AN ABSTRACT OF THE THESIS OF

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Title Fourier's Series for Practical Use

Abstract Approved: [redacted]
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In this thesis an attempt was made to collect, work out, and put in a general form, for practical use, a number of Fourier's series.

The investigation was limited to functions of one independent variable, and since the series were intended for practical use, all proofs of theorems were omitted. Only the theory needed to work out the results was explained.

The first part of the thesis was built around twelve series, called Fundamental Forms, consisting of six sine series, (A), and six cosine series, (B). Each series with its corresponding function was defined for a given interval. By repeated integrations new series and their functions were derived from the original forms. Forty-eight series make up this group.

The name of the next group—series derived from the fundamental forms by a change of the independent variable—tells how they were obtained. Only eighteen are listed, but a large number could be built up by the methods used there.

Several series given in miscellaneous groups were found by differentiating given series and their functions. In general, however, the derived series will not converge as rapidly as the given series; so the number obtained in this manner was small compared with the number obtained by repeated integrations.

To obtain the series representing functions defined by joined polynomial segments, a first degree function defined for a particular interval was taken. The coefficients (a_0, a_n, and b_n) were worked out, and the Fourier's series obtained. The function was then raised to higher degrees and the corresponding series found for each of the resulting functions. There are twenty series of this kind.

The miscellaneous series are listed in the following order: (1) Series for polynomial functions; (2) Series representing functions whose graphs are broken lines; (3) Series represented by hyperbolic, exponential, trigonometric, and logarithmic functions.

Drawings and graphs accompany each group.

To check the Fourier's series derived from the fundamental forms by repeated integrations, tables which show the degree of approximation of each of the series to its function were developed. These approximation tables also show how rapidly the rate of convergence increases with each successive integration. Data from these tables were used to plot the graphs found throughout the thesis.

Numerical tables, one to ten, inclusive, were worked out for the groups derived from the fundamental forms and later were used in obtaining the series for other groups. Whenever a cosine series of the above groups was integrated and the substitution of the limits made, an infinite series appeared as part of the left member of the equation. The numbers of Bernoulli and Euler (Table, number eleven) were used to obtain the numerical values of these infinite series. Formulas for deriving a few of them were found in Dwight's Tables of Integrals, but the general formulas were developed for several groups not found there. (See Numerical Tables, Number 12).

Tables of Integrals, one to eight, were obtained early, and used constantly in finding many of the series. These tables have advantages over other integral tables. For example, the powers of n are included in each term of the expression for a given integral. Also the results when n = 0, and when n is odd or even are given separately. The integral

$$\int_a^b x^k \begin{matrix}\sin\\ \cos\end{matrix}\, nx\, dx,$$

was evaluated for different intervals by assigning values to a and b and letting k and n range through all the values from 0 to 8, ... to k.

FOURIER'S SERIES FOR PRACTICAL USE

by

RUTH ELLEN PORTER

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FOURIER'S SERIES FOR PRACTICAL USE

INTRODUCTION

In this thesis an attempt was made to collect, work out, and put in a standard form for practical use a number of Fourier's Series.

A trigonometrical series $a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \ldots\ldots$ is said to be a Fourier's Series, if the constants a_0, a_1, a_2, b_1, b_2 ...... satisfy the equations

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx,$$

$$\left.\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx \end{aligned}\right\} n \geqq 1,$$

and the Fourier's Series is said to correspond to the function f(x). [The sine and cosine series being regarded as particular cases which arise when

$f(-x) = - f(x)$ (odd function), or

$f(-x) = f(x)$ (even function)

respectively.]

To determine the coefficients, let us assume the possibility of expanding f(x) throughout the interval $-\pi < x < \pi$ in a Fourier's Series, and let us also assume that the series may be integrated term by term. Let

1) $f(x) = a_0 + (a_1 \cos x + b_1 \sin x)$
$$+ (a_2 \cos 2x + b_2 \sin 2x) + \ldots\ldots$$

Integrate both sides of equation (1) with respect to x from $-\pi$ to π. The left-hand side becomes

$$\int_{-\pi}^{\pi} f(x)\, dx.$$

On the right-hand every term disappears except the first, which gives $2\pi a_0$. Hence

2) $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx.$

Next multiply both sides of equation (1) by cos nx, where n is any integer, and integrate with respect to x over the same interval. Then on the right side we have terms of the following types:

$a_0 \int \cos nx\, dx$, $a_n \int \cos^2 nx\, dx$, $a_m \int \cos mx \cos nx\, dx$,

$b_n \int \sin nx \cos nx\, dx$, $b_m \int \sin mx \cos nx\, dx$,

m being any integer except n. It is clear, that on integrating and substituting the limits $-\pi$ and π, every one of these terms will vanish except the second. It gives $a_n\pi$. The left side gives

$$\int_{-\pi}^{\pi} f(x) \cos nx\, dx,$$

therefore

3) $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx.$

Similarly, by multiplying both sides of (1) by sin nx and integrating between the same limits, it can be shown that

$$4) \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

If the interval $0 < x < 2\pi$ is taken instead of the above interval, the only difference in the formulas for the coefficients is that the limits of integration are from 0 to 2π.

In many important cases the sum of the Fourier's Series which corresponds to f(x) is equal to f(x); but if the function is arbitrary, there is no reason to assume that the series should converge at all in the interval $-\pi < x < \pi$, nor if it does converge at a point, is there any reason to assume that its sum for that value of x should be f(x). With limitations on f(x), Dirichlet gave a rigorous proof of the statement that it is possible to expand f(x) in a series of the type of equation (1). In this proof the sum of n terms of the series is taken, and it is shown that when n becomes infinitely great, the sum approaches f(x), provided that f(x) is single-valued and finite and has only a finite number of discontinuities and turning values in the interval $-\pi < x < \pi$. This gives the condition on which the expansion of f(x) in a Fourier's Series is possible. If there are no discontinuities in

$f(x)$, the series is equal to $f(x)$ between $-\pi$ and π. At a discontinuity in $f(x)$ the value of the series is the mean of the values of $f(x)$ on both sides of the discontinuity. At $-\pi$ and π the value of the series is the mean of the values of $f(x)$ at these two points.

It is not necessary that $f(x)$ should have the same mathematical expression throughout the range. For example, it may consist of several different and disconnected straight lines.

A Fourier's Series can always be integrated term by term, but cannot in general be differentiated term by term. For since a Fourier's Series converges only because the coefficients of successive terms decrease, it is obvious that differentiation must either destroy this rate of convergence or make it less rapid. It is also obvious that integration increases the rate of convergence.

This investigation was limited to functions of one independent variable; that is, to functions whose graphs may be represented by straight lines or curves, and whose integrals are represented by the areas under the curves.

Since the series here collected were intended for practical use by technical students, all proofs of the theorems were omitted. These proofs may be found in all standard texts on Fourier's Series. Only the theory needed to work out the results is explained. The writer hopes that some of these series may be of use to students of mathematical physics for solving problems of vibrating strings or particles, where the motion is simple harmonic.

Fourier's theorem shows that the composition of commensurate simple harmonic motions of suitable amplitudes and phases is able to produce a finite periodic motion of any form. The theorem also shows how to determine the amplitudes and phases of the components required to produce any given resultant. In other words, it shows how to analyze any given periodic motion, however complicated, into the simple harmonic components of which it may be made up. [For a particular example see p. 86, "Sound", by Barton.]

As far as known, no previous collection of Fourier's Series has been made. A collection of integrals, "Fourier's Integrals for Practical Application", has been compiled by Campbell and Foster of the American Telephone and Telegraph Company. They have worked on this for a period of several years, and have a much larger and better systematized collection of the integrals than was attempted to be made here for the series.

The general problem of Fourier's Series is an old one and many famous mathematicians have contributed to its development.

The question of the possibility of the expansion of an arbitrary function of x in a trigonometrical series of sines and cosines of multiples of x arose about 1750 in connection with the problem of the vibration of strings. The theory of these vibrations reduces to the solution of the differential equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}. \quad \text{(Wave equation in one dimension)}$$

Solutions in the functional form

$$y = \varphi(x + at) + \varphi(x - at)$$

were first obtained by d'Alembert and Euler.

"The principal difference between them lay in the fact that d'Alembert supposed the initial form of the string to be given by a single analytical expression, while Euler regarded it as lying along any arbitrary continuous curve, different parts of which might be given by different analytical expressions". (6:2)

Bernoulli gave the solution, when the string starts from rest, in the form of a trigonometrical series

$$y = A_1 \sin x \cos at + A_2 \sin 2x \cos at + \ldots\ldots$$

He asserted that since this solution was perfectly general that it must contain the solutions of Euler and d'Alembert.

LaGrange then entered the debate and proposed to examine the problem by means of his "finite number of particles stretched on a weightless string". A few changes in the solution of his problem would have given the development of the function in a sine series, and the coefficients would then have taken the definite integral forms of the Fourier Coefficients; but La Grange did not take this step.

About the same time (1757), however, while studying the problem of the expansion of the reciprocal of the distance between two planets expressed in a series of cosines of multiples of the angle between the radii, Clairant gave his results in a form which practically contained these coefficients. Euler in a paper published in 1793 multiplied both sides of the equation

$$f(x) = a_0 + 2a_1 \cos x + 2a_2 \cos 2x + \ldots\ldots$$

by cos nx and integrated the series term by term between the limits 0 and π, and found that

$$a_n = \frac{1}{\pi} \int_0^{\pi} f(x) \cos nx \, dx.$$

Fourier while working on the problems connected with the Mathematical Theory of the Conduction of Heat about 1807, in a number of special cases verified that a function $f(x)$ given in the interval $-\pi < x < \pi$, can be expressed as the sum of the series

$$a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \ldots$$

where a_0, a_n, and b_n are the coefficients given on page one.

He made no claim to the discovery of the values of the coefficients, but was the first to apply them to the representation of an entirely arbitrary function, in the sense in which the sum was then understood; and the first to see that, when a function is defined for a given range of the variable, its value outside that range is in no way determined.

Following Fourier, other writers on the "Theory of Heat", Poisson, Cauchy, and Dirichlet, made use of the trigonometrical series. A brief summary of the position as left by Dirichlet is as follows.

"When the function $f(x)$ is bounded in the interval $(-\pi, \pi)$, and this interval can be broken up into a finite number of partial intervals in each of which $f(x)$ is monotonic, the Fourier's Series converges at every point within the interval to

$1/2\ [f(x + 0) + f(x - 0)]$, and at the end points to
$1/2\ [f(-\pi + 0) + f(\pi - 0)]$." (6:9)

These sufficient conditions, and their extension to the unbounded function, cover most of the cases that are likely to be required in the application of Fourier's Series to the solution of the differential equations of mathematical physics.

Next we find Riemann giving his attention to this subject. His aim was to find a necessary and sufficient condition which the arbitrary function must satisfy so that, at a point x in the interval, the corresponding Fourier's Series shall converge to f(x). He was not able to solve this question. However, in the study of the problem he found that the concept of the definite integral (Cauchy's) should be widened, and the Riemann Integral we owe to his study of Fourier's Series.

A fundamental theorem proved by Riemann deals with the Fourier's Constants

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \, {\sin \atop \cos} \, nx dx.$$

He showed that for any bounded and integrable function f(x) these constants tend to zero as n tends to infinity. And this holds also for the integral

$$\int_a^b f(x) \, {\sin \atop \cos} \, nx \, dx.$$

This theorem shows that if f(x) is bounded and integrable in $(-\pi, \pi)$, the convergence of its Fourier's Series at a point in $(-\pi, \pi)$ depends only on the behavior of f(x) in the neighborhood of that point. (11:476)

In 1875 P. du BoisOReymond proved that if a trigonometrical series converges in the interval $-\pi < x < \pi$ to f(x), where f(x) is integrable, the series must be the Fourier's Series for f(x).

After the announcement by Stokes and Seidel (1847-48) of the idea of uniform convergence, the nature of the convergence of a Fourier's Series received greater attention. For some time it had been known that the series were, in general only conditionally convergent, if at all; and that their convergence depended upon the presence of positive and negative terms. At that time it was thought that term by term integration would only be permissible of the series converged uniformly. Later it was found that a Fourier Series could be integrated term by term even if the series itself did not converge.

Dini, Lipschitz, and Jordan continued the study of the convergence of the series during the '80's. They found that if a Fourier's Series for f(x) is not convergent, it may converge when one or other of the methods of summation applied to divergent series is used.

In 1904 Fejer discovered the theorem that, when the series is summed by the method of arithmetical means, its sum is

$$1/2\left[f(x+0)+f(x-0)\right]$$

at every point in $(-\pi, \pi)$ at which $f(x \pm 0)$ exist, the only condition attached to f(x) being that, if bounded, it shall be integrable in the interval $-\pi < x < \pi$, and if unbounded, that

$\int_{-\pi}^{\pi} f(x)\,dx$ shall be absolutely convergent.

Since 1905 the unity, symmetry and completeness of the theory for Fourier's Series has been improved. Much of this improvement has been due to the new definition of the definite integral given by Lebesque and published in 1902. The progress made in recent years lies in the field of the specialist; and in no department of pure mathematics has greater activity been shown than in the theory of trigonometrical series. Lebesque, Fejer, Hobson, Carslaw, Littlewood, and W. H. Young are important contributors.

Particular cases of Fourier's Series are usually studied by means of one of the two following methods.

First, the series may be given and it may be required to find the function corresponding to the series for a given interval.

Second, the function may be known for a given interval and it may be required to find a Fourier Series which will converge to the function in that interval.

The first part of this thesis is built around twelve series, called Fundamental Forms, consisting of six sine series (A) and six cosine series (B). Each series was defined for a given interval and the function corresponding to it was found. Each of the fundamental forms is a series that may be summed directly. By repeated integrations seven new series and their corresponding functions were derived (in theory) from each of the original forms. However, number four of the A forms was found to be a logarithmic function, and all of the B forms except number four were also found to be logarithmic functions. Since the integration of these functions could only be expressed as double, triple, and higher integrals, it was not thought worth while to work them out. The other six fundamental forms gave forty-two new series. The lists of these follow that of the fundamental forms. The limits of integration were usually taken from zero to x, from x to π, or from x to $\pi/2$. When a cosine series was integrated and the substitution of the limits made, it was found that an infinite series appeared as part of the left member of the equation. In many cases the numerical values of each of the infinite series was obtained by means of the numbers of Bernoulli and Euler. (See Numerical Tables, Number 11). Transposing this number to the right member of the equation gave a polynomial function for each of the series. Polynomial functions were also obtained for each of the sine series.

Formulas for several groups of these infinite series could not be found in tables. This made two integrations necessary in order to find the values of such series. For example, it was often possible to integrate from x to $\pi/2$ and make use of the values of infinite series already found or that could be derived from formulas given in

Dwight's Table of Integrals. As this integration was more difficult than that from zero to x, it was thought worth while, for future use, to work out formulas for obtaining all the infinite series needed for the integration of these forms from zero to x. Thus each Fourier Series was integrated from x to $\pi/2$. The value of the infinite series substituted into the equation gave a polynomial function for the value of the Fourier Series. Next, the same series was integrated from zero to x and a polynomial function plus the unknown infinite series obtained. These two polynomial expressions were equated to each other and the value of the unknown infinite series found. After three or four of the series were obtained, the general formula for deriving the group was written down. Numerical Tables, Number 12, contains the infinite series found in this way.

To check the Fourier Series derived from the fundamental forms by repeated integrations, tables which show the degree of approximation of each of the series to its corresponding function were developed. These approximation tables also show how rapidly the rate of convergence increases with each successive integration. Data from these tables were used to plot a graph for each of the first four functions of each group. (The graphs appear on the page opposite each group of series). The graphs of functions five and seven are practically the same as that of three, while those of six and eight are about like that of four. For that reason the last four were not plotted.

The name of the next group - Series derived from the fundamental forms by a change of the independent variable - explains how this group was obtained. While only eighteen different series are listed in the group, an infinite number could be built up by the methods explained there. Series 17 is often used in Applied Mathematics, for instance in the problem of the plucked string. (See Lord Rayleigh's Theory of Sound, Art. 127).

New series may also be found from given Fourier Series by differentiation. However, in general, the derived series will not converge as rapidly as the given series. For example, take equation 8, Form I A. Differentiate this equation with respect to x, subtract the value of the infinite series, $\sum_{n=1}^{\infty} \frac{1}{n^8}$, from the right member of the equation, reduce to lowest terms, divide both members of the equation by minus one, and obtain series 7 of the group. Similarly, by repeated differentiations and the subtraction of the numerical values for the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$, from the right members of the equations which are equal to the cosine series, we obtain each of the equations in succession and finally arrive at equation 1 of the group.

Several series listed in miscellaneous groups were found by differentiating given series and functions. The number obtained in this manner, however, is small compared with the number formed by repeated integrations. But a large number of new series might be built up by this method, if we were not interested in obtaining rapidly converging series. (After differentiating a few times, the series may diverge).

In the example given above, subtracting the numerical values of the infinite series each time kept the coefficients small, and the rate of convergence of the series while less rapid than the original series in each case, is not destroyed. Since differentiating causes the coefficients to become larger, in general, it either destroys the rate of convergence or makes it less rapid.

All the rest of the series listed were formed by method two. In the first group - Series representing functions defined by polynomial segments - the functions were joined functions. That is, they consisted of a certain algebraic expression for one part of an interval and the same or a different function (depending on whether the series to be found was to be a cosine or sine series) for the other part of the interval. For example, a first degree function, defined for a particular interval, was taken. The coefficients, a_0, a_n, and b_n (See page one of introduction) were worked out and the series obtained.

The function was then raised to the second, third, and higher degrees and the Fourier's Series found for each of the resulting functions. In general, as the degree of the function increased, if three or four or more of its derivatives were continuous in the interval, the Fourier series converged rapidly to the value of the function. Six sine series (A), six cosine series (B), and eight cosine series (C) with the functions from which each was derived make up the first group. The approximation tables show that as the degree of the function increases, the rate of convergence of the series to the function also increases quite rapidly. A graph of each of the functions (A) and (B) is found on the page opposite the list of series.

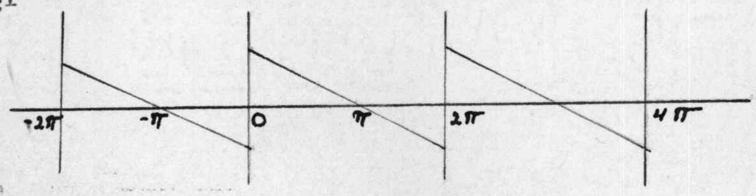
Integral tables 1-8 were worked out and used in integrating the functions of this and following groups. The same series (differing only in coefficients) occurred so often that Numerical Tables, No. 4 - No. 10 were made to save time in working out the approximations.

The remaining series are listed in the following order: miscellaneous series whose functions are polynomial and broken line segments, and miscellaneous series whose functions represent hyperbolic, exponential, trigonometric, and logarithmic functions. Drawings and graphs accompany each group.

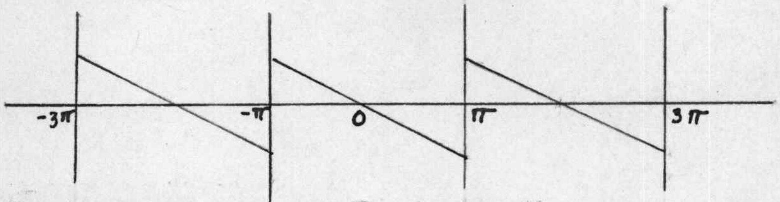
FUNDAMENTAL FORMS - A

Series which can be summed directly.

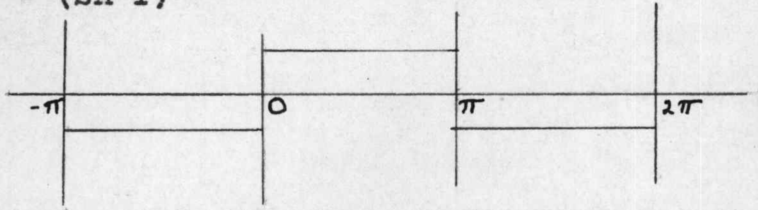
I A. $\sum_{n=1}^{\infty} \frac{1}{n} \sin nx = 1/2(\pi - x), \quad 0 < x < \pi.$



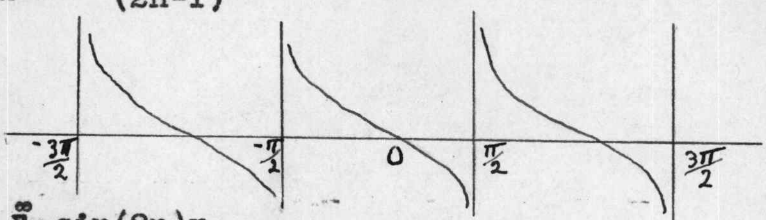
II A. $\sum_{n=1}^{\infty} \frac{(-1)^n \sin nx}{n} = -\,1/2x, \quad 0 < x < \pi$



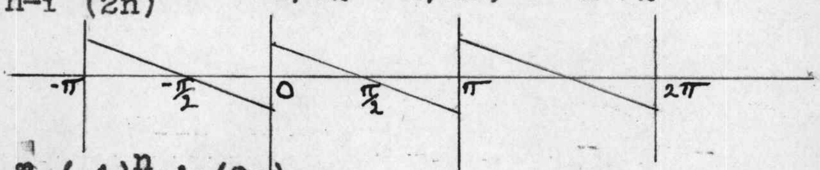
III A. $\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)} = 1/4\pi, \quad 0 < x < \pi$



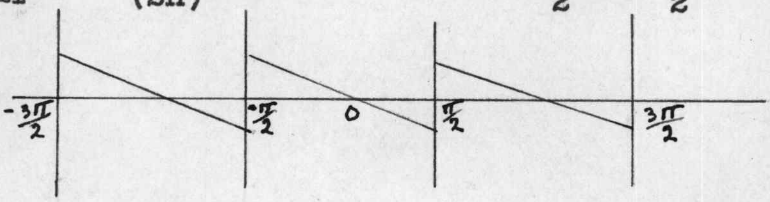
IV A. $\sum_{n=1}^{\infty} \frac{(-1)^n \sin(2n-1)x}{(2n-1)} = 1/2 \log(\sec x + \tan x), \; -\frac{\pi}{2} < x < \frac{\pi}{2}$



V A. $\sum_{n=1}^{\infty} \frac{\sin(2n)x}{(2n)} = 1/4\pi - 1/2x, \quad 0 < x < \pi.$



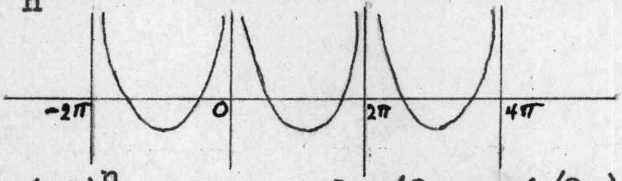
VI A. $\sum_{n=1}^{\infty} \frac{(-1)^n \sin(2n)x}{(2n)} = -\,1/2\, x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$



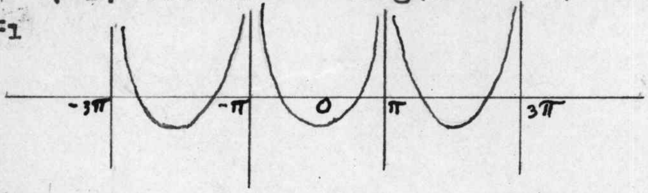
FUNDAMENTAL FORMS - B

Series which can be summed directly.

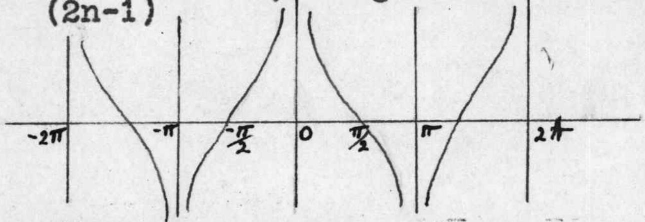
I B. $\sum_{n=1}^{\infty} \frac{1}{n} \cos nx = - \log(2 \sin 1/2\, x),\ 0 < x < \pi.$



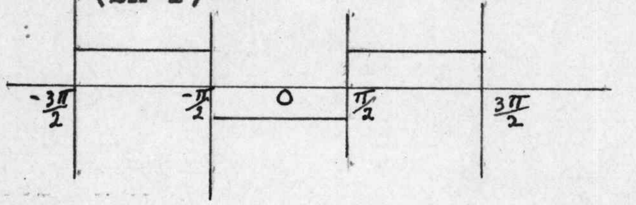
II B. $\sum_{n=1}^{\infty} (-1)^n \cos nx = - \log(2 \cos 1/2x),\ 0 < x < \pi$



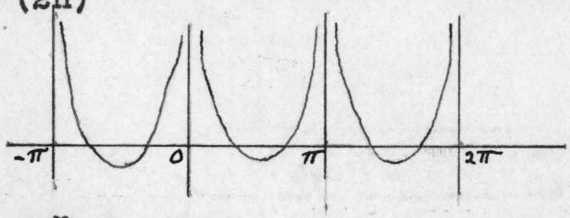
III B. $\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)} = 1/2 \log(\cot 1/2\, x),\ 0 < x < \pi$



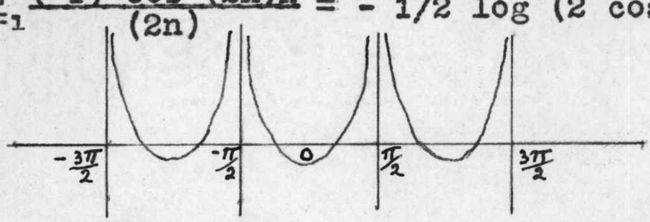
IV B. $\sum_{n=1}^{\infty} \frac{(-1)^n \cos(2n-1)x}{(2n-1)} = - 1/4\pi,\ -\pi/2 < x < \pi/2.$



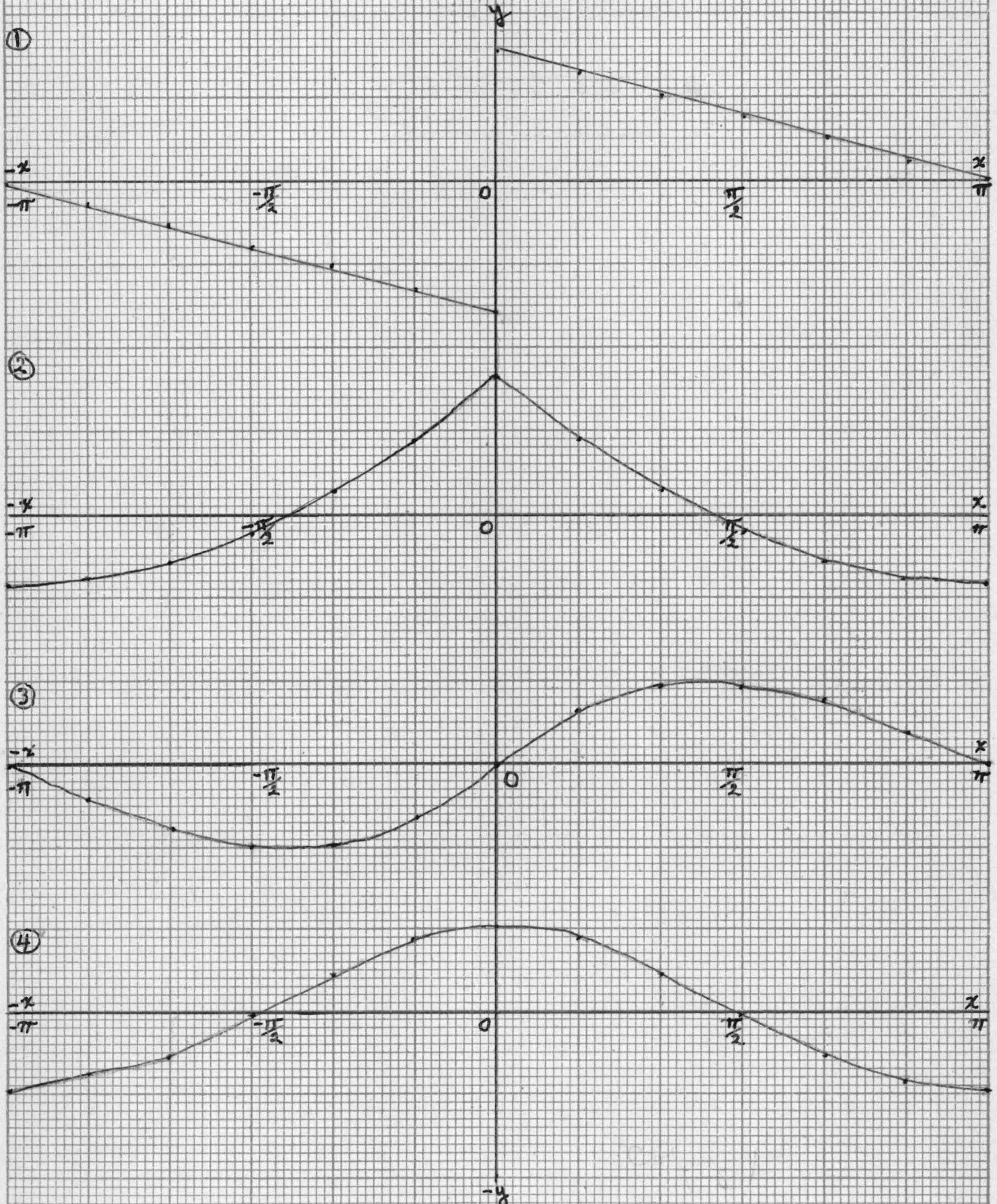
V B. $\sum_{n=1}^{\infty} \frac{\cos(2n)x}{(2n)} = - 1/2 \log(2 \sin x),\ 0 < x < \pi$



VI B. $\sum_{n=1}^{\infty} \frac{(-1)^n \cos(2n)x}{(2n)} = - 1/2 \log(2 \cos x),\ -\frac{\pi}{2} < x < \pi$



Graphs of Functions for I A



Graphs of ⑤ and ⑦ are practically the same as ③, and graphs of ⑥ and ⑧ are about the same as ④.

Opp. P. 19

LIST OF FOURIER SERIES OF TYPE $\sum_{n=1}^{\infty} \frac{1}{n^k} \begin{matrix} \sin nx \\ \cos nx \end{matrix}$

I A.

1. $\sum_{n=1}^{\infty} \frac{1}{n} \sin nx = 1/2(\pi - x), \quad 0 < x < \pi.$

2. $\sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx = \frac{1}{12} (2\pi^2 - 6\pi x + 3x^2)$

3. $\sum_{n=1}^{\infty} \frac{1}{n^3} \sin nx = \frac{1}{12} (2\pi^2 x - 3\pi x^2 + x^3)$

4. $\sum_{n=1}^{\infty} \frac{1}{n^4} \cos nx = \frac{1}{720}(8\pi^4 - 60\pi^2 x^2 + 60\pi x^3 - 15x^4)$

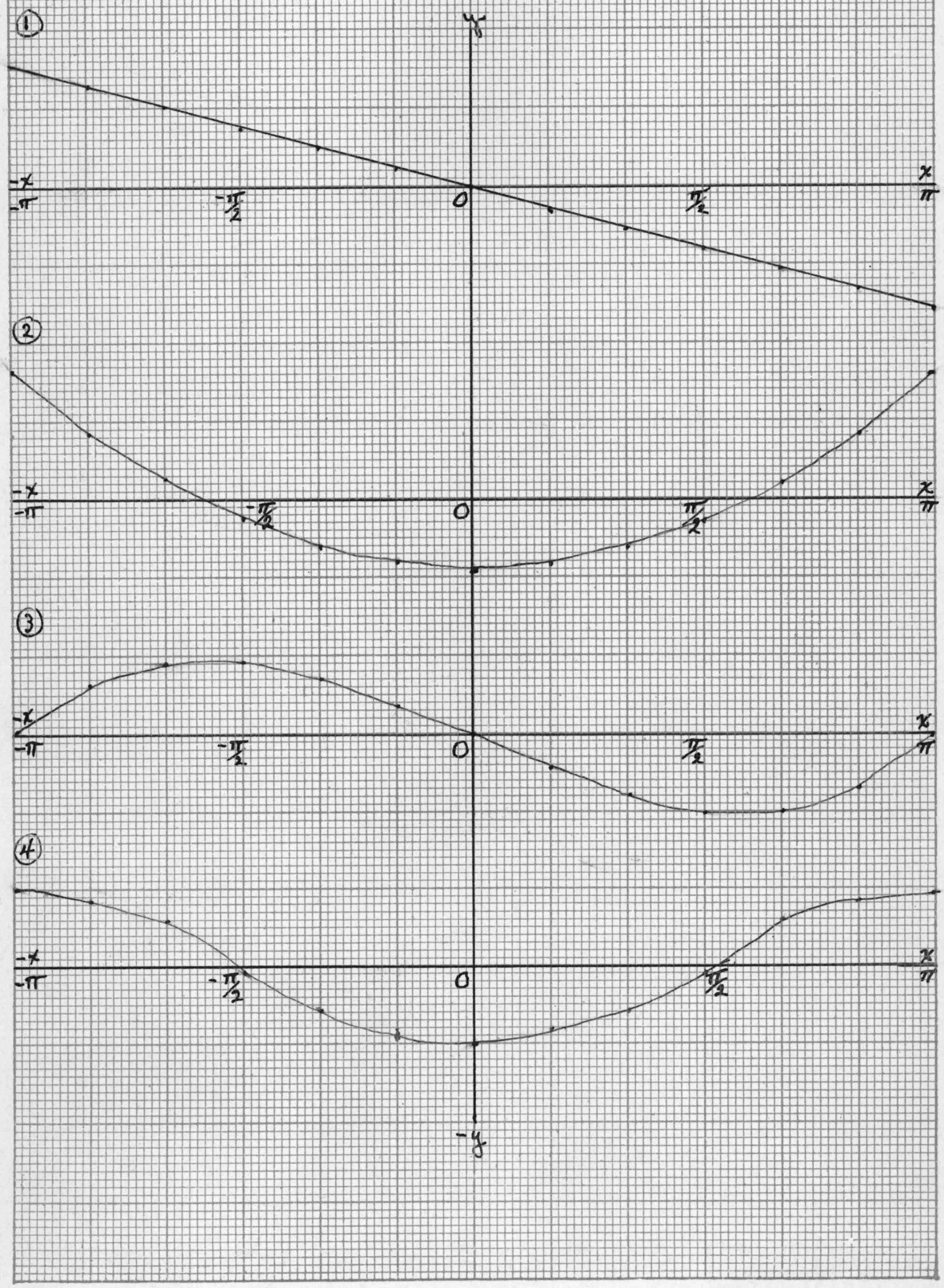
5. $\sum_{n=1}^{\infty} \frac{1}{n^5} \sin nx = \frac{1}{720}(8\pi^4 x - 20\pi^2 x^3 + 15\pi x^4 - 3x^5)$

6. $\sum_{n=1}^{\infty} \frac{1}{n^6} \cos nx = \frac{(32\pi^6 - 168\pi^4 x^2 + 210\pi^2 x^4 - 126\pi x^5 + 21x^6}{30240}$

7. $\sum_{n=1}^{\infty} \frac{1}{n^7} \sin nx = \frac{(32\pi^6 x - 56\pi^4 x^3 + 42\pi^2 x^5 - 21\pi x^6 + 3x^7).}{30240}$

8. $\sum_{n=1}^{\infty} \frac{1}{n^8} \cos nx = \frac{(128\pi^8 - 640\pi^6 x^2 + 560\pi^4 x^4 - 280\pi^2 x^6 + 120\pi x^7 - 15x^8}{1,209,600}$

Graphs of Functions for II A



Opp. P. 20

LIST OF FOURIER SERIES OF TYPE $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^k} \begin{matrix}\sin nx\\ \cos nx\end{matrix}$

II A.

