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In this dissertation, we use Fourier-analytic methods to study questions of equidistribution on the compact abelian group $\mathbb{Z}_p$ of p-adic integers. In particular, we prove a LeVeque-type Fourier analytic upper bound on the discrepancy of sequences. We establish p-adic analogues of the classical Dirichlet and Fejér kernels on $\mathbb{R}/\mathbb{Z}$, and investigate their properties. Finally, we compare notions of variation for functions on $\mathbb{Z}_p$ due to Beer and Taibleson. We also prove a p-adic Fourier analytic Koksma inequality.

Fourier Analysis and Equidistribution on the p-adic Integers

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I understand that my dissertation will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my dissertation to any reader upon request.

Naveen Somasunderam, Author

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"Wisdom cannot be imparted. The wisdom that a wise man attempts to impart always sounds like foolishness. Knowledge can be communicated, but not wisdom. One can find it, live it, do wonders through it, but one cannot communicate and teach it." - Siddartha, Herman Hesse.

Fourier Analysis and Equidistribution on the p-adic Integers

1 Introduction

1.1 Equidistribution theory on the circle group $\mathbb{R}/\mathbb{Z}$

The theory of equidistribution of sequences modulo one was initiated by Hermann Weyl in 1916. Since then, it has spurred a lot of interest in many areas of mathematics, including number theory, harmonic analysis, and ergodic theory. The standard reference in this subject is Kuipers and Niederreiter [14].

Consider the compact circle group $\mathbb{R}/\mathbb{Z}$, and recall that this quotient group may be identified with a natural fundamental domain, the interval $[0, 1)$ with addition modulo one. Let $\{x_n\}$ be a sequence in $\mathbb{R}/\mathbb{Z}$. It is interesting to investigate how this sequence distributes itself over the interval $[0, 1)$. That is, do the terms of the sequence fall in to each subinterval proportionally as n increases, or are they more concentrated in some parts and less so in others? Are there gaps in its distribution?

As a simple example, consider the sequence $\{x_n\} = \{1/n\}$. Since zero is its only limit point, the sequence gets clustered near zero. Hence, this would be an example of a sequence that does not distribute evenly over $\mathbb{R}/\mathbb{Z}$. On the other hand, consider the sequence $\{x_n\} = \{na\}$, where a is an irrational real number. We have the following theorem.

Theorem 1.1. *Let a be a real number. The sequence $\{x_n\} = \{na\}$ is dense in $\mathbb{R}/\mathbb{Z}$ if and only if a is irrational.*

Such sequences were investigated by Kronecker in relation to Diophantine approximations, who also proved Theorem 1.1 (see [11]). But, how does the sequence $\{x_n\} = \{na\}$ distribute itself as n increases? We are looking for a deeper notion than density. As we shall see in Theorem 1.5, the sequence $\{na\}$ is not just dense but also equidistributed.

Definition 1.1 (Equidistribution on $\mathbb{R}/\mathbb{Z}$)**.** The sequence $\{x_n\}$ is equidistributed in $\mathbb{R}/\mathbb{Z}$ if given any subinterval $A = [a, b]$ of $[0,\ 1)$ we have

$$\lim_{N\to\infty} \frac{|\{x_1, x_2, ..., x_N\} \cap [a,\ b]|}{N} = b - a.$$

In other words, Definition 1.1 says that the proportion of the first N elements of $\{x_n\}$ lying in A is equal to the length of A in the limit of large N, and this holds true for all subintervals $A = [a,\ b]$ in $\mathbb{R}/\mathbb{Z}$.

Note that the definition of equidistribution implies that any equidistributed sequence is also dense. Equidistribution is a property dependent on the ordering of the sequence, as the next theorem shows.

Theorem 1.2. *Any sequence has a rearrangement that is not equidistributed. Any dense sequence has an ordering that is equidistributed.*

Proof. Both statements follow from the pigeon hole principle. To prove the first statement, suppose that $\{x_n\}$ is a dense sequence (if $\{x_n\}$ is not dense, then there is nothing to prove). Since $\{x_n\}$ is dense, pick the first hundred terms in a rearrangement of the sequence to be in the interval $[0,\ 1/2)$. Then pick the next term in the rearrangement to be in $[1/2,\ 1)$, and repeat the process. This rearrangement is not equidistributed.

To prove the second assertion, suppose $\{x_n\}$ is dense in $\mathbb{R}/\mathbb{Z}$. Then consider a sequence of dyadic partitions P_k of the interval $[0, 1]$ of the form $P_k = \{j/2^k \,|\, j = 0, 1, ..., 2^k\}$, for $k \in \mathbb{N}$. For a fixed k, pick an element of $\{x_n\}$ from every subinterval $[j/2^k,\ (j+1)/2^k)$ for $\{j = 0, 1, ..., 2^k\}$. Repeat this procedure for all $k \in \mathbb{N}$. It can be shown that this leads to a rearrangement of $\{x_n\}$ that is equidistributed. $\square$

In particular, there is an ordering of the rational numbers in $[0,\ 1)$ that is equidistributed. Another interesting question is whether Definition 1.1 should be extended from subintervals to more general Borel measurable sets in $\mathbb{R}/\mathbb{Z}$, replacing length with the Lebesgue measure. Such an extension would make any ordering of rationals not be equidistributed, since it would admit the set of irrational numbers in $[0, 1)$ whose measure is one. However, it is possible to reasonably extend the definition to the algebra of Jordan sets. Pete Clark's expository article [3] develops these ideas further.

The connection between equidistribution and Fourier analysis was first established in the following foundational result of Weyl.

Theorem 1.3 (Weyl's Criterion)**.** *A sequence $\{x_n\}$ in $\mathbb{R}/\mathbb{Z}$ is equidistributed if and only if*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k x_n} = 0,$$

for all $k \in \mathbb{Z} - \{0\}$.

That is, a sequence is equidistributed if and only if the average value of any non-trivial Fourier mode $e^{2\pi i k x}$ over the first N terms of the sequence goes to zero in the limit of large N. Theorem 1.3 can easily be generalized to the space of all Riemann integrable functions.

Theorem 1.4 (Generalized Weyl's Criterion)**.** *A sequence $\{x_n\}$ in $\mathbb{R}/\mathbb{Z}$ is equidistributed if and only if for every Riemann integrable function $f : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)\, dx. \tag{1.1}$$

Proof. Given any interval A, let $\mathcal{X}_A(x)$ denote the characteristic function on A. Note that the definition of equidistribution implies that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mathcal{X}_A(x_n) = \int_0^1 \mathcal{X}_A(x)\, dx.$$

Therefore the condition in (1.1) is satisfied by $\mathcal{X}_A(x)$. The reverse implication of Theorem 1.4 follows immediately since characteristic functions of intervals are Riemann integrable. For the forward implication, suppose first that f is a real valued

Riemann integrable function and that $\{x_n\}$ is equidistributed. The condition in (1.1) also holds for step functions (i.e. finite linear combinations of characteristic functions of intervals), since it holds for characteristic functions of intervals. Note that any Riemann integrable f can be approximated by step functions s_1 and s_2, such that $s_1 \leq f \leq s_2$ and $\int_0^1 (s_2 - s_1)(x)\,dx \leq \epsilon$, for any arbitrarily small ϵ. Hence,

$$\begin{aligned} \int_0^1 f(x)\,dx - \frac{1}{N}\sum_{n=1}^N f(x_n) &\leq \int_0^1 s_2(x)\,dx - \frac{1}{N}\sum_{n=1}^N f(x_n) \\ &\leq \int_0^1 s_2(x)\,dx - \frac{1}{N}\sum_{n=1}^N s_1(x_n) \\ &= \int_0^1 (s_2 - s_1)(x)\,dx + \int_0^1 s_1(x)\,dx - \frac{1}{N}\sum_{n=1}^N s_1(x_n) \\ &\leq \epsilon + \int_0^1 s_1(x)\,dx - \frac{1}{N}\sum_{n=1}^N s_1(x_n). \end{aligned} \tag{1.2}$$

Similarly, we have

$$\begin{aligned} \int_0^1 f(x)\,dx - \frac{1}{N}\sum_{n=1}^N f(x_n) &\geq \int_0^1 s_1(x)\,dx - \frac{1}{N}\sum_{n=1}^N f(x_n) \\ &\geq \int_0^1 s_1(x)\,dx - \frac{1}{N}\sum_{n=1}^N s_2(x_n) \\ &= \int_0^1 (s_1 - s_2)(x)\,dx + \int_0^1 s_2(x)\,dx - \frac{1}{N}\sum_{n=1}^N s_2(x_n) \\ &\geq -\epsilon + \int_0^1 s_2(x)\,dx - \frac{1}{N}\sum_{n=1}^N s_2(x_n). \end{aligned} \tag{1.3}$$

Now taking the limit as $N \to \infty$ in (1.2) and (1.3), we get

$$-\epsilon \leq \int_0^1 f(x)\,dx - \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N f(x_n) \leq \epsilon.$$

Since ϵ was arbitrary, the result follows. The result also holds for complex valued functions f by considering the real and imaginary parts separately. $\square$

An immediate application of Theorem 1.3 is the following.

Theorem 1.5. *Let a and b be real numbers. The sequence $\{x_n\} = \{na + b\}$ is equidistributed in $\mathbb{R}/\mathbb{Z}$ if and only if a is irrational.*

Proof. For the forward implication suppose that a is rational, and $a = p/q$ for p, q integers. Then consider $e^{2\pi i q x_n}$. We have $e^{2\pi i q x_n} = e^{2\pi i q b}$ for all n, and therefore Weyl's criterion is not satisfied. For the reverse implication, let a be irrational. Then, for any non-trivial Fourier mode using the geometric series identity we have

$$\begin{aligned} \frac{1}{N}\left|\sum_{n=1}^{N} e^{2\pi i k(na+b)}\right| &= \frac{1}{N}\left|\frac{1-e^{2\pi i k N a}}{1-e^{2\pi i k a}}\right| \\ &= \frac{1}{N}\left|\frac{\sin(\pi k N a)}{\sin(\pi k a)}\right| \\ &\leq \frac{1}{N}\frac{1}{|\sin(\pi k a)|}. \end{aligned} \tag{1.4}$$

Since a is irrational, $|\sin(\pi k a)|$ is not equal to zero. Hence, the last expression in (1.4) goes to zero as $N \to \infty$. Therefore, by Weyl's criterion the sequence is equidistributed. $\square$

Remark 1.1. There is a natural connection between ergodic transformations on $\mathbb{R}/\mathbb{Z}$ and Theorem 1.4. For suppose that $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is ergodic. Then Birhoff's Ergodic Theorem states that for all $f \in L^1(\mathbb{R}/\mathbb{Z})$ we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f\left(T^n(x)\right) = \int_{\mathbb{R}/\mathbb{Z}} f(x)\,dx,$$

for almost all x. That is, if T is ergodic then its forward orbit is equidistributed for all most all x. In particular, it can be shown that the transformation $T(x) = a + x$ is ergodic in $\mathbb{R}/\mathbb{Z}$ if and only if a is irrational. The book by Walters [25] provides a good introduction to ergodic theory.

1.2 Discrepancy theory

Discrepancy theory is a quantitative study of the distribution of sequences. It attempts to measure the deviation of a finite sequence from visiting any interval proportionally.

Definition 1.2 (Discrepancy)**.** The discrepancy of a finite sequence $\{x_1, x_2, ..., x_N\}$ in $\mathbb{R}/\mathbb{Z}$ is

$$D_N = \sup_{0\leq a\leq b\leq 1}\left|\frac{|\{x_1, x_2, ..., x_N\} \cap [a,\, b]|}{N} - (b-a)\right|.$$

That is, D_N measures the maximal deviation of a set of N elements of $\mathbb{R}/\mathbb{Z}$ from visiting any interval in proportion to its length. Discrepancy theory was extensively studied by van der Corput and Pisot [24]. Some elementary arguments gives us the following theorem.

Theorem 1.6. *The discrepancy of a finite sequence satisfies*

$$\frac{1}{N} \leq D_N \leq 1.$$

Theorem 1.7. *A sequence $\{x_n\}$ in $\mathbb{R}/\mathbb{Z}$ is equidistributed if and only if* $\lim_{N\to\infty} D_N = 0$.

A proof of Theorem 1.7 is given by Weyl in [26]. The backward direction is immediate from the definitions, while the forward direction is not obvious a priori. However, it follows at once for example by Theorem 1.8 and the dominated convergence theorem.

There are two main theorems that establish upper bounds on the discrepancy D_N using the exponential sums that occur in Weyl's criterion (Theorem 1.3). The proof of both Theorems 1.8 and 1.9 are given in Kuipers and Niederreiter [14].

Theorem 1.8 (LeVeque's Inequality)**.** *The discrepancy D_N for a finite sequence $\{x_1, x_2, .., x_N\}$ satisfies*

$$D_N \leq \left(\frac{6}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i\, k x_n}\right|^2\right)^{1/3}.$$

It is easily seen that the constant appearing in Theorem 1.8 is the best possible. The sequence of zeros $\{x_n\} = \{0, 0, 0, \ldots\}$, satisfies the inequality exactly. In [15], LeVeque also shows that the exponent of $1/3$ is the best possible.

