24 rotations in $SL(3, \mathbb{O})$ are the 3 types of rotations with z, to which must be added the $27 - 1 = 26$ independent boosts—the diagonal zt boosts are not all independent.

Our 78 preferred generators of $SL(3, \mathbb{O})$ become a basis of the Lie algebra $\mathfrak{sl}(3, \mathbb{O})$ under differentiation, denoted by a dot; this preferred basis is summarized in Table I. For further details, see Refs. 5,7, and 8.

TABLE I. Our preferred basis for $\mathfrak{sl}(3, \mathbb{O})$.

	(2)	(3)	(21)
Boosts	$\dot{B}^1_{tz}$	$\dot{B}^1_{tx}$	$\dot{B}^1_{tq}$
	$\dot{B}^2_{tz}$	$\dot{B}^2_{tx}$	$\dot{B}^2_{tq}$
		$\dot{B}^3_{tx}$	$\dot{B}^3_{tq}$
	(7)	(3)	(21)
Simple rotations	$\dot{R}^1_{xq}$	$\dot{R}^1_{xz}$	$\dot{R}^1_{zq}$
		$\dot{R}^2_{xz}$	$\dot{R}^2_{zq}$
		$\dot{R}^3_{xz}$	$\dot{R}^3_{zq}$
Transverse rotations	(7)	(7)	(7)
	$\dot{A}_q$	$\dot{G}_q$	$\dot{S}^1_q$

B. Graded Lie algebras

A $\mathbb{Z}_2$-*grading* of a Lie algebra $\mathfrak{g}$ is a decomposition

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{m} \tag{4}$$

such that

$$\begin{aligned} [\mathfrak{p}, \mathfrak{p}] &\subset \mathfrak{p} \\ [\mathfrak{m}, \mathfrak{m}] &\subset \mathfrak{p} \\ [\mathfrak{p}, \mathfrak{m}] &\subset \mathfrak{m} \end{aligned} \tag{5}$$

so that $\mathfrak{p}$ is itself a Lie algebra, but $\mathfrak{m}$ is merely a vector space. We will assume further that $\mathfrak{m}$ is not Abelian (in which case it would be an ideal of $\mathfrak{g}$), and that the Killing form is nondegenerate on each of $\mathfrak{p}$ and $\mathfrak{m}$.[11] Each $\mathbb{Z}_2$-grading defines a map θ on $\mathfrak{g}$, given by

$$\theta(P + M) = P - M \qquad (P \in \mathfrak{p}, M \in \mathfrak{m}), \tag{6}$$

which is clearly an involution, that is, a Lie algebra automorphism whose square is the identity map. Conversely, an involution θ defines a $\mathbb{Z}_2$-grading in terms of the eigenspaces of θ with eigenvalues ± 1, which can be shown to satisfy (5).[12]

A *Cartan decomposition* of a real Lie algebra $\mathfrak{g}$ is a $\mathbb{Z}_2$-grading of $\mathfrak{g}$, with the further property that the Killing form B is negative definite on $\mathfrak{p}$ and positive definite on $\mathfrak{m}$. In this case, $(\mathfrak{p},\mathfrak{m})$ is called a *Cartan pair*, the signature of $\mathfrak{g}$ is $(|\mathfrak{p}|,|\mathfrak{m}|)$, and the associated involution θ is called a *Cartan involution*. Informally, $\mathfrak{p}$ consists of *rotations*, and $\mathfrak{m}$ of *boosts*.

We extend this terminology to the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$. Whereas each real form of $\mathfrak{g}^{\mathbb{C}}$ admits a unique (up to isomorphism) Cartan decomposition, $\mathfrak{g}^{\mathbb{C}}$ itself will admit several, one for each real form.

A slight modification of an involution θ on $\mathfrak{g}$ can be used to map one real form of $\mathfrak{g}^{\mathbb{C}}$ to another. Use θ to split $\mathfrak{g}$ into eigenspaces $\mathfrak{p}$ and $\mathfrak{m}$ as above, and then introduce the *associated Cartan map* ϕ^* on $\mathfrak{g}^{\mathbb{C}}$ via

$$\phi^*(P + M) = P + \xi M \qquad (P \in \mathfrak{p}^{\mathbb{C}}, M \in \mathfrak{m}^{\mathbb{C}}) \tag{7}$$

with $\xi^2 = -1$, that is, where ξ is a square root of -1 which commutes with all imaginary units used in the representation of $\mathfrak{g}$. The structure constants of a real form $\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$ are real by definition, and since

$$[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{p} \qquad [\mathfrak{p}, \xi\mathfrak{m}] \subset \xi\mathfrak{m} \qquad [\xi\mathfrak{m}, \xi\mathfrak{m},] \subset \xi^2\mathfrak{p} = (-1)\mathfrak{p} \tag{8}$$

then $\mathfrak{p} \oplus \xi\mathfrak{m}$ also has real structure constants, and is therefore also a real form of the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$. Note that ϕ^* is a vector space isomorphism, but not a Lie algebra isomorphism—as must be the case if it takes one real form to another. Further information regarding the interplay between involutive automorphisms, the Killing form, and real forms of a complex Lie algebra may be found in Ref. 12.

We claim that *any* $\mathbb{Z}_2$-grading of a real Lie algebra $\mathfrak{g}$ is in fact the Cartan map associated with (the restriction of) *some* Cartan decomposition of $\mathfrak{g}^{\mathbb{C}}$.

Lemma: Every $\mathbb{Z}_2$-grading of a real Lie algebra $\mathfrak{g}$ is the restriction of a Cartan decomposition of $\mathfrak{g}^{\mathbb{C}}$. Equivalently, the extension of the $\mathbb{Z}_2$-grading to $\mathfrak{g}^{\mathbb{C}}$ is the image of a Cartan decomposition of some (other) real form of $\mathfrak{g}^{\mathbb{C}}$ under an associated Cartan map.

Proof: Let $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{m}$ be a $\mathbb{Z}_2$-grading of $\mathfrak{g}$. Extend this grading to the $\mathbb{Z}_2$-grading $\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$, then restrict each component to the compact real form $\mathfrak{g}_c$ of $\mathfrak{g}^{\mathbb{C}}$. Since the compactness of $\mathfrak{g}$ depends entirely on the signature of the Killing form, $\mathfrak{g}_c$ is clearly isomorphic to $\mathfrak{g}$ *as a vector space*, and we therefore obtain the vector space decomposition

$$\mathfrak{g}_c = \mathfrak{p}_c \oplus \mathfrak{m}_c. \tag{9}$$

Since we started with a $\mathbb{Z}_2$-grading, and since $\mathfrak{g}_c$ is a Lie algebra by assumption, we must have

$$[\mathfrak{p}_c, \mathfrak{p}_c] \subset \mathfrak{p}^{\mathbb{C}} \cap \mathfrak{g}_c = \mathfrak{p}_c \tag{10}$$

with similar expressions holding for the other commutators in (5). Thus, (9) is a $\mathbb{Z}_2$-grading of $\mathfrak{g}_c$. It is now straightforward to invert the associated Cartan map, constructing the vector space

$$\mathfrak{g}' = \mathfrak{p}_c \oplus \xi\mathfrak{m}_c \tag{11}$$

with $\xi^2 = -1$. Even though associated Cartan maps are not Lie algebra isomorphisms, they do preserve the $\mathbb{Z}_2$-grading, so $\mathfrak{g}'$ is a Lie algebra, and the Killing form is negative definite on $\mathfrak{p}_c$, and positive definite on $\mathfrak{m}_c$, by construction (and the assumed nondegeneracy of the Killing form). Thus, (11) is a Cartan decomposition of the real form $\mathfrak{g}'$ of $\mathfrak{g}^{\mathbb{C}}$. □

The $\mathbb{Z}_2$-gradings of $\mathfrak{g}$ therefore correspond to the possible real forms of $\mathfrak{g}^{\mathbb{C}}$. We will use associated Cartan maps $\phi^* : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ not only to identify different real forms of $\mathfrak{g}^{\mathbb{C}}$, but also to identify real subalgebras of our particular real form $\mathfrak{g}$. When applied to the compact real form $\mathfrak{g}_c = \mathfrak{p}_c \oplus \mathfrak{m}_c$, the map ϕ^* changes the signature from $(|\mathfrak{p}| + |\mathfrak{m}|, 0)$ to $(|\mathfrak{p}|, |\mathfrak{m}|)$. Similar counting arguments give the signature of $\phi^*(\mathfrak{g})$ when $\mathfrak{g}$ is non-compact, since ϕ^* changes some compact generators into non-compact generators, and vice versa. We then use the rank, dimension, and signature of $\phi^*(\mathfrak{g})$ to identify the particular real form of the algebra. Using tables of real forms of $\mathfrak{g}^{\mathbb{C}}$ showing their maximal compact subalgebras,[12] we can also identify $\mathfrak{p} = \mathfrak{g} \cap \phi^*(\mathfrak{g})$ as a subalgebra of our original real form $\mathfrak{g}$. Since the maximal compact subalgebra of $\phi^*(\mathfrak{g})$ is known, we can further identify its pre-image as a non-compact subalgebra of $\mathfrak{g}$.

