AN ABSTRACT OF THE THESIS OF

Min-Chu Chen for the degree Doctor of Philosophy

in Civil Engineering presented on September 6, 1979

Title: NONLINEAR DIFFRACTION THEORY BY AN EIGENFUNCTION EXPANSION OF THE GREEN'S FUNCTION

Abstract approved: Redacted for Privacy

Dr. Robert T. Hudspeth

A second-order nonlinear diffraction theory is developed for a large diameter circular cylinder under the action of surface gravity waves of finite amplitude in water of finite depth. Boundary value problems are derived by using a perturbation expansion and the solutions for the diffracted waves are given in terms of Fredholm integral equations in which the resolvent kernel is a Green's function. Green's functions are obtained by the eigenfunction expansion method and the velocity potentials for the diffracted waves are recovered from Fredholm integral equations which incorporate the prescribed inhomogeneous boundary conditions and the Green's functions.

Green's functions are extended to the second-order boundary value problem by an expansion in a complete set of orthonormal eigenfunctions. The well-posed Sturm-Liouville problems at second-order yield two sets of eigenfunctions which result in weakly dispersive scattered waves. The results for the second-order velocity potential are shown to satisfy all the prescribed boundary
The dimensionless hydrodynamic forces and moments on the cylinder are presented. Graphs of the force and moment coefficients and the phase angles for both the first- and second-order solutions are presented. The results show that the second-order contribution may be significant near the free surface and indicate that the second-order effect of the hydrodynamic forces and moments are important in finite depth. Also, a comparison with limited model test results indicates that the nonlinear forces are in better agreement than the linear forces.
APPROVED:

Redacted for Privacy

Professor of Civil Engineering
in charge of major

Redacted for Privacy

Head of Department of Civil Engineering

Redacted for Privacy

Dean of Graduate School

Date thesis is presented: September 6, 1979

Typed by Yuh-Mei Chung for Min-Chu Chen
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| \( q_m \) | wave number for the second-order scattered wave due to the effect of inhomogeneous combined free surface boundary condition | \( \text{L}^{-1} \)
| \( Q \) | Bernoulli constant | |
| \((r,\theta,z)\) | cylindrical coordinates of a point in the fluid domain | |
| \((R,\theta,Z)\) | cylindrical coordinates of the source point | |
| \( R \) | bounded region of a three-dimensional space | |
| \( s \) | number of the line integral segment | |
| \( \sin \) | sine | |
| \( \sinh \) | hyperbolic sine | |
| \( \mathbf{S} \) | function describing the surface of the object | \( \text{L}^2 \)
| \( \mathbf{S}_b \) | bottom surface | \( \text{L}^2 \)
| \( \mathbf{S}_B \) | surface of structure | \( \text{L}^2 \)
| \( \mathbf{S}_f \) | free surface | \( \text{L}^2 \)
| \( \mathbf{S}_\infty \) | surface at \( r = \infty \) | \( \text{L}^2 \)
| \( \mathbf{S}^m_{nn} \) | one-sided sea surface spectral density | \( \text{L}^2\text{T} \)
| \( t \) | time variable | \( \text{T} \)
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<td>L$^3$</td>
</tr>
<tr>
<td>$V_0$</td>
<td>vector function</td>
<td></td>
</tr>
<tr>
<td>$V_\theta$,$V_r$,$V_z$</td>
<td>velocity components in the $r$, $\theta$, $z$ directions</td>
<td>LT$^{-1}$</td>
</tr>
<tr>
<td>$w_1$, $W_1$</td>
<td>nonlinear function of first-order scattered wave defined from the inhomogeneous second-order combined free surface boundary condition</td>
<td></td>
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<tr>
<td>$w_2$, $W_2$</td>
<td>nonlinear interaction of first-order incident and scattered waves defined from the inhomogeneous second-order combined free surface boundary condition</td>
<td></td>
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<tr>
<td>$w_3$</td>
<td>steady-state term in the inhomogeneous second-order combined free surface boundary condition</td>
<td></td>
</tr>
<tr>
<td>$(x,y,z)$</td>
<td>rectangular Cartesian coordinates</td>
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<tr>
<td>$Y_n$</td>
<td>Bessel function of the second kind of order $n$</td>
<td></td>
</tr>
<tr>
<td>$Z_j$, $Z_j^m$, $Z_j^m$</td>
<td>second-order Green's function for $z$-coordinate</td>
<td>L$^{-1}$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>separation constant for second-order partial differential equation</td>
<td>L$^{-1}$</td>
</tr>
<tr>
<td>$\beta$</td>
<td>random phase angle</td>
<td></td>
</tr>
<tr>
<td>$\gamma$</td>
<td>autocovariance function</td>
<td></td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>normalizing constant for orthonormal eigenfunction $\Lambda$</td>
<td>L$^{-1/2}$</td>
</tr>
<tr>
<td>Symbol</td>
<td>Definition</td>
<td>Dimension</td>
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<tr>
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</tr>
<tr>
<td>δ</td>
<td>Dirac delta function</td>
<td></td>
</tr>
<tr>
<td>δₘₙ</td>
<td>Kronecker delta</td>
<td></td>
</tr>
<tr>
<td>ε</td>
<td>perturbation parameter ( = ( \frac{H}{\ell} k ) )</td>
<td></td>
</tr>
<tr>
<td>( \hat{e}_F )</td>
<td>percentage of second-order force to first-order force</td>
<td></td>
</tr>
<tr>
<td>( \hat{e}_M )</td>
<td>percentage of second-order moment to first-order moment</td>
<td></td>
</tr>
<tr>
<td>ε̃</td>
<td>dimensionless relative error</td>
<td></td>
</tr>
<tr>
<td>η</td>
<td>water surface elevation above still water level</td>
<td>L</td>
</tr>
<tr>
<td>κ</td>
<td>wave number for propagating mode for the second-order scattered wave due to the effect of inhomogeneous structural boundary condition</td>
<td>L⁻¹</td>
</tr>
<tr>
<td>κₘ</td>
<td>wave number for evanescent modes for the second-order scattered wave due to the effect of inhomogeneous structural boundary condition</td>
<td>L⁻¹</td>
</tr>
<tr>
<td>λ</td>
<td>moment arm</td>
<td>L</td>
</tr>
<tr>
<td>Λ</td>
<td>second-order orthonormal eigenfunction</td>
<td>L⁻¹/₂</td>
</tr>
<tr>
<td>μ</td>
<td>phase shift</td>
<td></td>
</tr>
<tr>
<td>μₚ</td>
<td>force phase shift</td>
<td></td>
</tr>
<tr>
<td>μₘ</td>
<td>moment phase shift</td>
<td></td>
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<tr>
<td>ν</td>
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<td></td>
</tr>
<tr>
<td>ξ</td>
<td>first-order orthogonal eigenfunction</td>
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<tr>
<td>Ξ</td>
<td>time lag</td>
<td>T</td>
</tr>
<tr>
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<td>3.14159 .....</td>
<td></td>
</tr>
<tr>
<td>ρ</td>
<td>mass density of fluid</td>
<td>ML⁻³</td>
</tr>
<tr>
<td>σ</td>
<td>linear theory wave frequency</td>
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</tr>
<tr>
<td>σ₁</td>
<td>second-order wave frequency</td>
<td>T⁻¹</td>
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<td>Symbol</td>
<td>Definition</td>
<td>Dimension</td>
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<tr>
<td>( \tau )</td>
<td>dimensionless time parameter ( ( = \sigma t ) )</td>
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<tr>
<td>( \phi^i, \phi^i )</td>
<td>incident wave velocity potential</td>
<td>( L^2T^{-1} )</td>
</tr>
<tr>
<td>( \phi^S, \phi^S )</td>
<td>scattered wave velocity potential</td>
<td>( L^2T^{-1} )</td>
</tr>
<tr>
<td>( \phi_0, \phi_0 )</td>
<td>total velocity potential ( ( = \phi^i + \phi^S ) )</td>
<td>( L^2T^{-1} )</td>
</tr>
<tr>
<td>( \phi^2, \phi^2, \phi^2 )</td>
<td>second-order scattered wave velocity potential</td>
<td>( L^2T^{-1} )</td>
</tr>
<tr>
<td>( \psi )</td>
<td>first-order orthonormal eigenfunction</td>
<td>( L^{-\frac{1}{2}} )</td>
</tr>
<tr>
<td>( \Omega )</td>
<td>relative water depth ( ( = \frac{h}{L_0} ) )</td>
<td></td>
</tr>
<tr>
<td>( \Sigma )</td>
<td>summation</td>
<td></td>
</tr>
<tr>
<td>( % )</td>
<td>percentage</td>
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<tr>
<td>( \infty )</td>
<td>infinity</td>
<td></td>
</tr>
<tr>
<td>( * )</td>
<td>complex conjugate value</td>
<td></td>
</tr>
<tr>
<td>( \cdot )</td>
<td>dot-product indicator</td>
<td></td>
</tr>
<tr>
<td>( \vec{\nabla} )</td>
<td>vector gradient operator</td>
<td></td>
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<tr>
<td>( \nabla^2 )</td>
<td>Laplace operator</td>
<td></td>
</tr>
<tr>
<td>( \partial )</td>
<td>partial derivative operator</td>
<td></td>
</tr>
<tr>
<td>( \partial \nabla )</td>
<td>partial differentiation in the direction of the outward normal to ( S )</td>
<td></td>
</tr>
<tr>
<td>( \int )</td>
<td>integral</td>
<td></td>
</tr>
<tr>
<td>( \sim )</td>
<td>asymptotic value as ( r \to \infty )</td>
<td></td>
</tr>
<tr>
<td>Symbol</td>
<td>Definition</td>
<td>Dimension</td>
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<tr>
<td><strong>Subscripts</strong> :</td>
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<tr>
<td>c</td>
<td>cosine component</td>
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<tr>
<td>s</td>
<td>sine component</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>first-order</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>second-order</td>
<td></td>
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<tr>
<td><strong>Superscripts</strong> :</td>
<td></td>
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</tr>
<tr>
<td>i</td>
<td>incident wave</td>
<td></td>
</tr>
<tr>
<td>f</td>
<td>force</td>
<td></td>
</tr>
<tr>
<td>m</td>
<td>moment</td>
<td></td>
</tr>
<tr>
<td>s</td>
<td>scattered wave</td>
<td></td>
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<tr>
<td>'</td>
<td>derivative with respect to r</td>
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</table>
Drilling for oil on the continental shelf started in 1947 when a steel platform was erected in twenty feet of water in the Gulf of Mexico (Evans and Adamchak, 1969). Over the past thirty years, rapid advances have been made not only in the number of platforms constructed, but also in the greater water depths in which offshore structures have been placed.

Movable structures are preferred for drilling exploration, and these units have been constructed in great varieties; e.g., jack-up units, semi-submersible platforms and drilling ships. For production of crude oil and gas, fixed platforms are generally used. However, as operating depths became deeper, fixed platforms were no longer feasible, a situation that leads to the design of gravity platforms. Gravity platforms are usually constructed of concrete, and the diameters of the structures' surface piercing cylindrical legs are much larger than those used for the conventional steel jacketed space frame platforms. These larger leg diameters may not be considered as an insignificant factor when compared to the dimensions of the incident wave lengths.

The problem of the effect of finite amplitude waves on large structures is the purpose of this study. The determination of wave
forces on large structures in the sea remains of great interest to ocean engineering designers, but the forces and overturning moments induced by the very large amplitude storm waves are now of primary interest. The design criteria established for offshore structures are that they be able to withstand the environmental forces without significant damage, and environmental forces generally refer to those of waves, winds and currents, although earthquakes may also be included. The very large amplitude storm waves frequently encountered in the North Sea have now become part of the environmental conditions that affect offshore structure design criteria. Although the linear diffraction theory (MacCamy and Fuchs, 1954) has proven to be practical for many applications, the validity of the linear wave assumption decreases as the amplitude of the incident wave increases. Therefore, the main focus of this study is the investigation of wave forces and overturning moments on a large diameter circular cylinder subjected to a train of regular waves of finite amplitude.

1.1 Review of Previous Studies

1.1.1 Morison Equation

The most common formula used to calculate wave forces per unit length on a cylindrical structure under two-dimensional regular waves is the Morison equation (1950) which may be written as

\[ F(t) = C_M \rho a^2 \frac{du}{dt} + C_D \rho a |u| \]  

(1.1)
in which \( F \) = the instantaneous force per unit length of the fixed cylinder, \( C_D \) and \( C_M \) = drag and inertia coefficient, respectively, 
\[ \rho = \text{fluid density}, \ a = \text{cylinder radius}, \ u = \text{fluid particle velocity} \] 
and \( \frac{du}{dt} = \text{fluid particle acceleration}. \)

Eq. (1.1) requires two empirical coefficients and may be derived from the Froude-Krylov hypothesis which assumes that the object is so small that there will be no disturbance to the incident wave field. There are two major different opinions about the values of \( C_D \) and \( C_M \). The first is that the coefficients are given only as a function of Reynolds number (Shore Protection Manual, 1973), and the second is that these coefficients depend on a period parameter (Keulegan and Carpenter, 1958), called the Keulegan-Carpenter parameter. Experimental investigations to determine these two empirical coefficients have been performed elsewhere, notably by Keulegan and Carpenter (1958), Dean and Aagaard (1970), and Sarpkaya (1975), etc. A summary of these experimental findings follows.

Dean and Aagaard (1970) used measured pressure forces on an offshore platform and a least-squares regression analysis with theoretical nonlinear water particle kinematics to compute \( C_D \) and \( C_M \). They found that the value of drag coefficient varied with Reynolds number and that the inertia coefficient had a constant value. However, Horton (1977) found a significant linear dependence of drag coefficient on the inverse of the wave steepness parameter. The \( C_D \) correlation in terms of the wave steepness parameter affords a
significant reduction in the scatter that has been indicated in recently published wave force coefficient data.

Keulegan and Carpenter (1958) found that the average values of force coefficients depended on a period parameter but dependence on an oscillatory Reynolds number was very weak. In an effort to derive the best fit values for $C_D$ and $C_M$, Sarpkaya (1975) applied a variety of alternative data reduction procedures to the range of Reynolds number less than $5 \times 10^4$. Sarpkaya's conclusion that wave forces correlated reasonably well with the Keulegan-Carpenter parameter but not with the Reynolds number was similar to earlier investigations.

1.1.2 Diffraction Theory

As the diameter of the cylinder increases in comparison with the incident wave length, the incident wave is scattered and the assumption that the object does not affect the incident wave field is no longer valid. Thus, for a large diameter cylinder, the theory based on the Morison equation becomes invalid, and it should be replaced by a more basic approach which will include the effect of the scattered wave as well as the nonlinear free surface pressure distribution. Diffraction is the phenomenon by which energy is transmitted laterally along a wave crest when a portion of a wave train is interrupted by a barrier. The diffraction method is generally used for computing wave pressure forces on offshore structures having significantly large characteristic dimensions compared to the wave length of an incident wave field. In addition, the fundamental diffraction theory has widespread application in the general wave field theory of theoretical physics (cf.,
Two kinds of competitive methods are used with respect to the diffraction method for determining wave excited forces on large cylinders. The first is the analytical method which expands the wave motion in special well-known functions. The analytical methods are developed into computer programs which are capable of solving problems rather quickly at low cost. Their applications, however, are restricted to objects that have simple geometrical shapes; such as long rectangular breakwaters (Steimer, 1977), circular cylinders (MacCamy and Fuchs, 1954), etc. The second is a more general method based on a Green's function and a Fredholm integral equation. A general diffraction theory and uniqueness criteria based on a singular type of Green's function was developed by John (cf., Stoker (1957)), and extensive applications of this powerful method for monochromatic linear waves were reviewed by Wehausen (1960). A particularly elegant application of this method has been given by Ursell (1947) and by Mei (1969). Numerical results for objects with nonseparable geometries require methods of numerical quadrature in order to evaluate the resulting Fredholm integral. The wave interaction with a body of arbitrary geometry is computed by representing the body as a distribution of point wave sources over its submerged surface. The boundary surface of the structure body is simply represented as a set
of discrete points connected by straight line segments, and thus, form an irregular polygon. The integral equations are approximated by a finite difference technique which replaces the continuous integrals by discrete finite sums. The resulting sets of linear simultaneous algebraic equations are then solved to yield the required surface potentials from which the scattered field is computed. This method has a great advantage in that there are no severe restrictions on the geometry of the object that are necessary in order to apply the method of separation of variables for the fluid flow problem. However, it is an expensive and time consuming application. The wave forces acting on structures of arbitrary shape have been studied by Garrison and Seetharama Rao (1971), Milgram and Halkyard (1971), Garrison and Chow (1972), Ijima, et al (1974), Harms (1976), Isaacson (1978), and others. The diffraction method used in this study is of the analytical kind.

An exact solution of the diffraction method for a linearized monochromatic surface gravity wave field incident upon a right circular cylinder was given by Havelock (1940) for the case of infinite depth. MacCany and Fuchs (1954) extended Havelock's work to the case of finite depth and found that the closed-form solution asymptotically approached the inertia force in the Morison equation. This asymptotic approximation of the linear diffraction theory for small ratios of cylinder diameter to wave length, however, was not capable of including the nonlinear quadratic velocity force which was referred to as a drag force in the Froude-Krylov based Morison equation.
Chakrabarti (1972) applied the diffraction theory to each harmonic component separately by linear superposition and obtained a fifth-order solution satisfying the linearized free surface boundary conditions. As pointed out by Jen and Skjelbriea (1973) and Garrison (1973), Chakrabarti's solution for the scattered velocity potential satisfied all boundary conditions including the nonlinear dynamic free surface condition, but it failed to meet the kinematic free surface condition in the vicinity of the cylinder. Later, Yamaguchi and Tsuchiya (1974) presented a second-order solution in a closed-form for the wave interaction problem with the vertical cylinder. Raman, et al. (1975, 1976 and 1977) extended the diffraction theory to second-order for a circular cylinder which extended over the entire fluid depth and which was subjected to second-order Stokes' waves in water of finite depth. A closed-form solution was not possible due to the occurrence of an improper integral resulting from the inhomogeneous combined free surface boundary condition at second-order. They also found that the wave number of the second-order scattered wave was not an integer multiple of the first harmonic, and thus, the second-order scattered wave was weakly dispersive with respect to the second-order incident Stokes' wave. As pointed out by Chakrabarti (1978), all these solutions for the second-order scattered velocity potential derived by Yamaguchi and Tsuchiya (1974), and Raman, et al. (1976) also failed to satisfy the nonlinear kinematic free surface boundary condition.

Isaacson (1977) argued that the solution derived by Raman, et al. (1975) was mathematically inconsistent because the combined free
surface boundary condition for the second-order velocity potential implied that both the radial and the azimuthal velocities must vanish at the still water level on the cylinder boundary and, therefore, no second-order correction might be developed by a Stokes' perturbation method for circular obstacles which penetrated the free surface normally. The inconsistency derived by Isaacson assumes that the radial velocity at the still water level is also the normal velocity on the cylinder. For obstacles of arbitrary geometry which do not penetrate the free surface perpendicularly and/or which have an azimuthal dependent boundary at the still water level (e.g., an elliptical cylinder), the inconsistency is removed. The question of the uniqueness of eigenvalue solutions with respect to this inconsistency in the boundary condition has yet to be resolved. A similar inconsistency might be found in the plane beach problem (cf. Wehausen (1960) or Stoker (1957)). We also note that this second-order inconsistency given by Isaacson is similar to the second-order correction to the velocity potential for the incident Stokes' wave which may also be shown to vanish for the special case of deep water; however, this zero-correction condition does not invalidate the Stokes' perturbation method for higher order nonlinearities or for water of finite depth. Garrison (1978) and Shen (1977) suggested that these inconsistencies might be removed by decomposing the second-order scattered velocity potential, $\phi_2^s$, into a linear sum

$$\phi_2^s = \phi_2^{ss} + \phi_2^{sf}$$ (1.2)
in which \( \phi_{2}^{SS} \) denotes a velocity potential that satisfies an inhomogeneous structural boundary condition and \( \phi_{2}^{SF} \) denotes a velocity potential that satisfies an inhomogeneous combined free surface boundary condition. Then, the solutions of \( \phi_{2}^{SS} \) and \( \phi_{2}^{SF} \) might be solved by satisfying each prescribed inhomogeneous boundary condition.

Black (1975) presented especially convincing evidence that considerable computational savings might be realized for arbitrary structural geometries which have vertical axes of symmetry by expanding the Green's function in a set of nonsingular eigenfunctions. Although the particular application given by Black of the eigenfunction expansion of the Green's function is limited to linear wave theory and, therefore, is not capable of including the effects from the quadratic velocity terms in a mathematically consistent manner, the reduced numerical computations realized by the eigenfunction expansion method merit serious consideration in any effort to extend the diffraction theory to second-order. Both the singular type of Green's function summarized by Wehausen (1960) and the symmetric type of Green's function computed by Black (1975) from an eigenfunction expansion method employ Green's theorem to recover the unknown scattered velocity potential through a surface integral over the geometry of the scatterer. However, the computational savings which are realized by the eigenfunction expansion of Green's function is a direct result of reducing the two-dimensional facet surface integral required by the singular type of Green's function to a one-dimensional line integral for obstacles with vertical axes of symmetry. Both methods may be
checked by Haskind's theory once the scattered velocity potential has been computed through the Fredholm integral solution.

Fenton (1978) reexamined the work done by Black (1975) for the case of vertical bodies of revolution and found that the Green's function computed by Black possessed logarithmic singularities. After subtracting the singularities ignored by Black, Fenton found that the convergence rate of the resulting series form of the Green's function was extremely rapid. The amount of computational effort using Fenton's method compared to that with a surface integral method is reduced by a factor of $\frac{1}{16} \frac{(S)}{a^2}$ in which $a =$ the radius of cylinder and $s =$ the number of the line integral segments.

Garrison (1976) applied a more general singular Green's function method to evaluate the forces acting on a fixed body of arbitrary shape placed in a train of regular waves of finite amplitude correct to second-order, and showed that the second-order effects might be rather significant particularly when the body was near to or piercing the free surface. The extension by Garrison to second-order is not based on the eigenfunction expansion method; and therefore, the potential computational savings for a nonlinear diffraction theory which utilizes an eigenfunction expansion has yet to be investigated.

1.2 Scope of Present Study

A method is proposed to exploit the powerful eigenfunction expansion method (cf. Roach (1970), pp. 91-111) to obtain a Fredholm
integral representation for nonlinear pressure forces on objects with vertical axes of symmetry. This nonlinear extension will then make it possible to include in a mathematically consistent manner the quadratic velocity pressure forces which have been omitted in previous applications of the diffraction method, but which are included in the semi-empirical Froude-Krylov method of Morison. Although this quadratic velocity force in the Bernoulli equation will be computed from the linear velocity terms, they are of second-order in the perturbation parameter. Therefore, the velocity potential must also be computed to second-order in this same perturbation parameter in order to make the pressure force mathematically consistent since the temporal derivative of the velocity potential also appears linearly in the Bernoulli equation. The second-order velocity potential will not be required to compute the contribution to the total pressure force from the quadratic velocity component correct to second-order. We also note that while the quadratic drag force in the Morison equation is a real fluid effect which is due to the viscous separation of the boundary layer around the cylinder, it is proportional to the momentum flux surface integral in the momentum transport equation for the acceleration of a fluid particle (cf. Lamb (1945) or Muga and Wilson (1970)). Morison, et al (1950) argued that if the obstacle was small compared to some characteristic length scale in the pressure field (the Froude-Krylov hypothesis), then the force on the obstacle would be proportional to the force on a water particle which would be there in the absence of the obstacle. The force on the fluid particle in the absence of the
obstacle may be computed from the momentum transport equation and the
constants of proportionality are the empirically determined inertial
and drag coefficients. Since the quadratic velocity term in the
Bernoulli equation is derived from these same convective acceleration
terms in the momentum transport equation, the relationship between the
drag force in the Morison equation and the quadratic velocity pressure
force in the Bernoulli equation is evident. The real fluid effects are
treated empirically through the inertial and drag coefficients.

A detailed presentation of the nonlinear diffraction theory
correct to second-order by an eigenfunction expansion of the Green's
function is given in subsequent chapters. The formulations of the
boundary value problem are established for both the first- and
second-order velocity potentials in Chapter 2. Chapter 3 deals with the
Green's formulae which are then utilized to transform the first-order
boundary value problem. Chapter 4 solves for the first-order Green's
function which satisfies the pertinent boundary conditions established
in Chapter 3. Chapter 5 recovers the scattered velocity potential by
using the Green's function which is derived in Chapter 4. Chapter 6
extends the Green's function method to the second-order problem and
recovers the second-order scattered velocity potential. Chapter 7
presents the derivation of wave hydrodynamic forces and moments as well
as the numerical results of the first- and second-order force and
moment coefficients. The comparison between the experimental
measurements given by Chakrabarti (1975, 1978) and the numerical
results of wave forces from this study are also presented in this
chapter. In Chapter 8 the conclusions of the study are enunciated. Recommendations for further investigations of this subject conclude the study.
CHAPTER 2

FORMULATION OF THE DIFFRACTION PROBLEM

The diffraction by a right-circular cylinder of a surface gravity wave propagating in water of finite depth is formulated in the form of a boundary value problem. Section 2.1 formulates the basic equations governing the motion of surface gravity waves in water of finite depth. Section 2.2 formulates the linear boundary value problems for the first- and second-order by means of a perturbation expansion.

2.1 Problem Formulation

The problem to be considered now is the evaluation of the wave forces on a fixed rigid large circular body which has a constant cross section with respect to water depth and is placed in a train of regular waves.

A cylindrical coordinate system is employed with the origin at the intersection of the undisturbed still water surface and the vertical axis of symmetry of the cylinder as shown in Figure 2.1. The $r$-$\theta$ plane is the undisturbed free surface and the $z$-axis is directed vertically upward opposite to the direction of the force of gravity. The instantaneous free surface elevation above the still water level, $z = 0$, is denoted by the coordinate $z = n$. 
FIGURE 2.1: DEFINITION SKETCH FOR SURFACE GRAVITY WAVE IMPINGING ON A CYLINDRICAL STRUCTURE
The fluid is assumed to be incompressible and its motion to be irrotational in the domain shown in Figure 2.1. Conservation of mass may be expressed in terms of a scalar velocity potential, \( \phi(r, \theta, z, t) \), according to

\[
- \left( \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} \right) = \nabla^2 \phi(r, \theta, z, t) = 0 .
\] (2.1)

In Eq. (2.1) the three-dimensional gradient operator, \( \nabla \), is defined in cylindrical coordinates by

\[
\nabla = \frac{\partial}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{e}_\theta + \frac{\partial}{\partial z} \hat{e}_z \] (2.2)

in which \( \hat{e}_r, \hat{e}_\theta, \) and \( \hat{e}_z \) are the unit vectors in the direction of increasing \( r, \theta \) and \( z \), respectively (cf. Figure 2.1).

The velocity components may be determined by the following directional derivatives. Note our use of the negative gradient to define the velocity field;

\[
V_\theta = - \frac{1}{r} \frac{\partial \phi}{\partial \theta} \] (2.3a)

\[
V_r = - \frac{\partial \phi}{\partial r} \] (2.3b)

\[
V_z = - \frac{\partial \phi}{\partial z} . \] (2.3c)
The boundaries of the fluid (cf. Figure 2.1) are the free surface which extends infinitely along the $r$-axis; the horizontal bottom surface of the fluid at an impermeable, finite depth; and the immersed surface of the structural body.