A weaker version of the Theorem 1.8 was given by LeVeque in [15]. A more general version was presented by Elliot in [8]. The second main theorem is that of Erdös and Turán (see [9]).

Theorem 1.9 (Erdös and Turán)**.** *For a finite sequence $\{x_1, x_2, .., x_N\}$ of real numbers, and any positive integer m, we have*

$$D_N \leq \frac{6}{m+1} + \frac{4}{\pi} \sum_{k=1}^{m+1} \left(\frac{1}{k} - \frac{1}{m+1} \right) \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k x_n} \right|.$$

A proof of Theorem 1.9 is given in [14] and [16]. In practice, the inequality of Erdös and Turán gives a better bound than the LeVeque inequality. Montgomery in [16] provides a detailed discussion and considers some examples. In particular, consider the sequence $\{na\}$ where $a = \frac{1+\sqrt{5}}{2}$. The Erdös-Turán gives a bound of $D_N \ll (\log(N))^2/N$, whereas the use of the LeVeque inequality gives only $D_N \ll N^{-2/3}$.

Other notions of discrepancy also exist. In particular, the L_p discrepancy is defined as

$$D_N = \left(\int_0^1 \left| \frac{|\{x_1, x_2, ..., x_N\} \cap [0,\, x)|}{N} - x \right|^p dx \right)^{1/p}.$$

For more background and references on the theory of equidistribution in $\mathbb{R}/\mathbb{Z}$ see Kuipers and Niederreiter [14]. The survey article of Pete Clark [3] also provides a good introduction.

1.3 Summary of main results

Before stating our main results, we give a brief overview of the p-adic field $\mathbb{Q}_p$, define equidistribution and discrepancy on the p-adic unit ball $\mathbb{Z}_p$ analogous to Definitions 1.1 and 1.2 on $\mathbb{R}/\mathbb{Z}$, and define notation relating the Prüfer p-group $\mathbb{Z}(p^\infty)$

to the dual group of $\mathbb{Z}_p$. This will be sufficient for us to state our main results clearly and concisely. A more detailed discussion of this introductory material is done in Chapters 2, 3 and 4.

For a fixed prime p, let $|\cdot|_p$ denote the p-adic absolute value on the set of rationals $\mathbb{Q}$. Then the field $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the p-adic absolute value. Let

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\},$$

be the ring of p-adic integers. Any element of $\mathbb{Z}_p$ can be given a unique canonical expansion of the form $x = a_0 + a_1 p + a_2 p^2 + ...$, where the a_i are elements of $\{0, 1, 2, ..., p-1\}$ (see for example [12, 13]).

For $k \geq 0$, and $a \in \mathbb{Z}_p$, we denote by

$$\begin{aligned} D(a, 1/p^k) &= \{x \in \mathbb{Z}_p \mid |x - a|_p \leq 1/p^k\} \\ &= a + p^k \mathbb{Z}_p, \end{aligned}$$

a disk of radius $1/p^k$ centered at a. As a compact Hausdorff topological group, $\mathbb{Z}_p$ has a natural translation invariant measure μ called the Haar measure. The measure could be normalized so that $\mu(\mathbb{Z}_p) = 1$, and the measure of a disk is equal to its radius.

The following definition of equidistribution on $\mathbb{Z}_p$ using disks is an analogue to Definition 1.1 on $\mathbb{R}/\mathbb{Z}$, where the length of an interval is replaced by the measure of a disk.

Definition 1.3. A sequence $\{x_n\}$ is said to be equidistributed in $\mathbb{Z}_p$ if for every a in $\mathbb{Z}_p$ and every $k \in \mathbb{N}$, we have

$$\lim_{N \to \infty} \frac{\left|D(a, 1/p^k) \cap \{x_1, ..., x_N\}\right|}{N} = \frac{1}{p^k}.$$

That is, the proportion of the first N elements of $\{x_n\}$ lying in a disk $D(a, 1/p^k)$ is equal to its measure in the limit of large N, and this holds true for all such disks. The discrepancy of a set of N elements of $\mathbb{Z}_p$ is defined analogously to Definition 1.2.

Definition 1.4. The discrepancy of a finite sequence $\{x_1, x_2, ..., x_N\}$ in $\mathbb{Z}_p$ is

$$D_N = \sup_{a\in\mathbb{Z}_p,\, k\in\mathbb{N}} \left| \frac{\left|D(a, 1/p^k) \cap \{x_1, ..., x_N\}\right|}{N} - \frac{1}{p^k} \right|.$$

That is, D_N measures the maximal deviation of a set of N elements of $\mathbb{Z}_p$ from visiting any disk in proportion to its radius. As in the classical case, some elementary arguments show that

$$\frac{1}{N} \leq D_N \leq 1.$$

The main aim of this dissertation is to prove a Fourier analytic upper bound on the discrepancy D_N given by Definition 1.4.

Let $\mathbb{T}$ be the circle group under multiplication, $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. Given a compact abelian group G, the set of continuous group homomorphisms from G to $\mathbb{T}$ form a discrete group called the Pontryagin dual group of G. Let $\mathbb{Z}(p^\infty)$ denote the Prüfer p-group, the group of all p-th power roots of unity in $\mathbb{C}$. That is

$$\mathbb{Z}(p^\infty) = \left\{ e^{2\pi i \frac{m}{p^n}} \mid m, n \in \mathbb{N} \cup \{0\}, p \nmid m \right\}.$$

The dual group of $\mathbb{Z}_p$ is naturally isomorphic to $\mathbb{Z}(p^\infty)$, and its elements will play a role analogous to the functions $e^{2\pi ikx}, k \in \mathbb{Z}$, in classical Fourier analysis on $\mathbb{R}/\mathbb{Z}$.

Note that every element of $\mathbb{Z}(p^\infty)$ has finite order.

Notation 1.1. We denote the order of $\zeta \in \mathbb{Z}(p^\infty)$ by $\|\zeta\|$.

Notation 1.2. Suppose that $\zeta \in \mathbb{Z}(p^\infty)$ has order p^n, and let $x \in \mathbb{Z}_p$ have the canonical expansion $x = a_0 + a_1 p + a_2 p^2 + ... + a_{n-1}p^{n-1} + ...$ Then we interpret the notation ζ^x as

$$\zeta^x = \zeta^{a_0 + a_1 p + a_2 p^2 + ... + a_{n-1}p^{n-1}}.$$

The following theorem is a p-adic analogue to LeVeque's inequality.

Theorem 1.10 (Main Theorem)**.** *The discrepancy of a finite sequence* $\{x_1, ..., x_N\}$ *in* $\mathbb{Z}_p$ *is bounded by*

$$D_N \;\leq\; C(p) \left(\sum_{\zeta\in\mathbb{Z}(p^\infty)\setminus\{1\}} \frac{1}{\|\zeta\|^3} \left| \frac{1}{N} \sum_{n=1}^{N} \zeta^{x_n} \right|^2 \right)^{\frac{1}{4}},$$

where $C(p)$ is a constant dependent only on p.

Corollary 1.1. *A sequence $\{x_n\}$ in $\mathbb{Z}_p$ is equidistributed if and only if $\lim_{N\to\infty} D_N = 0$.*

Proof. The reverse direction follows immediately from Definition 1.4. The forward implication follows from Theorem 1.10, the p-adic Weyl's criterion as stated in Proposition 4.1, and the dominated convergence theorem. □

As an example application of Theorem 1.10, we have the following corollary.

Corollary 1.2. *Let $a, b \in \mathbb{Z}_p$, and assume a is a unit in $\mathbb{Z}_p$. Then the sequence $\{na + b\}$ has discrepancy*

$$D_N = O\left(N^{-1/2}\right).$$

Some quantitative results on the discrepancy of special p-adic sequences were found by Beer in [1] and [2]. In particular, she proves in [1] that the discrepancy of the sequence $\{na+b\}$ with a a unit is exactly equal to $D_N = N^{-1}$, the best possible. It is not surprising that the LeVeque-type inequality gives us a weaker bound, as this is the case in the classical setting on $\mathbb{R}/\mathbb{Z}$.

Let $\{x_1, x_2, ..., x_N\}$ be N points in $\mathbb{R}/\mathbb{Z}$, and $f : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}$. Theorem 1.4 motivates analyzing inequalities of the form

$$\left| \int_0^1 f(x)\,dx - \frac{1}{N}\sum_{n=1}^{N} f(x_n) \right| \leq C D_N,$$

where C is a constant. Let f be a function of bounded variation on $\mathbb{R}/\mathbb{Z}$, with variation $V(f)$. The classical Koksma inequality in $\mathbb{R}/\mathbb{Z}$ gives a bound of the form

$$\left| \int_0^1 f(x)\,dx - \frac{1}{N}\sum_{n=1}^{N} f(x_n) \right| \leq V(f) D_N^*,$$

where D_N^* is the star discrepancy, which is obtained by fixing $a = 0$ in Definition 1.2 (note that $D_N^* \leq D_N \leq 2D_N^*$, see [16]).

We derive a Fourier analytic Koksma inequality in $\mathbb{Z}_p$. For a function $f \in L^1(\mathbb{Z}_p)$, its Fourier coefficients $\hat{f}(\zeta)$ are given by

$$\hat{f}(\zeta) = \int_{\mathbb{Z}_p} f(x)\zeta^{-x}\, d\mu(x),$$

for ζ in $\mathbb{Z}(p^\infty)$. We have the following theorem.

Theorem 1.11. *Let f be a continuous complex-valued function on $\mathbb{Z}_p$ with Fourier coefficients $\hat{f}(\zeta)$ for $\zeta \in \mathbb{Z}(p^\infty)$. Then, for a set of N points $\{x_1, ..., x_N\}$ in $\mathbb{Z}_p$ with discrepancy D_N we have*

$$\left| \int_{\mathbb{Z}_p} f(x)\, d\mu(x) - \frac{1}{N}\sum_{n=1}^{N} f(x_n) \right| \leq D_N \sum_{\substack{\zeta \in \mathbb{Z}(p^\infty) \\ \zeta \neq 1}} \|\zeta\| \left| \hat{f}(\zeta) \right|. \tag{1.5}$$

Corollary 1.3. *Let $\zeta \in \mathbb{Z}(p^\infty)$ and $\{x_1, ..., x_N\}$ be N points in $\mathbb{Z}_p$ with discrepancy D_N, then*

$$\left| \frac{1}{N}\sum_{n=1}^{N} \zeta^{x_n} \right| \leq \|\zeta\| D_N.$$

In [1], Beer derives a Koksma inequality in $\mathbb{Z}_p$ based on a notion of bounded variation. We discuss this result in detail in Chapter 7, and compare it with Theorem 1.11.

1.4 Outline of dissertation

We start with an introduction to p-adic fields in Chapter 2. We define the p-adic absolute value, the p-adic field $\mathbb{Q}_p$, and the p-adic unit ball $\mathbb{Z}_p$ which is a compact topological group that is the primary set over which we work. We state and prove some elementary theorems on the topology of $\mathbb{Z}_p$. These theorems would make the reading of the succeeding chapters meaningful.

In Chapter 3, we develop the Fourier analytic tools that are necessary for investigating the distribution of sequences in $\mathbb{Z}_p$. In particular, we calculate the Pontryagin

Dual group of $\mathbb{Z}_p$. We also explicitly calculate the Fourier series representation of the characteristic function of a disk in $\mathbb{Z}_p$. Also included in this chapter is a discussion of the Dirichlet and Fejér kernels and the convergence of Fourier series in $\mathbb{Z}_p$. However, this latter part is not crucial for further reading.

In Chapter 4, we discuss Definition 1.3 and 1.4 in more detail and prove a version of Weyl's criterion for continuous functions using the density of trigonometric polynomials (the criterion also holds for a more general class of Riemann integrable functions).

The proof of Theorem 1.10 is given in Chapter 5, using the Fourier analysis established in Chapter 3.

The behavior of the linear sequence $\{x_n\} = \{na+b\}$ in $\mathbb{Z}_p$ is analyzed in Chapter 6. We give a Fourier analytic proof that this sequence is equidistributed in $\mathbb{Z}_p$ if and only if a is a unit in $\mathbb{Z}_p$. We also use Theorem 1.10 to find an upper bound on the discrepancy of this sequence.

In Chapter 7, we discuss some existing Koksma inequalities in both $\mathbb{R}/\mathbb{Z}$ and $\mathbb{Z}_p$ based on notions of functions of bounded variation. We also derive a Fourier analytic Koksma inequality in $\mathbb{Z}_p$, as given by Theorem 1.11.

2 p-adic Fields

In this chapter, we give a brief introduction to the p-adic fields and state some theorems that are useful. The p-adic numbers were first explicitly described by Kurt Hensel in 1897. Since then, the concept has grown to become an important aspect of modern number theory, including playing an important role in Wiles' proof of Fermat's Last Theorem. Applications have grown to such fields as mathematical physics and quantum mechanics (see for example [20] and the review article by Dragovich [6]). Two good elementary introductions to the topic are the books by Gouvêa [12] and Katok [13]. The results stated in this chapter can be found in Katok [13].