III. CARTAN DECOMPOSITIONS OF $\mathfrak{sl}(3, \mathbb{O})$

A. Some gradings of $\mathfrak{sl}(3, \mathbb{O})$

We first make some comments about our preferred basis for $\mathfrak{sl}(3, \mathbb{O})$, which is listed in Table I and further discussed in Ref. 8. Let $\mathfrak{b}$ and $\mathfrak{r}$ be the vector subspaces consisting of boosts and rotations, respectively. Our preferred choice of basis favors type 1 transformations, in the sense that we choose to represent $\mathfrak{so}(8)$, as well as its subalgebra $\mathfrak{g}_2$, in terms of transformations of type 1. Let $\mathfrak{t}_1$ be the subspace spanned by $\dot{B}^2_{tz} - \dot{B}^3_{tz}$ and all type 1 transformations, and $\mathfrak{t}_2$ and $\mathfrak{t}_3$ be the subspaces spanned by the type 2 and type 3 transformations in our preferred basis which are not in $\mathfrak{t}_1$.[13] Let $\mathfrak{t}_{23} = \mathfrak{t}_2 \oplus \mathfrak{t}_3$. Finally, let $\mathfrak{h}$ be the subspace corresponding to transformations that preserve the preferred quaternionic subalgebra $\mathbb{H} = \langle 1, k, k\ell, \ell \rangle$, and let $\mathfrak{h}^\perp$ be its orthogonal complement. The vector space $\mathfrak{h}$ is spanned by transformations with no labels in $\{i, j, j\ell, i\ell\}$ and is in fact a subalgebra, while $\mathfrak{h}^\perp$ is the subspace spanned by transformations having one label from $\{i, j, j\ell, i\ell\}$; $\mathfrak{h}$ contains *quaternionic* basis elements, such as $\dot{A}_{k\ell}$, $\dot{R}^1_{z\ell}$, and $\dot{B}^3_{tx}$, while $\mathfrak{h}^\perp$ contains *orthogonal-quaternionic* basis elements such as $\dot{A}_i$, $\dot{R}^1_{zi\ell}$, and $\dot{B}^3_{tj\ell}$.

Direct computation shows that

$$\begin{array}{lll} [\mathfrak{r},\mathfrak{r}]\subset\mathfrak{r} & [\mathfrak{t}_1,\mathfrak{t}_1]\subset\mathfrak{t}_1 & [\mathfrak{h},\mathfrak{h}]\subset\mathfrak{h} \\ [\mathfrak{r},\mathfrak{b}]\subset\mathfrak{b} & [\mathfrak{t}_1,\mathfrak{t}_{23}]\subset\mathfrak{t}_{23} & \left[\mathfrak{h},\mathfrak{h}^\perp\right]\subset\mathfrak{h}^\perp \\ [\mathfrak{b},\mathfrak{b}]\subset\mathfrak{r} & [\mathfrak{t}_{23},\mathfrak{t}_{23}]\subset\mathfrak{t}_1 & \left[\mathfrak{h}^\perp,\mathfrak{h}^\perp\right]\subset\mathfrak{h} \end{array} \tag{12}$$

so that each of the decompositions $\mathfrak{r}\oplus\mathfrak{b}$, $\mathfrak{t}_1\oplus\mathfrak{t}_{23}$, and $\mathfrak{h}\oplus\mathfrak{h}^\perp$ is a $\mathbb{Z}_2$-grading of $\mathfrak{sl}(3,\mathbb{O})$. We therefore introduce the involutions ϕ_s, ϕ_t, and $\phi_{\mathbb{H}}$ on $\mathfrak{sl}(3,\mathbb{O})$, given by

$$\begin{array}{ll} \phi_s(R+B)=R-B & R\in\mathfrak{r},\,B\in\mathfrak{b} \\ \phi_t(T_1+T_{23})=T_1-T_{23} & T_1\in\mathfrak{t}_1,\,T_{23}\in\mathfrak{t}_{23}\,, \\ \phi_{\mathbb{H}}(H+H')=H-H' & H\in\mathfrak{h},\,H'\in\mathfrak{h}^\perp \end{array} \tag{13}$$

and let ϕ_s^*, ϕ_t^*, and $\phi_{\mathbb{H}}^*$ be the associated Cartan maps on $\mathfrak{e}_6^{\mathbb{C}}$.

The associated Cartan map ϕ_s^* transforms $\mathfrak{sl}(3,\mathbb{O})$, which has signature (52, 26), into the compact real form $\phi_s^*(\mathfrak{sl}(3,\mathbb{O}))$, which has signature (78, 0). The subalgebra

$$\phi_s^*(\mathfrak{sl}(3,\mathbb{O}))\cap\mathfrak{sl}(3,\mathbb{O})=\mathfrak{r} \tag{14}$$

has dimension 52 and is therefore easily seen to be the compact real form $\mathfrak{su}(3,\mathbb{O})$ of $\mathfrak{f}_4$. For this case, the compact part of $\phi_s^*(\mathfrak{sl}(3,\mathbb{O}))$ is the entire algebra, so that its pre-image is already a known subalgebra of $\mathfrak{sl}(3,\mathbb{O})$, namely, $\mathfrak{sl}(3,\mathbb{O})$ itself.

B. Some subalgebras of $\mathfrak{sl}(3,\mathbb{O})$

We obtain two interesting subalgebras of $\mathfrak{sl}(3,\mathbb{O})$ when we apply the associated Cartan map ϕ_t^*. First, the signature of $\mathfrak{g}'=\phi_t^*(\mathfrak{sl}(3,\mathbb{O}))$ is again (52, 26), since $\mathfrak{t}_{23}$ contains the same number of boosts and rotations (16 of each). The real form $\mathfrak{g}'$ is therefore isomorphic to, but distinct from, $\mathfrak{sl}(3,\mathbb{O})$, with maximal compact subalgebra $\mathfrak{g}'_c=\mathfrak{f}_4$. Hence, the pre-image of $\mathfrak{g}'_c$ is a real form of $\mathfrak{f}_4$ in $\mathfrak{sl}(3,\mathbb{O})$. It has signature (36, 16), and the 16 non-compact generators identify this real form of $\mathfrak{f}_4$ as $\mathfrak{su}(2,1,\mathbb{O})=\mathfrak{f}_{4(-20)}$. The second subalgebra comes from looking at

$$\phi_t^*(\mathfrak{sl}(3,\mathbb{O}))\cap\mathfrak{sl}(3,\mathbb{O})=\mathfrak{t}_1. \tag{15}$$

Because $|\mathfrak{t}_1|=46$, it must be a real form of $\mathfrak{d}_5\oplus\mathfrak{d}_1$.[14] But $\mathfrak{t}_1$ contains $\mathfrak{so}(9,1)=\mathfrak{sl}(2,\mathbb{O})$, which has signature (36, 9), as well as the boost $\dot{B}_{tz}^2-\dot{B}_{tz}^3$. Writing $\mathfrak{u}(-1)$ for the non-compact real representation of $\mathfrak{d}_1$ generated by $\dot{B}_{tz}^2-\dot{B}_{tz}^3$, we identify $\mathfrak{t}_1$ as the subalgebra $\mathfrak{sl}(2,\mathbb{O})\oplus\mathfrak{u}(-1)$, with signature $(36,9)\oplus(0,1)$.