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx = -\frac{x}{2}, \quad 0 < x < \pi$

2. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx = \frac{3x^2 - \pi^2}{12}$

3. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin nx = \frac{x^3 - \pi^2 x}{12}$

4. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \cos nx = \frac{-7\pi^4 + 30\pi^2 x^2 - 15x^4}{720}$

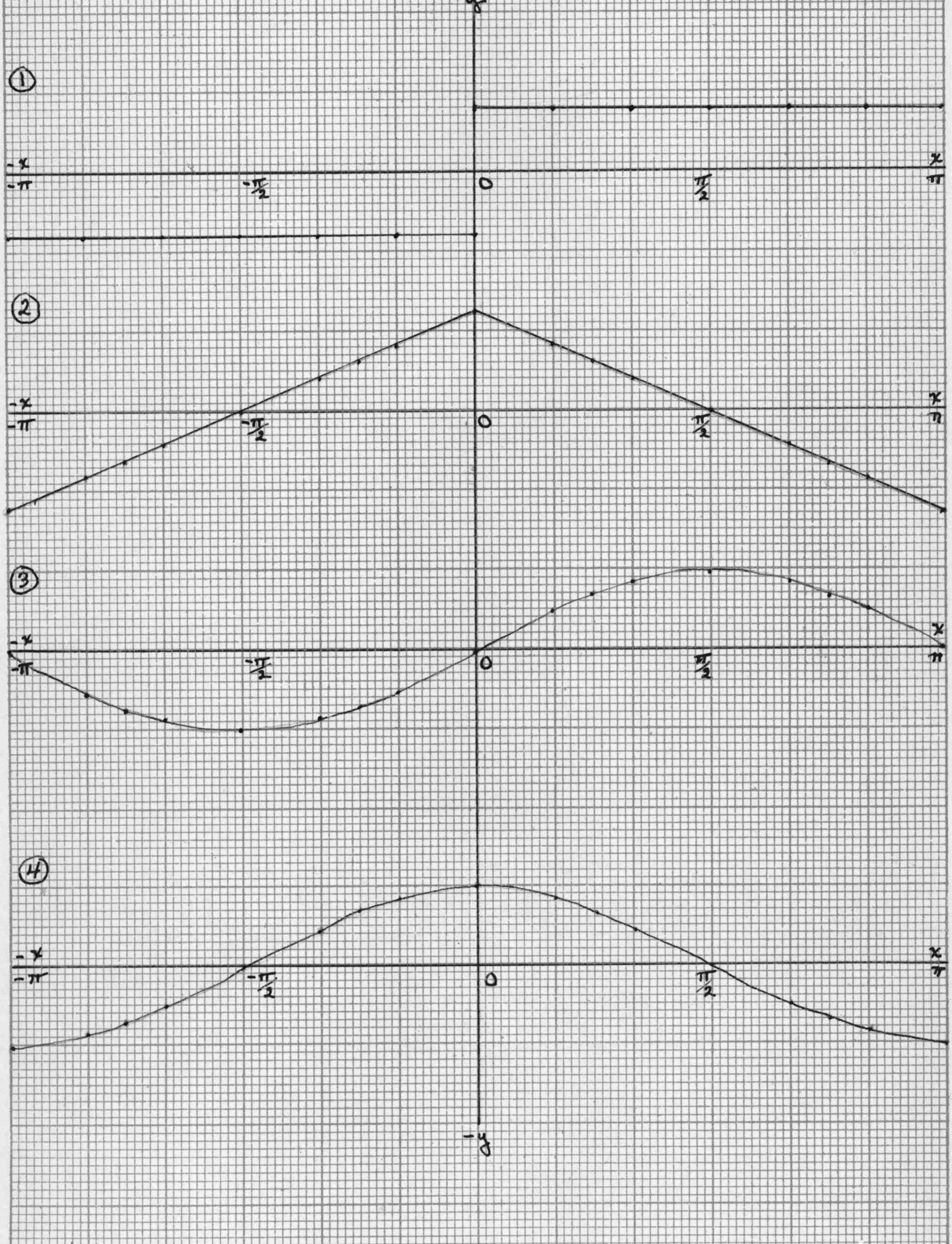
5. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5} \sin nx = \frac{-7\pi^4 x + 10\pi^2 x^3 - 3x^5}{720}$

6. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^6} \cos nx = \frac{-31\pi^6 + 147\pi^4 x^2 - 105\pi^2 x^4 + 21x^6}{30240}$

7. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^7} \sin nx = \frac{-31\pi^6 x + 49\pi^4 x^3 - 21\pi^2 x^5 + 3x^7}{30240}$

8. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^8} \cos nx = \frac{-127\pi^8 + 620\pi^6 x^2 - 490\pi^4 x^4 + 140\pi^2 x^6 - 15x^8}{1,209,600}$

Graphs of Functions for III A



Opp. P. 21

LIST OF FOURIER SERIES OF TYPE $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^k} \begin{matrix} \sin(2n-1)x \\ \cos(2n-1)x \end{matrix}$

III A.

1. $$\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)} = \frac{\pi}{4}, \quad 0 < x < \pi$$

2. $$\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2} = \frac{\pi^2 - 2\pi x}{8}$$

3. $$\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^3} = \frac{\pi^2 x - \pi x^2}{8}$$

4. $$\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^4} = \frac{\pi^4 - 6\pi^2 x^2 + 4\pi x^3}{96}$$

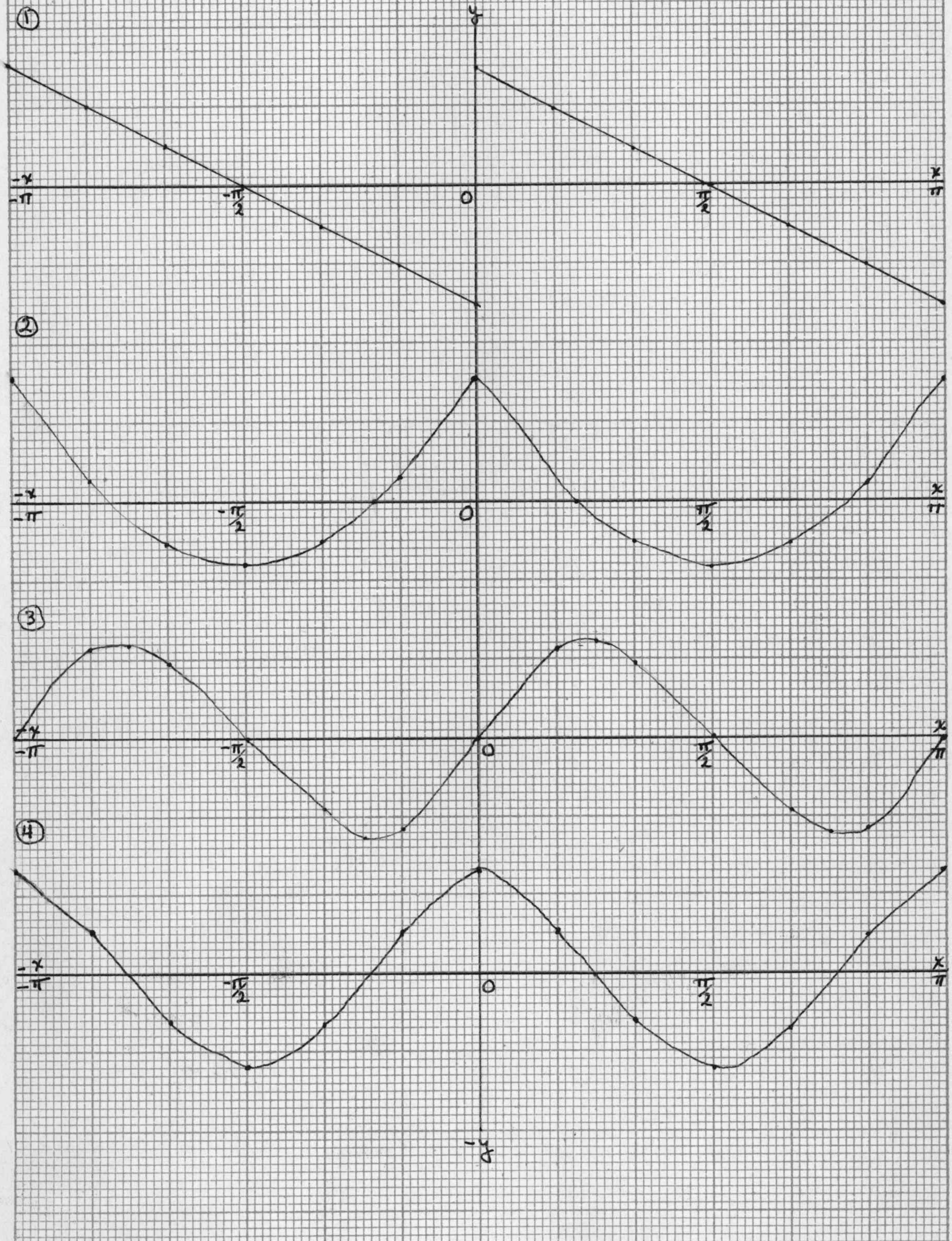
5. $$\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^5} = \frac{\pi^4 x - 2\pi^2 x^3 + \pi x^4}{96}$$

6. $$\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^6} = \frac{\pi^6 - 5\pi^4 x^2 + 5\pi^2 x^4 - 2\pi x^5}{960}$$

7. $$\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^7} = \frac{3\pi^6 x - 5\pi^4 x^3 + 3\pi^2 x^5 - \pi x^6}{2880}$$

8. $$\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^8} = \frac{17\pi^8 - 84\pi^6 x^2 + 70\pi^4 x^4 - 28\pi^2 x^6 + 8\pi x^7}{161,280}$$

Graphs of Functions for V A



Opp. P. 22

LIST OF FOURIER SERIES OF TYPE $\sum_{n=1}^{\infty} \frac{1}{(2n)^k} \begin{matrix} \sin(2n)x \\ \cos(2n)x \end{matrix}$

V A.

1. $\sum_{n=1}^{\infty} \frac{\sin(2n)x}{(2n)} = \frac{\pi}{4} - \frac{x}{2}, \quad 0 < x < \pi$

2. $\sum_{n=1}^{\infty} \frac{\cos(2n)x}{(2n)^2} = \frac{\pi^2-6\pi x+6x^2}{24}$

3. $\sum_{n=1}^{\infty} \frac{\sin(2n)x}{(2n)^3} = \frac{\pi^2 x-3\pi x^2+2x^3}{24}$

4. $\sum_{n=1}^{\infty} \frac{\cos(2n)x}{(2n)^4} = \frac{\pi^4-30\pi^2x^2+60\pi x^3-30x^4}{1440}$

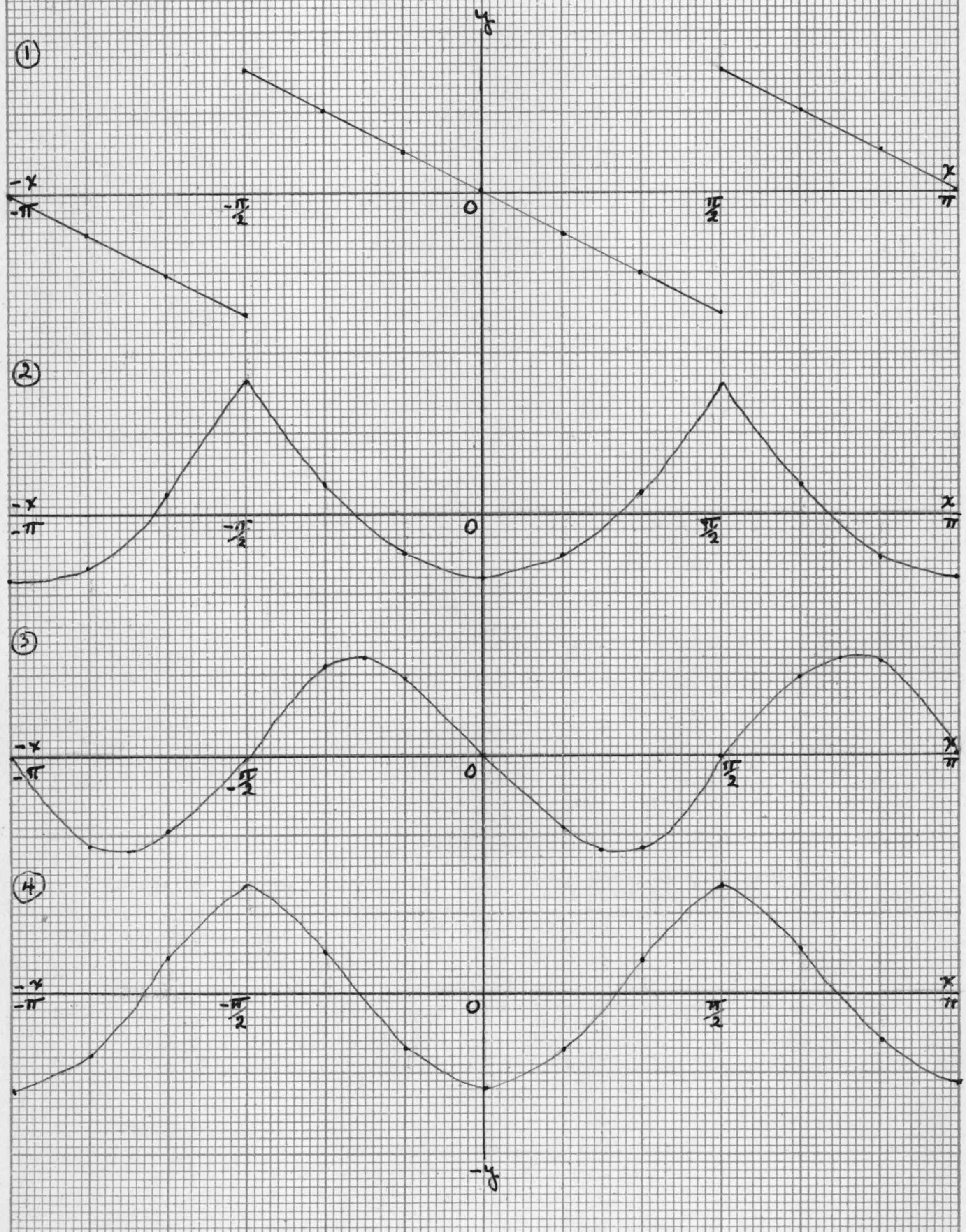
5. $\sum_{n=1}^{\infty} \frac{\sin(2n)x}{(2n)^5} = \frac{\pi^4x-10\pi^2x^3+15\pi x^4-6x^5}{1440}$

6. $\sum_{n=1}^{\infty} \frac{\cos(2n)x}{(2n)^6} = \frac{\pi^6-21\pi^4x^2+105\pi^2x^4-126\pi x^5+42x^6}{60,480}$

7. $\sum_{n=1}^{\infty} \frac{\sin(2n)x}{(2n)^7} = \frac{\pi^6x-7\pi^4x^3+21\pi^2x^5-21\pi x^6+6x^7}{60,480}$

8. $\sum_{n=1}^{\infty} \frac{\cos(2n)x}{(2n)^8} = \frac{\pi^8-20\pi^6x^2+70\pi^4x^4-140\pi^2x^6+120\pi x^7-30x^8}{2,419,200}$

Graphs of Functions for VI A



Opp. P. 23

LIST OF FOURIER SERIES OF TYPE $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)^k} \begin{matrix}\sin(2n)x \\ \cos(2n)x\end{matrix}$

VI A.

1. $\sum_{n=1}^{\infty} \frac{(-1)^n \sin(2n)x}{(2n)} = \frac{-x}{2}, \ -\frac{\pi}{2} < x < \frac{\pi}{2}$

2. $\sum_{n=1}^{\infty} \frac{(-1)^n \cos(2n)x}{(2n)^2} = \frac{-\pi^2+12x^2}{48}$

3. $\sum_{n=1}^{\infty} \frac{(-1)^n \sin(2n)x}{(2n)^3} = \frac{-\pi^2 x+4x^3}{48}$

4. $\sum_{n=1}^{\infty} \frac{(-1)^n \cos(2n)x}{(2n)^4} = \frac{-7\pi^4+120\pi^2x^2-240x^4}{11520}$

5. $\sum_{n=1}^{\infty} \frac{(-1)^n \sin(2n)x}{(2n)^5} = \frac{-7\pi^4x+40\pi^2x^3-48x^5}{11520}$

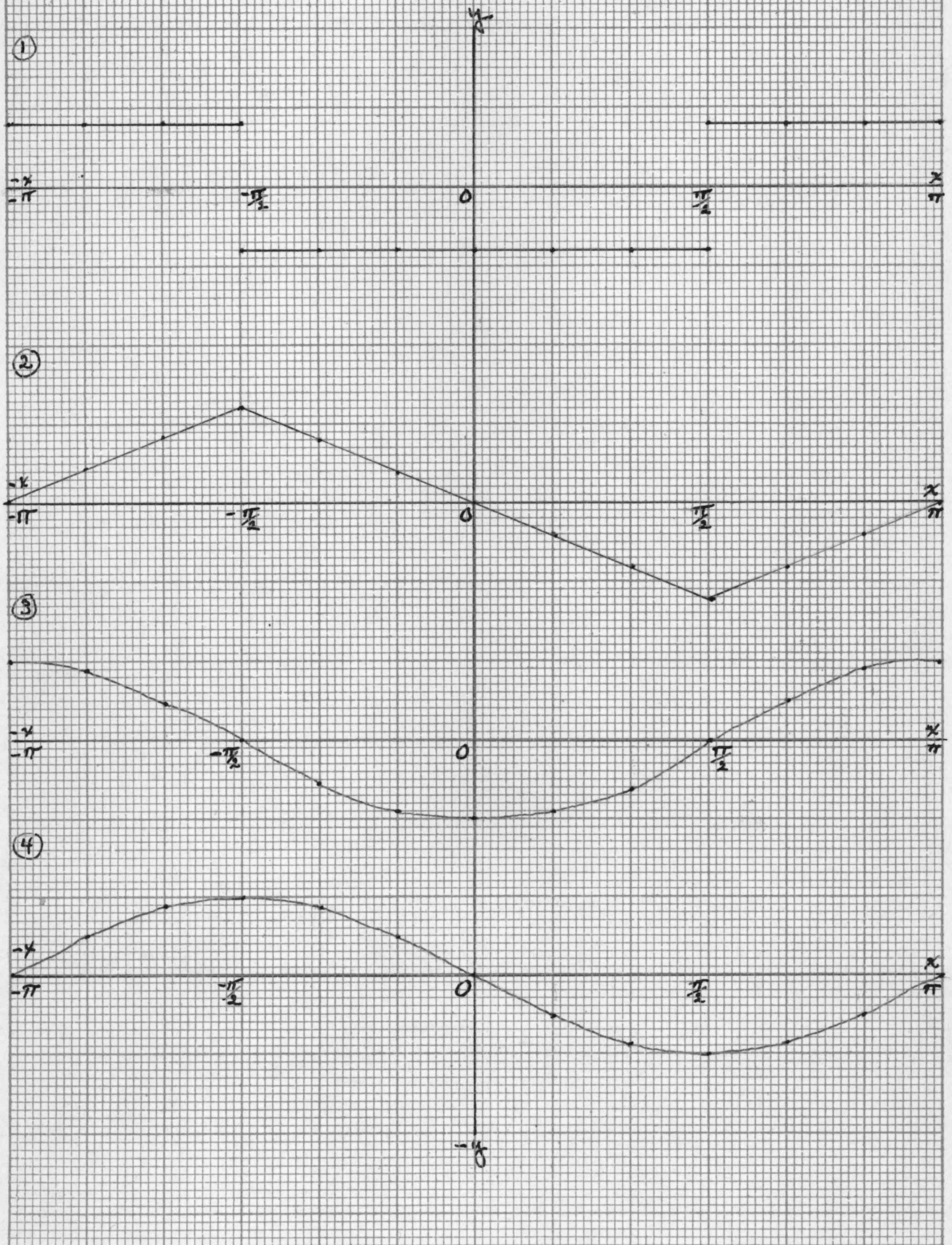
6. $\sum_{n=1}^{\infty} \frac{(-1)^n \cos(2n)x}{(2n)^6} = \frac{-31\pi^6+588\pi^4x^2-1680\pi^2x^4+1344x^6}{1,935,360}$

7. $\sum_{n=1}^{\infty} \frac{(-1)^n \sin(2n)x}{(2n)^7} = \frac{-31\pi^6x+196\pi^4x^3-336\pi^2x^5+192x^7}{1,935,360}$

8. $\sum_{n=1}^{\infty} \frac{(-1)^n \cos(2n)x}{(2n)^8} =$

$$\frac{-127\pi^8+2480\pi^6x^2-7840\pi^4x^4+8960\pi^2x^6-3840x^8}{309,657,600}$$

Graphs of Functions for IV B



Opp. p. 24

LIST OF FOURIER SERIES OF TYPE $\sum_{1}^{\infty} \frac{(-1)^n}{(2n-1)^k} \begin{matrix}\cos(2n-1)x \\ \sin(2n-1)x\end{matrix}$

IV B.

1. $\sum_{1}^{\infty} \frac{(-1)^n \cos(2n-1)x}{(2n-1)} = -\frac{\pi}{4}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

2. $\sum_{1}^{\infty} \frac{(-1)^n \sin(2n-1)x}{(2n-1)^2} = -\frac{\pi x}{4}$

3. $\sum_{1}^{\infty} \frac{(-1)^n \cos(2n-1)x}{(2n-1)^3} = \frac{-\pi^3+4\pi x^2}{32}$

4. $\sum_{1}^{\infty} \frac{(-1)^n \sin(2n-1)x}{(2n-1)^4} = \frac{-3\pi^3 x+4\pi x^3}{96}$

5. $\sum_{1}^{\infty} \frac{(-1)^n \cos(2n-1)x}{(2n-1)^5} = \frac{-5\pi^5+24\pi^3 x^2-16\pi x^4}{1536}$

6. $\sum_{1}^{\infty} \frac{(-1)^n \sin(2n-1)x}{(2n-1)^6} = \frac{-25\pi^5 x+40\pi^3 x^3-16\pi x^5}{7680}$

7. $\sum_{1}^{\infty} \frac{(-1)^n \cos(2n-1)x}{(2n-1)^7} = \frac{-61\pi^7+300\pi^5 x^2-240\pi^3 x^4+64\pi x^6}{184,320}$

8. $\sum_{1}^{\infty} \frac{(-1)^n \sin(2n-1)x}{(2n-1)^8} =$

$$\frac{-427\pi^7 x+700\pi^5 x^3-336\pi^3 x^5+64\pi x^7}{1,290,240}$$

FOURIER'S SERIES DERIVED FROM THE FUNDAMENTAL FORMS, A AND B, BY A CHANGE OF THE INDEPENDENT VARIABLE

I. Fourier's Series derived from I B

1. $$\sum_1^\infty \frac{\cos n(x-\alpha)}{n} = -\frac{1}{2}\log\left[4\sin^2\frac{1}{2}(x-\alpha)\right],\quad 0<x<2\pi$$

2. $$\sum_1^\infty \frac{\cos n(x+\alpha)}{n} = -\frac{1}{2}\log\left[4\sin^2\frac{1}{2}(x+\alpha)\right],\quad 0<x<2\pi$$

Valid for all values of x, except $x = \pm\alpha,\ 2\pi\pm\alpha, \ldots.$
Now add and subtract 1 and 2 and obtain

3. $$\sum_1^\infty \frac{\cos nx\cos n\alpha}{n} = -\frac{1}{4}\log\left[4(\cos x-\cos\alpha)^2\right],\quad 0<x<2\pi,$$

4. $$\sum_1^\infty \frac{\sin nx\sin n\alpha}{n} = +\frac{1}{4}\log\left[\frac{\sin^2\frac{1}{2}(x+\alpha)}{\sin^2\frac{1}{2}(x-\alpha)}\right],\quad 0<x<2\pi$$

Valid for all values of x except $x = \pm\alpha,\ 2\pi\pm\alpha, \ldots..$

II. Series derived from II B

5. $$\sum_1^\infty \frac{(-1)^n\cos nx\cos n\alpha}{n} = -\frac{1}{4}\log\left[4(\cos x+\cos\alpha)^2\right],\quad 0<x<2\pi$$

6. $$\sum_1^\infty \frac{(-1)^n\cos nx\cos n\alpha}{n} = -\frac{1}{4}\log\left[\frac{\cos^2\frac{1}{2}(x-\alpha)}{\cos^2\frac{1}{2}(x+\alpha)}\right],\quad 0<x<2\pi$$

Valid except for $x = \pi\pm\alpha,\ 3\pi\pm\alpha, \ldots..$

III. Fourier's Series derived from I A and II A.

7. $$\sum_1^\infty \frac{\sin n(x+\alpha)}{n} = +\frac{1}{2}\left[\pi-(x+\alpha)\right],\quad 0<x+\alpha<2\pi,\quad 0<\alpha<\pi$$

8. $$\sum_1^\infty \frac{\sin n(x-\alpha)}{n} = \frac{1}{2}\left[\pi-(x-\alpha)\right],\quad 0<x-\alpha<2\pi,\quad 0<\alpha<\pi$$

9. $$\sum_1^\infty \frac{(-1)^n\sin n(x+\alpha)}{n} = -\left(\frac{x+\alpha}{2}\right),\quad 0<x<\pi$$

10. $\sum_{1}^{\infty} \frac{(-1)^n \sin n(x-\alpha)}{n} = -\left(\frac{x-\alpha}{2}\right),\ 0 < x < \pi$

*11. $\sum_{1}^{\infty} \frac{\sin nx \cos n\alpha}{n} = f(x),$

*12. $\sum_{1}^{\infty} \frac{\cos nx \sin n\alpha}{n} = g(x),$

Where $\left.\begin{array}{l} f(x) = -\frac{1}{2}x \\ g(x) = \frac{1}{2}(\pi-\alpha) \end{array}\right\}$ if $0 < x < \alpha$ $\quad f(x) = \frac{1}{2}(\pi-x)$ and $g(x) = \frac{1}{2}\alpha$ if $\alpha < x < \pi$

$f(\alpha) = g(\alpha) = \frac{1}{4}(\pi-2\alpha)$

The values outside the range $x = 0$ to $x = \pi$ are given by the relations

$$f(-x) = -f(x), \quad g(-x) = g(x).$$

* Formulas 11 and 12 are very important. By assigning various values to α special series may be obtained. For example, when $\alpha = \pi$, 11 becomes

13. $\sum_{1}^{\infty} \frac{(-1)^n \sin nx}{n} = f(x) = -\frac{1}{2}\pi,\ -\pi < x < \pi$ Fundamental Form II A.

Similarly when $\alpha = \frac{\pi}{2}$, 12 takes the form

14. $\sum_{1}^{\infty} \frac{(-1)^n \cos(2n-1)x}{(2n-1)} = -gx = -\frac{1}{4}\pi,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$ Fundamental Form IV B.

If we integrate 11 and determine the constant of integration we get

15. $\sum_{1}^{\infty} \frac{\cos nx \cos n\alpha}{n^2} = \frac{1}{4}x^2 + \frac{1}{4}(\alpha-\pi)^2 - \frac{1}{12}\pi^2$, if $0 \leqq x \leqq \alpha$

or $\qquad \frac{1}{4}\alpha^2 + \frac{1}{4}(x-\pi)^2 - \frac{1}{12}\pi^2$, if $\alpha \leqq x \leqq \pi$.

These results remain valid from $-\alpha$ to $+\alpha$ and from $+\alpha$ to $2\pi-\alpha$ respectively.

Similarly we find

16. $\sum_{1}^{\infty} \frac{(-1)^n \cos nx \cos n\alpha}{n^2} =$

$$- \frac{1}{12}\pi^2 + \frac{1}{4}(\alpha^2+x^2), \quad -(\pi-\alpha) \leqq x \leqq (x-\alpha)$$

$$- \frac{1}{12}\pi^2 + \frac{1}{4} \quad (\alpha-x)^2 + (x-\pi)^2 \quad , (\pi-\alpha) \leqq x \leqq (x+\alpha)$$

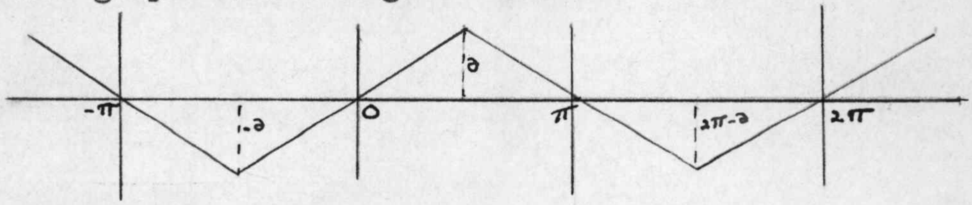
By integrating similarly the series of 12 and determining the constant of integration, we find that when (0 α π) 12 gives

*17. $\sum_{1}^{\infty} \frac{\sin nx \sin n\alpha}{n^2} = \frac{1}{2}x(\pi-\alpha), \quad -\alpha \leqq x \leqq \alpha$

or $\quad \frac{1}{2}\alpha(\pi-x), \quad \alpha \leqq x \leqq 2\pi-\alpha$

* Series 17 is often used in Applied Mathematics, for instance in the problem of the plucked string. (Lord Rayleigh's Theory of Sound, Art. 127).

The graph of 17 is given below.