If the equation for the free surface is expressed by the following material coordinate:

$$ S_f = z - n(r, \theta, t) = 0 \quad ; \quad r \geq a \quad (2.4) $$

then the kinematic and dynamic boundary conditions on the free surface may be given by

$$ \frac{\partial \eta}{\partial t} - \left( \frac{\partial \Phi}{\partial r} \frac{\partial \eta}{\partial r} + \frac{1}{r^2} \frac{\partial \Phi}{\partial \theta} \frac{\partial \eta}{\partial \theta} \right) = - \frac{\partial \Phi}{\partial z} \quad ; \quad z = \eta , r \geq a \quad (2.5) $$

and

$$ - \frac{\partial \Phi}{\partial t} + gn + \frac{1}{2} \left[ \left( \frac{\partial \Phi}{\partial r} \right)^2 + \left( \frac{\partial \Phi}{\partial \theta} \right)^2 + \left( \frac{1}{r} \frac{\partial \Phi}{\partial \theta} \right)^2 \right] = Q(t) \quad ; \quad z = \eta , r \geq a \quad (2.6) $$

respectively, in which a constant atmospheric pressure and an absence of surface tension on the free surface have been assumed and $Q = \text{the Bernoulli constant}$.

Let the equation of the cylinder surface be given by the following material coordinate:

$$ S_B = r - a = 0 \quad . \quad (2.7) $$
The kinematic condition to be satisfied on the cylinder surface is given by
\[ \hat{\Phi} \cdot \hat{\mathbf{S}}_B = 0 \quad (2.8) \]

In the case of a cylindrical structure, Eq. (2.8) becomes
\[ \frac{\partial \Phi}{\partial r} = 0 \quad ; \quad r = a \quad . \quad (2.9) \]

The remaining boundary condition specifies that the velocity normal to the impermeable bottom must be zero; thus,
\[ \frac{\partial \Phi}{\partial z} = 0 \quad ; \quad z = -h \quad . \quad (2.10) \]

To complete the specification of the boundary value problem, a radiation condition is required in order to insure that the diffracted wave represents an outgoing progressive wave as \( r \rightarrow \infty \).

2.2 Perturbation Expansion

The derivatives of the scalar velocity potential in the dynamic and kinematic free surface boundary conditions are to be evaluated at the unknown free surface elevation, \( z = h(r, \theta, t) \), which is a priori unknown. This requirement to evaluate the solution on the unknown free surface may be eliminated by expanding the potential function in a
Maclaurin series (Hildebrand, 1962) in the neighborhood of \( z = 0 \); e.g.,

\[
\frac{\partial \phi}{\partial x} \bigg|_{z=n} = \frac{\partial \phi}{\partial x} \bigg|_{z=0} + \frac{\partial^2 \phi}{\partial x \partial z} \bigg|_{z=0} n + \frac{\partial^3 \phi}{\partial x \partial z^2} \bigg|_{z=0} \frac{n^2}{2!} + \cdots \quad (2.11)
\]

Consequently, the kinematic and dynamic free surface boundary conditions, Eqs. (2.5) and (2.6), become

\[
\frac{\partial n}{\partial t} - \left( \frac{\partial \phi}{\partial r} \frac{\partial n}{\partial r} + \frac{\partial^2 \phi}{\partial r \partial z} \frac{\partial n}{\partial r} + \frac{1}{2} \frac{\partial^3 \phi}{\partial r \partial z^2} \frac{\partial n}{\partial r} \right)
+ \frac{1}{r^2} \frac{\partial \phi}{\partial \theta} \frac{\partial n}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta \partial z} \frac{\partial n}{\partial \theta} + \cdots \right) \\
= - \left( \frac{\partial \phi}{\partial z} + \frac{\partial^2 \phi}{\partial z^2} n + \frac{1}{2} \frac{\partial^3 \phi}{\partial z^3} n^2 \right) + \cdots \quad ; \quad z = 0, \quad r \geq a \quad (2.12)
\]

and

\[
g_n - \frac{\partial \phi}{\partial t} - \frac{\partial^2 \phi}{\partial t \partial z} n - \frac{1}{2} \frac{\partial^3 \phi}{\partial t \partial z^2} n^2 + \frac{1}{2} \left[ \left( \frac{\partial \phi}{\partial r} \right)^2 + \left( \frac{\partial \phi}{\partial \theta} \right)^2 + \left( \frac{1}{r} \frac{\partial \phi}{\partial \theta} \right)^2 \right] \\
+ \frac{\partial \phi}{\partial r} \frac{\partial^2 \phi}{\partial r \partial z} n + \frac{\partial \phi}{\partial \theta} \frac{\partial^2 \phi}{\partial \theta \partial z} n + \frac{1}{r^2} \frac{\partial \phi}{\partial \theta} \frac{\partial^2 \phi}{\partial \theta \partial z} n + \cdots = Q(t) \\
\quad ; \quad z = 0, \quad r \geq a \quad (2.13)
\]
Let a dimensionless small perturbation parameter \( \epsilon \) be defined as a ratio that is proportional to ratio of the wave amplitude to the wave length; viz.,

\[
\epsilon = \frac{H}{k}
\]

(2.14)

in which \( H \) = wave height of the incident wave, \( k \) = wave number of the incident wave (= \( \frac{2\pi}{L} \)), and \( L \) = wave length.

In perturbation theory (Van Dyke, 1964; Nayfeh, 1973; Lin and Segel, 1974), the velocity potential, water surface elevation, and the Bernoulli constant are assumed to be expandable in terms of the perturbation parameter \( \epsilon \) as follows:

\[
\phi = \epsilon \phi_1 + \epsilon^2 \phi_2 + \epsilon^3 \phi_3 + \cdots
\]

(2.15a)

\[
n = \epsilon n_1 + \epsilon^2 n_2 + \epsilon^3 n_3 + \cdots
\]

(2.15b)

\[
Q = \epsilon Q_1 + \epsilon^2 Q_2 + \epsilon^3 Q_3 + \cdots
\]

(2.15c)

In a nonlinear problem there is no reason to restrict the frequency of response to be the same as the wave frequency in the linear problem (Pierson, 1963). It is therefore necessary to use a
perturbation scheme to expand the frequency as follows:

\[ \tau = (\sigma + \varepsilon \sigma_1 + \varepsilon^2 \sigma_2 + \cdots ) t \]  \hspace{1cm} (2.16)

in which \( \sigma \) is the frequency parameter from linear wave theory.

The linear boundary value problem that results from the perturbation expansion from collecting terms with coefficient \( \varepsilon \) is established by the following equations:

\[ \nabla^2 \phi_1(r,\theta,z,\tau) = 0 \quad ; \quad a < r < \infty, -h \leq z \leq 0, -\pi \leq \theta \leq \pi \]  \hspace{1cm} (2.17a)

\[ \frac{\partial \phi_1}{\partial z} = 0 \quad ; \quad z = -h \]  \hspace{1cm} (2.17b)

\[ \sigma \frac{\partial \eta_1}{\partial \tau} = -\frac{\partial \phi_1}{\partial z} \quad ; \quad z = 0, r \geq a \]  \hspace{1cm} (2.17c)

\[ g\eta_1 - \sigma \frac{\partial \phi_1}{\partial \tau} = Q_1 \quad ; \quad z = 0, r \geq a \]  \hspace{1cm} (2.17d)

\[ \frac{\partial \phi_1}{\partial r} = 0 \quad ; \quad r = a \]  \hspace{1cm} (2.17e)

Eliminating the unknown free surface elevation from the kinematic free surface boundary condition given by Eq. (2.17c) and from the dynamic free surface boundary condition given by Eq. (2.17d), leaves the following linear homogeneous combined free surface boundary
condition for $\Phi_1$:

$$g \frac{\partial \Phi_1}{\partial z} + \sigma^2 \frac{\partial^2 \Phi_1}{\partial \tau^2} + \sigma \frac{\partial \Phi_1}{\partial \tau} = 0 \quad ; \quad z = 0 \ , \ r \geq a \ .$$ (2.17f)

The boundary value problem for the velocity potential correct to second-order, $\varepsilon^2$, is given by the following:

$$\nabla^2 \phi_2(r, \theta, z, \tau) = 0 \quad ; \quad a \leq r < \infty \ , \ -h \leq z \leq 0 \ , \ -\pi \leq \theta \leq \pi$$ (2.18a)

$$\frac{\partial \phi_2}{\partial z} = 0 \quad ; \quad z = -h$$ (2.18b)

$$\left( \sigma \frac{\partial \eta_2}{\partial \tau} + \sigma_1 \frac{\partial \eta_1}{\partial \tau} \right) - \frac{\partial \Phi_1}{\partial r} \frac{\partial \eta_1}{\partial \tau} - \frac{1}{r^2} \frac{\partial \Phi_1}{\partial \theta} \frac{\partial \eta_1}{\partial \theta} = - \left( \frac{\partial \Phi_2}{\partial z} + \frac{\partial^2 \phi_1}{\partial z^2} \eta_1 \right)$$

$$\quad ; \quad z = 0 \ , \ r \geq a$$ (2.18c)

$$g \eta_2 - \left( \sigma \frac{\partial \phi_2}{\partial \tau} + \sigma_1 \frac{\partial \phi_1}{\partial \tau} \right) - \sigma \frac{\partial^2 \phi_1}{\partial \tau \partial z} \eta_1 + \frac{1}{r^2} \left[ \left( \frac{\partial \Phi_1}{\partial r} \right)^2 + \right.$$

$$\left. \left( \frac{\partial \Phi_1}{\partial z} \right)^2 + \left( \frac{1}{r} \frac{\partial \Phi_1}{\partial \theta} \right)^2 \right] = Q_2 \quad ; \quad z = 0 \ , \ r \geq a$$ (2.18d)

$$\frac{\partial \phi_2}{\partial r} = 0 \quad ; \quad r = a \ .$$ (2.18e)
Eliminating the second-order correction to the unknown free surface, \( \eta_2 \), yields the following combined free surface boundary condition for \( \Phi_2 \):

\[
\frac{\sigma^2}{\partial \tau^2} \frac{\partial^2 \Phi_2}{\partial z^2} + g \frac{\partial \Phi_2}{\partial z} + \alpha \frac{\partial Q_2}{\partial \tau} = 2\sigma \left( \frac{\partial \Phi_1}{\partial r} \frac{\partial^2 \Phi_1}{\partial r \partial \tau} + \frac{\partial \Phi_1}{\partial \theta} \frac{\partial^2 \Phi_1}{\partial \theta \partial \tau} + \frac{\partial \Phi_1}{\partial z} \frac{\partial^2 \Phi_1}{\partial z \partial \tau} + \frac{1}{r^2} \frac{\partial \Phi_1}{\partial r} \frac{\partial^2 \Phi_1}{\partial r \partial \tau} \right)
\]

\[- \sigma \frac{\partial^2 \Phi_1}{\partial z^2} \frac{\partial \Phi_1}{\partial \tau} - \sigma^3 \frac{\partial^3 \Phi_1}{\partial r \partial \theta \partial \tau} - 2\sigma \sigma \frac{\partial^2 \Phi_1}{\partial r \partial \tau} \right) ; \quad z = 0 , \ r \geq a .
\]

(2.18f)

The boundary value problem correct to second-order will be discussed in Chapter 6. The remainder of this chapter will discuss the first-order boundary value problem only.

The time dependent parameter in Eq. (2.16) for the first-order problem may be written as

\[
\tau = \sigma t .
\]

(2.19)

Suppose a simple harmonic wave is approaching from a point of minus infinity; then,

\[
\Phi_1(r, \theta, z, \tau) = \Phi_1(z) \sin(k \cos \theta - \tau).
\]

(2.20)
From Eq. (2.20), the first-order velocity potential may be conveniently separated into spatial and temporal parts. In cylindrical coordinates the separation gives

$$\phi_1(r, \theta, z, \tau) = \{ \phi_1(r, \theta, z) \exp(-i\tau) \} + \{ \}^*$$

(2.21)

in which the asterisk * represents the complex conjugate (cf. Mei and Ünlüata (1972) for this notation convention).

For a simple harmonic wave with no structures present, the velocity potential describing the fluid motion is well known (Dean and Eagleson, 1966). When a structure is present, the total velocity potential may be linearly decomposed into the sum of the known incident wave velocity potential and the unknown scattered wave velocity potential due to the linearity of the boundary value problem. That is,

$$\phi_1 = \phi_1^i + \phi_1^s$$

(2.22)

in which $\phi_1^i$ is the velocity potential associated with the incident wave and $\phi_1^s$ is the velocity potential associated with the scattered wave.

From the definition given by Eq. (2.21), the incident and scattered wave velocity potential may be specified as

$$\phi_1^i(r, \theta, z, \tau) = \{ \phi_1^i(r, \theta, z) \exp(-i\tau) \} + \{ \}^*$$

(2.23)
\[
\phi_S^s(r, \theta, z, \tau) = \{ \phi_1^s(r, \theta, z) \exp(-i\tau) \} + \{ \}^*.
\]  
(2.24)

The scattered wave represents the disturbance of an incident wave caused by the presence of the cylindrical structure. At large distances from the cylinder, the waves must be propagating outwards. This is the so-called "Sommerfeld radiation condition" which may be expressed analytically by

\[
\frac{\partial \phi_1^s}{\partial r} - (ik - \frac{1}{2r}) \phi_1^s = 0 ; \quad r \to \infty.
\]  
(2.25)

This form of the radiation condition was introduced and applied by Bai (1972) for axi-symmetric bodies. Eq. (2.25) is consistent with the more classical form

\[
\sqrt{r} \left( \frac{\partial \phi_1^s}{\partial r} - ik\phi_1^s \right) = 0 \quad ; \quad r \to \infty
\]  
(2.26)

given by John (1950, p. 54). This statement is, furthermore, a sufficient condition to ensure the mathematical uniqueness of the solution to the Laplace equation for the case of monochromatic waves in water of constant depth and for structures of bounded extent (cf. John, 1950).
Substituting Eqs. (2.22), (2.23) and (2.24) into Eqs. (2.17), we may decompose the single boundary value problem into two separate problems for the incident velocity potential and for the scattered velocity potential by:

\[ \nabla^2 \phi_1^i(r, \theta, z) = 0 \quad ; \quad a \leq r < \infty, \quad -h \leq z \leq 0, \quad -\pi \leq \theta \leq \pi \]  \hspace{1cm} (2.27a)

\[ \frac{\partial \phi_1^i}{\partial z} = 0 \quad ; \quad z = -h \]  \hspace{1cm} (2.27b)

\[ \frac{\partial \phi_1^i}{\partial z} - \frac{\sigma^2}{g^2} \phi_1^i = 0 \quad ; \quad z = 0 \]  \hspace{1cm} (2.27c)

and

\[ \nabla^2 \phi_1^s(r, \theta, z) = 0 \quad ; \quad a \leq r < \infty, \quad -h \leq z \leq 0, \quad -\pi \leq \theta \leq \pi \]  \hspace{1cm} (2.28a)

\[ \frac{\partial \phi_1^s}{\partial z} = 0 \quad ; \quad z = -h \]  \hspace{1cm} (2.28b)

\[ \frac{\partial \phi_1^s}{\partial z} - \frac{\sigma^2}{g} \phi_1^s = 0 \quad ; \quad z = 0 \]  \hspace{1cm} (2.28c)
\[
\frac{\partial \phi^s}{\partial r} = - \frac{\partial \phi^i}{\partial r} \quad ; \quad r = a
\] (2.28d)

\[
\frac{\partial \phi^i}{\partial r} - (\imath k - \frac{1}{2r}) \phi^i = 0 \quad ; \quad r \rightarrow \infty
\] (2.28e)

provided that

\[
\frac{\partial \phi^i}{\partial t} = 0 \quad .
\] (2.29)

Note that the incident velocity potential represents only the incoming wave and, therefore, it must satisfy the boundary value problem when no structure is present. The entire problem has now been reduced to that of finding a solution of a scattered velocity potential from Laplace's equation that satisfies the combined free surface boundary condition, an inhomogeneous kinematic boundary condition on the structural boundary, a homogeneous bottom boundary condition, and a radiation condition.

Since the boundary conditions in the first-order boundary value problem at the undisturbed water level, \( z = 0 \), and at the horizontal finite depth, \( z = -h \), are homogeneous and linear, the resulting separable boundary value problem in the vertical \( z \) coordinate is a well-posed Sturm-Liouville problem which has eigenfunction solutions.
CHAPTER 3

GREEN'S FUNCTION FOR BOUNDARY VALUE PROBLEM

A solution to the wave diffraction boundary value problem may be obtained in terms of a Green's function. The formal analytical approach involves a distribution of source singularities over the submerged portion of the cylindrical structure. The magnitudes of the source singularities are chosen to satisfy the inhomogeneous boundary condition. By assuming that the incident wave may be represented by a Stoke's wave, the scattered potential may be recovered by an integral equation over the structure boundary on which an inhomogeneous boundary condition is prescribed. The desired solution for the velocity potential is recovered via an integral equation in which the kernel of the integral equation is specified by the Green's function.

This chapter will begin with Green's formulae in Section 3.1. It is then applied to express the boundary value problem in terms of the Green's function in Section 3.2.

3.1 Green's Formulae

For a given vector function, \( \mathbf{V}_o \), defined in a bounded region, \( \mathcal{R} \), of three-dimensional space, and for a given volume, \( \mathbf{V} \), bounded by a closed piece-wise smooth surface, \( \mathbf{S} \), Gauss' divergence theorem states
\[ \mathbf{V} \cdot \mathbf{V}_o \, dV = \int_S \mathbf{V}_o \cdot \mathbf{n}_o \, dS \]  
\[ \text{(3.1)} \]

in which \( \mathbf{n}_o \) is a unit inward vector normal to the surface of the structural body. This theorem may be used to transform volume integrals into surface integrals. From a physical point of view, the surface integral may be regarded as the flow or the rate of flux of the vector field \( \mathbf{V}_o \) through the surface \( S \) having a unit normal vector \( \mathbf{n}_o \). Thus the divergence in the volume integral is interpreted as the total source or creation of flow within \( \bar{R} \).

Let the vector quantity \( \mathbf{V}_o \) be given by

\[ \mathbf{V}_o = \phi^S \mathbf{v}_G \]  
\[ \text{(3.2)} \]

Substituting \( \mathbf{V}_o \) into Gauss' divergence theorem gives

\[ \mathbf{v} \cdot \{ \phi^S \mathbf{v}_G \} \, dV = \int_\mathbf{V} \{ \phi^S v^2_G + \mathbf{v}_G \cdot \mathbf{v}_G \} \, dV \]

\[ = \int_S \phi^S \mathbf{v}_G \cdot \mathbf{n}_o \, dS \]  
\[ \text{(3.3)} \]
Eq. (3.3) is the Green's first formula (Kellogg, 1953, p.212; Hildebrand, 1962, p.301; Butkov, 1968, p.536).

Substituting the symmetrical form of Eq. (3.2)

\[ \hat{V}_o = G \nabla \phi^S \]

into Eq. (3.1) gives,

\[
\int \hat{V} \cdot \{ G \nabla \phi^S \} \, dV = \int \{ G \nabla \phi^S + vG \cdot \nabla \phi^S \} \, dV
\]

\[
= \int_{\Sigma} G \nabla \phi^S \cdot n_o \, dS
\]

(3.5)

Subtracting Eq. (3.5) from Eq. (3.3) yields the following:

\[
\int \{ \phi^S \nabla^2 G - G \nabla \phi^S \} \, dV = \int \{ \phi^S \nabla G - G \nabla \phi^S \} \cdot n_o \, dS.
\]

(3.6)

\[ \Psi(\cdot) \cdot \hat{n}_o \] is the projection of the rate change of the scalar quantity \( \Psi(\cdot) \) in the normal direction to the surface boundary; i.e., the normal derivative of the function \( \Psi(\cdot) \) at a point on the surface boundary. Thus,

\[ \Psi(\cdot) \cdot \hat{n}_o = \frac{\partial \Psi(\cdot)}{\partial n}. \]

(3.7)
Substituting Eq. (3.7) into Eq. (3.6), Greens second formula (Kellogg, 1953, p.215; Hildebrand, 1962, p.301; Butkov, 1968, p.537) is determined as

\[
\int \{ \phi^S \nabla^2 G - G \nabla^2 \phi^S \} \, d\mathbf{r} = \int \{ \phi^S \frac{\partial G}{\partial n} - G \frac{\partial \phi^S}{\partial n} \} \, dS .
\] (3.8)

3.2 Application of Green's Formulae

We now proceed by means of the Green's second formula applied to the first-order boundary value problem. Figure 3.1 shows a region, \( \hat{R} \), bounded above by the free surface, \( S_f \), at \( z = 0 \); bounded below by the horizontal bottom surface, \( S_b \), at \( z = -h \); bounded at the body surface by \( S_B \); and bounded at a great distance from the body by a vertical cylinder \( S_\infty \). The scattered velocity potential \( \phi^S(r,\theta,z) \) and the Green's function \( G_1(r,\theta,z;R,\phi,Z) \) may be substituted into the Green's second formula, Eq. (3.8). The point located at \( (r,\theta,z) \) in the Green's function is a general observer point in either the fluid region or on one of its boundaries; and the source point located at \( (R,\phi,Z) \) is a particular point either in the interior or boundary of the fluid region at which a source point is located. The function \( G_1(r,\theta,z;R,\phi,Z) \) represents the Green's function for the first-order boundary value problem and may be interpreted as a point wave source located at the coordinates \( (R,\phi,Z) \) on the body surface.
FIGURE 3.1: REGION OF APPLICATION OF GREEN'S FORMULA
For the first-order scattered potential, Eq. (3.8) may be rewritten as

\[
\int \{ \phi_1^S(r,\theta,z) \, \nabla^2 G_1(r,\theta,z;R,\phi,Z) - G_1(r,\theta,z;R,\phi,Z) \, \nabla^2 \phi_1^S(r,\theta,z) \} \, dV(r,\theta,z)
\]

\[
= \int \left\{ \phi_1^S(r,\theta,z) \frac{\partial G_1(r,\theta,z;R,\phi,Z)}{\partial n} \right\} \, dS(r,\theta,z)
\]

\[
- G_1(r,\theta,z;R,\phi,Z) \frac{\partial \phi_1^S(r,\theta,z)}{\partial n} \right\} \, dS(r,\theta,z)
\]

(3.9)

in which \( \frac{\partial}{\partial n} \) designates the normal derivative of the function at a point on the surface.

In Eq. (3.9), both the Green's function and the scattered potential are unknown functions. Since the scattered potential \( \phi_1^S \) satisfies Laplace equation everywhere throughout the region \( \hat{R} \); i.e.,

\[
\nabla^2 \phi_1^S(r,\theta,z) = 0
\]

(3.10)

the second term of the integral on the left hand side of Eq. (3.9) vanishes. The Green's function \( G_1(r,\theta,z;R,\phi,Z) \) is chosen to be singular such that it satisfies the following differential equation:
\[ \nabla^2 G_1(r, \theta, z; R, \Theta, Z) = \frac{\partial^2 G_1}{\partial r^2} + \frac{1}{r} \frac{\partial G_1}{\partial r} + \frac{1}{r^2} \frac{\partial^2 G_1}{\partial \theta^2} + \frac{\partial^2 G_1}{\partial z^2} \]

\[ = - \frac{I(r)}{r} \delta(r - R) \delta(\theta - \Theta) \delta(z - Z) \quad (3.11) \]

in which

\[ I^+(r) = \begin{cases} 
4\pi & \text{source in fluid} \\
2\pi & \text{source on boundary} \\
0 & \text{source inside boundary} 
\end{cases} \quad (3.12) \]

This indicator function accounts for the fact that only half the source flow enters the fluid domain when the source is on the boundary.

Substituting Eq. (3.11) into Eq. (3.9) and using the substitution property of the \( \delta \)-function, where the substitution property is defined by

\[ \int_{V} \phi(s) \delta(s-s_0) \, dV(s) = \phi(s_0) \quad , \]

\[ (3.14) \]

the left hand side of Eq. (3.9) is reduced to

\[ - I \phi^S_1(R, \Theta, Z) \]
or

\[
\phi^S_1(R,\theta,Z) = -\frac{1}{I} \int \left\{ \phi^S_1(r,\theta,z) \frac{aG_1(r,\theta,z;R,\theta,Z)}{\partial n} \right\} aG_1(r,\theta,z;R,\theta,Z) \, dS(r,\theta,z) = \frac{a^S_1(r,\theta,z)}{\partial n} \right\} dS(R,\theta,Z) .
\]

(3.15)

The value of \( \phi^S_1(R,\theta,Z) \) is now expressed in terms of a surface integral which includes the prescribed boundary conditions.

Friedman (1956; pp. 148-174) shows that the Green's function is a solution to a self-adjoint operator, it follows then that the Green's function is a symmetric function with respect to the source point \((R,\theta,Z)\) and the observer coordinates \((r,\theta,z)\); i.e.,

\[
G_1(r,\theta,z;R,\theta,Z) = G_1(R,\theta,Z;r,\theta,z)
\]

(3.16)

Now making a change of the dummy variables in Eq. (3.15) yields

\[
\phi^S_1(r,\theta,z) = -\frac{1}{I} \int \left\{ \phi^S_1(r,\theta,z) \frac{aG_1(r,\theta,z;R,\theta,Z)}{\partial n} \right\} aG_1(r,\theta,z;R,\theta,Z) \, dS(R,\theta,Z) .
\]

(3.17)

in which the normal vector \( \mathbf{n} \) is taken to be positive when pointing out
of the fluid volume and \( S(R,\theta,Z) \) denotes the integration surface which contains the source point \((R,\theta,Z)\).

Eq. (3.17) states that the scattered potential at any point \((r,\theta,z)\) within the fluid is equal to the integration of the contributions of the individual sources located on the immersed surface. The integral is evaluated over four distinct boundary surfaces. The following discussion will be divided into these four distinct boundaries. Consider first the free surface boundary \( S_f \).

