

AN ABSTRACT OF THE THESIS OF

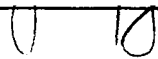
PATRICK C. ENDICOTT for the degree DOCTOR OF PHILOSOPHY

in MATHEMATICS presented on July 15, 1977

Title: SIMPLICIAL BUNDLES AND THE HOMOLOGY STRUCTURE
OF SUBMERSIONS

Signature redacted for privacy.

Abstract approved: _____

 J. Wolfgang Smith

In this dissertation we construct a homology spectral sequence attached to a submersion whose E^2 term takes values in a certain homology with local coefficients. The motivation for this work is that the spectral sequence provides an effective tool for the conjecture and proof of theorems regarding the global structure of submersions. The spectral sequence is first derived for certain combinatorial objects known as simplicial bundles which at once generalize the notion of fiber bundles (over polyhedra) and simplicial complexes. The spectral sequence of a submersion is then obtained by taking the direct limit of the spectral sequences associated with an approximating system of simplicial bundles.

Simplicial Bundles and the Homology
Structure of Submersions

by

Patrick C. Endicott

A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

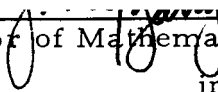
Doctor of Philosophy

Completed July 1977

Commencement June 1978

APPROVED:


Signature redacted for privacy.



Professor of Mathematics


in charge of major

Signature redacted for privacy.



Chairman of Department of Mathematics

Signature redacted for privacy.



Dean of Graduate School

Date thesis is presented July 15, 1977

Typed by Illa Atwood for Patrick C. Endicott

ACKNOWLEDGEMENT

The author would like to thank Professor J. Wolfgang Smith for suggesting the topic of this dissertation and for his help and encouragement in bringing it to completion. The author considers himself very fortunate to have worked with such a gifted and sensitive scholar.

TABLE OF CONTENTS

<u>Chapter</u>		<u>Page</u>
	INTRODUCTION	1
I	THE HOMOLOGY OF A SIMPLICIAL BUNDLE	10
	1-1. Simplicial Bundles	10
	1-2. The Homology of a Simplicial Bundle	12
II	THE SPECTRAL SEQUENCE OF A SIMPLICIAL BUNDLE	18
III	THE SPECTRAL SEQUENCE OF A SUBMER-SION	39
	BIBLIOGRAPHY	58

SIMPLICIAL BUNDLES AND THE HOMOLOGY STRUCTURE OF SUBMERSIONS

INTRODUCTION

Since the objects of study of this dissertation are submersions, we shall begin with a brief introduction to this topic. By a submersion we shall understand a surjective C^∞ -map $f:X \rightarrow Y$ between paracompact differentiable manifolds, with $\dim X \geq \dim Y$, whose differential has everywhere maximal rank. For each $y \in Y$, $f^{-1}(y)$ constitutes a regularly imbedded submanifold of X , with

$$\dim f^{-1}(y) = \dim X - \dim Y$$

(referred to as the codimension of the submersion). Each such submanifold is known as a fiber of the submersion.

If X is compact, or more generally, if each fiber is compact, it follows from a result of Ehresmann [1], that $f:X \rightarrow Y$ may be viewed as the projection of a fiber bundle, which implies in particular, that each fiber is then homeomorphic to a standard fiber F . Obviously an extensive theory already exists for that case. It is known, moreover, that the topological structures of X , Y , and F , as measured by means of homology groups, are related by an E^2 spectral sequence which converges to $H_*(X)$ and whose E^2 term is given by the formula:

$$E_{pq}^2 \cong H_p(Y; H_q(F)).$$

However, if the fibers are permitted to be noncompact, the submersion will in general fail to be a fiber bundle, and in this case the topological structure of the fibers may vary from point to point. In 1967 J. W. Smith began to investigate the question as to whether in this more general setting there exist relationships between the topological structures of X , Y and of the various fibers, and he found that in the case of codimension 1 a fairly extensive theory can be developed. One of the most striking results which was obtained is the following generalization of the classical Thom-Gysin sequence for sphere bundles [5].

Theorem. Let $f: X \rightarrow Y$ be an orientable submersion of codimension 1 whose fibers are path-connected, and let U denote the set of all $y \in Y$ such that $f^{-1}(y)$ is compact. There exists an exact sequence

$$\dots \rightarrow H_q(X) \rightarrow H_q(Y) \rightarrow H_{q-2}(U) \rightarrow H_{q-1}(X) \rightarrow \dots$$

relating the homology groups. In the case where the submersion is a fiber bundle the above sequence reduces to the classical Thom-Gysin sequence of a circle bundle, which may be regarded as an immediate consequence of the Serre spectral sequence for fiber bundles. The question naturally arises therefore whether the Serre spectral sequence can be generalized to the case of an arbitrary

submersion, with its E^2 term given in terms of an appropriate homology, which now must of necessity involve local coefficients. Such a homology theory has indeed been extensively developed by Sekino [4], based on the concept of a simplicial bundle, previously introduced by Smith [6], which is a combinatorial device by which a submersion may be approximated.

Given a simplicial complex K , a simplicial bundle over K is a function which assigns to each simplex $\sigma \in K$ a commutative triangle

$$\begin{array}{ccc} |\sigma| \times F_\sigma & \xrightarrow{\Phi_\sigma} & E_\sigma \\ \pi_\sigma \searrow & & \swarrow p_\sigma \\ & |\sigma| & \end{array}$$

where E_σ , F_σ are topological spaces, Φ_σ is a homeomorphism and π_σ is the natural projection, subject to the condition that for $\tau < \sigma$, $p_\sigma^{-1}(x) \subset p_\tau^{-1}(x)$ for all $x \in |\tau|$. Setting $E = \bigcup_{\sigma \in K} E_\sigma$ (the topological sum) there is a projection $p: E \rightarrow |K|$. The connection between simplicial bundles and submersions is given by the following approximation theorem.

Theorem 3.1. Given a submersion $f: X \rightarrow Y$ and a compact
subset $C \subset X$, there exists a simplicial bundle over K with
 $C \subset E \subset X$, $f(C) \subset |K| \subset Y$ and $p = f|_E$.

This suggests that a submersion may be viewed as the direct

limit of an approximating system of simplicial bundles, and moreover, that construction of a spectral sequence for a submersion may be accomplished in terms of a direct limit of spectral sequences associated with simplicial bundles. The present dissertation constitutes a direct approach leading to the realization of this program.

In the first chapter the homology of a simplicial bundle is defined. Let $p:E \rightarrow |K|$ be a simplicial bundle. The inclusions $p_\sigma^{-1}(x) \subset p_\tau^{-1}(x)$ given for all pairs $\sigma, \tau \in K$ with $\tau < \sigma$ induce fiber injections $i_\sigma^\tau: F_\sigma \rightarrow F_\tau$ satisfying the homotopy transitivity condition $i_\tau^\omega \circ i_\sigma^\tau \simeq i_\sigma^\omega$ for $\omega < \tau < \sigma$. Taking singular homology of the fibers with coefficients in some group G , we obtain an algebraic structure

$$\{H_*(F_\sigma; G); i_{\sigma*}^\tau\}_{\sigma, \tau \in K}$$

known (see [4]) as a local coefficient system for the simplicial complex K . Define

$$C_p(K; H_q(F; G)) = \bigoplus_{\sigma \in K^{(p)}} H_p(|\sigma|, |\dot{\sigma}|) \otimes H_q(F_\sigma; G)$$

where $K^{(p)}$ denotes the set of p -simplexes of K . A boundary operator

$$\partial: C_p(K; H_q(F; G)) \rightarrow C_{p-1}(K; H_q(F; G))$$

is defined on generators by the formula

$$\partial(a \otimes c) = \sum_{\tau < \sigma} \epsilon_{\sigma}^{\tau}(a) \otimes i_{\sigma*}^{\tau}(c)$$

where $\epsilon_{\sigma}^{\tau}: H_p(|\sigma|, |\dot{\sigma}|) \rightarrow H_{p-1}(|\tau|, |\dot{\tau}|)$ is given by the composition

$$\begin{aligned} H_p(|\sigma|, |\dot{\sigma}|) &\xrightarrow{j_{\sigma*}} H_p(|K^p|, |K^{p-1}|) \xrightarrow{\partial_*} \\ &H_{p-1}(|K^{p-1}|, |K^{p-2}|) \xrightarrow[\cong]{\{j_{\tau*}\}^{-1}} \bigoplus_{\tau \in K^{(p-1)}} H_{p-1}(|\tau|, |\dot{\tau}|) \\ &\xrightarrow{q_{\tau}} H_{p-1}(|\tau|, |\dot{\tau}|). \end{aligned}$$

Here K^p denotes the p -skeleton of K , j_{σ} the inclusion map, ∂_* the connecting homomorphism of the exact sequence of the triple $(|K^p|, |K^{p-1}|, |K^{p-2}|)$, q_{τ} the direct sum projection and

$$\{j_{\tau*}\}: \bigoplus_{\tau \in K^{(p-1)}} H_{p-1}(|\tau|, |\dot{\tau}|) \xrightarrow[\cong]{} H_{p-1}(|K^{p-1}|, |K^{p-2}|)$$

a direct sum representation.

The homology $H_*(K; H_*(F; G))$ of the resulting chain complex is defined to be the homology of the simplicial bundle.

In Chapter 2 we define the spectral sequence of a simplicial bundle and evaluate its E^2 term. Let $p: E \rightarrow |K|$ be a simplicial bundle and set $E_p = \bigcup_{\sigma \in K^p} E_{\sigma}$ for $p \geq 0$ and $E_p = \emptyset$ for $p < 0$. Then $\{E_p\}$ constitutes an increasing filtration of E , and

consequently one has the following (see Spanier [8]).

Theorem 2.1. For singular homology with any coefficient group G there is a convergent E^1 spectral sequence with $E^1 \cong H_p(E_p, E_{p-1}; G)$, d^1 the boundary operator of the triple (E_p, E_{p-1}, E_{p-2}) , and E^∞ isomorphic to the bigraded group associated to the filtration of $H_*(E; G)$ defined by

$$F_p H_*(E; G) = \text{im}[H_*(E_p; G) \rightarrow H_*(E; G)].$$

The problem of identifying the E^1 term of the spectral sequence thus reduces to the problem of computing the groups $H_n(E_p, E_{p-1}; G)$.

The following theorem will be proved in Chapter 2.

Theorem 2.5. There exists an isomorphism

$$\psi: C_p(K; H_q(F; G)) \rightarrow H_{p+q}(E_p, E_{p-1}; G)$$

given by the composition of isomorphisms

$$\begin{aligned} \bigoplus_{\sigma \in K^{(p)}} H_p(|\sigma|, |\dot{\sigma}|) \otimes H_q(F_\sigma; G) &\xrightarrow{\oplus \mu'_\sigma} \bigoplus_{\sigma \in K^{(p)}} H_{p+q}((|\sigma|, |\dot{\sigma}|) \times F_\sigma; G) \\ &\xrightarrow{\oplus \Phi_{\sigma_*}} \bigoplus_{\sigma \in K^{(p)}} H_{p+q}(E_\sigma, E_{\dot{\sigma}}; G) \xrightarrow{\{i_{\sigma_*}\}} H_{p+q}(E_p, E_{p-1}; G), \end{aligned}$$

where $E_{\dot{\sigma}} = p_\sigma^{-1}(|\dot{\sigma}|)$, μ'_σ is the homology cross product, Φ_{σ_*} is induced by the homeomorphisms of the simplicial bundle and $\{i_{\sigma_*}\}$

is an inclusion induced direct sum representation.

The decisive step in the evaluation of the E^2 terms, and the one involving the greatest technical difficulties encountered in the dissertation, may now be stated in terms of the following.

Theorem 2.6. The diagram

$$\begin{array}{ccc}
 C_p(K; H_q(F; G)) & \xrightarrow{\psi} & E_{pq}^1 \\
 \downarrow \partial & & \downarrow d^1 \\
 C_{p-1}(K; H_q(F; G)) & \xrightarrow{\psi} & E_{p-1, q}^1
 \end{array}$$

commutes.

The main theorem on the spectral sequence of simplicial bundles is obtained as an immediate consequence:

MAIN THEOREM. Given a simplicial bundle $p:E \rightarrow |K|$ there exists a convergent E^1 spectral sequence such that

$$E_{pq}^2 \cong H_p(K; H_q(F; G))$$

and E^∞ is isomorphic to the bigraded group associated to the filtration of $H_*(E; G)$ defined by

$$F_p H_*(E; G) = \text{im}[H_*(E_p; G) \rightarrow H_*(E; G)].$$

In the last chapter the spectral sequence of a submersion is obtained. As a first step in that direction, the following full

approximation theorem is proved.

Theorem 3.2. Let $f: X \rightarrow Y$ be a submersion. Then there is a sequence $\rho: {}_s E \rightarrow |{}_s K|$ of simplicial bundles such that

(1) Given $C \subset X$ compact there exists a positive integer s such that $C \subset {}_s E$, $f(C) \subset |{}_s K|$ and $\rho = f|_{{}_s E}$.