2.1 p-adic Absolute value

In number theory, it is of interest to measure the divisibility of rational numbers by a fixed prime number p. To this end, we define the p-adic absolute value on $\mathbb{Q}$ as follows.

Definition 2.1 (p-adic absolute value)**.** Suppose that $x \in \mathbb{Q}$, $x \neq 0$, and $x = p^r \frac{a}{b}$ where a, b, and r are integers with $p \nmid ab$. Then

$$|x|_p = p^{-r}.$$

If $x = 0$, we take $|x|_p = 0$.

Example 2.1 helps illustrate the definition.

Example 2.1. Let $p = 3$.

i. $|18|_3 = \left|2 \times 3^2\right|_3 = \dfrac{1}{9}$

ii. $|1/9|_3 = 9$

iii. $\left|\dfrac{18}{5}\right|_3 = \left|3^2 \dfrac{2}{5}\right|_3 = \dfrac{1}{9}$

iv. $|7|_3 = \left|3^0 \times 7\right|_3 = 1$

v. $\left|\dfrac{162}{45}\right|_3 = \left|\dfrac{2 \times 3^4}{5 \times 3^2}\right|_3 = \left|3^2 \times \dfrac{2}{5}\right|_3 = \dfrac{1}{9}$.

An important implication of Definition 2.1 is that the p-adic absolute value $|\cdot|_p$ takes on only discrete values of the form $p^k, k \in \mathbb{Z}$ or 0. Suppose that $x = \dfrac{s}{t} \in \mathbb{Q}$. Then $|x|_p$ is small when the numerator s is highly divisible by p. On the other hand, if the denominator t is highly divisible by p then $|x|_p$ is large. In particular, we have that $\lim_{k\to\infty} |p^k|_p = 0$, and $|1/p^k|_p$ grows without bound as k gets larger and larger. If p does not divide either s or t then $|x|_p = p^0 = 1$.

2.2 The field $\mathbb{Q}_p$

Definition 2.2 (Norm on a field F)**.** Let F be a field. Suppose that $\|\cdot\| : F \longrightarrow [0, \infty)$ satisfies the following properties.

1. $\|x\| = 0$ if and only if $x = 0$.
2. $\|xy\| = \|x\|\|y\|$, for all $x, y \in F$.
3. $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in F$.

Then, $\|\cdot\|$ is called a norm on F, and F is said to be a normed field.

Definition 2.3 (non-Archimedean norm)**.** A norm $\|\cdot\|$ on a field F is called non-Archimedean if it satisfies the additional property

$$\|x + y\| \leq \max(\|x\|, \|y\|). \tag{2.1}$$

Remark 2.1. The property given by (2.1) is called the strong triangle inequality, because it clearly implies the triangle inequality.

Proposition 2.1 (Strongest wins property)**.** *If F is a non-Archimedean field and $x, y \in F$ with $\|x\| < \|y\|$, then $\|x + y\| = \|y\|$.*

Proof. First, note that $\|x\| < \|y\|$ implies that $\max(\|x+y\|, \|x\|) = \|x+y\|$. For otherwise, we would have $\|y\| = \|x+y-x\| \leq \max(\|x+y\|, \|x\|) = \|x\|$ which contradicts our assumption that $\|x\| < \|y\|$.

Therefore, we have $\|y\| = \|x+y-x\| \leq \max(\|x+y\|, \|x\|) = \|x+y\| \leq \max(\|x\|, \|y\|) = \|y\|$. Hence, $\|x+y\| = \|y\|$. □

Proposition 2.2. *The p-adic absolute value $|\cdot|_p$ is a non-Archimedean norm on $\mathbb{Q}$.*

Proof. Suppose that x, y are in $\mathbb{Q}$ with $x = p^r \frac{a}{b}$, $y = p^s \frac{c}{d}$, with $p \nmid ab$ and $p \nmid cd$. Then $|x|_p = p^{-r}$ and $|y|_p = p^{-s}$.

First, we verify property 2 in Definition 2.2. We have $xy = p^{(r+s)} \frac{ac}{bd}$ and $p \nmid ac$, $p \nmid bd$. Hence, $|xy|_p = p^{-(r+s)} = |x|_p |y|_p$.

Next, we show that $|\cdot|_p$ satisfies the strong triangle inequality as given in (2.1) and the triangle inequality property in Definition 2.2 would follow. Without loss of generality assume that $s \leq r$, and thus $|x|_p \leq |y|_p$. We have

$$\begin{aligned} |x+y|_p &= |p^r \frac{a}{b} + p^s \frac{c}{d}|_p \\ &= |p^s (p^{r-s} \frac{a}{b} + \frac{c}{d})|_p \\ &= |p^s|_p |(\frac{p^{r-s} ad + bc}{bd})|_p \\ &\leq |p^s|_p, \end{aligned} \tag{2.2}$$

where the last line in (2.2) follows from the fact that $p \nmid bd$. Hence, $|x+y|_p \leq |y|_p = \max(|x|_p, |y|_p)$. By symmetry we would have $|x+y|_p \leq |x|_p = \max(|x|_p, |y|_p)$ if $r \leq s$. □

Suppose that F is a normed field with norm $\|\cdot\|$, that is not necessarily complete. Then F can be completed with respect to $\|\cdot\|$, to produce a new field $\hat{F}$. This is done by identifying all Cauchy sequences in F under an equivalence relation $\sim$, with two Cauchy sequences $\{x_n\}$ and $\{y_n\}$ being related under $\sim$ if $\|x_n - y_n\| \to 0$ as $n \to \infty$.

Then the set of all equivalence classes under this equivalence relation is $\hat{F}$.

Moreover, $\hat{F}$ has a norm induced by $\|\cdot\|$. For any $x \in F$,

$$\|x\| = \lim_{n\to\infty} \|a_n\|,$$

where $\{a_n\}$ is any Cauchy sequence belonging to the equivalence class x in $\hat{F}$. In addition, F can be identified with a dense subset of $\hat{F}$.

The real line $\mathbb{R}$ can be viewed as the completion of the rationals $\mathbb{Q}$ under the standard absolute value as the norm. It can be shown that $\mathbb{Q}$ is not complete with respect to the p-adic norm. The completion of $\mathbb{Q}$ with respect to $|\cdot|_p$ is denoted by $\mathbb{Q}_p$. The p-adic unit ball $\mathbb{Z}_p$ is the set

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}.$$

Every nonzero $x \in \mathbb{Q}_p$ has a unique canonical expansion of the form

$$x = \sum_{n=k}^{\infty} a_n p^n, \tag{2.3}$$

where k is an integer, the coefficients a_n belong to the set $\{0, 1, 2, ..., p-1\}$ and $a_k \neq 0$ [13]. The series in (2.3) diverges with respect to the ordinary absolute value, but converges in the p-adic norm.

Proposition 2.3. *Suppose that a nonzero $x \in \mathbb{Q}_p$ has the canonical expansion $x = a_k p^k + a_{k+1} p^{k+1} + ...$, for some $k \in \mathbb{Z}$, with $a_k \neq 0$. In other words, a_k is the first non-zero coefficient in the canonical expansion of x. Then $|x|_p = p^{-k}$.*

Proof. Consider the sequence $x_1 = a_k p^k$, $x_2 = a_k p^k + a_{k+1} p^{k+1}$, $x_3 = a_k p^k + a_{k+1} p^{k+1} + a_{k+2} p^{k+2}$,.... in $\mathbb{Q}$. That is, $\{x_n\}$ is the sequence of partial sums appearing in the canonical expansion of x. The sequence is Cauchy in $\mathbb{Q}$ since for any $m > n$ we have $|x_m - x_n|_p \leq 1/p^{n+1}$ which goes to zero as n, m go to infinity. By the strongest wins property, $|x_n|_p = p^{-k}$ for all n. Moreover, the sequence x_n belongs to the equivalence class represented by x and so

$$|x|_p = \lim_{n\to\infty} |x_n|_p = p^{-k}.$$

□

Remark 2.2. An important consequence of Proposition 2.3 is that the induced norm $|\cdot|_p$ on $\mathbb{Q}_p - \{0\}$ takes the same values as $|\cdot|_p$ does on $\mathbb{Q} - \{0\}$, on the discrete set $\{p^k, k \in \mathbb{Z}\}$.

Since $|x|_p \leq 1$ for all $x \in \mathbb{Z}_p$, every x in $\mathbb{Z}_p$ has a unique canonical expansion of the form

$$x = \sum_{n=0}^{\infty} a_n p^n, \tag{2.4}$$

where the a_n belong to the set $\{0, 1, 2, ..., p-1\}$. Therefore, $\mathbb{Z}_p$ can be viewed as the set of all canonical expansions in the non-negative powers of p

$$\mathbb{Z}_p = \left\{a_0 + a_1 p + a_2 p^2 + ... \,|\, a_n \in \{0, 1, 2, ..., p-1\}\right\}.$$

If $a_n = 0$ for $n = 0, 1, ..., m-1$ and $a_m \neq 0$, then

$$|x|_p = \frac{1}{p^m}.$$

In particular, if $a_0 \neq 0$ then $|x|_p = 1$.

It follows from the definition of $\mathbb{Z}_p$ that $\mathbb{Z} \subset \mathbb{Z}_p$. Moreover, the canonical expansions given by (2.4) implies that $\mathbb{Z}$ is densely embedded in $\mathbb{Z}_p$.

Proposition 2.4. *The p-adic unit ball $\mathbb{Z}_p$ is an abelian group under addition and and a ring under multiplication.*

Proof. The strong triangle inequality ensures that $\mathbb{Z}_p$ is closed under addition, making it an abelian group under addition. The multiplicative property (which is property 2 in Definition 2.2) ensures that $\mathbb{Z}_p$ is closed under multiplication. Hence, $\mathbb{Z}_p$ is a ring under multiplication. $\square$

Due to the algebraic structure given by Proposition 2.4 and the canonical expansions given by (2.4), $\mathbb{Z}_p$ is called the **ring of p-adic integers**.

By $\mathbb{Z}_p^\times$, we denote the set of all the elements of $\mathbb{Z}_p$ whose first term in the canonical expansion is non-zero. That is,

$$\mathbb{Z}_p^\times = \{a_0 + a_1 p + a_2 p^2 + ... \mid a_0 \neq 0\}.$$

Alternately,

$$\mathbb{Z}_p^\times = \{x \in \mathbb{Q}_p \mid |x|_p = 1\}.$$

The set of units in $\mathbb{Z}_p$ is precisely $\mathbb{Z}_p^\times$. As such, $\mathbb{Z}_p^\times$ is a multiplicative group.

2.3 The topology of $\mathbb{Q}_p$ and $\mathbb{Z}_p$

The p-adic absolute value naturally induces a metric $d(x, y)$ on $\mathbb{Q}_p$ such that $d(x, y) = |x - y|_p$. In this section, we shall look at the metric topology on $\mathbb{Q}_p$. The strong triangle inequality leads to a topology on $\mathbb{Q}_p$ that is very different from $\mathbb{R}$. In particular, $\mathbb{Q}_p$ is a totally disconnected space, and topologically $\mathbb{Z}_p$ is homeomorphic to a Cantor set. The details are given in [13].

Definition 2.4 (Disks and Circles)**.** Let $a \in \mathbb{Q}_p$ and $r > 0$ with $r = p^k$ for $k \in \mathbb{Z}$. By a disk centered at a with radius r in $\mathbb{Q}_p$, we mean the set

$$D(a, r) = \{x \in \mathbb{Q}_p \mid |x - a|_p \leq r\}.$$

By a circle centered at a with radius r in $\mathbb{Q}_p$, we mean the set

$$S(a, r) = \{x \in \mathbb{Q}_p \mid |x - a|_p = r\}.$$

We state and prove the following propositions, which are useful for our analysis.