The series

18. $\sum_{1}^{\infty} \frac{(-1)^n \cos(2n-1)x}{(2n-1)^3} = -\frac{1}{8}\pi(\frac{\pi^2}{4}-x^2), \quad -\frac{1}{2}\pi \leqq x \leqq \frac{1}{2}\pi$ IV B3

can be summed by writing $(\frac{1}{2}\pi-x)$ for x in

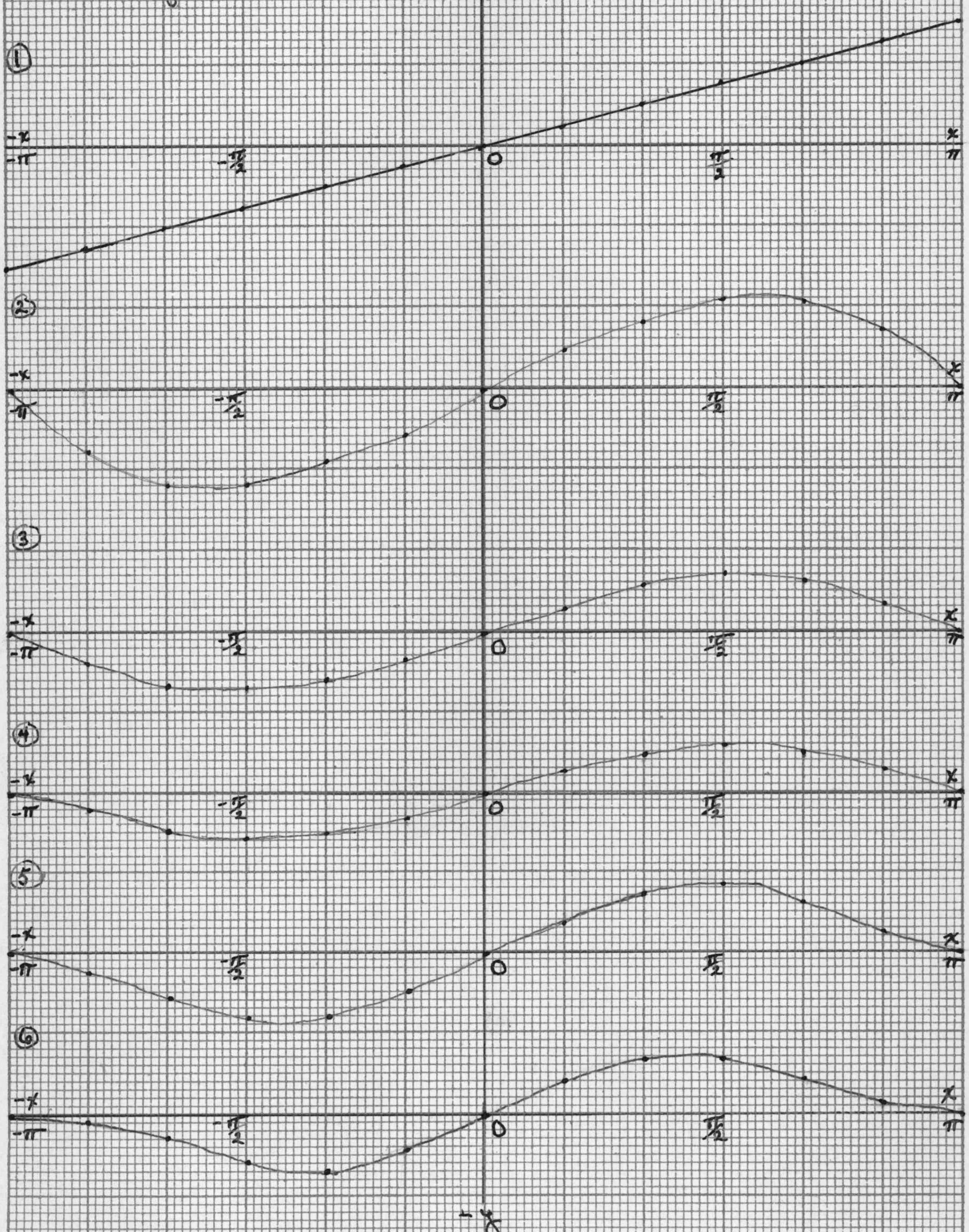
$$\sum_{1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^3} = \frac{1}{8}(\pi^2 x-\pi x^2), \quad 0 \leqq x \leqq \pi$$

III A 3

$$\frac{1}{8}(\pi^2 x+\pi x^2), \quad -\pi \leqq x \leqq 0.$$

Graphs for Functions Representing Polynomial Segments

A. Sine Graphs



Opp. P. 28

SERIES REPRESENTING FUNCTIONS DEFINED BY POLYNOMIAL SEGMENTS

A. Sine Series of Type $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2k-1}} \sin nx$.

1. $f(x) = x, \; 0 < x < \pi,$

$\quad = -x, \; -\pi < x < 0.$

$$S = -2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx, \; -\pi < x < \pi.$$

2. $f(x) = x(\pi^2 - x^2), \; 0 < x < \pi,$

$\quad = -x(\pi^2 - x^2), \; -\pi < x < 0.$

$$S = -12 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin nx, \; -\pi < x < \pi.$$

3. $f(x) = 7\pi^4 x - 10\pi^2 x^3 + 3x^5, \; 0 < x < \pi$

$\quad = -7\pi^4 x + 10\pi^2 x^3 - 3x^5, \; -\pi < x < 0.$

$$S = -720 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^5} \sin nx, \; -\pi < x < \pi$$

4. $f(x) = 3x^7 - 21\pi^2 x^5 + 49\pi^4 x^3 - 31\pi^6 x, \; 0 < x < \pi$

$\quad = -3x^7 + 21\pi^2 x^5 - 49\pi^4 x^3 + 31\pi^6 x, \; -\pi < x < 0$

$$S = 30240 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^7} \sin nx, \; -\pi < x < \pi$$

5. $f(x) = x(\pi^2 - x^2)^2, \; 0 < x < \pi,$

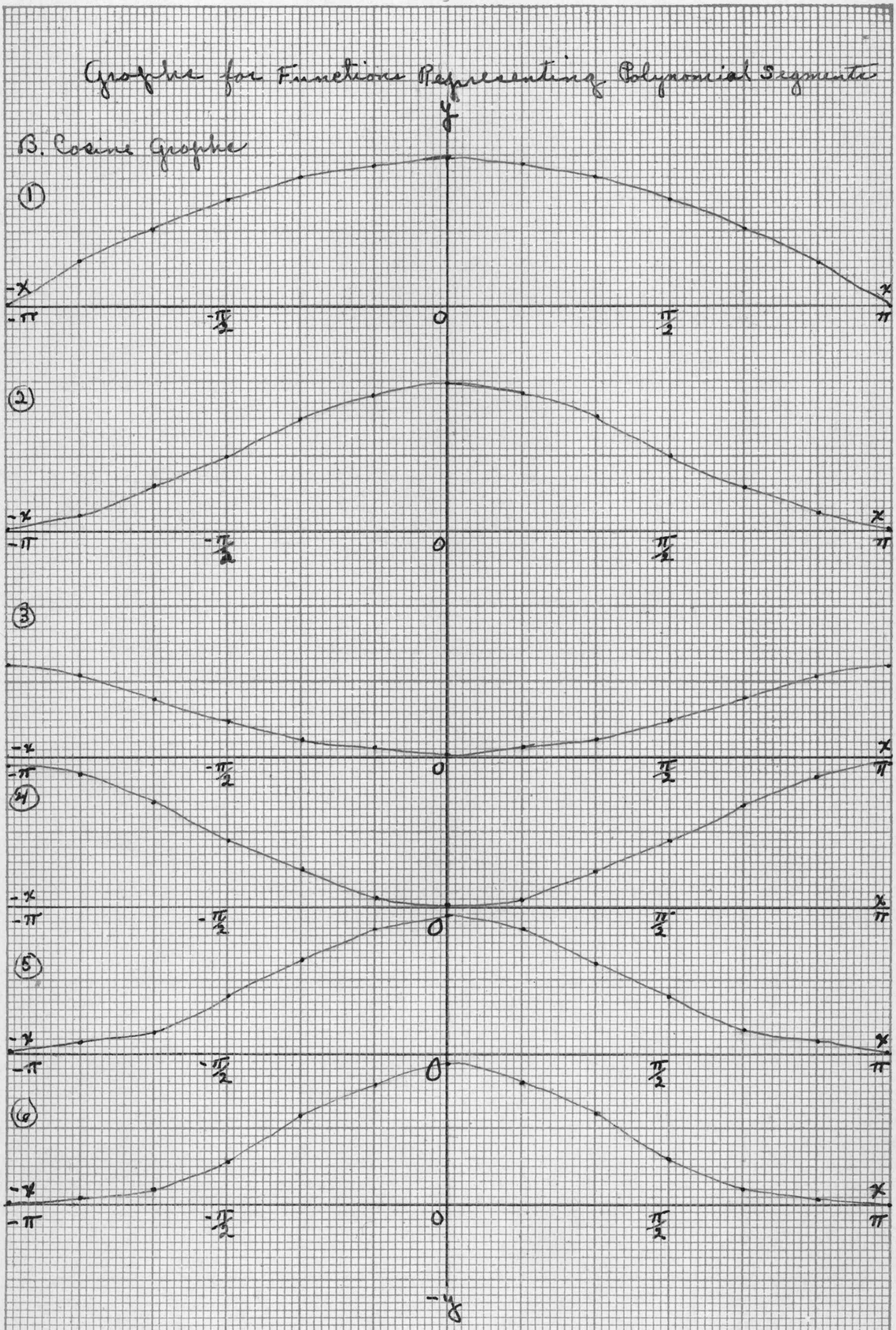
$\quad -x(\pi^2 - x^2)^2, \; -\pi < x < \pi.$

$$S = \sum_{n=1}^{\infty} (-1)^n \left(\frac{16\pi^2}{n^3} - \frac{240}{n^5}\right) \sin nx, \; -\pi < x < \pi.$$

6. $f(x) = x(\pi^2 - x^2)^3, \; 0 < x < \pi,$

$\quad = -x(\pi^2 - x^2)^3, \; -\pi < x < 0.$

$$S = \sum_{n=1}^{\infty} (-1)^n \left(\frac{960\pi^2}{n^5} - \frac{10080}{n^7}\right) \sin nx, \; -\pi < x < \pi$$


Graphs for Functions Representing Polynomial Segments
B. Cosine Graphs
y
①
②
③
④
⑤
⑥
-x
x
-π
-π/2
0
π/2
π
-y

Opp. P. 29

SERIES REPRESENTING FUNCTIONS DEFINED BY POLYNOMIAL SEGMENTS

B. Cosine Series of Type $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2k}} \cos nx$

1. $f(x) = (\pi^2 - x^2), \; -\pi < x < \pi.$

$$S = \frac{2\pi^2}{3} - 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx, \; -\pi < x < \pi.$$

2. $f(x) = (\pi^2 - x^2)^2, \; -\pi < x < \pi.$

$$S = \frac{8\pi^4}{15} - 48 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \cos nx, \; -\pi < x < \pi.$$

3. $f(x) = x^6 - 5\pi^2 x^4 + 7\pi^4 x^2, \; -\pi < x < \pi.$

$$S = \frac{31\pi^6}{21} + 1440 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^6} \cos nx, \; -\pi < x < \pi$$

4. $f(x) = 3x^8 - 28\pi^2 x^6 + 98\pi^4 x^4 - 124\pi^6 x^2, \; -\pi < x < \pi$

$$S = -\frac{127\pi^8}{5} - 241920 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^8} \cos nx, \; -\pi < x < \pi.$$

5. $f(x) = (\pi^2 - x^2)^3, \; -\pi < x < \pi.$

$$S = \frac{16\pi^6}{35} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{96\pi^2}{n^4} - \frac{1440}{n^6}\right) \cos nx, \; -\pi < x < \pi.$$

6. $f(x) = (\pi^2 - x^2)^4, \; -\pi < x < \pi.$

$$S = \frac{128\pi^8}{315} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{7680\pi^2}{n^6} - \frac{80640}{n^8}\right) \cos nx, \; -\pi < x < \pi$$

SERIES REPRESENTING FUNCTIONS DEFINED BY POLYNOMIAL SEGMENTS

C. Series of Type $\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^{2K}}$, etc.

1. $f(x) = (\pi - x), \quad 0 < x < \pi,$

$\quad = -(\pi - x), \quad -\pi < x < 0.$

$$S = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}, \quad -\pi < x < \pi.$$

2. $f(x) = \frac{\pi^3 - 3\pi x^2 + 2x^3}{\pi}, \quad 0 < x < \pi$

$\quad = \frac{\pi^3 - 3\pi x^2 - 2x^3}{\pi}, \quad -\pi < x < 0.$

$$S = \frac{\pi^2}{2} + \frac{48}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^4} \quad -\pi < x < \pi.$$

3. $f(x) = \frac{2x^3}{\pi^3} - \frac{3x^2}{\pi^2} + 1, \quad 0 < x < \pi,$

$\quad = \frac{2x^3}{\pi^3} - \frac{3x^2}{\pi^2} + 1, \quad -\pi < x < 0.$

$$S = \frac{1}{2} + \frac{48}{\pi^4}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^4}, \quad -\pi < x < \pi$$

4. $f(x) = \frac{\pi^4 - 6\pi^2 x + 8\pi x^3 - 3x^4}{\pi}, \quad 0 < x < \pi,$

$\quad = \frac{\pi^4 - 6\pi^2 x - 8\pi x^3 - 3\pi^4}{\pi}, \quad -\pi < x < 0.$

$$S = \frac{2\pi^3}{5} + \frac{48}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^4} + \frac{144}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n)x}{(2n)^4} \quad -\pi < x < \pi$$

5. $f(x) = (\pi - x)^3, \quad 0 < x < \pi,$

$\quad = (\pi - x)^3, \quad -\pi < x < 0.$

$$S = \frac{\pi^3}{4} - \frac{24}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^4} + 6\pi\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}, \quad -\pi < x < \pi.$$

SERIES REPRESENTING FUNCTIONS DEFINED BY POLYNOMIAL SEGMENTS

C. Series of Type $\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^{2k}}$

6. $f(x) = \frac{\pi^5-10\pi^3x^2+20\pi^2x^3-15\pi x^4+4x^5}{\pi}, \quad 0 < x < \pi,$

$$= \frac{\pi^5-10\pi^3x^2-20\pi^2x^3-15\pi x^4-4x^5}{\pi}, \quad -\pi < x < 0.$$

$$S = \frac{\pi^4}{3} - \frac{1920}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^6} + 240\sum_{n=1}^{\infty}\frac{\cos nx}{n^4}$$

7. $f(x) = \frac{\pi^6-15\pi^4x^2+40\pi^3x^3-45\pi^2x^4+24\pi x^5-5x^6}{\pi}, \quad 0 < x < \pi.$

$$= \frac{\pi^6-15\pi^4x^2-40\pi^3x^3-45\pi^2x^4-24\pi x^5-5x^6}{\pi}, \quad -\pi < x < 0.$$

$$S = \frac{2\pi^5}{7} - \frac{4320}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^6} - \frac{7200}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n)x}{(2n)^6}$$

$$+ 480\pi\sum_{n=1}^{\infty}\frac{\cos nx}{n^4}, \quad -\pi < x < \pi$$

8. $f(x) = \frac{\pi^7-21\pi^5x^2+70\pi^4x^3-105\pi^3x^4+84\pi^2x^5-35\pi x^6+6x^7}{\pi}, \quad 0 < x < \pi$

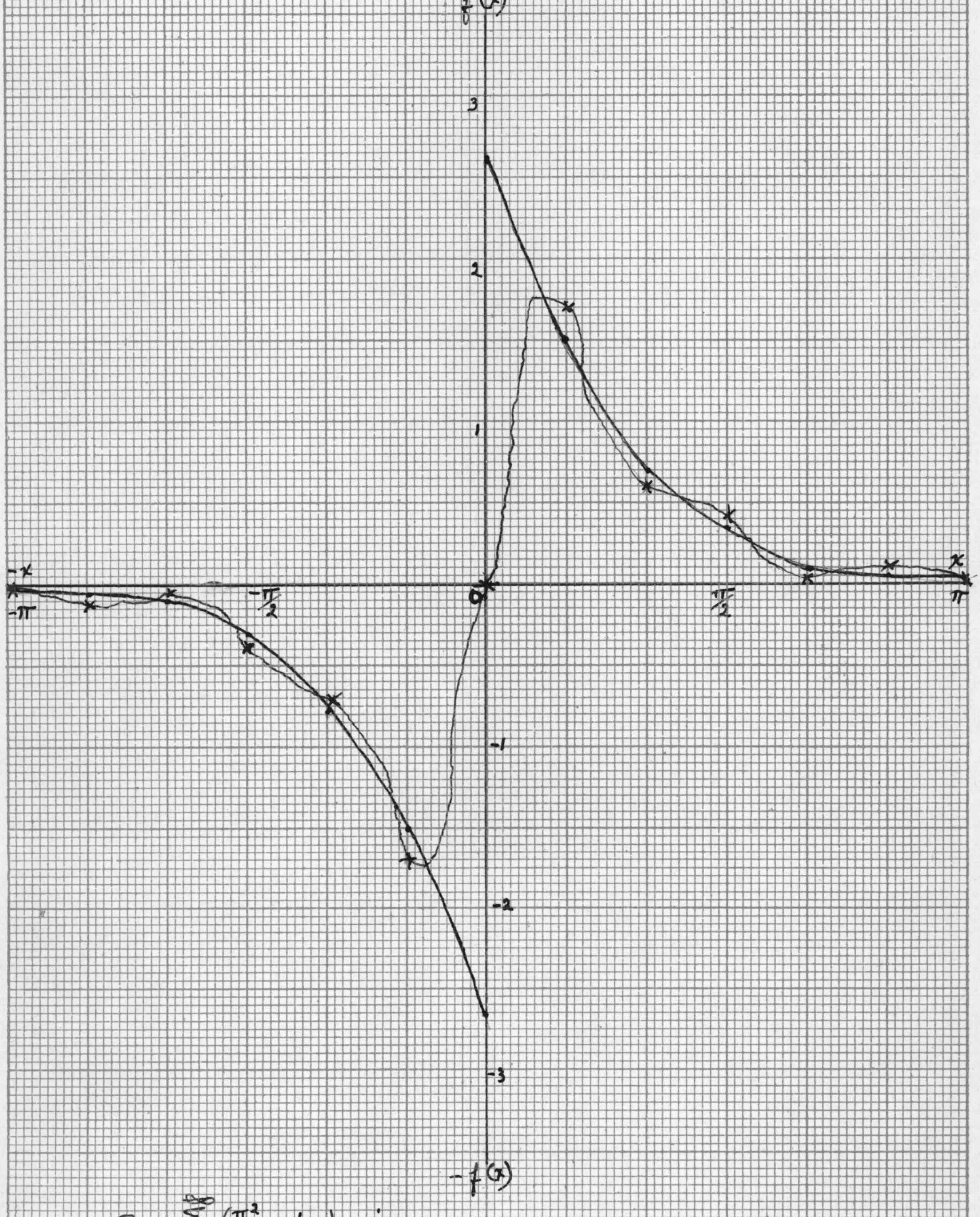
$$= \frac{\pi^7-21\pi^5x^2-70\pi^4x^3-105\pi^3x^4-84\pi^2x^5-35\pi x^6-6x^7}{\pi}, \quad -\pi < x < 0.$$

$$S = \frac{\pi^6}{4} + \frac{120960}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^8} - 20160\sum_{n=1}^{\infty}\frac{\cos nx}{n^6}$$

$$- 840\pi^2\sum_{n=1}^{\infty}\frac{\cos nx}{n^4}, \quad -\pi < x < \pi.$$

(6)

$f(x) = -\frac{1}{12}(\pi - x)^3, \quad -\pi < x < 0,$ $\qquad f(x) = \frac{1}{12}(\pi - x)^3, \quad 0 < x < \pi.$



$$S = \sum_{n=1}^{\infty}\left(\frac{\pi^2}{6n} - \frac{1}{n^3}\right)\sin nx, \qquad -\pi < x < \pi.$$

Opp. P. 32

MISCELLANEOUS SERIES FOR POLYNOMIAL FUNCTIONS

A. Sine Series

1. $f(x) = x, \; 0 < x < \frac{\pi}{2}$

$$= (\pi - x), \; \frac{\pi}{2} < x < \pi.$$

$$S = -\frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n \sin(2n-1)x}{(2n-1)^2}, \; 0 < x < \pi.$$

2. $f(x) = x^2, \; 0 < x < \pi$

$$S = -\frac{8}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^3} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n \pi^2}{n} \sin nx, \; 0 \quad x \quad \pi.$$

3. $f(x) = x^2, \; 0 \quad x \quad c$

$$S = -\frac{2c^2}{\pi^3}\left[4\sum_{n=1}^{\infty} \frac{\sin(2n-1)\frac{\pi x}{c}}{(2n-1)^3} + \sum_{n=1}^{\infty} \frac{(-1)^n \pi^2}{n} \sin \frac{n\pi x}{c}\right],$$

$$0 \quad x \quad c.$$

4. $f(x) = x(\pi - x)(2\pi - x)(\pi + x), \; -\pi < x < \pi.$

$$S = -48\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^3} - 4\pi^2 \sum_{n=1}^{\infty} \frac{\sin(2n)x}{(2n)^3}, \; -\pi < x < \pi.$$

5. $f(x) = x^3, \; 0 < x < \pi.$

$$S = -\frac{2}{\pi}\sum_{n=1}^{\infty} (-1)^n\left(\frac{\pi^3}{n} - \frac{6\pi}{n^3}\right) \sin nx, \; 0 < x < \pi.$$

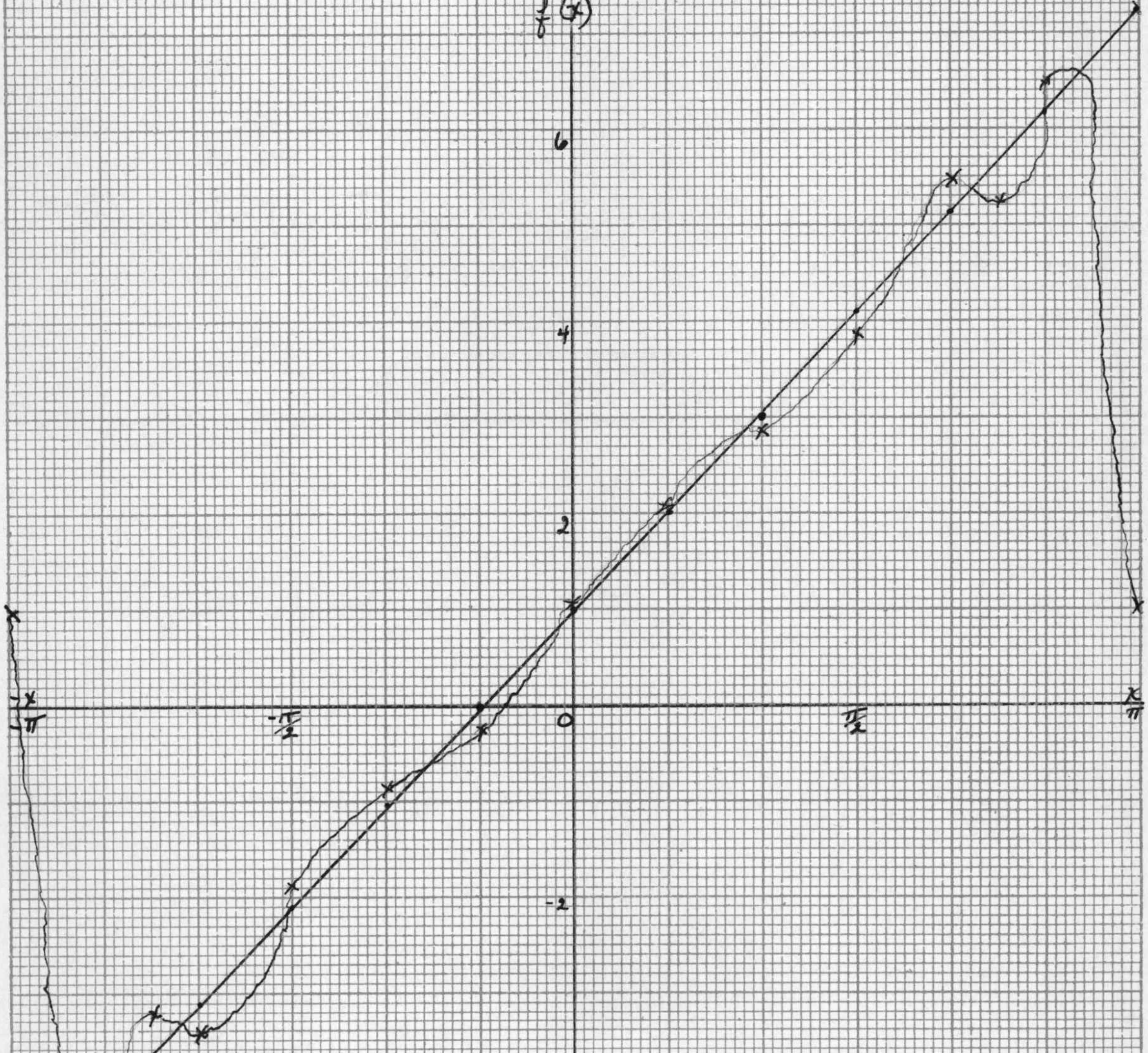
6. $f(x) = \frac{1}{12}(\pi - x)^3, \; 0 < x < \pi$, (See graph on ~~next~~ opposite page)

$$= -\frac{1}{12}(\pi - x)^3, \; -\pi < x < 0.$$

$$S = \sum_{n=1}^{\infty} \left(\frac{\pi^2}{6n} - \frac{1}{n^3}\right) \sin nx, \; -\pi < x < \pi.$$

7

$f(x) = 2x + 1$, $-\pi < x < \pi$.



$$S = 1 + \sum_{n=1}^{\infty} \frac{4}{n} (-1)^{n} \sin nx, \quad -\pi < x < \pi.$$

Opp. P. 33

MISCELLANEOUS SERIES FOR POLYNOMIAL FUNCTIONS

A. Sine Series

7. $f(x) = 2x + 1, \ -\pi < x < \pi$. (See graph on ~~next~~ opposite page)

$$S = 1 + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx, \ -\pi < x < \pi.$$

MISCELLANEOUS SERIES FOR POLYNOMIAL FUNCTIONS

B. Cosine Series.

1. $f(x) = x^2,\ 0 < x < \pi.$

$$S = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx,\ 0 < x < \pi$$

2. $f(x) = x^2,\ -c < x < c.$

$$S = \frac{c^2}{3} + \frac{4c^2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi x}{c},\ -c < x < c.$$

3. $f(x) = x^3,\ 0 < x < \pi$

$$S = \frac{\pi^3}{4} + \frac{6}{\pi}\left[\sum_{n=1}^{\infty} \frac{(-1)^n \pi^2}{n^2} \cos nx - 4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos(2n-1)x}{(2n-1)^4}\right],$$

4. $f(x) = -\frac{1}{4}(\pi - x)^2,\ 0 < x < \pi$

$$S = -\frac{\pi^2}{12} - \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx,\ 0 < x < \pi.$$

5. $f(x) = (\frac{1}{2} - x)^2,\ 0 \leqq x \leqq 1.$

$$S = \frac{1}{12} - \frac{1^2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{\cos 2n\pi x}{1},\ 0 \leqq x \leqq 1.$$

6. $f(x) = \frac{\pi^2 - x^2}{\pi},\ 0 < x < \pi$

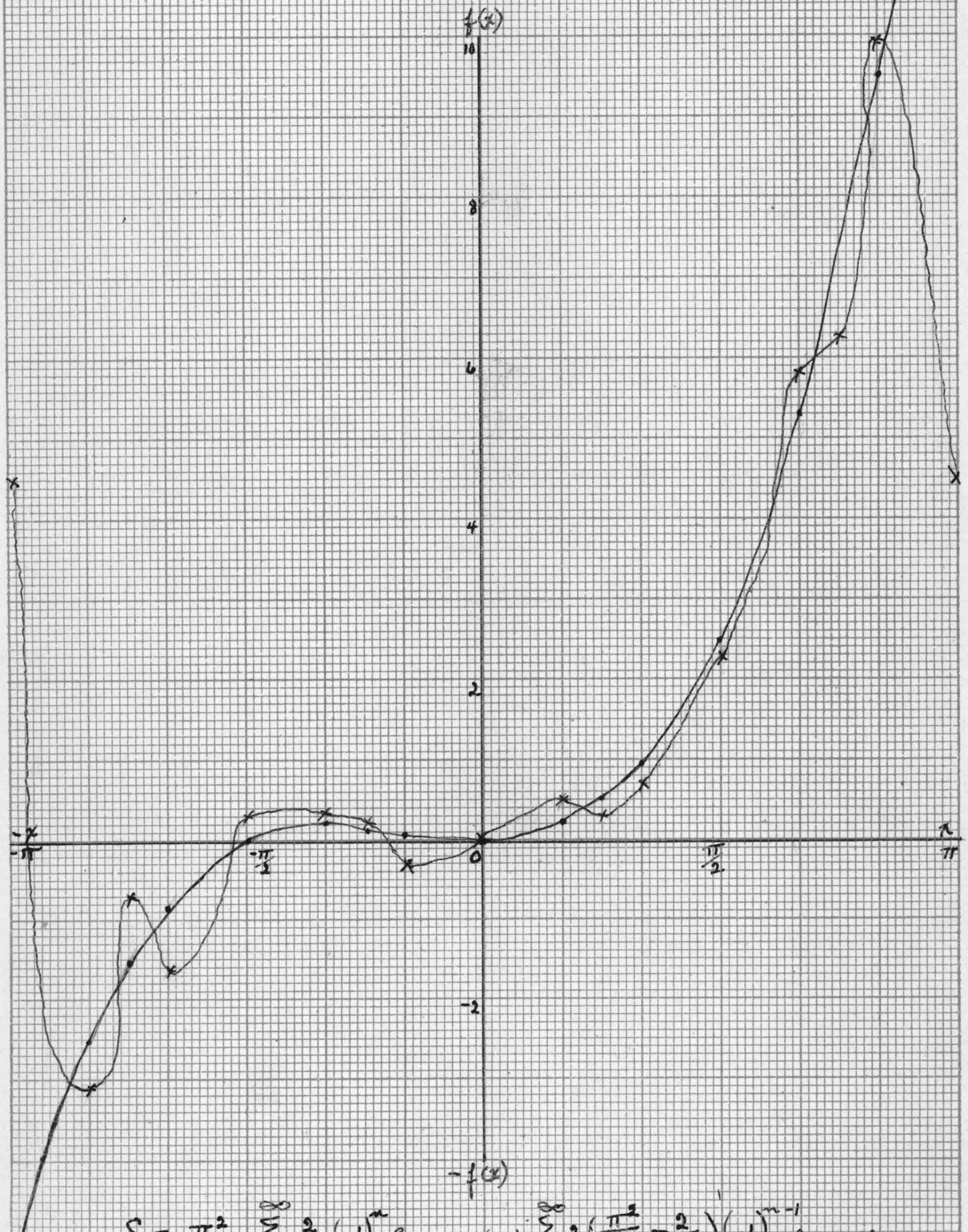
$$S = 2/3\pi - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos nx}{n^2},\ 0 < x < \pi$$

7. $f(x) = (\pi - x)^4,\ 0 < x < \pi$

$$S = \frac{\pi^4}{5} + 8\pi^2 \sum_{n=1}^{\infty} \frac{\cos nx}{n^2} - 48 \sum_{n=1}^{\infty} \frac{\cos nx}{n^4},\ 0 < x < \pi.$$

(3)

$f(x) = \frac{1}{2}x^2 + \frac{1}{3}x^3; \qquad -\pi < x < \pi.$



$$S = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2}(-1)^n \cos nx + \sum_{n=1}^{\infty} 2\left(\frac{\pi^2}{3n} - \frac{2}{n^3}\right)(-1)^{n-1} \sin nx.$$

Opp. P. 35

MISCELLANEOUS SERIES FOR POLYNOMIAL FUNCTIONS

C. Sine and Cosine Series.

1. $f(x) = x^2,\ 0 < x < 2\pi.$

$$S = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2} - 4\pi\sum_{n=1}^{\infty}\frac{\sin nx}{n},\ 0 < x < 2\pi$$

2. $f(x) = x + x^2,\ -\pi < x < \pi.$ (See graph following this list)

$$S = \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2} - 4\sum_{n=1}^{\infty}\frac{(-1)^n \sin nx}{2n}\quad -\pi < x < \pi.$$

3. $f(x) = \frac{1}{2}x^2 + \frac{1}{3}x^3,\ -\pi\ \ x\ \ \pi.$ (See graph on opposite page)

$$S = \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}(-1)^n\left(\frac{\pi^2}{3n} - \frac{2}{n^3}\right)\sin nx,$$
$$-\pi < x < \pi.$$

4. $f(x) = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 3x + 1,\ -\pi < x < \pi.$

$$S = \left(\frac{\pi^2}{2} + 1\right) + 6\sum_{n=1}^{\infty}\frac{(-1)^n \cos nx}{n^2}$$
$$- 2\sum_{n=1}^{\infty}(-1)^n\left(\frac{\pi^2}{3n} + \frac{3}{n} - \frac{2}{n^3}\right)\sin nx,$$
$$-\pi < x < \pi$$

5. $f(x) = \frac{1}{3}x^3 + \frac{3}{2}x^2 + x,\ -\pi < x < \pi.$

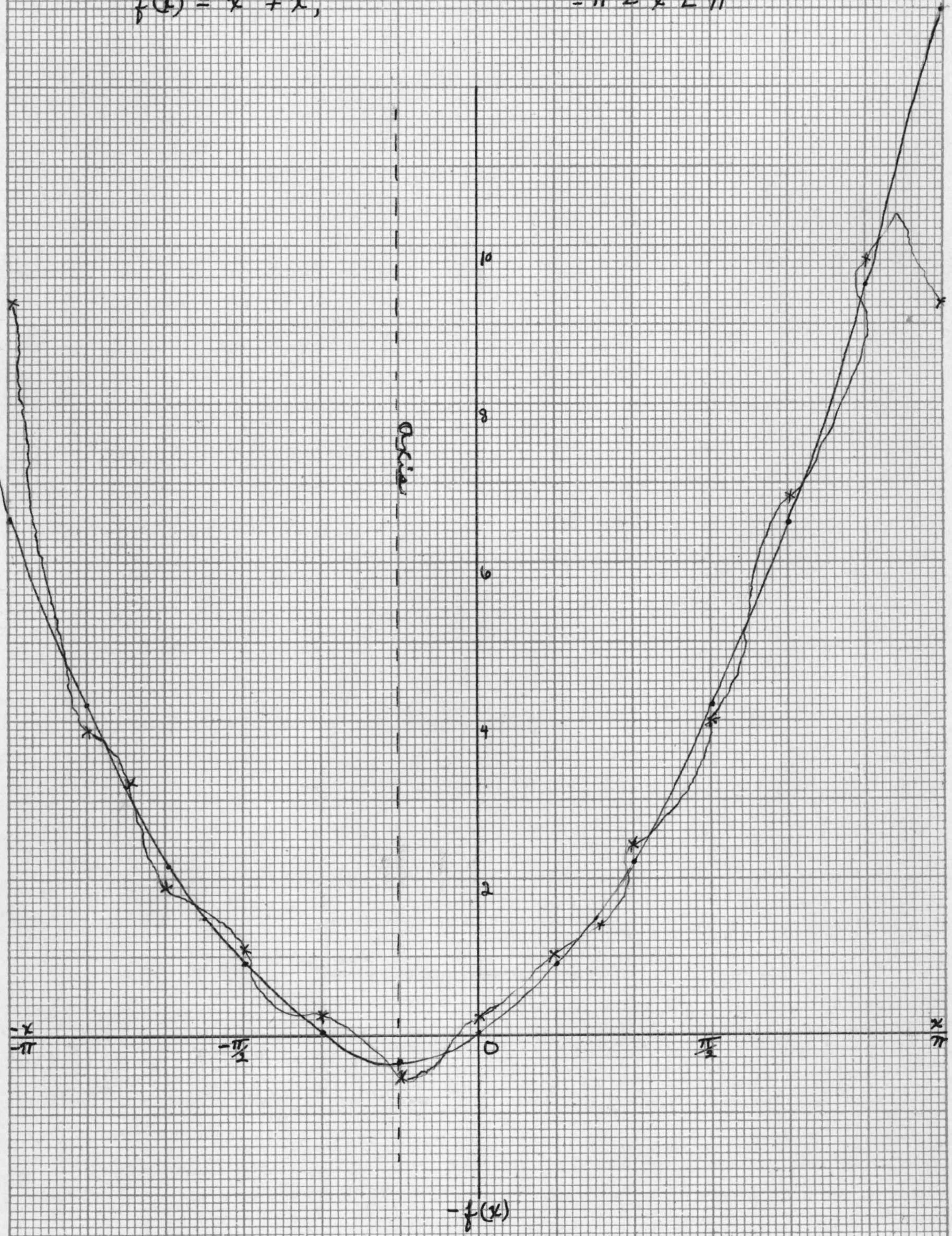
$$S = \frac{\pi}{2} + 6\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}(-1)^n\left(\frac{\pi^2}{3n} + \frac{1}{n} - \frac{2}{n^3}\right)\sin nx,$$
$$-\pi < x < \pi.$$

6. $f(x) = x^2 + 3x + 1,\ -\pi < x < \pi$

$$S = \left(\frac{\pi^2}{3} + 1\right) + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 6\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx.$$

(2)

$f(x) = x + x^2; \qquad -\pi < x < \pi$



$$S = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx + \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n-1} \sin nx, \quad -\pi < x < \pi.$$

MISCELLANEOUS SERIES FOR POLYNOMIAL FUNCTIONS

C. Sine and Cosine Series.

7. $f(x) = \frac{1}{12}(\pi - x)^3 + \frac{1}{4}(\pi - x)^2, \ 0 < x < \pi,$

$$= -\frac{1}{12}(\pi + x)^3 + \frac{1}{4}(\pi + x)^2, \ -\pi < x < 0.$$

$$S = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2} + \sum_{n=1}^{\infty} \left(\frac{\pi^2}{6n} - \frac{1}{n^3}\right) \sin nx, \ -\pi < x < \pi.$$

8. $f(x) = 8x^2 - 16x, \ -\pi < x < \pi.$

$$S = \frac{8\pi^2}{3} + 32\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx + 32\sum_{n=1}^{\infty} (-1)^n \sin nx, \ -\pi < x < \pi$$

9. $f(x) = 2x^3 - 4x^2 + 6x - 12, \ -\pi < x < \pi.$

$$S = -4\left(\frac{\pi^2}{3} + 3\right) - 16\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

$$- \sum_{n=1}^{\infty} (-1)^n \left(\frac{4\pi^2 + 12}{n} - \frac{24}{n^3}\right) \sin nx,$$

$$-\pi < x < \pi.$$

10. $f(x) = x^3 - 6x^2 + 11x - 6, \ -\pi < x < \pi.$

$$S = -2(\pi^2 + 3) - 24\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

$$- \sum_{n=1}^{\infty} (-1)^n \left(\frac{2\pi^2 + 22}{n} - \frac{12}{n^3}\right) \sin nx$$

11. $f(x) = x^3 + 2x^2 - 5x - 6, \ -\pi < x < \pi.$

$$S = \left(\frac{2\pi^2}{3} - 6\right) + 8\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

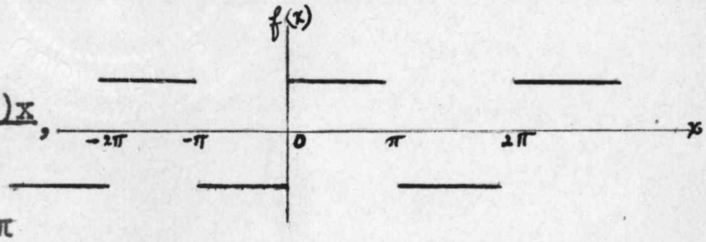
$$- \sum_{n=1}^{\infty} (-1)^n \left(\frac{2\pi^2 - 10}{n} - \frac{12}{n^3}\right) \sin nx,$$

$$-\pi < x < \pi.$$

SERIES REPRESENTING FUNCTIONS WHOSE GRAPHS ARE BROKEN LINES

A. Sine Series.

1. $f(x) = 1,\ 0 < x < \pi$

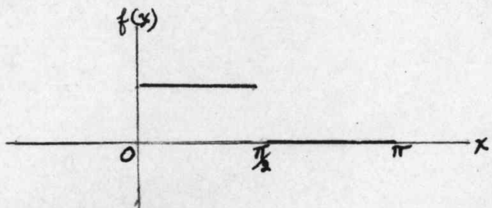


$$S = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)},\quad 0 < x < \pi$$

2. $f(x) = c,\ 0 < x < \pi$, where c is any constant.