(2) For $s < t$, ${}_s E \subset {}_t E$ and there exists a nonnegative integer n such that ${}_s^n K \subset {}_t K$ where ${}_s^n K$ denotes the n^{th} barycentric subdivision of ${}_s K$.

As a consequence of this theorem we obtain a direct system $\{{}_s E^r, {}_s d^r\}$ of spectral sequences indexed by the directed set J of positive integers and we define $\{E^r(f), d^r\}$, the spectral sequence of the submersion $f: X \rightarrow Y$, to be the direct limit of this system.

We show that the E^2 term of the spectral sequence of a submersion is isomorphic to the homology $H_*(Y; H_*(f_y; G))$ of a submersion defined in [4] as a direct limit of a suitable system of homology groups of simplicial bundles. The E^∞ term is then shown to be isomorphic to the bigraded group associated with a suitably defined filtration $FH_*(X; G)$ of $H_*(X; G)$. Thus we obtain the main theorem on the spectral sequence of a submersion.

MAIN THEOREM. Let $f: X \rightarrow Y$ be a submersion. There is a convergent E^1 spectral sequence such that

$$E_{pq}^2(f) \cong H_p(Y; H_q(f_y; G))$$

and $E^\infty(f)$ isomorphic to the bigraded group associated to the filtra-
tion $FH_*(X;G)$ of $H_*(X;G)$.

Symbols and Notations

We shall use the following symbols and notations: Let K denote a simplicial complex. Then

K^p = p -skeleton of K

$K^{(p)}$ = the set of p -simplexes of K

${}^n K$ = the n^{th} barycentric subdivision of K

σ = a p -simplex

$\overset{o}{\sigma}$ = the open simplex of σ

$|\sigma|$ = the closed space of σ

$|\dot{\sigma}|$ = the union of all proper faces of σ

$|\ddot{\sigma}|$ = the union of all faces of σ of dimension $< p - 1$.

\emptyset = the empty set

0 = the empty function, trivial group etc.

\mathbb{Z} = the group of integers

\approx = homeomorphism

\cong = isomorphism

\simeq = homotopy

1 = the identity morphism

G = an arbitrary coefficient group

I. THE HOMOLOGY OF A SIMPLICIAL BUNDLE

1-1. Simplicial Bundles

Let K be a finite simplicial complex. Suppose there is given a function Φ which assigns to each simplex $\sigma \in K$ a commutative triangle

$$\begin{array}{ccc} |\sigma| \times F_\sigma & \xrightarrow{\Phi_\sigma} & E_\sigma \\ \pi_\sigma \searrow & & \swarrow p_\sigma \\ & |\sigma| & \end{array}$$

where F_σ and E_σ are topological spaces, $|\sigma|$ is the underlying space of σ , π_σ is the natural projection, p_σ is a continuous surjection, and Φ_σ is a homeomorphism.

Assume the following descending face condition:

If $\tau < \sigma$ then for each $x \in |\tau|$, $p_\sigma^{-1}(x) \subset p_\tau^{-1}(x)$, and thus $p_\sigma^{-1}(|\tau|) \subset E_\tau$.

The function Φ shall be called a simplicial bundle over K , or more precisely, a descending simplicial bundle over K (in contrast to the "ascending" simplicial bundle considered by Smith in [6]).

E_σ shall be referred to as the total space over σ and p_σ as a projection. The spaces F_σ shall be called the fibers of the bundle. For each $x \in |\sigma|$ the restriction of Φ_σ^{-1} to $p_\sigma^{-1}(x)$ determines a homeomorphism $p_\sigma^{-1}(x) \approx F_\sigma$; $p_\sigma^{-1}(x)$ shall be called a fiber over x .

We shall refer to $|K|$ as the base space of the bundle. Define

$E = \bigcup_{\sigma \in K} E_\sigma$ (the topological sum) and observe that there is determined a well-defined function $p: E \rightarrow |K|$. We shall call E the total space of the bundle and often denote a simplicial bundle by the functional notation

$$E \xrightarrow{p} |K|$$

where p shall be called the projection of the bundle.

If $\tau < \sigma$, by the descending face condition, there exists a composition

$$|\tau| \times F_\sigma \xrightarrow[\approx]{\Phi_\sigma} p_\sigma^{-1}(|\tau|) \subset E_\tau \xrightarrow[\approx]{\Phi_\tau^{-1}} |\tau| \times F_\tau.$$

Let $\pi'_\tau: |\tau| \times F_\tau \rightarrow F_\tau$ denote the natural projection, and for each $x \in |\tau|$, define the injection $i_{\sigma, x}^\tau: F_\sigma \rightarrow F_\tau$ by the formula

$$i_{\sigma, x}^\tau(y) = \pi'_\tau \circ \Phi_\tau^{-1} \circ \Phi_\sigma(x, y).$$

We refer to $i_{\sigma, x}^\tau$ as a fiber injection from F_σ to F_τ .

The following properties of the fiber injections have been established in [4].

$$(1) \text{ If } b, c \in |\tau| \quad i_{\sigma, b}^\tau \simeq i_{\sigma, c}^\tau.$$

(2) If $\tau < \sigma < \rho$, $c \in |\sigma|$, and $b, d \in |\tau|$ then the fiber injections $i_{\sigma, b}^\tau$, $i_{\rho, c}^\sigma$ and $i_{\rho, d}^\tau$ satisfy the homotopy transitivity condition

$$i_{\sigma, b}^{\tau} \circ i_{\rho, c}^{\sigma} \simeq i_{\rho, d}^{\tau}.$$

From condition (1) it follows that the induced homomorphisms

$$(i_{\sigma, x}^{\tau})_*: H_*(F_{\sigma}; G) \rightarrow H_*(F_{\tau}; G)$$

agree for all $x \in |\tau|$. This common homomorphism shall be denoted by

$$i_{\sigma}^{\tau}: H_*(F_{\sigma}; G) \rightarrow H_*(F_{\tau}; G).$$

If $\tau < \sigma < \rho$ condition (2) implies that the induced homomorphisms i_{σ}^{τ} , i_{ρ}^{τ} , and i_{ρ}^{σ} satisfy the transitivity condition.

$$i_{\sigma}^{\tau} \circ i_{\rho}^{\sigma} = i_{\rho}^{\tau}.$$

1-2. The Homology of a Simplicial Bundle

Let K denote a finite simplicial complex and assume that:

(1) For each simplex $\sigma \in K$, there is given an abelian group

G_{σ} .

(2) If $\tau < \sigma$ there is given a homomorphism $w_{\sigma}^{\tau}: G_{\sigma} \rightarrow G_{\tau}$.

(3) If $\tau < \sigma < \rho$ then the homomorphisms w_{σ}^{τ} , w_{ρ}^{σ} and w_{ρ}^{τ} satisfy the transitivity condition

$$w_{\sigma}^{\tau} \circ w_{\rho}^{\sigma} = w_{\rho}^{\tau}.$$

The system of groups and homomorphisms

$$G = \{G_{\sigma}, w_{\sigma}^{\tau}\}_{\sigma, \tau \in K}$$

constitutes a local coefficient system for the complex K as defined by Sekino in [4].

Lemma 1.2.1

The inclusions $j_\sigma: (|\sigma|, |\dot{\sigma}|) \subset (|K^p|, |K^{p-1}|)$ induce a direct-sum representation

$$\{j_{\sigma_*}\}: \bigoplus_{\sigma \in K^{(p)}} H_p(|\sigma|, |\dot{\sigma}|) \cong H_p(|K^p|, |K^{p-1}|)$$

where $K^{(p)}$ is the set of p -simplexes of K .

Proof. The lemma is a consequence of lemma 2, p. 474 of Spanier [8] applied to the trivial fibration $1: |K| \rightarrow |K|$. //

For each $\sigma \in K^{(p)}$ we define a projection

$$q'_\sigma: H_p(|K^p|, |K^{p-1}|) \rightarrow H_p(|\sigma|, |\dot{\sigma}|)$$

to be the following composition

$$H_p(|K^p|, |K^{p-1}|) \xrightarrow[\cong]{\{j_{\sigma_*}\}^{-1}} \bigoplus_{\sigma \in K^{(p)}} H_p(|\sigma|, |\dot{\sigma}|) \xrightarrow{q_\sigma} H_p(|\sigma|, |\dot{\sigma}|)$$

where q_σ is the direct-sum projection.

Let ∂_* denote the connecting homomorphism of the exact sequence of the triple $(|K^p|, |K^{p-1}|, |K^{p-2}|)$ and consider the following diagram:

$$\begin{array}{ccc}
H_p(|K^p|, |K^{p-1}|) & \xrightarrow{\partial_*} & H_{p-1}(|K^{p-1}|, |K^{p-2}|) \\
\uparrow j_{\sigma_*} & & \downarrow q'_\tau \\
H_p(|\sigma|, |\dot{\sigma}|) & & H_{p-1}(|\tau|, |\dot{\tau}|)
\end{array}$$

Define $\epsilon_\sigma^\tau : H_p(|\sigma|, |\dot{\sigma}|) \rightarrow H_{p-1}(|\tau|, |\dot{\tau}|)$ to be the composition
 $\epsilon_\sigma^\tau = q'_\tau \circ \partial_* \circ j_{\sigma_*}$.

Lemma 1.2.2

Let $\tau < \sigma < \rho$ and let $a \in H_{p+1}(|\rho|, |\dot{\rho}|)$ then

$$\sum_{\sigma \in K^{(p)}} \epsilon_\sigma^\tau \circ \epsilon_\rho^\sigma(a) = 0.$$

Proof.

$$\begin{aligned}
\sum_{\sigma \in K^{(p)}} \epsilon_\sigma^\tau \circ \epsilon_\rho^\sigma(a) &= \sum_{\sigma \in K^{(p)}} q'_\tau \circ \partial_* \circ j_{\sigma_*} \circ q'_\sigma \circ \partial_* \circ j_{\rho_*}(a) \\
&= q'_\tau \circ \partial_* \left(\sum_{\sigma \in K^{(p)}} j_{\sigma_*} \circ q'_\sigma \right) (\partial_* \circ j_{\rho_*}(a)) \\
&= q'_\tau \circ \partial_* \left(\sum_{\sigma \in K^{(p)}} j_{\sigma_*} \circ q_\sigma \circ \{j_{\sigma_*}\}^{-1} \right) (\partial_* \circ j_{\rho_*}(a)) \\
&= q'_\tau \circ \partial_* \{j_{\sigma_*}\} \circ \{j_{\sigma_*}\}^{-1} (\partial_* \circ j_{\rho_*}(a))
\end{aligned}$$

$$= q'_\tau \circ \partial_* \circ \partial_* \circ j_{\rho_*} (a)$$

$$= 0 \quad (\text{since } \partial_* \circ \partial_* = 0) //$$

Let $G = \{G_\sigma, w_\sigma^\tau\}_{\sigma, \tau \in K}$ be a local coefficient system for K .

Define

$$C_p(K; G) = \bigoplus_{\sigma \in K^{(p)}} H_p(|\sigma|, |\dot{\sigma}|) \otimes G_\sigma$$

and define $\partial: C_p(K; G) \rightarrow C_{p-1}(K; G)$ on the σ -component of $C_p(K; G)$ by the formula

$$\partial(a \otimes c) = \sum_{\tau < \sigma} \epsilon_\sigma^\tau(a) \otimes w_\sigma^\tau(c)$$

where $a \in H_p(|\sigma|, |\dot{\sigma}|)$ and $c \in G_\sigma$.

∂ clearly determines a function $C_p(K; G) \rightarrow C_{p-1}(K; G)$

Lemma 1.2.3

∂ is a homomorphism and $\{C_p(K; G), \partial\}$ is a chain complex.

Proof. In order to show that ∂ is a homomorphism it suffices to show that ∂ is component wise a homomorphism. Let $\hat{\sigma} \in$

$H_p(|\sigma|, |\dot{\sigma}|)$ be a generator and $a, b \in G_\sigma$.

$$\begin{aligned}
\partial(\hat{\sigma} \otimes (a + b)) &= \sum_{\tau < \sigma} \epsilon_{\sigma}^{\tau}(\hat{\sigma}) \otimes w_{\sigma}^{\tau}(a + b) \\
&= \sum_{\tau < \sigma} \epsilon_{\sigma}^{\tau}(\hat{\sigma}) \otimes (w_{\sigma}^{\tau}(a) + w_{\sigma}^{\tau}(b)) \\
&= \sum_{\tau < \sigma} (\epsilon_{\sigma}^{\tau}(\hat{\sigma}) \otimes w_{\sigma}^{\tau}(a) + \epsilon_{\sigma}^{\tau}(\hat{\sigma}) \otimes w_{\sigma}^{\tau}(b)) \\
&= \sum_{\tau < \sigma} \epsilon_{\sigma}^{\tau}(\hat{\sigma}) \otimes w_{\sigma}^{\tau}(a) + \sum_{\tau < \sigma} \epsilon_{\sigma}^{\tau}(\hat{\sigma}) \otimes w_{\sigma}^{\tau}(b) \\
&= \partial(\hat{\sigma} \otimes a) + \partial(\hat{\sigma} \otimes b)
\end{aligned}$$

Suppose $\tau < \sigma < \rho$ and let $a \in H_{p+1}(|\rho|, |\dot{\rho}|)$ and $c \in G_{\rho}$.