Proposition 2.5. *Consider the disk $D(a, r)$ and suppose $b \in D(a, r)$. Then $D(b, r) = D(a, r)$. That is, every point of a disk may also play the role of a center.*

Proof. Suppose $c \in D(a, r)$. Then

$$\begin{aligned} |b - c|_p &= |b - a + a - c|_p \\ &\leq \max(|b - a|_p, |a - c|_p) \\ &\leq \max(r, r) \\ &= r. \end{aligned}$$

So, $c \in D(b, r)$. This means $D(a, r) \subseteq D(b, r)$. The result follows by symmetry. $\square$

Proposition 2.6. *Suppose $D(a,r) \cap D(b,s) \neq \emptyset$. Then either $D(a,r) \subseteq D(b,s)$ or $D(b,s) \subseteq D(a,r)$. That is, if two disks intersect then one contains the other.*

Proof. Without loss of generality assume that $r \leq s$, and let $c \in D(a,r) \cap D(b,s)$. Then using Proposition 2.5 we have $D(c,r) = D(a,r)$ and $D(c,s) = D(b,s)$. Therefore, $D(a,r) = D(c,r) \subseteq D(c,s) = D(b,s)$. □

Proposition 2.7. *A disk $D(a,r)$ in $\mathbb{Q}_p$ is both open and closed.*

Proof. We only need to show that $D(a,r)$ is open. Note that $r = p^k$, for some $k \in \mathbb{Z}$. So,

$$\begin{aligned} D(a,r) &= \{x \in \mathbb{Z}_p \mid |x-a|_p \leq p^k\} \\ &= \{x \in \mathbb{Z}_p \mid |x-a|_p < p^{k+1}\}, \end{aligned} \tag{2.5}$$

and the set in the last line of (2.5) is open by definition. □

Proposition 2.8. *The characteristic function $\mathcal{X}_{D(a,r)}(x)$ of a disk $D(a,r)$ in $\mathbb{Q}_p$ is continuous.*

Proof. Suppose that $x \in \mathbb{Q}_p$. If x is in $D(a,r)$, then for any $\epsilon > 0$, $D(x,r) = D(a,r)$ is an open neighborhood of x such that $|\mathcal{X}_{D(a,r)}(x) - \mathcal{X}_{D(a,r)}(y)| = 0 < \epsilon$, for all y in $D(x,r)$. On the other hand, suppose $x \notin D(a,r)$. Then pick an $s < r$, and observe that $D(x,s) \cap D(a,r) = \{\emptyset\}$ (otherwise, by Proposition 2.6 we will have $D(x,s) \subseteq D(a,r)$ which is not possible). Then for all y in the open neighborhood $D(x,s)$, we have $|\mathcal{X}_{D(a,r)}(x) - \mathcal{X}_{D(a,r)}(y)| = 0 < \epsilon$. □

Proposition 2.8 is surprising at first glance since the analogous function in $\mathbb{R}$ has jump discontinuities.

Proposition 2.9. *The set of all disks in $\mathbb{Q}_p$ is countable.*

Proof. This follows from the facts that the radii are discretely valued, the rational numbers are dense in $\mathbb{Q}_p$, and Proposition 2.5. □

Proposition 2.10. *For each $k \geq 0$, $\mathbb{Z}_p$ can be expressed as the disjoint union of p^k disks of radius $1/p^k$.*

Proof. Consider the integers $0, 1, 2, ..., p^k - 1$. Each of them are at a p-adic distance greater than $1/p^k$ from each other. Therefore, the disks $D(j, 1/p^k)$ for $j = 0, 1, 2, ..., p^{k-1}$ are disjoint. Then note that any element of $\mathbb{Z}_p$ can be written as $x = j + p^k y$, where $j \in \{0, 1, 2, ..., p^k - 1\}$ and $y \in \mathbb{Z}_p$. Hence, we have

$$\mathbb{Z}_p = \bigcup_{j=0}^{p^k-1} D(j, 1/p^k).$$

□

Proposition 2.11. *$\mathbb{Z}_p$ is compact.*

Proof. We follow Katok [13], and show that $\mathbb{Z}_p$ is sequentially compact. Let $\{x_n\}$ be a sequence in $\mathbb{Z}_p$, and suppose that each element of $\{x_n\}$ has canonical expansions of the form

$$\begin{aligned}
x_1 &= a_0^1 + a_1^1 p + a_2^1 p^2 + a_3^1 p^3 + \cdots \\
x_2 &= a_0^2 + a_1^2 p + a_2^2 p^2 + a_3^2 p^3 + \cdots \\
x_3 &= a_0^3 + a_1^3 p + a_2^3 p^2 + a_3^3 p^3 + \cdots \\
&\vdots
\end{aligned}$$

Since each a_i^j belongs to the finite set $\{0, 1, 2, ..., p - 1\}$, we can find a subsequence $\{x_{0k}\}$ of $\{x_k\}$ such that each element of x_{0k} has the same first coefficient b_0 in its canonical expansion. That is,

$$\begin{aligned}
x_{01} &= b_0 + a_1^{01} p + a_2^{01} p^2 + a_3^{01} p^3 + \cdots \\
x_{02} &= b_0 + a_1^{02} p + a_2^{02} p^2 + a_3^{02} p^3 + \cdots \\
x_{03} &= b_0 + a_1^{03} p + a_2^{03} p^2 + a_3^{03} p^3 + \cdots \\
&\vdots
\end{aligned}$$

We can now repeat this procedure to the sequence $\{x_{0k}\}$ to obtain a new sequence $\{x_{1k}\}$ such that the first two coefficients b_0 and b_1 match. Then the next sequence

$\{x_{2k}\}$ is a subsequence of $\{x_{1k}\}$ such that the first three coefficients b_0, b_1, and b_2 match and so on. We have a sequence of such sequences

$$\begin{aligned}
\{x_{0k}\} &= \{x_{00}, x_{01}, x_{02}, x_{03}, \cdots\} \\
\{x_{1k}\} &= \{x_{10}, x_{11}, x_{12}, x_{13}, \cdots\} \\
\{x_{2k}\} &= \{x_{20}, x_{21}, x_{22}, x_{23}, \cdots\} \\
&\vdots
\end{aligned}$$

Now pick the diagonal elements $\{x_{00}, x_{11}, x_{22}, ...\}$. This is a subsequence of the original sequence $\{x_n\}$ that converges to the point with canonical expansion $b_0 + b_1 p + b_2 p^2 +$ ...... $\square$

Proposition 2.12. *$\mathbb{Z}_p$ is a compact abelian group.*

Proof. This follows immediately from Propositions 2.4 and 2.11. $\square$

3 Fourier analysis on $\mathbb{Z}_p$

3.1 Harmonic analysis on groups

We begin with a brief discussion of Harmonic analysis on locally compact Hausdorff abelian groups (LCA groups). We state some of the basic theorems that help us develop the tools necessary for the succeeding chapters. We follow Rudin [21] closely.

Theorem 3.1 (Haar Measure). *Let G be a locally compact Hausdorff abelian group with group operation written additively. Then there exists a positive regular Borel measure μ on G, the Haar measure, that is translation invariant. That is,*

$$\mu(E + x) = \mu(E),$$

for every x in G and every Borel set E in G.

Remark 3.1. The Haar measure is unique up to scaling. That is if μ and ν are Haar measures on G, then $\nu = c\mu$, for some $c \in \mathbb{R}^+$.

Remark 3.2. If G is compact, then we have a normalized Haar measure such that $\mu(G) = 1$.

The usual construction of this measure is done by finding a positive linear functional T on $C_c(G)$, the space of continuous functions with compact support on G, with the property that T is invariant under precomposition by each translation operator $f(x) \mapsto f(x + a)$ on $C_c(G)$. The Reisz representation theorem then guarantees the existence of a measure μ such that $T(f) = \int_G f \, d\mu$, for every f in $C_c(G)$. The details can be found in some standard texts on real analysis, for example Folland [10].

Let $\mathbb{T} = \{z \in \mathbb{C} \,|\, |z| = 1\}$ be the circle group under multiplication. A continuous group homomorphism $\gamma : G \to \mathbb{T}$ is called a character of G. The set of all characters of G form a group called the Pontryagin dual group.

Proposition 3.1 (Dual group)**.** *Let G be an LCA group, and let $\hat{G}$ be the set of all characters of G. Then $\hat{G}$ is an abelian group, with the group operation $\cdot$ defined by*

$$(\gamma_1 \cdot \gamma_2)(x) = \gamma_1(x)\gamma_2(x),$$

for all x in G.

Note that the identity element of $\hat{G}$ (also called the trivial character of G) is given by $\gamma_0(x) = 1$, for all $x \in G$.

We have the following orthogonality condition.

Theorem 3.2. *Let G be a compact abelian group with normalized Haar measure μ, and let $\gamma \in \hat{G}$. Then*

$$\int_G \gamma(x)\, d\mu(x) = \begin{cases} 1 & \text{if } \gamma \text{ is trivial} \\ 0 & \text{else.} \end{cases}$$

Proof. Clearly, $\displaystyle\int_G \gamma_0(x)\, d\mu(x) = \mu(G) = 1$, as μ is the normalized Haar measure. Suppose that γ is not trivial. Then there exists some $x_0 \in G$ such that $\gamma(x_0) \neq 1$. Then,

$$\begin{aligned} \int_G \gamma(x)\, d\mu(x) &= \int_G \gamma(x + x_0)\, d\mu(x) \\ &= \gamma(x_0) \int_G \gamma(x)\, d\mu(x), \end{aligned} \tag{3.1}$$

where the first line of (3.1) follows from the translation invariance of the Haar measure. Since $\gamma(x_0) \neq 1$, the result follows. □

Definition 3.1 (Fourier Transform)**.** Let G be an LCA group. For any $f \in L^1(G)$ define the map

$$\begin{aligned} \hat{f} : \ \hat{G} &\longrightarrow \mathbb{C} \\ \gamma &\longmapsto \textstyle\int_G f(x)\gamma(-x)\, d\mu(x). \end{aligned}$$

The map $\hat{f}$ is called the Fourier transform of f.

Let $A(\hat{G})$ be the collection of all Fourier transforms as given in Definition 3.1. We then endow $\hat{G}$ with the weak topology induced by $A(\hat{G})$. That is, it is the weakest topology on $\hat{G}$ that makes each $\hat{f} \in A(\hat{G})$ continuous. We have the following theorem.

Theorem 3.3. *If G is compact, then the weak topology on $\hat{G}$ is discrete.*

Proof. We follow Rudin [21]. Define $f : G \to \mathbb{C}$ by $f(x) = 1$ for every x in G. Since G is compact, f is in $L^1(G)$. Let γ_0 denote the trivial character. Then using the orthogonality condition in Theorem 3.2, $\hat{f}(\gamma_0) = 1$ and $\hat{f}(\gamma)$ is zero otherwise. With $\hat{G}$ endowed with the weak topology, $\hat{f}$ is continuous. Therefore, $\hat{f}^{-1}(1)$ is an open set in $\hat{G}$ containing γ_0 alone. So, $\hat{G}$ is discrete. □

For any two functions $f, g : G \longrightarrow \mathbb{C}$, we define their convolution by

$$(f * g)(x) = \int_G f(x - y)g(y)\, d\mu(y), \tag{3.2}$$

provided the integral in (3.2) exists. In particular, if f and g are in $L^1(G)$ then (3.2) holds for almost all $x \in G$ and $(f * g)(x) = (g * f)(x)$. Moreover, for every $\gamma \in \hat{G}$ the Fourier transform of $f * g$ is $\hat{f}\hat{g}$.

The weak topology on $\hat{G}$ makes $\hat{G}$ an LCA group, with a Haar measure $\hat{\mu}$. This measure could be appropriately scaled so that the following theorem holds (see [21]).

Theorem 3.4 (Plancherel Theorem)**.** *Let $f \in (L^1 \cap L^2)(G)$. Then the map $f \mapsto \hat{f}$ is an isometry (with respect to the L^2 norms), from $(L^1 \cap L^2)(G)$ onto a dense linear subspace of $L^2(\hat{G})$. That is,*

$$\int_G |f(x)|^2\, d\mu(x) = \int_{\hat{G}} |\hat{f}(\gamma)|^2\, d\hat{\mu}(\gamma).$$

Moreover, this isometry may be uniquely extended to an isometry of $L^2(G)$ onto $L^2(\hat{G})$.

Remark 3.3. For the purpose of this dissertation, we consider $f \in (L^1 \cap L^2)(G)$ where G is compact. Hence $\hat{G}$ is discrete, and the Haar measure on $\hat{G}$ is the counting measure [21]. In this case, Theorem 3.4 is a version of Parseval's theorem as seen in elementary Fourier analysis [22].

3.2 Fourier series on $\mathbb{R}/\mathbb{Z}$

In order to emphasize the analogy between $\mathbb{R}/\mathbb{Z}$ and $\mathbb{Z}_p$ we first recall classical Fourier analysis on the circle $\mathbb{R}/\mathbb{Z}$. Some good references on this topic are the books by Dym and McKean [7] and Stein and Shakarchi [22].

Theorem 3.5. *For any $n \in \mathbb{Z}$, the map $\zeta_n : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{T}$ given by*

$$\begin{array}{rccl} \zeta_n : & \mathbb{R}/\mathbb{Z} & \longrightarrow & \mathbb{C} \\ & x & \longmapsto & e^{2\pi i n x}, \end{array}$$

is a character of $\mathbb{R}/\mathbb{Z}$. Moreover, the map $n \mapsto \zeta_n$ is an isomorphism between $\mathbb{Z}$ and the dual group of $\mathbb{R}/\mathbb{Z}$.

Let $f \in L^1(\mathbb{R}/\mathbb{Z})$. Then the Fourier coefficients of f are given by

$$\hat{f}(n) = \int_{\mathbb{R}/\mathbb{Z}} f(x) e^{-2\pi i n x}\, dx,$$

where we have identified ζ_n with n by Theorem 3.5. If in addition f is continuous and $\sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty$, then for all x in $\mathbb{R}/\mathbb{Z}$ we can express $f(x)$ as the Fourier series

$$f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{2\pi i n x}. \tag{3.3}$$

Remark 3.4. The pointwise convergence of the Fourier series to $f(x)$ is a rather intricate issue [7, 22]. In particular, if the assumption of absolute summability of the Fourier coefficients is relaxed, then the continuity of f at a point x is no guarantee for the convergence of its Fourier series to $f(x)$.