We also obtain two subalgebras by applying $\phi_{\mathbb{H}}^*$ to $\mathfrak{sl}(3,\mathbb{O})$. In this case, the signature of $\mathfrak{g}'=\phi_{\mathbb{H}}^*(\mathfrak{sl}(3,\mathbb{O}))$ is (36, 42), since $\mathfrak{h}^\perp$ contains 28 rotations and 12 boosts, resulting in a net of 16 changes in signature. The maximal compact subalgebra $\mathfrak{g}'_c$ of $\mathfrak{g}'$ is $\mathfrak{su}(4,\mathbb{H})$, the compact real form of $\mathfrak{c}_4$. The pre-image of $\mathfrak{g}'_c$ has signature (24, 12), and, due to the 12 non-compact generators, can be identified as $\mathfrak{su}(3,1,\mathbb{H})$. The invariant subalgebra

$$\phi_{\mathbb{H}}^*(\mathfrak{sl}(3,\mathbb{O}))\cap\mathfrak{sl}(3,\mathbb{O})=\mathfrak{h} \tag{16}$$

has dimension $|\mathfrak{h}|=38$ and signature (24, 14), and is therefore a real form of $\mathfrak{a}_5\oplus\mathfrak{a}_1$ with signature $(21,14)\oplus(3,0)$. Of the 24 rotations unaffected by $\phi_{\mathbb{H}}^*$, there are 21 which are quaternionic and form the subalgebra $\mathfrak{su}(3,\mathbb{H})$. The remaining three rotations $A_k, A_{k\ell}, A_\ell$ are the elements of $\mathfrak{g}_2$ that leave invariant the quaternionic subalgebra spanned by $\{k,\ell,k\ell\}$. The four elements i, j, $i\ell$, and $j\ell$ can be paired into two complex pairs (of which $i+i\ell, j+j\ell$ is one choice), and the transformations $\dot{A}_k$, $\dot{A}_{k\ell}$, $\dot{A}_\ell$ act as $\mathfrak{su}(2,\mathbb{C})$ transformations producing the other pairs of complex numbers. We henceforth refer to this copy of $\mathfrak{su}(2,\mathbb{C})$ as $\mathfrak{su}(2)_{\mathbb{H}}$. Hence, the 24 compact elements form the algebra $\mathfrak{su}(3,\mathbb{H})\oplus\mathfrak{su}(2)_{\mathbb{H}}$, and $\mathfrak{h}=\mathfrak{sl}(3,\mathbb{H})\oplus\mathfrak{su}(2)_{\mathbb{H}}$.

TABLE II. Intersections of subspaces $\mathfrak{t}_1$, $\mathfrak{t}_{23}$, $\mathfrak{h}$, and $\mathfrak{h}^\perp$.

$\cap$	$\mathfrak{t}_1$	$\mathfrak{t}_{23}$	$\cap$	$\mathfrak{t}_1$	$\mathfrak{t}_{23}$
$\mathfrak{h}$	$\mathfrak{h}_1$	$\mathfrak{h}_{23}$	$\mathfrak{h}$	(16, 6)	(8, 8)
$\mathfrak{h}^\perp$	$\mathfrak{h}_1^\perp$	$\mathfrak{h}_{23}^\perp$	$\mathfrak{h}^\perp$	(20, 4)	(8, 8)
	Subspace			Signature	

C. More subalgebras of $\mathfrak{sl}(3, \mathbb{O})$

Consider now the composition $\phi_t^* \circ \phi_{\mathbb{H}}^*$. We define the subspaces $\mathfrak{h}_1$, $\mathfrak{h}_{23}$, $\mathfrak{h}_1^\perp$, and $\mathfrak{h}_{23}^\perp$ of $\mathfrak{sl}(3, \mathbb{O})$ to be the intersections of pairs of subspaces $\mathfrak{t}_1$, $\mathfrak{t}_{23}$, $\mathfrak{h}$, and $\mathfrak{h}^\perp$, as indicated in Table II. For example, $\mathfrak{h}_1 = \mathfrak{h} \cap \mathfrak{t}_1$ and $\mathfrak{h}_{23}^\perp = \mathfrak{h}^\perp \cap \mathfrak{t}_{23}$. Table II also indicates the number of basis elements which are boosts and rotations in each of these spaces, whose commutation rules are given in Table III.

We see from Table III that, in addition to the subalgebras $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_{23}$ and $\mathfrak{t}_1 = \mathfrak{h}_1 \oplus \mathfrak{h}_1^\perp$ constructed previously, $\mathfrak{h}_1$ and $\mathfrak{h}_1 \oplus \mathfrak{h}_{23}^\perp$ are also subalgebras of $\mathfrak{sl}(3, \mathbb{O})$. Furthermore, $\phi_t^* \circ \phi_{\mathbb{H}}^*$ fixes

$$\phi_t^* \circ \phi_{\mathbb{H}}^* (\mathfrak{sl}(3, \mathbb{O})) \cap \mathfrak{sl}(3, \mathbb{O}) = \mathfrak{h}_1 \oplus \mathfrak{h}_{23}^\perp \tag{17}$$

as a subalgebra, since everything in $\mathfrak{h}_1$ is fixed by both maps, while everything in $\mathfrak{h}_{23}^\perp$ is multiplied by $\xi^2 = -1$. Thus, even though the composition of associated Cartan maps is not quite an associated Cartan map itself (due to the minus sign), it does lead to another $\mathbb{Z}_2$-grading; we return to this point in Sec. V A below.

The subalgebra $\mathfrak{p}$ in (17) has dimension 38 and signature (24, 14), and is thus a real form of $\mathfrak{a}_5 \oplus \mathfrak{a}_1$. However, it has a fundamentally different basis from $\mathfrak{h} = \mathfrak{sl}(3, \mathbb{H}) \oplus \mathfrak{su}(2)_{\mathbb{H}}$, as $\mathfrak{p}$ uses a mixture of the quaternionic transformations of type 1 with the orthogonal-quaternionic transformations of type 2 and type 3, while $\mathfrak{sl}(3, \mathbb{H})$ is comprised of only the quaternionic transformations. We therefore refer to this algebra as $\mathfrak{sl}(2, 1, \mathbb{H}) \oplus \mathfrak{su}(2)_2$. Explicitly,

$$\mathfrak{su}(2)_2 = \langle \dot{G}_k + 2\dot{S}_k^1, \dot{G}_{k\ell} + 2\dot{S}_{k\ell}^1, \dot{G}_\ell + 2\dot{S}_\ell^1 \rangle \tag{18}$$

again corresponds to permutations of $\{i, j, j\ell, i\ell\}$ and fixes $\{k, k\ell, \ell\}$, but is not in $\mathfrak{g}_2$. We also note that the maximal compact subalgebra $\mathfrak{g}_c'$ of $\phi_t^* \circ \phi_{\mathbb{H}}^* (\mathfrak{sl}(3, \mathbb{O}))$ has dimension 36, and its pre-image in $\mathfrak{sl}(3, \mathbb{O})$ has signature (24, 12). We identify this subalgebra as $\mathfrak{su}(3, 1, \mathbb{H})_2$, since this real algebra has a different basis than our previously identified $\mathfrak{su}(3, 1, \mathbb{H})$, which we henceforth refer to as $\mathfrak{su}(3, 1, \mathbb{H})_1$.

We summarize in Table IV the subalgebras constructed from our three associated Cartan maps, as well as compositions of these maps. For each map ϕ^*, the second column lists the signature of $\mathfrak{g}' = \phi^*(\mathfrak{sl}(3, \mathbb{O}))$, the third column identifies, and gives the signature of, the fixed subalgebra $\mathfrak{p} = \phi^* (\mathfrak{sl}(3, \mathbb{O})) \cap \mathfrak{sl}(3, \mathbb{O})$, and the fourth column identifies the pre-image of the maximal compact subalgebra of $\mathfrak{g}' = \phi^* (\mathfrak{sl}(3, \mathbb{O}))$, listing both the algebra and its signature. Each algebra listed in the third and fourth columns of Table IV is a subalgebra of $\mathfrak{sl}(3, \mathbb{O})$, constructed using the other real forms of $\mathfrak{e}_6$.

Despite recognizing $\mathfrak{h}_1$, $\mathfrak{h}_1 \oplus \mathfrak{h}_{23}^\perp$, $\mathfrak{h}_1 \oplus \mathfrak{h}_{23}$, and $\mathfrak{h}_1 \oplus \mathfrak{h}_1^\perp$ as subalgebras of $\mathfrak{sl}(3, \mathbb{O})$ using the commutation relations in Table III, we note that $\mathfrak{su}(3, 1, \mathbb{H})_2$ is not any of these subalgebras. In Sec. IV, we use another technique involving associated Cartan maps to give a finer refinement of

TABLE III. Commutation structure of $\mathfrak{h}_1$, $\mathfrak{h}_{23}$, $\mathfrak{h}_1^\perp$, and $\mathfrak{h}_{23}^\perp$.