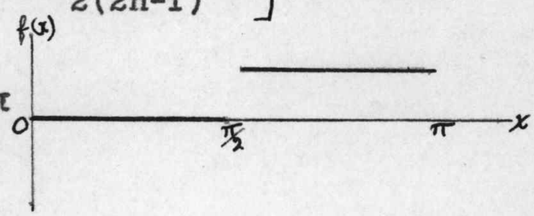
$$S = \frac{4c}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)},\quad 0 < x < \pi.$$

3. $f(x) = 1,\ 0 < x < \frac{\pi}{2}$,
$\quad = 0,\ \frac{\pi}{2} < x < \pi.$



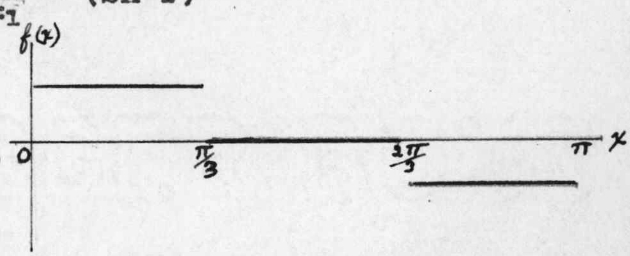
$$S = \frac{2}{\pi}\left[\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)} + \sum_{n=1}^{\infty}\frac{2\sin 2(2n-1)x}{2(2n-1)}\right],\quad 0 < x < \pi$$

4. $f(x) = 0,\ 0 \leq x < \frac{\pi}{2};\ f(\frac{1}{2}\pi) = \frac{1}{4}\pi$
$\quad = \frac{\pi}{2},\ \frac{\pi}{2} < x < \pi;\ f(\pi) = 0.$



$$S = \sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)} - \sum_{n=1}^{\infty}\frac{\sin 2(2n-1)}{(2n-1)},\quad 0 < x < \pi.$$

5. $f(x) = \frac{1}{3}\pi\ ,\ 0 < x < \frac{1}{3}\pi$
$\quad = 0\ ,\ \frac{1}{3}\pi < x < \frac{2}{3}\pi,$
$\quad = -\frac{1}{3}\pi,\ \frac{2}{3}\pi < x < \pi$



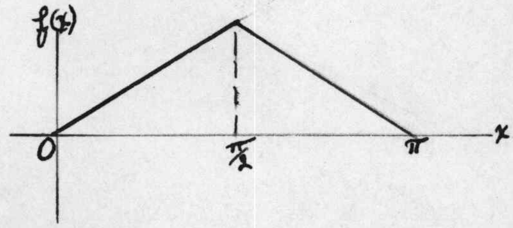
also $f(0) = f(\pi) = 0,\ f(\frac{\pi}{3}) = \frac{\pi}{6};\ f(\frac{2}{3}\pi) = -\frac{\pi}{6}$

$$S = \sum\frac{\sin(2n)x}{n},\quad 0 < x < \pi. \quad n = 1,2,4,5,7,8,11,13,\ldots.$$

Terms are zero when n is a multiple of three.

SERIES REPRESENTING FUNCTIONS WHOSE GRAPHS ARE BROKEN LINES

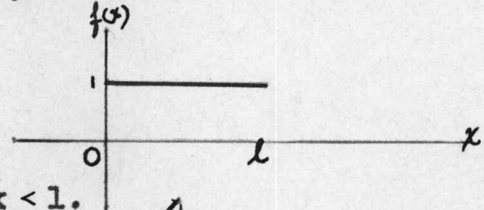
A. Sine Series. (Cont.)



6. $f(x) = \frac{1}{4}\pi x,\ 0 \leqq x \leqq \frac{\pi}{2},$

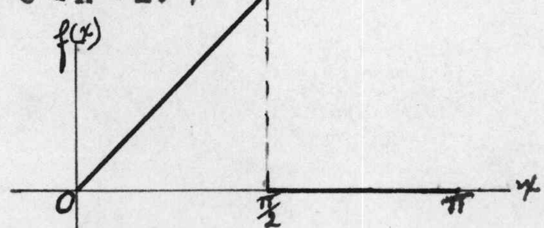
$= \frac{1}{4}\pi(\pi - x),\ \frac{\pi}{2} \leqq x \leqq \pi.$

$$S = -\sum_{n=1}^{\infty} \frac{(-1)^n \sin(2n-1)x}{(2n-1)^2},\ 0 < x < \pi$$



7. $f(x) = 1,\ 0 < x < l$

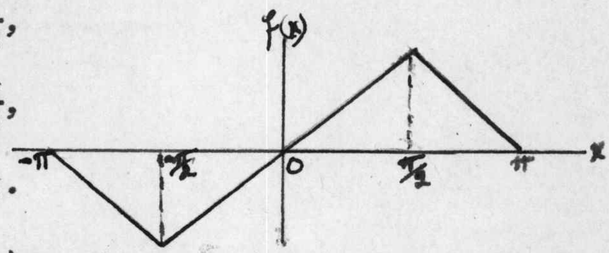
$$S = \frac{4}{\pi}\ \frac{\sin(2n-1)\frac{\pi x}{l}}{(2n-1)},\ 0 < x < l.$$



8. $f(x) = x,\ 0 < x < \frac{\pi}{2},$

$= 0,\ \frac{\pi}{2} < x < \pi$

$$S = \frac{2}{\pi}\left[\sum_{n=1}^{\infty} \frac{(-1)^n \sin(2n-1)x}{(2n-1)^2} - \sum_{n=1}^{\infty} \frac{(-1)^n n\pi\ \sin(2n)x}{(2n)^2}\right],\ 0 < x < \pi$$

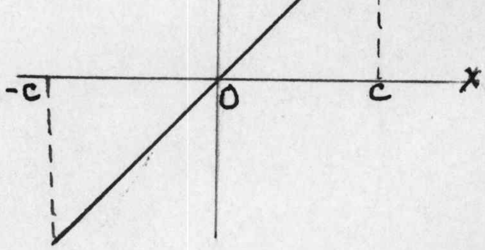


9. $f(x) = -\pi - x,\ -\pi < x < -\frac{\pi}{2},$

$= x,\quad -\frac{\pi}{2} < x < \frac{\pi}{2},$

$= \pi - x,\ \frac{\pi}{2} < x < \pi.$

$$S = -\frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n \sin(2n-1)x}{(2n-1)^2},\ -\pi < x < \pi.$$



10. $f(x) = x,\ -c < x < c.$

$$S = \frac{-2c}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\frac{n\pi x}{c}$$

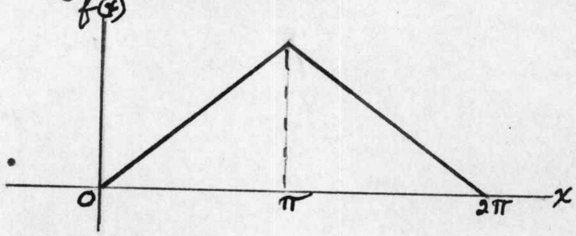
SERIES REPRESENTING FUNCTIONS WHOSE GRAPHS ARE BROKEN LINES

A. Sine Series. (Cont.)

11. $f(x) = x, \; 0 < x < \frac{c}{2}$,

$$= c - x, \; \frac{c}{2} < x < c.$$

$$S = -\frac{4c}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \sin \frac{n\pi x}{c}$$ (See Fig. 9)



12. $f(x) = x, \; 0 < x < \pi,$

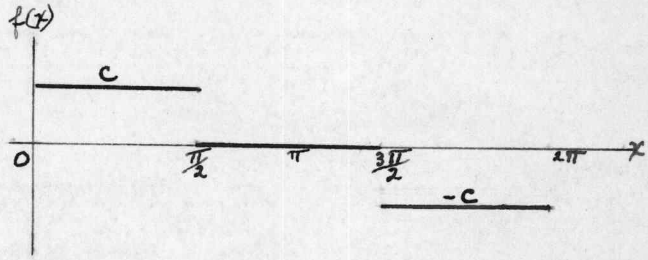
$$= x - 2\pi, \; \pi < x < 2\pi.$$

$$S = -2\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx, \; 0 < x < 2\pi$$

13. $f(x) = c, \; 0 < x < \frac{\pi}{2},$

$$= 0, \; \frac{\pi}{2} < x < \frac{3\pi}{2},$$

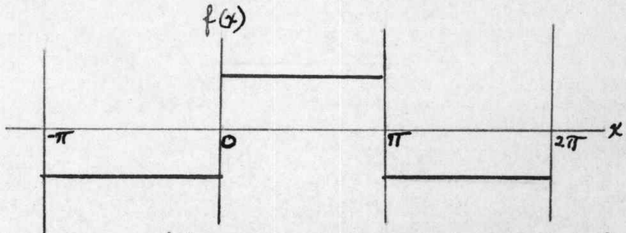
$$= -c \; \frac{3\pi}{2} < x < 2\pi$$



$$S = \frac{2c}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} \sin nx, \; 0 < x < 2\pi.$$

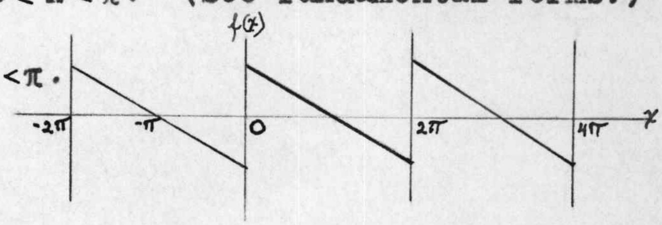
14. $f(x) = \frac{\pi}{4}, \; 0 < x < \pi.$

$$S = \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)},$$



$0 < x < \pi$. (See fundamental forms.)

15. $f(x) = \frac{1}{2}(\pi - x), \; 0 < x < \pi.$



$$S = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx,$$

$0 < x < \pi$. (See fundamental forms.)

SERIES REPRESENTING FUNCTIONS WHOSE GRAPHS ARE BROKEN LINES

A. Sine Series. (Cont.)

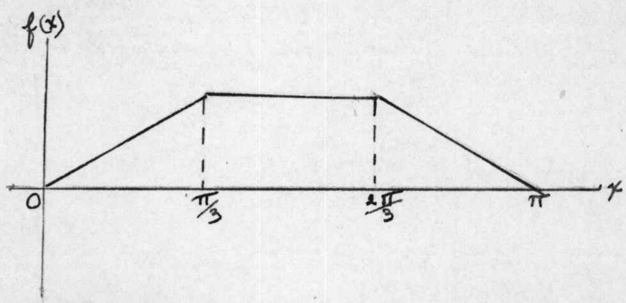
16. $f(x) = \frac{1}{2} - x, \ 0 < x < 1.$

$$S = \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{2n\pi x}{1}, \ 0 < x < 1.$$

17. $f(x) = \frac{3x}{2}, \ 0 \leqq x \leqq \frac{\pi}{3},$

$= \frac{\pi}{2}, \ \frac{\pi}{3} < x < \frac{2\pi}{3}$

$= \frac{3}{2}(\pi - x), \ \frac{2\pi}{3} \leqq x \leqq \pi.$

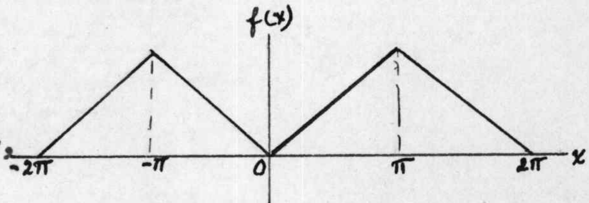


$$S = \frac{6}{\pi}\sum_{n=1}^{\infty} \frac{\sin \frac{1}{3}(2n-1)\pi \ \sin(2n-1)x}{(2n-1)^2}, \ 0 \leqq x \leqq \pi.$$

SERIES REPRESENTING FUNCTIONS WHOSE GRAPHS ARE BROKEN LINES

B. Cosine Series.

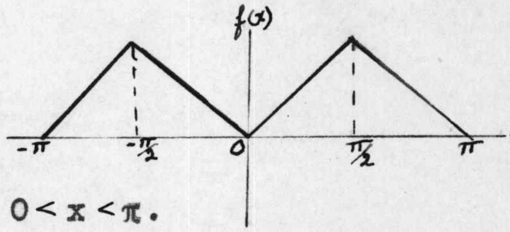
1. $f(x) = x, \; 0 < x < \pi$



$$S = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}, \quad 0 < x < \pi.$$

2. $f(x) = x, \; 0 < x < \frac{\pi}{2},$

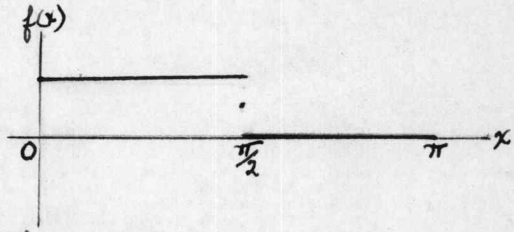
$= (\pi - x), \; \frac{\pi}{2} < x < \pi.$



$$S = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2(2n-1)x}{(2n-1)^2}, \quad 0 < x < \pi.$$

3. $f(x) = 1, \; 0 < x < \frac{\pi}{2}$

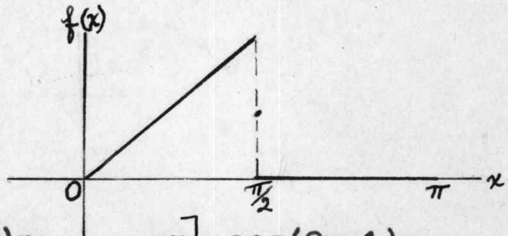
$= 0, \; \frac{\pi}{2} < x < \pi.$



$$S = \frac{1}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n \cos(2n-1)x}{(2n-1)}, \quad 0 < x < \pi.$$

4. $f(x) = x, \; 0 < x < \frac{\pi}{2}$

$= 0, \; \frac{\pi}{2} < x < \pi$



$$S = \frac{\pi}{8} - \frac{2}{\pi}\sum_{n=1}^{\infty} (-1)^n \left[(\frac{(2n-1)\pi}{2} + (-1)^n\right] \frac{\cos(2n-1)x}{(2n-1)^2}$$

$$- \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos\left[2(2n-1)\right]x}{\left[2(2n-1)\right]^2}, \quad 0 < x < \frac{\pi}{2}$$

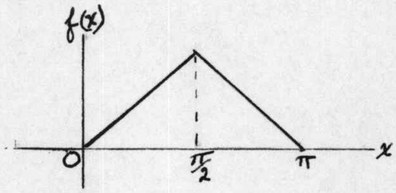
5. $f(x) = x, \; 0 < x < c$

$$S = \frac{c}{2} - \frac{4c}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} \cos\frac{n\pi x}{c}, \quad 0 < x < c.$$

SERIES REPRESENTING FUNCTIONS WHOSE GRAPHS ARE BROKEN LINES

6. $f(x) = \frac{1}{4}\pi x,\ 0 \leqq x \leqq \frac{\pi}{2},$

$= \frac{1}{4}\pi(\pi - x),\ \frac{\pi}{2} < x \leqq \pi.$

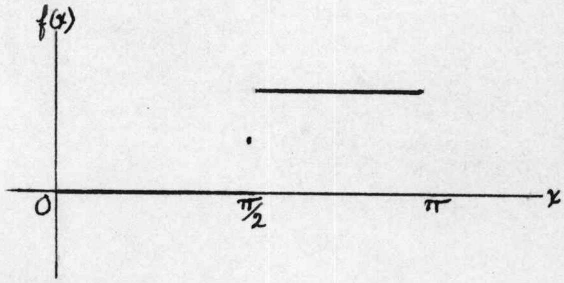


$$S = \frac{\pi^2}{16} - 2\sum_{n=1}^{\infty} \frac{\cos 2(2n-1)x}{2(2n-1)^2},\ 0 < x < \pi.$$

7. $f(x) = 0,\ 0 \leqq x \leqq \frac{\pi}{2},$

$f(\frac{\pi}{2}) = \frac{\pi}{4},$

$f(x) = \frac{\pi}{2},\ \frac{\pi}{2} < x \leqq \pi.$

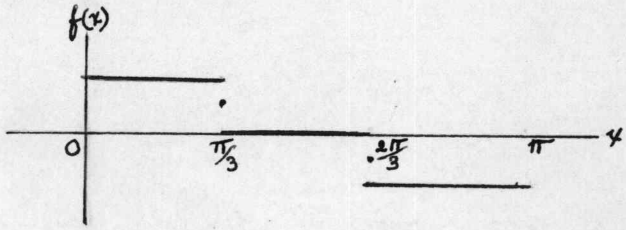


$$S = \frac{\pi}{4} - \sum_{n=1}^{\infty} (-1)^n \frac{\cos(2n-1)x}{(2n-1)},\ 0 < x < \pi.$$

8. $f(x) = \frac{\pi}{3},\ 0 \leqq x < \frac{\pi}{3}$

$= 0,\ \frac{\pi}{3} < x < \frac{2\pi}{3},$

$= -\frac{\pi}{3},\ \frac{2\pi}{3} < x \leqq \frac{3}{\pi}$

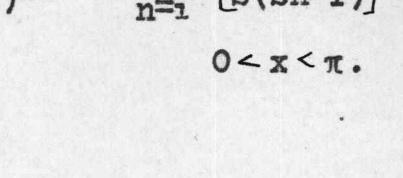


$$S = -\frac{2\sqrt{3}}{3}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cos nx,\ n = 1,5,7,11,13,\ldots.$$

Terms = 0, when n is a multiple of three.

9. $f(x) = x + \frac{3\pi}{2},\ 0 < x < \frac{\pi}{2},$

$= -2(x - \pi),\ \frac{\pi}{2} < x < \pi.$



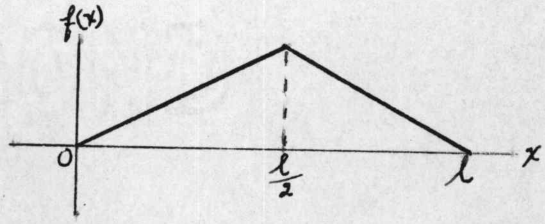
$$S = \frac{7}{8}\pi^2 + \frac{2}{\pi}\left[3\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2} - 2\sum_{n=1}^{\infty} \frac{\cos 2(2n-1)x}{[2(2n-1)]^2}\right],$$

$$0 < x < \pi.$$

SERIES REPRESENTING FUNCTIONS WHOSE GRAPHS ARE BROKEN LINES

B. Cosine Series (Cont.)

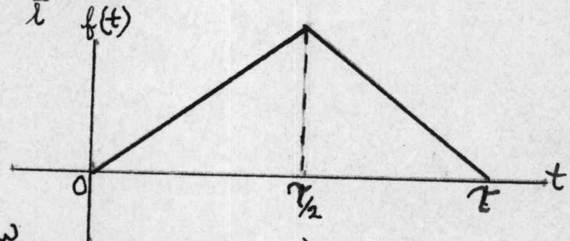
10. $f(x) = \frac{l}{4} - x,\ 0 < x < \frac{l}{2},$



$= x - \frac{3l}{4},\ \frac{l}{2} < x < l.$

$$S = \frac{2l}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\,\frac{\cos 2\pi(2n-1)x}{l},\quad 0 < x < l.$$

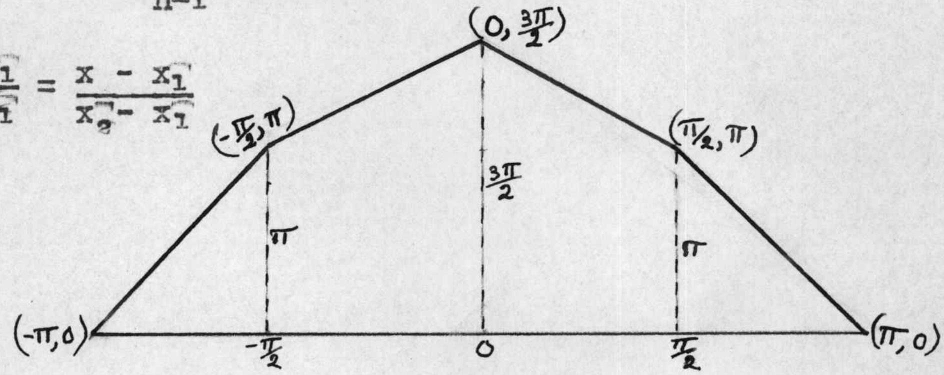
11. $f(t) = \frac{4kt}{\tau},\ 0 > t > \frac{\tau}{2},$



$= \frac{4k(\tau - t)}{\tau},\ \frac{\tau}{2} > t > \tau$

$$S = k - \frac{8k}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos(2n-1)\omega t}{(2n-1)^2},\quad 0 < t < \tau. \quad (\omega\tau = 2\pi)$$

12. $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$



$\frac{y - \frac{3\pi}{2}}{\pi - \frac{3\pi}{2}} = \frac{x - 0}{\frac{\pi}{2} - 0}$. $\therefore y = -x + \frac{3\pi}{2},\ 0 < x < \frac{\pi}{2},$

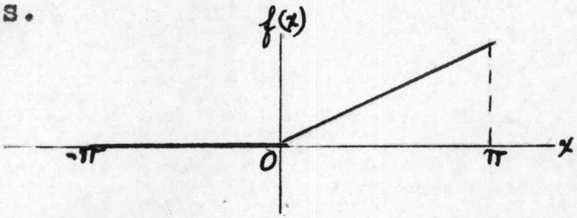
$\frac{y - \pi}{0 - \pi} = \frac{x - \frac{\pi}{2}}{\pi - \frac{\pi}{2}}$. $\therefore y = 2(x - \pi),\ \frac{\pi}{2} < x < \pi.$

$$S = \frac{7\pi}{8} + \frac{2}{\pi}\quad 3\left[\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} - 2\sum_{n=1}^{\infty} - 2\sum_{n=1}^{\infty}\frac{\cos 2(2n-1)x}{2(2n-1)^2}\right],$$

$-\pi < x < \pi.$

SERIES REPRESENTING FUNCTIONS WHOSE GRAPHS ARE BROKEN LINES

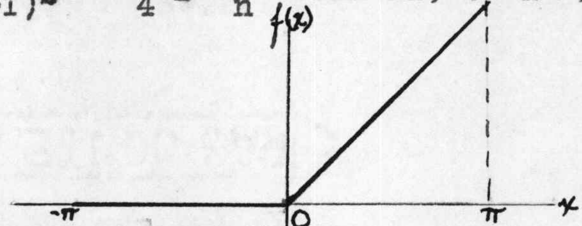
C. Sine and Cosine Series.



1. $f(x) = 0, \ -\pi < x \leqq 0,$

$\qquad = \frac{1}{4}\pi x, \ 0 < x < \pi.$

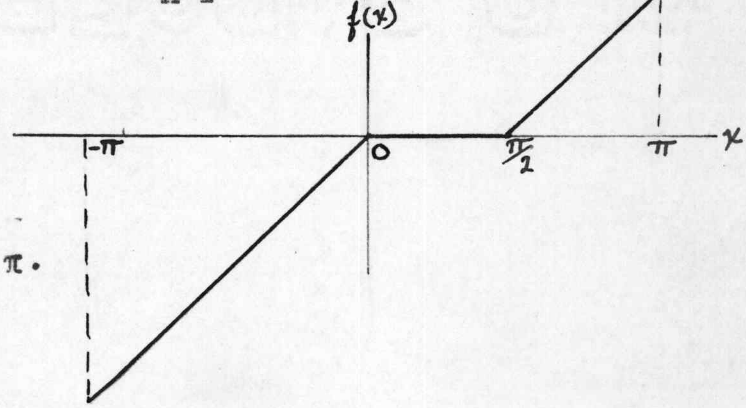
$$S = \frac{\pi^2}{16} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2} - \frac{\pi}{4}\sum \frac{(-1)^n}{n}\sin nx, \ 0 < x < \pi.$$



2. $f(x) = 0, \ -\pi < x < 0,$

$\qquad = x, \ 0 < x < \pi.$

$$S = \frac{\pi}{4} - \frac{2}{\pi}\left[\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n\pi}{2n}\sin nx\right], \ -\pi < x < \pi.$$



3. $f(x) = x, \ -\pi < x < 0,$

$\qquad = 0, \ 0 < x < \frac{\pi}{2},$

$\qquad = x - \frac{\pi}{2}, \ \frac{\pi}{2} < x < \pi.$

$$S = -\frac{3\pi}{16} + \frac{1}{\pi}\left[\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2} + \sum_{n=1}^{\infty} \cos(2n)x\right.$$

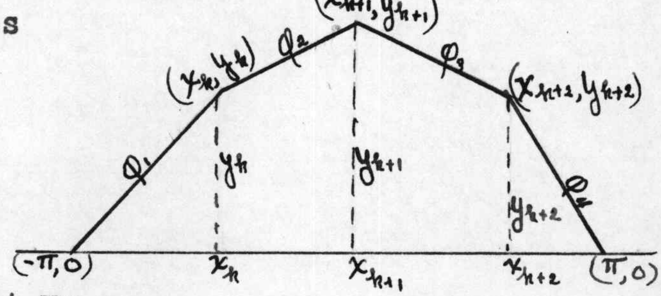
$$+ \sum_{n=1}^{\infty} \frac{3\pi}{2(2n-1)} + \frac{(-1)^n}{(2n-1)^2} \ \sin(2n-1)x$$

$$\left. - \sum_{n=1}^{\infty} \frac{3\pi}{(2n)^2}\sin(2n)x\right] -\pi < x < \pi.$$

SERIES REPRESENTING FUNCTIONS WHOSE GRAPHS ARE BROKEN LINES

C. Sine and Cosine Series

4. (Cont).



$$\frac{y - y_k}{y_{k+1} - y_k} = \frac{x - x_k}{x_{k+1} - x_k}$$

$$y = \frac{(x-x_k)(y_{k+1} - y_k)}{(x_{k+1} - x_k)} + y_k$$

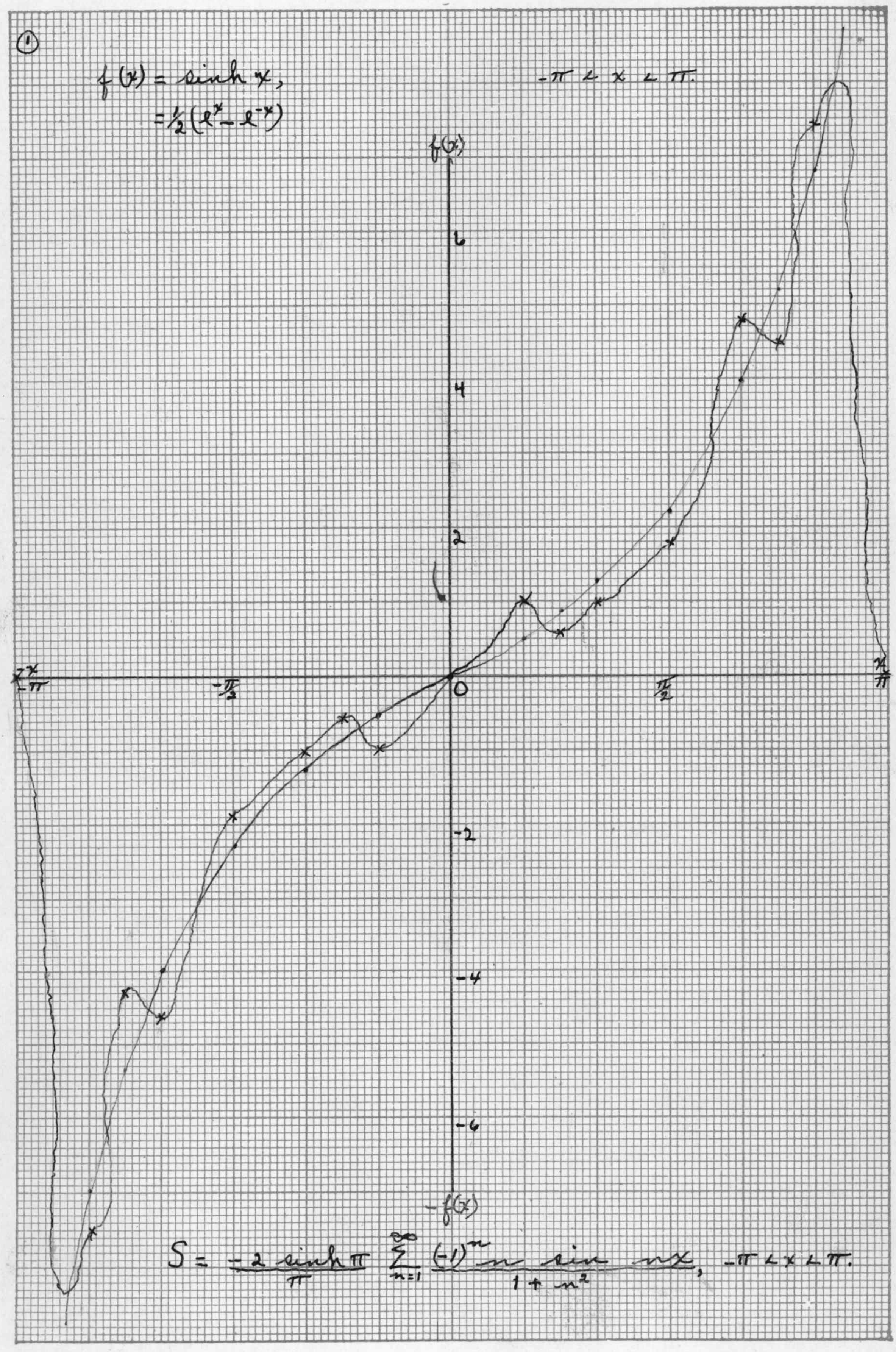
$$y = Ax + B, \quad \text{where } A = \frac{y_{k+1} - y_k}{x_{k+1} - x_k}$$

$$\text{and} \quad B = \frac{y_k x_{k+1} - x_k y_{k+1}}{x_{k+1} - x_k}$$

$$S = \frac{1}{4\pi}\Big[y_k(y_k(x_k+\pi) + (y_{k+1}+y_k)(x_{k+1} - x_k) + (y_{k+2} + y_{k+1})(x_{k+2} - x_{k+1}) + y_{k+2}(\pi - x_{k+2}) \Big]$$

$$+ \frac{1}{n\pi}\Big[(y_k \sin nx_k) + (y_{k+1} \sin nx_{k+1} - y_k \sin n\, x_k) + (y_{k+2} \sin nx_{k+2} - y_{k+1} \sin nx_{k+1}) + (-y_{k+2} \sin x_{k+2}) \Big]$$

$$+ \frac{1}{n^2\pi}\Bigg[\left(\frac{y_k}{x_k+\pi}\right)\Big[\cos nx_k - \cos n\pi\Big] + \left(\frac{y_{k+1} - y_k}{x_{k+1} - x_k}\right)\Big[\cos nx_{k+1} - \cos nx_k\Big] + \left(\frac{y_{k+2} - y_{k+1}}{y_{k+2} - x_{k+1}}\right)\Big[\cos nx_{k+2} - \cos nx_{k+1}\Big] + \left(\frac{y_{k+2}}{\pi - x_{k+2}}\right)\Big[\cos n\pi - \cos nx_{k+2}\Big] \Bigg]$$


①
f(x) = sinh x,
= ½(e^x − e^-x)
−π < x < π.
f(x)
6
4
2
0
−2
−4
−6
−f(x)
−x
−π
−π/2
π/2
x
π
S = −2 sinh π / π Σ_{n=1}^∞ (−1)^n n sin nx / (1 + n²), −π < x < π.