3.3 Convolution kernels

In classical Fourier analysis on $\mathbb{R}/\mathbb{Z}$, the N^{th} order Dirichlet kernel is given by

$$D_N(x) = \sum_{n=-(N-1)}^{N-1} e^{2\pi i n x}.$$

Let f be in $L^1(\mathbb{R}/\mathbb{Z})$. The convolution of the Dirichlet kernel with f gives the N^{th} partial Fourier sum

$$\begin{aligned} S_N(x) &= (f * D_N)(x) \\ &= \sum_{n=-(N-1)}^{N-1} \hat{f}(n) e^{2\pi i n x}, \end{aligned} \tag{3.4}$$

where $\hat{f}(n)$ is the n^{th} Fourier coefficient of f. The Fejér kernel is

$$F_N(x) = \frac{1}{N} \sum_{j=0}^{N-1} D_j(x).$$

The convolution of f with F_N gives the Cesaro means of the partial sums

$$f * F_N = \frac{S_0 + S_1 + ... + S_{N-1}}{N}. \tag{3.5}$$

Following Stein-Shakarchi [22], a sequence of functions $\{K_N\}$ in $\mathbb{R}/\mathbb{Z}$ that satisfy the conditions in the following definition are said to be good kernels because $(f * K_N)(x)$ converges to $f(x)$ at any point where f is continuous.

Definition 3.2 (Good kernels in $\mathbb{R}/\mathbb{Z}$)**.** A family of functions $\{K_N\}$ in $\mathbb{R}/\mathbb{Z}$ is said to be a family of good kernels if the following conditions are satisfied

i. For all $N \geq 1$

$$\int_0^1 K_N(x) dx = 1.$$

ii. There exists an $M > 0$ such that for all $N \geq 1$,

$$\int_0^1 |K_N(x)| dx \leq M.$$

iii. For every $0 < \delta < 1/2$,

$$\lim_{N \to \infty} \int_{\delta < x \leq 1/2} |K_N(x)| dx = 0.$$

Theorem 3.6. *Let $\{K_N\}$ be a family of good kernels, and f be a bounded function in $L^1(\mathbb{R}/\mathbb{Z})$. Then*

$$\lim_{N\to\infty}(K_N * f)(x) = f(x),$$

whenever f is continuous at x. If f is continuous everywhere, then the convergence is uniform.

For a proof of Theorem 3.6 see [22]. The Dirichlet kernels are not a family of good kernels in $\mathbb{R}/\mathbb{Z}$ because they violate condition (ii) in Definition 3.2, validating the comments we made in Remark 3.4. Examples can be found to show that the partial sum in (3.4) is not guaranteed to converge to f pointwise, even at a point where f is continuous. A Lipschitz condition on f is enough to guarantee convergence [22]. In particular, Stein-Shakarchi has a construction of an everywhere continuous function whose Fourier series diverges at a point (see Page 83 of [22]). On the other hand, the Fejér kernels are a family of good kernels. Therefore, the Cesaro means in (3.5) converges to $f(x)$ at all points x where f is continuous.

3.4 p-adic Haar measure and a change of variables formula

By Proposition 2.12, $\mathbb{Z}_p$ is a compact abelian group and hence there exists a normalized Haar measure μ on $\mathbb{Z}_p$ such that $\mu(\mathbb{Z}_p) = 1$ (see for example [10]).

Lemma 3.1 (Measure of a disk)**.** *The μ-measure of a disk $D(a, r)$ in $\mathbb{Z}_p$ centered at a with radius r is equal to its radius r.*

Proof. Let $k \geq 0$. By Proposition 2.10, $\mathbb{Z}_p$ can be written as the disjoint union of p^k disks $\mathbb{Z}_p = \cup_{j=0}^{p^k-1} D(j, 1/p^k)$. Hence,

$$\begin{aligned}
1 &= \mu(\mathbb{Z}_p) \\
&= \sum_{j=0}^{p^k-1} \mu(D(j, 1/p^k)) \\
&= p^k \mu(D(0, 1/p^k)), \qquad (3.6)
\end{aligned}$$

where the last line of (3.6) follows from the translation invariance of the Haar measure μ. Therefore, we conclude that $\mu(D(a,1/p^k)) = \mu(D(0,1/p^k)) = 1/p^k$. $\square$

Note that the measure of a circle $S(a,1/p^k)$ is $\mu(S(a,1/p^k)) = 1/p^k - 1/p^{k+1}$, since $S(a,1/p^k)$ can be written as the difference of the two disks $D(a,1/p^k)$ and $D(a,1/p^{k+1})$.

Lemma 3.2. *Let E be a Borel subset of $\mathbb{Z}_p$, and $k \geq 0$. Then $\mu(p^k E) = \frac{1}{p^k}\mu(E)$.*

Proof. For every Borel subset E of $\mathbb{Z}_p$, define $\nu(E) = \mu(\alpha E)$ where $\alpha \in \mathbb{Z}_p$ is a fixed non-zero constant. It is easily seen that ν is a translation invariant measure on $\mathbb{Z}_p$ since

$$\nu(E+x) = \mu(\alpha E + \alpha x) = \mu(\alpha E) = \nu(E).$$

By the uniqueness of Haar measure up to scaling, we have that $\nu = c\mu$, for some constant $c \in \mathbb{R}^+$. Suppose we take α to be p^k. Then using Lemma 3.1

$$\nu(\mathbb{Z}_p) = \mu(p^k\mathbb{Z}_p) = 1/p^k,$$

and we have $c = 1/p^k$, so that $\nu = \frac{1}{p^k}\mu$. $\square$

Notation 3.1. We denote by $\mathcal{X}_E(x)$ the characteristic function of a subset E of $\mathbb{Z}_p$.

We have the following change of variables formula.

Proposition 3.2. *Let $a \in \mathbb{Z}_p$ and $k \geq 0$, and let $f : \mathbb{Z}_p \to \mathbb{C}$ be an integrable function. Then*

$$\int_{\mathbb{Z}_p} \mathcal{X}_{D(a,1/p^k)}(x) f(x)\, d\mu(x) = \frac{1}{p^k}\int_{\mathbb{Z}_p} f(a+p^k x)\, d\mu(x).$$

Proof. First we show that the above formula is valid for any characteristic function $\mathcal{X}_E(x)$ where E is measurable. Let $D = D(a,1/p^k)$, since $\mathcal{X}_D(x)\mathcal{X}_E(x) = \mathcal{X}_{D\cap E}(x)$ it follows that

$$\int_{\mathbb{Z}_p} \mathcal{X}_D(x)\mathcal{X}_E(x)\, d\mu(x) = \mu(D \cap E).$$

On the other hand, define

$$A = \{x \in \mathbb{Z}_p \,|\, a + p^k x \in E\}.$$

Then $a+p^kA = D \cap E$, and $\frac{1}{p^k}\mu(A) = \mu(p^kA) = \mu(a+p^kA) = \mu(D\cap E)$ using Lemma 3.2. In particular,

$$\begin{aligned}\int_{\mathbb{Z}_p} \mathcal{X}_E(a+p^kx)\,d\mu(x) &= \mu(A)\\ &= p^k\mu(D\cap E)\\ &= p^k\int_{\mathbb{Z}_p} \mathcal{X}_D(x)\mathcal{X}_E(x)\,d\mu(x).\end{aligned}$$

Since the result is true for characteristic functions, the general case follows by an approximation argument and the dominated convergence theorem. □

3.5 Dual group of $\mathbb{Z}_p$

As we have described in section 3.1, if G is a compact abelian group then the set of all continuous group homomorphisms (or characters) from G to the multiplicative unit circle $\mathbb{T} = \{z \in \mathbb{C} \,|\, |z| = 1\}$ forms a discrete group under multiplication, the Pontryagin dual group $\widehat{G}$. Hence, by Proposition 2.12, $\mathbb{Z}_p$ has a corresponding dual group $\widehat{\mathbb{Z}}_p$ which we calculate in this section.

The Prüfer p-group

$$\mathbb{Z}(p^\infty) = \left\{e^{2\pi i \frac{m}{p^n}} \mid m, n \in \mathbb{N}\cup\{0\}, p \nmid m\right\},$$

is the group of all p-th power roots of unity in $\mathbb{C}$. Recall Notation 1.2 from Chapter 1 for the meaning of ζ^x. Suppose that $\zeta \in \mathbb{Z}(p^\infty)$ has order p^n, and let $x \in \mathbb{Z}_p$ have the canonical expansion $x = a_0 + a_1p + a_2p^2 + ... + a_{n-1}p^{n-1} + ...$, then we interpret the notation ζ^x as

$$\zeta^x = \zeta^{a_0+a_1p+a_2p^2+....+a_{n-1}p^{n-1}}.$$

Lemma 3.3. *Let $\zeta, \omega \in \mathbb{Z}(p^\infty)$. Then for any x, y in $\mathbb{Z}_p$, we have*

$$\zeta^{x+y} = \zeta^x\zeta^y,$$

and

$$(\zeta\omega)^x = \zeta^x\omega^x.$$

Proof. Suppose that the order of ζ is $\|\zeta\| = p^n$. Let x and y be in $\mathbb{Z}_p$. The unique canonical expansions of x and y in $\mathbb{Z}_p$ allows us to write $x = m + p^n s$, $y = r + p^n t$ for $m, r \in \mathbb{N} \cup \{0\}$ and $s, t \in \mathbb{Z}_p$. Then note that by the definition of ζ^x we have $\zeta^x = \zeta^{m+p^n s} = \zeta^m$, and similarly $\zeta^y = \zeta^r$. Therefore,

$$\begin{aligned} \zeta^{(x+y)} &= \zeta^{m+p^n s+r+p^n t} \\ &= \zeta^{m+r+p^n(s+t)} \\ &= \zeta^{m+r} \\ &= \zeta^m\zeta^r \\ &= \zeta^x\zeta^y. \end{aligned}$$

This proves the first assertion. To prove the second assertion, let ζ have order p^n and ω have order p^q. Without loss of generality assume that $n \leq q$, and thus $\zeta\omega$ has order dividing p^q. Write $x = m + p^n s = l + p^q t$, for some $0 \leq m < p^n$ and $0 \leq l < p^q$. Then,

$$\begin{aligned} (\zeta\omega)^x &= (\zeta\omega)^{l+p^q t} \\ &= (\zeta\omega)^l \\ &= \zeta^l\omega^l \\ &= \zeta^x\omega^x, \end{aligned}$$

where we have used the fact that $l \equiv m \mod p^n$. □

The next lemma states that the dual group $\widehat{\mathbb{Z}}_p$ of $\mathbb{Z}_p$ is isomorphic to the Prüfer p-group $\mathbb{Z}(p^\infty)$. The result is known, but we include a proof due to the lack of a suitable reference. We use some ideas from Conrad's calculation of the character group of $\mathbb{Q}_p$ in [4].

Lemma 3.4. *For each $\zeta \in \mathbb{Z}(p^\infty)$, the map $x \mapsto \zeta^x$ is a character of $\mathbb{Z}_p$. Moreover, the map*

$$\Psi : \mathbb{Z}(p^\infty) \longrightarrow \widehat{\mathbb{Z}}_p$$

$$\zeta \quad \longmapsto \quad (x \mapsto \zeta^x)$$

is an isomorphism from the Prüfer p-group $\mathbb{Z}(p^\infty)$ to the Pontryagin dual group of $\mathbb{Z}_p$.

Proof. It follows from Lemma 3.3 that for each $\zeta \in \mathbb{Z}(p^\infty)$ the map $x \mapsto \zeta^x$ is a character of $\mathbb{Z}_p$, and that Ψ is a group homomorphism. To show the injectivity of Ψ, suppose that $\zeta_1^x = \zeta_2^x$ for all x in $\mathbb{Z}_p$. Then picking $x = 1$, we get $\zeta_1 = \zeta_2$.

We need to argue that Ψ is surjective. Let γ be in the dual group of $\mathbb{Z}_p$. Since $\gamma(0) = 1$ and γ is continuous, there exists a disk $D = p^n\mathbb{Z}_p$ of radius $1/p^n$ centered at zero, such that $|\gamma(x) - \gamma(0)| < 1$ for all x in D, and we can pick a smallest n such that this is true. Moreover, since D is a subgroup of $\mathbb{Z}_p$ we must have that the image $\gamma(D)$ is a subgroup of $\mathbb{T}$.

Note that there does not exist any non-trivial subgroup of $\mathbb{T}$ satisfying the condition $|x - y| < 1$ for all elements x and y in the subgroup. Hence, we conclude that $\gamma(D) = \{1\}$.

Now suppose that $\gamma(1) = \zeta = e^{2\pi i\theta}$ for some θ in $[0, 1)$. Then, $1 = \gamma(p^n) = \gamma(1)^{p^n} = e^{2\pi i p^n\theta}$ and thus $p^n\theta \in \mathbb{Z}$. We conclude that $\theta = \frac{m}{p^n}$, for $m \in \mathbb{Z}$ where $p \nmid m$ by the minimality of n.