[,]	$\mathfrak{h}_1$	$\mathfrak{h}_{23}$	$\mathfrak{h}_1^\perp$	$\mathfrak{h}_{23}^\perp$
$\mathfrak{h}_1$	$\mathfrak{h}_1$	$\mathfrak{h}_{23}$	$\mathfrak{h}_1^\perp$	$\mathfrak{h}_{23}^\perp$
$\mathfrak{h}_{23}$	$\mathfrak{h}_{23}$	$\mathfrak{h}_1$	$\mathfrak{h}_{23}^\perp$	$\mathfrak{h}_1^\perp$
$\mathfrak{h}_1^\perp$	$\mathfrak{h}_1^\perp$	$\mathfrak{h}_{23}^\perp$	$\mathfrak{h}_1$	$\mathfrak{h}_{23}$
$\mathfrak{h}_{23}^\perp$	$\mathfrak{h}_{23}^\perp$	$\mathfrak{h}_1^\perp$	$\mathfrak{h}_{23}$	$\mathfrak{h}_1$

TABLE IV. Compositions of associated Cartan maps and the corresponding subalgebras of $\mathfrak{sl}(3, \mathbb{O})$.

Map	Signature of $\mathfrak{g}' = \phi^*\left(\mathfrak{sl}(3, \mathbb{O})\right)$	Signature of $\mathfrak{p} = \mathfrak{g}' \cap \mathfrak{sl}(3, \mathbb{O})$	Signature of $(\phi^*)^{-1}(\mathfrak{g}'_c)$
1	(52, 26)	(52, 26) $\mathfrak{sl}(3, \mathbb{O})$	(52, 0) $\mathfrak{su}(3, \mathbb{O})$
ϕ_s^*	(78, 0)	(52, 0) $\mathfrak{su}(3, \mathbb{O})$	(52, 26) $\mathfrak{sl}(3, \mathbb{O})$
ϕ_t^*	(52, 26)	(36, 10) $\mathfrak{sl}(2, \mathbb{O}) \oplus \mathfrak{u}(-1)$	(36, 16) $\mathfrak{su}(2, 1, \mathbb{O})$
$\phi_{\mathbb{H}}^*$	(36, 42)	(24, 14) $\mathfrak{sl}(3, \mathbb{H}) \oplus \mathfrak{su}(2)_{\mathbb{H}}$	(24, 12) $\mathfrak{su}(3, 1, \mathbb{H})_1$
$\phi_t^* \circ \phi_s^*$	(46, 32)	(36, 16) $\mathfrak{su}(2, 1, \mathbb{O})$	(36, 10) $\mathfrak{sl}(2, \mathbb{O}) \oplus \mathfrak{u}(-1)$
$\phi_{\mathbb{H}}^* \circ \phi_s^*$	(38, 40)	(24, 12) $\mathfrak{su}(3, 1, \mathbb{H})_1$	(24, 14) $\mathfrak{sl}(3, \mathbb{H}) \oplus \mathfrak{su}(2)_{\mathbb{H}}$
$\phi_t^* \circ \phi_{\mathbb{H}}^*$	(36, 42)	(24, 14) $\mathfrak{sl}(2, 1, \mathbb{H}) \oplus \mathfrak{su}(2)_2$	(24, 12) $\mathfrak{su}(3, 1, \mathbb{H})_2$
$\phi_t^* \circ \phi_{\mathbb{H}}^* \circ \phi_s^*$	(38, 40)	(24, 12) $\mathfrak{su}(3, 1, \mathbb{H})_2$	(24, 14) $\mathfrak{sl}(2, 1, \mathbb{H}) \oplus \mathfrak{su}(2)_2$

subspaces of $\mathfrak{sl}(3, \mathbb{O})$, allowing us to provide a nice basis for $\mathfrak{su}(3, 1, \mathbb{H})_2$ and other subalgebras of $\mathfrak{sl}(3, \mathbb{O})$.

IV. CONSTRUCTING SUBALGEBRAS OF $\mathfrak{sl}(3, \mathbb{O})$

A. Using composition of associated Cartan maps

We have already used associated Cartan maps to identify the maximal subalgebras of $\mathfrak{sl}(3, \mathbb{O})$. This was done by separating the algebra into two separate spaces, one of which was left invariant by the map. In this section, we use composition of associated Cartan maps to separate $\mathfrak{sl}(3, \mathbb{O})$ into four or more subspaces spanned by either the compact or non-compact generators, with the condition that the map either preserves the entire subspace or changes the character of all the basis elements in the subspace. We identify additional subalgebras of $\mathfrak{sl}(3, \mathbb{O})$ by taking various combinations of these subspaces.

We continue to use the associated Cartan maps ϕ_s^*, ϕ_t^*, and $\phi_{\mathbb{H}}^*$, as well as the subspaces $\mathfrak{r}$, $\mathfrak{b}$, $\mathfrak{t}_1$, $\mathfrak{t}_{23}$, $\mathfrak{h}$, and $\mathfrak{h}^\perp$ defined in Sec. III.

We first consider the composition $\phi_{\mathbb{H}}^* \circ \phi_s^*$. This map fixes the subspaces $\mathfrak{r}_{\mathbb{H}} = \mathfrak{r} \cap \mathfrak{h}$, consisting of quaternionic rotations, as well as $\mathfrak{b}_\perp = \mathfrak{b} \cap \mathfrak{h}^\perp$, consisting of orthogonal-quaternionic boosts. Under $\phi_{\mathbb{H}}^* \circ \phi_s^*$, the two subspaces $\mathfrak{b}_{\mathbb{H}}$ and $\mathfrak{r}_\perp = \mathfrak{r} \cap \mathfrak{h}^\perp$ change signature. These spaces consist of orthogonal-quaternionic rotations and quaternionic boosts, respectively. The dimensions of these four spaces are displayed in Table V.

We list the signature of these spaces under $\phi_{\mathbb{H}}^* \circ \phi_s^*$, and can thus identify subalgebras of $\phi_{\mathbb{H}}^* \circ \phi_s^*\,(\mathfrak{sl}(3, \mathbb{O}))$. However, we are primarily interested in the pre-image of these subalgebras in our

TABLE V. Splitting of $\mathfrak{e}_6$ basis under $\phi_{\mathbb{H}}^* \circ \phi_s^*$.

$\cap$	$\mathfrak{h}$	$\mathfrak{h}^\perp$
$\mathfrak{r}$	$\lvert\mathfrak{r}_{\mathbb{H}}\rvert = 24$	$\lvert\mathfrak{r}_\perp\rvert = 28$
$\mathfrak{b}$	$\lvert\mathfrak{b}_{\mathbb{H}}\rvert = 14$	$\lvert\mathfrak{b}_\perp\rvert = 12$

TABLE VI. Splitting of $\mathfrak{e}_6$ basis under $\phi_t^* \circ \phi_s^*$.

$\cap$	$\mathfrak{t}_1$	$\mathfrak{t}_{23}$
$\mathfrak{r}$	$\lvert\mathfrak{r}_1\rvert = 36$	$\lvert\mathfrak{r}_{23}\rvert = 16$
$\mathfrak{b}$	$\lvert\mathfrak{b}_1\rvert = 10$	$\lvert\mathfrak{b}_{23}\rvert = 16$

preferred algebra $\mathfrak{sl}(3, \mathbb{O})$. Determining the signature of these spaces in $\mathfrak{sl}(3, \mathbb{O})$ is straightforward, as the rotations are compact and the boosts are not.

We use the subspaces represented in Table V to identify subalgebras of $\mathfrak{sl}(3, \mathbb{O})$. As previously identified, the 24 rotations in $\mathfrak{r}_{\mathbb{H}}$ fixed by the automorphism form the subalgebra $\mathfrak{su}(3, \mathbb{H})_1 \oplus \mathfrak{su}(2)_{\mathbb{H}}$, where $\mathfrak{su}(3, \mathbb{H})_1 \subset \mathfrak{su}(3, 1, \mathbb{H})_1$. The entries in the first column of Table V represent all quaternionic rotations and boosts, and form the subalgebra $\mathfrak{sl}(3, \mathbb{H}) \oplus \mathfrak{su}(2)_{\mathbb{H}}$. Of course, the entries in the first row, $\mathfrak{r}_{\mathbb{H}}$ and $\mathfrak{r}_{\perp}$, form $\mathfrak{f}_4$. We finally consider the entries $\mathfrak{r}_{\mathbb{H}}$ and $\mathfrak{b}_{\perp}$ on the main diagonal of the table. Two orthogonal-quaternionic boosts commute to a quaternionic rotation, and an orthogonal-quaternionic boost commuted with a quaternionic rotation is again an orthogonal-quaternionic boost. Hence, the subspace $\mathfrak{r}_{\mathbb{H}} \oplus \mathfrak{b}_{\perp}$ closes under commutation and is a subalgebra with signature (24, 12). While both $\mathfrak{so}(9 - n, n, \mathbb{R})$ and $\mathfrak{su}(4 - n, n, \mathbb{H})$ have dimension 36, only $\mathfrak{su}(3, 1, \mathbb{H})$ has 12 boosts, so $\mathfrak{r}_{\mathbb{H}} \oplus \mathfrak{b}_{\perp}$ is the previously identified $\mathfrak{su}(3, 1, \mathbb{H})_1$.