Opp. P 46

MISCELLANEOUS SERIES FOR EXPONENTIAL, HYPERBOLIC, TRIGONOMETRIC, AND LOGARITHMIC FUNCTIONS

A. Sine Series.

on opposite page

1. $f(x) = \sinh x, \ -\pi < x < \pi.$ (See graph on opposite page)

$$S = \frac{-2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n n\sin nx}{1+n^2}, \ -\pi < x < \pi.$$

1'. $f(x) = \dfrac{\pi}{\sinh\pi}\sinh x, \ -\pi < x < \pi.$

$$S = -2\sum_{n=1}^{\infty}\frac{(-1)^n\, n\sin nx}{1+n^2}, \ -\pi < x < \pi.$$

2. $f(x) = e^x, \ 0 < x < \pi.$

$$S = \frac{2}{\pi}\left[\sum_{n=1}^{\infty}\frac{(2n-1)}{1+(2n-1)^2}(1+e^{\pi})\sin(2n-1)x + \sum_{n=1}^{\infty}\frac{(2n)}{1+(2n)^2}(1-e^{\pi})\sin(2n)x.\right]$$

$0 < x < \pi.$

3. $f(x) = \cosh x, \ 0 < x < \pi.$ (See graph ~~at end of list~~ opposite P. 49.)

$$S = \frac{2}{\pi}\left[\sum_{n=1}^{\infty}\frac{(2n-1)}{1+(2n-1)^2}(1+\cosh\pi)\sin(2n-1)x + \sum_{n=1}^{\infty}\frac{(2n)}{1+(2n)^2}(1-\cosh\pi)\sin(2n)x\right]$$

$0 < x < \pi.$

4. $f(x) = e^x, \ 0 < x < c.$

MISCELLANEOUS SERIES FOR EXPONENTIAL, HYPERBOLIC, TRIGONOMETRIC, AND LOGARITHMIC FUNCTIONS

A. Sine Series.

4. (Cont).

$$S = 2\pi\left[\sum_{n=1}^{\infty}\frac{(2n-1)(1+e^{c})}{c^2+\pi^2(2n-1)^2}\sin\frac{\pi(2n-1)x}{c}\right.$$

$$\left.+\sum_{n=1}^{\infty}\frac{2n(1-e^{c})}{c^2+\pi^2(2n)^2}\sin\frac{2n\pi x}{c}\right],\ 0<x<\pi.$$

5. $f(x) = \frac{\pi}{2}\,\frac{\sinh a(\pi-x)}{\sinh a\pi},\ 0<x<2\pi.$

$$S = \sum_{n=1}^{\infty}\frac{n\sin nx}{a^2+n^2},\ 0<x<2\pi.$$

6. $f(x) = \cos x,\ -\pi<x<\pi$

$$S = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n)x}{(2n)^2-1},\ -\pi<x<\pi.$$

7. $f(x) = \sin mx,\ -\pi<x<\pi.$

$$S = \frac{2}{\pi}\sin m\pi\sum_{n=1}^{\infty}\frac{n\sin nx}{n^2-m^2},\ -\pi<x<\pi.$$

8. $f(x) = \sin\mu x,\ 0<x<\pi,$ [If μ is a fraction.]

$$S = \frac{-2\sin\mu x}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n\sin nx}{n^2-\mu^2},\ 0<x<\pi.$$

9. If $f(x) = \frac{\pi}{2}\sin x,\ 0\leq x\leq\frac{\pi}{2},$

$$= \frac{\pi}{2},\qquad \frac{\pi}{2}<x\leq\pi$$

then when $0\leq x<\pi,$

$$f(x) = \frac{\pi}{4}\sin x + S_1 + S_2 + S_3$$

where $S_1 = -\sum_{n=1}^{\infty}\frac{(-1)^n 2n}{(2n)^2-1}\sin(2n)x,$

MISCELLANEOUS SERIES FOR EXPONENTIAL,
HYPERBOLIC, TRIGONOMETRIC, AND LOGARITHMIC FUNCTIONS

A. Sine Series.

9. (Cont).

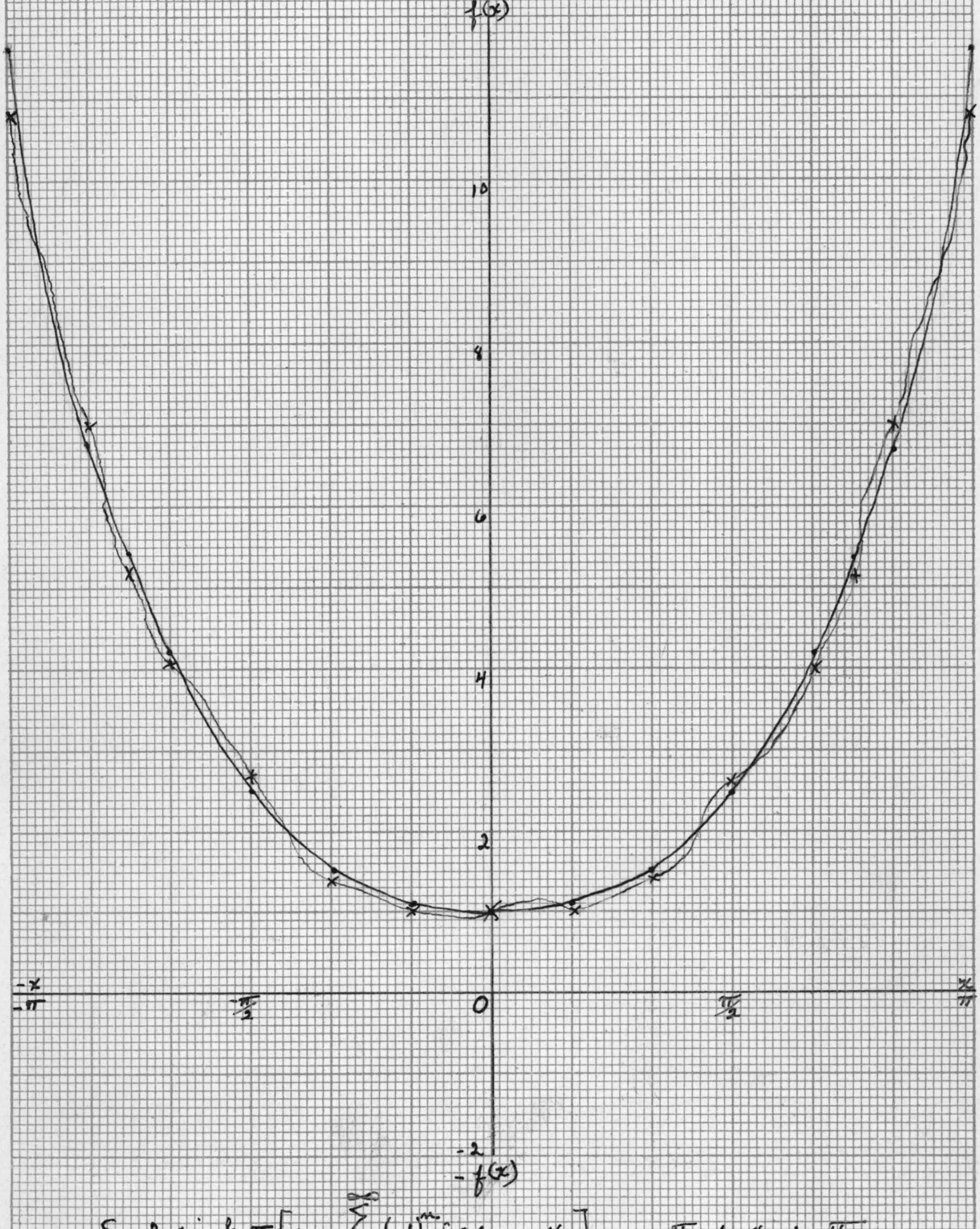
$$S_2 = \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)},$$

$$S_3 = \sum_{n=1}^{\infty} \frac{\sin 2(2n-1)x}{(2n-1)}.$$

①

$f(x) = \cosh x$

$= \frac{1}{2}(e^{x} + e^{-x})$ $\qquad -\pi < x < \pi.$



$$S = \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^{n}\cos nx}{1+n^{2}}\right], \quad -\pi < x < \pi.$$

Opp. P. 49

MISCELLANEOUS SERIES FOR EXPONENTIAL, HYPERBOLIC, TRIGONOMETRIC, AND LOGARITHMIC FUNCTIONS

B. Cosine Series.

1. $f(x) = \cosh x, \ -\pi < x < \pi.$

$$S = \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{1+n^2}\right], \ -\pi < x < \pi.$$

1' $f(x) = \dfrac{\pi}{\sinh\pi}\cosh x, \ -\pi < x < \pi.$

$$S = 2\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{1+n^2}\right], \ -\pi < x < \pi.$$

2. $f(x) = e^x, \ 0 < x < \pi.$

$$S = \frac{2}{\pi}\left[\frac{e^x-1}{2} + \sum_{n=1}^{\infty}\frac{e^x-1}{1+(2n)^2}\cos(2n)x - \sum_{n=1}^{\infty}\frac{e^x+1}{1+(2n-1)^2}\cos(2n-1)x\right], \ 0 < x < \pi.$$

3. $f(x) = e^{kx}, \ 0 < x < \pi.$

$$S = \frac{2k}{\pi}(e^{k\pi}-1)\left[\frac{1}{2k^2} + \sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{k^2+n^2}\right], \ 0 < x < \pi.$$

4. $f(x) = \sinh x, \ 0 < x < \pi.$

$$S = \left[\frac{\cosh\pi-1}{2} + \sum_{n=1}^{\infty}\frac{(\cosh\pi-1)\cos(2n)x}{1+(2n)^2} - \sum_{n=1}^{\infty}\frac{(\cosh\pi+1)\cos(2n-1)x}{1+(2n-1)^2}\right],$$

$$0 < x < \pi.$$

MISCELLANEOUS SERIES FOR EXPONENTIAL, HYPERBOLIC, TRIGONOMETRIC, AND LOGARITHMIC FUNCTIONS

B. Cosine Series. (Cont)

5. $f(x) = e^x$, $0 < x < c$.

$$S = 2c\left[\frac{e^c-1}{2} + \sum_{n=1}^{\infty}\frac{e^c-1}{c^2+\pi^2(2n)^2}\cos(2n)x - \sum_{n=1}^{\infty}\frac{e^c+1}{c^2+\pi^2(2n-1)^2}\cos(2n-1)x\right], \quad 0 < x < c.$$

6. $f(x) = \dfrac{\cosh mx}{\sinh m\pi}$, $-\pi < x < \pi$.

$$S = \frac{2}{\pi}\left[\frac{1}{2m} + \sum_{n=1}^{\infty}\frac{(-1)^n m\cos nx}{n^2+m^2}\right], \quad -\pi < x < \pi.$$

7. $f(x) = \sin x$, $0 < x < \pi$,

$= \sin x$, $-\pi < x < 0$.

$$S = \frac{4}{\pi}\left[\frac{1}{2} - \sum_{n=1}^{\infty}\frac{\cos(2n)x}{(2n)^2-1}\right], \quad -\pi < x < \pi.$$

8. $f(x) = \cos mx$, $-\pi < x < \pi$.

$$S = \frac{2}{\pi}\sin m\pi\left[\frac{1}{2m} - \sum_{n=1}^{\infty}\frac{(-1)^n m\cos nx}{n^2-m^2}\right], \quad -\pi < x < \pi.$$

9. $f(x) = x\sin x$, $0 < x < \pi$.

$$S = 1 - \frac{\cos x}{2} + 2\sum_{n=2}^{\infty}\frac{(-1)^n\cos nx}{n^2-1}, \quad 0 < x < \pi.$$

10. $f(x) = \cos \mu x$, $0 < x < \pi$. (If μ is a fraction).

$$S = \frac{2\mu\sin\mu\pi}{\pi}\left[\frac{1}{2\mu^2} + \sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{\mu^2-n^2}\right]$$

MISCELLANEOUS SERIES FOR EXPONENTIAL, HYPERBOLIC, TRIGONOMETRIC, AND LOGARITHMIC FUNCTIONS

B. Cosine Series (Cont)

11. $f(x) = \frac{1}{2}(\pi - x)\sin x,\ 0 < x < \pi.$

$$S = \frac{1}{2} + \frac{1}{4}\cos x - \sum_{n=2}^{\infty} \frac{\cos nx}{n^2-1},\ 0 < x < \pi.$$

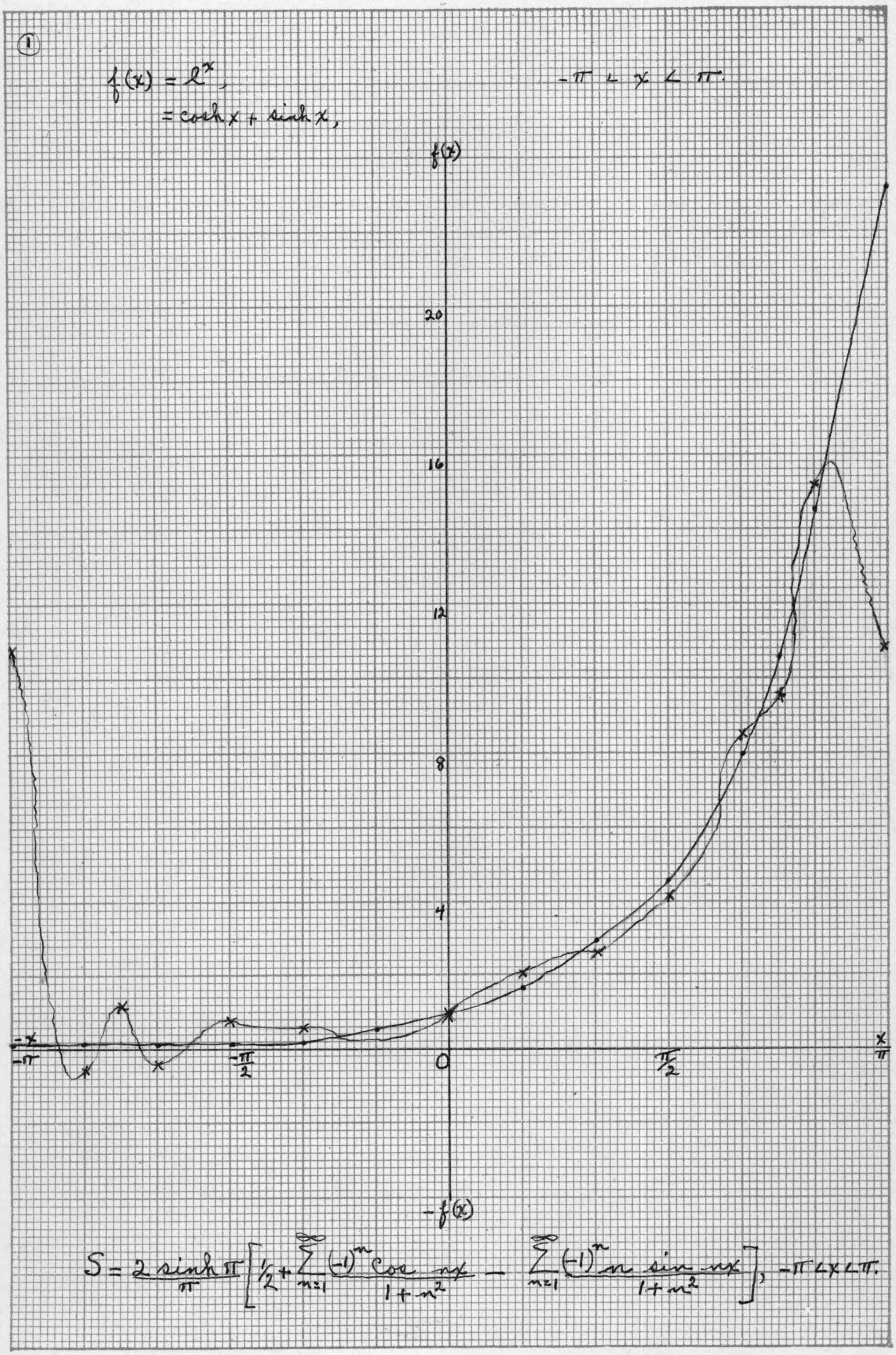
12. $f(x) = \log\left(2\sin\frac{1}{2}x\right),\ 0 < x < \pi.$

$$S = -\sum_{n=1}^{\infty} \frac{\cos nx}{n},\ 0 < x < \pi.$$

13. $f(x) = \log\left(2\cos\frac{1}{2}x\right),\ 0 \leq x < \pi.$

$$S = -\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n},\ 0 < x < \pi.$$

(See fundamental forms for these and other logarithmic series.)



1
$f(x) = e^x,$
$= \cosh x + \sinh x,$
$-\pi < x < \pi.$
$f(x)$
20
16
12
8
4
$-x$
$-\pi$
$-\frac{\pi}{2}$
0
$\frac{\pi}{2}$
x
π
$-f(x)$
$S = \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n \cos nx}{1+n^2} - \sum_{n=1}^{\infty}\frac{(-1)^n n \sin nx}{1+n^2}\right], -\pi < x < \pi.$

Opp. P. 52

MISCELLANEOUS SERIES FOR EXPONENTIAL, HYPERBOLIC, TRIGONOMETRIC, AND LOGARITHMIC FUNCTIONS

C. Sine and Cosine Series.

1. $f(x) = e^x, \ -\pi < x < \pi$. (See graph opposite this page)

$$S = \frac{2 \sinh \pi}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{1+n^2} - \sum_{n=1}^{\infty} \frac{(-1)^n n \sin nx}{1+n^2}\right], \quad -\pi < x < \pi$$

1' $f(x) = \dfrac{\pi}{2 \sinh \pi} e^x, \ -\pi < x < \pi$.

$$S = \left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{1+n^2} - \sum_{n=1}^{\infty} \frac{(-1)^n n \sin nx}{1+n^2}\right], \quad -\pi < x < \pi$$

2. $f(x) = e^{-x}, \ -\pi < x < \pi$. (See graph at end of list).

$$S = \frac{2 \sinh \pi}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{1+n^2} + \sum_{n=1}^{\infty} \frac{(-1)^n n \sin nx}{1+n^2}\right],$$

$$-\pi < x < \pi.$$

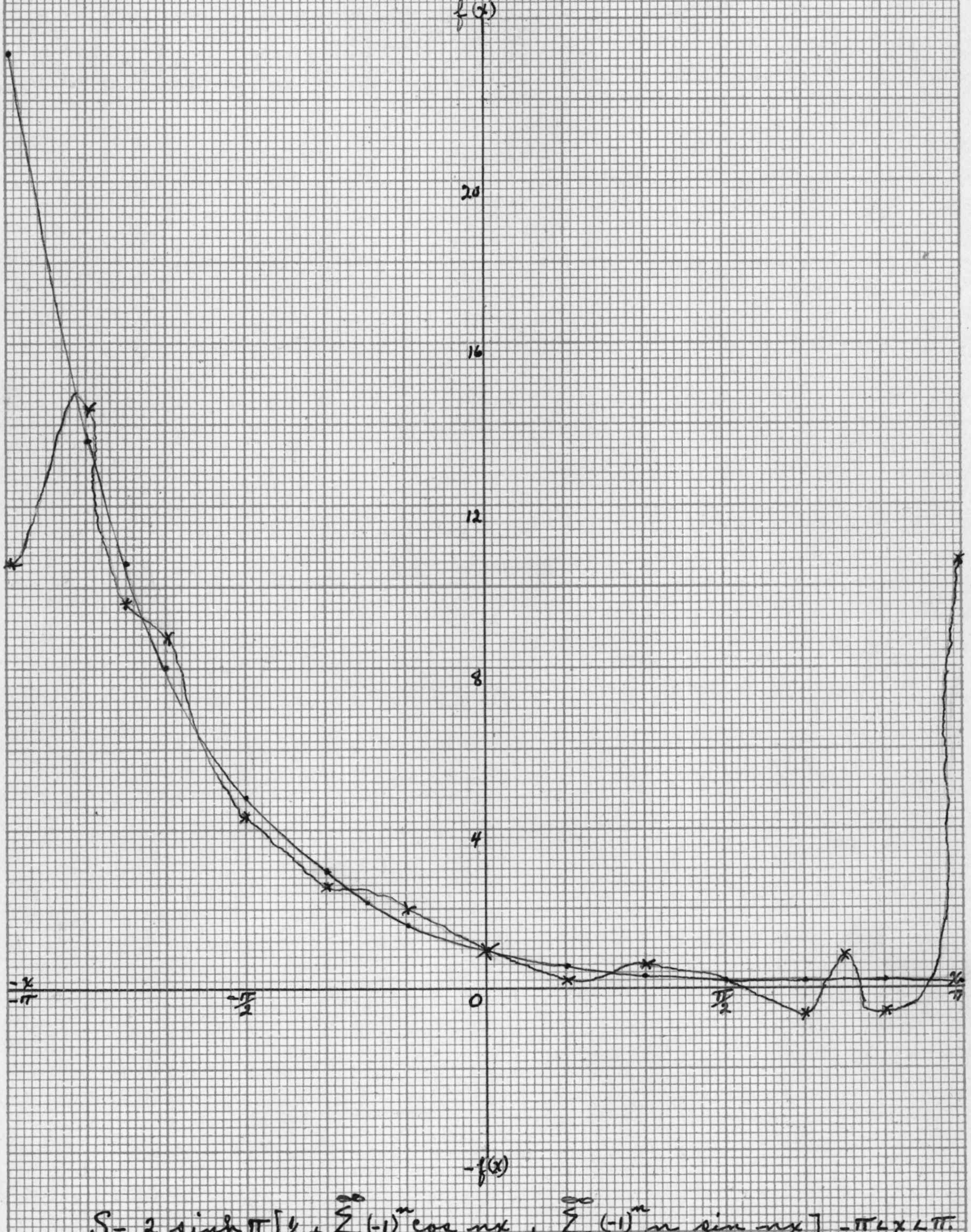
2' $f(x) = \dfrac{\pi}{2 \sinh \pi} e^{-x}, \ -\pi < x < \pi$.

$$S = \left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{1+n^2} + \sum_{n=1}^{\infty} \frac{(-1)^n n \sin nx}{1+n^2}\right],$$

$$-\pi < x < \pi.$$

② $f(x) = e^{-x}, \qquad -\pi < x < \pi.$

$\quad = \cosh x - \sinh x,$



$$S = \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n \cos nx}{1+n^2} + \sum_{n=1}^{\infty}\frac{(-1)^n n \sin nx}{1+n^2}\right], \quad -\pi < x < \pi.$$

Following P. 52

APPROXIMATIONS OF SERIES TO FUNCTIONS

IA

1. $f(x) = \frac{1}{2}(\pi - x)$, $0 < x < \pi$.

$$S = \sum_1^\infty \frac{1}{n} \sin nx, \; 0 < x < \pi.$$

x	f(x)	S_8	$(f(x) - S_8)$
0°	1.57079	0	1.57079
30	1.30898	1.40317	0.09419
45	1.17809	1.03371	0.14438
60	1.04719	1.14129	0.09410
90	0.78539	0.72381	0.06158
120	0.52359	0.49179	0.03180
135	0.39269	0.36705	0.02564
150	0.26179	0.32063	0.05884
180	0	0	0

2. $f(x) = \frac{1}{12}(2\pi^2 - 6\pi x + 3x^2)$, $0 < x < \pi$.

$$S = \sum_1^\infty \frac{1}{n^2} \cos nx, \; 0 < x < \pi.$$

x	f(x)	S_9	$(f(x) - S_9)$
0°	1.64493	1.52740	0.11753
30	0,89101	0.87189	0.01912
45	0.56545	0.56781	0.00236
60	0.27416	0.28280	0.00864
90	-0.20561	-0.19965	0.00596
120	-0.54830	-0.55538	0.00708
135	-0.66824	-0.66157	0.00667
150	-0.75391	-0.75555	0.00164
180	-0.82246	-0.81562	0.00684

APPROXIMATIONS OF SERIES TO FUNCTIONS

IA

3. $f(x) = \frac{1}{12}(2\pi^2 x - 3\pi x^2 + x^3)$, $0 < x < \pi$

$$S = \sum_{1}^{\infty} \frac{1}{n^3} \sin nx, \quad 0 < x < \pi.$$

x	f(x)	S_8	$\lvert f(x) - S_8 \rvert$
0°	0	0	0
30	0.65791	0.65967	0.00176
45	0.84782	0.84596	0.00186
60	0.95697	0.95804	0.00107
90	0.96894	0.96806	0.00088
120	0.76559	0.76522	0.00037
135	0.60560	0.60520	0.00040
150	0.41871	0.41949	0.00078
180	0	0	0

4. $f(x) = \frac{1}{720}(8\pi^4 - 60\pi^2 x^2 + 60\pi x^3 - 15x^4)$, $0 < x < \pi$.

$$S = \sum_{1}^{\infty} \frac{1}{n^4} \cos nx, \quad 0 < x < \pi.$$

x	f(x)	S_9	$\lvert f(x) - S_9 \rvert$
0°	1.08232	1.08176	0.00056
30	0.89285	0.89269	0.00016
45	0.69389	0.69388	0.00001
60	0.45598	0.45611	0.00013
90	-0.05918	-0.05913	0.00005
120	-0.52110	-0.52121	0.00011
135	-0.70127	-0.70120	0.00007
150	-0.83583	-0.83587	0.00004
180	-0.94703	-0.94694	0.00009

APPROXIMATIONS OF SERIES TO FUNCTIONS

IA

5. $f(x) = \frac{8\pi^4 x - 20\pi^2 x^3 + 15\pi x^4 - 3x^5}{720}$, $0 < x < \pi$.

$$S = \sum_1^\infty \frac{1}{n^5} \sin nx, \quad 0 < x < \pi.$$

x	f(x)	S_8	$\lvert f(x) - S_8 \rvert$
0°	0	0	0
30	0.53209	0.53211	0.00002
45	0.74088	0.74087	0.00001
60	0.89202	0.89203	0.00001
90	0.99615	0.99615	0.00000
120	0.83956	0.83957	0.00001
135	0.67864	0.67863	0.00001
150	0.47638	0.47635	0.00003
180	0	0	0

6. $f(x) = \frac{32\pi^6 - 168\pi^4 x^2 + 210\pi^2 x^4 - 126x^5 + 21x^6}{30240}$, $0 < x < \pi$.

$$S = \sum_1^\infty \frac{1}{n^6} \cos nx, \quad 0 < x < \pi.$$

x	f(x)	S_9	$\lvert f(x) - S_9 \rvert$
0°	1.01733	1.01731	0.00002
30	0.87362	0.87365	0.00003
45	0.70585	0.70586	0.00001
60	0.49074	0.49075	0.00001
90	-0.01539	-0.01540	0.00001
120	-0.50656	-0.50657	0.00001
135	-0.70639	-0.70634	0.00005
150	-0.85827	-0.85831	0.00004
180	-0.98554	-0.98555	0.00001

APPROXIMATIONS OF SERIES TO FUNCTIONS

IA

7. $f(x) = \dfrac{32\pi^6x-56\pi^4x^3+42\pi^2x^5-21\pi x^6+3x^7}{30240}$, $0 < x < \pi$.

$S = \sum_1^\infty \frac{1}{n^7} \sin nx$, $0 < x < \pi$.

x	f(x)	S_8	$\lvert f(x) - S_8 \rvert$
0°	0	0	0
30	0.50727	0.50728	0.00001
45	0.71522	0.71522	0.00000
60	0.87272	0.87273	0.00001
90	0.99955	0.99955	0.00000
120	0.85930	0.85932	0.00002
135	0.69962	0.69962	0.00000
150	0.49367	0.49364	0.00003
180	0	0	0

8. $f(x) = \dfrac{128\pi^8-640\pi^6x^2+560\pi^4x^4-280\pi^2x^6+120\pi x^7-15x^8}{1209600}$, $0 < x < \pi$

$S = \sum_1^\infty \frac{1}{n^8} \cos nx$, $0 < x < \pi$.

x	f(x)	S_9	$\lvert f(x) - S_9 \rvert$
0°	1.00407	1.00406	0.00001
30	0.86796	0.86797	0.00001
45	0.70698	0.70699	0.00001
60	0.49788	0.49789	0.00001
90	-0.00388	-0.00389	0.00001
120	-0.50179	-0.50180	0.00001
135	-0.70699	-0.70701 ~~-70701~~	0.00002
150	-0.86405	-0.86408	0.00003
180	-0.99622	-0.99624	0.00002

APPROXIMATIONS OF SERIES TO FUNCTIONS

II A

1. $f(x) = -\frac{x}{2}$, $0 < x < \pi$

$$S = \sum_{1}^{\infty} \frac{(-1)^n}{n} \sin nx, \quad 0 < x < \pi.$$

x	f(x)	S_8	$\vert f(x) - S_8 \vert$
0°	0	0	0
30	-0.26179	-0.32064	0.05885
45	-0.39269	-0.36705	0.02564
60	-0.52359	-0.49180	0.03179
90	-0.78539	-0.72382	0.06157
120	-1.04719	-1.14130	0.09411
135	-1.17809	-1.03371	0.14438
150	-1.30898	-1.40318	0.09420
180	-1.57079	0	1.57079

2. $f(x) = \frac{3x^2 - \pi^2}{12}$, $0 < x < \pi$.

$$S = \sum_{1}^{\infty} \frac{(-1)^n \cos nx}{n^2}, \quad 0 < x < \pi.$$

x	f(x)	S_9	$\vert f(x) - S_9 \vert$
0°	-0.82246	-0.81562	0.00684
30	-0.75392	-0.75555	0.00163
45	-0.66825	-0.66157	0.00668
60	-0.54831	-0.55538	0.00707
90	-0.20562	-0.19965	0.00597
120	+0.27414	+0.28280	0.00866
135	+0.56541	+0.56781	0.00240
150	+0.89094	+0.87189	0.01905
180	+1.64493	+1.52740	0.11753

APPROXIMATIONS OF SERIES TO FUNCTIONS

II A

3. $f(x) = \frac{x^3 - \pi^2 x}{12}$, $0 < x < \pi$.

$$S = \sum_1^\infty \frac{(-1)^n}{n^3} \sin nx, \quad 0 < x < \pi.$$

x	f(x)	S_8	$\lvert f(x) - S_8 \rvert$
0°	0	0	0
30	-0.41867	-0.41948	0.00081
45	-0.60558	-0.60521	0.00037
60	-0.76558	-0.76523	0.00035
90	-0.96894	-0.96804	0.00090
120	-0.95698	-0.95803	0.00105
135	-0.84783	-0.84595	0.00188
150	-0.65796	-0.65968	0.00172
180	0	0	0

4. $f(x) = \frac{-7\pi^4 + 30\pi^2 x^2 - 15x^4}{720}$, $0 < x < \pi$.

$$S = \sum_1^\infty \frac{(-1)^n}{n^4} \cos nx, \quad 0 < x < \pi.$$

x	f(x)	S_9	$\lvert f(x) - S_9 \rvert$
0°	-0.94703	-0.94694	0.00009
30	-0.83585	-0.83587	0.00002
45	-0.70129	-0.70120	0.00009
60	-0.52112	-0.52121	0.00009
90	-0.05919	-0.05913	0.00006
120	+0.45596	+0.45611	0.00015
135	+0.69387	+0.69388	0.00001
150	+0.89282	+0.89269	0.00013
180	+1.08232	+1.08176	0.00056

APPROXIMATIONS OF SERIES TO FUNCTIONS

II A

5. $f(x) = \dfrac{-7\pi^4 x + 10\pi^2 x^3 - 3x^5}{720}$, $0 < x < \pi$.

$$S = \sum_{1}^{\infty} \frac{(-1)^n}{n^5} \sin nx, \quad 0 < x < \pi.$$

x	f(x)	S_8	(f(x) - S_8)
0°	0	0	0
30	-0.47634	-0.47635	0.00001
45	-0.67862	-0.67863	0.00001
60	-0.83956	-0.83957	0.00001
90	-0.99615	-0.99615	0.00000
120	-0.89202	-0.89203	0.00001
135	-0.74088	-0.74087	0.00001
150	-0.53213	-0.53211	0.00002
180	0	0	0

6. $f(x) = \dfrac{-31\pi^6 + 147\pi^4 x^2 - 105\pi^2 x^4 + 21x^6}{30240}$, $0 < x < \pi$.

$$S = \sum_{1}^{\infty} \frac{(-1)^n}{n^6} \cos nx, \quad 0 < x < \pi.$$

x	f(x)	S_9	(f(x) - S_9)
0°	-0.98554	-0.98555	0.00001
30	-0.85829	-0.85831	0.00002
45	-0.70633	-0.70634	0.00001
60	-0.50656	-0.50657	0.00001
90	-0.01539	-0.01540	0.00001
120	+0.49074	+0.49075	0.00001
135	+0.70585	+0.70586	0.00001
150	+0.87362	+0.87365	0.00003
180	+1.01733	+1.01731	0.00002

APPROXIMATIONS OF SERIES TO FUNCTIONS

II A

7. $f(x) = \dfrac{-31\pi^6 x + 49\pi^4 x^3 - 21\pi^2 x^5 + 3x^7}{30240}, \; 0 < x < \pi$

$$S = \sum_1^\infty \frac{(-1)^n}{n^7} \sin nx, \; 0 < x < \pi$$

x	f(x)	S_8	$\lvert f(x) - S_8 \rvert$
0°	0	0	0
30	-0.49363	-0.49364	0.00001
45	-0.69962	-0.69961	0.00001
60	-0.85930	-0.85931	0.00001
90	-0.99955	-0.99955	0.00000
120	-0.87272	-0.87273	0.00001
135	-0.71522	-0.71522	0.00000
150	-0.50727	-0.50728	0.00001
180	0	0	0

8. $f(x) = \dfrac{-127\pi^8 + 620\pi^2 x^2 - 490\pi^4 x^4 + 140\pi^2 x^6 - 15x^8}{1,209,600}, \; 0 < x < \pi$

$$S = \sum_1^\infty \frac{(-1)^n}{n^8} \cos nx, \; 0 < x < \pi.$$

x	f(x)	S_9	$\lvert f(x) - S_9 \rvert$
0°	-0.99622	-0.99624	0.00002
30	-0.86405	-0.86408	0.00003
45	-0.70699	-0.70701	0.00002
60	-0.50179	-0.50180	0.00001
90	-0.00388	-0.00389	0.00001
120	+0.49788	+0.49789	0.00001
135	+0.70698	+0.70699	0.00001
150	+0.86796	+0.86798	0.00002
180	+1.00407	+1.00405	0.00002

APPROXIMATIONS OF SERIES TO FUNCTIONS

III A

1. $f(x) = \frac{\pi}{4}$, $0 < x < \pi$

$$S = \sum_{1}^{\infty} \frac{\sin (2n-1)x}{(2n-1)}, \quad 0 < x < \pi.$$

x	f(x)	S_8	$(f(x) - S_8)$
0°	0.78539	0	0.78539
30	0.78539	0.86190	0.07651
45	0.78539	0.70038	0.08501
60	0.78539	0.81654	0.03115
90	0.78539	0.72380	0.06159
120	0.78539	0.81654	0.03115
135	0.78539	0.70038	0.08501
150	0.78539	0.86190	0.07651
180	0.78539	0	0.78539

2. $f(x) = \frac{\pi^2 - 2\pi x}{8}$, $0 < x < \pi$.

$$S = \sum_{1}^{\infty} \frac{\cos (2n-1)x}{(2n-1)^2}, \quad 0 < x < \pi.$$

x	f(x)	S_9	$(f(x) - S_9)$
0°	1.23369	1.17151	0.06218
30	0.82247	0.81372	0.00875
45	0.61685	0.61469	0.00216
60	0.41123	0.41909	0.00786
90	0	0	0
120	-0.41122	-0.41909	0.00787
135	-0.61683	-0.61469	0.00214
150	-0.82243	-0.81372	0.00871
180	-1.23369	-1.17151	0.06218

APPROXIMATIONS OF SERIES TO FUNCTIONS

III A

3. $f(x) = \dfrac{\pi^2 x - \pi x^2}{8}$, $0 < x < \pi$.

$$S = \sum_1^\infty \frac{\sin (2n-1)x}{(2n-1)^3}, \quad 0 < x < \pi.$$

x	f(x)	S_8	$(f(x) - S_8)$
0°	0	0	0
30	0.53829	0.53958	0.00129
45	0.72670	0.72558	0.00112
60	0.86128	0.86163	0.00035
90	0.96894	0.96806	0.00088
120	0.86128	0.86163	0.00035
135	0.72672	0.72558	0.00114
150	0.53833	0.53958	0.00125
180	0	0	0

4. $f(x) = \dfrac{\pi^4 - 6\pi^2 x^2 + 4\pi x^3}{96}$, $0 < x < \pi$.

$$S = \sum_1^\infty \frac{\cos(2n-1)x}{(2n-1)^4}$$

x	f(x)	S_9	$(f(x) - S_9)$
0°	1.01467	1.01435	0.00032
30	0.86435	0.86428	0.00007
45	0.69759	0.69754	0.00005
60	0.48855	0.48866	0.00011
90	0	0	0
120	-0.48853	-0.48866	0.00013
135	-0.69757	-0.69754	0.00003
150	-0.86432	-0.86428	0.00004
180	-1.01467	-1.01435	0.00032

APPROXIMATIONS OF SERIES TO FUNCTIONS

III A

5. $f(x) = \dfrac{\pi^4 x - 2\pi^2 x^3 + \pi x^4}{96}$, $0 < x < \pi$.

$$S = \sum_{1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^5}, \quad 0 < x < \pi.$$

x	f(x)	S_8	$(f(x) - S_8)$
0°	0	0	0
30	0.50421	0.50424	0.00003
45	0.70975	0.70975	0.00000
60	0.86578	0.86580	0.00002
90	0.99615	0.99615	0.00000
120	0.86579	0.86580	0.00001
135	0.70977	0.70975	0.00002
150	0.50426	0.50424	0.00002
180	0	0	0

6. $f(x) = \dfrac{\pi^5 - 5\pi^4 x^2 + 5\pi^2 x^4 - 2\pi x^5}{960}$, $0 < x < \pi$.

$$S = \sum_{1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^6}, \quad 0 < x < \pi.$$

x	f(x)	S_9	$(f(x) - S_9)$
0°	1.00144	1.00143	0.00001
30	0.86596	0.86598	0.00002
45	0.70609	0.70610	0.00001
60	0.49866	0.49866	0.00000
90	0	0	0
120	-0.49866	-0.49866	0.00000
135	-0.70609	-0.70610	0.00001
150	-0.86596	-0.86598	0.00002
180	-1.00143	-1.00143	0.00000

APPROXIMATIONS OF SERIES TO FUNCTIONS

III A

7. $f(x) = \dfrac{3\pi^6 x - 5\pi^4 x^3 + 3\pi^2 x^5 - \pi x^6}{2880}$, $0 < x < \pi$.

$$S = \sum_{1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^7}, \quad 0 < x < \pi.$$

x	f(x)	S_8	$(f(x) - S_8)$
0°	0	0	0
30	0.50045	0.50045	0.00000
45	0.70741	0.70743	0.00002
60	0.86600	0.86603	0.00003
90	0.99954	0.99955	0.00001
120	0.86601	0.86603	0.00002
135	0.70741	0.70743	0.00002
150	0.50045	0.50045	0.00000
180	0	0	0

8. $f(x) = \dfrac{17\pi^8 - 84\pi^6 x^2 + 70\pi^4 x^4 - 28\pi^2 x^6 + 8\pi x^7}{161{,}280}$, $0 < x < \pi$.

$$S = \sum_{1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^8}, \quad 0 < x < \pi.$$

x	f(x)	S_9	$(f(x) - S_9)$
0°	1.00014	1.00015	0.00001
30	0.86602	0.86603	0.00001
45	0.70699	0.70700	0.00001
60	0.49985	0.49985	0.00000
90	0	0	0
120	-0.49985	-0.49985	0.00000
135	-0.70699	-0.70700	0.00001
150	-0.86602	-0.86603	0.00001
180	-1.00015	-1.00015	0.00000

APPROXIMATIONS OF SERIES TO FUNCTIONS

V A

1. $f(x) = \frac{\pi}{4} - \frac{x}{2}$, $0 < x < \pi$.

$$S = \sum_{1}^{\infty} \frac{\sin(2n)x}{(2n)}, \quad 0 < x < \pi.$$

x	f(x)	S_8	$\lvert f(x) - S_8 \rvert$
0°	0.78539	0	0.78539
30	0.52360	0.54127	0.01767
45	0.39270	0.33333	0.05937
60	0.26180	0.32475	0.06295
90	0	0	0
120	-0.26179	-0.32475	0.06296
135	-0.39268	-0.33333	0.05935
150	-0.52357	-0.54127	0.01770
180	-0.78539	0	0.78539

2. $f(x) = \frac{\pi^2 - 6\pi x + 6x^2}{24}$, $0 < x < \pi$.

$$S = \sum_{1}^{\infty} \frac{\cos(2n)x}{(2n)^2}, \quad 0 < x < \pi.$$

x	f(x)	S_9	$\lvert f(x) - S_9 \rvert$
0°	+0.41123	+0.35589	0.05534
30	+0.06854	+0.05817	0.01037
45	-0.05139	-0.04688	0.00451
60	-0.13707	-0.13629	0.00078
90	-0.20561	-0.19965	0.00596
120	-0.13707	-0.13629	0.00078
135	-0.05141	-0.04688	0.00453
150	+0.06851	+0.05817	0.01034
180	+0.41123	+0.35589	0.05534

APPROXIMATIONS OF SERIES TO FUNCTIONS

V A

3. $f(x) = \dfrac{\pi^2 x - 3\pi x^2 + 2x^3}{24}$, $0 < x < \pi$.