(1) \( S_f \):

\[
\int_{-\pi}^{\pi} \int_{-\pi}^{a} \left( \phi^S_1 \frac{\partial G_1}{\partial \theta} - \frac{\phi^S_1}{g} \right) R \, dR \, d\theta \quad ; \quad Z = 0
\]  

(3.18)

Since \( \phi^S_1 \) satisfies the combined free surface boundary condition given by Eq. (2.28c); i.e.,

\[
\frac{\partial \phi^S_1}{\partial Z} = \frac{\sigma^2}{g} \phi^S_1 \quad ; \quad Z = 0
\]  

(3.19)

Eq. (3.18) is reduced to

\[
\int_{-\pi}^{\pi} \int_{-\pi}^{a} \phi^S_1 \left( \frac{\partial G_1}{\partial \theta} - \frac{\sigma^2}{g} G_1 \right) R \, dR \, d\theta \quad ; \quad Z = 0
\]  

(3.20)
If the Green's function, $G_1$, is chosen to satisfy the following combined free surface boundary condition

$$\frac{\partial G_1}{\partial Z} - \frac{\sigma^2}{g} G_1 = 0 \quad ; \quad Z = 0$$

(3.21)

on the free surface plane $Z = 0$; then, the integral across this free surface plane will vanish and

$$\int_{S_f} \left\{ \phi_1 \frac{\partial G_1}{\partial Z} - G_1 \frac{\partial \phi_1}{\partial Z} \right\} dS_f = 0 \quad ; \quad Z = 0 \quad .$$

(3.22)

Since the radiation condition given by Eq. (2.28e) requires that

$$\frac{\partial \phi_1^s}{\partial R} - \left( i k - \frac{1}{2R} \right) \phi_1^s = 0 \quad ; \quad R \rightarrow \infty \quad .$$

(3.24)
substitution reduces Eq. (3.23) to

\[
\int_{-\pi}^{\pi} \int_{0}^{-h} \phi_1^S \left( \frac{\partial G_1}{\partial R} - (ik - \frac{1}{ZR}) G_1 \right) R \, dZ \, d\theta \quad ; \quad R \to \infty \quad (3.25)
\]

If the Green's function is again required to satisfy the same radiation condition; viz.,

\[
\frac{\partial G_1}{\partial R} - (ik - \frac{1}{ZR}) G_1 = 0 \quad ; \quad R \to \infty \quad (3.26)
\]

then the integral across the surface \( S_\infty \) will also vanish; i.e.,

\[
\int_{S_\infty} \left\{ \phi_1 \frac{\partial G_1}{\partial R} - G_1 \frac{\partial \phi_1}{\partial R} \right\} \, dS_\infty = 0 \quad ; \quad R \to \infty \quad (3.27)
\]

(3) \( S_b \):

\[
\int_{-\pi}^{\pi} \int_{-\infty}^{a} \left\{ \phi_1 \frac{\partial G_1}{\partial Z} - G_1 \frac{\partial \phi_1}{\partial Z} \right\} R \, dR \, d\theta \quad ; \quad Z = -h \quad (3.28)
\]

The second term in the bracket vanishes due to the prescribed bottom boundary condition given by Eq. (2.28b); viz.,
\[
\frac{\partial \phi^S}{\partial Z} = 0 \quad ; \quad Z = -h \tag{3.29}
\]

and if the Green's function is also required to satisfy the bottom boundary condition

\[
\frac{\partial G_1}{\partial Z} = 0 \quad ; \quad Z = -h \tag{3.30}
\]

then the integral becomes

\[
\int_{S_b} \left\{ \phi \frac{\partial G_1}{\partial Z} - G_1 \frac{\partial \phi^S}{\partial Z} \right\} dS_b = 0 \quad ; \quad Z = -h \tag{3.31}
\]

(4) \( S_B \):

\[
\int_{-\pi}^{\pi} \int_{-h}^{0} \left\{ \phi \frac{\partial G_1}{\partial \alpha n} - G_1 \frac{\partial \phi^S}{\partial \alpha n} \right\} R \, dZ \, d\phi \quad ; \quad R = a \quad \tag{3.32}
\]

In the case of the cylindrical obstacles, we have

\[
\frac{\partial}{\partial \alpha n} = -\frac{\partial}{\partial R} \tag{3.33}
\]
and

\[ \frac{\partial \phi_1^s}{\partial R} = - \frac{\partial \phi_1^i}{\partial R} \quad ; \quad R = a \]  \hspace{1cm} (3.34)

which alters Eq. (3.32) to

\[
- \int_{-\pi}^{\pi} \int_{-h}^{0} \left\{ \phi_1 \frac{\partial G_1}{\partial R} + G_1 \frac{\partial \phi_1^i}{\partial R} \right\} R \, dZ \, d\Theta \quad ; \quad R = a . \]  \hspace{1cm} (3.35)

Since

\[ \frac{\partial \phi_1^i}{\partial R} \neq 0 \]  \hspace{1cm} (3.36)

we may require that

\[ \frac{\partial G_1}{\partial R} = 0 \quad ; \quad R = a \]  \hspace{1cm} (3.37)

which is the structural boundary condition for the Green's function.

Eq. (3.35) is now reduced to

\[
- \int_{-\pi}^{\pi} \int_{-h}^{0} G_1(r,\Theta,z;R,\Theta,Z) \frac{\partial \phi_1^i(R,\Theta,Z)}{\partial R} R \, dZ \, d\Theta \quad ; \quad R = a . \]  \hspace{1cm} (3.38)
From Eqs. (3.22), (3.27), (3.31) and (3.38), we may find that Eq. (3.17) becomes

\[
\phi_2(r, \theta, z) = \frac{1}{I} \int_{-\pi}^{\pi} \int_{-h}^{0} G_1(r, \theta, z; R, \phi, Z) \frac{\partial \phi_1(R, \phi, Z)}{\partial R} R \, dZ \, d\phi
\]

; \quad R = a \quad . \quad (3.39)

3.3 Boundary Value Problem for the Green's Function

To give a prospective view of the boundary value problem for the Green's function, the requirements obtained from the boundary integral are summarized as follows:

(1) partial differential equation

\[
\frac{\partial^2 G_1}{\partial r^2} + \frac{1}{r} \frac{\partial G_1}{\partial r} + \frac{1}{r^2} \frac{\partial^2 G_1}{\partial \theta^2} + \frac{\partial^2 G_1}{\partial z^2} = - \frac{I}{r} \delta(r-R) \delta(\theta-\phi) \delta(z-Z)
\]

; \quad a \leq r < \infty, -h \leq z \leq 0, -\pi \leq \theta \leq \pi

(3.40a)

(2) bottom boundary condition

\[
\frac{\partial G_1}{\partial z} = 0 \quad ; \quad z = -h \quad (3.40b)
\]
(3) combined free surface boundary condition

\[ \frac{\partial G_1}{\partial z} - \frac{\sigma^2}{g} G_1 = 0 \quad ; \quad z = 0 \]  

(3.40c)

(4) structural condition

\[ \frac{\partial G_1}{\partial r} = 0 \quad ; \quad r = a \]  

(3.40d)

(5) Sommerfeld radiation condition

\[ \frac{\partial G_1}{\partial r} - (ik - \frac{1}{2r}) G_1 = 0 \quad ; \quad r = \infty \] 

(3.40e)

Mathematically, a homogeneous partial differential equation with an inhomogeneous structural boundary condition for the scattered velocity potential \( \phi_s^1 \); i.e.,

\[ \nabla^2 \phi_s^1(r,\theta,z) = 0 \]  

(3.41a)

and

\[ \frac{\partial \phi_s^1(r,\theta,z)}{\partial r} = - \frac{\partial \phi_f^1(r,\theta,z)}{\partial r} \quad ; \quad r = a \]  

(3.41b)
may be transformed into an inhomogeneous partial differential equation
with a homogeneous boundary condition; i.e.,

$$\nabla^2 G_1(r, \theta, z; R, \phi, Z) = - \frac{I}{r} \delta(r-R) \delta(\theta-\phi) \delta(z-Z) \quad (3.42a)$$

and

$$\frac{\partial G_1(r, \theta, z; R, \phi, Z)}{\partial r} = 0 \quad \text{; } r = a \quad (3.42b)$$

The motivation for constructing a scattered potential from the
Green's function is that, once the Green's function has been
constructed for any separable coordinate system, the desired velocity
potential is recovered from an integral equation over the boundary
surface on which the inhomogeneous boundary conditions are prescribed;
i.e.,

$$\Phi_s(r, \theta, z) = \frac{1}{I} \int_{-\pi}^{\pi} \int_{-h}^{h} G_1(r, \theta, z; R, \phi, Z) \frac{\partial \phi_1(R, \phi, Z)}{\partial R} R \, dZ \, d\phi \quad (3.43)$$

; \quad R = a \ .

The kernel of this integral equation is the Green's function. This
expression means that the scattered potential represents a disturbance
which responds to the incident wave field.
CHAPTER 4
A GREEN'S FUNCTION

Chapter 3 demonstrated that a partial differential equation may be transformed into an integral equation by means of a Green's function in which the kernel of the integral equation was a Green's function,

\[ G_1(r,\theta,z;R,\Theta,Z) \]. The Green's function for the scattered wave problem is required to satisfy an inhomogeneous Helmholtz partial differential equation; i.e.,

\[
\frac{\partial^2 G_1}{\partial r^2} + \frac{1}{r} \frac{\partial G_1}{\partial r} + \frac{1}{r^2} \frac{\partial^2 G_1}{\partial \theta^2} + \frac{\partial^2 G_1}{\partial z^2} = - \frac{I}{r} \delta(r-R) \delta(\theta-\Theta) \delta(z-Z). \tag{4.1}
\]

Again, the point \((R,\Theta,Z)\) indicates the location of the source point and, thus, appears in the partial differential equation as a parameter.

The Fourier cosine transform pair (Hildebrand, 1962; p.219) with respect to \(\theta\) is defined as:

\[
G_1(r,\theta,z;R,\Theta,Z) = \frac{1}{2\pi} \sum_{n=0}^{\infty} (2 - \delta_{n0}) \tilde{G}_n(r,z;R,\Theta,Z) \cos n\theta \tag{4.2a}
\]

\[
\tilde{G}_n(r,z;R,\Theta,Z) = \int_{-\pi}^{\pi} G_1(r,\theta,z;R,\Theta,Z) \cos n\theta \, d\theta \tag{4.2b}
\]
in which the Kronecker delta symbol, $\delta_{nm}$, implies

\[
\delta_{nm} = \begin{cases} 
1 & ; \ n = m \\
0 & ; \ n \neq m 
\end{cases}
\]  

(4.3)

The Fourier cosine integral transform of Eq. (4.1) yields the following transformed equation:

\[
\frac{\partial^2 \tilde{G}}{\partial r^2} + \frac{1}{r} \frac{\partial \tilde{G}}{\partial r} - \frac{n^2}{r^2} \tilde{G} + \frac{\partial^2 \tilde{G}}{\partial z^2} = -\frac{I}{r} \delta(r - R) \cos n\theta \delta(z - Z). 
\]  

(4.4)

A Green's function satisfying appropriate homogeneous boundary conditions for the wave diffraction problem is presented in this chapter. Section 4.1 deals with the first-order solution of a Green's function for the evanescent modes. Section 4.2 deals with the first-order solution of a Green's function for the propagating mode. Section 4.3 summarizes the final result for the Green's function for the first-order problem which is a linear combination of the evanescent modes and propagating mode.

### 4.1 Eigenfunction Expansion of the Green's Function for the Evanescent Modes

One of the methods to approach the problem of constructing the Green's function is by determining its expansions in a series of suitably chosen orthogonal functions (Hildebrand, 1962; p.207).
We expand the Green's function for the evanescent modes by the following infinite series:

\[ G_n(r,z;R,\theta,z) = \sum_{m=2}^{\infty} g_n(r;R,\theta,z) \psi_m(z) \quad (4.5) \]

in which the orthonormal eigenfunctions \( \psi_m(z) \) are defined by

\[ \psi_m(z) = U_m \xi_m(z) \quad (4.6) \]

in which

\[ U_m = \frac{2 \sqrt{k_m}}{(2k_m h + \sin 2k_m h)^{3/2}} \quad (4.7) \]

and

\[ \xi_m(z) = \cos k_m (z+h) \quad (4.8) \]

provided that the eigenvalues, \( k_m \), are determined from

\[ \sigma^2 = -gk_m \tan k_m h \quad (4.9) \]
in which

\[
(2m - 3) \frac{\pi}{2} < k_m h < (m - 1) \pi \quad ; \quad m \geq 2 .
\]  

(4.10)

We note that the eigenvalues computed from Eq. (4.9) are all real and non-negative and that the eigenfunctions \( \xi(z) \) given by Eq. (4.8) are also all real. These properties along with the fact that the boundary conditions that are prescribed along the horizontal boundaries for the Green's function are all homogeneous indicate that the boundary value problem in the separable \( z \) coordinate is a proper Sturm-Liouville problem (Hildebrand, 1962; p.205). One of the properties of a solution to a Sturm-Liouville problem is that the set of eigenfunctions \( \psi_m \) are orthogonal over a given interval of orthogonality which, in this case, is the water depth. It is convenient to normalize a set of orthogonal eigenfunctions to give a set of orthonormal eigenfunctions which satisfy the following condition:

\[
\int_{-h}^{0} \psi_n(z) \psi_m(z) \, dz = \delta_{nm}
\]

(4.11)

or

\[
\int_{-h}^{0} \xi_n(z) \xi_m(z) \, dz = \frac{\delta_{nm}}{v_m^2}
\]

(4.12)
in which \( \delta_{nm} \) is the Kronecker delta defined in Eq. (4.3).

Substituting Eq. (4.5) into Eq. (4.4) yields

\[
\sum_{m=2}^{\infty} \left\{ \frac{a^2 g_m}{\ar^2} + \frac{1}{r} \frac{a g_m}{\ar} - \frac{n^2}{r^2} g_m \right\} \psi_m(z) + g_m \frac{a^2 \psi_m}{\az^2} = - \frac{T}{r} \delta(r-R) \delta(z-Z) \cos \theta . \tag{4.13}
\]

Multiplying both sides of Eq. (4.13) by the orthonormal eigenfunction \( \psi_1(z) \) and integrating over the interval of orthogonality \([-h, 0] \) yields

\[
\frac{a^2 g_m}{\ar^2} + \frac{1}{r} \frac{a g_m}{\ar} - \left( \frac{n^2}{r^2} + k_m^2 \right) g_m = - \frac{T}{r} \psi_m(Z) \delta(r-R) \cos \theta \tag{4.14} .
\]

Next, an integration over \( r \) yields

\[
\int_a^\infty \left\{ \frac{a^2 g_m}{\ar^2} + \frac{1}{r} \frac{a g_m}{\ar} - \left( \frac{n^2}{r^2} + k_m^2 \right) g_m \right\} \, dr = - \int_a^\infty \frac{T}{r} \delta(r-R) \psi_m(Z) \cos \theta \, dr \tag{4.15} .
\]
In order to evaluate the integral on the left of Eq. (4.15), we separate the integral into three parts according to the following linear decomposition:

\[
\int_{a}^{R} \left\{ \frac{a^2 g_n^m}{r^2} + \frac{1}{r} \frac{a g_n^m}{ar} - \left( \frac{n^2}{r^2} + k_m^2 \right) g_n^m \right\} \, dr
\]

\[
= \int_{a}^{R-} \left\{ \frac{a^2 g_n^m}{r^2} + \frac{1}{r} \frac{a g_n^m}{ar} - \left( \frac{n^2}{r^2} + k_m^2 \right) g_n^m \right\} \, dr
\]

\[
+ \int_{R-}^{R+} \left\{ \frac{a^2 g_n^m}{r^2} + \frac{1}{r} \frac{a g_n^m}{ar} - \left( \frac{n^2}{r^2} + k_m^2 \right) g_n^m \right\} \, dr
\]

\[
+ \int_{R+}^{\infty} \left\{ \frac{a^2 g_n^m}{r^2} + \frac{1}{r} \frac{a g_n^m}{ar} - \left( \frac{n^2}{r^2} + k_m^2 \right) g_n^m \right\} \, dr
\]

\[
= - \frac{I}{R} \psi_m(Z) \cos n\phi
\]

which gives the following three equations:

\[
\frac{a^2 g_n^m}{r^2} + \frac{1}{r} \frac{a g_n^m}{ar} - \left( \frac{n^2}{r^2} + k_m^2 \right) g_n^m = 0 \quad ; \quad a \leq r < R < \infty
\]

\[
\frac{a^2 g_n^m}{r^2} + \frac{1}{r} \frac{a g_n^m}{ar} - \left( \frac{n^2}{r^2} + k_m^2 \right) g_n^m = 0 \quad ; \quad a \leq R < r < \infty
\]
and
\[
\int_{R_{-}}^{R_{+}} \left\{ \frac{a^{2}g^{m}_{n}}{ar^{2}} + \frac{1}{r} \frac{ag^{m}_{n}}{ar} - \left( \frac{n^{2}}{r^{2}} + k^{2}_{m} \right) g^{m}_{n} \right\} dr
\]
\[= - \frac{I}{R} \psi_{m}(Z) \cos n\theta \quad (4.17c)
\]
in the limit as \( R_{-} \rightarrow R_{+} \).

Noting that
\[
\frac{a^{2}g^{m}_{n}}{ar^{2}} + \frac{1}{r} \frac{ag^{m}_{n}}{ar} = \frac{1}{r} \frac{a}{r} \left\{ \frac{ag^{m}_{n}}{ar} \right\}
\]
\[\quad (4.18)
\]
Eq. (4.17c) may be rewritten as
\[
\int_{R_{-}}^{R_{+}} \frac{1}{r} \frac{a}{r} \left\{ \frac{ag^{m}_{n}}{ar} \right\} dr - \int_{R_{-}}^{R_{+}} \left( \frac{n^{2}}{r^{2}} + k^{2}_{m} \right) g^{m}_{n} dr
\]
\[= - \frac{I}{R} \psi_{m}(Z) \cos n\theta \quad . \quad (4.19)
\]

Integrating twice by parts then gives
\[
\frac{ag^{m}_{n}}{ar} \bigg|_{R_{-}}^{R_{+}} + \frac{1}{r} g^{m}_{n} \bigg|_{R_{-}}^{R_{+}} + \int_{R_{-}}^{R_{+}} \left[ \frac{(1-n^{2})}{r^{2}} - k^{2}_{m} \right] g^{m}_{n} dr
\]
\[= - \frac{I}{R} \psi_{m}(Z) \cos n\theta \quad . \quad (4.20)
\]
From Eq. (4.20), we obtain the following two conditions:

\[ g^m_\infty \left|_{r = R_+} - g^m_\infty \left|_{r = R_-} \right. = 0 \qquad ; \; r = \lim R_\pm \to R \]  

(4.21)

and

\[ \frac{ag^m_\infty}{ar} \left|_{r = R_+} - \frac{ag^m_\infty}{ar} \left|_{r = R_-} \right. = - \frac{I}{R} \psi_m(Z) \cos n\theta \]  

; \; r = \lim R_\pm \to R . \]  

(4.22)

Eqs. (4.21) and (4.22) imply that \( g^m_\infty \) is continuous at \( r = R \) and that \( \frac{ag^m_\infty}{ar} \) has a jump discontinuity at \( r = R \).

The boundary value problem for the Green's function has now been reduced to that of finding a solution \( g^m_\infty \) that will satisfy the radiation condition and the zero-velocity boundary condition on the structure. The requirements of continuity, Eq. (4.21), and of jump discontinuity, Eq. (4.22), for the Green's function at the source point \( r = R \) have been added to the boundary value problem.

Let \( g^m_\infty \) represent \( g^m_\infty \) for \( a \leq r < R < \infty \) and \( g^m_\infty \) represent \( g^m_\infty \) for \( a \leq R < r < \infty \). A summary of the boundary value problem for \( g^m_\infty \) is provided as follows:

\[ \frac{a^2 g^m_\infty}{ar^2} + \frac{1}{r} \frac{ag^m_\infty}{ar} - \left( \frac{n^2}{r^2} + k^2_m \right) g^m_\infty = 0 \; ; \; a \leq r < R < \infty \]  

(4.23a)
\[
\frac{a^2 g_n^m}{r^2} + \frac{1}{r} \frac{ag_n^m}{ar} - \left( \frac{n^2}{r^2} + k_m^2 \right) g_n^m = 0 ; \quad a \leq R < r < \infty
\]  
(4.23b)

\[
\frac{ag_n^m}{ar} = 0 ; \quad r = a
\]  
(4.23c)

bounded as \( r \to \infty \) ; \( r \to \infty \)  
(4.23d)

\[
\hat{g}_n^m(r) - g_n^m(r) = 0 ; \quad r = R
\]  
(4.23e)

\[
\frac{ag_n^m(r)}{ar} - \frac{ag_n^m(r)}{ar} = - \frac{1}{R} \psi_m(Z) \cos \Theta ; \quad r = R
\]  
(4.23f)

The differential equation of Eqs.(4.23a,b) is precisely the modified Bessel equation; hence, its solution may be written in the form of

\[
g_n^m = a_n^m I_n(k_m r) + b_n^m K_n(k_m r)
\]  
(4.24a)

and

\[
\hat{g}_n^m = c_n^m I_n(k_m r) + d_n^m K_n(k_m r)
\]  
(4.24b)
in which \( I_n(k_m r) \) = the modified Bessel function of the first kind of order \( n \); \( K_n(k_m r) \) = the modified Bessel function of the second kind of order \( n \); and \( a^m_n, b^m_n, c^m_n \) and \( d^m_n \) are arbitrary coefficients to be determined from the prescribed boundary conditions.

The asymptotic behavior of the modified Bessel function for large \( r \) (Duff and Naylor, 1966; p.330) is expressed by

\[
I_n(k_m r) \sim \frac{1}{\sqrt{2\pi k_m r}} \left\{ e^{-k_m r + i(n+\frac{1}{2})\pi} + e^{k_m r} \right\} ; \quad r \to \infty
\]

(4.25a)

and

\[
K_n(k_m r) \sim \frac{\pi}{\sqrt{2k_m r}} e^{-k_m r} ; \quad r \to \infty.
\]

(4.25b)

The bounded requirement for Eq. (4.23d) requires that

\[
c^m_n = 0
\]

(4.26)

for all values of \( m \).

The structural boundary condition at \( r = a \), the continuity and the jump discontinuity of \( g^m_n \) at the source point \( r = R \) lead to the following three simultaneous equations:
\[ a_n I_n(k_m a) + b_n K_n(k_m a) = 0 \]  
(4.27a)

\[ a_n I_n(k_m R) + b_n K_n(k_m R) - d_n K_n(k_m R) = 0 \]  
(4.27b)

\[ a_n I'_n(k_m R) + b_n K'_n(k_m R) - d_n K'_n(k_m R) = \frac{I}{k_{mR}} \psi_m(Z) \cos n \phi \]  
(4.27c)

In which "'" indicates the derivative with respect to \( k_m r \). The definition for the derivative of the Bessel functions is defined as

\[ \frac{\partial y_n(kr)}{\partial r} = k y'_n(kr) ; \quad y = I, J, K, Y, H(1), H(2). \]  
(4.28a)

The Wronskian identity (Hildebrand, 1962; p. 178),

\[ I_n(k_m R) K_n(k_m R) - I_n(k_m R) K'_n(k_m R) = \frac{1}{k_m R} \]  
(4.28b)

allows \( a_n^m, b_n^m, \) and \( d_n^m \) of Eqs. (4.27) to be solved uniquely, if and only if the determinant of their coefficients is not zero; viz.,

\[
\begin{vmatrix}
I'_n(k_m a) & K'_n(k_m a) & 0 \\
I_n(k_m R) & K_n(k_m R) & -K_n(k_m R) \\
I'_n(k_m R) & K'_n(k_m R) & -K'_n(k_m R)
\end{vmatrix} = -\frac{K'_n(k_m a)}{k_m R} \neq 0
\]  
(4.29)

Therefore, the unknown coefficients may be obtained by standard matrix inversion methods and are given by the following:
\[ a_n^m = I \psi_m(Z) K_n(k_m R) \cos n\theta \]  
\[ (4.30a) \]

\[ b_n^m = - I \psi_m(Z) \frac{K_n(k_m R)}{K_n(k_m a)} I_n(k_m a) \cos n\theta \]  
\[ (4.30b) \]

\[ c_n^m = I \psi_m(Z) \cos n\theta \frac{f_1(R)}{K_n(k_m a)} \]  
\[ (4.30c) \]

in which

\[ f_1(R) = I_n(k_m R) K_n(k_m a) - I_n(k_m a) K_n(k_m R) \]  
\[ (4.31) \]

Substituting Eqs. (4.30) into Eqs. (4.24) yields the following:

\[ g_{2n}^m(r;R,\theta,Z) = I \psi_m(Z) \cos n\theta \frac{K_n(k_m R)}{K_n(k_m a)} f_1(r) \]  
\[ ; \quad a \leq r < R < \infty \]  
\[ (4.32a) \]

and

\[ \hat{g}_{2n}^m(r;R,\theta,Z) = I \psi_m(Z) \cos n\theta \frac{K_n(k_m r)}{K_n(k_m a)} f_1(R) \]  
\[ ; \quad a \leq R < r < \infty \]  
\[ (4.32b) \]
Substituting Eqs. (4.32) into Eqs. (4.5) and (4.2a) yields the solution of the Green's function for the evanescent modes:

\[ G_1(r, \theta, z; R, \varphi, \Omega) = \sum_{n=0}^{\infty} \frac{I_0(z - \delta n_0)}{2\pi} \cos n\theta \cdot \cos n\varphi \cdot \sum_{m=2}^{\infty} \psi_m(z) \psi_m(\Omega) \cdot \]

\[ \cdot \frac{K_n(k_mR)}{K_n(k_m\alpha)} f_1(r) \]

; \ a \leq r < R < \infty \quad (4.33a) \]

and

\[ G_1(r, \theta, z; R, \varphi, \Omega) = \sum_{n=0}^{\infty} \frac{I_0(z - \delta n_0)}{2\pi} \cos n\theta \cdot \cos n\varphi \cdot \sum_{m=2}^{\infty} \psi_m(z) \psi_m(\Omega) \cdot \]

\[ \cdot \frac{K_n(k_mR)}{K_n(k_m\alpha)} f_1(R) \]

; \ a \leq R < r < \infty \quad (4.33b) \]

in which

\[ f_1(R) = I_n(k_mR) K_n(k_m\alpha) - I_n(k_m\alpha) K_n(k_mR) \]

(4.34)

and
\[ \psi_m(z) = \frac{2\sqrt{k_m} \cos k_m(z+h)}{(2k_m h + \sin 2k_m h)^{1/2}}. \] (4.35)

4.2 Eigenfunction Expansion of the Green's Function for the Propagating Mode

Following the procedures outlined in Section 4.1, the Green's function may be expanded in terms of an orthonormal eigenfunction for the propagating mode; i.e.,

\[ \tilde{G}_n(r,z;R,\Theta,Z) = g_n(r;R,\Theta,Z) \psi_1(z) \] (4.36)

in which the orthonormal eigenfunction \( \psi_1(z) \) is defined by

\[ \psi_1(z) = U_1 \xi_1(z) \] (4.37)

in which

\[ U_1 = \frac{2\sqrt{k}}{(2kh + \sinh 2kh)^{1/2}} \] (4.38)

and

\[ \xi_1(z) = \cosh k(z+h), \] (4.39)
provided that the eigenvalue \( k \) is computed from

\[
\sigma^2 = gk \tanh kh .
\]  
(4.40)

By definition, the eigenvalue for \( m = 1 \)

\[
k = ik_1 .
\]  
(4.41)

Substituting Eq. (4.41) into Eq. (4.9) leads to Eq. (4.40).

We define \( g_n \) to represent \( g^1_n \) for \( a \leq r \leq R < \infty \) and \( \hat{g}_n \) to represents \( g^1_n \) for \( a \leq R < r < \infty \). We note that \( R \) is the point where the source is located.