The algebra $\mathfrak{r}_{\mathbb{H}} \oplus \mathfrak{b}_{\perp} = \mathfrak{su}(3, 1, \mathbb{H})_1$ is a real form of the complex Lie algebra $\mathfrak{c}_4$. Since $\mathfrak{c}_4$ contains $\mathfrak{c}_3$ but not $\mathfrak{b}_3$, the 21-dimensional subalgebra contained within $\mathfrak{r}_{\mathbb{H}}$ is a real form of $\mathfrak{c}_3$, not of $\mathfrak{b}_3$. In addition, any simple 21-dimensional subalgebra of $\mathfrak{r}_{\mathbb{H}} \oplus \mathfrak{b}_{\perp}$ is a real from of $\mathfrak{c}_3$. Eliminating the boosts from $\mathfrak{su}(3, 1, \mathbb{H})_1$ leaves $\mathfrak{su}(3, \mathbb{H})_1$.

We next consider the composition $\phi_t^* \circ \phi_s^*$. This automorphism separates the basis for $\mathfrak{sl}(3, \mathbb{O})$ into the subspaces

$$\begin{aligned} \mathfrak{r}_1 &= \mathfrak{r} \cap \mathfrak{t}_1 \qquad & \mathfrak{b}_1 &= \mathfrak{b} \cap \mathfrak{t}_1, \\ \mathfrak{b}_{23} &= \mathfrak{b} \cap \mathfrak{t}_{23} \qquad & \mathfrak{r}_{23} &= \mathfrak{r} \cap \mathfrak{t}_{23.} \end{aligned} \tag{19}$$

As shown in Table VI, this map leaves the signatures of $\mathfrak{r}_1$ and $\mathfrak{b}_{23}$ alone, while it reverses the signatures of $\mathfrak{r}_{23}$ and $\mathfrak{b}_1$.

We again use the subspaces represented in Table VI to identify subalgebras of $\mathfrak{sl}(3, \mathbb{O})$. The subspace $\mathfrak{r}_1$ is the subalgebra $\mathfrak{so}(9, \mathbb{R})$, containing all subalgebras $\mathfrak{so}(n, \mathbb{R})$ for $n \leq 9$, and we have already seen that the subspace $\mathfrak{t}_1 = \mathfrak{r}_1 \oplus \mathfrak{b}_1$ is $\mathfrak{so}(9, 1, \mathbb{R}) \oplus \mathfrak{u}(-1)$. Again, the complete set of rotations $\mathfrak{r}_1 \oplus \mathfrak{r}_{23}$ form the subalgebra $\mathfrak{su}(3, \mathbb{O})$, which is a real form of $\mathfrak{f}_4$. Interestingly, the subspace $\mathfrak{r}_1 \oplus \mathfrak{b}_{23}$ on the main diagonal is another form of $\mathfrak{f}_4$. The 16 boosts in $\mathfrak{r}_1 \oplus \mathfrak{b}_{23}$ identify this form of $\mathfrak{f}_4$ as $\mathfrak{su}(2, 1, \mathbb{O})$.

We finally consider the composition $\phi_t^* \circ \phi_{\mathbb{H}}^* \circ \phi_s^*$, which creates a finer refinement than compositions of two maps. The resulting subspaces are listed in Table VII. We continue with our previous conventions for designating intersections of subspaces, that is, $\mathfrak{r}_{23,\perp} = \mathfrak{r} \cap \mathfrak{t}_{23} \cap \mathfrak{h}^{\perp}$.

Using this division of $\mathfrak{sl}(3, \mathbb{O})$, we find a large list of subalgebras of $\mathfrak{sl}(3, \mathbb{O})$ simply by combining certain subspaces. The subspace description of these algebras, as well as their identity and signature in $\mathfrak{sl}(3, \mathbb{O})$, is listed in Table VIII. This fine refinement of $\mathfrak{sl}(3, \mathbb{O})$ provides a description of the basis for $\mathfrak{su}(3, 1, \mathbb{H})_1$ and $\mathfrak{su}(3, 1, \mathbb{H})_2$, as well as $\mathfrak{sl}(3, \mathbb{H})$ and $\mathfrak{sl}(2, 1, \mathbb{H})$.

TABLE VII. Splitting of $\mathfrak{e}_6$ basis under $\phi_t^* \circ \phi_{\mathbb{H}}^* \circ \phi_s^*$.

$\cap$	$\mathfrak{h}_1$	$\mathfrak{h}_{23}$	$\mathfrak{h}_1^{\perp}$	$\mathfrak{h}_{23}^{\perp}$
$\mathfrak{r}$	$\lvert\mathfrak{r}_{1,\mathbb{H}}\rvert = 16$	$\lvert\mathfrak{r}_{23,\mathbb{H}}\rvert = 8$	$\lvert\mathfrak{r}_{1,\perp}\rvert = 20$	$\lvert\mathfrak{r}_{23,\perp}\rvert = 8$
$\mathfrak{b}$	$\lvert\mathfrak{b}_{1,\mathbb{H}}\rvert = 6$	$\lvert\mathfrak{b}_{23,\mathbb{H}}\rvert = 8$	$\lvert\mathfrak{b}_{1,\perp}\rvert = 4$	$\lvert\mathfrak{b}_{23,\perp}\rvert = 8$

TABLE VIII. Subalgebras of $\mathfrak{sl}(3, \mathbb{O})$ using Cartan decompositions.