$[\partial(a \otimes c)]_{\sigma}$ shall denote the σ -component of $\partial(a \otimes c)$

$$\partial([\partial(a \otimes c)]_{\sigma}) = \partial(\epsilon_{\rho}^{\sigma}(a) \otimes w_{\rho}^{\sigma}(c))$$

$$\begin{aligned}
[\partial(\epsilon_{\rho}^{\sigma}(a) \otimes w_{\rho}^{\sigma}(c))]_{\tau} &= \epsilon_{\sigma}^{\tau} \circ \epsilon_{\rho}^{\sigma}(a) \otimes w_{\sigma}^{\tau} \circ w_{\rho}^{\sigma}(c) \\
&= \epsilon_{\sigma}^{\tau} \circ \epsilon_{\rho}^{\sigma}(a) \otimes w_{\rho}^{\tau}(c)
\end{aligned}$$

$$\begin{aligned}
[\partial(\partial(a \otimes c))]_{\tau} &= \sum_{\sigma \in K(p)} \epsilon_{\sigma}^{\tau} \circ \epsilon_{\rho}^{\sigma}(a) \otimes w_{\rho}^{\tau}(c) \\
&= \{ \sum_{\sigma \in K(p)} \epsilon_{\sigma}^{\tau} \circ \epsilon_{\rho}^{\sigma}(a) \} \otimes w_{\rho}^{\tau}(c)
\end{aligned}$$

$$\begin{aligned}
&= 0 \otimes w_{\rho}^{\tau}(c) \quad \text{by lemma 1.2.2} \\
&= 0
\end{aligned}$$

Thus $[\partial(\partial(a \otimes c))]_{\tau} = 0$ for all $\tau \in K^{(p-1)}$. Therefore $\partial\partial = 0$. //

We define the homology of K with coefficients in the local coefficient system G to be the homology of the above chain complex.

Let $p:E \rightarrow |K|$ be a simplicial bundle, let G be an arbitrary group and let $G_{\sigma} = H_*(F_{\sigma}; G)$ for each $\sigma \in K$. If $\tau < \sigma$ let $w_{\sigma}^{\tau}: G_{\sigma} \rightarrow G_{\tau}$ be $i_{\sigma}^{\tau}: H_*(F_{\sigma}; G) \rightarrow H_*(F_{\tau}; G)$. We observe that $\{H_*(F_{\sigma}; G), i_{\sigma}^{\tau}\}$ is a local coefficient system for K . The resulting chain complex $\{C_p(K; H_*(F; G)), \partial_p\}$ we define to be the chain complex of the simplicial bundle. The homology of this chain complex is defined to be the homology of the simplicial bundle, and we shall denote this group by $H_*(K; H_*(F; G))$.

II. THE SPECTRAL SEQUENCE OF A SIMPLICIAL BUNDLE

Let $p:E \rightarrow |K|$ be a simplicial bundle and define

$$E_p = \bigcup_{\sigma \in K^p} E_\sigma,$$

where K^p denotes the p -skeleton of K , for $p \geq 0$ and let $E_p = \emptyset$ if $p < 0$. Then $E_p \subset E_{p+1}$, so $\{E_p\}$ is an increasing filtration of E . Furthermore, $E_{-1} = \emptyset$, $\bigcup_p E_p = E$, and $E = E_m$ where $m = \dim K$. Our starting point will be the following classical result [see Spanier [8], page 473, theorem 1].

Theorem 2.1. Let $p:E \rightarrow |K|$ be a simplicial bundle. For singular homology with any coefficient group G there is a convergent E^1 spectral sequence with $E_{pq}^1 \cong H_{p+q}(E_p, E_{p-1}; G)$, d^1 the boundary operator of the triple (E_p, E_{p-1}, E_{p-2}) , and E^∞ isomorphic to the bigraded group associated to the filtration of $H_*(E; G)$ defined by

$$F_p H_*(E; G) = \text{im}[H_*(E_p; G) \rightarrow H_*(E; G)]. //$$

Our aim shall be to evaluate the E^2 term of the given spectral sequence by identifying the E^1 chain complex with the chain complex $C_p(K; H_q(F; G))$ attached to the simplicial bundle. The first step in this identification is effected by the following lemma, corresponding to Spanier ([8], page 474 lemma 2) .

Lemma 2.2

The inclusion maps $i_\sigma: (p^{-1}(|\sigma|), p^{-1}(|\dot{\sigma}|)) \subset (E_p, E_{p-1})$
induce a direct-sum representation

$$\{i_{\sigma_*}\}: \bigoplus_{\sigma \in K^{(p)}} H_n(p^{-1}(|\sigma|), p^{-1}(|\dot{\sigma}|)) \xrightarrow{\cong} H_n(E_p, E_{p-1})$$

where $\dot{\sigma}$ denotes the simplicial complex consisting of all proper faces of σ .

Proof. For each $\sigma \in K^{(p)}$ let e_σ be a p -cell contained in the interior of $|\sigma|$ and let $\overset{o}{e}_\sigma = e_\sigma - \dot{e}_\sigma$. There exists a deformation retraction $r_\sigma: (|\sigma| - \overset{o}{e}_\sigma) \rightarrow |\dot{\sigma}|$. Let $\bar{\Phi}_\sigma^{-1}$ denote the restriction of Φ_σ^{-1} to $p_\sigma^{-1}(|\sigma| - \overset{o}{e}_\sigma)$. Then $\Phi_\sigma \circ (r_\sigma \times 1_{F_\sigma}) \circ \bar{\Phi}_\sigma^{-1}: p_\sigma^{-1}(|\sigma| - \overset{o}{e}_\sigma) \rightarrow p_\sigma^{-1}(|\dot{\sigma}|)$ determines a deformation retraction which extends by the inclusion to a deformation retraction $\bar{r}_\sigma: p^{-1}(|\sigma| - \overset{o}{e}_\sigma) \rightarrow p^{-1}(|\dot{\sigma}|)$. The \bar{r}_σ combine to give a deformation retraction

$$r: (E_p - \bigcup_{\sigma \in K^{(p)}} p^{-1}(\overset{o}{e}_\sigma)) \rightarrow E_{p-1}.$$

Thus the inclusions

$$(p^{-1}(|\sigma|), p^{-1}(|\dot{\sigma}|)) \subset (p^{-1}(|\sigma|), p^{-1}(|\sigma| - \overset{o}{e}_\sigma))$$

and

$$(E_p, E_{p-1}) \subset (E_p, E_p - \bigcup_{\sigma \in K^{(p)}} p^{-1}(\overset{o}{e}_\sigma))$$

are homotopy equivalences. There is a commutative diagram induced by inclusion maps

$$\begin{array}{ccc}
 \bigoplus_{\sigma \in K^{(p)}} H_n(p^{-1}(|\sigma|), p^{-1}(|\dot{\sigma}|)) & \xrightarrow{\{i_{\sigma_*}\}} & H_n(E_p, E_{p-1}) \\
 \downarrow & & \downarrow \\
 \bigoplus_{\sigma \in K^{(p)}} H_n(p^{-1}(|\sigma|), p^{-1}(|\sigma| - e_\sigma^0)) & \rightarrow & H_n(E_p, E_p - \bigcup_{\sigma \in K^{(p)}} p^{-1}(e_\sigma^0)) \\
 \uparrow & & \uparrow \\
 \bigoplus_{\sigma \in K^{(p)}} H_n(p^{-1}(e_\sigma), p^{-1}(\dot{e}_\sigma)) & \rightarrow & H_n(\bigcup_{\sigma \in K^{(p)}} p^{-1}(e_\sigma), \bigcup_{\sigma \in K^{(p)}} p^{-1}(\dot{e}_\sigma))
 \end{array}$$

in which the vertical maps are isomorphisms, the top two because they are induced by homotopy equivalences and the bottom two because they are induced by suitable excision maps. Since e_σ is disjoint from e_ρ if $\sigma \neq \rho$, the bottom map is an isomorphism because it is induced by a chain isomorphism. //

Lemma 2.3

For $\sigma \in K^{(p)}$ the inclusion map

$$i_\sigma: (p_\sigma^{-1}(|\sigma|), p_\sigma^{-1}(|\dot{\sigma}|)) \subset (p^{-1}(|\sigma|), p^{-1}(|\dot{\sigma}|))$$

induces an isomorphism

$$i_{\sigma_*}: H_n(p_\sigma^{-1}(|\sigma|), p_\sigma^{-1}(|\dot{\sigma}|)) \cong H_n(p^{-1}(|\sigma|), p^{-1}(|\dot{\sigma}|)).$$

Proof. Let e be a p -cell contained in the interior of $|\sigma|$.

There is a commutative diagram induced by inclusion maps

$$\begin{array}{ccc}
H_n(p_\sigma^{-1}(|\sigma|), p_\sigma^{-1}(|\dot{\sigma}|)) & \longrightarrow & H_n(p_\sigma^{-1}(|\sigma|), p_\sigma^{-1}(|\sigma| - \overset{\circ}{e})) \\
\downarrow i_{\sigma_*} & & \downarrow \\
H_n(p^{-1}(|\sigma|), p^{-1}(|\dot{\sigma}|)) & \longrightarrow & H_n(p^{-1}(|\sigma|), p^{-1}(|\sigma| - \overset{\circ}{e}))
\end{array}$$

The top map is an isomorphism since $p_\sigma^{-1}(|\dot{\sigma}|)$ is a deformation retract of $p_\sigma^{-1}(|\sigma| - \overset{\circ}{e})$ and the bottom map is an isomorphism since $p^{-1}(|\dot{\sigma}|)$ is a deformation retract of $p^{-1}(|\sigma| - \overset{\circ}{e})$. Since the closure of $p^{-1}(|\dot{\sigma}|) - p_\sigma^{-1}(|\dot{\sigma}|)$ is contained in the interior of $p^{-1}(|\sigma| - \overset{\circ}{e})$ the vertical map to the right is induced by a suitable excision map and hence is an isomorphism. //

Lemma 2.4

Assume $\sigma \in K^{(p)}$ then there exists an isomorphism

$$\mu'_\sigma: H_p(|\sigma|, |\dot{\sigma}|) \otimes H_q(F_\sigma; G) \rightarrow H_{p+q}(|\sigma| \times F_\sigma, |\dot{\sigma}| \times F_\sigma; G)$$

where μ'_σ is the homology cross product.

Proof. In the relative Künneth theorem (Spanier [8], theorem 10, p. 235), take $X = |\sigma|$, $A = |\dot{\sigma}|$, $Y = F_\sigma$ and $B = \emptyset$. Since $\{\emptyset, |\dot{\sigma}| \times F_\sigma\}$ is an excisive couple in $|\sigma| \times F_\sigma$, there is a short exact sequence

$$\begin{aligned}
0 \rightarrow [H(|\sigma|, |\dot{\sigma}|) \otimes H(F_\sigma, \phi; G)]_n &\xrightarrow{\mu'_\sigma} H_n((|\sigma|, |\dot{\sigma}|) \times (F_\sigma, \phi); G) \\
&\rightarrow [H(|\sigma|, |\dot{\sigma}|) * H(F_\sigma, \phi; G)]_{n-1} \rightarrow 0
\end{aligned}$$

Since $H(|\sigma|, |\dot{\sigma}|)$ is free the torsion product vanishes and the following isomorphism is obtained

$$0 \rightarrow [H(|\sigma|, |\dot{\sigma}|) \otimes H(F_\sigma, \phi; G)]_n \xrightarrow{\mu'_\sigma} H_n((|\sigma|, |\dot{\sigma}|) \times (F_\sigma, \phi); G) \rightarrow 0.$$

Taking $n = p+q$ and recalling the definition of the tensor product chain complex one obtains

$$[H(|\sigma|, |\dot{\sigma}|) \otimes H(F_\sigma, \phi; G)]_{p+q} = \bigoplus_{i+j=p+q} H_i(|\sigma|, |\dot{\sigma}|) \otimes H_j(F_\sigma, \phi; G)$$

Since $H_i(|\sigma|, |\dot{\sigma}|) = 0$ if $i \neq p$

$$[H(|\sigma|, |\dot{\sigma}|) \otimes H(F_\sigma, \phi; G)]_{p+q} = H_p(|\sigma|, |\dot{\sigma}|) \otimes H_q(F_\sigma; G)$$

where the pair (F_σ, ϕ) is identified with F_σ . Also by the definition of the product of topological pairs

$$H_{p+q}((|\sigma|, |\dot{\sigma}|) \times (F_\sigma, \phi); G) = H_{p+q}(|\sigma| \times F_\sigma, |\dot{\sigma}| \times F_\sigma; G)$$

Thus the following isomorphism is obtained

$$\mu'_\sigma: H_p(|\sigma|, |\dot{\sigma}|) \otimes H_q(F_\sigma; G) \rightarrow H_{p+q}(|\sigma| \times F_\sigma, |\dot{\sigma}| \times F_\sigma; G). //$$

We summarize the identification of E_{pq}^1 in the following theorem.