$S = \sum_1^\infty \dfrac{\sin(2n)x}{(2n)^3}$, $0 < x < \pi$.

x	f(x)	S_8	$\lvert f(x) - S_8 \rvert$
0°	0	0	0
30	0.11962	0.12010	0.00048
45	0.12111	0.12037	0.00074
60	0.09569	0.09640	0.00071
90	0	0	0
120	-0.09569	-0.09640	0.00071
135	-0.12111	-0.12037	0.00074
150	-0.11962	-0.12010	0.00048
180	0	0	0

4. $f(x) = \dfrac{\pi^4 - 30\pi^2 x^2 + 60\pi x^3 - 30x^4}{1440}$, $0 < x < \pi$.

$S = \sum_1^\infty \dfrac{\cos(2n)x}{(2n)^4}$, $0 < x < \pi$.

x	f(x)	S_9	$\lvert f(x) - S_9 \rvert$
0°	+0.06764	+0.06741	0.00023
30	+0.02849	+0.02841	0.00008
45	-0.00369	-0.00366	0.00003
60	-0.03256	-0.03255	0.00001
90	-0.05918	-0.05913	0.00005
120	-0.03256	-0.03255	0.00001
135	-0.00369	-0.00366	0.00003
150	+0.02849	+0.02841	0.00008
180	+0.06764	+0.06741	0.00023

APPROXIMATIONS OF SERIES TO FUNCTIONS

V A

5. $f(x) = \dfrac{\pi^4 x - 10\pi^2 x^3 + 15\pi x^4 - 6x^5}{1440}$, $0 < x < \pi$.

$$S = \sum_1^{\infty} \frac{\sin(2n)x}{(2n)^5},\quad 0 < x < \pi.$$

x	f(x)	S_8	$\lvert f(x) - S_8 \rvert$
0°	0	0	0
30	0.02787	0.02789	0.00002
45	0.03112	0.03112	0.00000
60	0.02623	0.02623	0.00000
90	0	0	0
120	-0.02623	-0.02623	0.00000
135	-0.03112	-0.03112	0.00000
150	-0.02787	-0.02789	0.00002
180	0	0	0

6. $f(x) = \dfrac{\pi^6 - 21\pi^4 x^2 + 105\pi^2 x^4 - 126\pi x^5 + 42x^6}{60{,}480}$, $0 < x < \pi$.

$$S = \sum_1^{\infty} \frac{\cos(2n)x}{(2n)^6},\quad 0 < x < \pi.$$

x	f(x)	S_9	$\lvert f(x) - S_9 \rvert$
0°	+0.01589	+0.01588	0.00001
30	+0.00766	+0.00767	0.00001
45	-0.00024	-0.00024	0.00000
60	-0.00791	-0.00791	0.00000
90	-0.01539	-0.01542	0.00003
120	-0.00791	-0.00791	0.00000
135	-0.00024	-0.00024	0.00000
150	+0.00766	+0.00767	0.00001
180	+0.01589	+0.01588	0.00001

APPROXIMATIONS OF SERIES TO FUNCTIONS

V A

7. $f(x) = \dfrac{\pi^6 x - 7\pi^4 x^3 + 21\pi^2 x^5 - 21\pi x^6 + 6x^7}{60,480}$, $0 < x < \pi$.

$S = \sum_{1}^{\infty} \dfrac{\sin(2n)x}{(2n)^7}$, $0 < x < \pi$.

x	f(x)	S_8	$(f(x) - S_8)$
0°	0	0	0
30	0.00681	0.00681	0.00000
45	0.00780	0.00781	0.00001
60	0.00671	0.00671	0.00000
90	0	0	0
120	-0.00671	-0.00671	0.00000
135	-0.00780	-0.00781	0.00001
150	-0.00681	-0.00681	0.00000
180	0	0	0

8. $f(x) = \dfrac{\pi^8 - 20\pi^6 x^2 + 70\pi^4 x^4 - 140\pi^2 x^6 + 120\pi x^7 - 30x^8}{2,419,200}$, $0 < x < \pi$

$S = \sum_{1}^{\infty} \dfrac{\cos(2n)x}{(2n)^8}$, $0 < x < \pi$.

x	f(x)	S_9	$(f(x) - S_9)$
0°	+0.00392	+0.00391	0.00001
30	+0.00194	+0.00194	0.00000
45	-0.00001	-0.00001	0.00000
60	-0.00196	-0.00196	0.00000
90	-0.00389	-0.00390	0.00001
120	-0.00196	-0.00196	0.00000
135	-0.00001	-0.00001	0.00000
150	+0.00194	+0.00194	0.00000
180	+0.00391	+0.00391	0.00000

APPROXIMATIONS OF SERIES TO FUNCTIONS

VI A

1. $f(x) = -\frac{x}{2}, \ -\frac{\pi}{2} < x < \frac{\pi}{2}$

$$S = \sum_{1}^{\infty} \frac{(-1)^n \sin(2n)x}{(2n)} = -\frac{x}{2}, \ -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

x	f(x)	S_8	$(f(x) - S_8)$
-90°	0	0	0
-60	+0.52359	+0.54127	0.01768
-45	+0.39269	+0.33333	0.05936
-30	+0.26179	+0.32475	0.06296
0	0	0	0
30	-0.26179	-0.32475	0.06296
45	-0.39269	-0.33333	0.05936
60	-0.52359	-0.54127	0.01768
90	0	0	0

2. $f(x) = \frac{-\pi^2 + 12x^2}{48}, \ -\frac{\pi}{2} < x < \frac{\pi}{2}$

$$S = \sum_{1}^{\infty} \frac{(-1)^n \cos(2n)x}{(2n)^2}, \ -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

x	f(x)	S_9	$(f(x) - S_9)$
-90°	+0.41122	+0.35589	0.05533
-60	+0.06853	+0.05817	0.01036
-45	-0.05140	-0.04688	0.00452
-30	-0.13707	-0.13629	0.00078
0	-0.20561	-0.19965	0.00596
30	-0.13707	-0.13629	0.00078
45	-0.05140	-0.04688	0.00452
60	+0.06853	+0.05817	0.01036
90	+0.41122	+0.35589	0.05533

APPROXIMATIONS OF SERIES TO FUNCTIONS

VI A

3. $f(x) = \frac{-\pi^2 x + 4x^3}{48}, \; -\frac{\pi}{2} < x < \frac{\pi}{2}$

$$S = \sum_1^\infty \frac{(-1)^n \sin(2n)x}{(2n)^3}, \; -\frac{\pi}{2} < x < \frac{\pi}{2}$$

x	f(x)	S_8	$\lvert f(x) - S_8 \rvert$
-90	0	0	0
-60	+0.11962	+0.12009	0.00047
-45	+0.12111	+0.12038	0.00073
-30	+0.09569	+0.09641	0.00072
0	0	0	0
30	-0.09569	-0.09641	0.00072
45	-0.12111	-0.12038	0.00073
60	-0.11962	-0.12009	0.00047
90	0	0	0

4. $f(x) = \frac{-7\pi^4 + 120\pi^2 x^2 - 240x^4}{11520}, \; -\frac{\pi}{2} < x < \frac{\pi}{2}$

$$S = \sum_1^\infty \frac{(-1)^n \cos(2n)x}{(2n)^4}, \; -\frac{\pi}{2} < x < \frac{\pi}{2} \,.$$

x	f(x)	S_9	$\lvert f(x) - S_9 \rvert$
-90	+0.06744	+0.06741	0.00003
-60	+0.02849	+0.02841	0.00008
-45	-0.00370	-0.00366	0.00004
-30	-0.03257	-0.03255	0.00002
0	-0.05918	-0.05913	0.00005
30	-0.03257	-0.03255	0.00002
45	-0.00370	-0.00366	0.00004
60	+0.02849	+0.02841	0.00008
90	+0.06744	+0.06741	0.00003

APPROXIMATIONS OF SERIES TO FUNCTIONS

VI A

5. $f(x) = -\dfrac{7\pi^4 x + 40\pi^2 x^3 - 48x^5}{11520}, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}.$

$$S = \sum_{1}^{8} \frac{(-1)^n \sin(2n)x}{(2n)^5}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

x	f(x)	S_8	$\lvert f(x) - S_8 \rvert$
-90	0	0	0
-60	+0.02787	+0.02789	0.00002
-45	+0.03112	+0.03112	0.00000
-30	+0.02623	+0.02623	0.00000
0	0	0	0
30	-0.02623	-0.02623	0.00000
45	-0.03112	-0.03112	0.00000
60	-0.02787	-0.02789	0.00002
90	0	0	0

6. $f(x) = -\dfrac{31\pi^6 + 588\pi^4 x^2 - 1680\pi^2 x^4 + 1344x^6}{1,935,360}, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}.$

$$S = \sum_{1}^{8} \frac{(-1)^n \cos(2n)x}{(2n)^6}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

x	f(x)	S_9	$\lvert f(x) - S_9 \rvert$
-90	+0.01589	+0.01588	0.00001
-60	+0.00766	+0.00767	0.00001
-45	-0.00024	-0.00024	0.00000
-30	-0.00791	-0.00791	0.00000
0	-0.01539	-0.01540	0.00001
30	-0.00791	-0.00791	0.00000
45	-0.00024	-0.00024	0.00000
60	+0.00766	+0.00767	0.00001
90	+0.01589	+0.01588	0.00001

APPROXIMATIONS OF SERIES TO FUNCTIONS

VI A

7. $f(x) = -\dfrac{31\pi^6 x + 196\pi^4 x^3 - 336\pi^2 x^5 + 192x^7}{1{,}935{,}360}, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

$$S = \sum_{1}^{\infty} \frac{(-1)^n \sin(2n)x}{(2n)^7}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

x	f(x)	S_8	$(f(x) - S_8)$
-90	0	0	0
-60	+0.00681	+0.00681	0.00000
-45	+0.00780	+0.00781	0.00001
-30	+0.00671	+0.00671	0.00000
0	0	0	0
30	-0.00671	-0.00671	0.00000
45	-0.00780	-0.00781	0.00001
60	-0.00681	-0.00681	0.00000
90	0	0	0

8. $f(x) = -\dfrac{127\pi^8 + 2480\pi^6 x^2 - 7840\pi^4 x^4 + 8960\pi^2 x^6 - 3840x^8}{309{,}657{,}600}, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

$$S = \sum_{1}^{\infty} \frac{(-1)^n \cos(2n)x}{(2n)^8}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

x	f(x)	S_9	$(f(x) - S_9)$
-90	+0.00392	+0.00392	0.00000
-60	+0.00194	+0.00195	0.00001
-45	-0.00001	-0.00001	0.00000
-30	-0.00196	-0.00196	0.00000
0	-0.00389	-0.00389	0.00000
30	-0.00196	-0.00196	0.00000
45	-0.00001	-0.00001	0.00000
60	+0.00194	+0.00195	0.00001
90	+0.00392	+0.00392	0.00000

APPROXIMATIONS OF SERIES TO FUNCTIONS

IV B

1. $f(x) = -\frac{\pi}{4}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

$$S = \sum_{1}^{\infty} \frac{(-1)^n \cos(2n-1)x}{(2n-1)}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

x	f(x)	S_9	$\lvert f(x) - S_9 \rvert$
-90°	-0.78539	0	0.78539
-60	-0.78539	-0.86191	0.07652
-45	-0.78539	-0.70038	0.08501
-30	-0.78539	-0.81654	0.03115
0	-0.78539	-0.72382	0.06157
30	-0.78539	-0.81654	0.03115
45	-0.78539	-0.70038	0.08501
60	-0.78539	-0.86191	0.07652
90	-0.78539	0	0.78539

2. $f(x) = -\frac{\pi x}{4}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

$$S = \sum_{1}^{\infty} \frac{(-1)^n \sin(2n-1)x}{(2n-1)^2}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

x	f(x)	S_8	$\lvert f(x) - S_8 \rvert$
-90	+1.23369	+1.17151	0.06218
-60	+0.82246	+0.81372	0.00874
-45	+0.61684	+0.61470	0.00214
-30	+0.41122	+0.41909	0.00787
0	0	0	0
30	-0.41122	-0.41909	0.00787
45	-0.61684	-0.61470	0.00214
60	-0.82246	-0.81372	0.00874
90	-1.23369	-1.17151	0.06218

APPROXIMATIONS OF SERIES TO FUNCTIONS

IV B

3. $f(x) = \dfrac{-\pi^3+4\pi x^2}{32}, \; -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}.$

$$S = \sum_1^{\infty} \frac{(-1)^n \cos(2n-1)x}{(2n-1)^3}, \; -\frac{\pi}{2} < x < \frac{\pi}{2}$$

x	f(x)	S_9	$(f(x) - S_9)$
-90	0	0	0
-60	-0.53830	-0.53958	0.00128
-45	-0.72671	-0.72558	0.00113
-30	-0.86128	-0.86163	0.00035
0	-0.96894	-0.96806	0.00088
30	-0.86128	-0.86163	0.00035
45	-0.72671	-0.72558	0.00113
60	-0.53830	-0.53958	0.00128
90	0	0	0

4. $f(x) = \dfrac{-\pi^3 x+4\pi x^3}{96}, \; -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

$$S = \sum_1^{\infty} \frac{(-1)^n \sin(2n-1)x}{(2n-1)^4}, \; -\frac{\pi}{2} < x < \frac{\pi}{2}$$

x	f(x)	S_8	$(f(x) - S_8)$
-90	+1.01467	+1.01435	0.00032
-60	+0.86434	+0.86429	0.00005
-45	+0.69758	+0.69775	0.00017
-30	+0.48854	+0.48866	0.00012
0	0	0	0
30	-0.48854	-0.48866	0.00012
45	-0.69758	-0.69775	0.00017
60	-0.86434	-0.86429	0.00005
90	-1.01467	-1.01435	0.00032

APPROXIMATIONS OF SERIES TO FUNCTIONS

IV B

5. $f(x) = -\dfrac{5\pi^5+24\pi^3x^2-16\pi x^4}{1536}, \; -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

$$S = \sum_{1}^{\infty} \frac{(-1)^n \cos(2n-1)x}{(2n-1)^5}, \; -\frac{\pi}{2} < x < \frac{\pi}{2}$$

x	f(x)	S	$(f(x) - S_9)$
-90	0	0	0
-60	-0.50423	-0.50425	0.00002
-45	-0.70976	-0.70975	0.00001
-30	-0.86579	-0.86581	0.00002
0	-0.99615	-0.99616	0.00001
30	-0.86579	-0.86581	0.00002
45	-0.70976	-0.70975	0.00001
60	-0.50423	-0.50425	0.00002
90	0	0	0

6. $f(x) = -\dfrac{25\pi^5x+40\pi^3x^3-16\pi x^5}{7680}, \; -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

$$S = \sum_{1}^{\infty} \frac{(-1)^n \sin(2n-1)x}{(2n-1)^6}, \; -\frac{\pi}{2} < x < \frac{\pi}{2}$$

x	f(x)	S	$(f(x) - S_8)$
-90	+1.00144	+1.00143	0.00001
-60	+0.86595	+0.86598	0.00003
-45	+0.70608	+0.70611	0.00003
-30	+0.49865	+0.49866	0.00001
0	0	0	0
30	-0.49865	-0.49866	0.00001
45	-0.70608	-0.70611	0.00003
60	-0.86595	-0.86598	0.00003
90	-1.00144	-1.00143	0.00001

APPROXIMATIONS OF SERIES TO FUNCTIONS

IV B

7. $f(x) = -\dfrac{61\pi^7+300\pi^5x^2-240\pi^3x^4+64\pi x^6}{184,320}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

$$S = \sum_{1}^{\infty} \frac{(-1)^n\cos(2n-1)x}{(2n-1)^7}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

x	f(x)	S	(f(x) - S_9)
-90	0	0	0
-60	-0.50046	-0.50045	0.00001
-45	-0.70742	-0.70743	0.00001
-30	-0.86604	-0.86603	0.00002
0	-0.99954	-0.99956	0.00001
30	-0.86604	-0.86603	0.00001
45	-0.70742	-0.70743	0.00001
60	-0.50046	-0.50045	0.00001
90	0	0	0

8. $f(x) = -\dfrac{427\pi^7x+700\pi^5x^3-336\pi^3x^5+64\pi x^7}{1,290,240}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

$$S = \sum_{1}^{\infty} \frac{(-1)^n\sin(2n-1)x}{(2n-1)^8}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

x	f(x)	S	(f(x) - S_8)
-90	+1.00014	+1.00015	0.00001
-60	+0.86601	+0.86603	0.00002
-45	+0.70698	+0.70701	0.00003
-30	+0.49983	+0.49985	0.00002
0	0	0	0
30	-0.49983	-0.49985	0.00002
45	-0.70698	-0.70701	0.00003
60	-0.86601	-0.86603	0.00002
90	-1.00014	-1.00015	0.00001

APPROXIMATIONS OF SERIES TO FUNCTIONS
- POLYNOMIAL SEGMENTS

A

1. $f(x) = x,\ 0 < x < \pi,$

$\quad = -x,\ -\pi < x < 0.$

$$S = -2\sum_{1}^{\infty} \frac{(-1)^n}{n} \sin nx,\ -\pi < x < \pi$$

x	$f(x)^*$	S_8^*	$\lvert f(x) - S_8 \rvert$
0°	0	0	0
30	0.05236	0.06413	0.01177
45	0.07854	0.07341	0.00513
60	0.10472	0.09836	0.00636
90	0.15708	0.14476	0.0123 2
120	0.20944	0.22826	0.01882
135	0.23562	0.20674	0.02888
150	0.26180	0.20864	0.05316
180	0.31416	0	0.31416

(*Original function and series divided by 10 in every case.)

APPROXIMATIONS OF SERIES TO FUNCTIONS
- POLYNOMIAL SEGMENTS

A

2. $f(x) = x(\pi^2-x^2), \quad 0 < x < \pi,$

$\quad = -x(\pi^2-x^2), \quad -\pi < x < 0.$

$$S = -12\sum_1^{\infty} \frac{(-1)^n}{n^3} \sin nx, \quad -\pi < x < \pi.$$

x	f(x)	S_8*	$(f(x) - S_8)$
0°	0 *	0	0
30	0.05024	0.05034	0.00010
45	0.07267	0.07263	0.00004
60	0.09187	0.09183	0.00004
90	0.11627	0.11616	0.00011
120	0.11484	0.11496	0.00012
135	0.10174	0.10151	0.00023
150	0.07896	0.07916	0.00020
180	0	0	0

(*Original function and series divided by 100 in every case)

APPROXIMATIONS OF SERIES TO FUNCTIONS
- POLYNOMIAL SEGMENTS

A

3. $f(x) = 7\pi^4 x - 10\pi^2 x^3 + 3x^5, \quad 0 < x < \pi,$

$\quad = -7\pi^4 x + 10\pi^2 x^3 - 3x^5, \quad -\pi < x < 0.$

$$S = -720\sum_{1}^{\infty} \frac{(-1)^n}{n^5} \sin nx, \; -\pi < x < \pi.$$

x	f(x)*	S_8*	$\lvert(f(x) - S_8)\rvert$
0°	0	0	0
30	0.34297	0.34297	0.00000
45	0.48861	0.48861	0.00000
60	0.60448	0.60449	0.00001
90	0.71723	0.71723	0.00000
120	0.64227	0.64226	0.00001
135	0.53345	0.53343	0.00002
150	0.38314	0.38312	0.00002
180	0	0	0

(*Original function and series divided by 1,000 in every case.)

APPROXIMATIONS OF SERIES TO FUNCTIONS - POLYNOMIAL SEGMENTS

A

4. $f(x) = 3x^7 - 21\pi^2x^5 + 49\pi^4x^3 - 31\pi^6x, \; 0 < x < \pi.$

$= -3x^7 + 21\pi^2x^5 - 49\pi^4x^3 + 31\pi^6x, -\pi < x < \pi.$

$$S = 30240\sum_1^\infty \frac{(-1)^n}{n^7} \sin nx, \; -\pi < x < \pi.$$

x	f(x)*	S_8*	$\lvert f(x) - S_8 \rvert$
0°	0	0	0
30	0.14928	0.14928	0.00000
45	0.21156	0.21156	0.00000
60	0.25985	0.25985	0.00000
90	0.30226	0.30226	0.00000
120	0.26391	0.26391	0.00000
135	0.21629	0.21628	0.00001
150	0.15341	0.15340	0.00001
180	0	0	0

(*Original function and series divided by 100,000 in each case.)

APPROXIMATIONS OF SERIES TO FUNCTIONS - POLYNOMIAL SEGMENTS

A

5. $f(x) = x(\pi^2-x^2)^2, \quad 0 < x < \pi,$

$= -x(\pi^2-x^2)^2, \quad -\pi < x < \pi.$

$$S = \sum_{1}^{\infty} (-1)^n\left(\frac{16\pi^2}{n^3} - \frac{240}{n^5}\right) \sin nx, \quad -\pi < x < \pi.$$

x	f(x)*	S_8	$(f(x) - S_8)$
0°	0	0	0
30	0.48208	0.48082	0.00126
45	0.67240	0.67300	0.00060
60	0.80597	0.80657	0.00060
90	0.86068	0.86209	0.00141
120	0.62968	0.62801	0.00167
135	0.43932	0.44222	0.00290
150	0.23812	0.23539	0.00273
180	0	0	0

(*Original function and series divided by 100 in each case)

APPROXIMATIONS OF SERIES TO FUNCTIONS - POLYNOMIAL SEGMENTS

A

6. $f(x) = x(\pi^2-x^2)^3,\ 0 < x < \pi,$

$= -x(\pi^2-x^2)^3, -\pi < x < 0.$

$$S = \sum_1^\infty (-1)^n \left(\frac{960\pi^2}{n^5} - \frac{10080}{n^7}\right) \sin nx,\ -\pi < x < \pi.$$

x	f(x)	S_8	$\lvert f(x) - S_8 \rvert$
0°	0	0	0
30	0.46258	0.46257	0.00001
45	0.62215	0.62218	0.00003
60	0.70708	0.70708	0.00000
90	0.63709	0.63714	0.00005
120	0.34526	0.34531	0.00005
135	0.18970	0.18982	0.00012
150	0.07182	0.07174	0.00008
180	0	0	0

(*Original function and series divided by 100 in each case.)

APPROXIMATIONS OF SERIES TO FUNCTIONS - POLYNOMIAL SEGMENTS

B

1. $f(x) = (\pi^2 - x^2), \; -\pi < x < \pi.$

$$S = \frac{2\pi^2}{3} - 4\sum_{1}^{\infty} \frac{(-1)^n}{n^2} \cos nx, \; -\pi < x < \pi.$$

x	f(x)*	S_9*	$\lvert f(x) - S_9 \rvert$
0°	0.98696	0.98422	0.00274
30	0.95954	0.96019	0.00065
45	0.92528	0.92260	0.00268
60	0.87730	0.88012	0.00282
90	0.74022	0.73783	0.00239
120	0.54832	0.54485	0.00347
135	0.43181	0.43085	0.00096
150	0.30159	0.30922	0.00763
180	0	0.04701	0.04701

(*Original function and series divided by 10 in every case)

APPROXIMATIONS OF SERIES TO FUNCTIONS - POLYNOMIAL SEGMENTS

B

2. $f(x) = (\pi^2 - x^2)^2, \; -\pi < x < \pi.$

$$S = \frac{8\pi^4}{15} - 48 \sum_{1}^{\infty} \frac{(-1)^n}{n^4} \cos nx, \; -\pi < x < \pi.$$

x	f(x)*	S_9*	$\lvert f(x) - S_9 \rvert$
0	0.97409	0.97404	0.00005
30	0.92072	0.92073	0.00001
45	0.85613	0.85609	0.00004
60	0.76965	0.76969	0.00004
90	0.54793	0.54790	0.00003
120	0.30065	0.30058	0.00007
135	0.18646	0.18645	0.00001
150	0.09096	0.09102	0.00006
180	0	0.00269	0.00269

(*Original function and series divided by 100 in every case.)

APPROXIMATIONS OF SERIES TO FUNCTIONS
- POLYNOMIAL SEGMENTS

B

3. $f(x) = x^6 - 5^2x^4 + 7\pi^4x^2, \quad -\pi < x < \pi.$

$$S = \frac{31\pi^6}{21} + 1440 \sum_1^\infty \frac{(-1)^n}{n^6} \cos nx, \quad -\pi < x < \pi.$$

x	f(x)*	S_9*	$\lvert f(x) - S_9 \rvert$
0°	0	0	0
30	0.01832	0.01832	0.00000
45	0.04021	0.04021	0.00000
60	0.06897	0.06897	0.00000
90	0.13970	0.13970	0.00000
120	0.21258	0.21259	0.00001
135	0.24356	0.24356	0.00000
150	0.26773	0.26772	0.00001
180	0.28842	0.28842	0.00000

(*Original function and series divided by 10,000 in each case.)

APPROXIMATIONS OF SERIES TO FUNCTIONS - POLYNOMIAL SEGMENTS

B

4. $f(x) = 3x^8 - 28\pi^2x^6 + 98\pi^4x^4 - 124\pi^6x^2, \quad -\pi < x < \pi.$

$$S = -\frac{127\pi^8}{5} - 241920 \sum_1^\infty \frac{(-1)^n}{n^8} \cos nx, \quad -\pi < x < \pi.$$

x	f(x) *	S_9 *	(f(x) - S_9)
0°	0	0	0
30	0.03198	0.03197	0.00001
45	0.06996	0.06997	0.00001
60	0.11962	0.11961	0.00001
90	0.24007	0.24007	0.00000
120	0.36145	0.36145	0.00000
135	0.41204	0.41204	0.00000
150	0.45098	0.45098	0.00000
180	0.48391	0.48391	0.00000

(*Original function and series divided by 1,000,000 in each case.)

APPROXIMATIONS OF SERIES TO FUNCTIONS
- POLYNOMIAL SEGMENTS

B

5. $f(x) = (\pi^2 - x^2)^3, \quad -\pi < x < \pi.$

$$S = \frac{16\pi^6}{35} + \sum_1^\infty (-1)^n\left(\frac{96\pi^2}{n^4} - \frac{1440}{n^6}\right) \cos nx, \quad -\pi < x < \pi.$$

x	f(x)*	S_9*	$\lvert f(x) - S_9\rvert$
0°	0.96139	0.96147	0.00008
30	0.88348	0.88349	0.00001
45	0.79216	0.79225	0.00009
60	0.67521	0.67511	0.00010
90	0.40559	0.40564	0.00005
120	0.16485	0.16497	0.00012
135	0.08051	0.08049	0.00002
150	0.02743	0.02724	0.00019
180	0	-0.00049	0.00049

(*Original function and series divided by 1000 in each case.)

APPROXIMATIONS OF SERIES TO FUNCTIONS - POLYNOMIAL SEGMENTS

B

6. $f(x) = (\pi^2 - x^2)^4, \quad -\pi < x < \pi.$

$$S = \frac{128\pi^8}{315} + \sum_1^\infty (-1)^n \left(\frac{7680\pi^2}{n^6} - \frac{80640}{n^8}\right) \cos nx, \quad -\pi < x < \pi.$$

x	f(x)*	S_9	$\lvert f(x) - S_9 \rvert$
0°	0.94885	0.94885	0.00000
30	0.84773	0.84773	0.00000
45	0.73297	0.73294	0.00003
60	0.59236	0.59236	0.00000
90	0.30022	0.30020	0.00002
120	0.09039	0.09039	0.00000
135	0.03477	0.03471	0.00006
150	0.00827	0.00830	0.00003
180	0	-0.00004	0.00004

(*Original function and series divided by 10,000 in each case.)

APPROXIMATIONS OF SERIES TO FUNCTIONS
- POLYNOMIAL SEGMENTS

C

1. $f(x) = (\pi - x), \quad 0 < x < \pi,$

$= -(\pi - x), \; -\pi < x < 0.$

$$S = \frac{\pi}{2} + \frac{4}{\pi}\sum_{1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}, \; -\pi < x < \pi.$$

x	$f(x)^*$	S_9^*	$\lvert f(x) - S_9 \rvert$
0°	0.31416	0.30624	0.00792
30	0.26180	0.26069	0.00111
45	0.23562	0.23534	0.00028
60	0.20944	0.21044	0.00100
90	0.15708	0.15708	0.00000
120	0.10472	0.10372	0.00100
135	0.07854	0.07881	0.00027
150	0.05236	0.05347	0.00111
180	0	0.00792	0.00792

(*Original function and series divided by 10 in each case)

APPROXIMATIONS OF SERIES TO FUNCTIONS
- POLYNOMIAL SEGMENTS

C

2. $f(x) = \dfrac{\pi^3 - 3\pi x^2 + 2x^3}{\pi}, \quad 0 < x < \pi,$

$\quad = \dfrac{\pi^3 - 3\pi x^2 - 2x^3}{\pi}, \quad -\pi < x < 0.$

$$S = \frac{\pi^2}{2} + \frac{48}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^4}, \quad -\pi < x < \pi.$$

x	f(x)*	S_9*	$\lvert f(x) - S_9 \rvert$
0°	0.98696	0.98680	0.00016
30	0.91392	0.91382	0.00010
45	0.83275	0.83272	0.00003
60	0.73108	0.73114	0.00006
90	0.49348	0.49348	0.00000
120	0.25588	0.25582	0.00006
135	0.15422	0.15424	0.00002
150	0.07312	0.07314	0.00002
180	0	0.00016	0.00016

(*Original function and series divided by 10 in each case)

APPROXIMATIONS OF SERIES TO FUNCTIONS - POLYNOMIAL SEGMENTS

C

3. $f(x) = \frac{2x^3}{\pi^3} - \frac{3x^2}{\pi^2} + 1,\ 0 < x < \pi,$

$$= -\frac{2x^3}{\pi^3} - \frac{3x^2}{\pi^2} + 1,\ -\pi < x < 0.$$

$$S = \frac{1}{2} + \frac{48}{\pi^4}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^4},\ -\pi < x < \pi.$$

x	$f(x)^*$	$S_9{}^*$	$\lvert f(x) - S_9 \rvert$
0°	1.00000	0.99983	0.00017
30	0.92592	0.92588	0.00004
45	0.84375	0.84372	0.00003
60	0.74074	0.74079	0.00005
90	0.50000	0.50000	0.00000
120	0.25926	0.25921	0.00005
135	0.15625	0.15628	0.00003
150	0.07408	0.07412	0.00004
180	0	0.00017	0.00017

(*Original function and series divided by 100 in each case.)