Basis	Subalgebra of $\mathfrak{sl}(3, \mathbb{O})$	Signature
$\mathfrak{r}_{1,\mathbb{H}}$	$\mathfrak{su}(2, \mathbb{H}) \oplus \mathfrak{su}(2)_{\mathbb{H}} \oplus \mathfrak{su}(2)$	$(10 + 3 + 3, 0)$
$\mathfrak{r}_{1,\mathbb{H}} \oplus \mathfrak{r}_{23,\perp}$	$\mathfrak{su}(3, \mathbb{H})_2 \oplus \mathfrak{su}(2)_{\mathbb{H}}$	$(21 + 3, 0)$
$\mathfrak{h}_1 = \mathfrak{r}_{1,\mathbb{H}} \oplus \mathfrak{b}_{1,\mathbb{H}}$	$\mathfrak{sl}(2, \mathbb{H}) \oplus \mathfrak{su}(2)_{\mathbb{H}}$ $\oplus \mathfrak{su}(2) \oplus \mathfrak{u}(-1)$	$(10 + 3 + 3, 5 + 1)$
$\mathfrak{r}_{1,\mathbb{H}} \oplus \mathfrak{b}_{23,\perp}$	$\mathfrak{su}(2, 1, \mathbb{H})_1 \oplus \mathfrak{su}(2)_{\mathbb{H}}$	$(13 + 3, 8)$
$\mathfrak{r}_{1,\mathbb{H}} \oplus \mathfrak{b}_{23,\mathbb{H}}$	$\mathfrak{su}(2, 1, \mathbb{H})_2 \oplus \mathfrak{su}(2)_{\mathbb{H}}$	$(13 + 3, 8)$
$\mathfrak{r}_{\mathbb{H}} = \mathfrak{r}_{1,\mathbb{H}} \oplus \mathfrak{r}_{23,\mathbb{H}}$	$\mathfrak{su}(3, \mathbb{H}) \oplus \mathfrak{su}(2)_{\mathbb{H}}$	$(21 + 3, 0)$
$\mathfrak{r}_{1,\mathbb{H}} \oplus \mathfrak{b}_{1,\perp}$	$\mathfrak{so}(5, \mathbb{R}) \oplus \mathfrak{so}(4, 1, \mathbb{R})$	$(10 + 6, 4)$
$\mathfrak{r}_1 = \mathfrak{r}_{1,\mathbb{H}} \oplus \mathfrak{r}_{1,\perp}$	$\mathfrak{so}(9) = \mathfrak{su}(2, \mathbb{O})$	$(36, 0)$
$\mathfrak{h} = \mathfrak{r}_{1,\mathbb{H}} \oplus \mathfrak{b}_{1,\mathbb{H}}$ $\oplus \mathfrak{b}_{23,\mathbb{H}} \oplus \mathfrak{r}_{23,\mathbb{H}}$	$\mathfrak{sl}(3, \mathbb{H}) \oplus \mathfrak{su}(2)_{\mathbb{H}}$	$(21 + 3, 14)$
$\mathfrak{h}_1 \oplus \mathfrak{h}_{23}^{\perp} = \mathfrak{r}_{1,\mathbb{H}} \oplus \mathfrak{b}_{1,\mathbb{H}}$ $\oplus \mathfrak{r}_{23,\perp} \oplus \mathfrak{b}_{23,\perp}$	$\mathfrak{sl}(2, 1, \mathbb{H})_1 \oplus \mathfrak{su}(2)_2$	$(21 + 3, 14)$
$\mathfrak{t}_1 = \mathfrak{r}_{1,\mathbb{H}} \oplus \mathfrak{b}_{1,\mathbb{H}}$ $\oplus \mathfrak{b}_{1,\perp} \oplus \mathfrak{r}_{1,\perp}$	$\mathfrak{sl}(2, \mathbb{O}) \oplus \mathfrak{u}(-1)$	$(36, 9 + 1)$
$\mathfrak{r}_{\mathbb{H}} \oplus \mathfrak{b}_{\perp} = \mathfrak{r}_{1,\mathbb{H}} \oplus \mathfrak{b}_{23,\perp}$ $\oplus \mathfrak{r}_{23,\mathbb{H}} \oplus \mathfrak{b}_{1,\perp}$	$\mathfrak{su}(3, 1, \mathbb{H})_1$	$(24, 12)$
$\mathfrak{r}_{1,\mathbb{H}} \oplus \mathfrak{r}_{23,\perp}$ $\oplus \mathfrak{b}_{23,\mathbb{H}} \oplus \mathfrak{b}_{1,\perp}$	$\mathfrak{su}(3, 1, \mathbb{H})_2$	$(24, 12)$
$\mathfrak{r} = \mathfrak{r}_{1,\mathbb{H}} \oplus \mathfrak{r}_{23,\perp}$ $\oplus \mathfrak{r}_{23,\mathbb{H}} \oplus \mathfrak{r}_{1,\perp}$	$\mathfrak{su}(3, \mathbb{O})$	$(52, 0)$
$\mathfrak{r}_1 \oplus \mathfrak{b}_{23} = \mathfrak{r}_{1,\mathbb{H}} \oplus \mathfrak{b}_{23,\perp}$ $\oplus \mathfrak{b}_{23,\mathbb{H}} \oplus \mathfrak{r}_{1,\perp}$	$\mathfrak{su}(2, 1, \mathbb{O})$	$(36, 16)$

B. Chains of subalgebras of $\mathfrak{sl}(3, \mathbb{O})$

We have used associated Cartan maps to produce large simple subalgebras of $\mathfrak{sl}(3, \mathbb{O})$, ranging in dimension from 52, for $\mathfrak{f}_4$, to 21, for $\mathfrak{c}_3$. Each of these subalgebras in turn has its own associated Cartan maps, which we could use to find even smaller subalgebras, thereby giving a catalog of subalgebra chains contained within $\mathfrak{sl}(3, \mathbb{O})$. However, having identified the real form of the large subalgebras of $\mathfrak{sl}(3, \mathbb{O})$, it is not too difficult a task to find the smaller algebras simply by looking for simple subalgebras of smaller dimension and/or rank, using the tables of real forms listed in Ref. 12 when needed. Furthermore, we can choose our smaller subalgebras and their bases so that they use a subset of our preferred basis for the Cartan subalgebra of $\mathfrak{sl}(3, \mathbb{O})$, namely $\{\dot{B}^1_{tz}, \dot{B}^2_{tz} - \dot{B}^3_{tz}, \dot{R}^1_{x\ell}, \dot{S}^1_{\ell}, \dot{G}_{\ell}, \dot{A}_{\ell}\}$, henceforth referred to as our *Cartan basis*.

We display these chains of subalgebras of $\mathfrak{sl}(3, \mathbb{O})$ in the following tables. Each table is built from a $\mathfrak{u}(1)$ algebra, generated by a single element of our Cartan basis. We extend each algebra $\mathfrak{g}$ to a larger algebra $\mathfrak{g}'$ by adding elements to the basis for $\mathfrak{g}$. In particular, each algebra of higher rank must add new elements of our Cartan basis, as indicated along the arrows (with dots suppressed). Figure 1 is built from the algebra $\mathfrak{u}(1) = \langle G_l - S^1_l \rangle$, leading to

$$\mathfrak{u}(1) \subset \mathfrak{su}(1, \mathbb{H}) \subset \mathfrak{su}(2, \mathbb{H}) \subset \mathfrak{su}(3, \mathbb{H})_1 \subset \mathfrak{sl}(3, \mathbb{H}). \tag{20}$$

Additional subalgebras of $\mathfrak{sl}(3, \mathbb{O})$ can be inserted into this chain of subalgebras. For instance, we can insert $\mathfrak{sl}(2, \mathbb{H})$ between $\mathfrak{su}(2, \mathbb{H})$ and $\mathfrak{sl}(3, \mathbb{H})$, and extend $\mathfrak{sl}(2, \mathbb{H})$ to $\mathfrak{sl}(2, \mathbb{O})$. We can also expand $\mathfrak{su}(1, \mathbb{H})$ to $\mathfrak{su}(1, \mathbb{O}) = \mathfrak{so}(7)$ and insert the $\mathfrak{so}(n, \mathbb{R})$ chain for $n \geq 7$ into the figure. However, $\mathfrak{so}(7)$ uses a basis in this chain that is not compatible with $\mathfrak{g}_2 = \mathrm{aut}(\mathbb{O})$. We do not list all of the possible $\mathfrak{u}(1)$ subalgebras, but do add the subalgebra $\mathfrak{u}(1) = \langle R^1_{x\ell} \rangle$ into the chain and extend it to $\mathfrak{su}(2, \mathbb{C})_s$ and $\mathfrak{sl}(2, \mathbb{C})_s$, which use the standard (type 1) matrix definition of $\mathfrak{su}(2, \mathbb{C})$ and $\mathfrak{sl}(2, \mathbb{C})$.

With one exception, the algebras in Figure 1 are built from subalgebras by extending the basis at each step; we do not allow any changes of basis. The one exception is the inclusion of the chain

$$\mathfrak{su}(2, \mathbb{C})_s \subset \mathfrak{su}(3, \mathbb{C})_s \subset \mathfrak{su}(3, \mathbb{H})_1 \tag{21}$$