Let $E_{\dot{\sigma}} = p_{\sigma}^{-1}(|\dot{\sigma}|)$.

Theorem 2.5.

$$E_{pq}^1 \cong \bigoplus_{\sigma \in K^{(p)}} H_p(|\sigma|, |\dot{\sigma}|) \otimes H_q(F_{\sigma}; G)$$

where the isomorphism is given by the following composition of isomorphisms:

$$\begin{aligned} \bigoplus_{\sigma \in K^{(p)}} H_p(|\sigma|, |\dot{\sigma}|) \otimes H_q(F_{\sigma}; G) &\xrightarrow{\mu'_{\sigma}} \bigoplus_{\sigma \in K^{(p)}} H_{p+q}(|\sigma| \times F_{\sigma}, |\dot{\sigma}| \times F_{\sigma}; G) \\ &\xrightarrow{\Phi_{\sigma*}} \bigoplus_{\sigma \in K^{(p)}} H_{p+q}(E_{\sigma}, E_{\dot{\sigma}}; G) \xrightarrow{\{i_{\sigma}\}} H_{p+q}(E_p, E_{p-1}; G) \cong E_{pq}^1. \end{aligned}$$

Here $\{i_{\sigma}\}$ is inclusion induced, $\Phi_{\sigma*}$ is induced by the homeomorphism of pairs

$$\Phi_{\sigma}: (|\sigma| \times F_{\sigma}, |\dot{\sigma}| \times F_{\sigma}) \rightarrow (E_{\sigma}, E_{\dot{\sigma}}),$$

and μ'_{σ} is the homology cross product.

Proof. Throughout we use the fact that a direct sum of isomorphisms is again an isomorphism. Lemma 2.4 gives the first identification, $\Phi_{\sigma*}$ is induced by a homeomorphism of pairs and is thus an isomorphism, lemmas 2.2 and 2.3 combined give the next identification, and theorem 1.2 gives the final identification. //

In order to identify the chain complex (E^1, d^1) it remains to identify the differential d^1 . By the definition of the product of topological pairs $(|\sigma|, |\dot{\sigma}|) \times (F_\sigma, \phi) = (|\sigma| \times F_\sigma, |\dot{\sigma}| \times F_\sigma)$. Thus

$$H_{p+q}(|\sigma| \times F_\sigma, |\dot{\sigma}| \times F_\sigma; G) = H_{p+q}((|\sigma|, |\dot{\sigma}|) \times F_\sigma; G)$$

and we may regard

$$\Phi_{\sigma_*}: H_{p+q}(|\sigma| \times F_\sigma, |\dot{\sigma}| \times F_\sigma; G) \rightarrow H_{p+q}(E_\sigma, E_{\dot{\sigma}}; G)$$

as an isomorphism

$$\Phi_{\sigma_*}: H_{p+q}((|\sigma|, |\dot{\sigma}|) \times F_\sigma; G) \rightarrow H_{p+q}(E_\sigma, E_{\dot{\sigma}}; G).$$

Define

$$\psi_\sigma: H_p(|\sigma|, |\dot{\sigma}|) \otimes H_q(F_\sigma; G) \rightarrow H_{p+q}(E_p, E_{p-1}; G)$$

to be the following composition:

$$\begin{aligned} H_p(|\sigma|, |\dot{\sigma}|) \otimes H_q(F_\sigma; G) &\xrightarrow{\mu'_\sigma} H_{p+q}((|\sigma|, |\dot{\sigma}|) \times F_\sigma; G) \xrightarrow{\Phi_{\sigma_*}} \\ &H_{p+q}(E_\sigma, E_{\dot{\sigma}}; G) \xrightarrow{i_{\sigma_*}} H_{p+q}(E_p, E_{p-1}; G). \end{aligned}$$

Define

$$\psi: \bigoplus_{\sigma \in K(p)} H_p(|\sigma|, |\dot{\sigma}|) \otimes H_q(F_\sigma; G) \rightarrow H_{p+q}(E_p, E_{p-1}; G)$$

to be the composition:

$$\bigoplus_{\sigma \in K^{(p)}} H_p(|\sigma|, |\dot{\sigma}|) \otimes H_q(F_{\sigma}; G) \xrightarrow{\bigoplus \mu'_{\sigma}} \bigoplus_{\sigma \in K^{(p)}} H_{p+q}((|\sigma|, |\dot{\sigma}|) \times F_{\sigma}; G)$$

$$\begin{aligned} & \xrightarrow{\bigoplus \Phi_{\sigma*}} \bigoplus_{\sigma \in K^{(p)}} H_{p+q}(E_{\sigma}, E_{\dot{\sigma}}; G) \xrightarrow{\{i_{\sigma*}\}} H_{p+q}(E_p, E_{p-1}; G). \end{aligned}$$

Observe that $\mu'_{\sigma}(a \otimes c) = a \times c$ is the homology cross product. Also

observe that ψ is precisely the isomorphism under which we identify

E_{pq}^1 with

$$\bigoplus_{\sigma \in K^{(p)}} H_p(|\sigma|, |\dot{\sigma}|) \otimes H_q(F_{\sigma}; G).$$

The identification of d^1 will be completed once the following theorem is established.

Theorem 2.6. The following diagram is commutative

$$\begin{array}{ccc} \bigoplus_{\sigma \in K^{(p)}} H_p(|\sigma|, |\dot{\sigma}|) \otimes H_q(F_{\sigma}; G) & \xrightarrow{\psi} & H_{p+q}(E_p, E_{p-1}; G) \\ \downarrow \partial & & \downarrow d^1 \\ \bigoplus_{\tau \in K^{(p-1)}} H_{p-1}(|\tau|, |\dot{\tau}|) \otimes H_q(F_{\tau}; G) & \xrightarrow{\psi} & H_{p+q-1}(E_{p-1}, E_{p-2}; G) \end{array}$$

where the boundary operator d^1 to the right is the boundary operator of the triple (E_p, E_{p-1}, E_{p-2}) .

The proof of the above theorem will conclude the identification of d^1 with the boundary operator of the chain complex of the simplicial bundle $p:E \rightarrow |K|$. For theorem 2.1 identifies E_{pq}^1 with $H_{p+q}(E_p, E_{p-1}; G)$ and d^1 with the boundary operator of the triple (E_p, E_{p-1}, E_{p-2}) . Also theorem 2.5 identifies $H_{p+q}(E_p, E_{p-1}; G)$, under the composition of isomorphisms which determine ψ , with

$$\bigoplus_{\sigma \in K^{(p)}} H_p(|\sigma|, |\dot{\sigma}|) \otimes H_q(F_\sigma; G).$$

The above theorem identifies the boundary operator of the triple (E_p, E_{p-1}, E_{p-2}) with the boundary operator of the chain complex $\{C_p(K; H_q(F; G)), \partial\}$ which is the chain complex of the simplicial bundle. The proof of theorem 2.6 will depend upon the following key lemma.

Lemma 2.7

Let $\sigma \in K^{(p)}$ and τ be a $(p-1)$ -face of σ . Let $x_0 \in |\tau|$ and define $\Phi_\sigma|_\tau = \Phi_\sigma| |\tau| \times F_\sigma$. Then $i_{\tau*} \circ \Phi_{\tau*} \circ (1_\tau \times i_{\sigma, x_0}^\tau) = i_{\sigma*} \circ (\Phi_\sigma|_\tau)_* : H_*((|\tau|, |\dot{\tau}|) \times F_\sigma; G) \rightarrow H_*(E_{p-1}, E_{p-2}; G)$.

Proof. Since homotopic maps induce the same homomorphism on homology it suffices to show

$$i_\tau \circ \Phi_\tau \circ (1_\tau \times i_{\sigma, x_0}^\tau) \simeq i_\sigma \circ \Phi_\sigma|_\tau. \text{ Define}$$

$$\rho: (|\tau|, |\dot{\tau}|) \times F_{\sigma} \rightarrow (|\tau|, |\dot{\tau}|) \times F_{\tau}$$

by the formula

$$\rho(x, y) = (x, i_{\sigma, x}^{\tau}(y)).$$

First it will be shown that the following diagram commutes:

$$\begin{array}{ccc} (|\tau|, |\dot{\tau}|) \times F_{\sigma} & \xrightarrow{\rho} & (|\tau|, |\dot{\tau}|) \times F_{\tau} \\ \downarrow \Phi_{\sigma}|_{\tau} & & \downarrow \Phi_{\tau} \\ (p_{\sigma}^{-1}(|\tau|), \rho_{\sigma}^{-1}(|\dot{\tau}|)) \overset{j}{\subset} & & (E_{\tau}, E_{\dot{\tau}}) \end{array}$$

where j is an inclusion. Let $(x, y) \in |\tau| \times F_{\sigma}$

$$\begin{aligned} \Phi_{\tau} \circ \rho(x, y) &= \Phi_{\tau}(x, i_{\sigma, x}^{\tau}(y)) \\ &= \Phi_{\tau}(x, \pi'_{\tau} \circ \Phi_{\tau}^{-1} \circ j \circ \Phi_{\sigma}(x, y)) \\ &= \Phi_{\tau}(x, \pi'_{\tau} \circ \Phi_{\tau}^{-1}(z)) \end{aligned}$$

where $\Phi_{\sigma}(x, y) = z$, whence $j \circ \Phi_{\sigma}(x, y) = j(z) = z$. Thus

$$\Phi_{\tau} \circ \rho(x, y) = \Phi_{\tau}(x, \pi'_{\tau}(x, y'))$$

where

$$\Phi_{\tau}^{-1}(z) = (x, y').$$

Thus

$$\Phi_{\tau} \circ \rho(x, y) = \Phi_{\tau}(x, y') = z = j \circ \Phi_{\sigma}(x, y).$$

Next we demonstrate $\rho \simeq 1_{\tau} \times i_{\sigma, x_0}^{\tau}$ for $x_0 \in |\tau|$. For each $x \in |\tau|$ there exists a canonical homotopy [see Sekino [4] page 39]

$$H_x: i_{\sigma, x_0}^{\tau} \simeq i_{\sigma, x}^{\tau}.$$

More precisely, $H_x: F_{\sigma} \times I \rightarrow F_{\tau}$ is defined by identifying I with the line segment joining x_0 to x and setting

$$H_x(y, t) = \pi'_{\tau} \circ \Phi_{\tau}^{-1} \circ \Phi_{\sigma}(y, t).$$

We may then define the desired homotopy

$$\bar{H}: (|\tau|, |\dot{\tau}|) \times F_{\sigma} \times I \rightarrow (|\tau|, |\dot{\tau}|) \times F_{\tau}$$

by the formula

$$\bar{H}(x, y, t) = (x, H_x(y, t)).$$

$$\begin{aligned} \bar{H}(x, y, 0) &= (x, H_x(y, 0)) \\ &= (x, i_{\sigma, x_0}^{\tau}(y)) \\ &= (1_{|\tau|} \times i_{\sigma, x_0}^{\tau})(x, y). \end{aligned}$$

$$\begin{aligned} \bar{H}(x, y, 1) &= (x, H_x(y, 1)) \\ &= (x, i_{\sigma, x}^{\tau}(y)) \\ &= \rho(x, y). \end{aligned}$$

Thus \bar{H} establishes a homotopy between ρ and $1_{|\tau|} \times i_{\sigma, x_0}^\tau$.

By commutativity of the preceding diagram we have:

$$j \circ \Phi_\sigma|_\tau = \Phi_\tau \circ \rho.$$

Thus

$$i_\tau \circ j \circ \Phi_\sigma|_\tau = i_\tau \circ \Phi_\tau \circ \rho.$$

Observing that $i_\tau \circ j = i_\sigma$ one obtains

$$i_\sigma \circ \Phi_\sigma|_\tau = i_\tau \circ \Phi_\tau \circ \rho.$$

But $\rho \simeq 1_{|\tau|} \times j_{\sigma, x_0}^\tau$. Thus

$$i_\tau \circ \Phi_\tau \circ \rho \simeq i_\tau \circ \Phi_\tau \circ (1_{|\tau|} \times i_{\sigma, x_0}^\tau).$$

Therefore

$$i_\sigma \circ \Phi_\sigma|_\tau \simeq i_\tau \circ \Phi_\tau \circ (1_{|\tau|} \times i_{\sigma, x_0}^\tau). //$$

Before proceeding with the proof of theorem 2.6, we cite for the reader's convenience certain results from Spanier.

(1) Spanier, p. 235 #11.

Let $f: (X, A) \rightarrow (X', A')$ and $g: (Y, B) \rightarrow (Y', B')$ be maps and let $z \in H_p(X, A)$ and $z' \in H_q(Y, B)$. Then in the group $H_{p+q}((X', A') \times (Y', B'))$ we have

$$(f \times g)_*(z \times z') = f_*z \times g_*z'.$$

(2) Spanier, pp. 180-190.

If (X_1, A_1) and (X_2, A_2) are pairs in a space X , we say that $\{(X_1, A_1), (X_2, A_2)\}$ is an excisive couple of pairs if $\{X_1, X_2\}$ and $\{A_1, A_2\}$ are both excisive couples of subsets.