Substitution of Eq. (4.41) into Eqs. (4.23) yields the following boundary value problem for the propagating mode:

\[
\frac{\partial^2 g_n}{\partial r^2} + \frac{1}{r} \frac{\partial g_n}{\partial r} + \left( k^2 - \frac{n^2}{r^2} \right) g_n = 0 ; \quad a \leq r < R < \infty \]  
(4.42a)

\[
\frac{\partial^2 \hat{g}_n}{\partial r^2} + \frac{1}{r} \frac{\partial \hat{g}_n}{\partial r} + \left( k^2 - \frac{n^2}{r^2} \right) \hat{g}_n = 0 ; \quad a \leq R < r < \infty \]  
(4.42b)

\[
\frac{\partial g_n}{\partial r} = 0 \quad ; \quad r = a \]  
(4.42c)


\[ \frac{\dot{g}_n}{ar} - \left( ik - \frac{1}{2r} \right) g_n = 0 \quad ; \quad r \to \infty \]  

(4.42d)

\[ \hat{g}_n(r) - g_n(r) = 0 \quad ; \quad r = R \]  

(4.42e)

\[ \frac{\dot{g}_n(r)}{ar} - \frac{\dot{g}_n(r)}{ar} = - \frac{T}{R} \psi_1(Z) \cos \theta \quad ; \quad r = R \]  

(4.42f)

Solutions to Eqs. (4.42a) and (4.42b) are readily seen to be Bessel functions of order \( n \) that are given by

\[ g_n = a_n J_n(\kappa r) + b_n Y_n(\kappa r) \]  

(4.43a)

and

\[ \hat{g}_n = c_n H_n^{(1)}(\kappa r) + d_n H_n^{(2)}(\kappa r) \]  

(4.43b)

in which \( J_n(\kappa r) \) = the Bessel function of the first kind of order \( n \); \( Y_n(\kappa r) \) = the Bessel function of the second kind of order \( n \); \( H_n^{(1)}(\kappa r) \) and \( H_n^{(2)}(\kappa r) \) = Hankel functions of the first and second kind of order \( n \), respectively, and \( a_n, b_n, c_n, d_n \) are arbitrary constant coefficients.

The asymptotic behavior of the Hankel functions for large values of \( r \) (cf., Duff and Naylor, 1966; p. 322) is:
\[ H_n^{(1)}(kr) \sim \sqrt{\frac{2}{\pi kr}} \exp \left\{ kr - \left(2n+1\right) \frac{\pi}{4} \right\} ; \quad r \to \infty \]  

(4.44a)

and

\[ H_n^{(2)}(kr) \sim \sqrt{\frac{2}{\pi kr}} \exp \left\{ -1\{ kr - \left(2n+1\right) \frac{\pi}{4} \right\} ; \quad r \to \infty \]  

(4.44b)

Since the incident wave is coming from infinity at \( \theta = -\pi \), we must require that

\[ d_n = 0. \]  

(4.45)

Substituting Eqs. (4.43) into the structural boundary condition, the continuity condition for \( g_n \) and the jump discontinuity for \( \frac{\partial g_n}{\partial r} \) at the source point leads to the following three simultaneous equations for the coefficients \( a_n, b_n, \) and \( c_n \):

\[ a_n J_n'(ka) + b_n Y_n'(ka) = 0 \]  

(4.46a)

\[ a_n J_n(kR) + b_n Y_n(kR) - c_n H_n^{(1)}(kR) = 0 \]  

(4.46b)

\[ a_n J_n'(kR) + b_n Y_n'(kR) - c_n H_n^{(1)}(kR) = \frac{1}{kR} \psi_1(Z) \cos n\theta . \]  

(4.46c)
The solutions are

\[ a_n = -\frac{I \pi}{2} \psi_1(Z) \cos n \phi Y_n(ka) \frac{H_n^{(1)}(kR)}{H_n^{(1)'}(ka)} \]  
(4.47a)

\[ b_n = \frac{I \pi}{2} \psi_1(Z) \cos n \phi J_n(ka) \frac{H_n^{(1)}(kR)}{H_n^{(1)'}(ka)} \]  
(4.47b)

\[ c_n = \frac{I \pi}{2} \psi_1(Z) \cos n \phi \frac{f_2(R)}{H_n^{(1)'}(ka)} \]  
(4.47c)

in which

\[ f_2(R) = J_n'(ka) Y_n(kR) - J_n(kR) Y_n'(ka) \]  
(4.48)

where we have used the following definitions for Hankel functions of the first kind:

\[ H_n^{(1)}(ka) = J_n(ka) + i Y_n(ka) \]  
(4.49a)

and

\[ H_n^{(1)'}(ka) = J_n'(ka) + Y_n'(ka) \]  
(4.49b)
and the Wronskian identities (Duff and Naylor, 1966; p. 322)

\[ J_n(kR) H_n^{(1)}(kR) - J_n^\prime(kR) H_n^{(1)}(kR) = \frac{2i}{\pi kR} \]  

(4.50a)

and

\[ Y_n(kR) H_n^{(1)}(kR) - Y_n^\prime(kR) H_n^{(1)}(kR) = -\frac{2}{\pi kR} \]  

(4.50b)

Thus, the solutions of Eqs. (4.42a) and (4.42b) which satisfy the prescribed boundary conditions given by Eqs. (4.42c) - (4.42f) may be written as the following:

\[ g_n(r; R, \theta, Z) = \frac{T}{2} \psi_1(Z) \cos n\theta \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)} f_2(r) \]  

(4.51a)

and

\[ \hat{g}_n(r; R, \theta, Z) = \frac{T}{2} \psi_1(Z) \cos n\theta \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)} f_2(R) \]  

(4.51b)
Substituting Eqs. (4.51) into Eqs. (4.36) and (4.2a) yields the solution of the Green's function for the propagating mode:

\[ G_1(r, \theta, z; R, \phi, Z) = \frac{T}{4} \sum_{n=0}^{\infty} (2 - \delta_{n0}) \psi_1(z) \psi_1(Z) \cos \phi \cos \phi \cdot \]

\[ f_2(r) \frac{H^{(1)}_n(kr)}{H^{(1)}_n'(ka)} ; \quad a \leq r < R < \infty \]

(4.52a)

and

\[ G_1(r, \theta, z; R, \phi, Z) = \frac{T}{4} \sum_{n=0}^{\infty} (2 - \delta_{n0}) \psi_1(Z) \psi_1(z) \cos \phi \cos \phi \cdot \]

\[ f_2(R) \frac{H^{(1)}_n(kr)}{H^{(1)}_n'(ka)} ; \quad a \leq R < r < \infty \]

(4.52b)

in which

\[ f_2(R) = J_n'(ka) Y_n(kr) - J_n(kr) Y_n'(ka) \]

(4.53)

and

\[ \psi_1(z) = \frac{2\sqrt{k} \cosh k(z+h)}{(2kh + \sinh 2kh)^{1/2}} . \]

(4.54)
The symmetry of the Green's function in the variables \((r, \theta, z)\) and \((R, \Theta, Z)\) is obvious. This is generally true for the Green's function of a self-adjoint operator (Friedman, 1956; p. 161).

4.3 Summary of First-order Green's Function

The total first-order Green's function may be written as a summation of the propagating mode and the evanescent modes; viz.,

\[
G_1(r, \theta, z; R, \Theta, Z) = G_1(r, \theta, z; R, \Theta, Z) \mid_{\text{propagating mode}} + G_1(r, \theta, z; R, \Theta, Z) \mid_{\text{evanescent modes}}
\]

in which the propagating mode for the Green's function is defined in Eqs. (4.52) and the evanescent mode is defined in Eqs. (4.33); i.e.,

\[
G_1(r, \theta, z; R, \Theta, Z) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{\cos n\theta \cdot \cos n\Theta}{2} \psi_m(z) \psi_m(Z) \left[ \frac{\delta_{m1}}{2} \frac{H_n^{(1)}(kR)}{H_n^{(1)'}(ka)} f_2(r) + \frac{(1 - \delta_{m1})}{\pi} \frac{K_n(k_m R)}{K_n(k_m a)} f_1(r) \right]
\]

\[; \ a \leq r < R < \infty \]

\[(4.56a)\]
\[ G_1(r, \theta, z; R, \Theta, Z) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} I(2 - \delta_{m n}) \frac{\psi_m(Z) \psi_m(z)}{2} \cos n\theta \cdot \cos n\phi \cdot \cos ne \cdot \]

\[ \cdot \left[ \frac{\delta_{m 1}}{2} \frac{H_n^{(1)}(kr)}{H_n^{(1)'}(ka)} f_2(R) + \frac{1 - \delta_{m 1}}{\pi} \frac{K_n(k_m r)}{K_n'(k_m a)} f_1(R) \right] \]

; \text{ } a \leq R < r < \infty \hspace{1cm} (4.56b)

in which \( I, f_1(R) \) and \( f_2(R) \) are defined in Eqs. (3.12), (4.31) and (4.48), respectively.
CHAPTER 5

DIFFRACTION THEORY CORRECT TO FIRST-ORDER

The solution of the Green's function for the first-order scattered velocity potential is given by Eqs. (4.56). The velocity potential for a circular cylinder based on an eigenfunction expansion of the Green's function will be presented in this chapter.

The scattered velocity potential defined by Eq. (3.43) in the form of a homogeneous Fredholm integral equation of the second kind is expressed as

\[
\phi_1^S(r, \theta, z) = \frac{1}{I} \int_{-\pi}^{\pi} \int_{-h}^{0} G_1(r, \theta, z; R, \Theta, Z) \frac{\partial \phi_1^I(R, \Theta, Z)}{\partial R} R \, dZ \, d\Theta
\]

; \quad R = a \quad . \quad (5.1)

This equation gives the scattered velocity potential, \( \phi_1^S \), in terms of a distribution of source points with source strength densities given by the inhomogeneous boundary conditions.

In order to evaluate the scattered velocity potential explicitly for any separable structural geometry, the selection of the appropriate Green's function to substitute for the kernel in the integral equation given by Eq. (5.1) becomes the critical step.
Since the source points are distributed over the submerged surface of the cylinder on which \( R < r \) for \( R = a \), we must, therefore, select

\[
G_1(r, \theta, z; R, \Theta, Z) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{I(2 - \delta_{n0})}{2 \pi a} \psi_m(Z) \psi_m(z) \cos \theta \cos \Theta .
\]

\[
\cdot \left[ \frac{\delta_{m1}}{2} \frac{H_n^{(1)}(kr)}{H_n^{(1)'}(ka)} f_2(R) + \frac{(1 - \delta_{m1})}{\pi} \frac{K_n(k_m r)}{K_n'(k_m a)} f_1(R) \right] \text{ ; } R = a . \tag{5.2}
\]

Evaluating Eq. (5.2) at \( R = a \) and using the Wronskian identities (Hildebrand, 1962; p.178)

\[
K_n'(k_m a) I_n(k_m a) - K_n(k_m a) I_n'(k_m a) = f_1(a) = - \frac{1}{k_m a} \tag{5.3}
\]

and

\[
J_n'(ka) Y_n(ka) - J_n(ka) Y_n'(ka) = f_2(a) = - \frac{2}{\pi k a} , \tag{5.4}
\]

we obtain

\[
G_1(r, \theta, z; a, \Theta, Z) = - \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{I(2 - \delta_{n0})}{2 \pi a} \psi_m(Z) \psi_m(z) \cos \theta \cos \Theta .
\]

\[
\cdot \left[ \frac{\delta_{m1}}{k} \frac{H_n^{(1)}(kr)}{H_n^{(1)'}(ka)} + \frac{(1 - \delta_{m1})}{k_m} \frac{K_n(k_m r)}{K_n'(k_m a)} \right] . \tag{5.5}
\]
In order to apply the boundary condition at the cylinder given by Eq. (5.1) using a simple harmonic incident wave velocity potential, $\phi_1^i$, it is first necessary that the incident plane wave be expressed in terms of a Bessel function using the following Jacobi expansion (Morse and Feshbach, 1953; p. 1371):

\[ e^{ikx} = e^{ikr \cos \theta} \]

\[ = \sum_{l=0}^{\infty} (2 - \delta_{10}) i^{l+1} J_1(kr) \cos \theta \]  \hspace{1cm} (5.6a)

From the definition given by Eqs. (2.23), the complex form of the incident wave velocity potential may be represented by the following first-order Stokes' wave:

\[ \phi_1^i(r, \theta, z) = \frac{i}{2} \frac{g}{k_0u_1} \cdot \psi_1(z) \frac{\sinh kh}{\cosh kh} \exp(ikr \cos \theta) \]

\[ = \frac{i}{2} \frac{g}{k_0u_1} \cdot \psi_1(z) \frac{\sinh kh}{\cosh kh} \exp(ikx) \]  \hspace{1cm} (5.7a)

\[ = \frac{i}{2} \frac{g}{k_0u_1} \cdot \frac{\psi_1(z)}{\cosh kh} \sum_{l=0}^{\infty} (2 - \delta_{10}) i^{l+1} J_1(kr) \cos \theta \]  \hspace{1cm} (5.7b)

Changing the coordinate $(r, \theta, z)$ to $(R, \theta, Z)$, taking the derivative with respect to $R$, and evaluating at $R = a$ gives

\[ \frac{\partial \phi_1^i(a, \theta, Z)}{\partial R} = \frac{1}{2} \frac{g}{\sigma u_1} \frac{\psi_1(Z)}{\cosh kh} \sum_{l=0}^{\infty} (2 - \delta_{10}) i^{l+1} J_1(ka) \cos \theta \]  \hspace{1cm} (5.8)
Substitution of Eq. (5.5) and Eq. (5.8) into Eq. (5.1) yields the following scattered velocity potential for the first-order problem:

\[
\phi^s_1(r, \theta, z) = -\frac{1}{I} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{I(2 - \delta_{n0})}{2\pi a} \psi_m(z) \cos n\theta \cdot \frac{1}{2} \frac{g}{U_{10}} \frac{1}{\cosh kh} \cdot
\]

\[
\cdot \left\{ \frac{\delta_{m1}}{k} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)} + \frac{(1 - \delta_{m1})}{k_m} \frac{K_n(k_m r)}{K_n(k_m a)} \right\} .
\]

\[
\cdot \sum_{l=0}^{\infty} (2 - \delta_{l0}) l^{l+1} J_{l+1}'(ka) \cdot \int_{-\pi}^{\pi} \cos n\theta \cos l\theta \, d\theta .
\]

\[
\cdot \int_{-h}^{0} \psi_m(Z) \psi_1(Z) \, dZ . \quad (5.9)
\]

Since \( \psi_m(Z) \) and \( \psi_1(Z) \) form an orthonormal set over the interval \([-h, 0] \), we have

\[
\int_{-h}^{0} \psi_m(Z) \psi_1(Z) \, dZ = \delta_{m1} . \quad (5.10)
\]

Thus, the scattered velocity potential resulting from the evanescent modes of the Green's function is zero over the interval of orthogonality \([-h, 0]\) since the incident wave eigenfunction \( \psi_1(Z) \) is a member of the complete set of eigenfunctions, \( \psi_m(Z) \).
Again, because of orthogonality (periodicity) of \( \cos n\theta \) and \( \cos l\theta \) in the interval of orthogonality (periodicity) \([-\pi, \pi]\), all terms except \( l = n \) integrate to zero; i.e.

\[
n \int_{-\pi}^{\pi} \cos n\theta \cos l \theta \, d\theta = \frac{2\pi}{2 - \delta_{n0}} \delta_{nl} \quad \text{for all } n
\]

in which \( \delta_{nl} \) is defined by Eq. (4.3).

Substituting Eqs. (5.10) and (5.11) into Eq. (5.9) yields

\[
\phi_1^S(r,\theta,z) = -\frac{1}{2} \frac{g}{k\sigma} \frac{\psi_1(z)}{U_1} \frac{1}{\cosh kh} \sum_{n=0}^{\infty} (2 - \delta_{n0}) \, i^{n+1} J_n(ka) \cdot \frac{H_n^{(1)}(kr)}{H_n^{(1)'}(ka)} \cos n\theta.
\]

Accordingly, the total velocity potential governing the motion when the incident wave is given by a first-order Stokes' wave may be expressed as

\[
\phi_1(r,\theta,z,t) = \{ (\phi_1^1 + \phi_1^S) \exp(-i\omega t) \} + \{ \}^*
\]

in which
Multiplying Eq. (5.14) by $\epsilon (= \frac{H}{2} k)$ reduces exactly to the diffraction solution given by MacCamy and Fuchs (1954).
CHAPTER 6

DIFFRACTION BY A CIRCULAR CYLINDER CORRECT TO SECOND-ORDER

The boundary value problem for the second-order velocity potential is defined by Eqs. (2.18) from a perturbation expansion. The time dependent parameter in Eq. (2.16) for the second-order may be written as

\[ \tau = (\sigma + \varepsilon \sigma_1) t \]  \hspace{1cm} (6.1)

The second-order velocity potential may be expressed as

\[ \phi_2(r,\theta,z,\tau) = \phi_2(z) \sin 2(k r \cos \theta - \tau) \]  \hspace{1cm} (6.2)

Similar to the first-order solution, the solution for the second-order velocity potential may also be conveniently separated into the complex-valued spatial and temporal parts; i.e.

\[ \phi_2(r,\theta,z,\tau) = \left\{ \phi_2(r,\theta,z) \exp(-i2\tau) \right\} + \left\{ \right\}^* \]  \hspace{1cm} (6.3)

in which * represents the complex conjugate.

Based on the linearity of the diffraction problem at each perturbation order, we may define \( \phi_2 \) by a linear sum of incident \( \phi_2^i \) and scattered \( \phi_2^s \) components by

\[ \phi_2 = \phi_2^i + \phi_2^s \]  \hspace{1cm} (6.4)
in which \( \phi^i_2 \) denotes the second-order velocity potential for the incident wave field, and \( \phi^s_2 \) denotes the second-order velocity potential for the scattered wave field. Substituting Eq. (6.4) into Eqs. (2.18), we obtain the following boundary value problem for the second-order velocity potential for the incident wave field only:

\[
\nabla^2 \phi^i_2(r, \theta, z, \tau) = 0 \quad ; \quad a \leq r < \infty, \quad -h \leq z \leq 0, \quad -\pi \leq \theta \leq \pi, \quad \tau > 0
\]

\[\text{(6.5a)}\]

\[
\frac{\partial \phi^i_2}{\partial z} = 0 \quad ; \quad z = -h
\]

\[\text{(6.5b)}\]

\[
\sigma^2 \frac{\partial^2 \phi^i_2}{\partial \tau^2} + g \frac{\partial \phi^i_2}{\partial z} = 2\sigma \left[ \frac{\partial \phi^1_1}{\partial r} \frac{\partial \phi^1_1}{\partial \theta \partial \tau} + \frac{\partial \phi^1_1}{\partial z} \frac{\partial \phi^1_1}{\partial \tau} + \frac{1}{r^2} \frac{\partial \phi^1_1}{\partial \theta} \frac{\partial \phi^1_1}{\partial \theta \partial \tau} \right]
\]

\[
- \sigma \frac{\partial^2 \phi^1_1}{\partial z^2} \frac{\partial \phi^1_1}{\partial \tau} - \frac{\sigma^3}{g} \frac{\partial \phi^1_1}{\partial \tau} \frac{\partial^2 \phi^1_1}{\partial z^2} - 2\sigma_1 \sigma \frac{\partial^2 \phi^1_1}{\partial \tau^2}
\]

\[; \quad z = 0, \quad \text{(6.5c)}\]

and the following boundary value problem for second-order velocity potential for the scattered wave field only:

\[
\nabla^2 \phi^s_2(r, \theta, z, \tau) = 0 \quad ; \quad a \leq r < \infty, \quad -h \leq z \leq 0, \quad -\pi \leq \theta \leq \pi, \quad \tau > 0
\]

\[\text{(6.6a)}\]
\[
\frac{\partial^2 \phi^s_2}{\partial z^2} = 0 ; \quad z = -h \quad (6.6b)
\]

\[
\sigma^2 \frac{\partial^2 \phi^s_2}{\partial \tau^2} + g \frac{\partial \phi^s_2}{\partial z} + \sigma \frac{\partial Q_2}{\partial \tau} = (W_1 + W_2) ; \quad z = 0 \quad (6.6c)
\]

\[
\frac{\partial \phi^s_2}{\partial r} = - \frac{\partial \phi^s_2}{\partial r} ; \quad r = a \quad (6.6d)
\]

\[
\frac{\partial \phi^s_2}{\partial r} - (i\alpha - \frac{1}{2a}) \phi^s_2 = 0 \text{ for propagating mode}
\]

bounded as \( r \to \infty \) for evanescent modes

\[
(6.6e)
\]

in which \( W_1(\phi^s_1) \) is a nonlinear function of \( \phi^s_1 \) only; viz.,

\[
W_1(\phi^s_1) = \sigma \left\{ \frac{\partial}{\partial \tau} \left[ \frac{(\partial \phi^s_1)}{\partial r}^2 + \frac{(\partial \phi^s_1)}{\partial z}^2 + \left( \frac{1}{r} \frac{\partial \phi^s_1}{\partial \theta} \right)^2 \right] - \frac{\partial^2 \phi^s_1}{\partial z^2} \frac{\partial \phi^s_1}{\partial \tau} + \frac{\sigma^2}{g} \frac{\partial \phi^s_1}{\partial \tau} - \frac{\partial^3 \phi^s_1}{\partial \tau^2 \partial z} \right\} \quad (6.7a)
\]

and \( W_2(\phi^i_1, \phi^s_1) \) is a nonlinear interaction function of both \( \phi^i_1 \) and \( \phi^s_1 \); viz.,

\[
W_2(\phi^i_1, \phi^s_1) = \sigma \left\{ 2 \frac{\partial}{\partial \tau} \left[ \frac{\partial \phi^i_1 \partial \phi^s_1}{\partial r \partial r} + \frac{\partial \phi^i_1 \partial \phi^s_1}{\partial r \partial z} + \frac{\partial \phi^i_1 \partial \phi^s_1}{\partial z \partial z} \right] + \left( \frac{r^2}{\sigma} \right) \frac{\partial \phi^i_1 \partial \phi^i_1}{\partial \theta \partial \theta} \right] \]

\[
- \left[ \frac{\partial^2 \phi^i_1}{\partial z^2} \frac{\partial \phi^s_1}{\partial \tau} + \frac{\partial^2 \phi^i_1}{\partial z^2} \frac{\partial \phi^s_1}{\partial \tau} + \frac{\sigma^2}{g} \frac{\partial \phi^i_1 \partial \phi^s_1}{\partial \tau} - \frac{\partial^3 \phi^i_1 \partial \phi^i_1}{\partial \tau^2 \partial z} \right] \right\} , \quad (6.7b)
and \( \alpha \) is the separation constant for the second-order scattered wave. The inhomogenous term in Eq. (6.6c), which is defined in Eqs. (6.7), may be seen to be a linear combination of: (1) a nonlinear function of only the first-order scattered wave potential, and (2) a nonlinear interaction function of both the first-order incident and scattered waves potentials.

A periodic solution to the second-order incident wave boundary value problem which satisfies Eqs. (6.5a) and (6.5b) exactly may be assumed from a Stokes' wave perturbation expansion similar to Eq. (2.20) by the following:

\[
\phi_2^i = C_2 \cosh 2k(z+h) \sin 2(kr\cos \varphi - \tau) .
\] (6.8)

The first-order incident wave velocity potential may be represented as

\[
\phi_1^i = - \frac{2g}{k_0} \frac{\cosh k(z+h)}{\cosh kh} \sin(kr\cos \varphi - \tau) .
\] (6.9)

Substituting Eqs. (6.8) and (6.9) into Eq. (6.5c) yields

\[
C_2 = - \frac{3\sigma}{8k^2 \sinh^4 kh} ,
\] (6.10)

and

\[
\sigma_1 = 0 .
\] (6.11)
The resulting second-order velocity potential for the incident wave field defined by Eq. (6.8) may be written in complex-valued form as

\[ \phi_2^i(r,\theta,z,\tau) = \{ \phi_2^i(r,\theta,z) \exp(-iz\tau) \} + \{ \}^* \]  

(6.12)

in which

\[ \phi_2^i = -\frac{i}{2} C_2 \cosh 2k(z+h) \exp i(2krcos \theta) \]  

(6.13a)

\[ = \frac{3\sigma \cosh 2k(z+h)}{16k^2 \sinh^4 \omega_0} \sum_{n=0}^{\infty} (2 - \delta_{n0}) i^{n+1} J_n(2kr) \cos n\theta . \]  

(6.13b)

Eq. (6.13b) may be shown to be identical to the second-order Stokes' wave solution (cf., Dean and Eagleson, Eq. (2.39), p. 102), if the exponential phase function is expanded in the Jacobi expansion given by Eq. (5.6b).

Similarly, both the second-order scattered velocity potential and the inhomogenous combined free surface boundary condition may also be separated into the complex-valued form of spatial and temporal parts given by Eq. (6.3); i.e.,

\[ \phi_2^s(r,\theta,z,\tau) = \{ \phi_2^s(r,\theta,z) \exp(-i2\tau) \} + \{ \}^* \]  

(6.14)

and

\[ W_1 + W_2 = \{ F_s \exp(-i2\tau) \} + \{ \}^* + W_3 \]  

(6.15)
in which

\[ F_s = w_1 (\phi_1^S) + w_2 (\phi_1^S, \phi_1^S) \quad ; \quad z = 0 \quad . \] (6.16)

Substituting Eqs. (6.14) and (6.15) into Eqs. (6.7) yields

\[ w_1 = -i \sigma \{ 2 \left[ \frac{\partial \phi_1^S}{\partial r} \right]^2 + \left( \frac{1}{r} \frac{\partial \phi_1^S}{\partial \theta} \right)^2 - \frac{\partial^2 \phi_1^S}{\partial z^2} \phi_1^S + \frac{\sigma^2}{g} \phi_1^S \frac{\partial \phi_1^S}{\partial z} \} , \]

(6.17)

\[ w_2 = -i \sigma \{ 4 \left[ \frac{\partial \phi_1^i}{\partial r} \frac{\partial \phi_1^S}{\partial r} + \frac{\partial \phi_1^i}{\partial z} \frac{\partial \phi_1^S}{\partial z} + \left( \frac{1}{r} \frac{\partial \phi_1^i}{\partial \theta} \right) \left( \frac{1}{r} \frac{\partial \phi_1^S}{\partial \theta} \right) \right] - \left[ \frac{\partial^2 \phi_1^i}{\partial z^2} \phi_1^i + \frac{\partial^2 \phi_1^S}{\partial z^2} \phi_1^S - \frac{\sigma^2}{g} \left( \phi_1^i \frac{\partial \phi_1^i}{\partial z} + \phi_1^S \frac{\partial \phi_1^S}{\partial z} \right) \right] \} \]

(6.18)

and

\[ w_3 = -i \sigma \{ \frac{\partial^2 \phi_1^i}{\partial z^2} (\phi_1^i + \phi_1^i)^* + \frac{\partial^2 \phi_1^i}{\partial z^2} (\phi_1^i)^* \]

\[ + \frac{\sigma^2}{g} \left[ \phi_1^i \left( \frac{\partial \phi_1^i}{\partial z} + \frac{\partial \phi_1^i}{\partial z} \right)^* + \phi_1^i \left( \frac{\partial \phi_1^i}{\partial z} \right)^* \right] \}

\[ + i \sigma \{ \left( \frac{\partial \phi_1^i}{\partial z} \right)^* (\phi_1^i + \phi_1^i) + \left( \frac{\partial \phi_1^i}{\partial z} \right)^* \phi_1^i \]

\[ + \frac{\sigma^2}{g} \left[ (\phi_1^i)^* \left( \frac{\partial \phi_1^i}{\partial z} + \frac{\partial \phi_1^i}{\partial z} \right) + (\phi_1^i)^* \left( \frac{\partial \phi_1^i}{\partial z} \right) \right] \} \] (6.19a)
Equating the temporal derivative of the Bernoulli constant in Eq. (6.6c) with \( w_3 \) and substituting Eqs. (6.14) and (6.15) into Eqs. (6.6) yields the following boundary value problem for the spatial scattered velocity potential, \( \phi_s^2 \):

\[
\nabla^2 \phi_s^2(r, \theta, z) = 0 \quad ; \quad a \leq r < \infty, \quad -h \leq z \leq 0,
\]
\[
-\pi \leq \theta \leq \pi
\]

\[
\frac{\partial \phi_s^2}{\partial z} = 0 \quad ; \quad z = -h
\]

\[
\frac{\partial \phi_s^2}{\partial z} - \frac{4a^2}{g} \phi_s^2 = \frac{F_s^2}{g} \quad ; \quad z = 0
\]

\[
\frac{\partial \phi_s^2}{\partial r} = -\frac{\partial \phi_2^1}{\partial r} \quad ; \quad r = a
\]

\[
\frac{\partial \phi_s^2}{\partial r} - \left( i\alpha - \frac{1}{2r} \right) \phi_s^2 = 0 \quad \text{for propagating mode}
\]
\[
\text{bounded as } r \to \infty \quad \text{for evanescent modes}
\]

In order to complete the derivation of the boundary value problem for the second-order velocity potential for the scattered wave field, we substitute the first-order velocity potentials for both the
incident wave field, Eq. (5.7b), and the scattered wave field, Eq. (5.12), into Eq. (6.16). The details of this substitution are presented in Appendix A.