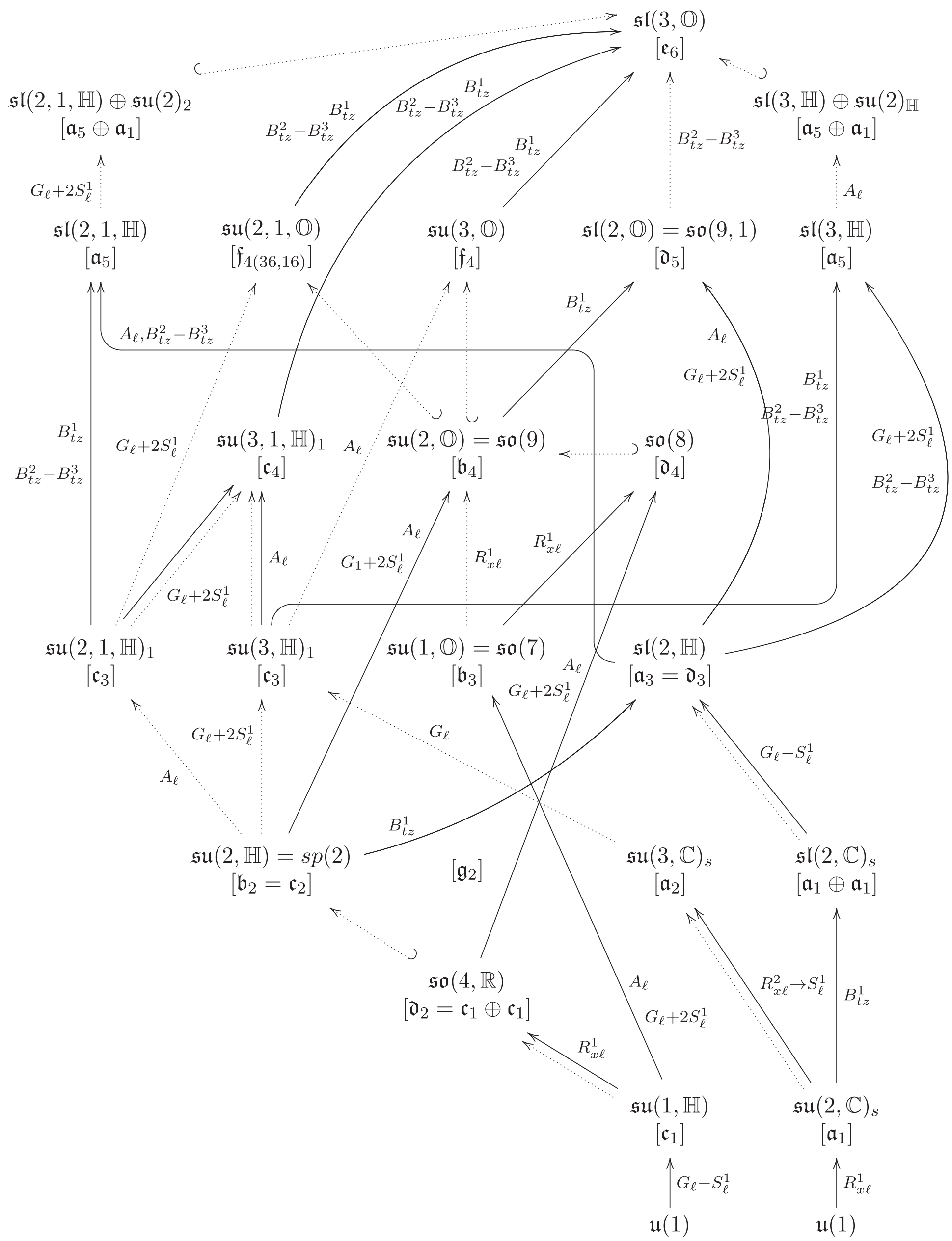

FIG. 1. Preferred subalgebra chains of $\mathfrak{e}_6$.

using the Cartan basis elements $\dot{S}^1_\ell$ and $\dot{G}_\ell$. We also use different notations to indicate possible methods used to identify the subalgebras, with dashed and solid arrows indicating that the root diagram of the smaller algebra can be obtained as a *slice* or as a *projection* of that of the larger algebra; for details, see Ref. 15.

Finally, we have identified four different real forms of $\mathfrak{c}_3$, all of which contain $\mathfrak{su}(2,\mathbb{H})$. Space constraints limit us to listing only $\mathfrak{su}(2,1,\mathbb{H})_1$ and $\mathfrak{su}(3,\mathbb{H})_1$ in Figure 1, but the algebras $\mathfrak{su}(2,1,\mathbb{H})_2$, $\mathfrak{su}(3,\mathbb{H})_2$, and $\mathfrak{su}(3,1,\mathbb{H})_2$ should also be in this table. We list these four real forms of $\mathfrak{c}_3$ algebras, all built from $\mathfrak{su}(2,\mathbb{H})$, in Figure 2, and include all the algebras which are built from the $\mathfrak{c}_3$ algebras. Figure 2 can be incorporated into Figure 1 without having to adjust our choice of Cartan basis.

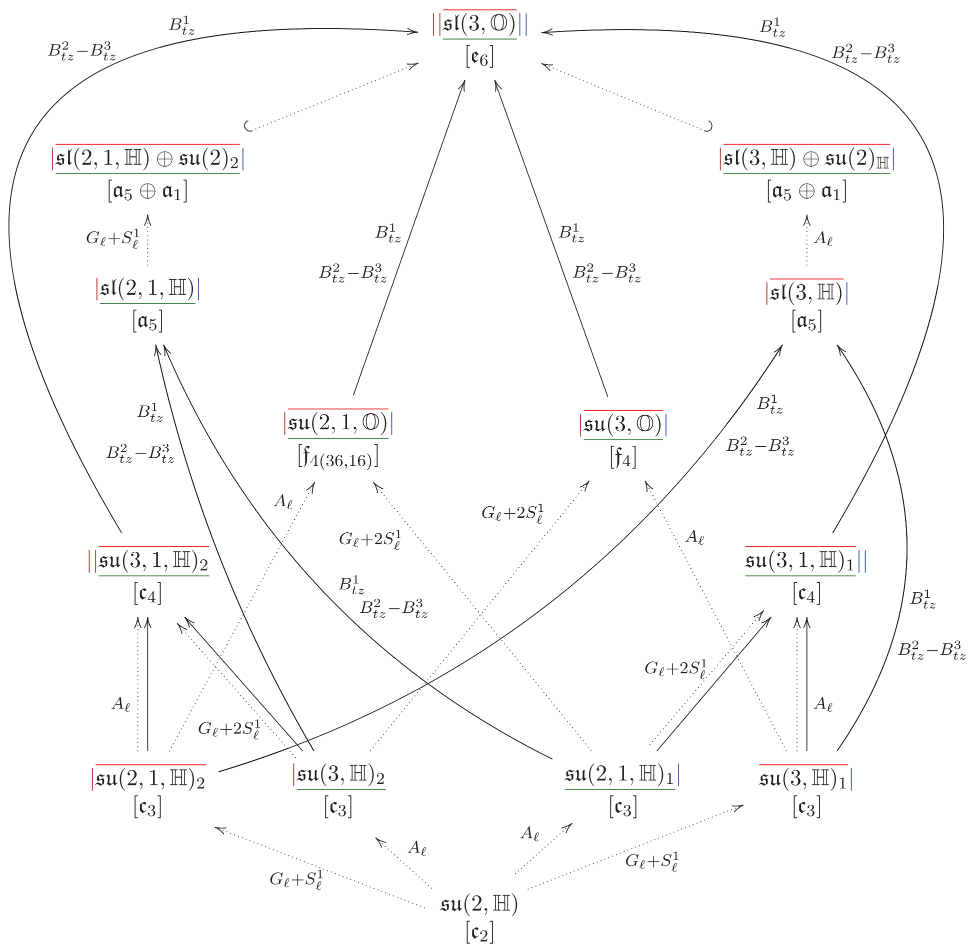

FIG. 2. Four real forms of $\mathfrak{c}_3$. An underline indicates the presence of $\mathfrak{su}(2)_2$, an overline indicates the presence of $\mathfrak{su}(2)_{\mathbb{H}}$, each initial vertical line indicates the presence of one component of $\mathfrak{b}_{23,\mathbb{H}}\oplus\mathfrak{r}_{23,\perp}$, and each final vertical line indicates the presence of one component of $\mathfrak{r}_{23,\mathbb{H}}\oplus\mathfrak{b}_{23,\perp}$.

V. DISCUSSION

A. The group of associated Cartan maps

We return to the structure of associated Cartan maps (7). The square of ϕ^* is clearly the original involution ϕ, and the inverse of ϕ^* is obtained by replacing ξ with $-\xi$, or equivalently as the cube of ϕ^*. But the composition of two associated Cartan maps is not quite an associated Cartan map as we have defined them above, although it does lead to a graded Lie algebra structure, which we use to define a group operation as follows.

Given two associated Cartan maps, written symbolically as

$$\phi_1^*(\mathfrak{p}_1+\mathfrak{m}_1)=\mathfrak{p}_1+\xi\mathfrak{m}_1, \tag{22}$$

$$\phi_2^*(\mathfrak{p}_2+\mathfrak{m}_2)=\mathfrak{p}_2+\xi\mathfrak{m}_2, \tag{23}$$

where

$$\mathfrak{p}_1\oplus\mathfrak{m}_1=\mathfrak{g}^{\mathbb{C}}=\mathfrak{p}_2\oplus\mathfrak{m}_2 \tag{24}$$

we define their product to be

$$(\phi_1^*\star\phi_2^*)(q_{pp}+q_{pm}+q_{mp}+q_{mm})=q_{pp}+\xi q_{pm}+\xi q_{mp}+q_{mm}, \tag{25}$$