For any integer k, we have $\gamma(k) = \gamma(1)^k = \zeta^k$. This completely determines γ, since we can write $\mathbb{Z}_p$ as the union of p^n disjoint balls $\mathbb{Z}_p = \cup_{k=0}^{p^n-1}(k + p^n\mathbb{Z}_p)$ and for any $x \in \mathbb{Z}_p$ we have $x = k + p^n y$, for some $0 \le k < p^n$ and $y \in \mathbb{Z}_p$, and hence $\gamma(x) = \gamma(k)\gamma(p^n y) = \gamma(k)$ because $\gamma \equiv 1$ on $D(0, 1/p^n)$. We conclude that

$$\gamma(x) = \zeta^x,$$

for all x in $\mathbb{Z}_p$. $\square$

Using Lemma 3.4, we shall express the Fourier series of any $f \in L^1(\mathbb{Z}_p)$ in terms of the elements of $\mathbb{Z}(p^\infty)$. Let $f \in L^1(\mathbb{Z}_p)$. For each $\zeta \in \mathbb{Z}(p^\infty)$, the ζ^{th} Fourier coefficient of f is given by

$$\hat{f}(\zeta) = \int_{\mathbb{Z}^p} f(x)\zeta^{-x}\, d\mu(x).$$

The Fourier inversion formula gives

$$f(x) = \sum_{\zeta\in\mathbb{Z}(p^\infty)} \hat{f}(\zeta)\zeta^x, \tag{3.7}$$

whenever $\sum_{\zeta\in\mathbb{Z}(p^\infty)} |\hat{f}(\zeta)| < \infty$ and f is continuous. As we show in Corollary 3.2, this condition can be relaxed in $\mathbb{Z}_p$ so that 3.7 holds for all continuous functions f.

3.6 Fourier coefficients of a disk in $\mathbb{Z}_p$

We use Proposition 3.2 to calculate the Fourier coefficients of the characteristic function of a disk.

Lemma 3.5. *Let $a \in \mathbb{Z}_p$, $k \geq 0$, and $\zeta \in \mathbb{Z}(p^\infty)$. Then*

$$\widehat{\mathcal{X}}_{D(a,1/p^k)}(\zeta) = \begin{cases} \zeta^{-a}p^{-k} & \text{if } \|\zeta\| \leq p^k, \\ \\ 0 & \text{if } \|\zeta\| > p^k. \end{cases}$$

Proof. Suppose that $\|\zeta\| \leq p^k$, then $\zeta^{p^k x} = 1$ for all x in $\mathbb{Z}_p$. Therefore, we have

$$\begin{aligned} \int_{\mathbb{Z}_p} \mathcal{X}_{D(a,1/p^k)}(x)\zeta^{-x}\, d\mu(x) &= p^{-k}\int_{\mathbb{Z}_p} \zeta^{-(a+p^k x)}\, d\mu(x) \\ &= \zeta^{-a}p^{-k}\int_{\mathbb{Z}_p} \zeta^{-p^k x}\, d\mu(x) \\ &= \zeta^{-a}p^{-k}. \end{aligned}$$

On the other hand suppose $\|\zeta\| > p^k$, and let $\omega = \zeta^{p^k}$. Then $\|\omega\| = \|\zeta\|/p^k > 1$ and hence

$$\int_{\mathbb{Z}_p} \mathcal{X}_{D(a,1/p^k)}(x)\zeta^{-x}\, d\mu(x) = \zeta^{-a}p^{-k}\int_{\mathbb{Z}_p} \zeta^{-p^k x}\, d\mu(x)$$

$$\begin{aligned} &= \zeta^{-a} p^{-k} \int_{\mathbb{Z}_p} \omega^{-x}\, d\mu(x) \\ &= 0, \end{aligned}$$

using the orthogonality of characters. □

Corollary 3.1. *Let $a \in \mathbb{Z}_p$ and $k \geq 0$. Then*

$$\mathcal{X}_{D(a,1/p^k)}(x) = p^{-k} \sum_{\|\zeta\| \leq p^k} \zeta^{-a} \zeta^{x}.$$

Proof. Let $f(x) = \mathcal{X}_{D(a,1/p^k)}(x)$, and $S_k(x) = p^{-k} \sum_{\|\zeta\| \leq p^k} \zeta^{-a}\zeta^{x}$. First, we claim that

$$\int_{\mathbb{Z}_p} |f(x) - S_k(x)|^2 \, d\mu(x) = 0. \tag{3.8}$$

Then, the result would follow since (3.8) would imply that $f(x) = S_k(x)$ at any point where f is continuous, and we know by Proposition 2.8 that the characteristic function of a disk is continuous in $\mathbb{Z}_p$. Note that

$$\int_{\mathbb{Z}_p} |f(x) - S_k(x)|^2 \, d\mu(x) = \int_{\mathbb{Z}_p} |f(x)|^2 \, d\mu(x) + \int_{\mathbb{Z}_p} |S_k(x)|^2 \, d\mu(x) - 2 \int_{\mathbb{Z}_p} f(x)\overline{S_k(x)} \, d\mu(x).$$

The orthogonality of characters gives us

$$\begin{aligned} \int_{\mathbb{Z}_p} |S_k(x)|^2 \, d\mu(x) &= \frac{1}{p^{2k}} \sum_{\|\zeta\| \leq p^k} 1 \\ &= \frac{1}{p^k}. \end{aligned}$$

Now since $\hat{f}(\zeta) = p^{-k}\zeta^{-a}$, we have

$$\begin{aligned} \int_{\mathbb{Z}_p} f(x)\overline{S_k(x)} \, d\mu(x) &= \sum_{\|\zeta\| \leq p^k} p^{-k} \zeta^{a} \int_{\mathbb{Z}_p} f(x) \zeta^{-x} \, d\mu(x) \\ &= \frac{1}{p^{2k}} \sum_{\|\zeta\| \leq p^k} 1 \\ &= \frac{1}{p^k}. \end{aligned}$$

Finally, note that $\int_{\mathbb{Z}_p} |f(x)|^2 \, d\mu(x) = 1/p^k$, using the fact that the measure of a disk is equal to its radius. This proves (3.8). □

3.7 The Dirichlet and Fejér kernels in $\mathbb{Z}_p$

In this section, we look at the convergence properties of Fourier series in $\mathbb{Z}_p$. After we proved the following results on the convergence of Fourier series using p-adic analogues of Dirichlet and Fejér kernels, we learned that similar results had been obtained by Taibleson for the ring of formal power series over the finite field $\mathbb{F}_p$ [23], which, like $\mathbb{Z}_p$, is the ring of integers in a local field.

Analogous to the circle, we define the Dirichlet and Fejér kernels on $\mathbb{Z}_p$ as follows

Definition 3.3. The N^{th} order Dirichlet kernel on $\mathbb{Z}_p$ is defined to be

$$D_N(x) = \sum_{\|\zeta\| \leq p^N} \zeta^x$$

and the N^{th} order Fejér kernel on $\mathbb{Z}_p$ is

$$F_N(x) = \frac{1}{N} \sum_{k=0}^{N-1} D_k(x),$$

where $\zeta \in \mathbb{Z}(p^\infty)$.

Note that if $f \in L^1(\mathbb{Z}_p)$, then

$$(f * D_N)(x) = \sum_{\|\zeta\| \leq p^N} \hat{f}(\zeta)\zeta^x,$$

which is the N^{th} partial Fourier sum. Similarly, $f * F_N$ gives the N^{th} Cesaro means,

$$(f * F_N)(x) = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{\|\zeta\| \leq p^k} \hat{f}(\zeta)\zeta^x.$$

An alternate expression for the Dirichlet kernel can be derived using Corollary 3.1, for $N \geq 0$ we have

$$\mathcal{X}_{D(0;1/p^N)}(x) = p^{-N} \sum_{\|\zeta\| \leq p^N} \zeta^x,$$

and hence

$$D_N(x) = p^N \mathcal{X}_{D(0;1/p^N)}(x).$$

That is, the Dirichlet kernel of order N is the scaled characteristic function of a disk of radius p^{-N} centered at zero. From this, we also get the following expression for the Fejér kernel

$$F_N(x) = \frac{1}{N}\sum_{k=0}^{N-1} p^k \mathcal{X}_{D(0;1/p^k)}(x).$$

This expression helps us visualize the Fejér kernel in $\mathbb{Z}_p$ as a **'wedding cake'**. From this we have

$$f * F_N(x) = \frac{1}{N}\sum_{k=0}^{N-1} p^k f * \mathcal{X}_{D(0;1/p^k)}(x). \tag{3.9}$$

Motivated by Definition 3.2 in $\mathbb{R}/\mathbb{Z}$, we define analogous notions of good kernels in $\mathbb{Z}_p$.

Definition 3.4 (Good kernels in $\mathbb{Z}_p$)**.** A family of functions $\{K_N\}$ in $\mathbb{Z}_p$ is said to be a family of good kernels if the following conditions are satisfied

i. For all $N \geq 1$

$$\int_{\mathbb{Z}_p} K_N(x)\, d\mu(x) = 1.$$

ii. There exists an $M > 0$ such that for all $N \geq 1$,

$$\int_{\mathbb{Z}_p} |K_N(x)|\, d\mu(x) \leq M.$$

iii. For every $k > 0$,

$$\lim_{N\to\infty} \int_{\mathbb{Z}_p - D(0;1/p^k)} |K_N(x)|\, d\mu(x) = 0.$$

Definition 3.5 (Excellent kernels in $\mathbb{Z}_p$)**.** A family of good kernels $\{K_N\}$ in $\mathbb{Z}_p$ is said to be excellent kernels if they satisfy the following condition, which is stronger than condition (iii) in Definition 3.4. For every $k \geq 1$, we have

$$\lim_{N\to\infty} \sup_{x\in\mathbb{Z}_p - D(0,1/p^k)} |K_N(x)| = 0. \tag{3.10}$$

Theorem 3.7. *Let $\{K_N\}$ be a family of good kernels, and $f \in L^1(\mathbb{Z}_p)$ be a bounded function. Then*

$$\lim_{N\to\infty} (K_N * f)(x) = f(x),$$

whenever f is continuous at x. If f is continuous everywhere, then the convergence is uniform. If the family of kernels $\{K_N\}$ is excellent, then the same conclusions hold with the boundedness assumption on f removed.

Proof. Since f is bounded, there exists a B such that $|f(x)| \leq B$ for all x in $\mathbb{Z}_p$. Note that if f is continuous at x, then given an $\epsilon > 0$, we can find an k large enough such that for all $|y|_p \leq 1/p^k$, we have $|f(x-y) - f(x)| < \frac{\epsilon}{2M}$. We have

$$\begin{aligned}
|f * K_N(x) - f(x)| &= \left| \int_{\mathbb{Z}_p} f(x-y) K_N(y)\, d\mu(y) - f(x) \right| \\
&= \left| \int_{\mathbb{Z}_p} f(x-y) K_N(y)\, d\mu(y) - \int_{\mathbb{Z}_p} f(x) K_N(y)\, d\mu(y) \right| \\
&= \left| \int_{\mathbb{Z}_p} (f(x-y) - f(x)) K_N(y) d\mu(y) \right| \\
&\leq \left| \int_{D(0;1/p^k)} (f(x-y) - f(x)) K_N(y) d\mu(y) \right| \\
&\quad + \left| \int_{\mathbb{Z}_p - D(0;1/p^k)} (f(x-y) - f(x)) K_N(y) d\mu(y) \right| \\
&\leq \frac{\epsilon}{2} + 2B \int_{\mathbb{Z}_p - D(0;1/p^k)} |K_N(y)|\, d\mu(y). \qquad (3.11)
\end{aligned}$$

Now pick an L such that the integral on the right hand side is less than $\frac{\epsilon}{4B}$ for all $N > L$. This gives the desired result. If f is continuous everywhere on $\mathbb{Z}_p$, then it is uniformly continuous since $\mathbb{Z}_p$ is compact. Therefore, our pick of k, and hence L can be made independent of x giving us uniform convergence. Now suppose that $\{K_N\}$ is a family of excellent kernels, and assume that f is not necessarily bounded. Then the last line of the estimate 3.11 can be replaced by

$$|f * K_N(x) - f(x)| \leq \frac{\epsilon}{2} + 2\|f\|_1 \sup_{y \in \mathbb{Z}_p - D(0;1/p^k)} |K_N(y)|.$$

The remainder of the proof follows analogously. $\square$

Theorem 3.8. *The Dirichlet and Fejér kernels are a family of excellent kernels in $\mathbb{Z}_p$.*

Proof. First, consider the Dirichlet kernel. Writing $D_N(x) = p^N \mathcal{X}_{D(0,1/p^N)}(x)$ we get

$$\begin{aligned}\int_{\mathbb{Z}_p} D_N(x)\, d\mu(x) &= p^N \int_{\mathbb{Z}_p} \mathcal{X}_{D(0,1/p^N)}(x)\, d\mu(x) \\ &= 1.\end{aligned}$$

Since $D_N(x)$ is non-negative, condition (ii) of Definition 3.4 is also satisfied.