By comparing the first-order boundary value problem (Eqs. (2.28)) and the second-order boundary value problem (Eqs. (6.20)) for the scattered velocity potential, the striking similarity of the first- and second-order boundary value problems is evident. The only major difference is the combined free surface boundary condition; viz., the second-order problem has an inhomogeneous combined free surface boundary condition while the first-order problem has a homogeneous combined free surface boundary condition.

This problem may be further simplified by decomposing the boundary value problem for the second-order scattered potential with two separate inhomogeneous boundary conditions into two separate boundary value problems with each problem having only one inhomogeneous boundary condition (cf. Garrison, 1978; Shen, 1977).

We begin by defining the separate scattered potentials by

\[ \phi_2^s = \phi_2^{ss} + \phi_2^{sf} \]  

(6.21)

in which \( \phi_2^{ss} \) denotes the second-order scattered velocity potential associated with the inhomogeneous term that occurs in the structural boundary condition and \( \phi_2^{sf} \) denotes the second-order velocity potential associated with the inhomogeneous term that occurs in the combined free surface boundary condition.
Thus, the boundary value problem for $\phi^s_2$ may be replaced by two boundary value problems defined by

\[
\nabla^2 \phi^s_2 (r, \theta, z) = 0 \quad ; \quad 0 \leq r < \infty, \quad -h \leq z \leq 0, \\
-\pi \leq \theta \leq \pi  \quad (6.22a)
\]

\[
\frac{\partial \phi^s_2}{\partial z} = 0 \quad ; \quad z = h  \quad (6.22b)
\]

\[
\frac{\partial \phi^s_2}{\partial z} - \frac{4\sigma^2}{g} \phi^s_2 = 0 \quad ; \quad z = 0  \quad (6.22c)
\]

\[
\frac{\partial \phi^s_2}{\partial r} = -\frac{\partial \phi^s_2}{\partial r} \quad ; \quad r = a  \quad (6.22d)
\]

\[
\frac{\partial \phi^s_2}{\partial r} - \left( i\kappa - \frac{1}{2r} \right) \phi^s_2 = 0 \text{ for propagating mode} \quad \left\{ \begin{array}{l}
\text{bounded as } r \to \infty \text{ for evanescent modes} \\
\end{array} \right. \quad (6.22e)
\]

and by

\[
\nabla^2 \phi^s_f (r, \theta, z) = 0 \quad ; \quad a \leq r < \infty, \quad -h \leq z \leq 0, \\
-\pi \leq \theta \leq \pi  \quad (6.23a)
\]

\[
\frac{\partial \phi^s_f}{\partial z} = 0 \quad ; \quad z = -h  \quad (6.23b)
\]
\[
\frac{\partial \phi_{2}^sf}{\partial z} - \frac{4\sigma^2}{g} \phi_{2}^sf = \frac{F_s}{g}
\]
; \(z = 0\) \hspace{1cm} (6.23c)

\[
\frac{\partial \phi_{2}^sf}{\partial r} = 0
\]
; \(r = a\) \hspace{1cm} (6.23d)

\[
\frac{\partial \phi_{2}^sf}{\partial r} - (iq_m - \frac{1}{2r}) \phi_{2}^sf = 0
\]
; \(r \to \infty\) \hspace{1cm} (6.23e)

in which the eigenvalues \(\kappa\) and \(q_m\) in Eqs. (6.22e) and (6.23e) are to be determined from the homogeneous boundary conditions in the \(z\)- and \(r\)-coordinates, respectively.

The boundary value problem defined by Eqs. (6.22) accounts for the inhomogeneous boundary condition on the surface of the cylinder but contains a homogeneous boundary condition on the free surface. In contrast, the boundary value problem that is defined by Eqs. (6.23) results in a homogeneous structural boundary condition on the surface of the cylinder but is inhomogeneous in the combined free surface boundary condition. Physically, these two linearly decomposed problems represent two separate inhomogeneous boundary conditions. The first is a scattered wave potential that is generated by a periodic pressure distribution on the surface of the cylinder but without a forced pressure distribution on the free surface. The second is a scattered wave potential that is generated by a periodic pressure distribution on the free surface only.
Since the boundary conditions in $\psi^S_{s2}$ at the undisturbed water level, $z = 0$, and the horizontal finite depth, $z = -h$, are homogeneous and linear, the resulting separable boundary value problem in the vertical $z$ coordinate is a well-posed Sturm-Liouville problem which has eigenfunction solutions in $z$ coordinate for $\psi^S_{s2}$. This is essentially the analytical method employed by Raman, et al (1975, 1976, and 1977).

The boundary conditions in $\psi^sf_{s2}$ at the structural boundary, $r = a$, and the infinite boundary, $r \to \infty$, are also homogeneous and linear. Hence, this boundary value problem is also a well-posed Sturm-Liouville problem and the eigenvalues for $\psi^sf_{s2}$ are determined from the homogeneous boundary conditions in the horizontal $r$ coordinate.

Referring to Chapters 4 and 5, the eigenfunction expansion of the Green's function is a powerful method for solving for the Green's function. The Green's function for the second-order problem is evaluated in a similar fashion as the first-order problem (see Chapter 4). Accordingly, the Green's function for the second-order scattered velocity potentials $\psi^S_{s2}$ and $\psi^sf_{s2}$ may be expanded in eigenfunction expansions that are derived from the homogeneous boundary conditions for each separate boundary value problem.

The problem solution technique for the Green's function is to transform the homogeneous partial differential equation with an inhomogeneous boundary condition into an inhomogeneous partial differential equation with homogeneous boundary conditions. Then, the desired velocity potentials for each of the two scattered waves are recovered from an integral equation evaluated over the boundaries where the inhomogeneous boundary conditions are prescribed. Each of the
second-order scattered velocity potential components $\phi^s_2$ and $\phi^f_2$ now have only one inhomogeneous boundary condition. To be more precise, the inhomogeneous boundary condition for $\phi^s_2$ is derived only from the prescribed velocity distribution on the cylinder boundary; while that for $\phi^f_2$ is derived only from the prescribed pressure and velocity distribution given by the combined free surface boundary condition. Therefore, the solutions for $\phi^s_2$ and $\phi^f_2$ in terms of the Green's function method are evaluated by the inhomogeneous structural boundary condition and by the inhomogeneous combined free surface boundary condition, respectively.

Section 6.1 deals with the solution of $\phi^s_2$; Section 6.2 deals with the solution of $\phi^f_2$; and Section 6.3 summarizes the final result for the scattered velocity potential correct to second-order.

6.1 $\phi^s_2$

By applying the Green's second formula defined by Eq. (3.8), the boundary value problem for $\phi^s_2$ defined by Eqs. (6.22) may be suitably transformed into a boundary value problem in terms of a Green's function $G^s_2$; i.e.,

$$\frac{a^2 G^s_2}{a^2} + \frac{1}{r} \frac{ag^s_2}{ar} + \frac{1}{r^2} \frac{ag^s_2}{a\theta} + \frac{2}{r^2} \frac{ag^s_2}{az} = -\frac{I}{r} \delta(r-R) \delta(\theta-\Theta) \delta(z-Z)$$

$$; \quad a \leq r < \infty, \quad -h \leq z \leq 0, \quad -\pi \leq \theta \leq \pi$$

(6.24a)
\[
\frac{\partial G_2^s}{\partial z} = 0 \quad ; \quad z = -h \quad (6.24b)
\]
\[
\frac{\partial G_2^s}{\partial z} - \frac{4\sigma^2}{g} G_2^s = 0 \quad ; \quad z = 0 \quad (6.24c)
\]
\[
\frac{\partial G_2^s}{\partial r} = 0 \quad ; \quad r = a \quad (6.24d)
\]
\[
\frac{\partial G_2^s}{\partial r} - (i\kappa - \frac{1}{2r}) G_2^s = 0 \text{ for propagating mode} \]
\[
\text{bounded as } r \to \infty \text{ for evanescent modes} \quad (6.24e)
\]

in which \( G_2^s(r, \theta, z; R, \phi, Z) \) is defined as the second-order Green's function for the scattered wave structural boundary condition. The scattered velocity potential \( \phi_2^{ss} \) may be recovered from the following integral equation:

\[
\phi_2^{ss}(r, \theta, z) = \frac{1}{I} \int_{-\pi}^{\pi} \int_{-h}^{0} G_2^s(r, \theta, z; R, \phi, Z) \frac{\partial \phi_2^{ss}(R, \phi, Z)}{\partial R} R \, dZ \, d\theta \quad ; \quad R = a.
\] (6.25)

6.1.1 Green's Function for \( \phi_2^{ss} \)

The boundary value problem defined by Eqs. (6.24) is similar in form to the first-order boundary value problem as indicated in Eqs. (3.40). The only significant difference is that the factor \( \sigma^2 \) in Eq. (3.40c) is replaced by \( 4\sigma^2 \) which occurs in the combined free surface boundary condition in Eq. (6.24c). Accordingly, the same method of solution is valid and the solution for the Green's function of \( \phi_2^{ss} \) may be obtained by replacing \( k \) by \( \kappa \) in Eq. (4.56); i.e.,
\[
G_s^\varepsilon(r,\theta,z;R,\Theta,Z) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{\Lambda_m(z) \Lambda_m(Z)}{2} \frac{\Lambda_m(z) \Lambda_m(Z)}{2} \cos \theta \cdot \cos \theta - \\
\delta_{m1} \frac{H_n^{(1)}(\kappa R)}{H_n^{(1)}(\kappa a)} f_4(r) + \frac{(1 - \delta_{m1})}{\pi} \frac{K_n(\kappa_m R)}{K_n(\kappa_m a)} f_3(r) \\
; \quad a \leq r < R = \infty
\]

(6.26a)

\[
G_s^\varepsilon(r,\theta,z;R,\Theta,Z) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{\Lambda_m(z) \Lambda_m(Z)}{2} \frac{\Lambda_m(z) \Lambda_m(Z)}{2} \cos \theta \cdot \cos \theta - \\
\delta_{m1} \frac{H_n^{(1)}(\kappa R)}{H_n^{(1)}(\kappa a)} f_4(R) + \frac{(1 - \delta_{m1})}{\pi} \frac{K_n(\kappa_m R)}{K_n(\kappa_m a)} f_3(R) \\
; \quad a \leq R < \infty
\]

(6.26b)

in which

\[
f_3(R) = I_n(\kappa_m R) K_n'(\kappa_m a) - I_n'(\kappa_m R) K_n(\kappa_m a)
\]

(6.27)

\[
f_4(R) = J_n'(\kappa a) Y_n(\kappa R) - J_n(\kappa R) Y_n'(\kappa a)
\]

(5.28)

and the orthonormal eigenfunctions \( \Lambda_m(z) \) are defined by
\[ \Lambda_m(z) = \Gamma_m \psi_m(z) \]  

(6.29)

in which the normalizing constant is

\[ \Gamma_m = \frac{2\sqrt{\kappa_m}}{(2\kappa_m h + \sin 2\kappa_m h)^{\frac{1}{2}}} \]  

(6.30)

and the eigenfunctions are

\[ \psi_m(z) = \cos \kappa_m(z + h) \]  

(6.31)

provided that the eigenvalues, \( \kappa_m \), are determined from

\[ 4\sigma^2 = -g\kappa \tan \kappa_m h \]  

(6.32)

in which

\[ (2m - 3) \frac{\pi}{2} < \kappa_m h < (m - 1) \pi \]  

; \( m \geq 2 \)  

(6.33)

Similar to Eq. (4.41), the eigenvalue for \( m = 1 \) is defined as

\[ \kappa_{m=1} = i\kappa_1 = i\kappa \]  

(6.34)

Substituting Eq. (6.34) into Eqs. (6.30)-(6.32) yields

\[ \Gamma_1 = \frac{2\sqrt{\kappa}}{(2\kappa h + \sinh 2\kappa h)^{\frac{1}{2}}} \]  

(6.35)
and

\[ \nu_1(z) = \cosh \kappa(z + h) \quad , \tag{6.36} \]

provided that the eigenvalue \( \kappa \) is given by

\[ 4 \sigma^2 = g \kappa \tanh \kappa h \quad . \tag{6.37} \]

It is apparent that the source singularities are now distributed over the submerged surface of the circular cylinder and that the Green's function must satisfy the zero-flow condition on the surface of cylinder; i.e. \( R < r \) and \( R = a \), we may now select the Green's function given by Eq. (6.26b); viz.,

\[
G_2^S(r, \theta, z; R, \Theta, Z) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{I(2 - \delta_{n0})}{2} \Lambda_m(Z) \Lambda_n(Z) \cos \eta \cos \Theta \cdot \\
\left[ \frac{\delta_{m1}}{2} \frac{H_{1}^{(1)}(\kappa r)}{H_1^{(1)}(\kappa a)} f_4(R) + \frac{(1 - \delta_{m1})}{\pi} \frac{K_n(\kappa_m r)}{K_n(\kappa_m a)} f_3(R) \right] \\
; \quad R = a \quad . \tag{6.38}
\]

Evaluating Eq. (6.38) at \( R = a \) and using the following Wronskian identities [cf., Eqs. (5.3) and (5.4)]:

\[ f_3(a) = - \frac{1}{\kappa_m a} \quad (6.39) \]
and

\[ f_4(a) = - \frac{2}{\pi \kappa a} \quad (6.40) \]

we obtain

\[ G_s^2(r, \theta, z; a, \Theta, Z) = - \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{I(2 - \delta_{n0}^1)}{2\pi a} \Lambda_n(z) \Lambda_m(Z) \cos \theta \cos \Theta \]

\[ \cdot \left\{ \delta_{m1} \frac{H_n^{(1)}(\kappa r)}{\kappa H_n^{(1)'}(\kappa a)} + \frac{1 - \delta_{m1}}{\kappa_m} \frac{K_n(\kappa_m r)}{K_n^{(1)'}(\kappa_m a)} \right\} \quad (6.41) \]

\section*{6.1.2 Solution for $\phi_2^{5S}$}

The second-order incident wave velocity potential in terms of the Jacobi expansion was defined in Eq. (6.13b). Changing the coordinate $(r, \theta, z)$ to $(R, \Theta, Z)$, taking the derivative with respect to $R$ and evaluating at $R = a$ yields

\[ \frac{\partial \phi_2^1(a, \Theta, Z)}{\partial R} = \frac{3\sigma \cosh 2k(Z + h)}{8k \sinh^4kh} \sum_{l=0}^{\infty} (2 - \delta_{l0}^1) j_{l+1}^1(2ka) \cos \Theta. \quad (6.42) \]

Substituting Eq. (6.41) and Eq. (6.42) into Eq. (6.25) yields the following second-order scattered velocity potential associated with the inhomogeneous structural boundary condition:
\[ \phi_{2}^{SS}(r, \theta, z) = -\frac{3\sigma \Lambda_1(z)}{16\pi k \sinh^4 kh} \sum_{n=0}^{\infty} (2 - \delta_{n0}) \frac{H_{n1}(kr)}{H_{n1}(ka)} \cos n\theta \cdot \]

\[ \cdot \sum_{l=0}^{\infty} (2 - \delta_{10}) i^{l+1} J_{l1}^{r}(2ka) \int_{-h}^{\infty} \cosh 2k(Z + h) \Lambda_1(Z) dZ \]

\[ \int_{-\pi}^{\pi} \cos n\theta \cos l\theta \, d\theta \]

\[ = \frac{3\sigma}{16\pi k \sinh^4 kh} \sum_{n=0}^{\infty} (2 - \delta_{n0}) \cos n\theta \sum_{m=2}^{\infty} \frac{\Lambda_m(z)}{\kappa_m K_{n1}(\kappa_m r)} \]

\[ \cdot \sum_{l=0}^{\infty} (2 - \delta_{10}) i^{l+1} J_{l1}^{r}(2ka) \int_{-h}^{\infty} \cosh 2k(Z + h) \Lambda_m(Z) dZ \]

\[ \int_{-\pi}^{\pi} \cos n\theta \cos l\theta \, d\theta \quad . \]

(6.43)

The integral of \( \cos n\theta \) and \( \cos l\theta \) in the interval of orthogonality \([-\pi, \pi]\) will contribute only when \( n = 1 \) as shown in Eq. (5.11). Therefore, one of the two summations shown in Eq. (6.43) is eliminated by the orthogonality of \( \cos n\theta \) and common subscripts may be used.

Integrating by parts, the following integral results:

\[ \int_{-h}^{\infty} \cosh 2k(Z + h) \Lambda_m(Z) dZ = \frac{r_1 \cosh kh \cosh 2kh}{4k^2 - \kappa^2} (2k \tanh 2kh - \frac{4\sigma^2}{g}) \]

\[ ; \quad m = 1 \quad \text{(6.44a)} \]
\[ \phi_{2}^{ss}(r, \theta, z) = \frac{3\sigma^2 I_1 \cosh \kappa h \cosh 2kh}{8k \sinh^4 kh (4k^2 - \kappa^2)} (2k \tanh 2kh - \frac{4\sigma^2}{g}) \Lambda_1(z) \cdot \]

\[ \cdot \sum_{n=0}^{\infty} (2 - \delta_{n0}) i^{n+1} \frac{J_n'(2ka)}{H_n^{(1)}(\kappa a)} H_n^{(1)}(\kappa r) \cos n\theta \]

\[ - \frac{3\sigma \cosh 2kh}{8k \sinh^4 kh} (2k \tanh 2kh - \frac{4\sigma^2}{g}) \cdot \]

\[ \cdot \sum_{n=0}^{\infty} (2 - \delta_{n0}) i^{n+1} \cos n\theta \sum_{m=2}^{\infty} \frac{I_m \cos \kappa_m h}{\kappa_m (4k^2 + \kappa_m^2)} \Lambda_m(z) \cdot \]

\[ \cdot \frac{J_n'(2ka)}{K_n'(\kappa_m a)} K_n(\kappa_m r) \cdot \]

(6.45)

Note that the first term in Eq. (6.45) is the second-order scattered velocity potential \( \phi_{2}^{ss} \) for the propagating mode and the second term represents the evanescent modes. In view of the similarity between the solution for the first-order and for the second-order scattered
velocity potentials, it is worth noting here that the coefficients for
the evanescent modes for the scattered velocity potential in the
first-order solution vanish due to the property of orthogonality. In
contrast, the coefficients for the evanescent modes for the
second-order scattered velocity potential do not vanish due to the
difference between the eigenvalue equations for the incident and
scattered waves in second-order solution by eigenfunction expansion.

6.2 \( \phi_2^{sf} \)

The governing equation for \( \phi_2^{sf} \) is defined in Eq. (6.23a). By
applying Green's second formula, Eq. (3.8), the boundary value problem
for \( \phi_2^{sf} \), Eqs. (6.23), may be suitably transformed to a boundary
value problem in terms of the Green's function, \( G_2^f \); i.e.,

\[
\frac{\partial^2 G_2^f}{\partial r^2} + \frac{1}{r} \frac{\partial G_2^f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 G_2^f}{\partial \theta^2} + \frac{\partial^2 G_2^f}{\partial z^2} = - \frac{1}{r} \delta(r-R) \delta(\theta-\hat{\theta}) \delta(z-z')
\]

\[
; \quad a \leq r < \infty, \quad -h \leq z \leq 0, \quad -\pi \leq \theta \leq \pi
\]

(6.46a)

\[
\frac{\partial G_2^f}{\partial z} = 0 \quad ; \quad z = -h
\]

(6.46b)

\[
\frac{\partial G_2^f}{\partial z} - \frac{1}{g} G_2^f = 0 \quad ; \quad z = 0
\]

(6.46c)
\[
\frac{\partial G_f^2}{\partial r} = 0 \quad ; \quad r = a \quad (6.46d)
\]

\[
\frac{\partial G_f^2}{\partial r} - (i\omega - \frac{1}{2r}) G_f^2 = 0 \quad ; \quad r \rightarrow \infty \quad (6.46e)
\]

Then, \(\phi_2^{sf}\) may be recovered from the following integral equation (cf., Appendix D):

\[
\phi_2^{sf}(r,\theta,z) = \frac{1}{I} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{1}{a} G_f^2 F_s R \, dR \, d\Theta \quad ; \quad Z = 0 \quad (6.47)
\]

in which \(F_s\), given by Eq. (A.8) in Appendix A, is the inhomogeneous term in Eq. (6.23c) and \(G_f^2(r,\theta,z;R,\theta,0)\) is the Green's function for the scattered velocity potential, \(\phi_2^{sf}\).

Our next step is to try to find a solution for the Green's function, \(G_f^2\), based on the boundary value problem defined in Eqs. (6.46).

### 6.2.1 Green's Function For \(\phi_2^{sf}\)

The problem solution technique to solve for the Green's function for \(\phi_2^{sf}\), defined by Eqs. (6.46), is based on an eigenfunction expansion of the Green's function. Since Chapters 4 and 5 have demonstrated the successful application of the method of an
eigenfunction expansion of the Green's function for the first-order
diffraction problem, it is expected that the scheme may be successfully
applied to the second-order problem.

We begin by taking the following Fourier cosine transform pair
(Hildebrand, 1962; p.219) with respect to \( \theta \) for the Green's function
\( G^f_2 \):

\[
G^f_2(r, \theta, z; R, \theta, Z) = \frac{1}{2\pi} \sum_{j=0}^{\infty} (2 - \delta_{j0}) \tilde{G}_j(r, z; R, \theta, Z) \cos j\theta , \quad (6.48a)
\]

\[
\tilde{G}_j(r, z; R, \theta, Z) = \int_{-\pi}^{\pi} G^f_2(r, \theta, z; R, \theta, Z) \cos j\theta \, d\theta . \quad (6.48b)
\]

Consequently, Eq. (6.46a) becomes

\[
\frac{\partial^2 \tilde{G}_j}{\partial z^2} + \frac{\partial^2 \tilde{G}_j}{\partial r^2} + \frac{1}{r} \frac{\partial \tilde{G}_j}{\partial r} - \frac{j^2}{r^2} \tilde{G}_j = - \frac{I}{r} \delta(r-R) \cos j\theta \delta(z-Z) . \quad (6.49)
\]

The Hankel transform (Miles, 1971; p. 72) is defined as

\[
z^m_j(z; R, \theta, Z) = \int_a^\infty \tilde{G}_j(r, z; R, \theta, Z) J_j(q_m r) \, r \, dr \tag{6.50a}
\]

and its inverse as

\[
\tilde{G}_j(r, z; R, \theta, Z) = \sum_{m=1}^{\infty} \frac{1}{a^2} J_j(q_m r) z^m_j(z; R, \theta, Z) . \tag{6.50b}
\]
The Bessel functions \( J_j(q_m r) \) form a complete orthogonal set with respect to the weighting function \( r \) over the interval of orthogonality \([a, \infty)\) (see Appendix B); i.e.,

\[
\int_a^\infty r J_j(q_m r) J_j(q_1 r) \, dr = a^2 \delta^m_j \delta_{m1} \tag{6.51}
\]

in which

\[
\delta^m_j = \frac{1}{(q_m a)^2} \int_a^\infty (q_m r) J_j^2(q_m r) \, dq_m r. \tag{6.52}
\]

Note that the Hankel transform only involves the Bessel function of the first kind.

The Bessel functions, \( J_j(q_m r) \), are the eigenfunctions and the eigenvalues, \( q_m \), are to be determined from the homogeneous boundary condition on the structure in Eq. (6.46d). Hence the eigenvalues, \( q_m \), are the zeros of

\[
J'_n(q_m a) = 0. \tag{6.53}
\]

Taking the Hankel transform of Eq. (6.49) based on Eq. (6.50a), we obtain
\frac{\partial^2 z_m}{\partial z^2} - q_m \frac{\partial z_m}{\partial z} = - I J_j(q_m R) \cos j\Theta \delta(z - Z) . \quad (6.54)

Integrating over the total depth yields

\int_{-h}^{Z-} \left( \frac{\partial^2 z_j}{\partial z^2} - q_m \frac{\partial z_m}{\partial z} \right) dz + \int_{Z-}^{Z+} \left( \frac{\partial^2 z_j}{\partial z^2} - q_m \frac{\partial z_m}{\partial z} \right) dz

+ \int_{Z+}^{0} \left( \frac{\partial^2 z_j}{\partial z^2} - q_m \frac{\partial z_m}{\partial z} \right) dz = \int_{-h}^{0} I J_j(q_m R) \delta(z - Z) \cos j\Theta \, dz

(6.55)

which gives for \( z \neq Z \)

\frac{\partial^2 z_j}{\partial z^2} = \frac{\partial z_m}{\partial z} = 0 \quad ; \quad -h \leq z < Z \leq 0 \quad (6.56a)

\frac{\partial^2 z_j}{\partial z^2} = \frac{\partial z_m}{\partial z} = 0 \quad ; \quad -h \leq Z < z \leq 0 \quad (6.56b)

and for \( z \to \lim Z_\pm \)

\frac{\partial z_m}{\partial z} \bigg|_{z=Z_+} - \frac{\partial z_m}{\partial z} \bigg|_{z=Z_-} - q_m \frac{\partial z_m}{\partial z} \bigg|_{Z-} = - I J_j(q_m R) \cos j\Theta

\quad ; \quad z = \lim Z_\pm \to Z . \quad (6.56c)
From Eq. (6.56c), we obtain the following continuity condition:

\[ \frac{z_j^m}{z = Z^+} - \frac{z_j^m}{z = Z^-} = 0 \quad ; \quad z = \lim_{z \to Z^+} Z \quad \text{(6.57)} \]

and the following jump condition:

\[ \frac{\partial z_j^m}{\partial z} \bigg|_{z = Z^+} - \frac{\partial z_j^m}{\partial z} \bigg|_{z = Z^-} = - I J_j(q R) \cos \phi \quad ; \quad z = \lim_{z \to Z^+} Z \quad \text{(6.58)} \]

Eqs. (6.57) and (6.58) represent the continuity and jump conditions, respectively, for \( z_j^m \) at the source coordinate \( z = Z \).