If $\{(X_1, A_1), (X_2, A_2)\}$ is an excisive couple of pairs, there is an exact sequence of the form

$$\begin{aligned} \dots \xrightarrow{\partial_*} H_q(X_1 \cap X_2, A_1 \cap A_2) &\xrightarrow{i_*} H_q(X_1, A_1) \oplus H_q(X_2, A_2) \\ &\xrightarrow{j_*} H_q(X_1 \cup X_2, A_1 \cup A_2) \xrightarrow{\partial_*} \dots \end{aligned}$$

called the relative Mayer-Vietoris sequence of $\{(X_1, A_1), (X_2, A_2)\}$.

Also given the triple (X, A, B) , $\{(X, B), (A, A)\}$ is always an excisive couple of pairs, and the relative Mayer-Vietoris sequence of $\{(X, B), (A, A)\}$ coincides with the homology sequence of the triple (X, A, B) .

(3) Spanier, p. 235 #15.

Let $\{(X_1, A_1), (X_2, A_2)\}$ be an excisive couple of pairs in X and let $z \in H_p(X_1 \cup X_2, A_1 \cup A_2)$ and $z' \in H_q(Y, B)$. For the connecting homomorphism of the appropriate Mayer-Vietoris sequence we have

$$\partial_*(z \times z') = \partial_* z \times z'$$

where $z \times z'$ is the homology cross product.

Proof of Theorem 2.7. It suffices to show $\psi \circ \partial = \partial \circ \psi_\sigma$ for each $\sigma \in K^{(p)}$. Let $a \in H_p(|\sigma|, |\dot{\sigma}|)$ and $c \in H_q(F_\sigma)$.

$$\begin{aligned}
\psi \circ \partial(a \otimes c) &= \psi \left(\sum_{\tau} \epsilon_{\sigma}^{\tau}(a) \otimes i_{\sigma}^{\tau}(c) \right) \\
&= \{i_{\tau_*}\} \circ \oplus \Phi_{\tau_*} \circ \oplus \mu'_{\tau} \left(\sum_{\tau} \epsilon_{\sigma}^{\tau}(a) \otimes i_{\sigma}^{\tau}(c) \right) \\
&= \{i_{\tau_*}\} \circ \oplus \Phi_{\tau_*} \left(\sum_{\tau} \epsilon_{\sigma}^{\tau}(a) \times i_{\sigma}^{\tau}(c) \right) \\
&= \{i_{\tau_*}\} \left(\sum_{\tau} \Phi_{\tau_*} (\epsilon_{\sigma}^{\tau}(a) \times i_{\sigma}^{\tau}(c)) \right) \\
&= \sum_{\tau} i_{\tau_*} \circ \Phi_{\tau_*} (\epsilon_{\sigma}^{\tau}(a) \times i_{\sigma}^{\tau}(c)).
\end{aligned}$$

By the first property of the fiber injections $i_{\sigma}^{\tau} = (i_{\sigma, x_0}^{\tau})_*$ where $x_0 \in |\tau|$. Thus we may continue the calculation as:

$$\begin{aligned}
\psi \circ \partial(a \otimes c) &= \sum_{\tau} i_{\tau_*} \circ \Phi_{\tau_*} (\epsilon_{\sigma}^{\tau}(a) \times i_{\sigma}^{\tau}(c)) \\
&= \sum_{\tau} i_{\tau_*} \circ \Phi_{\tau_*} (\epsilon_{\sigma}^{\tau}(a) \times (i_{\sigma, x_0}^{\tau})_*(c)) \\
&= \sum_{\tau} i_{\tau_*} \circ \Phi_{\tau_*} (1_{|\tau|_*} (\epsilon_{\sigma}^{\tau}(a)) \times (i_{\sigma, x_0}^{\tau})_*(c))
\end{aligned}$$

In the first result from Spanier cited above, take $X = X' = |\tau|$, $A = A' = |\dot{\tau}|$, $Y = F_{\sigma}$, $Y' = F_{\tau}$, $B = B' = \emptyset$, $f = 1_{|\tau|}: (|\tau|, |\dot{\tau}|) \rightarrow (|\tau|, |\dot{\tau}|)$ and $g = i_{\sigma, x_0}^{\tau}: F_{\sigma} \rightarrow F_{\tau}$.

Since $\epsilon_{\sigma}^{\tau}(a) \in H_{p-1}(|\tau|, |\dot{\tau}|)$ and $c \in H_q(F_{\sigma}; G)$, in

$H_{p+q-1}(|\tau|, |\dot{\tau}|) \times F_{\tau}; G$ we have

$$(1_{|\tau|} \times i_{\sigma, x_0}^{\tau})_* (\epsilon_{\sigma}^{\tau}(a) \times c) = 1_{|\tau|_*} (\epsilon_{\sigma}^{\tau}(a)) \times (i_{\sigma, x_0}^{\tau})_*(c)$$

Thus

$$\begin{aligned} \psi \circ \partial(a \otimes c) &= \sum_{\tau} i_{\tau_*} \circ \Phi_{\tau_*} (1_{|\tau|_*} (\epsilon_{\sigma}^{\tau}(a)) \times (i_{\sigma, x_0}^{\tau})_*(c)) \\ &= \sum_{\tau} i_{\tau_*} \circ \Phi_{\tau_*} \circ (1_{|\tau|} \times i_{\sigma, x_0}^{\tau})_* (\epsilon_{\sigma}^{\tau}(a) \times c). \end{aligned}$$

Also by lemma 2.7

$$\begin{aligned} \psi \circ \partial(a \otimes c) &= \sum_{\tau} i_{\tau_*} \circ \Phi_{\tau_*} \circ (1_{|\tau|} \times i_{\sigma, x_0}^{\tau})_* (\epsilon_{\sigma}^{\tau}(a) \times c) \\ &= \sum_{\tau} i_{\sigma_*} \circ (\Phi_{\sigma}|_{\tau})_* (\epsilon_{\sigma}^{\tau}(a) \times c) \end{aligned}$$

Next we compute $\partial\psi_{\sigma}(a \otimes c)$. View σ as a simplicial complex and let $\ddot{\sigma}$ denote the $(p-2)$ - skeleton of σ and define $E_{\ddot{\sigma}} = p_{\sigma}^{-1}(|\ddot{\sigma}|)$.

First it is to be observed that by naturality of the connecting homomorphism of a triple the following diagram commutes.

$$\begin{array}{ccc} H_{p+q}(E_{\sigma}, E_{\ddot{\sigma}}; G) & \xrightarrow{i_{\sigma_*}} & H_{p+q}(E_p, E_{p-1}; G) \\ \downarrow \partial & & \downarrow \partial \\ H_{p+q-1}(E_{\dot{\sigma}}, E_{\ddot{\sigma}}; G) & \xrightarrow{i_{\dot{\sigma}_*}} & H_{p+q-1}(E_{p-1}, E_{p-2}; G) \end{array}$$

where we are considering the triples (E_p, E_{p-1}, E_{p-2}) and $(E_\sigma, E_{\dot{\sigma}}, E_{\ddot{\sigma}})$ and where i_σ and $i_{\dot{\sigma}}$ are inclusions.

Thus

$$\begin{aligned} \partial \circ \psi_\sigma(a \otimes c) &= \partial \circ i_{\sigma*} \circ \Phi_{\sigma*}(a \times c) \\ &= i_{\dot{\sigma}*} \circ \partial \circ \Phi_{\sigma*}(a \times c). \end{aligned}$$

Recall that $(|\sigma|, |\dot{\sigma}|) \times F_\sigma = (|\sigma| \times F_\sigma, |\dot{\sigma}| \times F_\sigma)$. Thus

$$H_{p+q}((|\sigma|, |\dot{\sigma}|) \times F_\sigma; G) = H_{p+q}(|\sigma| \times F_\sigma, |\dot{\sigma}| \times F_\sigma; G).$$

Again by naturality of the connecting homomorphisms of a triple we have the following commutative diagram.

$$\begin{array}{ccc} H_{p+q}(|\sigma| \times F_\sigma, |\dot{\sigma}| \times F_\sigma; G) & \xrightarrow{\Phi_{\sigma*}} & H_{p+q}(E_\sigma, E_{\dot{\sigma}}; G) \\ \downarrow \partial & & \downarrow \partial \\ H_{p+q-1}(|\dot{\sigma}| \times F_\sigma, |\ddot{\sigma}| \times F_\sigma; G) & \xrightarrow{\Phi_{\dot{\sigma}*}} & H_{p+q-1}(E_{\dot{\sigma}}, E_{\ddot{\sigma}}; G) \end{array}$$

where $\Phi_{\dot{\sigma}*}$ is induced by Φ_σ and the triples under consideration are $(E_\sigma, E_{\dot{\sigma}}, E_{\ddot{\sigma}})$ and $(|\sigma| \times F_\sigma, |\dot{\sigma}| \times F_\sigma, |\ddot{\sigma}| \times F_\sigma)$. Thus

$$\begin{aligned} \partial \circ \psi_\sigma(a \otimes c) &= i_{\dot{\sigma}*} \circ \partial \circ \Phi_{\sigma*}(a \times c) \\ &= i_{\dot{\sigma}*} \circ \Phi_{\dot{\sigma}*} \circ \partial(a \times c). \end{aligned}$$

In the second result from Spanier take $X = |\sigma|$, $B = |\dot{\sigma}|$, $A = (|\ddot{\sigma}|)$ then $\{(|\sigma|, |\dot{\sigma}|), (|\ddot{\sigma}|, |\dot{\sigma}|)\}$ is an excisive couple of pairs and the associated relative Mayer-Vietoris sequence reduces to the exact sequence of the triple $(|\sigma|, |\dot{\sigma}|, |\ddot{\sigma}|)$ and the connecting homomorphisms coincide. In the third result from Spanier take $Y = F_{\sigma}$, $B = \emptyset$, $X_1 = |\sigma|$, $X_2 = |\dot{\sigma}|$, $A_1 = A_2 = |\ddot{\sigma}|$. Then in

$$H_{p+q-1}((|\dot{\sigma}|, |\ddot{\sigma}|) \times F_{\sigma}; G) = H_{p+q-1}(|\dot{\sigma}| \times F_{\sigma}, |\ddot{\sigma}| \times F_{\sigma}; G)$$

we have $\partial_*(z \times z') = \partial_* z \times z'$ where the second ∂_* is the connecting homomorphism of the triple $(|\sigma|, |\dot{\sigma}|, |\ddot{\sigma}|)$. Adapting this to the present situation one obtains:

$$\begin{aligned} \partial \circ \psi_{\sigma}(a \otimes c) &= i_{\dot{\sigma}*} \circ \Phi_{\dot{\sigma}*} \circ \partial(a \times c) \\ &= i_{\dot{\sigma}*} \circ \Phi_{\dot{\sigma}*} (\partial_*(a) \times c). \end{aligned}$$

To establish commutativity we must compare

$$\sum_{\tau} i_{\sigma}* \circ (\Phi_{\sigma}|_{\tau})_* (\epsilon_{\sigma}^{\tau}(a) \times c)$$

and

$$i_{\dot{\sigma}*} \circ \Phi_{\dot{\sigma}*} (\partial_*(a) \times c).$$

Let $j_{\tau}^{\dot{\sigma}} : (|\tau|, |\dot{\tau}|) \subset (|\dot{\sigma}|, |\ddot{\sigma}|)$ denote the inclusion and consider the following commutative diagram.

$$\begin{array}{ccc}
(|\tau|, |\dot{\tau}|) \times F_{\sigma} & \xrightarrow{\Phi_{\sigma}|_{\tau}} & (p_{\sigma}^{-1}(|\tau|), p_{\sigma}^{-1}(|\dot{\tau}|)) \\
\downarrow j_{\tau}^{\dot{\sigma}} \times 1_{F_{\sigma}} & & \downarrow i_{\sigma} \\
(|\dot{\sigma}|, |\ddot{\sigma}|) \times F_{\sigma} & & \\
\downarrow \Phi_{\sigma}^{\dot{\sigma}} & \xrightarrow{i_{\sigma}^{\dot{\sigma}}} & (E_{p-1}, E_{p-2}) \\
(E_{\sigma}^{\dot{\sigma}}, E_{\sigma}^{\ddot{\sigma}}) & &
\end{array}$$

by functoriality we obtain the commutative diagram:

$$\begin{array}{ccc}
H_{p+q-1}((|\tau|, |\dot{\tau}|) \times F_{\sigma}; G) & \xrightarrow{(\Phi_{\sigma}|_{\tau})_*} & H_{p+q-1}(p_{\sigma}^{-1}(|\tau|), p_{\sigma}^{-1}(|\dot{\tau}|); G) \\
\downarrow (j_{\tau}^{\dot{\sigma}} \times 1_{F_{\sigma}})_* & & \downarrow i_{\sigma_*} \\
H_{p+q-1}((|\dot{\sigma}|, |\ddot{\sigma}|) \times F_{\sigma}; G) & & \\
\downarrow \Phi_{\sigma_*}^{\dot{\sigma}} & \xrightarrow{i_{\sigma_*}^{\dot{\sigma}}} & H_{p+q-1}(E_{p-1}, E_{p-2}; G) \\
H_{p+q-1}(E_{\sigma}^{\dot{\sigma}}, E_{\sigma}^{\ddot{\sigma}}; G) & &
\end{array}$$