Given a ball of radius $1/p^k$ centered at zero, for all $N > k$ we have $D(0, 1/p^N) \subset D(0, 1/p^k)$ and hence

$$\begin{aligned}\lim_{N\to\infty} \sup_{\mathbb{Z}_p - D(0;1/p^k)} |D_N(x)| &= \lim_{N\to\infty} p^N \sup_{\mathbb{Z}_p - D(0;1/p^k)} \mathcal{X}_{D(0,1/p^N)}(x) \\ &= 0,\end{aligned}$$

which shows the condition in 3.10. For the Fejér kernel we have

$$\begin{aligned}\int_{\mathbb{Z}_p} F_N(x)\, d\mu(x) &= \frac{1}{N}\sum_{j=0}^{N-1} \int_{\mathbb{Z}_p} D_j(x)\, d\mu(x) \\ &= 1.\end{aligned}$$

Since each D_j is non-negative, F_N is non-negative and hence condition (ii) for a good kernel is also satisfied. Finally we need to check the condition of Definition 3.5. Given ball $D(0, 1/p^k)$ we have

$$\begin{aligned}\sup_{\mathbb{Z}_p - D(0;1/p^k)} |F_N(x)| &= \frac{1}{N}\sum_{j=0}^{N-1} \sup_{\mathbb{Z}_p - D(0;1/p^k)} D_j(x) \\ &= \frac{1}{N}\sum_{j=0}^{N-1} p^j \sup_{\mathbb{Z}_p - D(0;1/p^k)} \mathcal{X}_{D(0,1/p^j)}(x) \\ &= \frac{1}{N}\sum_{j=0}^{k-1} p^j \sup_{\mathbb{Z}_p - D(0;1/p^k)} \mathcal{X}_{D(0,1/p^j)}(x).\end{aligned}$$

Letting N go to infinity, the term on the right hand side goes to zero. □

Corollary 3.2. *The following hold*

1. *Let $f : \mathbb{Z}_p \to \mathbb{C}$ be any integrable function. Then the Fourier series of f converges to f at any point where f is continuous. Similarly, the Cesaro means also converge to f at any point of continuity.*

2. *If f is continuous on $\mathbb{Z}_p$, then its Fourier series converges uniformly to f on $\mathbb{Z}_p$. Similarly for the Cesaro means.*

3. *Consider the set $C(\mathbb{Z}_p)$ of continuous functions from $\mathbb{Z}_p$ to $\mathbb{C}$ with the sup metric. Denote by Φ the set of trigonometric polynomials. That is Φ is the set of all finite linear combinations of the form*

$$\Phi = \{\, a_1\zeta_1^x + ... + a_n\zeta_n^x \,|\, a_i \in \mathbb{C}, \zeta_i \in \mathbb{Z}(p^\infty) \,\}\,.$$

   Then Φ is dense in $C(\mathbb{Z}_p)$.

Proof. The statements follow from Theorems 3.7 and 3.8. $\square$

Note that (3) in Corollary 3.2 is also implied by the Stone-Weierstrass theorem.

4 Equidistribution theory on $\mathbb{Z}_p$

Equidistribution of sequences on the ring of p-adic integers was previously studied in [1, 2, 5]. In particular, Cugiani in [5] defines equidistribution and shows that the sequence $na+b$ is equidistributed if a is a unit. Beer does a quantitative analysis in [1] and [2]. Our aim is to derive a LeVeque-type inequality on the discrepancy of a finite sequence using Fourier analysis. Recall from Proposition 2.10 that $\mathbb{Z}_p$ can be written as the union of p^k disjoint disks of the form

$$\mathbb{Z}_p = \bigcup_{j=0}^{p^k-1} D(j, 1/p^k).$$

Hence, it is natural to define a notion of equidistribution using such sets.

Definition 4.1. A sequence $\{x_n\}$ is said to be equidistributed in $\mathbb{Z}_p$ if for every a in $\mathbb{Z}_p$ and every $k \in \mathbb{N}$, we have

$$\lim_{N\to\infty} \frac{\left|D(a, 1/p^k) \cap \{x_1, ..., x_N\}\right|}{N} = \frac{1}{p^k}.$$

That is, the proportion of the first N elements of $\{x_n\}$ lying in a disk $D(a, 1/p^k)$ is equal to its measure in the limit of large N, and this holds true for all such disks.

This definition of equidistribution in $\mathbb{Z}_p$ was first given by Cugiani in [5], where Propositions 4.1 and 6.1 were also proved. The details are also given in Kuipers and Niederreiter [14]. One also wants to measure how well a sequence distributes itself. To this end, we define the notion of discrepancy to quantify the idea that some sequences are better equidistributed than others.

Definition 4.2. The discrepancy of a finite sequence $\{x_1, x_2, ..., x_N\}$ in $\mathbb{Z}_p$ is

$$D_N = \sup_{a\in\mathbb{Z}_p,\, k\in\mathbb{N}} \left| \frac{\left|D(a, 1/p^k) \cap \{x_1, ..., x_N\}\right|}{N} - \frac{1}{p^k} \right|.$$

Some elementary arguments show that

$$\frac{1}{N} \le D_N \le 1.$$

The main aim of this dissertation is to prove a Fourier analytic upper bound on the discrepancy of a set of N elements $\{x_1, x_2, ..., x_N\}$ in $\mathbb{Z}_p$.

We begin with the observation that a Weyl type criterion holds for equidistribution in $\mathbb{Z}_p$.

Proposition 4.1 (p-adic Weyl's Criterion)**.** *A sequence $\{x_n\}$ is equidistributed in $\mathbb{Z}_p$ if and only if for every non-trivial ζ in $\mathbb{Z}(p^\infty)$ we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \zeta^{x_n} = 0.$$

Proof. First, we show the forward implication. Note that from the definition of equidistribution, for any disk $D(a, 1/p^k)$ we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mathcal{X}_{D(a,1/p^k)}(x_n) = p^{-k}.$$

For every $\zeta \in \mathbb{Z}(p^\infty)$, the character ζ^x is locally constant on $\mathbb{Z}_p$. That is, ζ^x can be written as a finite linear combination of characteristic functions of disks

$$\zeta^x = \sum_{i=1}^{\|\zeta\|} \zeta^i \mathcal{X}_{D(i,1/\|\zeta\|)}(x). \tag{4.1}$$

This follows from the definition of ζ^x as given in Chapter 3. Indeed, if ζ has order $\|\zeta\| = p^n$, then the value of ζ^x depends only on $x \mod p^n$. Therefore, ζ^x is constant on each disk of radius $1/\|\zeta\|$, and (4.1) holds.

Also, using the orthogonality of characters for any $\zeta \neq 1$ we have

$$\begin{aligned} 0 &= \int_{\mathbb{Z}_p} \zeta^x \, d\mu \\ &= \sum_{i=1}^{\|\zeta\|} \zeta^i \int_{\mathbb{Z}_p} \mathcal{X}_{D(i,1/\|\zeta\|)}(x)\, d\mu \\ &= \frac{1}{\|\zeta\|}\sum_{i=1}^{\|\zeta\|} \zeta^i. \end{aligned} \tag{4.2}$$

Hence,

$$\frac{1}{N}\sum_{n=1}^{N} \zeta^{x_n} = \frac{1}{N}\sum_{n=1}^{N}\sum_{i=1}^{\|\zeta\|} \zeta^i \mathcal{X}_{D(i,1/\|\zeta\|)}(x_n)$$

$$
\begin{aligned}
&= \frac{1}{N}\sum_{i=1}^{\|\zeta\|}\zeta^i\sum_{n=1}^{N}\mathcal{X}_{D(i,1/\|\zeta\|)}(x_n) \\
&= \sum_{i=1}^{\|\zeta\|}\zeta^i\left(\frac{1}{N}\sum_{n=1}^{N}\mathcal{X}_{D(i,1/\|\zeta\|)}(x_n)-\frac{1}{\|\zeta\|}\right), \qquad (4.3)
\end{aligned}
$$

where the last line follows by making use of (4.2). Taking the limit as N goes to infinity the term in the last line of (4.3) goes to zero, by the definition of equidistribution.

To prove the reverse implication, assume that $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\zeta^{x_n}=0$. Note that for any disk $D(a,1/p^k)$, its characteristic function $\mathcal{X}_{D(a,1/p^k)}(x)$ can be expressed as a finite Fourier series as we have shown in Corollary 3.1. Therefore,

$$
\begin{aligned}
\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mathcal{X}_{D(a,1/p^k)}(x_n) &= \lim_{N\to\infty}\frac{1}{Np^k}\sum_{n=1}^{N}\sum_{\|\zeta\|\le p^k}\zeta^{-a}\zeta^{x_n} \\
&= \lim_{N\to\infty}\frac{1}{Np^k}\sum_{n=1}^{N}\sum_{1<\|\zeta\|\le p^k}\zeta^{-a}\zeta^{x_n}+\frac{1}{p^k} \\
&= \frac{1}{p^k}\sum_{1<\|\zeta\|\le p^k}\zeta^{-a}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\zeta^{x_n}+\frac{1}{p^k} \\
&= \frac{1}{p^k}, \qquad (4.4)
\end{aligned}
$$

and hence $\{x_n\}$ is equidistributed. □

Proposition 4.1 could be extended to a more general class of Riemann integrable functions on $\mathbb{Z}_p$. The details are given in Kuipers and Niederreiter [14]. We give here instead, a proof for the class of continuous functions $f:\mathbb{Z}_p\to\mathbb{C}$. Our proof is different from the one presented in [14], since we argue using the density of trigonometric polynomials.

Theorem 4.1. *The sequence $\{x_n\}$ is equidistributed in $\mathbb{Z}_p$ if and only if*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}f(x_n)=\int_{\mathbb{Z}_p}f(x)\,d\mu, \qquad (4.5)$$

for all continuous functions $f:\mathbb{Z}_p\to\mathbb{C}$.

Proof. The reverse implication is trivial since characteristic functions of disks are continuous in $\mathbb{Z}_p$.

For the forward implication, assume $\{x_n\}$ is equidistributed. By the p-adic Weyl criterion, (4.5) holds for all characters $f(x) = \zeta^x$ of $\mathbb{Z}_p$. We know from Corollary 3.2 that trigonometric polynomials are dense in $C(\mathbb{Z}_p)$. Since (4.5) holds when $f(x)$ is any character, it also holds for any trigonometric polynomial. Hence, given an $\epsilon > 0$, pick a trigonometric polynomial ψ such that $\|f-\psi\|_\infty < \frac{\epsilon}{2}$. We then have,

$$\begin{aligned}
\left|\int_{\mathbb{Z}_p} f(x)\,d\mu(x) - \frac{1}{N}\sum_{n=1}^{N} f(x_n)\right| &\leq \left|\int_{\mathbb{Z}_p} f(x)\,d\mu(x) - \frac{1}{N}\sum_{n=1}^{N} \psi(x_n)\right| \\
&\quad + \left|\frac{1}{N}\sum_{n=1}^{N} \psi(x_n) - \frac{1}{N}\sum_{n=1}^{N} f(x_n)\right| \\
&\leq \left|\int_{\mathbb{Z}_p} (f-\psi)(x)\,d\mu(x)\right| \\
&\quad + \left|\int_{\mathbb{Z}_p} \psi(x)\,d\mu(x) - \frac{1}{N}\sum_{n=1}^{N} \psi(x_n)\right| \\
&\quad + \left|\frac{1}{N}\sum_{n=1}^{N} (\psi(x_n) - f(x_n))\right| \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} + \left|\int_{\mathbb{Z}_p} \psi(x)\,d\mu(x) - \frac{1}{N}\sum_{n=1}^{N} \psi(x_n)\right|,
\end{aligned}$$

and letting N go to infinity we get

$$\limsup_{N\to\infty}\left|\int f\,d\mu - \frac{1}{N}\sum_{n=1}^{N} f(x_n)\right| \leq \epsilon.$$

Since ϵ was arbitrary this completes the proof.

$\square$

5 A LeVeque-type inequality

We are now ready to prove Theorem 1.10, the main theorem of this dissertation. For the ease of reading, we restate the theorem below.

Theorem 5.1 (LeVeque-type Inequality)**.** *The discrepancy of a finite sequence* $\{x_1, ..., x_N\}$ *in* $\mathbb{Z}_p$ *is bounded by*

$$D_N \leq C(p) \left(\sum_{\zeta \in \mathbb{Z}(p^\infty)\setminus\{1\}} \frac{1}{\|\zeta\|^3} \left| \frac{1}{N} \sum_{n=1}^{N} \zeta^{x_n} \right|^2 \right)^{\frac{1}{4}}, \tag{5.1}$$

where $C(p)$ *is a constant dependent only on* p.