APPROXIMATIONS OF SERIES TO FUNCTIONS
- POLYNOMIAL SEGMENTS

C

4. $$f(x) = \frac{\pi^4 - 6\pi^2 x + 8\pi x^3 - 3x^4}{\pi}, \quad 0 < x < \pi,$$

$$= \frac{\pi^4 - 6\pi^2 x - 8\pi x^3 - 3x^4}{\pi}, \quad -\pi < x < \pi$$

$$S = \frac{2\pi^3}{5} + \frac{48}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^4} + \frac{144}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n)x}{(2n)^4}$$

$$-\pi < x < \pi$$

x	f(x)*	S_9*	$\lvert f(x) - S_9 \rvert$
0°	0.31006	0.30990	0.00016
30	0.26915	0.26910	0.00005
45	0.22891	0.22892	0.00001
60	0.18374	0.18377	0.00003
90	0.09690	0.09692	0.00002
120	0.03445	0.03444	0.00001
135	0.01576	0.01577	0.00001
150	0.00503	0.00499	0.00004
180	0	-0.00006	0.00006

(*Original function and series divided by 100 in each case.)

APPROXIMATIONS OF SERIES TO FUNCTIONS
- POLYNOMIAL SEGMENTS

C

5. $f(x) = (\pi - x)^3, \ 0 < x < \pi$

$= -(\pi - x)^3, \ -\pi < x < 0.$

$$S = \frac{\pi^3}{4} - \frac{24}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^4} + 6\pi \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}, \ -\pi < x < \pi.$$

x	$f(x)^*$	$S_9{}^*$	$\lvert f(x) - S_9 \rvert$
0°	0.31006	0.28859	0.02147
30	0.17944	0.17584	0.00360
45	0.13081	0.13126	0.00045
60	0.09187	0.09349	0.00162
90	0.03876	0.03988	0.00112
120	0.01148	0.01016	0.00132
135	0.00485	0.00610	0.00125
150	0.00144	0.00113	0.00031
180	0	0.00127	0.00127

(*Original function and series divided by 100.)

APPROXIMATIONS OF SERIES TO FUNCTIONS - POLYNOMIAL SEGMENTS

C

6. $f(x) = \pi^5 - 10\pi^3x^2 + 20\pi^2x^3 - 15\pi x^4 + 4x^5, \; 0 < x < \pi.$

$= \pi^5 - 10\pi^3x^2 - 20\pi^2x^3 - 15\pi x^4 - 4x^5, \; -\pi < x < 0.$

$$S = \frac{\pi^4}{3} - \frac{1920}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^6} + 240\sum_{n=1}^{\infty} \frac{\cos nx}{n^4}$$

$-\pi < x < \pi.$

x	f(x)	S_9	$\lvert f(x) - S_9 \rvert$
0°	0.97409	0.97277	0.00132
30	0.78293	0.78250	0.00043
45	0.61642	0.61638	0.00004
60	0.44897	0.44928	0.00031
90	0.18265	0.18278	0.00013
120	0.04410	0.04387	0.00023
135	0.01522	0.01544	0.00022
150	0.00326	0.00326	0.00000
180	0	0.01914	0.01914

APPROXIMATIONS OF SERIES TO FUNCTIONS - POLYNOMIAL SEGMENTS

C

7. $$f(x) = \frac{\pi^6-15\pi^4x^2+40\pi^3x^3-45\pi^2x^4+24\pi x^5-5x^6}{\pi},\quad 0<x<\pi,$$

$$= \frac{\pi^6-15\pi^4x^2-40\pi^3x^3-45\pi^2x^4-24\pi x^5-5x^6}{\pi},\quad -\pi<x<0.$$

$$S = \frac{2\pi^5}{7} - \frac{4320}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^6} - \frac{7200}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n)x}{(2n)^6} + 480\pi\sum_{n=1}^{\infty}\frac{\cos nx}{n^4}.$$

x	f(x)*	S_9*	$\lvert f(x) - S_9\rvert$
0°	0.30602	0.30522	0.00080
30	0.22547	0.22519	0.00028
45	0.16339	0.16337	0.00002
60	0.10746	0.10765	0.00019
90	0.03347	0.03361	0.00014
120	0.00546	0.00530	0.00016
135	0.00142	0.00156	0.00014
150	0.00020	0.00020	0.00000
180	0	0.00016	0.00016

(*Original function and series divided by 1000 in each case.)

APPROXIMATIONS OF SERIES TO FUNCTIONS - POLYNOMIAL SEGMENTS

C

8. $$f(x) = \frac{\pi^7-21\pi^5x^2+70\pi^4x^3-105\pi^3x^4+84\pi^2x^5-35\pi x^6+6x^7}{\pi}, \quad 0 < x < \pi,$$

$$= \frac{\pi^7-21\pi^5x^2-70\pi^4x^3-105\pi^3x^4-84\pi^2x^5-35\pi x^6-6x^7}{\pi}, \quad -\pi < x < 0.$$

$$S = \frac{\pi^6}{4} + \frac{120960}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^8} - 20160 \sum_{n=1}^{\infty} \frac{\cos nx}{n^6} + 840\pi^2 \sum_{n=1}^{\infty} \frac{\cos nx}{n^4}$$

x	f(x)*	S_9	$\lvert f(x) - S_9 \rvert$
0°	0.96138	0.95733	0.00405
30	0.64727	0.64228	0.00499
45	0.42777	0.42778	0.00001
60	0.25321	0.25426	0.00105
90	0.06009	0.06100	0.00091
120	0.00659	0.00566	0.00093
135	0.00129	0.00191	0.00062
150	0.00125	0.00023	0.00102
180	0	0.00081	0.00081

(*Original function and series divided by 1000 in each case.)

NUMERICAL TABLES

Table 1. Multiple Values of cosine x.

Table 2. Multiple Values of sine x.

Table 3. Exponential and Hyperbolic Functions.

Table 4. Multiple Powers of x.

Table 5. $\sum_{1}^{\infty} \frac{1}{n^k} \begin{matrix} \sin nx \\ \cos nx \end{matrix}$

Table 6. $\sum_{1}^{\infty} (-1)^n \frac{1}{n^k} \begin{matrix} \sin nx \\ \cos nx \end{matrix}$

Table 7. $\sum_{1}^{\infty} \frac{1}{(2n-1)^k} \begin{matrix} \sin(2n-1)x \\ \cos(2n-1)x \end{matrix}$

Table 8. $\sum_{1}^{\infty} \frac{1}{(2n)^k} \begin{matrix} \sin(2n)x \\ \cos(2n)x \end{matrix}$

Table 9. $\sum_{1}^{\infty} (-1)^n \frac{1}{(2n)^k} \begin{matrix} \sin(2n)x \\ \cos(2n)x \end{matrix}$

Table 10. $\sum_{1}^{\infty} (-1)^n \frac{1}{(2n-1)^k} \begin{matrix} \cos(2n-1)x \\ \sin(2n-1)x \end{matrix}$

Table 11. Bernoulli's Numbers - B_n
Euler's Numbers - E_n
Factorial Numbers - $n!$

Table 12. Infinite Series.

NUMERICAL TABLES

TABLE 1. MULTIPLE VALUES OF COSINE x

x		cos x	cos 2x	cos 3x	cos 4x	cos 5x	cos 6x	cos 7x	cos 8x
0°	0^r	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
10		.98481	.93969	.86603	.76604	.64279	.50000	.34202	.17365
20		.93969	.76604	.50000	.17365	- .17365	- .50000	- .76604	- .93969
30	$\pi/6$	.86603	.50000	0	- .50000	- .86603	-1.00000	- .86603	- .50000
40		.76604	.17365	.50000	- .93969	- .93969	- .50000	.17365	.76604
45	$\pi/4$	.70711	0	- .70711	-1.00000	- .70711	0	.70711	1.00000
50		.64279	- .17365	- .86603	- .93969	- .34202	.50000	.98481	.76604
60	$\pi/3$	.50000	- .50000	-1.00000	- .50000	.50000	1.00000	.50000	- .50000
70		.34202	- .76604	- .86603	.17365	.98481	.50000	- .64279	- .93969
80		.17365	- .93969	- .50000	.76604	.76604	- .50000	- .93969	.17365
90	$\pi/2$	0	-1.00000	0	1.00000	0	-1.00000	0	1.00000
100		- .17365	- .93969	.50000	.76604	- .76604	- .50000	.93969	.17365
110		- .34202	- .76604	.86603	.17365	- .98481	.50000	.64279	- .93969
120	$2\pi/3$	- .50000	- .50000	1.00000	- .50000	- .50000	1.00000	- .50000	- .50000
130		- .64279	- .17365	.86603	- .93969	.34202	.50000	- .98481	.76604
135	$3\pi/4$	- .70711	0	.70711	-1.00000	.70711	0	- .70711	1.00000
140		- .76604	.17365	.50000	- .93969	.93969	- .50000	- .17365	.76604
150	$5\pi/6$	- .86603	.50000	0	- .50000	.86603	-1.00000	.86603	- .50000
160		- .93969	.76604	- .50000	.17365	.17365	- .50000	.76604	- .93969
170		- .98481	.93969	- .86603	.76604	- .64279	.50000	- .34202	.17365
180	π	-1.00000	1.00000	-1.00000	1.00000	-1.00000	1.00000	-1.00000	1.00000

TABLE 2. MULTIPLE VALUES OF SINE x

x		Sin x	Sin 2x	Sin 3x	Sin 4x	Sin 5x	Sin 6x	Sin 7x	Sin 8x
0°	0^r	0	0	0	0	0	0	0	0
10		.17365	.34202	.50000	.64279	.76604	.86603	.93969	.98481
20		.34202	.64279	.86603	.98481	.98481	.86603	.64279	.34202
30	$\pi/6$	.50000	.86603	1.00000	.86603	.50000	0	- .50000	- .86603
40		.64279	.98481	.86603	.34202	- .34202	- .86603	- .98481	- .64279
45	$\pi/4$	.7071	1.00000	.70711	0	- .70711	-1.00000	- .70711	0
50		.76604	.98481	.50000	- .34202	- .93969	- .86603	- .17365	.64279
60	$\pi/3$	.86603	.86603	0	- .86603	- .86603	0	.86603	.86603
70		.93969	.64279	- .50000	- .98481	- .17365	.86603	.76604	- .34202
80		.98481	.34202	- .86603	- .64279	.64279	.86603	- .34202	- .98481
90	$\pi/2$	1.00000	0	-1.00000	0	1.00000	0	-1.00000	0
100		.98481	- .34202	- .86603	.64279	.64279	- .86603	- .34202	.98481
110		.93969	- .64279	- .50000	.98481	- .17365	- .86603	.76604	.34202
120	$2\pi/3$	.86603	- .86603	0	.86603	- .86603	0	.86603	- .86603
130		.76604	- .98481	.50000	.34202	- .93969	.86603	- .17365	- .64279
135	$3\pi/4$	.70711	-1.00000	.70711	0	- .70711	1.00000	- .70711	0
140		.64279	- .98481	.86603	- .34202	- .34202	.86603	- .98481	.64279
150	$5\pi/6$	.50000	- .86603	1.00000	- .86603	.50000	0	- .50000	.86603
160		.34202	- .64279	.86603	- .98481	.98481	- .86603	.64279	- .34202
170		.17365	- .34202	.50000	- .64279	.76604	- .86603	.93969	- .98481
180	π	0	0	0	0	0	0	0	0

TABLE 3. EXPONENTIAL AND HYPERBOLIC FUNCTIONS

x		e^x	e^{-x}	sinh x	cosh x	tanh x	ctnh x
0°	0^r	1.00000	1.00000	0.00000	1.00000	0	∞
30	$\pi/6$	1.68809	0.59239	0.54786	1.14025	0.48047	2.08127
45	$\pi/4$	2.19328	0.45594	0.86867	1.32461	0.65579	1.52487
60	$\pi/3$	2.84965	0.35093	1.24937	1.60035	0.78068	1.28092
90	$\pi/2$	4.81048	0.20788	2.30130	2.50918	0.91715	1.09033
120	$2\pi/3$	8.12053	0.12300	3.99875	4.12175	0.97015	1.03075
135	$3\pi/4$	10.55072	0.09478	5.22797	5.32275	0.98219	1.01812
150	$5\pi/6$	13.70820	0.07295	6.81746	6.88941	0.98955	1.01055
180	π	23.14069	0.04321	11.54876	11.59195	0.99627	1.00373

TABLE 4. MULTIPLE POWERS OF x

x	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8
0	0	0	0	0	0	0	0	0
$\pi/6$	0.52359	0.27415	0.14354	0.07516	0.03935	0.02060	0.01079	0.00565
$\pi/4$	0.78539	0.61684	0.48446	0.38049	0.29883	0.23470	0.18433	0.14477
$\pi/3$	1.04719	1.09661	1.14836	1.20255	1.25930	1.31873	1.38096	1.44613
$\pi/2$	1.57079	2.46738	3.87574	6.08797	9.56292	15.02134	23.59537	37.06337
$2\pi/3$	2.09438	4.38643	9.18685	19.24075	40.29744	84.39815	176.76180	370.20638
$3\pi/4$	2.35617	5.55154	13.08037	30.81958	72.61617	171.09604	403.13136	949.84602
$5\pi/6$	2.61795	6.85366	17.94254	46.97267	122.97210	321.93481	842.80924	2206.43245
π	3.14159	9.86959	31.00621	97.40880	306.01851	961.38469	3020.27653	9488.47054

TABLE 5. $\sum_{1}^{8} \frac{1}{n^k} \begin{matrix} \sin nx \\ \cos nx \end{matrix}$, $0 < x < \pi$ $(k = 1.......8)$

x	$\sum_1^8 \frac{1}{n} \sin nx$	$\sum_1^8 \frac{1}{n^2} \cos nx$	$\sum_1^8 \frac{1}{n^3} \sin nx$	$\sum_1^8 \frac{1}{n^4} \cos nx$	$\sum_1^8 \frac{1}{n^5} \sin nx$	$\sum_1^8 \frac{1}{n^6} \cos nx$	$\sum_1^8 \frac{1}{n^7} \sin nx$	$\sum_1^8 \frac{1}{n^8} \cos nx$
0	0	1.52740	0	1.08176	0	1.01731	0	1.00406
$\frac{\pi}{6}$	1.40317	0.87189	0.65967	0.89269	0.53211	0.87365	0.50728	0.86797
$\frac{\pi}{4}$	1.03371	0.56781	0.84596	0.69388	0.74087	0.70586	0.71522	0.70699
$\frac{\pi}{3}$	1.14129	0.28280	0.95804	0.45611	0.89203	0.49075	0.87274	0.49789
$\frac{\pi}{2}$	0.72381	0.19965	0.96806	-0.05913	0.99615	-0.01540	0.99955	-0.00389
$\frac{2\pi}{3}$	0.49179	-0.55538	0.76522	-0.52121	0.83957	-0.50657	0.85931	-0.50180
$\frac{3\pi}{4}$	0.36705	-0.66157	0.60520	-0.70120	0.67863	-0.70634	0.69962	-0.70701
$\frac{5\pi}{6}$	0.32063	-0.75555	0.41949	-0.83587	0.47635	-0.85831	0.49364	-0.86408
π	0	-0.81562	0	-0.94694	0	-0.98555	0	-0.99624

TABLE 6. $\sum_{1}^{8} \frac{(-1)^n}{n^k} \begin{matrix}\sin nx\\ \cos nx\end{matrix}, 0 < x < \pi$ (k = 1......8)

x	$\sum_1^8 \frac{(-1)^n}{n} \sin nx$	$\sum_1^8 \frac{(-1)^n}{n^2} \cos nx$	$\sum_1^8 \frac{(-1)^n}{n^3} \sin nx$	$\sum_1^8 \frac{(-1)^n}{n^4} \cos nx$
0	0	-0.81562	0	-0.94694
π/6	-0.32064	-0.7555	-0.41948	-0.83587
π/4	-0.36705	-0.66157	-0.60521	-0.70120
π/3	-0.49180	-0.55538	-0.76523	-0.52121
π/2	-0.72382	-0.19965	-0.96804	-0.05913
2π/3	-1.14130	+0.28280	-0.95803	+0.45611
3π/4	-1.03371	+0.56781	-0.84595	+0.69388
5π/6	-1.40318	+0.87189	-0.65968	+0.89269
π	0	+1.52740	0	+1.08176

x	$\sum_1^8 \frac{(-1)^n}{n^5} \sin nx$	$\sum_1^8 \frac{(-1)^n}{n^6} \cos nx$	$\sum_1^8 \frac{(-1)^n}{n^7} \sin nx$	$\sum_1^8 \frac{(-1)^n}{n^8} \cos nx$
0	0	-0.98555	0	-0.99624
π/6	-0.47635	-0.85831	-0.49364	-0.86408
π/4	-0.67863	-0.70634	-0.69961	-0.70701
π/3	-0.83957	-0.50657	-0.85931	-0.50180
π/2	-0.99615	-0.01540	-0.99955	-0.00389
2π/3	-0.89203	+0.49075	-0.87273	+0.49789
3π/4	-0.74087	+0.70586	-0.71522	+0.70699
5π/6	0.53211	+0.87365	-0.50728	+0.86798
π	0	+1.01731	0	+1.00405

TABLE 7. $\sum_{1}^{8} \frac{1}{(2n-1)^k} \begin{matrix} \sin(2n-1)x \\ \cos(2n-1)x, 0 < x < \pi \end{matrix}$ (k = 1.....8)

x	$\sum_1^8 \frac{\sin(2n-1)x}{(2n-1)}$	$\sum_1^8 \frac{\cos(2n-1)x}{(2n-1)^2}$	$\sum_1^8 \frac{\sin(2n-1)x}{(2n-1)^3}$	$\sum_1^8 \frac{\cos(2n-1)x}{(2n-1)^4}$
0	0	1.17151	0	1.01435
$\pi/6$	0.86190	0.81372	0.53958	0.86428
$\pi/4$	0.70038	0.61469	0.72558	0.69754
$\pi/3$	0.81654	0.41909	0.86163	0.48866
$\pi/2$	0.72380	0	0.96806	0
$2\pi/3$	0.81654	-0.41909	0.86163	-0.48866
$3\pi/4$	0.70038	-0.61469	0.72558	-0.69754
$5\pi/6$	0.86190	-0.81372	0.53958	-0.86428
π	0	-1.17151	0	-1.01435

x	$\sum_1^8 \frac{\sin(2n-1)x}{(2n-1)^5}$	$\sum_1^8 \frac{\cos(2n-1)x}{(2n-1)^6}$	$\sum_1^8 \frac{\sin(2n-1)x}{(2n-1)^7}$	$\sum_1^8 \frac{\cos(2n-1)x}{(2n-1)^8}$
0	0	1.00143	0	1.00015
$\pi/6$	0.50424	0.86598	0.50045	0.86603
$\pi/4$	0.70975	0.70610	0.70743	0.70700
$\pi/3$	0.86580	0.49866	0.86603	0.49985
$\pi/2$	0.99615	0	0.99955	0
$2\pi/3$	0.86580	-0.49866	0.86603	-0.49985
$3\pi/4$	0.70975	-0.70610	0.70743	-0.70700
$5\pi/6$	0.50424	0.86598	0.50045	-0.86603
π	0	-1.00143	0	-1.00015

TABLE 8. $\sum_{1}^{8} \frac{1}{(2n)^k} \begin{matrix} \sin(2n)x \\ \cos(2n)x, \end{matrix} \quad 0 < x < \pi \qquad (k = 1.......8)$

x	$\sum_1^8 \frac{\sin(2n)x}{2n}$	$\sum_1^8 \frac{\cos(2n)x}{(2n)^2}$	$\sum_1^8 \frac{\sin(2n)x}{(2n)^3}$	$\sum_1^8 \frac{\cos(2n)x}{(2n)^4}$	$\sum_1^8 \frac{\sin(2n)x}{(2n)^5}$	$\sum_1^8 \frac{\cos(2n)x}{(2n)^6}$	$\sum_1^8 \frac{\sin(2n)x}{(2n)^7}$	$\sum_1^8 \frac{\cos(2n)x}{(2n)^8}$
0	0	0.35589	0	0.06741	0	0.01588	0	0.00391
$\pi/6$	0.54127	0.05817	0.12010	0.02841	0.02789	0.00767	0.00681	0.00194
$\pi/4$	0.33333	-0.04688	0.12037	-0.00366	0.03112	-0.00024	0.00781	-0.00001
$\pi/3$	0.32475	-0.13629	0.09640	-0.03255	0.02623	-0.00791	0.00671	-0.00196
$\pi/2$	0	-0.19965	0	-0.05913	0	-0.01542	0	-0.00390
$2\pi/3$	-0.32475	-0.13629	0.09640	-0.03255	-0.02623	-0.00791	-0.00671	-0.00196
$3\pi/4$	-0.33333	-0.04688	0.12037	-0.00366	-0.03112	-0.00024	-0.00781	-0.00001
$5\pi/6$	-0.54127	0.05817	0.12010	0.02841	-0.02789	0.00767	-0.00681	0.00194
π	0	0.35589	0	0.06741	0	0.01588	0	0.00391

TABLE 9. $\sum_{1}^{8}\frac{(-1)^n}{(2n)^k}\begin{matrix}\sin(2n)x\\ \cos(2n)x,\end{matrix}$ $-\frac{\pi}{2} < x < \frac{\pi}{2}$ (k = 1......8)

x	$\sum_{1}^{8}(-1)^n\frac{\sin(2n)x}{(2n)}$	$\sum_{1}^{8}(-1)^n\frac{\cos(2n)x}{(2n)^2}$	$\sum_{1}^{8}(-1)^n\frac{\sin(2n)x}{(2n)^3}$	$\sum_{1}^{8}(-1)^n\frac{\cos(2n)x}{(2n)^4}$
-π/2	0	+0.35589	0	+0.06741
-π/3	+0.54127	+0.05817	+0.12009	+0.02841
-π/4	+0.33333	-0.04688	+0.12038	-0.00366
-π/6	+0.32475	-0.13629	+0.09641	-0.03255
0	0	-0.19965	0	-0.05913
π/6	-0.32475	-0.13629	-0.09641	-0.03255
π/4	-0.33333	-0.04688	-0.12038	-0.00366
π/3	-0.54127	+0.05817	-0.12009	+0.02841
π/2	0	+0.35589	0	+0.06741

x	$\sum_{1}^{8}(-1)^n\frac{\sin(2n)x}{(2n)^5}$	$\sum_{1}^{8}(-1)^n\frac{\cos(2n)x}{(2n)^6}$	$\sum_{1}^{8}(-1)^n\frac{\sin(2n)x}{(2n)^7}$	$\sum_{1}^{8}(-1)^n\frac{\cos(2n)x}{(2n)^8}$
-π/2	0	+0.01588	0	+0.00392
-π/3	+0.02789	+0.00767	+0.00681	+0.00195
-π/4	+0.03112	-0.00024	+0.00781	-0.00001
-π/6	+0.02623	-0.00791	+0.00671	-0.00196
0	0	-0.01540	0	-0.00389
π/6	-0.02623	-0.00791	-0.00671	-0.00196
π/4	-0.03112	-0.00024	-0.00781	-0.00001
π/3	-0.02789	+0.00767	-0.00681	+0.00195
π/2	0	+0.01588	0	+0.00392

TABLE 10. $\sum_{1}^{8} \frac{(-1)^n}{(2n-1)^k} \begin{matrix} \cos(2n-1)x \\ \sin(2n-1)x, \end{matrix} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$, (k = 1.....8) IV B

x	$\sum_{1}^{8} \frac{(-1)^n \cos(2n-1)x}{(2n-1)}$	$\sum_{1}^{8} \frac{(-1)^n \sin(2n-1)x}{(2n-1)^2}$	$\sum_{1}^{8} \frac{(-1)^n \cos(2n-1)x}{(2n-1)^3}$	$\sum_{1}^{8} \frac{(-1)^n \sin(2n-1)x}{(2n-1)^4}$
$-\pi/2$	0	+1.17151	0	+1.01435
$-\pi/3$	-0.86191	+0.81372	-0.53958	+0.86429
$-\pi/4$	-0.70038	+0.61470	-0.72558	+0.69775
$-\pi/6$	-0.81654	+0.41909	-0.86163	+0.48866
0	-0.72382	0	-0.96806	0
$\pi/6$	-0.81654	-0.41909	-0.86163	-0.48866 -0.48866
$\pi/4$	-0.70038	-0.61470	-0.72558	-0.69775
$\pi/3$	-0.86191	-0.81372	-0.53958	-0.86429
$\pi/2$	0	-1.17151	0	-1.01435

TABLE 10. $\sum_{1}^{8} \frac{(-1)^n}{(2n-1)^k} \begin{matrix} \cos(2n-1)x \\ \sin(2n-1)x, \end{matrix} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$, (k = 1.....8) Cont'd

x	$\sum_{1}^{8} \frac{(-1)^n \cos(2n-1)x}{(2n-1)^5}$	$\sum_{1}^{8} \frac{(-1)^n \sin(2n-1)x}{(2n-1)^6}$	$\sum_{1}^{8} \frac{(-1)^n \cos(2n-1)x}{(2n-1)^7}$	$\sum_{1}^{8} \frac{(-1)^n \sin(2n-1)x}{(2n-1)^8}$
$-\pi/2$	0	+1.00143	0	+1.00015
$-\pi/3$	-0.50425	+0.86598	-0.50045	+0.86603
$-\pi/4$	-0.70975	+0.70611	-0.70743	+0.70701
$-\pi/6$	-0.86581	+0.49866	-0.86603	+0.49985
0	-0.99616	0	-0.99956	0
$\pi/6$	-0.86581	-0.49866	-0.86603	-0.49985
$\pi/4$	-0.70975	-0.70611	-0.70743	-0.70701
$\pi/3$	-0.50425	-0.86598	-0.50045	-0.86603
$\pi/2$	0	-1.00143	0	-1.00015

TABLE 11. Bernoulli's Numbers - B_n
Euler's Numbers - E_n
Factorial Numbers - n!

B_n	E_n	n!
$B_1 = \frac{1}{6}$	$E_0 = 1$	$0! = 1$
$B_2 = \frac{1}{30}$	$E_1 = 1$	$2! = 2$
$B_3 = \frac{1}{42}$	$E_2 = 5$	$3! = 6$
$B_4 = \frac{1}{30}$	$E_3 = 61$	$4! = 24$
$B_5 = \frac{5}{66}$	$E_4 = 1,385$	$5! = 120$
$B_6 = \frac{691}{2730}$	$E_5 = 50,521$	$6! = 720$
$B_7 = \frac{7}{6}$	$E_6 = 2,702,765$	$7! = 5040$
$B_8 = \frac{3617}{510}$	$E_7 = 199,360,981$	$8! = 40320$

$$E_n = \frac{(2n)!}{(2n-2)!2!} E_{n-1} - \frac{(2n)!}{(2n-4)!4!} E_{n-2} + \dots + (-1)^{n-1}$$

$$B_n = \frac{2n}{2^{2n}(2^{2n}-1)} \quad \frac{(2n-1)!}{(2n-2)!1!} E_{n-1} - \frac{(2n-1)!}{(2n-4)!3!} E_{n-2} + \dots + (-1)^{n-1}$$

$$E_n = \frac{2^{2n+2}(2n)!}{\pi^{2n+1}} \sum_1^{\infty} \frac{(-1)^n}{(2n-1)^{2n+1}} (-1)$$

$$B_n = \frac{(2n)!}{\pi^{2n} 2^{2n-1}} \sum_1^{\infty} \frac{1}{n^{2n}}$$

TABLE 12. INFINITE SERIES

I A. $\sum_{1}^{\infty} \frac{1}{n^{2k}}$

1 $\sum_{1}^{\infty} \frac{1}{n^2} = B_1\pi^2 = \frac{\pi^2}{6}$

2 $\sum_{1}^{\infty} \frac{1}{n^4} = \frac{B_2\pi^4}{3} = \frac{\pi^4}{90}$

3 $\sum_{1}^{\infty} \frac{1}{n^6} = \frac{2B_3\pi^6}{45} = \frac{\pi^6}{945}$

4 $\sum_{1}^{\infty} \frac{1}{n^8} = \frac{B_4\pi^8}{315} = \frac{\pi^8}{9450}$

⋮

k $\sum_{1}^{\infty} \frac{1}{n^{2k}} = \frac{Bk\pi^{2k}2^{2k-1}}{(2k)!}$

II A. $\sum_{1}^{\infty} \frac{(-1)^n}{n^{2k}}$

1 $\sum_{1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{B_1\pi^2}{2} = -\frac{\pi^2}{12}$

2 $\sum_{1}^{\infty} \frac{(-1)^n}{n^4} = -\frac{7B_2\pi^4}{24} = -\frac{7\pi^4}{720}$

3 $\sum_{1}^{\infty} \frac{(-1)^n}{n^6} = -\frac{31B_3\pi^6}{720} = -\frac{31\pi^6}{30240}$

4 $\sum_{1}^{\infty} \frac{(-1)^n}{n^8} = -\frac{127B_4\pi^8}{40320} = -\frac{127\pi^8}{1,209,600}$

⋮

k $\sum_{1}^{\infty} \frac{(-1)^n}{n^{2k}} = -\frac{B_k\pi^{2k}(2^{2k-1}-1)}{(2k)!}$

TABLE 12. INFINITE SERIES CONT'D

III A. $\displaystyle\sum_{1}^{\infty} \frac{1}{(2n-1)^{2k}}$

$$1 \quad \sum_{1}^{\infty} \frac{1}{(2n-1)^2} = \frac{3B_1\pi^2}{4} = \frac{\pi^2}{8}$$

$$2 \quad \sum_{1}^{\infty} \frac{1}{(2n-1)^4} = \frac{15B_2\pi^4}{48} = \frac{\pi^4}{96}$$

$$3 \quad \sum_{1}^{\infty} \frac{1}{(2n-1)^6} = \frac{7B_3\pi^6}{160} = \frac{\pi^6}{960}$$

$$4 \quad \sum_{1}^{\infty} \frac{1}{(2n-1)^8} = \frac{17B_4\pi^8}{2688} = \frac{17\pi^8}{161{,}280}$$

$\vdots$

$$k \quad \sum_{1}^{\infty} \frac{1}{(2n-1)^{2k}} = \frac{B_k\pi^{2k}(2^{2k}-1)}{2(2k)!}$$

IV A. $\displaystyle\sum_{1}^{\infty} \frac{1}{(2n)^{2k}}$

$$1 \quad \sum_{1}^{\infty} \frac{1}{(2n)^2} = \frac{B_1\pi^2}{4} = \frac{\pi^2}{24}$$

$$2 \quad \sum_{1}^{\infty} \frac{1}{(2n)^4} = \frac{B_2\pi^4}{48} = \frac{\pi^4}{1440}$$

$$3 \quad \sum_{1}^{\infty} \frac{1}{(2n)^6} = \frac{B_3\pi^6}{1440} = \frac{\pi^6}{60480}$$

$$4 \quad \sum_{1}^{\infty} \frac{1}{(2n)^8} = \frac{B_4\pi^8}{80640} = \frac{\pi^8}{2{,}419{,}200}$$

$\vdots$

$$k \quad \sum_{1}^{\infty} \frac{1}{(2n)^{2k}} = \frac{B_k\pi^{2k}\left[2.2^{2k-1} - 2^{2k} + 1\right]}{2(2k)!}$$

TABLE 12. INFINITE SERIES CONT'D

VI A. $\sum_{1}^{\infty} \frac{(-1)^n}{(2n)^{2k}}$

1 $\sum_{1}^{\infty} \frac{(-1)^n}{(2n)^2} = -\frac{B_1\pi^2}{8} = -\frac{\pi^2}{48}$

2 $\sum_{1}^{\infty} \frac{(-1)^n}{(2n)^4} = -\frac{7B_2\pi^4}{384} = -\frac{7\pi^4}{11520}$

3 $\sum_{1}^{\infty} \frac{(-1)^n}{(2n)^6} = -\frac{31B_3\pi^6}{46,080} = -\frac{31\pi^6}{1,935,360}$

4 $\sum_{1}^{\infty} \frac{(-1)^n}{(2n)^8} = \frac{127B_4\pi^8}{10,321,920} = -\frac{127\pi^8}{309,657,600}$

⋮

(k) $\sum_{1}^{\infty} \frac{(-1)^n}{(2n)^{2k}} = -\frac{B_k\pi^{2k}\left[2^{2k-1}-1\right]}{2^{2k-1}2(2k)!}$

IV B. $\sum_{1}^{\infty} \frac{(-1)^n}{(2n-1)^{2k+1}}$

1 $\sum_{1}^{\infty} \frac{(-1)^n}{(2n-1)} = -\frac{E_0\pi}{4} = -\frac{\pi}{4}$ (k=0)

2 $\sum_{1}^{\infty} \frac{(-1)^n}{(2n-1)^3} = -\frac{E_1\pi^3}{32} = -\frac{\pi^3}{32}$ (k=1)

3 $\sum_{1}^{\infty} \frac{(-1)^n}{(2n-1)^5} = -\frac{E_2\pi^5}{1536} = -\frac{5\pi^5}{1536}$ (k=2)

4 $\sum_{1}^{\infty} \frac{(-1)^n}{(2n-1)^7} = -\frac{E_3\pi^7}{184,320} = -\frac{61\pi^7}{184,320}$ (k=3)

⋮

(k) $\sum_{1}^{\infty} \frac{(-1)^n}{(2n-1)^{2k+1}} = -\frac{E_k\pi^{2k+1}}{2^{2k+2}(2k)!}$

TABLES OF INTEGRALS

1. $\int_0^{\pi} x^k \cos nx \, dx$

2. $\int_0^{\pi} x^k \sin nx \, dx$

3. $\int_0^{\pi/2} x^k \cos nx \, dx$

4. $\int_0^{\pi/2} x^k \sin nx \, dx$

5. $\int_0^{2\pi} x^k \cos nx \, dx$

6. $\int_0^{2\pi} x^k \sin nx \, dx$

7. $\int_{-\pi}^{\pi} x^k \cos nx \, dx$

8. $\int_{-\pi}^{\pi} x^k \sin nx \, dx$

TABLES OF INTEGRALS

TABLE 1. $\int_0^\pi x^k \cos nx\,dx$

0a. $\int_0^\pi \cos nx\,dx = 0$

0b. $\int_0^\pi \cos nx\,dx = \pi$, when $n = 0$

1a. $\int_0^\pi x \cos nx\,dx = \begin{cases} -2/n^2, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases}$

1b. $\int_0^\pi x \cos nx\,dx = \frac{\pi^2}{2}$, when $n = 0$

2a. $\int_0^\pi x^2 \cos nx\,dx = \begin{cases} -\frac{2\pi}{n^2}, & \text{when } n \text{ is odd} \\ +\frac{2\pi}{n^2}, & \text{when } n \text{ is even} \end{cases}$

2b. $\int_0^\pi x^2 \cos nx\,dx = \frac{\pi^3}{3}$, when $n = 0$

3a. $\int_0^\pi x^3 \cos nx\,dx = \begin{cases} -\frac{3\pi^2}{n^2} + \frac{12}{n^4}, & \text{when } n \text{ is odd} \\ +\frac{3\pi}{n^2}, & \text{when } n \text{ is even} \end{cases}$

3b. $\int_0^\pi x^3 \cos nx\,dx = \frac{\pi^4}{4}$, when $n = 0$

4a. $\int_0^\pi x^4 \cos nx\,dx = \begin{cases} -\frac{4\pi^3}{n^2} + \frac{24\pi}{n^4}, & \text{when } n \text{ is odd} \\ +\frac{4\pi^3}{n^2} - \frac{24\pi}{n^4}, & \text{when } n \text{ is even} \end{cases}$

4b. $\int_0^\pi x^4 \cos nx\,dx = \frac{\pi^5}{5}$, when $n = 0$

5a. $\int_0^\pi x^5 \cos nx\,dx = \begin{cases} -\frac{5\pi^4}{n^2} + \frac{60\pi^2}{n^4} - \frac{240}{n^6}, & \text{when } n \text{ is odd} \\ \frac{+5\pi^4}{n^2} - \frac{60\pi^2}{n^4}, & \text{when } n \text{ is even} \end{cases}$

5b. $\int_0^{\pi} x^5 \cos nx \, dx = \frac{\pi^6}{6}$, when $n = 0$

6a. $\int_0^{\pi} x^6 \cos nx \, dx = \begin{cases} -\frac{6\pi^5}{n^2} + \frac{120\pi^3}{n^4} - \frac{720\pi}{n^6}, \text{ when } n \text{ is odd} \\ +\frac{6\pi^5}{n^2} - \frac{120\pi^3}{n^4} + \frac{720\pi}{n^6}, \text{ when } n \text{ is even} \end{cases}$

6b. $\int_0^{\pi} x^6 \cos nx \, dx = \frac{\pi^7}{7}$, when $n = 0$

7a. $\int_0^{\pi} x^7 \cos nx \, dx = \begin{cases} -\frac{7\pi^6}{n^2} + \frac{210\pi^4}{n^4} - \frac{2520\pi^2}{n^6} + \frac{10080}{n^8}, \\ \qquad \text{when } n \text{ is odd} \\ +\frac{7\pi^6}{n^2} - \frac{210\pi^4}{n^4} + \frac{2520\pi^2}{n^6}, \text{ when } n \text{ is even} \end{cases}$

7b. $\int_0^{\pi} x^7 \cos nx \, dx = \frac{\pi^8}{8}$, when $n = 0$

8a. $\int_0^{\pi} x^8 \cos nx \, dx = \begin{cases} -\frac{8\pi^7}{n^2} + \frac{336\pi^5}{n^4} - \frac{6720\pi^3}{n^6} + \frac{40320\pi}{n^8}, \\ \qquad \text{when } n \text{ is odd} \\ +\frac{8\pi^7}{n^2} - \frac{336\pi^5}{n^4} + \frac{6720\pi^3}{n^6} - \frac{40320\pi}{n^8}, \\ \qquad \text{when } n \text{ is even} \end{cases}$

8b. $\int_0^{\pi} x^8 \cos nx \, dx = \frac{\pi^9}{9}$, when $n = 0$

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..

ka. $$\int_0^\pi x^k \cos nx \, dx = \begin{cases} (-1)^n \left[\dfrac{k\pi^{k-1}}{n^2} - \dfrac{k(k-1)(k-2)\pi^{k-3}}{n^4} + \dfrac{k(k-1)(k-2)(k-3)(k-4)\pi^{k-5}}{n^6} \right. \\ \left. \quad - \dfrac{k(k-1)(k-2)(k-3)(k-4)(k-5)(k-6)\pi^{k-7}}{n^8} + \ldots\ldots\ldots\ldots \right] \\ \quad + \dfrac{(-1)^{\frac{k+1}{2}} \cdot k!}{n^{k+1}}, \text{ if k is odd} \end{cases}$$

$$= \begin{cases} (-1)^n \left[\dfrac{k\pi^{k-1}}{n^2} - \dfrac{k(k-1)(k-2)\pi^{k-3}}{n^4} + \dfrac{k(k-1)(k-2)(k-3)(k-4)\pi^{k-5}}{n^6} \right. \\ \left. \quad - \dfrac{k(k-1)(k-2)(k-3)(k-4)(k-5)(k-6)\pi^{k-7}}{n^8} + \ldots\ldots\ldots\ldots \right] \\ \quad + 0, \text{ when k is even.} \end{cases}$$

kb. $$\int_0^\pi x^k \cos nx \, dx = \frac{\pi^{k+1}}{k+1}, \text{ when } n = 0$$

[Change signs of all odd powers of x for interval $-\pi < x < 0$.]