Let \( z_j^m \) represent \( z_j^m \) for \(-h \leq z < Z \leq 0 \) and \( z_j^m \) represent \( z_j^m \) for \(-h \leq Z < z \leq 0 \). A summary of the boundary value problem for \( z_j^m \) is provided below:

\[ \frac{\partial^2 z_j^m}{\partial z^2} - q_m z_j^m = 0 \quad ; \quad -h \leq z < Z \leq 0 \quad \text{(6.59a)} \]

\[ \frac{\partial^2 z_j^m}{\partial z^2} - q_m z_j^m = 0 \quad ; \quad -h \leq Z < z \leq 0 \quad \text{(6.59b)} \]

\[ \frac{\partial z_j^m}{\partial z} = 0 \quad ; \quad z = -h \quad \text{(6.59c)} \]
\[
\frac{\partial z_j^m}{\partial z} - \frac{4\sigma^2}{g} z_j^m = 0 \quad ; \quad z = 0 \quad (6.59d)
\]

\[
\frac{\partial z_j^m(z)}{\partial z} - \frac{\partial z_j^m(z)}{\partial z} = 0 \quad ; \quad z = Z \quad (6.59e)
\]

\[
\frac{\partial z_j^m(z)}{\partial z} - \frac{\partial z_j^m(z)}{\partial z} = - I J_j(q_m R) \cos j\theta \quad ; \quad z = Z . \quad (6.59f)
\]

The solutions to the differential equations [Eqs. (6.59a) and (6.59b)] are

\[
z_j^m(z) = a_j^m \cosh q_m(z + h) + b_j^m \sinh q_m(z + h) \quad ; \quad -h \leq z < z \leq 0
\]

(6.60a)

and

\[
z_j^m(z) = c_j^m \cosh q_m(z + h) + d_j^m \sinh q_m(z + h) \quad ; \quad -h \leq z < Z \leq 0 .
\]

(6.60b)

Substituting the derivative of Eq. (6.60b) into Eq. (6.59c) yields

\[
d_j^m = 0 . \quad (6.61)
\]

The three remaining coefficients, \( a_j^m, b_j^m, \) and \( c_j^m \), may be determined from the properties of Green's function given by the continuity of \( z_j^m(z) \) and by the jump condition of \( \frac{\partial z_j^m(z)}{\partial z} \) at \( z = Z \) and by the
combined free surface boundary condition. Substituting Eqs. (6.60) into
Eqs. (6.59d), (6.59e) and (6.59f) leads to the following three
simultaneous equations for the coefficients \( a_j^m \), \( b_j^m \), and \( c_j^m \):

\[
a_j^m \left( \frac{4\sigma^2}{g} \cosh q_m h - q_m \sinh q_m h \right) + b_j^m \left( \frac{4\sigma^2}{g} \sinh q_m h - q_m \cosh q_m h \right) = 0 \tag{6.62a}
\]

\[
a_j^m \cosh q_m (Z + h) + b_j^m \sinh q_m (Z + h) - c_j^m \cosh q_m (Z + h) = 0 \tag{6.62b}
\]

\[
a_j^m q_m \sinh q_m (Z + h) + b_j^m q_m \cosh q_m (Z + h) - c_j^m q_m \sinh q_m (Z + h) = - I \ J_j(q_m R) \cos \theta \tag{6.62c}
\]

The solutions are:

\[
a_j^m = \frac{I}{q_m} \ L_m \ J_j(q_m R) \cos \theta \cosh q_m (Z + h) \tag{6.63a}
\]

\[
b_j^m = - \frac{I}{q_m} \ J_j(q_m R) \cos \theta \cosh q_m (Z + h) \tag{6.63b}
\]
\[ c_j^m = \frac{I}{q_m^2} J_j(q_m R) \cos j\theta \left[ L_m \cosh q_m (Z + h) - \sinh q_m (Z + h) \right] \]  

(6.63c)

in which

\[ L_m = \frac{4\sigma^2 \tanh q_m h - gq_m}{4\sigma^2 - gq_m \tanh q_m h} \]  

(6.64)

Note that the denominator does not vanish for \( q_m h \). Thus, the solution of Eqs. (6.59a) and (6.59b) satisfying the boundary conditions Eqs. (6.59c)-(6.59f) may be written as

\[ Z_j^m(z;R,\phi,Z) = \frac{I}{q_m} J_j(q_m R) \cos j\phi \cosh q_m (Z + h) \varphi_m(z) \]  

; \(-h \leq z < Z \leq 0\)  

(6.65a)

and

\[ Z_j^m(Z;R,\phi,Z) = \frac{I}{q_m} J_j(q_m R) \cos j\phi \cosh q_m (Z + h) \varphi_m(z) \]  

; \(-h \leq Z < z \leq 0\)  

(6.65b)

in which

\[ \varphi_m(z) = L_m \cosh q_m (z + h) - \sinh q_m (z + h) \]  

(6.66)

Substituting Eqs. (6.65) into Eqs. (6.50b) and (6.48a), the solution of the Green's function may be expressed as
\[ G_2^f(r, \theta, z; R, \theta, Z) = \sum_{j=0}^{\infty} \sum_{m=1}^{\infty} \frac{I(2 - \delta_{j0})}{2\pi q_m a^2 \sigma_j^m} J_j(q_m r) J_j(q_m R) \cosh q_m(z + h) \cdot \mathcal{D}_m(Z) \cos j\theta \cos j\theta \; ; \; -h \leq z < Z \leq 0 \]  

(6.67a)

and

\[ G_2^f(r, \theta, z; R, \theta, Z) = \sum_{j=0}^{\infty} \sum_{m=1}^{\infty} \frac{I(2 - \delta_{j0})}{2\pi q_m a^2 \sigma_j^m} J_j(q_m r) J_j(q_m R) \cosh q_m(Z + h) \cdot \mathcal{D}_m(Z) \cos j\theta \cos j\theta \; ; \; -h \leq Z < z \leq 0 \]  

(6.67b)

provided that

\[ J'_n(q_m a) = 0 \]  

(6.67c)

6.2.2 Solution for \( \phi_2^{sf} \)

The scattered velocity potential correct to second-order that is due to the free surface effect, \( \phi_2^{sf} \), is determined by the integral equation given in Eq. (6.47). The appropriate Green's function to be placed in Eq. (6.47) must be decided from the boundary conditions. Since the inhomogeneous term for \( \phi_2^{sf} \) occurs in the combined free surface boundary condition where the source points are distributed over the free surface, we may select \( z < Z \) and \( Z = 0 \). Eq. (6.67a) then gives

\[ G_2^f(r, \theta, z; R, \theta, 0) = \sum_{j=0}^{\infty} \sum_{m=1}^{\infty} \frac{I(2 - \delta_{j0})}{2\pi q_m a^2 \sigma_j^m} J_j(q_m r) J_j(q_m R) \cosh q_m(z + h) \cdot \mathcal{D}_m(0) \cos j\theta \cos j\theta \]  

(6.68)
in which

\[ \mathcal{D}_m(0) = L_m \cosh q_m h - \sinh q_m h \]  

(6.69a)

\[ = \frac{gq_m \operatorname{sech} q_m h}{gq_m \tanh q_m h - 4\sigma^2} \]  

(6.69b)

Accordingly,

\[ \psi^s_f(r, \theta, z) = \frac{1}{I g} \sum_{j=0}^{\infty} \sum_{m=1}^{2\pi q_m a^2 \sigma^2} J_j(q_m r) \cos j\theta \cosh q_m (z + h) \]

\[ \cdot \mathcal{D}_m(0) \int_{-\pi}^{\pi} \int_0^{\infty} J_j(q_m R) F_s(R, \theta) \cos j\theta R \, dR \, d\theta \]  

(6.70)

Substituting the expression for \( F_s \) given by Eq. (A.8) in Appendix A into Eq. (6.70) and rearranging terms will yield the following type of integral:

\[ \int_{-\pi}^{\pi} \cos j\theta \cos n\theta \, d\theta = \frac{2\pi}{(2 - \delta_{jn})} \delta_{jn} \quad \text{for all } j \text{ and } n. \]  

(6.71)

which will contribute only when \( n = j \). Thus, the double summation in \( F_s \) will be replaced by a single summation as a consequence of \( \delta_{jn} \).

Eq. (6.70) now reduces to

\[ \psi^s_f(r, \theta, z) = \frac{-i\sigma}{4ak^3 \tanh kh} \sum_{n=0}^{\infty} \sum_{m=1}^{2\pi q_m a^2 \sigma^2} \mathcal{D}_m(0) \frac{N^m_n}{n^m} J_n(q_m r) \cos n\theta \]

\[ \cdot \cosh q_m (z + h) \]  

(6.72)
in which

\[
M_n^m = \left\{ \sum_{\ell=0}^{n} B_{\ell} B_{n-\ell} \left[ k^2 \int_{a}^{\infty} J^1'_{\ell}(kR) J^1_{n-\ell}(kR) J_n(q_m R) R \, dR \right.ight.
\]

\[
+ \int_{a}^{\infty} \left( \frac{1}{R^2} \ell(\ell - n) + \frac{k^2}{2} \left( 3\tanh^2 kh - 1 \right) \right) H^1_{\ell}(kR) H^1_{n-\ell}(kR) \cdot J_n(q_m R) R \, dR \bigg] \left( k^2 \right) \cdot J_n(q_m R) R \, dR \bigg] \left( 2 - \delta_{n0} \right) \left\{ \sum_{\ell=n}^{\infty} B_{\ell} B_{n-\ell} \left[ k^2 \int_{a}^{\infty} J^1'_{\ell}(kR) J^1_{n-\ell}(kR) J_n(q_m R) R \, dR \right.ight.
\]

\[
+ \int_{a}^{\infty} \left( \frac{1}{R^2} \ell(\ell - n) + \frac{k^2}{2} \left( 3\tanh^2 kh - 1 \right) \right) H^1_{\ell}(kR) H^1_{n-\ell}(kR) \cdot J_n(q_m R) R \, dR \bigg] \left( k^2 \right) \cdot J_n(q_m R) R \, dR \bigg] \left( 2 - \delta_{n0} \right) \left\{ \sum_{\ell=n}^{\infty} B_{\ell} B_{n-\ell} \left[ k^2 \int_{a}^{\infty} J^1'_{\ell}(kR) J^1_{n-\ell}(kR) J_n(q_m R) R \, dR \right.ight.
\]

\[
+ \sum_{\ell=n}^{\infty} \frac{k^2}{2cosh^2 kh} \sum_{\ell=0}^{n} A_{n-\ell} B_{\ell} \int_{a}^{\infty} J_n(q_m R) R \, dR \bigg] \left( 2 \right) \cdot J_n(q_m R) R \, dR \bigg] \left( k^2 \right) \cdot J_n(q_m R) R \, dR \bigg] \left( 2 - \delta_{n0} \right) \left\{ \sum_{\ell=n}^{\infty} B_{\ell} B_{n-\ell} \left[ k^2 \int_{a}^{\infty} J^1'_{\ell}(kR) J^1_{n-\ell}(kR) J_n(q_m R) R \, dR \right.ight.
\]

\[
+ \int_{a}^{\infty} \left( \frac{1}{R^2} \ell(\ell - n) + \frac{k^2}{2} \left( 3\tanh^2 kh - 1 \right) \right) H^1_{\ell}(kR) H^1_{n-\ell}(kR) \cdot J_n(q_m R) R \, dR \bigg] \left( k^2 \right) \cdot J_n(q_m R) R \, dR \bigg] \left( 2 - \delta_{n0} \right) \left\{ \sum_{\ell=n}^{\infty} B_{\ell} B_{n-\ell} \left[ k^2 \int_{a}^{\infty} J^1'_{\ell}(kR) J^1_{n-\ell}(kR) J_n(q_m R) R \, dR \right.ight.
\]
\[
\begin{align*}
+ \int_{\text{a}}^{\infty} \left( \frac{1}{R^2} \ell(\ell - n) + \frac{k^2}{4} \left( 5\tanh^2 \kh - 1 \right) - \frac{k^2}{4\cosh^2 \kh} \right) J_{\ell}(kR) dR \\
\cdot H_{\ell-n}^{(1)}(kR) J_n(qmR) R dR \right]
\end{align*}
\]

\[
\begin{align*}
+ \sum_{\ell=n}^{\infty} A_{\ell-n} B_\ell \left[ k^2 \int_{\text{a}}^{\infty} J_{\ell-n}^\prime(kR) H_{\ell-n}^{(1)}(kR) J_n(qmR) R dR \right]
\end{align*}
\]

\[
\begin{align*}
+ \int_{\text{a}}^{\infty} \left( \frac{1}{R^2} \ell(\ell - n) + \frac{k^2}{4} \left( 5\tanh^2 \kh - 1 \right) - \frac{k^2}{4\cosh^2 \kh} \right) \\
\cdot J_{\ell-n}(kR) H_{\ell-n}^{(1)}(kR) J_n(qmR) R dR \right] 
\end{align*}
\]

in which \(A_n\) and \(B_n\) are defined in Eqs. (A.2) and (A.4) in Appendix A, respectively.

6.3 Second-order Total Velocity Potential

The second-order total velocity potential is defined by Eqs. (6.12), (6.14), and (6.21). As demonstrated previously, the solution to the scattered velocity potential \(\phi_2^s\) may be expressed in terms of linear combinations of \(\phi_2^{ss}\) and \(\phi_2^{sf}\). A summary of the velocity potential expressions for the second-order is provided below:

\[
\begin{align*}
\phi_2(r, \theta, z, t) &= \left\{ \phi_2^i(r, \theta, z) + \phi_2^{ss}(r, \theta, z) + \phi_2^{sf}(r, \theta, z) \right\} \exp(-2i\omega t)
\end{align*}
\]

\[
\begin{align*}
+ \{ \}^* 
\end{align*}
\]

in which
\[
\phi^s_2(r, \theta, z) = \frac{3\sigma \cosh 2k(z + h)}{16k^2 \sinh^4 kh} \sum_{n=0}^{\infty} (2 - \delta_{n0}) J_{n+1}(2kr) \cos n\theta,
\]

(6.75a)

\[
\phi^{SS}_2(r, \theta, z) = -\frac{3\sigma I_1 \cosh kh \cosh 2kh}{8k \sinh^4 kh (4k^2 - \kappa^2)} \left( 2k \tanh 2kh - \frac{4\sigma^2}{g} \right) \Lambda_1(z),
\]

\[
\cdot \sum_{n=0}^{\infty} (2 - \delta_{n0}) \frac{J'_{n}(2ka)}{H'_{n}(1)(ka)} H_{n}^{(1)}(\kappa r) \cos n\theta
\]

\[
- \frac{3\sigma \cosh 2k}{8k \sinh^4 kh} \left( 2k \tanh 2kh - \frac{4\sigma^2}{g} \right),
\]

\[
\cdot \sum_{n=0}^{\infty} (2 - \delta_{n0}) \cos n\theta \sum_{m=2}^{\infty} \frac{\Gamma_m \cos \kappa_m h}{\kappa_m (4k^2 + \kappa_m^2)} \Lambda_m(z),
\]

\[
\cdot \frac{J'_{n}(2ka)}{K'_{n}(\kappa_m a)} K_{n}(\kappa_m r),
\]

(6.75b)

provided that \(4\sigma^2 + g\kappa_m \tan \kappa_m h = 0\), and

\[
\phi^s_2(r, \theta, z) = -\frac{i\sigma}{4ak^3 \tanh kh} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{D_m(0) H_m^n}{(q_m a) \partial^m_n} J_n(q_m r) \cos n\theta \cosh q_m(z+h),
\]

(6.75c)
provided that $J'_n(q_m) = 0$, in which $\Lambda_m$, $\Gamma_m$, $\sigma^m_n$, $\varphi_m(0)$ and $\chi^m_n$ are defined in Eqs. (6.29), (6.30), (6.52), (6.69) and (6.73), respectively.
CHAPTER 7

HYDRODYNAMIC WAVE FORCES AND MOMENTS

The total hydrodynamic pressure may be expanded into the following series form in the perturbation parameter $\varepsilon$ by:

\[
P = P_s + \varepsilon P_1 + \varepsilon^2 P_2 \ldots \tag{7.1a}
\]

in which $P_s$ = hydrostatic pressure, $P_1$ = first-order hydrodynamic pressure, $P_2$ = second-order hydrodynamic pressure and $\varepsilon = \frac{H}{2} k$.

Hence, the hydrodynamic pressure distribution about the cylinder correct at each perturbation order is evaluated by substitution of the velocity potential into the perturbed Bernoulli's equation; i.e.,

\[
\varepsilon^0 : \quad P_s = -\rho g z \tag{7.1b}
\]

\[
\varepsilon^1 : \quad P_1 = \rho \frac{\partial \phi_1}{\partial t} ; \quad r = a \tag{7.2}
\]

and

\[
\varepsilon^2 : \quad P_2 = \rho \left\{ \frac{\partial \phi_2}{\partial t} - \frac{1}{2} \left[ (\frac{\partial \phi_1}{\partial z})^2 + \left(\frac{1}{r} \frac{\partial \phi_1}{\partial \theta}\right)^2 \right] \right\} ; \quad r = a \tag{7.3}
\]

since

\[
\frac{\partial \phi_1}{\partial r} = 0 \quad ; \quad r = a \tag{7.4}
\]
The resultant in-line horizontal hydrodynamic force exerted by the fluid on the surface of the cylinder is obtained by integration of the hydrodynamic pressure distribution over the entire wetted surface of the cylinder; i.e.,

\[ F = - \int_{-h}^{\eta} \int_{-\pi}^{\pi} P \cos\theta \, d\theta \, dz \quad ; \quad r = a \quad . \quad (7.5) \]

By substituting the pressure perturbation expansion, Eq. (7.1), into Eq. (7.5) and collecting terms equal in power of \( \varepsilon \), equations for the in-line pressure force component are obtained for each perturbation order. Let \( F_1 \) and \( F_2 \) be defined as the first- and second-order dynamic pressure forces, respectively; then,

\[ F_1 = - \int_{-h}^{\eta} \int_{-\pi}^{\pi} P_1 \cos\theta \, d\theta \, dz \quad ; \quad r = a \quad . \quad (7.6) \]

and

\[ F_2 = - \int_{-h}^{\eta} \int_{-\pi}^{\pi} P_2 \cos\theta \, d\theta \, dz \]

\[ - \int_{-\pi}^{\pi} \frac{1}{2} \rho \eta_1^2 \cos\theta \, d\theta \quad ; \quad r = a \quad . \quad (7.7) \]

We note that objects with a vertical axis of symmetry such as the cylinder of the present study in the first-order problem experience zero net in-line horizontal hydrostatic forces. However, for the second-order problem the second term in Eq. (7.7) is a consequence of hydrostatic forces.
The positive moment vector is defined by \( \mathbf{M} = \mathbf{\dot{x}} \times \mathbf{F} \) in which \( \mathbf{\dot{x}} \) represents the positive moment arm vector. The moment about the ocean floor is therefore defined by integrating the product of the pressure force and the distance from the bottom.

\[
M = - \int_{-h}^{h} \int_{-\pi}^{\pi} \mathbf{a} \cos \theta \, d\theta \, dz; \quad r = a .
\] (7.8)

The moment equation may be treated in a similar manner by expanding in a perturbation series \( \epsilon \) according to

\[
M = \epsilon^0 M_0 + \epsilon M_1 + \epsilon^2 M_2 + \ldots
\] (7.9)

in which \( M_0 \) = hydrostatic moment, \( M_1 \) = first-order hydrodynamic moment and \( M_2 \) = second-order hydrodynamic moment. Then

\[
M_1 = - \int_{-h}^{h} \int_{-\pi}^{\pi} \mathbf{a} \cos \theta \, d\theta \, dz; \quad r = a
\] (7.10)

and

\[
M_2 = - \int_{-h}^{h} \int_{-\pi}^{\pi} \mathbf{a} \cos \theta \, d\theta \, dz - \int_{-\pi}^{\pi} \frac{1}{2} \rho g n_1^2 (z + h) \cos \theta \, d\theta \, dz; \quad r = a
\] (7.11)

Note that the second term in Eq. (7.11) is from hydrostatic contribution.
The hydrodynamic pressures $P_1$ and $P_2$ in Eqs. (7.6) - (7.7) and (7.10) - (7.11) are complex-valued and, therefore, the amplitude of the force and moment are also complex-valued and contain both phase and magnitude information. Using the equation for the first-order incident wave velocity potential given in Eqs. (2.23) and (5.7), it may be shown that the horizontal water particle velocity of the incident wave is in-phase with $\cos \omega t$ and the horizontal water particle acceleration of the incident wave is in-phase with $\sin \omega t$. The hydrodynamic force and moment on the circular cylinder may each be expressed as a linear sum of two components; one which is in-phase with the acceleration and another which is in-phase with the velocity of the incident wave field water particles.

7.1 First-order Force and Moment

The total first-order hydrodynamic pressure force may be obtained by substituting the expression for the total first-order velocity potential, Eq. (5.13), into Eqs. (7.2) and (7.6) and integrating to obtain

$$F_1 = F_{1c} \cos \omega t + F_{1s} \sin \omega t$$

(7.12a)

$$= |F_1| \cos (\omega t - \nu_{f1})$$

(7.12b)

The component of the first-order hydrodynamic pressure force that is in-phase with the cosine component is given by

$$F_{1c} = \frac{4\rho g \tanh kh \cdot J_1'(ka)}{k^3\{[J_1'(ka)]^2 + [Y_1'(ka)]^2\}}$$

(7.13)
and the component of the first-order hydrodynamic pressure force that is in-phase with the sine component is given by

\[
F_{1S} = -\frac{4\rho g \tanh k h \cdot Y_1'(ka)}{k^3\left[ J_1'(ka)\right]^2 + \left[ Y_1'(ka)\right]^2},
\]

(7.14)

and

\[
| F_1 | = \left( (F_{1C})^2 + (F_{1S})^2 \right)^{\frac{1}{2}}.
\]

(7.15)

The phase angle, \( \mu_{F1} \), is given by

\[
\mu_{F1} = \text{ARCTAN} \left\{ \frac{F_{1S}}{F_{1C}} \right\}.
\]

(7.16)

Similarly, the total first-order hydrodynamic moment may be obtained by the following

\[
M_1 = M_{1C} \cos \sigma t + M_{1S} \sin \sigma t
\]

(7.17a)

\[
= | M_1 | \cos (\sigma t - \mu_{M1})
\]

(7.17b)

The component of the first-order hydrodynamic moment that is in-phase with the cosine component is given by

\[
M_{1C} = \frac{4\rho g (kh \sinh kh - \cosh kh + 1) J_1'(ka)}{k^4 \cosh kh \left\{ \left[ J_1'(ka)\right]^2 + \left[ Y_1'(ka)\right]^2 \right\}},
\]

(7.18)
and the component of the first-order hydrodynamic moment that is in-phase with the sine component is given by

\[ M_{1s} = -\frac{4\rho g (kh \sinh kh - \cosh kh + 1) Y_1'(ka)}{k'\cosh kh \{ [J_1'(ka)]^2 + [Y_1'(ka)]^2 \}} , \]  

(7.19)

and

\[ | M_1 | = \sqrt{\left( \frac{M_{1c}}{M_{1s}} \right)^2 + \left( \frac{M_{1s}}{M_{1c}} \right)^2} . \]  

(7.20)

The phase angle, \( \nu_{m1} \), is given by

\[ \nu_{m1} = \arctan \left\{ \frac{M_{1s}}{M_{1c}} \right\} . \]  

(7.21)

It then follows that

\[ \nu_{f1} = \nu_{m1} = \arctan \left\{ -\frac{Y_1'(ka)}{J_1'(ka)} \right\} . \]  

(7.22)

Since a sinusoidal incident wave has been assumed in Eq. (2.23), the forces and moments in Eqs. (7.12) and (7.17), respectively, must also vary sinusoidally. Hence, the maximum forces may be normalized with respect to the weight of the water displaced by the cylinder and the dimensionless wave height. The dimensionless first-order force coefficients may be defined as
\[ C_1^f = \left( (C_{1c}^f)^2 + (C_{1s}^f)^2 \right)^{\frac{1}{2}} \]  

(7.23)

In which

\[ C_{1c}^f = \frac{\varepsilon \cdot |F_{1c}|}{\rho g a^2 h \left( \frac{H}{2a} \right)} \]  

(7.24a)

And

\[ C_{1s}^f = \frac{\varepsilon \cdot |F_{1s}|}{\rho g a^2 h \left( \frac{H}{2a} \right)} \]  

(7.24b)

In which \( \rho = \) mass density of the fluid, \( g = \) gravitational constant, \( a = \) radius of the cylinder, \( h = \) water depth, \( \varepsilon = \left( \frac{H}{2} k \right) \) perturbation parameter defined in Eq. (2.14).

Similarly to the force coefficients, the first-order moment coefficients may be defined as

\[ C_1^m = \left( (C_{1c}^m)^2 + (C_{1s}^m)^2 \right)^{\frac{1}{2}} \]  

(7.25)
in which

\[ C_{1C}^m = \frac{\epsilon \cdot \lvert M_{1C} \rvert}{\rho \pi a^2 h^2 \left( \frac{H}{2a} \right)} \]  \hfill (7.26a)

and

\[ C_{1s}^m = \frac{\epsilon \cdot \lvert M_{1s} \rvert}{\rho \pi a^2 h^2 \left( \frac{H}{2a} \right)} . \]  \hfill (7.26b)

Then,

\[ C_{1C}^f = \frac{4 \tanh kh \cdot J_1'(ka)}{\pi(ka)(kh) \left\{ \lvert J_1'(ka) \rvert^2 + \lvert Y_1'(ka) \rvert^2 \right\}} , \]  \hfill (7.27)

\[ C_{1s}^f = \frac{4 \tanh kh \cdot Y_1'(ka)}{\pi(ka)(kh) \left\{ \lvert J_1'(ka) \rvert^2 + \lvert Y_1'(ka) \rvert^2 \right\}} , \]  \hfill (7.28)

\[ C_{1C}^m = \frac{4 \left( kh \sinh kh - \cosh kh + 1 \right) J_1'(ka)}{\pi(ka)(kh)^2 \cosh kh \left\{ \lvert J_1'(ka) \rvert^2 + \lvert Y_1'(ka) \rvert^2 \right\}} . \]  \hfill (7.29)
The theoretical dimensionless first-order moment arm \( \lambda_1/h \), which is normalized by the water depth, may be estimated from a ratio of the maximum amplitude of the hydrodynamic moment to the magnitude of the hydrodynamic force, which is given by

\[
\lambda_1 = \frac{M_1}{F_1} \quad .
\]

(7.31)

7.2 Second-order Force and Moment

Using a similar procedure as in Section 7.1, the second-order horizontal in-line component of the hydrodynamic force and moment may be obtained by substituting the appropriate expressions for the first- and second-order velocity potentials (Eqs. (5.13) and (6.74) into Eqs. (7.3), (7.7) and (7.11)). Substitution leads to

\[
F_2 = F_{2c} \cos 2\alpha t + F_{2s} \sin 2\alpha t \quad .
\]

(7.32a)

\[
= |F_2| \cos (2\alpha t - \mu_{f2}) \quad .
\]

(7.32b)

The component of the second-order hydrodynamic pressure force that is in-phase with the cosine component is given by
in which $\bar{\tau}_1^m$ is the real part of $\tau_n^m$ for $n = 1$ defined in Eq. (6.73).