From this it follows that:

$$i_{\sigma_*} \circ (\Phi_{\sigma}|_{\tau})_*(\epsilon_{\sigma}^{\tau}(a) \times c) = i_{\sigma_*}^{\dot{\sigma}} \circ \Phi_{\sigma_*}^{\dot{\sigma}} \circ (j_{\tau}^{\dot{\sigma}} \times 1_{F_{\sigma}})_*(\epsilon_{\sigma}^{\tau}(a) \times c)$$

$$\begin{aligned}
&= i_{\dot{\sigma}*} \circ \Phi_{\dot{\sigma}*} \circ (j_{\tau*}^{\dot{\sigma}} \times 1_{H(F_{\sigma})})(\epsilon_{\sigma}^{\tau}(a) \times c) \\
&= i_{\dot{\sigma}*} \circ \Phi_{\dot{\sigma}*} (j_{\tau*}^{\dot{\sigma}} \circ \epsilon_{\sigma}^{\tau}(a) \times c)
\end{aligned}$$

Whence:

$$\begin{aligned}
\sum_{\tau} i_{\sigma*} \circ (\Phi_{\sigma}|_{\tau})_*(\epsilon_{\sigma}^{\tau}(a) \times c) &= \sum_{\tau} i_{\dot{\sigma}*} \circ \Phi_{\dot{\sigma}*} (j_{\tau*}^{\dot{\sigma}} \circ \epsilon_{\sigma}^{\tau}(a) \times c) \\
&= i_{\dot{\sigma}*} \circ \Phi_{\dot{\sigma}*} \left(\sum_{\tau} (j_{\tau*}^{\dot{\sigma}} \circ \epsilon_{\sigma}^{\tau}(a) \times c) \right) \\
&= i_{\dot{\sigma}*} \circ \Phi_{\dot{\sigma}*} \left(\sum_{\tau} j_{\tau*}^{\dot{\sigma}} \circ \epsilon_{\sigma}^{\tau}(a) \times c \right) \\
&\quad \dots \dots \dots (1)
\end{aligned}$$

Consider the following commutative diagram:

$$\begin{array}{ccc}
H_p(|\sigma|, |\dot{\sigma}|) & \xrightarrow{\partial_*} & H_{p-1}(|\dot{\sigma}|, |\ddot{\sigma}|) \\
\downarrow j_{\sigma*} & & \downarrow j_{\dot{\sigma}*} \\
H_p(|K^p|, |K^{p-1}|) & \xrightarrow{\partial_*} & H_{p-1}(|K^{p-1}|, |K^{p-2}|) \\
\uparrow j_{\sigma*} & & \downarrow q'_{\tau} \\
H_p(|\sigma|, |\dot{\sigma}|) & \xrightarrow{\epsilon_{\sigma}^{\tau}} & H_{p-1}(|\tau|, |\dot{\tau}|)
\end{array}$$

where j_{σ} and $j_{\dot{\sigma}}$ are inclusion maps.

From this diagram it follows that:

$$\partial_* \circ j_{\sigma_*} = j_{\dot{\sigma}_*} \circ \partial_*$$

Thus

$$q'_\tau \circ \partial_* \circ j_{\sigma_*} = q'_\tau \circ j_{\dot{\sigma}_*} \circ \partial_* = \epsilon_\sigma^\tau$$

Hence

$$\begin{aligned} \sum_\tau j_{\tau_*}^{\dot{\sigma}} \circ \epsilon_\sigma^\tau(a) &= \sum_\tau j_{\tau_*}^{\dot{\sigma}} \circ q'_\tau \circ j_{\dot{\sigma}_*} \circ \partial_*(a) \\ &= \sum_\tau j_{\tau_*}^{\dot{\sigma}} \circ q_\tau \circ \{j_{\tau_*}\}^{-1} \circ j_{\dot{\sigma}_*} \circ \partial_*(a) \\ &= \{j_{\tau_*}^{\dot{\sigma}}\} \circ \{j_{\tau_*}\}^{-1} \circ j_{\dot{\sigma}_*} \circ \partial_*(a) \\ &= \mathbf{1}_{H_{p-1}(|\dot{\sigma}|, |\ddot{\sigma}|)} \circ \partial_*(a) \\ &= \partial_*(a). \end{aligned}$$

Thus we may conclude the relation:

$$\partial_*(a) = \sum_\tau j_{\tau_*}^{\dot{\sigma}} \circ \epsilon_\sigma^\tau(a).$$

Thus

$$\begin{aligned} i_{\dot{\sigma}_*} \circ \Phi_{\dot{\sigma}_*}(\partial_*(a) \times c) &= i_{\dot{\sigma}_*} \circ \Phi_{\dot{\sigma}_*} \left(\sum_\tau j_{\tau_*}^{\dot{\sigma}} \circ \epsilon_\sigma^\tau(a) \times c \right) \\ &\quad \dots \dots \dots (2) \end{aligned}$$

Comparing (1) and (2) we obtain:

$$\sum_\tau i_{\sigma_*}(\Phi_\sigma|_\tau)_*(\epsilon_\sigma^\tau(a) \times c) = i_{\dot{\sigma}_*} \circ \Phi_{\dot{\sigma}_*}(\partial_*(a) \times c)$$

Thus $\psi \circ \partial = \partial \circ \psi_\sigma$ and this concludes the proof of theorem 2.6. //

We may summarize this chapter in the following theorem.

MAIN THEOREM. Given a simplicial bundle $p:E \rightarrow |K|$ and
G an arbitrary group. There exists a convergent E^1 spectral se-
quence such that

$$E_{pq}^2 \cong H_p(K; H_q(F; G))$$

and E^∞ isomorphic to the bigraded group associated to the filtration
of $H_*(E; G)$ defined by

$$F_p H_*(E; G) = \text{im} [H_*(E_p; G) \rightarrow H_*(E; G)].$$

III. THE SPECTRAL SEQUENCE OF A SUBMERSION

In this chapter we define the spectral sequence of a submersion and evaluate the E^2 and E^∞ terms. A simplicial bundle $p:E \rightarrow |K|$ is said to approximate a submersion $f:X \rightarrow Y$ with respect to a compact subset $C \subset X$ provided $C \subset E \subset X$, $f(C) \subset |K| \subset Y$ and $p = f|_E$. The proof of the basic approximation theorem, theorem 3.1, rests upon the following tubular neighborhood theorem which is proved in [7].

TUBULAR NEIGHBORHOOD THEOREM

Let $f:X \rightarrow Y$ be a submersion and let $y \in Y$. Given a compact subset $C \subset X$ and a compact neighborhood $F \subset f^{-1}(y)$ of $C \cap f^{-1}(y)$ there exists a tubular neighborhood

$$\theta: D \times F \rightarrow V$$

where D is a neighborhood of $y \in Y$, $V \subset X$ and θ is a homeomorphism such that

- (1) $f \circ \theta = \pi$ (where $\pi: D \times F \rightarrow D$ is the natural projection).
- (2) $\theta(y, x) = x$ for all $x \in F$.
- (3) $C \cap f^{-1}(D) \subset V$.

The basic approximation theorem may be formulated as follows.

Theorem 3.1. Let $f:X \rightarrow Y$ be a submersion and let $C \subset X$ be compact. Let M be a simplicial complex which triangulates Y ,

i.e. assume without loss of generality (by [3]) that $Y = |M|$. Then there exists a simplicial bundle $p:E \rightarrow |K|$ which approximates f with respect to C and such that K is a barycentric subdivision of a subcomplex N of M .

Proof. Let N be the smallest subcomplex of M such that $|N| \supset f(C)$. Since $f(C)$ is compact N is a finite simplicial complex. For every $y \in |N|$, let F_y be a nonempty compact neighborhood of $f^{-1}(y) \cap C$ in $f^{-1}(y)$. By the tubular neighborhood theorem there exist tubular neighborhoods $\theta_y: D_y \times F_y \rightarrow V_y$ such that $C \cap f^{-1}(D_y) \subset V_y$. By the compactness of $|N|$ one obtains a finite family V_1, \dots, V_r of these neighborhoods which cover C . The set $\{D_1, \dots, D_r\}$ forms an open covering of $|N|$. Let L_0 be a barycentric subdivision of N subordinate to this covering. Thus there exists a function j which associates to every m -simplex $\sigma \in L_0$ (where $m = \dim L_0$) a tubular neighborhood

$$\theta_{j(\sigma)}: D_{j(\sigma)} \times F_{j(\sigma)} \rightarrow V_{j(\sigma)}$$

such that $|\sigma| \subset D_{j(\sigma)}$. In this way we obtain a collection of tubular neighborhoods

$$\{\theta_{\sigma_j^m}: D_{\sigma_j^m} \times F_{\sigma_j^m} \rightarrow V_{\sigma_j^m} \mid j \in I_m\}$$

where I_m indexes the set of m -simplexes of L_0 . Define a subset $C_\sigma \subset X$ by $C_\sigma = V_{j(\sigma)} \cap f^{-1}(|\sigma|)$ and let C_0 denote the union of

the C_σ as σ ranges over all m -simplexes of L_0 . Clearly $C \subset C_0$ and C_0 is compact. For every $y \in |L_0^{m-1}|$ let F'_y be a nonempty compact neighborhood of $f^{-1}(y) \cap C_0$ in $f^{-1}(y)$. By the tubular neighborhood theorem there exist tubular neighborhoods $\theta'_y: D'_y \times F'_y \rightarrow V'_y$ such that $C_0 \cap f^{-1}(D'_y) \subset V'_y$. By the compactness of $|L_0^{m-1}|$ one obtains a finite family V'_1, \dots, V'_s of these neighborhoods which cover $C_0 \cap |L_0^{m-1}| \equiv C_0 \cap f^{-1}(|L_0^{m-1}|)$. The set $\{D'_1, \dots, D'_s\}$ forms an open covering of $|L_0^{m-1}|$. Let L_1 be a barycentric subdivision of L_0^{m-1} subordinate to this covering. Thus there exists a function j' which associates to every $(m-1)$ -simplex $\sigma \in L_1$ a tubular neighborhood

$$\theta'_{j'(\sigma)}: D'_{j'(\sigma)} \times F'_{j'(\sigma)} \rightarrow V'_{j'(\sigma)}$$

such that $|\sigma| \subset D'_{j'(\sigma)}$. In this way we obtain a collection of tubular neighborhoods

$$\{\theta_{\sigma_j^{m-1}}: D_{\sigma_j^{m-1}} \times F_{\sigma_j^{m-1}} \rightarrow V_{\sigma_j^{m-1}} \mid j \in I_{m-1}\}$$

where I_{m-1} indexes the set of $(m-1)$ -simplexes of L_1 . Define a compact subset $C_\sigma \subset X$ by $C_\sigma = V'_{j'(\sigma)} \cap f^{-1}(|\sigma|)$ and let C_1 denote the union of the C_σ as σ ranges over all $(m-1)$ -simplexes of L_1 . Clearly C_1 is compact and $C_0 \cap f^{-1}(|L_0^{m-1}|) \subset C_1$. Repeat this construction with C_1 replacing C_0 and $|L_1^{m-2}|$ replacing $|L_0^{m-1}|$. The construction will now involve a suitable barycentric

subdivision L_2 of L_1^{m-2} and will furnish a compact set C_2 such that $C_1 \cap f^{-1}(|L_1^{m-2}|) \subset C_2$ and a collection of tubular neighborhoods

$$\{\theta_{\sigma_j^{m-2}} : D_{\sigma_j^{m-2}} \times F_{\sigma_j^{m-2}} \rightarrow V_{\sigma_j^{m-2}} \mid j \in I_{m-2}\}$$

where I_{m-2} indexes the set of $(m-2)$ -simplexes of L_2 . Continuing in this manner one obtains a finite sequence C_0, C_1, \dots, C_m of compact subsets whose union \overline{C} contains C and such that

$$C_k \cap f^{-1}(|L_k^{m-k-1}|) \subset C_{k+1}$$

for $k = 0, 1, \dots, m-1$. Also one obtains collections of tubular neighborhoods

$$\{\theta_{\sigma_j^i} : D_{\sigma_j^i} \times F_{\sigma_j^i} \rightarrow V_{\sigma_j^i} \mid j \in I_i\}$$

where $I_i, i = m, m-1, \dots, 0$, indexes the set of i -simplexes of L_{m-i} .

Let K be a barycentric subdivision of L_0 such that $L_m \subset K^0$ (i.e.

each L_i admits a barycentric subdivision sL_i such that ${}^sL_i \subset K$).

Let $\sigma \in K$ and consider the following sequence

$$|K| = |L_0| \supset |L_1| \supset \dots \supset |L_m| \subset |K^0|.$$

There exists a largest index i such that $|\sigma| \subset |L_i|$. Also there

exists $p \geq 0$ such that ${}^pL_i \subset K$ for all i . Since $\sigma \in {}^pL_i$ there

exists $\overline{\sigma} \in L_i$ such that $|\sigma| \subset |\overline{\sigma}|$. By the maximality of i

$\dim \bar{\sigma} = m-i$, for otherwise $|\sigma| \subset L_{i+1}$. Thus there exists a unique pair of integers (i, j) such that $|\sigma| \subset |\sigma_j^i|$ for some $j \in I_i$.