TABLE 2. $\int_0^\pi x^k \sin nx \, dx$

0. $$\int_0^\pi \sin nx \, dx = \begin{cases} \frac{2}{n}, \text{ when n is odd} \\ 0, \text{ when n is even} \end{cases}$$

1. $$\int_0^\pi x \sin nx \, dx = \begin{cases} \frac{\pi}{n}, \text{ when n is odd} \\ -\frac{\pi}{n}, \text{ when n is even} \end{cases}$$

2. $$\int_0^\pi x^2 \sin nx \, dx = \begin{cases} \frac{\pi^2}{n} - \frac{4}{n^3}, \text{ when n is odd} \\ -\frac{\pi^2}{n}, \text{ when n is even} \end{cases}$$

3. $$\int_0^\pi x^3 \sin nx \, dx = \begin{cases} \frac{\pi^3}{n} - \frac{6\pi}{n^3}, \text{ when n is odd} \\ -\frac{\pi^3}{n} + \frac{6\pi}{n^3}, \text{ when n is even} \end{cases}$$

4. $$\int_0^\pi x^4 \sin nx \, dx = \begin{cases} \frac{\pi^4}{n} - \frac{12\pi^2}{n^3} + \frac{48}{n^5}, \text{ when n is odd} \\ -\frac{\pi^4}{n} + \frac{12\pi^2}{n^3}, \text{ when n is even} \end{cases}$$

5. $$\int_0^\pi x^5 \sin nx \, dx = \begin{cases} \frac{\pi^5}{n} - \frac{20\pi^3}{n^3} + \frac{120\pi}{n^5}, \text{ when n is odd} \\ -\frac{\pi^5}{n} + \frac{20\pi^3}{n^3} - \frac{120\pi}{n^5}, \text{ when n is even} \end{cases}$$

6. $$\int_0^\pi x^6 \sin nx \, dx = \begin{cases} \frac{\pi^6}{n} - \frac{30\pi^4}{n^3} + \frac{360\pi^2}{n^5} - \frac{1440}{n^7}, \text{ when n is odd} \\ -\frac{\pi^6}{n} + \frac{30\pi^4}{n^3} - \frac{360\pi^2}{n^5}, \text{ when n is even} \end{cases}$$

7. $$\int_0^\pi x^7 \sin nx\, dx = \begin{cases} \dfrac{\pi^7}{n} - \dfrac{42\pi^5}{n^3} + \dfrac{840\pi^3}{n^5} - \dfrac{5040\pi}{n^7}, & \text{when } n \text{ is odd} \\[2ex] -\dfrac{\pi^7}{n} + \dfrac{42\pi^5}{n^3} - \dfrac{840\pi^3}{n^5} + \dfrac{5040\pi}{n^7}, & \text{when } n \text{ is even} \end{cases}$$

8. $$\int_0^\pi x^8 \sin nx\, dx = \begin{cases} \dfrac{\pi^8}{n} - \dfrac{56\pi^6}{n^3} + \dfrac{1680\pi^4}{n^5} - \dfrac{20160\pi^2}{n^7} + \dfrac{80640}{n^9}, & \text{when } n \text{ is odd} \\[2ex] -\dfrac{\pi^8}{n} + \dfrac{56\pi^6}{n^3} - \dfrac{1680\pi^4}{n^5} + \dfrac{20160\pi^2}{n^7}, & \text{when } n \text{ is even} \end{cases}$$

$\vdots$

k. $$\int_0^\pi x^k \sin nx\, dx = \begin{cases} (-1)^n \left[\dfrac{-\pi^k}{n} + \dfrac{k(k-1)\pi^{k-2}}{n^3} - \dfrac{k(k-1)(k-2)(k-3)\pi^{k-4}}{n^5} + \dfrac{k(k-1)(k-2)(k-3)(k-4)(k-5)\pi^{k-6}}{n^7} \cdots\cdots\cdots\cdots \right] \\[2ex] + 0, \text{ when } k \text{ is odd.} \end{cases}$$

$$= \begin{cases} (-1)^n \left[-\dfrac{\pi k}{n} + \dfrac{k(k-1)\pi^{k-2}}{n^3} - \dfrac{k(k-1)(k-2)(k-3)\pi^{k-4}}{n^5} + \dfrac{k(k-1)(k-2)(k-3)(k-4)(k-5)\pi^{k-6}}{n^7} \cdots\cdots\cdots\cdots \right] \\[2ex] + \dfrac{(-1)^{\frac{k}{2}} \,.\, k!}{k+1}, \text{ when } k \text{ is even} \end{cases}$$

[Change signs of all even powers of x for interval $-\pi < x < 0$.]

TABLE 3. $\int_0^{\frac{\pi}{2}} x^k \cos nx \, dx$

0a. $$\int_0^{\frac{\pi}{2}} \cos nx \, dx = \begin{cases} (-1)^{\frac{n-1}{2}} \cdot \frac{1}{n}, \text{ when n is odd} \\ 0, \text{ when n is even} \end{cases}$$

0b. $$\int_0^{\frac{\pi}{2}} \cos nx \, dx = \frac{\pi}{2}, \text{ when } n = 0$$

1a. $$\int_0^{\frac{\pi}{2}} x \cos nx \, dx = \begin{cases} (-1)^{\frac{n-1}{2}} \cdot \frac{\pi}{n}, \text{ when n is odd} \\ \frac{1}{n} (-1)^{\frac{n}{2}} - 1 \quad, \text{ when n is even} \end{cases}$$

1b. $$\int_0^{\frac{\pi}{2}} x \cos nx \, dx = \frac{\pi^2}{8}, \text{ when } n = 0$$

2a. $$\int_0^{\frac{\pi}{2}} x^2 \cos nx \, dx = \begin{cases} (-1)^{\frac{n-1}{2}} \left[\frac{\pi^2}{4n} \quad \frac{2}{n^3}\right], \text{ when n is odd} \\ \frac{\pi}{n^2}(-1)^{\frac{n}{2}}, \text{ when n is even} \end{cases}$$

2b. $$\int_0^{\frac{\pi}{2}} x^2 \cos nx \, dx = \frac{\pi^3}{24}, \text{ when } n = 0$$

3a. $$\int_0^{\frac{\pi}{2}} x^3 \cos nx \, dx = \begin{cases} (-1)^{\frac{n-1}{2}} \left[\frac{\pi^3}{8n} - \frac{3\pi}{n^3}\right], \text{ when n is odd} \\ \left[\left(\frac{3\pi^2}{4n^2} - \frac{6}{n^4}\right)(-1)^{\frac{n}{2}} + \frac{6}{n^4}\right], \text{ when n is even} \end{cases}$$

3b. $$\int_0^{\frac{\pi}{2}} x^3 \cos nx \, dx = \frac{\pi^4}{64}, \text{ when } n = 0$$

4a. $$\int_0^{\frac{\pi}{2}} x^4 \cos nx \, dx = \begin{cases} (-1)^{\frac{n-1}{2}} \left[\frac{\pi^4}{16n} - \frac{3\pi^2}{n^3} + \frac{24}{n^5}\right], \text{ when n is odd} \\ \left[\frac{\pi^3}{2n^2} - \frac{12\pi}{n^4}\right] (-1)^{\frac{n}{2}}, \text{ when n is even} \end{cases}$$

4b. $$\int_0^{\frac{\pi}{2}} x^4 \cos nx \, dx = \frac{\pi^5}{160}, \text{ when } n = 0$$

5a. $$\int_0^{\frac{\pi}{2}} x^5 \cos nx \, dx = \begin{bmatrix} (-1)^{\frac{n-1}{2}} \left[\frac{\pi^5}{32n} - \frac{5\pi^3}{2n^3} + \frac{60\pi}{n^5}\right], \text{ when n is odd,} \\ \left[\left(\frac{5\pi^4}{16n^2} - \frac{15\pi^2}{n^4} + \frac{120}{n^6}\right)(-1)^{\frac{n}{2}} - \frac{120}{n^6}\right], \text{ when n is even.} \end{bmatrix}$$

5b. $$\int_0^{\frac{\pi}{2}} x^5 \cos nx \, dx = \frac{\pi^6}{384}, \text{ when } n = 0.$$

6a. $$\int_0^{\frac{\pi}{2}} x^6 \cos nx \, dx = \begin{bmatrix} (-1)^{\frac{n-1}{2}} \left[\frac{\pi^6}{64n} - \frac{15\pi^4}{8n^3} + \frac{90\pi^2}{n^5} - \frac{720}{n^7}\right], \text{ when n is odd.} \\ \left[\frac{3\pi^5}{16n^2} - \frac{15\pi^3}{n^4} + \frac{360\pi}{n^6}\right] (-1)^{\frac{n}{2}}, \text{ when n is even} \end{bmatrix}$$

6b. $$\int_0^{\frac{\pi}{2}} x^6 \cos nx \, dx = \frac{\pi^7}{896}, \text{ when } n = 0.$$

7a. $$\int_0^{\frac{\pi}{2}} x^7 \cos nx \, dx = \begin{bmatrix} (-1)^{\frac{n-1}{2}} \left[\frac{\pi^7}{128n} - \frac{21\pi^5}{16n^3} + \frac{105\pi^3}{n^5} - \frac{2520\pi}{n^7}\right], \text{ when n is odd.} \\ \left[\left(\frac{7\pi^6}{64n^2} - \frac{105\pi^4}{8n^4} + \frac{630\pi^2}{n^6} - \frac{5040}{n^8}\right)(-1)^{\frac{n}{2}} + \frac{5040}{n^8}\right], \text{ when n is even.} \end{bmatrix}$$

7b. $$\int_0^{\frac{\pi}{2}} x^7 \cos nx \, dx = \frac{\pi^8}{2048}, \text{ when } n = 0.$$

8a. $$\int_0^{\frac{\pi}{2}} x^8 \cos nx \, dx = \begin{bmatrix} (-1)^{\frac{n-1}{2}} \left[\frac{\pi^8}{256n} - \frac{7\pi^6}{8n^3} + \frac{105\pi^4}{n^5} - \frac{5040\pi^2}{n^7} + \frac{40320}{n^9}\right], \text{ when n is odd.} \\ \left[\frac{\pi^7}{16n^2} - \frac{21\pi^5}{2n^4} + \frac{840\pi^3}{n^6} - \frac{20160\pi}{n^8}\right] (-1)^{\frac{n}{2}}, \text{ when n is even} \end{bmatrix}$$

8b. $$\int_0^{\frac{\pi}{2}} x^8 \cos nx \, dx = \frac{9}{4608}, \text{ when } n = 0$$

⋮

ka. $$\int_0^{\frac{\pi}{2}} x^k \cos nx \, dx = \left[(-1)^{\frac{n-1}{2}} \left[\frac{\left(\frac{\pi}{2}\right)^k}{n} - \frac{k(k-1)(\pi/2)^{k-2}}{n^3} + \frac{k(k-1)(k-2)(k-3)(\pi/2)^{k-4}}{n^5} - \frac{k(k-1)(k-2)(k-3)(k-4)(k-5)(\pi/2)^{k-6}}{n^7} + \ldots\ldots\ldots\ldots \right] + 0, \text{ when } n \text{ is odd. (Sine Series).} \right]$$

$$= \left[(-1)^{\frac{n}{2}} \left[\frac{k(\pi/2)^{k-1}}{n^2} - \frac{k(k-1)(k-2)(\pi/2)^{k-3}}{n^4} + \frac{k(k-1)(k-2)(k-3)(k-4)\left(\frac{\pi}{2}\right)^{k-5}}{n^6} - \frac{k(k-1)(k-2)(k-3)(k-4)(k-5)(k-6)(\pi/2)^{k-7}}{n^8} + \ldots\ldots \right] + \frac{(-1)^{\frac{k+1}{2}} \cdot k!}{n^{k+1}}, \text{ when } k \text{ is odd; } + 0, \text{ when } k \text{ is even. When } n \text{ is even. (Cosine Series).} \right]$$

kb. $$\int_0^{\frac{\pi}{2}} x^k \cos nx \, dx = \left[\frac{(\pi/2)^{k+1}}{k+1}, \text{ if } n = 0. \right]$$

TABLE 4. $\int_0^{\frac{\pi}{2}} x^k \sin nx\, dx$

0. $$\int_0^{\frac{\pi}{2}} \sin nx\, dx = \left[\begin{array}{l} 0, \text{ when n is odd} \\ -\frac{1}{n}\left[(-1)^{\frac{n}{2}} - 1\right], \text{ when n is even} \end{array}\right]$$

1. $$\int_0^{\frac{\pi}{2}} x \sin nx\, dx = \left[\begin{array}{l} (-1)^{\frac{n-1}{2}} \cdot \frac{1}{n^2}, \text{ when n is odd} \\ -\frac{\pi}{2n}(-1)^{\frac{n}{2}}, \text{ when n is even} \end{array}\right]$$

2. $$\int_0^{\frac{\pi}{2}} x^2 \sin nx\, dx = \left[\begin{array}{l} (-1)^{\frac{n-1}{2}} \cdot \frac{\pi}{n^2}, \text{ when n is odd} \\ \left[-\left(\frac{\pi^2}{4n} - \frac{2}{n^3}\right)(-1)^{\frac{n}{2}} - \frac{2}{n^3}\right], \text{ when n is even.} \end{array}\right]$$

3. $$\int_0^{\frac{\pi}{2}} x^3 \sin nx\, dx = \left[\begin{array}{l} (-1)^{\frac{n-1}{2}}\left(\frac{3\pi^2}{4n^2} - \frac{6}{n^4}\right), \text{ when n is odd} \\ -\left(\frac{\pi^3}{8n} - \frac{3\pi}{n^3}\right)(-1)^{\frac{n}{2}}, \text{ when n is even} \end{array}\right]$$

4. $$\int_0^{\frac{\pi}{2}} x \sin nx\, dx = \left[\begin{array}{l} (-1)^{\frac{n-1}{2}}\left(\frac{\pi^3}{2n^2} - \frac{12\pi^2}{n^4}, \text{ when n is odd} \\ \left[-\left(\frac{\pi^4}{16n} - \frac{3\pi^2}{n^3} + \frac{24}{n^5}\right)(-1)^{\frac{n}{2}} + \frac{24}{n^5}\right], \text{ when n is even} \end{array}\right]$$

5. $$\int_0^{\frac{\pi}{2}} x^5 \sin nx\, dx = \left[\begin{array}{l} (-1)^{\frac{n-1}{2}}\left(\frac{5\pi^4}{16n^2} - \frac{15\pi^2}{n^4} + \frac{120}{n^6}\right), \text{ when n is odd} \\ -\left(\frac{\pi^5}{32n} - \frac{5\pi^3}{2n^3} + \frac{60\pi}{n^5}\right)(-1)^{\frac{n}{2}}, \text{ when n is even} \end{array}\right]$$

6. $$\int_0^{\frac{\pi}{2}} x^6 \sin nx\, dx = \left[\begin{array}{l} (-1)^{\frac{n-1}{2}}\left(\frac{3\pi^5}{16n^2} - \frac{15\pi^3}{n^4} + \frac{360\pi}{n^6}\right), \text{ when n is odd} \\ \left[-\left(\frac{\pi^6}{64n} - \frac{15\pi^4}{8n^3} + \frac{90\pi^2}{n^5} - \frac{720}{n^7}\right)(-1)^{\frac{n}{2}} - \frac{720}{n^7}\right], \end{array}\right]$$

when n is even

7. $$\int_0^{\frac{\pi}{2}} x^7 \sin nx\, dx = \left[\begin{array}{l} (-1)^{\frac{n-1}{2}} \left(\dfrac{7\pi^6}{64n^2} - \dfrac{105\pi^4}{8n^4} + \dfrac{630\pi^2}{n^6} - \dfrac{5040}{n^8}\right), \text{ if n is odd} \\[2ex] -\left(\dfrac{\pi^7}{128n} - \dfrac{21\pi^5}{16n^3} + \dfrac{105\pi^3}{n^5} - \dfrac{2520\pi}{n^7}\right)(-1)^{\frac{n}{2}}, \text{ if n is even} \end{array}\right]$$

8. $$\int_0^{\frac{\pi}{2}} x^8 \sin nx\, dx = \left[\begin{array}{l} (-1)^{\frac{n-1}{2}} \left(\dfrac{\pi^7}{16n^2} - \dfrac{21\pi^5}{2n^4} + \dfrac{840\pi^3}{n^6} - \dfrac{20160}{n^8}\right), \text{ if n is odd} \\[2ex] \left[-\left(\dfrac{\pi^8}{256n} - \dfrac{7\pi^6}{8n^3} + \dfrac{105\pi^4}{n^5} - \dfrac{5040\pi^2}{n^7} + \dfrac{40320}{n^9}\right)(-1)^{\frac{n}{2}} + \dfrac{40320}{n^9}\right], \text{ if n is even} \end{array}\right]$$

$\vdots$

k. $$\int_0^{\frac{\pi}{2}} x^k \sin nx\, dx = \left[\begin{array}{l} (-1)^{\frac{n-1}{2}} \left[\dfrac{k(\pi/2)^{k-1}}{n^2} - \dfrac{k(k-1)(k-2)(\pi/2)^{k-3}}{n^4} + \dfrac{k(k-1)(k-2)(k-3)(k-4)(\frac{\pi}{2})^{k-5}}{n^6} \right. \\[2ex] \left. \qquad \dfrac{-k(k-1)(k-2)(k-3)(k-4)(k-5)(k-6)(\pi/2)^{k-7}}{n^8} + \ldots\ldots\ldots\ldots\ldots \right] \\[2ex] + 0, \text{ when n is } \underline{\text{odd}}. \end{array}\right]$$

$$= \left[\begin{array}{l} (-1)^{\frac{n}{2}} \left[-\dfrac{(\frac{\pi}{2})^k}{n} + \dfrac{k(k-1)(\frac{\pi}{2})^{k-2}}{n^3} - \dfrac{k(k-1)(k-2)(k-3)(\frac{\pi}{2})^{k-4}}{n^5} \right. \\[2ex] \left. \qquad + \dfrac{k(k-1)(k-2)(k-3)(k-4)(k-5)(\frac{\pi}{2})^{k-6}}{n^7} - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \right] \\[2ex] + \dfrac{(-1)^{\frac{k}{2}} \cdot k!}{n^{k+1}}, \text{ if k is even; } + 0, \text{ if k is odd.} \\ \qquad \text{When n is } \underline{\text{even}}. \end{array}\right]$$

TABLE 5. $\int_0^{2\pi} x^k \cos nx \, dx$

0a. $\int_0^{2\pi} \cos nx \, dx = 0.$

0b. $\int_0^{2\pi} \cos nx \, dx = 2\pi$, if $n = 0$.

1a. $\int_0^{2\pi} x \cos nx \, dx = 0.$

1b. $\int_0^{2\pi} x \cos nx \, dx = 2\pi^2$, when $n = 0$.

2a. $\int_0^{2\pi} x^2 \cos nx \, dx = \frac{4\pi}{n^2}.$

2b. $\int_0^{2\pi} x^2 \cos nx \, dx = \frac{8\pi^3}{3}$, when $n = 0$.

3a. $\int_0^{2\pi} x^3 \cos nx \, dx = \frac{12\pi^2}{n^2}$

3b. $\int_0^{2\pi} x^3 \cos nx \, dx = 4\pi^4$, when $n = 0$.

4a. $\int_0^{2\pi} x^4 \cos nx \, dx = \frac{32\pi^3}{n^2} - \frac{48\pi}{n^4}$

4b. $\int_0^{2\pi} x^4 \cos nx \, dx = \frac{32\pi^5}{5}$, when $n = 0$.

5a. $\int_0^{2\pi} x^5 \cos nx \, dx = \frac{80\pi^4}{n^2} - \frac{240\pi^2}{n^4}$

5b. $\int_0^{2\pi} x^5 \cos nx \, dx = \frac{32\pi^6}{3}$, when $n = 0$.

6a. $\int_0^{2\pi} x^6 \cos nx \, dx = \frac{192\pi^5}{n^2} - \frac{960\pi^3}{n^4} + \frac{1440\pi}{n^6}$

6b. $\int_0^{2\pi} x^6 \cos nx \, dx = \frac{128\pi^7}{7}$, when $n = 0$.

7a. $\int_0^{2\pi} x^7 \cos nx\, dx = \frac{448\pi^6}{n^2} - \frac{3360\pi^4}{n^4} + \frac{10080\pi^2}{n^6}$

7b. $\int_0^{2\pi} x^7 \cos nx\, dx = 32\pi^8$, when $n = 0$.

8a. $\int_0^{2\pi} x^8 \cos nx\, dx = \frac{1024\pi^7}{n^2} - \frac{10752\pi^5}{n^4} + \frac{53760\pi^3}{n^6} - \frac{80640\pi}{n^8}$

8b. $\int_0^{2\pi} x^8 \cos nx\, dx = \frac{512\pi^9}{9}$, when $n = 0$.

$\vdots$

ka. $$\int_0^{2\pi} x^k \cos nx\, dx = \left[\frac{k(2\pi)^{k-1}}{n^2} - \frac{k(k-1)(k-2)(2\pi)^{k-3}}{n^4} + \frac{k(k-1)(k-2)(k-3)(k-4)(2\pi)^{k-5}}{n^6} - \frac{k(k-1)(k-2)(k-3)(k-4)(k-5)(2\pi)^{k-7}}{n^8} + \dots\right]$$

$$+ \left[\frac{(-1)^{\frac{k+1}{2}} \cdot k!}{n^{k+1}}, \text{ when } k \text{ is odd.}\right]$$

$+ 0$, if k is even.

kb. $\int_0^{2\pi} x^k \cos nx\, dx = \frac{(2\pi)^{k+1}}{k+1}$, when $n = 0$.

TABLE 6. $\int_0^{2\pi} x^k \sin nx \, dx$

0. $\int_0^{2\pi} \sin nx \, dx = 0.$

1. $\int_0^{2\pi} x \sin nx \, dx = -\frac{2\pi}{n}$

2. $\int_0^{2\pi} x^2 \sin nx \, dx = -\frac{4\pi^2}{n}$

3. $\int_0^{2\pi} x^3 \sin nx \, dx = -\left(\frac{8\pi^3}{n} - \frac{12\pi}{n^3}\right).$

4. $\int_0^{2\pi} x^4 \sin nx \, dx = -\left(\frac{16\pi^4}{n} - \frac{48\pi^2}{n^3}\right).$

5. $\int_0^{2\pi} x^5 \sin nx \, dx = -\left(\frac{32\pi^5}{n} - \frac{160\pi^3}{n^3} + \frac{240\pi}{n^5}\right).$

6. $\int_0^{2\pi} x^6 \sin nx \, dx = -\left(\frac{64\pi^6}{n} - \frac{480\pi^4}{n^3} + \frac{1440\pi^2}{n^5}\right.$

7. $\int_0^{2\pi} x^7 \sin nx \, dx = -\left(\frac{128\pi^7}{n} - \frac{1344\pi^5}{n^3} + \frac{6720\pi^3}{n^5} - \frac{10080\pi}{n^7}\right)$

8. $\int_0^{2\pi} x^8 \sin nx \, dx = -\left(\frac{256\pi^8}{n} - \frac{3584\pi^6}{n^3} + \frac{26880\pi^4}{n^5} - \frac{80640\pi^2}{n^7}\right).$

⋮

k. $$\int_0^{2\pi} x^k \sin nx \, dx = (-1)^n \left[-\left(\frac{(2\pi)^k}{n} + \frac{k(k-1)(2\pi)^{k-2}}{n^3} - \frac{k(k-1)(k-2)(k-3)(2\pi)^{k-4}}{n^5} + \frac{k(k-1)(k-2)(k-3)(k-4)(k-5)(2\pi)^{k-6} \ldots}{n^7}\right.\right]$$

$+ \ 0$, if k is odd.

$+ \ \frac{(-1)^{\frac{n}{2}} \cdot k!}{k+1}$, if k is even.

TABLE 7. $\int_{-\pi}^{\pi} x^k \cos nx \, dx$

0a. $\int_{-\pi}^{\pi} \cos nx \, dx = 0$

0b. $\int_{-\pi}^{\pi} \cos nx \, dx = 2\pi$, when $n = 0$.

1a. $\int_{-\pi}^{\pi} x \cos nx \, dx = 0$.

1b. $\int_{-\pi}^{\pi} x \cos nx \, dx = 0$, when $n = 0$.

2a. $\int_{-\pi}^{\pi} x^2 \cos nx \, dx = (-1)^n . \frac{4\pi}{n^2}$.

2b. $\int_{-\pi}^{\pi} x^2 \cos nx \, dx = \frac{2\pi^3}{3}$, when $n = 0$.

3a. $\int_{-\pi}^{\pi} x^3 \cos nx \, dx = 0$.

3b. $\int_{-\pi}^{\pi} x^3 \cos nx \, dx = 0$, when $n = 0$.

4a. $\int_{-\pi}^{\pi} x^4 \cos nx \, dx = (-1)^n \left[\frac{8\pi^3}{n^2} - \frac{48\pi}{n^4}\right]$

4b. $\int_{-\pi}^{\pi} x^4 \cos nx \, dx = \frac{2\pi^5}{5}$, when $n = 0$.

5a. $\int_{-\pi}^{\pi} x^5 \cos nx \, dx = 0$.

5b. $\int_{-\pi}^{\pi} x^5 \cos nx \, dx = 0$, when $n = 0$.

6a. $\int_{-\pi}^{\pi} x^6 \cos nx \, dx = (-1) \left[\frac{12\pi^5}{n^2} - \frac{240\pi^3}{n^4} + \frac{1440\pi}{n^6} \right]$

6b. $\int_{-\pi}^{\pi} x^6 \cos nx \, dx = \frac{2\pi^7}{7}$, when $n = 0$.

7a. $\int_{-\pi}^{\pi} x^7 \cos nx \, dx = 0$.

7b. $\int_{-\pi}^{\pi} x^7 \cos nx \, dx = 0$. when $n = 0$.

8a. $\int_{-\pi}^{\pi} x^8 \cos nx \, dx = (-1)^n \left[\frac{16\pi^7}{n^2} - \frac{672\pi^5}{n^4} + \frac{13440\pi^3}{n^6} - \frac{80640\pi}{n^8} \right]$

8b. $\int_{-\pi}^{\pi} x^8 \cos nx \, dx = \frac{2\pi^9}{9}$, when $n = 0$.

ka. $$\int_{-\pi}^{\pi} x^k \cos nx \, dx = 2(-1)^n \left[\frac{k\pi^{k-1}}{n^2} - \frac{k(k-1)(k-2)\pi^{k-3}}{n^4} + \frac{k(k-1)(k-2)(k-3)(k-4)\pi^{k-5}}{n^6} - \frac{k(k-1)(k-2)(k-3)(k-4)(k-5)(k-6)\pi^{k-7}}{n^8} + \cdots\cdots\cdots\cdots \right.$$

when n is even. $\Big]$

$= 0$, when n is odd.

kb. $\int_{-\pi}^{\pi} x^k \cos nx \, dx = \frac{2\pi^{k+1}}{k+1}$, when $n = 0$, and k is even.

$= 0$, when $n = 0$, and k is odd.

TABLE 8. $\int_{-\pi}^{\pi} x^k \sin nx \, dx$

0. $\int_{-\pi}^{\pi} \sin nx \, dx = 0.$

1. $\int_{-\pi}^{\pi} x \sin nx \, dx = (-1)^n \left[-\frac{2\pi}{n} \right] \left[= 0 \text{ when } n = 0 \right]$

2. $\int_{-\pi}^{\pi} x^2 \sin nx \, dx = 0.$

3. $\int_{-\pi}^{\pi} x^3 \sin nx \, dx = -(-1)^n \left[\frac{2\pi^3}{n} - \frac{12\pi}{n^3} \right]$

4. $\int_{-\pi}^{\pi} x^4 \sin nx \, dx = 0.$

5. $\int_{-\pi}^{\pi} x^5 \sin nx \, dx = -(-1)^n \left[\frac{2\pi^5}{n} - \frac{40\pi^3}{n^3} + \frac{240\pi}{n^5} \right]$

6. $\int_{-\pi}^{\pi} x^6 \sin nx \, dx = 0.$

7. $\int_{-\pi}^{\pi} x^7 \sin nx \, dx = -(-1)^n \left[\frac{2\pi^7}{n} - \frac{84\pi^5}{n^3} + \frac{1680\pi^3}{n^5} - \frac{10080\pi}{n^7} \right]$

8. $\int_{-\pi}^{\pi} x^8 \sin nx \, dx = 0.$

$\vdots$

$$\text{k.}\quad \int_{-\pi}^{\pi} x^k \sin nx \, dx = \Bigg[(-1)^n \Bigg[-\frac{\pi k}{n} + \frac{k(k-1)\pi^{k-2}}{n^3} - \frac{k(k-1)(k-2)(k-3)\pi^{k-4}}{n^5} + \frac{k(k-1)(k-2)(k-3)(k-4)(k-5)\pi^{k-6}}{n^7} - \ldots\ldots \Bigg] + \frac{(-1)^{\frac{k}{2}}.\ k!}{k+1}, \text{ if k is even; } + 0, \text{ if k is odd.}$$

$$\text{k.}\quad \int_{-\pi}^{\pi} x^k \sin nx \, dx = \Bigg[(-1)^n \quad -\frac{\pi^k}{n} \quad \frac{k(k-1)\pi^{k-2}}{n^3} - \frac{k(k-1)(k-2)(k-3)\pi^{k-4}}{n^5} + \frac{k(k-1)(k-2)(k-3)(k-4)(k-5)\pi^{k-6}}{n^7} - \ldots\ldots + \frac{(-1)^{\frac{k}{2}}.\ k!}{k+1}, \text{ if k is even; } + 0, \text{ if k is odd.}$$

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