The component of the second-order hydrodynamic pressure force that is in-phase with the sine component is given by

$$F_{2c} = \frac{12\sigma^2 \rho r_1^2 \cosh \kappa h \sinh \kappa h \cosh 2\kappa h (\tanh 2\kappa h - \frac{2\sigma^2}{g}) J_1'(2ka)}{k \kappa^3 \sinh^4 \kappa h (4k^2 - \kappa^2) \{[ J_1'(\kappa a)]^2 + [ Y_1'(\kappa a)]^2\}}$$

$$+ \frac{\pi \rho \sigma^2}{ak^3 \tanh \kappa h} \sum_{m=1}^{\infty} \frac{\mathcal{D}_m^m}{\eta^2_{0,1}} \int J_1(q_m a) \sinh q_m h$$

$$= \frac{3\rho a \pi g J_1(2ka)}{2k^2 \sinh^2 \kappa h}$$

$$- \frac{6\sigma^2 \rho r_1^2 \cosh \kappa h \sinh \kappa h \cosh 2\kappa h (\tanh 2\kappa h - \frac{2\sigma^2}{g}) J_1'(2ka)}{k \kappa^4 \sinh^4 \kappa h (4k^2 - \kappa^2) \{[ J_1'(\kappa a)]^2 + [ Y_1'(\kappa a)]^2\}}$$

$$+ \frac{6\sigma^2 \rho a \cosh 2\kappa h (\tanh 2\kappa h - \frac{2\sigma^2}{g}) J_1'(2ka)}{k \sinh^4 \kappa h}$$

$$\cdot \sum_{m=2}^{\infty} \frac{r_m^2 \cos \kappa_m h \sin \kappa_m h}{\kappa_m^2 (4k^2 + \kappa_m^2)} \frac{K_1(\kappa_m a)}{K_1'(\kappa_m a)}$$

$$+ \frac{\pi \rho \sigma^2}{ak^3 \tanh \kappa h} \sum_{m=1}^{\infty} \frac{\mathcal{D}_m^m}{\eta^2_{0,1}} \int J_1(q_m a) \sinh q_m h$$

in which $\bar{\tau}_1^m$ is the imaginary part of $\tau_n^m$ for $n = 1$ defined in Eq. (6.73).

The amplitude of the second-order horizontal in-line force is given by
\[ | F_2 | = \left( (F_{2c})^2 + (F_{2S})^2 \right)^{\frac{1}{2}}. \]  

(7.35)

The corresponding phase angle, \( \nu_{f2} \), is given by

\[ \nu_{f2} = \text{ARCTAN} \left\{ \frac{F_{2S}}{F_{2c}} \right\}. \]  

(7.36)

Similarly, the total second-order hydrodynamic moment may be obtained from the following:

\[ M_2 = M_{2c} \cos 2\sigma t + M_{2S} \sin 2\sigma t \]  

(7.37a)

\[ = | M_2 | \cos (2\sigma t - \nu_{m2}). \]  

(7.37b)

The component of the second-order hydrodynamic moment that is in-phase with the cosine component is given by

\[ M_{2c} = \frac{12\sigma^2 \rho \tau^2 \cosh \kappa h (\kappa h \sinh \kappa h - \cosh \kappa h + 1) \cosh 2\kappa h}{\kappa^4 \sinh^4 \kappa h (4\kappa^2 - \kappa^2)}. \]  

(7.38)
and the component of the second-order hydrodynamic moment that is in-phase with the sine component is given by

\[
M_{2s} = -\frac{3\rho g a \pi J_1(2ka)(2kh \sinh 2kh - \cosh 2kh + 1)}{8k^3 \sinh^3 kh \cosh kh}
\]

\[
-\frac{6\sigma^2 \rho \Gamma_1^2 \cosh kh (\kappa h \sinh \kappa h - \cosh \kappa h + 1) \cosh 2kh}{ak^5 \sinh^4 kh (4k^2 - \kappa^2)}
\]

\[
\cdot (\mathrm{ktanh} 2kh - \frac{2\sigma^2}{g}) \cdot \frac{J'_1(2ka)}{\{[J'_1(\kappa a)]^2 + [Y'_1(\kappa a)]^2\}}
\]

\[
+ \frac{6\sigma^2 \rho \pi a \cosh 2kh}{k \sinh^4 kh} (\mathrm{ktanh} 2kh - \frac{2\sigma^2}{g}) J'_1(2ka)
\]

\[
\cdot \sum_{m=2}^{\infty} \frac{\pi^2 \cos \kappa_m h (\kappa_m h \sin \kappa_m h + \cos \kappa_m h - 1) K_1(\kappa_m a)}{\kappa_m^3 \sinh^2 \kappa_m (4k^2 + \kappa_m^2) K'_1(\kappa_m a)}
\]

\[
+ \frac{\pi \sigma^2}{ak^3 \mathrm{tanh} kh} \sum_{m=1}^{\infty} \frac{\rho_1^m}{q_1^m} J_1(q_1 a)(q_1 h \sinh q_1 h - \cosh q_1 h + 1),
\]

\[
(7.39)
\]

and

\[
| M_2 | = \sqrt{(M_{2c})^2 + (M_{2s})^2}
\]

\[
(7.40)
\]

The corresponding phase shift for the second-order hydrodynamic moment, \( \nu_{m2} \), is given by
\[ \mu_{m2} = \text{ARCTAN} \left\{ \frac{M_{2s}}{M_{2c}} \right\} . \]  \quad (7.41)

The dimensionless second-order force coefficients are defined as

\[ C_2^f = \{(C_{2c}^f)^2 + (C_{2s}^f)^2\}^{\frac{1}{2}} \]  \quad (7.42)

in which

\[ C_{2c}^f = \frac{e^2 | F_{2c} |}{\rho g \pi a^2 h(\frac{H}{2a})} , \]  \quad (7.43)

\[ C_{2s}^f = \frac{e^2 | F_{2s} |}{\rho g \pi a^2 h(\frac{H}{2a})} , \]  \quad (7.44)

and the dimensionless second-order moment coefficients are defined as

\[ C_2^m = \{(C_{2c}^m)^2 + (C_{2s}^m)^2\}^{\frac{1}{2}} \]  \quad (7.45)

in which

\[ C_{2c}^m = \frac{e^2 | M_{2c} |}{\rho g \pi a^2 h^2(\frac{H}{2a})} , \]  \quad (7.46)

\[ C_{2s}^m = \frac{e^2 | M_{2s} |}{\rho g \pi a^2 h^2(\frac{H}{2a})} , \]  \quad (7.47)
Then

\[ C_{2c}^f = \left( \frac{H}{a} \right) (ka) \left\{ \frac{48 \Omega \cosh \kappa h \sinh \kappa h \cosh 2kh}{(\kappa a)(\kappa h) \sinh^2 \kappa h [4(\kappa h)^2 - (\kappa h)^2]} \right\} \]

\[ \cdot \frac{[kh \tanh 2kh - 4\pi \Omega]}{\sinh 4\kappa h} \left\{ \frac{J'_{1}(2ka)}{(2\kappa h + \sinh 2\kappa h)\left\{ [J'_{1}(\kappa a)]^2 + [Y'_{1}(\kappa a)]^2 \right\}} \right\} \]

\[ + \frac{\pi \Omega}{(kh)^2 \tanh \kappa h} \sum_{m=1}^{\infty} \frac{D_m}{\left( q_m a \right)^2} \frac{N_1^m}{J_{1}(q_m a) \sinh q_m h}, \quad (7.48) \]

\[ C_{2s}^f = \left( \frac{H}{a} \right) (ka) \left\{ \frac{-3J_{1}(2ka)}{4(\kappa h) \sinh^2 \kappa h} \right\} \]

\[ \frac{24 \Omega \cosh \kappa h \sinh \kappa h \cosh 2kh}{(\kappa h)(\kappa a)^2 \sinh^2 \kappa h [4(\kappa h)^2 - (\kappa h)^2]} \right\} \]

\[ \cdot \frac{[kh \tanh 2kh - 4\pi \Omega]}{\sinh 4\kappa h} \left\{ \frac{J'_{1}(2ka)}{(2\kappa h + \sinh 2\kappa h)\left\{ [J'_{1}(\kappa a)]^2 + [Y'_{1}(\kappa a)]^2 \right\}} \right\} \]

\[ + \frac{24 \pi \Omega \cosh 2kh}{\sinh^2 \kappa h} \left\{ \frac{\cos \kappa_m h \sin \kappa_m h}{\kappa_m a} \right\} \frac{K_1(\kappa_m a)}{\kappa_m a} \left\{ \frac{[J'_{1}(\kappa_m a)]^2 + [Y'_{1}(\kappa_m a)]^2}{\sinh \kappa_m h \sinh \kappa_m h} \right\} \]

\[ + \frac{\pi \Omega}{(kh)^2 \tanh \kappa h} \sum_{m=1}^{\infty} \frac{D_m}{\left( q_m a \right)^2} \frac{N_1^m}{J_{1}(q_m a) \sinh q_m h}, \quad (7.49) \]
\[ C_{2c}^m = \left( \frac{\hbar}{a} \right)(ka) \left\{ \frac{48 \Omega \cosh \kappa h (\kappa h \sinh \kappa h - \cosh \kappa h + 1) \cosh 2\kappa h}{(\kappa a)(\kappa h)^2 \sinh^4 \kappa h [4(\kappa h)^2 - (\kappa h)^2]} \right\} \cdot \]

\[
\left[ \frac{\kappa h \tanh 2\kappa h - 4\pi \Omega}{\kappa h} \right] J_1'(2\kappa a) \cdot \frac{\pi \Omega}{(\kappa h)^2 \tanh \kappa h} \sum_{m=1}^{\infty} \frac{D_m N_1^m J_1(q_m a)}{(q_m a)^2 (q_m h) C_1^m} (q_m h \sinh q_m h - \cosh q_m h + 1),
\]

(7.50)

\[ C_{2s}^m = \left( \frac{\hbar}{a} \right)(ka) \left\{ \frac{-3J_1(2\kappa a)(2\kappa h \sinh 2\kappa h - \cosh 2\kappa h + 1)}{16(\kappa h)^2 \sinh^3 \kappa h \cosh \kappa h} \right\} \cdot 
\]

\[
- \frac{24 \Omega \cosh \kappa h (\kappa h \sinh \kappa h - \cosh \kappa h + 1) \cosh 2\kappa h}{(\kappa a)^2(\kappa h)^2 \sinh^4 \kappa h [4(\kappa h)^2 - (\kappa h)^2]}(2\kappa h + \sinh 2\kappa h)
\]

\[
\left[ \frac{\kappa h \tanh 2\kappa h - 4\pi \Omega}{\kappa h} \right] J_1'(2\kappa a) \cdot \frac{\pi \Omega}{(\kappa h)^2 \tanh \kappa h} \sum_{m=1}^{\infty} \frac{D_m N_1^m J_1(q_m a)}{(q_m a)^2 (q_m h) C_1^m} (q_m h \sinh q_m h - \cosh q_m h + 1).
\]

(7.51)
in which

$$\Omega = \frac{2nmh}{gT^2}.$$  \hspace{1cm} (7.52)

The theoretical dimensionless second-order moment arm is defined as

$$\lambda_2 = \left| \frac{M_2}{F_2} \right|.$$  \hspace{1cm} (7.53)

A discussion of the numerical results for the first-order (Section 7.1) and the second-order (Section 7.2) dimensionless force and moment coefficients are presented in the following section.

7.3 Discussion of Numerical Results

Computations of the dimensionless second-order force and moment coefficients require the evaluation of an infinite summation of a sequence of semi-infinite integrals, $N_n^m$, defined by Eq. (6.73). The variable of integration in these semi-infinite integrals were nondimensionalized by $R = kR$ and were evaluated numerically by Simpson's first rule in which

$$\Delta R = R(\text{max}) - R(\text{min}) \quad \text{and} \quad N = 597.$$ 

The upper limit of integration was first approximated by twenty wave lengths. By numerical comparison, it was later determined that the upper limit could be approximated quite accurately with only three
wave lengths instead of twenty wave lengths. A typical semi-infinite integral was approximated for the twenty wave lengths case by

\[ \sum_{\ell=n}^{\infty} k^2 \int_{\alpha}^{\infty} H_{\ell}^{(1)}(kR) H_{\ell-n}^{(1)'}(kR) J_1(q_m R) R \, dR \]

\[ = \int_{\alpha}^{\infty} \frac{40\pi}{ka} H_{\ell}^{(1)}(R) H_{\ell-n}^{(1)'}(R) J_1\left(\frac{q_m}{k} R\right) R \, dR \]  

(7.54)

in which

\[ \frac{q_m}{k} = \frac{q_m a}{(kh)(\frac{a}{h})} \]  

(7.55)

The summations of the infinite series derived from the Jacobi expansion for \( N^m_n \) in Eq. (6.73) and from the eigenvalue evanescent modes for \( \phi^ss \) in Eq. (6.75b) were truncated when the addition of a single term failed to change each sum by more than 1%. Most of the numerical computations required only a single eigenvalue, \( q_1 \), to achieve the 1% accuracy criteria.

In order to discuss the force and moment coefficients obtained in the numerical computations, the following values for the design parameters have been chosen as representative of ocean engineering applications:

\[ \frac{a}{h} ; 0.3 \quad \frac{H}{a} ; 1.0 \quad \Omega ; 0.01 - 0.5 \]
The choice of values for \( \omega \) covers a reasonably complete range of design wave conditions from shallow water (\( \omega = 0.01 \)) to deep water (\( \omega = 0.5 \)). The range from deep water to shallow water conditions is determined by the magnitude of \( h/L \). It has been found that 
\[ h/L = 1/2 = h/L_0 = \omega \] for deep water and 
\[ h/L = 1/25 \] for shallow water (cf., Shore Protection Manual, Vol. I, 1973). Using the linear dispersion equation for the incident wave [Eq. (4.40)], it may be shown that \( \omega < 0.01 \) for shallow water, and \( \omega > 0.5 \) for deep water.

Since our interests are mainly in water of finite depth, the following discussion will be restricted within the range of

\[ 0.01 \leq \omega \leq 0.5 \quad (7.56) \]

In Figure 7.1, the first- and second-order dimensionless force coefficients are plotted versus \( \omega \) for the case of \( a/h = 0.3 \) and \( H/a = 1.0 \) (cf., Table 7.1). Figure 7.2 shows the ratio of the second-order force coefficient to the first-order force coefficient expressed as a percentage. Also summarized in Table 7.1 are the relative percentages, \( \hat{\varepsilon}_F \), of the second-order contribution compared to the linear force coefficient. In deep water the percentage is approximately zero, which means that the second-order force has essentially no effect on the resultant in-line force. It can be explained by the fact that there exists no second-order correction for the velocity potential in the deep water. Reducing \( \omega \), in other words, decreasing the water depth, makes the second-order effect become non-negligible. For instance, the percentage of the second-order
Figure 7.1 First- and second-order force coefficients,
\( \{C_1^f, \text{Eq.}(7.23); C_2^f, \text{Eq.}(7.42)\}; \frac{H}{a} = 1.0, \frac{a}{h} = 0.3. \)
Table 7.1 Summary of first- and second-order force coefficients and percentage of second-order force effect;
\( \frac{H}{a} = 1.0, \frac{a}{h} = 0.3. \)

<table>
<thead>
<tr>
<th>( \Omega )</th>
<th>0.05</th>
<th>0.10</th>
<th>0.20</th>
<th>0.30</th>
<th>0.40</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_1^f \times \frac{H}{2a} )</td>
<td>0.16262</td>
<td>0.21927</td>
<td>0.27262</td>
<td>0.27828</td>
<td>0.25353</td>
</tr>
<tr>
<td>( C_2^f \times \frac{H}{2a} )</td>
<td>0.06862</td>
<td>0.04373</td>
<td>0.01540</td>
<td>0.00538</td>
<td>0.00199</td>
</tr>
<tr>
<td>{ \hat{e}_F }</td>
<td>{42.20}</td>
<td>{19.94}</td>
<td>{5.65}</td>
<td>{1.93}</td>
<td>{0.78}</td>
</tr>
<tr>
<td>( C_{2E}^f \times \frac{H}{2a} )</td>
<td>0.06862</td>
<td>0.04374</td>
<td>0.01542</td>
<td>0.00541</td>
<td>0.00171</td>
</tr>
<tr>
<td>{ \hat{e}_F }</td>
<td>{42.20}</td>
<td>{19.95}</td>
<td>{5.60}</td>
<td>{1.94}</td>
<td>{0.67}</td>
</tr>
</tbody>
</table>

Note: \( \hat{e}_F = \left( \frac{C_2^f}{C_1^f} \right) \times 100\% \).

\( C_{2E}^f = C_2^f \) less the infinite integral term given by Eq. (6.73).
Figure 7.2 Percentage of second-order force effect, $\varepsilon_F$;
\[ \frac{H}{a} = 1.0, \frac{a}{h} = 0.3. \]
contribution can be as high as 42.2% at $\Omega = 0.05$.

The phase angle of the first- and second-order dimensionless force coefficients is shown in Figure 7.3. It should be pointed out that in Figure 7.3 the component in-phase with the acceleration component of the incident wave is always dominant.

Figure 7.4 shows the first- and second-order dimensionless moment arms versus $\Omega$ for $\frac{a}{h} = 0.3$. The results show that the second-order effects may be rather significant near the free surface.

The second-order effects, excluding the contribution from the pressure distribution due to the inhomogeneous combined free surface boundary condition, $C_{2E}^f$, are also tabulated in Table 7.1. In calculating the second-order forces, it is assumed that the effect of the second-order contribution from the inhomogeneous free surface boundary condition may be neglected. This appears to be true from Table 7.1. It can be seen that even without taking into account the effect of the second-order term contribution from the inhomogeneous free surface boundary condition the second-order force is quite realistic, which is primarily to be attributed to the large second-order term contribution from the inhomogeneous structural boundary condition. Thus, the influence of the $\phi_{2f}^s$ term appears to be very small, nearly negligible.

### 7.4 Comparison with Experiments

The results of the computations are compared with the model measurements given by Chakrabarti (1975, 1978) in Figure 7.5 and Tables
Figure 7.3 First- and second-order force phase angles, 

$\{\mu_{f1}, \text{Eq.(7.16)}; \mu_{f2}, \text{Eq.(7.36)}\}; \frac{H}{a} = 1.0, \frac{a}{h} = 0.3.$
Figure 7.4 First- and second-order dimensionless moment arms,

\[ \Omega = \frac{h}{L_0} \]

\[ \left\{ \frac{\lambda_1}{h}, \text{Eq.}(7.31); \frac{\lambda_2}{h}, \text{Eq.}(7.53) \right\}; \frac{H}{a} = 1.0, \frac{a}{h} = 0.3. \]
Figure 7.5 Comparison between linear and second-order force coefficients with experimental data;

\[
\frac{H}{a} = 0.261, \quad \frac{a}{h} = 0.862.
\]

- * Measured (Chakrabarti)
- □ Linear
- ○ Nonlinear
- + Nonlinear (\(C_2E\))
7.2 - 7.3. The force coefficients and abscissa used by Chakrabarti are converted into the notation used in this study in Table 7.2. The relationship is a result of transforming abscissa of ka to $\Omega$ in order to compare his results to the ones presented in this study. The relationship between the two is given by the following equations:

$$\Omega = \frac{ka}{2\pi} \left( \frac{h}{a} \right) \tanh \left[ (ka) \frac{h}{a} \right]$$ (7.57)

and

$$C_M^f = \frac{1}{2\pi} \frac{H}{a} \frac{a}{h} C_f^c$$ (7.58)

in which $C_M^f$ = measured wave force coefficient in this study and $C_f^c$ = measured wave force coefficient given by Chakrabarti (1975, 1978). Dimensionless relative errors are tabulated in Table 7.4 by

$$\varepsilon = \left\{ \frac{\text{Theoretical} - \text{Measured}}{\text{Theoretical}} \right\} \times 100 \%.$$ (7.59)

Negative values for these relative errors indicate an underprediction by theoretical calculation while positive values of relative errors indicate overprediction by theoretical calculation. In general the agreement is satisfactory for the nonlinear wave forces comparison. The computations are also carried out without taking into account the effect of the second-order term contribution from the inhomogeneous free surface boundary condition. Due to a small difference between $C_M^f$ and
Table 7.2 Conversion of measured wave force coefficients defined by Chakrabarti (1975, 1978); $\frac{H}{a} = 0.261$, $\frac{a}{h} = 0.862$.

<table>
<thead>
<tr>
<th>$ka$</th>
<th>$C_{Chakrabarti}^f$</th>
<th>$\Omega$</th>
<th>$C_M^{f} * \frac{H}{2a}$ (converted value)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4.23</td>
<td>0.12</td>
<td>0.15148</td>
</tr>
<tr>
<td>0.86</td>
<td>3.54</td>
<td>0.19</td>
<td>0.12677</td>
</tr>
<tr>
<td>1.17</td>
<td>3.29</td>
<td>0.22</td>
<td>0.11781</td>
</tr>
<tr>
<td>1.30</td>
<td>2.80</td>
<td>0.25</td>
<td>0.10027</td>
</tr>
<tr>
<td>1.46</td>
<td>2.25</td>
<td>0.30</td>
<td>0.08057</td>
</tr>
<tr>
<td>1.70</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: $C_{M}^{f} = \frac{1}{2\pi} \frac{H}{a} \frac{a}{h} C_{Chakrabarti}^f$

$\Omega = \frac{ka}{2\pi} \left( \frac{h}{a} \right) \tanh \left[ \left( \frac{ka}{a} \right) \frac{h}{a} \right]$
Table 7.3 Summary of first- and second-order force coefficients and percentage of second-order force effect;

\[ \frac{H}{a} = 0.261, \frac{a}{h} = 0.862. \]

<table>
<thead>
<tr>
<th>( \Omega )</th>
<th>0.12</th>
<th>0.19</th>
<th>0.22</th>
<th>0.25</th>
<th>0.30</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_1^f \cdot H/2a )</td>
<td>0.13458</td>
<td>0.11356</td>
<td>0.10278</td>
<td>0.09246</td>
<td>0.07716</td>
</tr>
<tr>
<td>( (C_1^f + C_2^f) \cdot H/2a )</td>
<td>0.14399</td>
<td>0.11670</td>
<td>0.10477</td>
<td>0.09339</td>
<td>0.07755</td>
</tr>
<tr>
<td>{ \hat{\varepsilon}_F }</td>
<td>{6.99}</td>
<td>{2.76}</td>
<td>{1.93}</td>
<td>{1.01}</td>
<td>{0.50}</td>
</tr>
<tr>
<td>( (C_1^f + C_2^f \varepsilon) \cdot H/2a )</td>
<td>0.14415</td>
<td>0.11676</td>
<td>0.10458</td>
<td>0.09337</td>
<td>0.07744</td>
</tr>
<tr>
<td>{ \hat{\varepsilon}_F }</td>
<td>{7.11}</td>
<td>{2.81}</td>
<td>{1.75}</td>
<td>{0.99}</td>
<td>{0.35}</td>
</tr>
</tbody>
</table>

Note: \( \hat{\varepsilon}_F \) = percentage of nonlinear force to linear force

\[ = \left( \frac{C_2^f}{C_1^f} \right) \times 100\%. \]
The effect of the inhomogeneous free surface term again is negligibly small. This may easily be shown by comparing the results shown in Tables 7.3 and 7.4. The calculations also show the nonlinear forces are in better agreement when compared to model test results than the linear forces.

Similar results for moment coefficients, phase angle and percentage of the second-order effect which have exactly the same tendency as those force coefficients are presented in Figures 7.6 - 7.8 and Table 7.5.
Table 7.4 Summary of relative errors for experimental test; \( \frac{H}{a} = 0.261, \frac{a}{h} = 0.862. \)

<table>
<thead>
<tr>
<th>( \Omega )</th>
<th>0.12</th>
<th>0.19</th>
<th>0.22</th>
<th>0.25</th>
<th>0.30</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varepsilon_{F1} )</td>
<td>-12.56</td>
<td>-11.63</td>
<td>-14.62</td>
<td>-8.45</td>
<td>-4.42</td>
</tr>
<tr>
<td>( \varepsilon_{F2} )</td>
<td>-5.20</td>
<td>-8.63</td>
<td>-12.45</td>
<td>-7.37</td>
<td>-3.89</td>
</tr>
<tr>
<td>( \varepsilon_{F2E} )</td>
<td>-5.08</td>
<td>-8.57</td>
<td>-12.65</td>
<td>-7.39</td>
<td>-4.04</td>
</tr>
</tbody>
</table>

Note: \( \varepsilon = \left( \frac{\text{Theoretical} - \text{Measured}}{\text{Theoretical}} \right) \times 100\% \).

\[ \varepsilon_{F2E} = \left( \frac{C_{2E}^f - C_{N1}^f}{C_{2E}^f} \right) \times 100\% \]
Figure 7.6 First- and second-order moment coefficients,
\( \{ C_1^m, \text{Eq.}(7.25); C_2^m, \text{Eq.}(7.45) \}; \ \frac{H}{a} = 1.0, \ \frac{a}{h} = 0.3. \)
Figure 7.7 Percentage of second-order moment effect, $\hat{e}_M$;

$$\Omega = \frac{h}{L_0}$$

$H/a = 1.0$, $a/h = 0.3$. 
Figure 7.8 First- and second-order moment phase angles,

(\(\mu_m^1\), Eq. (7.21); \(\mu_m^2\), Eq. (7.41)); \(\frac{H}{a} = 1.0, \frac{a}{h} = 0.3\).
Table 7.5 Summary of first- and second-order moment coefficients and percentage of second-order moment effect;

\( \frac{H}{a} = 1.0, \frac{a}{h} = 0.3 \).