Set

$$E_\sigma = V_{\sigma_j^i} \cap f^{-1}(|\sigma|)$$

$$F_\sigma = F_{\sigma_j^i}$$

$\Phi_\sigma: |\sigma| \times F_\sigma \rightarrow E_\sigma$ to be the restriction of $\theta_{\sigma_j^i}$ to $|\sigma| \times F_{\sigma_j^i}$.

$p_\sigma: E_\sigma \rightarrow |\sigma|$ to be $f|_{E_\sigma}$.

The following diagram commutes by property 1 of the tubular neighborhood theorem

$$\begin{array}{ccc} |\sigma| \times F_\sigma & \xrightarrow{\Phi_\sigma} & E_\sigma \\ \pi_\sigma \searrow & & \swarrow p_\sigma \\ & |\sigma| & \end{array}$$

Next suppose $\tau < \sigma$; then there exists a unique pair of integers (p, q) such that $|\tau| \subset |\sigma_q^p|$ for some $q \in I_p$. It is clear that $p \geq i$, where $|\sigma| \subset |\sigma_j^i|$, and since $C_k \cap f^{-1}(|L_k^{m-k-1}|) \subset C_{k+1}$, $E_\sigma \cap f^{-1}(|\tau|) \subset E_\tau$. Also

$$E = \bigcup_{\sigma \in k} E_\sigma \supset C$$

and $|K| \supset f(c)$. Thus we obtain a simplicial bundle satisfying the

conditions of the theorem. //

We are now able to state and prove the full approximation theorem.

Theorem 3.2. Let $f: X \rightarrow Y$ be a submersion. Then there is a sequence $p: E \rightarrow |K|$ of simplicial bundles such that

- (1) Given $C \subset X$ compact there exists a positive integer s such that $p: E \rightarrow |K|$ approximates f with respect to C .
- (2) For $s < t$, $E \subset_t E$ and there exists a nonnegative integer n such that ${}_s K \subset_t K$.

Proof. Without loss of generality we assume X to be connected. X admits a complete Riemannian metric [2]. Let $x \in X$ and for $s = 1, 2, \dots$, let C_s denote the closed ball in X centered at x . Each C_s is compact and $C_s \subset C_{s+1}$, moreover, $\bigcup_s C_s = X$. Let $C \subset X$ be compact then C is bounded thus $C \subset C_s$ for some s . By theorem 3.1 there exists a simplicial bundle $p: E \rightarrow |K|$ which approximates f with respect to C_1 and ${}_1 K = {}^{n_1} N_1$ for $n_1 \geq 0$, where N_1 is the smallest subcomplex of M such that $|N_1| \supset f(C_1)$. Consider ${}_1 E \cup C_2$ which is compact; then by theorem 3.1, there exists a simplicial bundle ${}_2 p: {}_2 E \rightarrow |{}_2 K|$ which approximates f with respect to ${}_1 E \cup C_2$ and such that ${}_2 K = {}^{n_2} N_2$ for $n_2 \geq n_1$, where N_2 is the smallest subcomplex of M such that $|N_2| \supset f({}_1 E \cup C_2)$. Since $f(C_1) \subset f(C_2)$, $f(C_1) \subset f({}_1 E \cup C_2)$ and thus $|N_1| \subset |N_2|$. Continuing in this manner we obtain for each C_s a simplicial bundle

$p_s: E \rightarrow |_s K|$ approximating f with respect to $E_{s-1} \cup C_s$ and such that $|_s K| = N_s^{n_s}$ for $n_s \geq n_{s-1}$, where N_s is the smallest subcomplex of M such that $|N_s| \supset f(E_{s-1} \cup C_s)$. Let $C \subset X$ be compact; then $C \subset C_s \subset E_s \subset X$ for some s . Moreover,

$$f(C) \subset f(C_s) \subset f(E_{s-1} \cup C_s) \subset Y$$

and $p_s = f|_s E$. Thus $p_s: E \rightarrow |_s K|$ approximates f with respect to C . For $s < t$, $E_s \subset E_t$ by construction and taking $n = n_t - n_s$ we obtain $|_s K| \subset |_t K|$. Thus we obtain a sequence of simplicial bundles satisfying the conditions of the theorem. //

We are now in a position to construct a direct system of spectral sequences (over the directed set of natural numbers J). Thus, let $p_s: E \rightarrow |_s K|$ denote a sequence of simplicial bundles satisfying the conditions of theorem 3.2. Then, for each $s \in J$, we obtain a spectral sequence $\{^s E^r, ^s d^r\}$ associated with $p_s: E \rightarrow |_s K|$ satisfying the conditions of the main theorem on the spectral sequence of a simplicial bundle. For $s < t$, the inclusion

$|_s K| \subset |_t K|$ preserves skeleta and therefore the inclusion

$j: E_s \subset E_t$ is a filtration-preserving map. Hence j induces a homomorphism ${}^t \phi^r: {}^s E^r \rightarrow {}^t E^r$ of spectral sequences. One readily verifies that

$$\{ \{ {}^s E^r, {}^s d^r \}, {}^t \phi^r \}_{s < t}$$

forms a direct system of spectral sequences indexed by the directed

set J . Define

$$E^r(f) = \varinjlim_{s \in J} sE^r$$

and

$$d^r = \varinjlim_{s \in J} s_d^r.$$

Next we verify that $\{E^r(f), d^r\}$ is a spectral sequence.

Theorem 3.3. $\{E^r(f), d^r\}$ is a spectral sequence.

Proof. The following diagram commutes since the projections ${}^t_s\phi^r$ are homomorphisms of spectral sequences.

$$\begin{array}{ccc} sE^r & \xrightarrow{{}^t_s\phi^r} & tE^r \\ \downarrow s_d^r & & \downarrow t_d^r \\ sE^r & \xrightarrow{{}^t_s\phi^r} & tE^r \end{array}$$

Thus the differentials s_d^r determine an endomorphism of the direct system $\{sE^r, {}^t_s\phi^r\}_{s < t}$ and hence d^r is defined. Since $s_d^r \circ s_d^r = 0$ for all $s \in J$, $d^r \circ d^r = 0$ and thus $E^r(f)$ is a bigraded group and d^r is a differential of bidegree $(-r, r-1)$. Again, since ${}^t_s\phi^r$ is a homomorphism of spectral sequences, we have the following commutative diagram:

$$\begin{array}{ccc}
s_E^{r-1} & \xrightarrow{s_\phi^{r-1}} & t_E^{r-1} \\
||| & & ||| \\
H(s_E^r) & \xrightarrow{(s_\phi^r)_*} & H(t_E^r)
\end{array}$$

where the vertical morphisms are the isomorphisms of the respective spectral sequences. Thus

$$\lim_{s \in J} H(s_E^r) \cong \lim_{s \in J} s_E^{r-1}.$$

Hence

$$\begin{aligned}
H(E^r(f)) &= H\left(\lim_{s \in J} s_E^r\right) \\
&= \lim_{s \in J} H(s_E^r) \\
&\cong \lim_{s \in J} s_E^{r-1} \\
&= E^{r-1}(f).
\end{aligned}$$

Thus $\{E^r(f), d^r\}$ is a spectral sequence. //

We now direct our attention to the evaluation of $E^2(f)$. In order to do this we shall require various results obtained in [4]. Let $p: E \rightarrow |K|$ and $\bar{p}: \bar{E} \rightarrow |\bar{K}|$ be simplicial bundles such that ${}^n K \subset K$ and $E \subset \bar{E}$. Then there exists a chain map

$$\Phi: C_p(K; H_q(F; G)) \rightarrow C_p(\bar{K}; H_q(\bar{F}; G))$$

defined on generators by the formula

$$\Phi(\hat{\sigma} \otimes c) = \zeta(\hat{\sigma}) \otimes j(c)$$

where $\hat{\sigma}$ is a generator of $H_p(|\sigma|, |\dot{\sigma}|)$, $c \in H_q(F_{\sigma})$, $j: F_{\sigma} \subset \bar{F}_{\sigma}$ is inclusion, and $\zeta: C_p(K) \rightarrow C_p({}^n K)$ is the subdivision chain map. Let ${}^n K$ denote the n^{th} barycentric subdivision of the simplicial complex K . Let $p: E \rightarrow |K|$ be a simplicial bundle $\{E_{\sigma}, p_{\sigma}, \Phi_{\sigma}, F_{\sigma}\}_{\sigma \in K}$; then for each $\tau \in {}^n K$ there are two possibilities

(1) $\tau \in K$ i.e., τ is a vertex of K

(2) $\tau \notin K$ and therefore, $|\tau| \subset |\sigma|$ where σ is the unique simplex of K such that $|\tau| \subset |\overset{\circ}{\sigma}|$.

If (1) is the case, we have spaces and maps $E_{\tau}, p_{\tau}, \Phi_{\tau}$ and F_{τ} . If (2) is the case, we define

$$E_{\tau} = p_{\sigma}^{-1}(|\tau|) \subset E_{\sigma}$$

$$p_{\tau} = p_{\sigma}|_{E_{\tau}}: E_{\tau} \rightarrow |\tau|$$

$$F_{\tau} = F_{\sigma}$$

$$\Phi_{\tau} = \Phi_{\sigma}|_{|\tau| \times F_{\tau}}: |\tau| \times F_{\tau} \rightarrow E_{\tau}$$

Thus one obtains the family

$$\{E_{\tau}, p_{\tau}, \Phi_{\tau}, F_{\tau}\}_{\tau \in {}^n K}$$

which is called the canonical n^{th} subdivision of $p: E \rightarrow |K|$ and shall be denoted by $p: {}^n E \rightarrow |{}^n K|$. It is shown in [4] that $p: {}^n E \rightarrow |{}^n K|$

is a simplicial bundle.

Theorem 3.4. Let $s_p: E \rightarrow |_s K|$ be a sequence of simplicial bundles as given by theorem 3.2. If $s < t$ then the following diagram commutes:

$$\begin{array}{ccc}
 C_p(sK; H_q(sF; G)) & \xrightarrow{\psi} & sE_{pq}^1 \\
 \downarrow \Phi_s^t & & \downarrow t_s \phi_{pq}^1 \\
 C_p(tK; H_q(tF; G)) & \xrightarrow{\psi} & tE_{pq}^1
 \end{array}$$

Proof. In order to simplify notation let $p: E \rightarrow |K|$ denote $s_p: sE \rightarrow |_s K|$ and $\bar{p}: \bar{E} \rightarrow |\bar{K}|$ denote $t_p: tE \rightarrow |_t K|$. It clearly suffices to show that the diagram is componentwise commutative. Consider the following diagram

$$\begin{array}{ccc}
 H_p(|\sigma|, |\dot{\sigma}|) \otimes H_q(F_\sigma; G) & \xrightarrow{\mu'_\sigma} & H_{p+q}((|\sigma|, |\dot{\sigma}|) \times F_\sigma; G) \\
 \downarrow k_* \otimes 1 & & \downarrow (k \times 1)_* \\
 H_p(|\sigma|, |[{}^n \sigma]^{p-1}|) \otimes H_q(F_\sigma; G) & \xrightarrow{\mu'_\sigma} & H_{p+q}((|\sigma|, |[{}^n \sigma]^{p-1}|) \times F_\sigma; G) \\
 \uparrow \{j_{\tau_*}^\sigma\} \otimes 1 & & \uparrow \{(j_\tau^\sigma \times 1)_*\} \\
 \bigoplus_{|\tau| \subset |\sigma|} H_p(|\tau|, |\dot{\tau}|) \otimes H_q(F_\tau; G) & \xrightarrow{\oplus \mu'_\tau} & \bigoplus_{|\tau| \subset |\sigma|} H_{p+q}((|\tau|, |\dot{\tau}|) \times F_\tau; G) \\
 \downarrow \oplus (1 \otimes j_*) & & \downarrow \oplus (1 \times j)_* \\
 \bigoplus_{|\tau| \subset |\sigma|} H_p(|\tau|, |\dot{\tau}|) \otimes H_q(\bar{F}_\tau; G) & \xrightarrow{\oplus \mu'_\tau} & \bigoplus_{|\tau| \subset |\sigma|} H_{p+q}((|\tau|, |\dot{\tau}|) \times \bar{F}_\tau; G)
 \end{array}$$

Here j , j_{τ}^{σ} , and k are inclusion maps. This diagram commutes by the naturality of the homology cross product. Also $\{j_{\tau*}^{\sigma}\}^{-1} \circ k_* = \zeta$, and thus the composition

$$(\oplus 1 \otimes j) \circ (\{j_{\tau*}^{\sigma}\}^{-1} \otimes 1) \circ (k_* \otimes 1)$$

coincides with Φ_s^t .