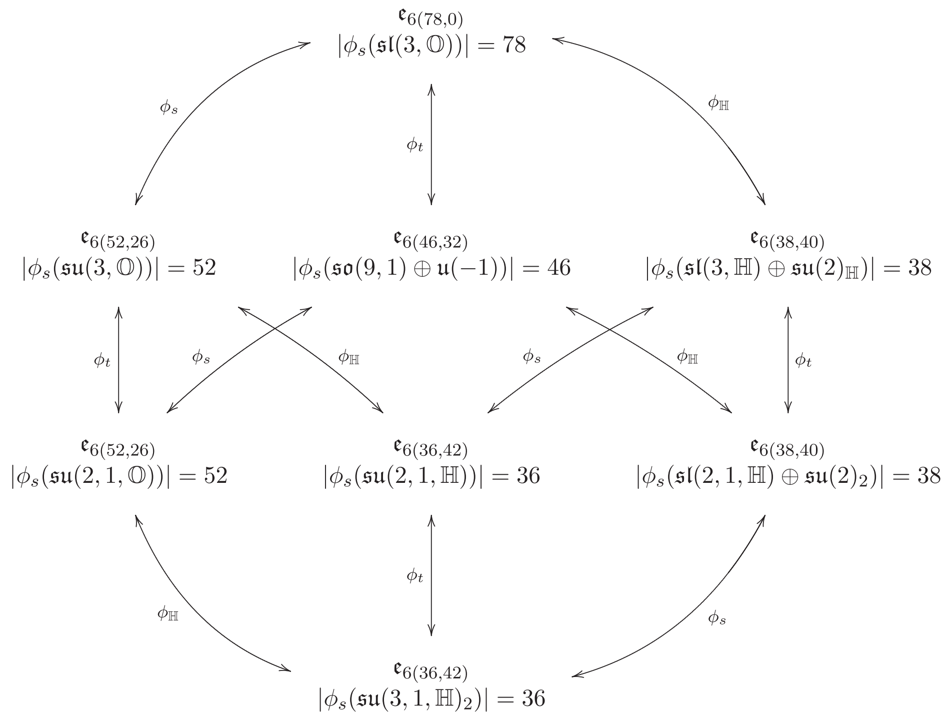

FIG. 3. Composition of associated Cartan maps of $\mathfrak{e}_6$ acting on real forms of $\mathfrak{e}_6$, showing the maximal compact subalgebra under ϕ_s.

where

$$q_{pp} \in \mathfrak{p}_1 \cap \mathfrak{p}_2 \qquad q_{pm} \in \mathfrak{p}_1 \cap \mathfrak{m}_2 \qquad q_{mp} \in \mathfrak{m}_1 \cap \mathfrak{p}_2 \qquad q_{mm} \in \mathfrak{m}_1 \cap \mathfrak{m}_2 \tag{26}$$

and which differs from composition by the sign of the last term. It is easily verified that

$$\mathfrak{g}^{\mathbb{C}} = (q_{pp} \oplus q_{mm}) \oplus (q_{pm} \oplus q_{mp}) \tag{27}$$

is a $\mathbb{Z}_2$-grading of $\mathfrak{g}^{\mathbb{C}}$, so that $\phi_1^* \star \phi_2^*$ is indeed an associated Cartan map. The operation $\star$ is commutative, and $\phi \star \phi$ is the identity map for any ϕ.

The set of associated Cartan maps therefore forms a group under the operation $\star$. We consider in particular the group generated by the associated Cartan maps ϕ_s^*, ϕ_t^*, and $\phi_{\mathbb{H}}^*$, which is easily seen to be a copy of $(\mathbb{Z}_2)^3$, and hence of order 8. The orbit of $\mathfrak{sl}(3, \mathbb{O}) = \mathfrak{e}_{6(52,26)}$ under this group is shown in Figure 3, from which the multiplication table can be inferred.

The multiplication table of the finite group $(\mathbb{Z}_2)^3$ is identical to that of the octonionic units, without the minus signs, and can therefore also be represented using the 7-point projective plane. Using commutation as the operation, the same multiplication table applies directly to the 8 subspaces listed in Table VII, as shown in Figure 4. Using octonionic language for this multiplication table, the 15 proper subalgebras of $sl(3, \mathbb{O})$ listed in Table VIII consist precisely of 1 "real" subalgebra, corresponding to the "identity element" $\mathfrak{r}_{1,\mathbb{H}}$, 7 "complex" subalgebras, formed by adding any one other subspace, corresponding to the "points" in the multiplication table, and 7 "quaternionic" subalgebras, formed by adding any one additional subspace (and ensuring the algebra closes), corresponding to the "lines" in the multiplication table. There is of course one subalgebra missing from this description, namely the "octonionic" algebra $sl(3, \mathbb{O})$ itself, corresponding to the entire Fano "plane."

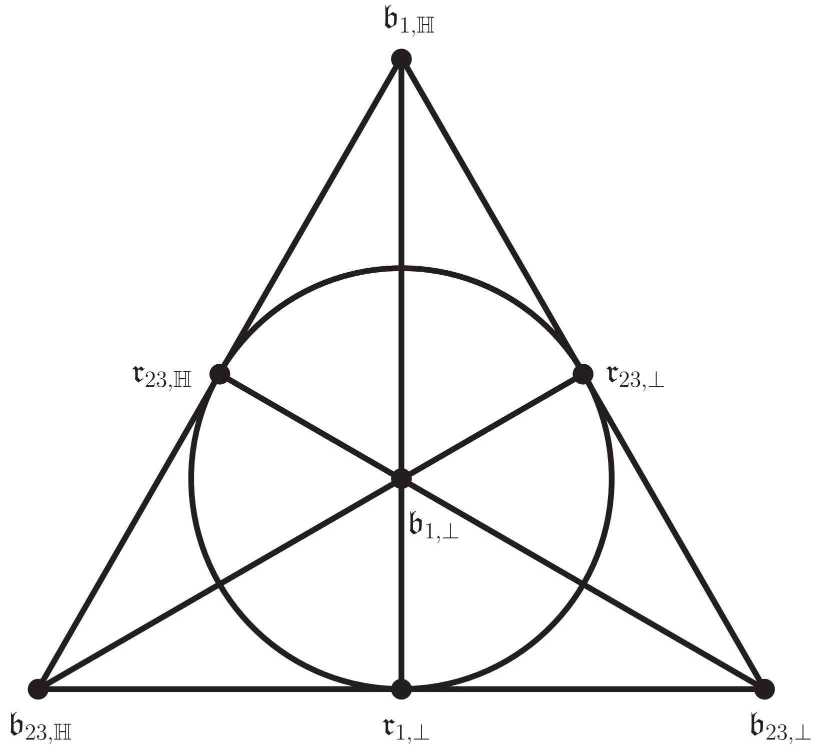

FIG. 4. The $(\mathbb{Z}_2)^3$ multiplication table under commutation for the subspaces of $\mathfrak{e}_6$ determined by the associated Cartan maps ϕ_t^*, $\phi_{\mathbb{H}}^*$, and ϕ_s^*. All elements square to $\mathfrak{r}_{1,\mathbb{H}}$.

B. Real forms of $\mathfrak{e}_6$

There are 5 real forms of $\mathfrak{e}_6$, all of which appear in Figure 3, although it is at first sight somewhat surprising that several of them appear more than once. However, our interpretation of $SL(3, \mathbb{O})$ is tied to a particular choice of basis, so different copies of a given real form yield different decompositions of $\mathfrak{sl}(3, \mathbb{O})$, not necessarily with the same signature.

We regard Figure 3 itself as an indication that there are really 8 "real forms" of $\mathfrak{e}_6$ of relevance to the structure of $SL(3, \mathbb{O})$, and hence of possible relevance to physics. At the very least, all 8 of these real forms played a role in the construction of the "maps" of $\mathfrak{sl}(3, \mathbb{O})$ given in Figures 1 and 2.

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[11] For an example of a $\mathbb{Z}_2$-grading that does not satisfy these conditions, see Ref. 8.
[12] R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications* (Wiley, 1974), reprinted by Dover Publications, Mineola, New York, 2005.
[13] Since $\dot{B}^1_{tz} + \dot{B}^2_{tz} + \dot{B}^3_{tz} = 0$, and since $\mathfrak{t}_1$ contains $\dot{B}^1_{tz}$ and $\dot{B}^2_{tz} - \dot{B}^3_{tz}$, we regard both $\dot{B}^2_{tz}$ and $\dot{B}^3_{tz}$ as being elements of $\mathfrak{t}_1$.
[14] The labels $\mathfrak{a}_m$, $\mathfrak{b}_m$, $\mathfrak{c}_m$, and $\mathfrak{d}_m$ refer to the standard, Cartan-Killing classification of (complex!) simple Lie algebras, as, of course, do $\mathfrak{e}_6$, $\mathfrak{f}_4$, and $\mathfrak{g}_2$.
[15] A. Wangberg, "The structure of E_6," Ph.D. thesis (Oregon State University, 2007), available at http://xxx.lanl.gov/abs/0711.3447.