<table>
<thead>
<tr>
<th>( \Omega )</th>
<th>0.05</th>
<th>0.10</th>
<th>0.20</th>
<th>0.30</th>
<th>0.40</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_1^m \times \frac{H}{2a} )</td>
<td>0.08360</td>
<td>0.11628</td>
<td>0.15526</td>
<td>0.17138</td>
<td>0.16839</td>
</tr>
<tr>
<td>( C_2^m \times \frac{H}{2a} )</td>
<td>0.03762</td>
<td>0.02624</td>
<td>0.01026</td>
<td>0.00394</td>
<td>0.00159</td>
</tr>
<tr>
<td>( { \hat{\varepsilon}_M } )</td>
<td>{44.99}</td>
<td>{22.56}</td>
<td>{6.61}</td>
<td>{2.30}</td>
<td>{0.95}</td>
</tr>
<tr>
<td>( C_{2E}^m \times \frac{H}{2a} )</td>
<td>0.03762</td>
<td>0.02624</td>
<td>0.01029</td>
<td>0.00394</td>
<td>0.00135</td>
</tr>
<tr>
<td>( { \hat{\varepsilon}_M } )</td>
<td>{44.99}</td>
<td>{22.56}</td>
<td>{6.63}</td>
<td>{2.30}</td>
<td>{0.80}</td>
</tr>
</tbody>
</table>

Note: \( \hat{\varepsilon}_M = ( \frac{C_{2E}^m}{C_2^m} / \frac{C_1^m}{C_1^m} ) \times 100\% \).

\( C_{2E}^m = C_2^m \) less the infinite integral term given by Eq. (6.73)
CHAPTER 8

CONCLUSIONS AND RECOMMENDATIONS

8.1 Conclusions

This dissertation deals with a nonlinear diffraction theory correct to second-order for a fixed vertical circular cylinder subjected to a two-dimensional sinusoidal incident wave. The nonlinear effects are accounted for by a perturbation procedure.

The method applied in this work is based on the Green's function method which gives an integral representation for the solution of the wave potential. In applying Green's second formula to the differential equation, the spatial dimensions of the problem are reduced by one. Hence, the advantage of formulating the problem in terms of an integral equation is that the integral is a surface integral around the structure boundary which directly incorporates the structural geometry. With an appropriate Green's function for a particular differential equation, structural geometry, and prescribed boundary conditions, the integral equation becomes a simple numerical quadrature; i.e., an integration of known quantities.

The method of attacking the nonlinear diffraction problem is to try to construct a Green's function at each perturbation order which will give the solution as a boundary integral which includes the product of the Green's function and the prescribed boundary conditions. The Green's function is solved by eigenfunction expansion method at
each perturbation order. The desired eigenfunctions are then obtained from a well-posed Sturm-Liouville problem.

The solution of the first-order velocity potential based on an eigenfunction expansion of the Green's function is of some interest. It not only represents the exact solution of the linear diffraction theory; but it also indicates that the Green's function method is capable of solving a complex problem. However, the main concern of the present study is to extend the linear diffraction theory to a second-order problem based on the Green's function method.

Closed form solutions for the first- and second-order hydrodynamic force and moment coefficients are presented. Results giving the phase shift and dimensionless force and moment coefficients on a fixed circular cylinder are presented. The nonlinear effects in the total hydrodynamic forces and moments are graphically illustrated.

The results obtained from the present work indicate that the second-order contributions to the hydrodynamic forces and moments are significant in an intermediate water region and may become more important in a shallow water region. This implies that as cylindrical structures are subjected to hydrodynamic forces and moments in finite water depth, the second-order nonlinear effects should be taken into consideration.

8.2 Recommendations

The present investigations of the nonlinear diffraction problem
have been limited to cylindrical structures whose boundaries coincide with a separable coordinate system in which the Helmholtz equation separates. This study has shown that the eigenfunction expansion of the Green's function method is a powerful approach for solving a nonlinear diffraction problem for a circular cylinder. This method may, in fact, serve the purpose of solving the nonlinear diffraction problem for arbitrary shapes of cylinders and, thereby, remove the geometric limitation.

The essential difference between the diffraction problem and the radiation problem is that the former deals with fixed structure, whereas the latter deals with a floating structure in otherwise still fluid. The diffraction of an incident wave caused by the presence of a floating structure is represented by the sum of the diffracted wave potential and six degrees of freedom radiated wave potentials, which are to be determined by satisfying the kinematic boundary conditions on the structural surface. The diffracted wave potential is nothing but the velocity potential for estimating the wave-exciting force. The radiated wave potential is the velocity potential induced by the forced oscillation with the given velocity $V_n$ in calm water in which $V_n$ is the prescribed normal velocity component of a point on the structural surface. It should be possible to extend the method of an eigenfunction expansion of the Green's function to solve the radiation problem for a large floating vertical cylinder.

The main suggestion for future work is to extend the nonlinear diffraction theory to non-Gaussian waves. The extension to random waves...
will follow the general methodology outlined by Morse and Ingard (1968), pp. 863-873). The more complicated extension to nonlinear random waves will exploit the spectral interaction extension developed by Longuet-Higgins (1964) which linearly decomposes the second-order spectral interactions into first-order self-interactions and cross spectral interactions. This method has been used by Hudspeth and Chen (1979) to simulate nonlinear random sea surface realizations by a Finite Fourier Transform algorithm.

Following the procedure outlined by Morse and Ingard (1968), loc. cit., the pressure force on a vertical circular cylinder may be computed by the first-order solution given by MacCamy and Fuchs (1954) from the real part of the following integral:

$$ F_1(t) = i\rho \sigma \exp(-i\sigma t) \int_{\pi}^{0} \int_{-\pi}^{-h} \{\phi_1^1 + \phi_1^S\} r \, dz \, d\theta ; \quad r = a \quad (8.1a) $$

$$ = T(\sigma) n(t) \quad (8.1b) $$

in which the free surface profile is given by the real part of

$$ n(t) = \frac{H}{2} \exp(-i\sigma t) \quad (8.2) $$

and the frequency domain transfer function is given by

$$ T(\sigma) = \frac{\rho g}{\frac{4\tanh kh}{k^2 H_1'(1)}(ka)} \quad (8.3) $$
For the case of a linear, Gaussian sea having spectral representation, the instantaneous free surface profile may be synthesized from a linear sum of \( N \) sinusoids each having a random phase angle according to

\[
\eta(t) = \sum_{n=-N}^{N} \left( S_{n}(\sigma_n) \Delta \sigma \right)^{1/2} \exp \left\{ -i(\sigma_n t + \beta_n) \right\}
\]  

(8.4)

in which \( S_{n}(\sigma_n) \) is the spectral density of the free surface at frequency \( \sigma_n \). By further restricting the stochastic sea to a random process which is spatially homogeneous and temporally stationary, the autocovariance function for the linear pressure force reduces to

\[
\gamma_{FF}(\Xi) = E \left\{ F_1(t) F_1^*(t + \Xi) \right\}
\]

(8.5a)

\[
= \sum_{n=-N}^{N} \phi^2(\sigma_n) S_{n}(\sigma_n) \Delta \sigma \exp \left\{ i(\sigma_n \Xi) \right\}
\]

(8.5b)

since

\[
E(\exp \left\{ -(i\beta_n) \exp (i\beta_m) \right\}) = \delta_{mn}
\]

(8.6)

The autocovariance function for the total pressure force correct to second-order which is due to a weakly nonlinear, slightly non-Gaussian sea may be determined from

\[
\gamma_{FF}(\Xi) = E\left\{ [F_1(t) + F_2(t)][F_1(t + \Xi) + F_2(t + \Xi)]^* \right\}
\]

(8.7)
For a nonlinear random sea which is derived from a linear Gaussian process, it may be shown that

\[ \gamma_{FF}(\xi) = E\{F_1(t)F_1^*(t + \xi) + E\{F_2(t) F_2^*(t + \xi)\}\} \]  

(8.8)

since

\[ E\{F_1(t)F_2(t + \xi)\} = E\{F_2(t) F_1^*(t + \xi)\} = 0 \]  

(8.9)

for a Gaussian linear sea. In principle, we may now compute the first-order and second-order pressure forces independently and linearly sum them to determine the total contribution for this example of a vertical cylinder.

The pressure force problem now becomes one of determining the spectral transfer function from the second-order Green's function for a particular structural geometry and the nonlinear interaction matrix for the nonlinear random sea. The method for computing the spectral transfer function has been previously outlined.

It is anticipated that the experience gained from solving the deterministic Stokes' wave problem by an eigenfunction expansion of the Green's function correct to second-order will provide the insight and analytical tools in the case of nonlinear random waves. The nonlinear interaction matrix correct to second-order for a random sea in water of finite depth has been given by Hasselmann (1962); viz,
The second-order contribution given in Eq. (8.8) may now be computed by a three-fold product of Eq. (8.10), of the spectral representation of the linear sea, and of a complex numerical integral from the second-order scattered velocity potential. Since the nonlinear spectral interaction methods are well established (cf. Hasselmann (1962)), the extension of a nonlinear diffraction method by an eigenfunction expansion of the Green's function may be further extended to include nonlinear spectral interactions.
REFERENCES


APPENDICES
APPENDIX A

DERIVATION OF $F_s$

The expression for $F_s$ is given in Eq. (6.16). The first-order incident wave potential specified in Eq. (5.7b) may be rewritten as

$$\phi^i_1(r,\theta,z) = \sum_{n=0}^{\infty} \frac{g A_n}{2k\sigma \cosh kh} J_n(kr) \cos \theta \cosh (z + h) \quad (A.1)$$

in which

$$A_n = (2 - \delta_{n0}) i^{n+1}. \quad (A.2)$$

The first-order scattered wave potential expressed in Eq. (5.12) may be rewritten as

$$\phi^s_1(r,\theta,z) = \sum_{n=0}^{\infty} \frac{g B_n}{2k\sigma \cosh kh} H^{(1)}_n(kr) \cos \theta \cosh (z + h) \quad (A.3)$$

in which

$$B_n = - (2 - \delta_{n0}) i^{n+1} \frac{J'_n(ka)}{H^{(1)}_n(ka)}. \quad (A.4)$$

After substituting Eqs. (A.1) and (A.3) into Eqs. (6.17) and (6.18), the resultant expression contains a number of products of two infinite series. For each of these products, one of the summation
indexes is divided from 0 to \( n \) plus from \( n \) to \( \infty \) (Arfken, 1966, p.253; Raman, et al, 1976). For illustrative purpose, the algebraic manipulations applied to the first term in Eq. (6.17) is presented below

\[
\left( \frac{\partial \phi_1^S}{\partial r} \right)^2 = \frac{g}{4k^3 \tanh k h} \sum_{n=0}^{\infty} B_n k H_1^{(1)}(kr) \cos n \theta \sum_{\ell=0}^{\infty} B_{\ell} k H_\ell^{(1)}(kr) \cos \ell \theta
\]

\[
= \frac{g}{4k \tanh k h} \sum_{n=0}^{\infty} \sum_{\ell=0}^{n} B_n B_{n-\ell} H_1^{(1)}(kr) H_\ell^{(1)}(kr) \cos \ell \theta \cos (n-\ell) \theta
\]

\[
= \frac{g}{4k \tanh k h} \sum_{n=0}^{\infty} \sum_{\ell=0}^{n} B_n B_{n-\ell} H_\ell^{(1)}(kr) H_{\ell-n}^{(1)}(kr) \cdot [\cos n \theta + \cos (n-2\ell) \theta]
\]

\[
= \frac{g}{8k \tanh k h} \sum_{n=0}^{\infty} \sum_{\ell=0}^{n} B_n B_{n-\ell} H_\ell^{(1)}(kr) H_{\ell-n}^{(1)}(kr) + (2 - \delta_{n\ell}) \sum_{\ell=n}^{\infty} \sum_{n=0}^{\infty} B_n B_{n-\ell} H_{\ell-n}^{(1)}(kr) H_{\ell-n}^{(1)}(kr) \cos n \theta
\]

(A.5)

where the prime denotes the derivative with respect to \( r \).

The final series forms of \( w_1(\phi_1^S) \) and \( w_2(\phi_1^S, \phi_1^S) \) result in

\[
w_1(\phi_1^S) = -i \frac{g}{4k^3 \tanh k h} \sum_{n=0}^{\infty} \left( k^2 \left[ \sum_{\ell=0}^{n} B_n B_{n-\ell} H_\ell^{(1)}(kr) H_{\ell-n}^{(1)}(kr) \right] + (2 - \delta_{n\ell}) \sum_{\ell=n}^{\infty} \sum_{n=0}^{\infty} B_n B_{n-\ell} H_{\ell-n}^{(1)}(kr) H_{\ell-n}^{(1)}(kr) \right)
\]
\begin{align*}
+ \frac{1}{r^2} \left[ \sum_{\ell=0}^{n} \ell (\ell - n) A_{\ell} B_{\ell - n} J_{\ell}^{(1)} (kr) H_{n-\ell}^{(1)} (kr) \\
+ (2 - \delta_{n0}) \sum_{\ell=0}^{n} \ell (\ell - n) B_{\ell} B_{\ell - n} H_{\ell}^{(1)} (kr) H_{n-\ell}^{(1)} (kr) \right] \\
+ \frac{k^2}{2} (3 \tanh^2 kh - 1) \left[ \sum_{\ell=0}^{n} B_{\ell} B_{\ell - n} H_{\ell}^{(1)} (kr) H_{n-\ell}^{(1)} (kr) \right] \cos \eta e^2 (1.6)
\end{align*}

and

\begin{align*}
w_2(\phi_1^1, \phi_1^s) = -i \frac{\sigma g}{2 k^3 \tanh kh} \sum_{n=0}^{\infty} \left\{ k^2 \left[ \sum_{\ell=0}^{n} A_{\ell} B_{\ell - n} J_{\ell}^{(1)} (kr) H_{n-\ell}^{(1)} (kr) \\
+ \frac{1}{2} (2 - \delta_{n0}) \sum_{\ell=0}^{n} (A_{\ell} B_{\ell - n} J_{\ell}^{(1)} (kr) H_{n-\ell}^{(1)} (kr)) \right] \\
+ \frac{1}{r^2} \left[ \sum_{\ell=0}^{n} \ell (\ell - n) A_{\ell} B_{\ell - n} J_{\ell}^{(1)} (kr) H_{n-\ell}^{(1)} (kr) \\
+ \frac{1}{2} (2 - \delta_{n0}) \sum_{\ell=0}^{n} \ell (\ell - n) (A_{\ell} B_{\ell - n} J_{\ell}^{(1)} (kr) H_{n-\ell}^{(1)} (kr)) \right] \cos \eta e^2 (1.6)\right)
\end{align*}
\[ + A_{\ell-n} B_{\ell} J_{\ell-n}(kr) H_{\ell}^{(1)}(kr) ]
\[ + \frac{k^2}{4} (5 \tanh^2 kh - 1) \left[ \sum_{\ell=0}^{n} A_{\ell} B_{n-\ell-n} J_{\ell}(kr) H_{n-\ell}^{(1)}(kr) \right. \]
\[ + \frac{1}{2} (2 - \delta_{n0}) \sum_{\ell=n}^{\infty} (A_{\ell} B_{\ell-n} J_{\ell}(kr) H_{\ell-n}^{(1)}(kr)) \]
\[ - \frac{k^2}{4 \cosh^2 kh} \left[ \sum_{\ell=0}^{n} B_{\ell} A_{n-\ell-n} H_{\ell}^{(1)}(kr) J_{n-\ell}(kr) \right. \]
\[ + \frac{1}{2} (2 - \delta_{n0}) \sum_{\ell=n}^{\infty} (B_{\ell} A_{\ell-n} H_{\ell}^{(1)}(kr) J_{\ell-n}(kr)) \]
\[ + B_{\ell-n} A_{\ell-n} H_{\ell-n}^{(1)}(kr) J_{\ell}(kr)) ) \right] \cos n \theta , \quad (A.7) \]

and

\[ F_s = w_1(\phi^S_1) + w_2(\phi_{\dagger_1}, \phi^S_1) \quad . \quad (A.8) \]
APPENDIX B

ORTHOGONALITY OF BESSEL FUNCTIONS IN A SEMI-INFINITE INTERVAL

A Sturm-Liouville problem arises from the solution of a second-order differential equation of the form

\[ \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \left( q^2 - \frac{j^2}{r^2} \right) \phi = 0. \]  

(B.1)

over the semi-infinite interval \( a \leq r < \infty \).

Let \( J_j(q_m r) \) and \( J_j(q_\lambda r) \) be eigenfunctions of the Sturm-Liouville problem, and

\[ \int_a^\infty r J_j(q_m r) J_j(q_\lambda r) \, dr = \int_0^a r J_j(q_m r) J_j(q_\lambda r) \, dr - \int_0^a r J_j(q_m r) J_j(q_\lambda r) \, dr. \]  

(B.2)

The first term on the right hand side of Eq. (B.2) is orthogonal with respect to the weighting function \( r \) on the interval \( 0 \leq r < \infty \) (Titchmarsh, 1962; Richtmyer, 1978). The second term on the right hand side of Eq. (B.2) is also shown to be orthogonal with respect to the weighting function \( r \) on an interval \( 0 \leq r \leq a \) (Hildebrand, 1962). Then, the Bessel functions are said to form an orthogonal set with respect to the weighting function \( r \) on a semi-infinite interval \( a \leq r < \infty \).

Accordingly, the orthogonality relation for Bessel functions is
\[ \int_{a}^{\infty} r J_{j}(q_{m} r) J_{j}(q_{1} r) \, dr = \delta_{j}^{m} \delta_{m_{1}} \, , \]  

(8.3)

in which \( \delta_{m_{1}} \) is the Kronecker delta.
APPENDIX C

FURTHER COMMENT ON SOLUTION OF $\phi_2^{sf}$

It is easy to show that the solutions of $\phi_2^i$ and $\phi_2^{ss}$ satisfy their boundary conditions, respectively. The most critical requirement is to show that the solution of $\phi_2^{sf}$ satisfies the inhomogeneous combined free surface boundary condition defined in Eq. (6.23c); i.e.,

$$\frac{\partial \phi_2^{sf}}{\partial z} - \frac{4\sigma^2}{g} \phi_2^{sf} = \frac{F_s}{g} ; \quad z = 0$$  \hspace{1cm} (C.1)

in which $F_s$ is given by Eq. (A.8) in Appendix A.

Substituting the expression for the second-order scattered velocity potential, $\phi_2^{sf}$ given by Eq. (6.75c), into Eq. (C.1) yields

$$\frac{\partial \phi_2^{sf}}{\partial z} - \frac{4\sigma^2}{g} \phi_2^{sf} = -i\sigma \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{D_m(0)}{4ak^3\tanh kh} \frac{a_n^m}{q_m} \cos n\theta \cdot \left( q_m \sinh q_m h - \frac{4\sigma^2}{g} \cosh q_m h \right) \quad (C.2a)$$

$$= -i\sigma \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{N_n^m}{4k^3\tanh kh} \frac{a_n^m}{q_m^2} \cos n\theta \cdot \quad (C.2b)$$

by Eq. (6.69b). Substituting Eq. (C.2b) into Eq. (C.1) yields
\[ \sum_{m=1}^{\infty} \frac{N_m}{a^2} \left[ \sum_{n} J_n(q_m r) \right] = \]

\[ k^2 \left[ \sum_{\ell=0}^{n} B_{\ell} B_{n-\ell} H^{(1)\prime}_{\ell}(kr) H^{(1)}_{n-\ell}(kr) + (2 - \delta_{n0}) \sum_{\ell=n}^{\infty} B_{\ell} B_{\ell-n} H^{(1)\prime}_{\ell}(kr) H^{(1)}_{\ell-n}(kr) \right] \]

\[ + \frac{1}{r^2} \left[ \sum_{\ell=0}^{n} \ell(\ell - n) B_{\ell} B_{n-\ell} H^{(1)\prime}_{\ell}(kr) H^{(1)}_{n-\ell}(kr) \right] + (2 - \delta_{n0}) \left[ \sum_{\ell=n}^{\infty} \ell(\ell - n) B_{\ell} B_{\ell-n} H^{(1)\prime}_{\ell}(kr) H^{(1)}_{\ell-n}(kr) \right] \]

\[ + \frac{k^2}{2} \left( 3 \tanh^2 kh - 1 \right) \left[ \sum_{\ell=0}^{n} B_{\ell} B_{n-\ell} H^{(1)\prime}_{\ell}(kr) H^{(1)}_{n-\ell}(kr) \right] \]

\[ + (2 - \delta_{n0}) \left[ \sum_{\ell=n}^{\infty} B_{\ell} B_{\ell-n} H^{(1)\prime}_{\ell}(kr) H^{(1)}_{\ell-n}(kr) \right] \]

\[ + 2k^2 \left[ \sum_{\ell=0}^{n} A_{\ell} B_{n-\ell} J^{\prime}_{\ell}(kr) H^{(1)\prime}_{n-\ell}(kr) \right] \]

\[ + \frac{1}{2} (2 - \delta_{n0}) \left[ \sum_{\ell=n}^{\infty} \left( A_{\ell} B_{\ell-n} J^{\prime}_{\ell}(kr) H^{(1)\prime}_{\ell-n}(kr) + A_{\ell-n} B_{\ell} J^{\prime}_{n}(kr) H^{(1)\prime}_{\ell}(kr) \right) \right] \]

\[ + \frac{2}{r^2} \left[ \sum_{\ell=0}^{n} \ell(\ell - n) A_{\ell} B_{n-\ell} J^{\prime}_{\ell}(kr) H^{(1)}_{n-\ell}(kr) \right] \]

\[ + \frac{1}{2} (2 - \delta_{n0}) \left[ \sum_{\ell=n}^{\infty} \ell(\ell - n) \left( A_{\ell} B_{\ell-n} J^{\prime}_{\ell}(kr) H^{(1)}_{\ell-n}(kr) \right) \right] \]

\[ + A_{\ell-n} B_{\ell} J^{\prime}_{n}(kr) H^{(1)}_{\ell}(kr) \]
\[ + \frac{k^2}{2} (5 \tanh^2 kh - 1) \left[ \sum_{\ell=0}^{n} A_{\ell} B_{n-\ell} J_{\ell}(kr) H_{n-\ell}^{(1)}(kr) \right. \\
\left. + \frac{1}{2} (2 - \delta_{n0}) \sum_{\ell=n}^{\infty} (A_{\ell} B_{n-\ell} J_{\ell}(kr) H_{n-\ell}^{(1)}(kr) + A_{\ell} B_{n-\ell} J_{\ell-n}(kr) H_{n-\ell}^{(1)}(kr)) \right] \\
- \frac{k^2}{2 \cosh^2 kh} \left[ \sum_{\ell=0}^{n} B_{\ell} A_{n-\ell} H_{\ell}^{(1)}(kr) J_{n-\ell}(kr) \right. \\
\left. + \frac{1}{2} (2 - \delta_{n0}) \sum_{\ell=n}^{\infty} (B_{\ell} A_{n-\ell} H_{\ell}^{(1)}(kr) J_{n-\ell}(kr) + B_{\ell} A_{n-\ell} H_{n-\ell}^{(1)}(kr) J_{\ell}(kr)) \right]. \]

(C.3)

Multiplying Eq. (C.3) by \( r J_n(q_m r) \) and integrating over \( r \) from \( a \) to \( \infty \), we obtain the expression for \( N_n^m \) which is exactly the same as Eq. (6.73) due to

\[ \int_a^\infty r J_n(q_m r) J_n(q_{\ell} r) \, dr = a^2 \delta_n^m \delta_{m\ell}. \]

(C.4)

Note that \( N_n^m \) is simply like an eigenfunction expansion of \( F_s \) in terms of \( J_n(q_m r) \), which are second-order eigenfunctions.

It is also easy to show that \( \phi_s^{sf} \) satisfies the remaining boundary conditions.

The foregoing discussion shows that the solution of \( \phi_s^{sf} \) satisfies all the boundary conditions for \( \phi_s^{sf} \).
Isaacson (1977, 1978) concluded that it was not possible for a second-order solution to simultaneously satisfy the kinematic boundary conditions on both the cylinder body surface and the mean free surface. However, he reached this conclusion by differentiating the combined free surface boundary condition given by Eq. (C.1) with respect to $r$. The physical meaning of the radial derivative of Eq. (C.1) is not known and, therefore, as long as a solution to a well-posed boundary value problem satisfies all at the physical boundary conditions, the solution may be assumed to be a correct physical solution to the second-order diffraction problem.
APPENDIX D

DERIVATION OF EQ. (6.47)

The Green's second formula defined by Eq. (3.8) is applied to the second-order scattered velocity potential, $\phi_{2}^{sf}$. The Green's function, $G_{2}^{f}$, is chosen to be singular so that it satisfies Eq. (6.46a). Hence, the velocity potential, $\phi_{2}^{sf}$, is now expressed in terms of a surface integral which includes the prescribed boundary conditions defined in Eqs. (6.23). The velocity potential is expressed as

$$\phi_{2}^{sf} = \frac{-1}{k} \int S_{f} + S_{\omega} + S_{b} + S_{B} \left\{ \phi_{2}^{sf}(R, \theta, Z) \frac{\partial G_{2}^{f}(r, \theta, z; R, \theta, Z)}{\partial n} \right. $$

$$ \left. - G_{2}^{f}(r, \theta, z; R, \theta, Z) \frac{\partial \phi_{2}^{sf}}{\partial n} \right\} dS(R, \theta, Z) \quad (D.1)$$

in which the normal vector $\hat{n}$ is taken to be positive when pointing out of the fluid volume and $S(R, \theta, Z)$ denotes the integration surface which contains the source point $(R, \theta, Z)$.

Since the velocity potential, $\phi_{2}^{sf}$, satisfies the homogeneous bottom boundary condition, the structural boundary condition and the radiation condition, we may select the Green's function, $G_{2}^{f}$, also to satisfy the homogeneous bottom boundary condition, the structural boundary condition and the radiation condition in Eqs. (6.46b), (6.46d) and (6.46e), respectively [cf., Section 3.2]. Then Eq. (D.1) becomes
\[
\phi_{2f} = - \frac{1}{I} \int_{S_f} \left\{ \phi_{2f} \frac{\partial G_f^f}{\partial n} - G_f^f \frac{\partial \phi_{2f}}{\partial n} \right\} dS \\
= - \frac{1}{I} \int_{-\pi}^{\pi} \int_{a}^{\infty} \left\{ \phi_{2f} \frac{\partial G_f^f}{\partial z} - G_f^f \frac{\partial \phi_{2f}}{\partial z} \right\} R dR d\theta ; \quad Z = 0 . \quad (D.2)
\]

Since \( \phi_{2f} \) satisfies the combined free surface boundary condition given by Eq. (6.23c); i.e.,
\[
\frac{\partial \phi_{2f}}{\partial Z} = \frac{4a^2}{g} \phi_{2f} + \frac{F_s}{g} ; \quad Z = 0 , \quad (D.3)
\]

Eq. (D.2) is reduced to
\[
\phi_{2f} = - \frac{1}{I} \int_{-\pi}^{\pi} \int_{a}^{\infty} \left\{ \left( \frac{\partial G_f^f}{\partial Z} - \frac{4a^2}{g} G_f^f \right) \phi_{2f} - G_f^f \frac{F_s}{g} \right\} R dR d\theta \\
; \quad Z = 0 . \quad (D.4)
\]

If the Green's function, \( G_f^f \), is chosen to satisfy the following combined free surface boundary condition:
\[
\frac{\partial G_f^f}{\partial Z} - \frac{4a^2}{g} G_f^f = 0 ; \quad Z = 0 ,
\]

the velocity potential, \( \phi_{2f} \), is recovered from the following integral equation:
\[
\phi_{2f}(r, \theta, z) = \frac{1}{I} \int_{-\pi}^{\pi} \int_{a}^{\infty} \frac{1}{g} G_f^f F_s R dR d\theta ; \quad Z = 0 . \quad (D.5)
\]