Next consider the diagram:

$$\begin{array}{ccc}
 H_{p+q}(|\sigma|, |\dot{\sigma}|) \times F_{\sigma}; G & \xrightarrow{\Phi_{\sigma*}} & H_{p+q}(E_{\sigma}, E_{\dot{\sigma}}; G) \\
 \downarrow (k \times 1)_* & & \downarrow l_* \\
 H_{p+q}(|\sigma|, |[{}^n\sigma]^{p-1}|) \times F_{\sigma}; G & \xrightarrow{\Phi_{\sigma*}} & H_{p+q}(E_{\sigma}, p_{\sigma}^{-1}([{}^n\sigma]^{p-1}); G) \\
 \uparrow \{(j_{\tau}^{\sigma} \times 1)_*\} & & \uparrow \{l_{\tau}^{\sigma}\} \\
 |\tau| \overset{\oplus}{\subset} |\sigma| \quad H_{p+q}(|\tau|, |\dot{\tau}|) \times F_{\tau}; G & \xrightarrow{\oplus \Phi_{\tau*}} & |\tau| \overset{\oplus}{\subset} |\sigma| \quad H_{p+q}(E_{\tau}, E_{\dot{\tau}}; G) \\
 \downarrow \oplus (1 \times j)_* & & \downarrow \oplus m_* \\
 |\tau| \overset{\oplus}{\subset} |\sigma| \quad H_{p+q}(|\tau|, |\dot{\tau}|) \times \bar{F}_{\tau}; G & \xrightarrow{\oplus \bar{\Phi}_{\tau*}} & |\tau| \overset{\oplus}{\subset} |\sigma| \quad H_{p+q}(\bar{E}_{\tau}, \bar{E}_{\dot{\tau}}; G)
 \end{array}$$

Here l , l_{τ}^{σ} , and m are inclusion maps. The top two rectangles commute by functoriality, since the corresponding diagrams on the space level commute. Although the lower rectangle does not commute on

the space level, it has been shown by J. W. Smith to be homotopy commutative, which suffices. Consider the diagram:

$$\begin{array}{ccc}
 H_{p+q}(E_\sigma, E_\sigma; G) & \xrightarrow{i\sigma_*} & H_{p+q}(E_p, E_{p-1}; G) \\
 \downarrow l_* & & \downarrow n_* \\
 H_{p+q}(E_\sigma, E_\sigma^{-1}([{}^n\sigma]^{p-1}); G) & \xrightarrow{i\sigma_*} & H_{p+q}({}^nE_p, {}^nE_{p-1}; G) \\
 \uparrow \{l\tau_*^\sigma\} & & \uparrow l \\
 |\tau| \bigoplus |\sigma| H_{p+q}(E_\tau, E_\tau; G) & \xrightarrow{\{i\tau_*\}} & H_{p+q}({}^nE_p, {}^nE_{p-1}; G) \\
 \downarrow \oplus m_* & & \downarrow r_* \\
 |\tau| \bigoplus |\sigma| H_{p+q}(\bar{E}_\tau, \bar{E}_\tau; G) & \xrightarrow{\{\bar{i}\tau_*\}} & H_{p+q}(\bar{E}_p, \bar{E}_{p-1}; G)
 \end{array}$$

Here n and r are inclusion maps and the diagram commutes since it is inclusion induced.

Also

$$\begin{array}{ccc}
 H_{p+q}(E_p, E_{p-1}; G) & \cong & E_{pq}^1 \\
 \downarrow r_* \circ n_* & & \downarrow \phi_{pq}^1 \\
 H_{p+q}(\bar{E}_p, \bar{E}_{p-1}; G) & \cong & \bar{E}_{pq}^1
 \end{array}$$

where $r_* \circ n_*$ corresponds to ϕ_{pq}^1 .

From the above we conclude that the diagram

$$\begin{array}{ccc}
 C_p(K; H_q(F; G)) & \xrightarrow{\psi} & E_{pq}^1 \\
 \downarrow \Phi_s^t & & \downarrow t_s^1 \phi_{pq}^1 \\
 C_p(\bar{K}; H_q(\bar{F}; G)) & \xrightarrow{\psi} & \bar{E}_{pq}^1
 \end{array}$$

commutes. //

In order to complete the identification of $E^2(f)$, we summarize below the definition of the homology of a submersion as given in [4]. Let $f: X \rightarrow Y$ be a submersion and let \mathcal{B} denote the set of all simplicial bundles $p: E \rightarrow |K|$ such that $E \subset X$, $|K| \subset Y$, and $p = f|_E$. It is shown in [4] that \mathcal{B} forms a directed set, where $p \leq \bar{p}$ for $p, \bar{p} \in \mathcal{B}$ provided $E \subset \bar{E}$. With each simplicial bundle in this directed set there is associated its homology $H_*(K; H_*(F; G))$, and if $p \leq \bar{p}$ there exists a homomorphism

$$\Phi_*: H_*(K; H_*(F; G)) \rightarrow H_*(\bar{K}; H_*(\bar{F}; G)).$$

It is further shown in [4] that the set $\{H_*(K; H_*(F; G)); \Phi_*\}$ forms a direct system of homology groups. The homology $H_*(Y; H_*(f_y; G))$ of the submersion $f: X \rightarrow Y$ is defined to be the direct limit of this system. We observe that the sequence of simplicial bundles guaranteed by theorem 3.2 forms a cofinal set in \mathcal{B} . Thus the direct limit taken over the directed set \mathcal{B} is isomorphic to the direct limit of the

sequence ${}_s p: {}_s E \rightarrow |{}_s K|$ taken over the directed set J and hence

$$H_p(Y; H_q(f_y; G)) \cong \lim_{s \in J} H_p({}_s K; H_q({}_s F; G)).$$

As a consequence of theorem 3.4 it follows that the following diagram is commutative for $s < t$:

$$\begin{array}{ccc} H_p({}_s K; H_q({}_s F; G)) & \xrightarrow{\psi_*} & {}_s E_{pq}^2 \\ \downarrow {}_s \Phi_* & & \downarrow {}_s \phi_{pq}^2 \\ H_p({}_t K; H_q({}_t F; G)) & \xrightarrow{\psi_*} & {}_t E_{pq}^2 \end{array}$$

Thus we have that ψ_* is an isomorphism of direct systems and hence

$$\begin{aligned} E_{pq}^2(f) &= \lim_{s \in J} {}_s E_{pq}^2 \\ &\cong \lim_{s \in J} H_p({}_s K; H_q({}_s F; G)) \\ &\cong \lim_{p \in \mathcal{B}} H_p(K; H_q(F; G)) \\ &= H_p(Y; H_q(f_y; G)) \end{aligned}$$

Thus the E^2 term of the spectral sequence of a submersion is isomorphic to the homology of the submersion.

It remains to interpret the E^∞ term of the spectral sequence of

a submersion. To this end let

$$\{ {}_s p: {}_s E \rightarrow |{}_s K| \mid s \in J \}$$

be an approximating system of simplicial bundles for the submersion $f: X \rightarrow Y$ as given by theorem 3.2 and consider the following diagram:

$$\begin{array}{ccccccc} & \downarrow & & \downarrow & & \downarrow & \\ \rightarrow & F_p H_*({}_s E; G) & \xrightarrow{j_p} & F_{p+1} H_*({}_s E; G) & \rightarrow \dots \rightarrow & H_*({}_s E; G) & \\ & \downarrow i_{s*} & & \downarrow & & \downarrow & \\ \rightarrow & F_p H_*({}_{s+1} E; G) & \rightarrow & F_{p+1} H_*({}_{s+1} E; G) & \rightarrow \dots \rightarrow & H_*({}_{s+1} E; G) & \\ & \downarrow & & \downarrow & & \downarrow & \end{array}$$

The horizontal arrows are inclusions and the vertical arrows are inclusion induced and thus the diagram commutes. For every

p , $\{F_p H_*({}_s E; G); i_{s*}\}$ clearly forms a direct system indexed by J .

Define

$$F_p H_*({}_s X; G) = \lim_{s \in J} F_p H_*({}_s E; G).$$

The j_p constitute a monomorphism of direct systems and therefore determines a monomorphism

$$F_p H_*({}_s X; G) \rightarrow F_{p+1} H_*({}_s X; G)$$

of direct limits. Since ${}_s E_p = {}_s E$ for $p = \dim {}_s K$ and

$$H_*(X; G) = \varinjlim_{s \in J} H_*({}^s E; G), \quad \{F_p H_*(X; G)\}$$

constitutes an increasing filtration of $H_*(X; G)$. Each ${}^s E^\infty$ is naturally isomorphic to the bigraded group associated to the filtration $\{F_p H_*({}^s E; G)\}$ of $H_*({}^s E; G)$, more explicitly, there exists a natural isomorphism ρ

$$\frac{F_p H_{p+q}({}^s E; G)}{F_{p-1} H_{p+q}({}^s E; G)} \xrightarrow[\cong]{\rho} {}^s E_{pq}^\infty.$$

Equivalently, there is a short exact sequence

$$0 \rightarrow F_{p-1} H_{p+q}({}^s E; G) \xrightarrow{j_{p-1}} F_p H_{p+q}({}^s E; G) \xrightarrow{\beta} {}^s E_{pq}^\infty \rightarrow 0$$

where β is the composition

$$F_p H_{p+q}({}^s E; G) \rightarrow \frac{F_p H_{p+q}({}^s E; G)}{F_{p-1} H_{p+q}({}^s E; G)} \xrightarrow{\rho} {}^s E_{pq}^\infty.$$

The inclusions $i_s: {}^s E \rightarrow {}^{s+1} E$ induce a homomorphism of short exact sequences:

$$\begin{array}{ccccccc} 0 & \rightarrow & F_{p-1} H_{p+q}({}^s E; G) & \xrightarrow{j_{p-1}} & F_p H_{p+q}({}^s E; G) & \xrightarrow{\beta} & {}^s E_{pq}^\infty \rightarrow 0 \\ & & \downarrow i_{s*} & & \downarrow i_{s*} & & \downarrow i^\infty \\ 0 & \rightarrow & F_{p-1} H_{p+q}({}^{s+1} E; G) & \rightarrow & F_p H_{p+q}({}^{s+1} E; G) & \rightarrow & {}^{s+1} E_{pq}^\infty \rightarrow 0 \end{array}$$

Taking the direct limit of these short exact sequences we obtain a short exact sequence:

$$0 \rightarrow F_{p-1} H_*(X; G) \rightarrow F_p H_*(X; G) \rightarrow \lim_{s \in J} {}^s E_{pq}^\infty \rightarrow 0.$$

Since each $\{{}^s E^r, {}^s d^r\}$ is a first quadrant spectral sequence and therefore $\{E^r(f), d^r\}$ is likewise a first quadrant spectral sequence, we have ${}^s E_{pq}^\infty = {}^s E_{pq}^r$ for some $r = r(p, q)$ and thus

$$\lim_{s \in J} {}^s E_{pq}^\infty = \lim_{s \in J} {}^s E_{pq}^r = E_{pq}^r(f) = E_{pq}^\infty(f).$$

Therefore, we obtain the short exact sequence

$$0 \rightarrow F_{p-1} H_*(X; G) \rightarrow F_p H_*(X; G) \rightarrow E_{pq}^\infty(f) \rightarrow 0$$

and thus

$$E_{pq}^\infty(f) \cong \frac{F_p H_{p+q}(X; G)}{F_{p-1} H_{p+q}(X; G)}.$$

Thus we conclude that the E^∞ term of the spectral sequence of a submersion is isomorphic to the bigraded group associated to the filtration of $H_*(X; G)$ defined above. We may now summarize this chapter in the following main theorem on the spectral sequence of a submersion.

MAIN THEOREM. Let $f: X \rightarrow Y$ be a submersion and let G be an arbitrary group. There is a convergent E^1 spectral sequence

such that

$$E_{pq}^2(f) \cong H_p(Y; H_q(f_y; G))$$

and $E^\infty(f)$ isomorphic to the bigraded group associated to the filtra-
tion $FH_*(X; G)$ of $H_*(X; G)$.

BIBLIOGRAPHY

1. Ehresmann, Charles. Sur les Espaces Fibrés Différentiables. Académie des sciences, Comptes rendus hebdomadaires des séances, Paris 224:1611-1612. 1947.
2. Gordon, W. B. An Analytic Criterion for the Completeness of Riemannian Manifolds. Proceedings American Mathematical Society 37:221-225. 1973.
3. Munkres, James R. Elementary Differential Topology. Annals of Mathematics Studies. Number 54. Princeton University Press, 1966. 112 p.
4. Sekino, Junpei. Homology Theory of Submersions. Corvallis, Oregon State University, Ph.D. Thesis.
5. Smith, J. Wolfgang. Submersions of Codimension 1. Journal of Mathematics and Mechanics 18:437-444. 1968.
6. Smith, J. Wolfgang. Submersions with p-Connected Fibers. Mathematische Zeitschrift 121:288-294. 1971.
7. Smith, J. Wolfgang and Short, Donald R. A Vietoris-Begle Theorem for Submersions. Indiana Univ. Math J. 20:327-336. 1970/71.
8. Spanier, Edwin H. Algebraic Topology. New York, McGraw-Hill, 1966. 528 p.