Let $\{x_1, x_2, ..., x_N\}$ be a finite sequence in $\mathbb{Z}_p$. Define the function $f : \mathbb{Z}_p \times \mathbb{Z}_p \longrightarrow \mathbb{R}$

$$f(x, y) = \frac{|\{x_1, x_2, ..., x_N\} \cap D(x, |y|_p)|}{N} - |y|_p,$$

where $D(x, |y|_p)$ is a disk of radius $|y|_p$ centered at x. The discrepancy of the points $\{x_1, ..., x_N\}$ is then

$$D_N = \sup_{x,y\in\mathbb{Z}_p} |f(x, y)|.$$

We suppress the p in $|\cdot|_p$ as it would be clear from the context. We can also write

$$\begin{aligned} f(x, y) &= \frac{1}{N} \left(\sum_{n=1}^{N} \mathcal{X}_{D(x,|y|)}(x_n) \right) - |y| \\ &= \frac{1}{N} \left(\sum_{n=1}^{N} \mathcal{X}_{D(x_n,|y|)}(x) \right) - |y|. \end{aligned}$$

Our proof of Theorem 1.10 proceeds as follows. We shall bound the L^2 norm

$$\|f\|_2^2 = \iint_{\mathbb{Z}_p^2} |f(x, y)|^2 \, d\mu(x) d\mu(y)$$

from below by a constant multiple of D_N^4 using geometrical arguments, and from above by using Theorem 3.4 (Plancherel theorem). The two steps are given below as lemmas.

Lemma 5.1. *The discrepancy D_N is bounded by*

$$D_N^4 \leq C_1(p)\|f\|_2^2,$$

where $C_1(p)$ is a constant dependent on p.

Lemma 5.2. *The L^2 norm of the function f is bounded by*

$$\|f\|_2^2 \leq C_2(p) \sum_{\zeta \in \mathbb{Z}(p^\infty)\setminus\{1\}} \frac{1}{\|\zeta\|^3} \left| \frac{1}{N} \sum_{n=1}^{N} \zeta^{x_n} \right|^2 .$$

where $C_2(p)$ is a constant dependent on p.

The proof of Theorem 1.10 then follows by combining Lemmas 5.1 and 5.2.

Remark 5.1. For $x > 0$, we use the notation $\lfloor x \rfloor$ and $\lceil x \rceil$ to denote

$$\lfloor x \rfloor = \max\{p^k \,|\, k \in \mathbb{Z},\, p^k \leq x\}$$

$$\lceil x \rceil = \min\{p^k \,|\, k \in \mathbb{Z},\, x \leq p^k\}.$$

We consider these to be the floor and ceiling functions on $(0, \infty)$ relative to the value group $p^{\mathbb{Z}}$ of $\mathbb{Q}_p$. Note that $\lfloor x \rfloor \leq x < p\lfloor x \rfloor$ and $\frac{1}{p}\lceil x \rceil < x \leq \lceil x \rceil$.

Proof of Lemma 5.1. Pick a point (x_0, y_0) for which $f(x_0, y_0)$ is not zero. We consider each of the two possibilities $f(x_0, y_0) > 0$ and $f(x_0, y_0) < 0$ separately. Our strategy in each case is to find a small neighborhood around the point (x_0, y_0) where $|f(x, y)|$ is bounded away from zero. Using this fact and integrating over this neighborhood, we produce a bound of the form $\|f\|_2^2 \geq C(p)|f(x_0, y_0)|^4$, where $C(p)$ is a constant depending only on p.

Case 1

Suppose that $\Delta = f(x_0, y_0) > 0$. This case occurs when the disk $D(x_0, |y_0|)$ contains more then the expected number of points x_n.

Let $R = \lfloor \Delta + |y_0| \rfloor$. Since, $|y_0| < |y_0| + \Delta$ and $|y_0|$ is in the value group of $\mathbb{Q}_p$, we have $|y_0| \leq R$. We consider the two cases $|y_0| < R$ and $|y_0| = R$.

Case 1.1:

Suppose that $|y_0| < R$. We must then have $|y_0| \leq \frac{1}{p}R$. If we fix $|y| = \frac{1}{p}R$ and $|x - x_0| \leq \frac{1}{p}R$, then $D(x_0, |y_0|) \subseteq D(x, |y|)$. We get a nonnegative lower bound on $f(x, y)$ as follows

$$
\begin{aligned}
f(x,y) &= \frac{1}{N}\sum_{n=1}^{N} \mathcal{X}_{D(x,|y|)}(x_n) - |y| \\
&\geq \frac{1}{N}\sum_{n=1}^{N} \mathcal{X}_{D(x_0,|y_0|)}(x_n) - |y| \\
&= |y_0| + f(x_0, y_0) - |y| \\
&= |y_0| + \Delta - |y| \\
&\geq \left(1 - \frac{1}{p}\right) R.
\end{aligned}
$$

We can bound the L^2 norm of f from below by evaluating the required integral only on the set $|y| = \frac{1}{p}R, |x - x_0| \leq \frac{1}{p}R$

$$
\begin{aligned}
\|f\|_2^2 &= \iint_{\mathbb{Z}_p^2} |f(x,y)|^2 \, d\mu(x) d\mu(y) \\
&\geq \iint_{|y|=\frac{1}{p}R, |x-x_0|\leq\frac{1}{p}R} |f(x,y)|^2 \, d\mu(x) d\mu(y) \\
&\geq \iint_{|y|=\frac{1}{p}R, |x-x_0|\leq\frac{1}{p}R} \left(1 - \frac{1}{p}\right)^2 R^2 \, d\mu(x) d\mu(y) \\
&= \left(1 - \frac{1}{p}\right)^3 \frac{1}{p^2} R^4 \\
&\geq \left(1 - \frac{1}{p}\right)^3 \frac{1}{p^6} \Delta^4 \\
&= \frac{(p-1)^3}{p^9} \Delta^4,
\end{aligned}
$$

using $R = \lfloor \Delta + |y_0| \rfloor \geq \frac{1}{p}(|y_0| + \Delta) \geq \frac{1}{p}\Delta$.

Case 1.2:

Suppose that $|y_0| = R$. If we let $|y| = R$ and $|x - x_0| \leq R$, then $D(x_0, |y_0|) = D(x, |y|)$. From this, we get

$$\begin{aligned}
f(x, y) &= \frac{1}{N}\sum_{n=1}^{N} \mathcal{X}_{D(x,|y|)}(x_n) - |y| \\
&= \frac{1}{N}\sum_{n=1}^{N} \mathcal{X}_{D(x_0,|y_0|)}(x_n) - |y_0| \\
&= |y_0| + f(x_0, y_0) - |y| \\
&= f(x_0, y_0) \\
&= \Delta.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|f\|_2^2 &= \iint_{\mathbb{Z}_p^2} |f(x, y)|^2 \, d\mu(x) d\mu(y) \\
&\geq \iint_{|y|=R, |x-x_0|\leq R} \Delta^2 \, d\mu(x) d\mu(y) \\
&= \left(1 - \frac{1}{p}\right) R^2 \Delta^2 \\
&\geq \left(1 - \frac{1}{p}\right) \frac{1}{(p-1)^2} \Delta^4 \\
&= \frac{1}{p(p-1)} \Delta^4,
\end{aligned}$$

using $R + \Delta = |y_0| + \Delta < p\lfloor |y_0| + \Delta \rfloor = pR$ and therefore $\Delta < (p - 1)R$.

Finally, since $\frac{(p-1)^3}{p^9} < \frac{1}{p(p-1)}$ we conclude

$$\|f\|^2 \geq \frac{(p-1)^3}{p^9} \Delta^4$$

holds in both cases 1.1 and 1.2, so it holds in general for case 1.

Case 2:

Suppose that $f(x_0, y_0) < 0$ and $\Delta = |f(x_0, y_0)| = -f(x_0, y_0)$. In other words, the disk $D(x_0, |y_0|)$ contains fewer than the expected number of points x_n.

Now let $R = |y_0|$. Then if $|y| = R$ and $|x - x_0| \leq R$, by the strong triangle inequality $D(x, |y|) = D(x_0, |y_0|)$ and we have

$$\begin{aligned} f(x,y) &= \frac{1}{N}\sum_{n=1}^{N} \mathcal{X}_{D(x,|y|)}(x_n) - |y| \\ &= \frac{1}{N}\sum_{n=1}^{N} \mathcal{X}_{D(x_0,|y_0|)}(x_n) - |y_0| \\ &= f(x_0, y_0). \end{aligned}$$

Therefore,

$$\begin{aligned} \|f\|_2^2 &= \iint_{\mathbb{Z}_p^2} |f(x,y)|^2 \, d\mu(x) d\mu(y) \\ &\geq \iint_{|y|=R, |x-x_0|\leq R} |f(x,y)|^2 \, d\mu(x) d\mu(y) \\ &= \iint_{|y|=R, |x-x_0|\leq R} \Delta^2 \, d\mu(x) d\mu(y) \\ &= \left(1 - \frac{1}{p}\right) R^2 \Delta^2 \\ &\geq \left(1 - \frac{1}{p}\right) \Delta^4, \end{aligned}$$

where the last line follows because $\Delta \leq R$. To see this, note that

$$\begin{aligned} \Delta &= -f(x_0, y_0) \\ &= |y_0| - \frac{1}{N}\sum_{n=1}^{N} \mathcal{X}_{D(x_0,|y_0|)}(x_n) \\ &\leq |y_0| \\ &= R. \end{aligned}$$

$\square$

Next, we need to prove Lemma 5.2. Our goal is to find an upper bound on the L^2-norm of $f(x, y)$ using Theorem 3.4. Suppose $f(x, y)$ has a Fourier series in two p-adic variables given by

$$f(x,y) = \sum_{\zeta,\omega \in \mathbb{Z}(p^\infty)} \hat{f}(\zeta, \omega) \zeta^x \omega^y.$$

With $f \in (L^1 \cap L^2)(\mathbb{Z}_p^2)$, the Plancherel Theorem gives us

$$\|f\|_2^2 = \sum_{\zeta,\omega \in \mathbb{Z}(p^\infty)} |\hat{f}(\zeta,\omega)|^2.$$

Therefore, we need to bound the Fourier coefficients of $f(x,y)$. The Fourier coefficients are

$$\begin{aligned}
\hat{f}(\zeta,\omega) &= \iint_{\mathbb{Z}_p^2} f(x,y)\, \zeta^{-x}\omega^{-y}\ d\mu(x)d\mu(y) \\
&= \frac{1}{N}\sum_{n=1}^{N} \iint_{\mathbb{Z}_p^2} \mathcal{X}_{D(x_n,|y|)}(x)\, \zeta^{-x}\omega^{-y}\ d\mu(x)d\mu(y) \\
&\quad - \iint_{\mathbb{Z}_p^2} |y|\, \zeta^{-x}\omega^{-y}\ d\mu(x)d\mu(y).
\end{aligned} \tag{5.2}$$

Note that if $\zeta = 1$, we use Theorem 3.2 to obtain

$$\begin{aligned}
\hat{f}(1,\omega) &= \frac{1}{N}\sum_{n=1}^{N} \iint_{\mathbb{Z}_p^2} \mathcal{X}_{D(x_n,|y|)}(x)\, \omega^{-y}\ d\mu(x)d\mu(y)\ -\int_{\mathbb{Z}_p} |y|\, \omega^{-y}\, d\mu(y) \\
&= \frac{1}{N}\sum_{n=1}^{N} \int_{\mathbb{Z}_p} |y|\omega^{-y}\, d\mu(y)\ - \int_{\mathbb{Z}_p} |y|\, \omega^{-y}\, d\mu(y) \\
&= 0.
\end{aligned} \tag{5.3}$$

When $\zeta \neq 1$, it follows from the orthogonality of characters that the second integral in line 2 of (5.2) is zero

$$\begin{aligned}
\iint_{\mathbb{Z}_p^2} |y|\, \zeta^{-x}\omega^{-y}\ d\mu(x)d\mu(y) &= \int_{\mathbb{Z}_p} |y|\omega^{-y} \left(\int_{\mathbb{Z}_p} \zeta^{-x}\, d\mu(x) \right) d\mu(y) \\
&= 0.
\end{aligned}$$

Therefore,

$$\hat{f}(\zeta,\omega) = \frac{1}{N}\sum_{n=1}^{N} \iint_{\mathbb{Z}_p^2} \mathcal{X}_{D(x_n,|y|)}(x)\, \zeta^{-x}\omega^{-y}\ d\mu(x)d\mu(y).$$

Using Lemma 3.5, we have

$$\int_{\mathbb{Z}_p} \mathcal{X}_{D(x_n,|y|)}(x)\zeta^{-x}\, d\mu(x) = \begin{cases} \zeta^{-x_n}\, |y| & \text{if } \|\zeta\| \leq 1/|y|, \\ \\ 0 & \text{else.} \end{cases}$$

Hence, for $\zeta \neq 1$,

$$\hat{f}(\zeta, \omega) = \frac{1}{N} \sum_{n=1}^{N} \zeta^{-x_n} \int_{|y| \leq 1/\|\zeta\|} |y| \, \omega^{-y} \, d\mu(y).$$

The following lemma provides some estimates that are useful in our succeeding calculations.

Lemma 5.3. *Let* $R = 1/p^k, k \in \mathbb{Z}$ *satisfy* $0 < R < 1$, *and let* $\omega \in \mathbb{Z}(p^\infty)$. *Then,*

$$\left| \int_{|y| \leq R} |y| \, \omega^{-y} \, d\mu(y) \right| \leq \frac{p}{\max(1/R, \|\omega\|)^2}. \tag{5.4}$$

Moreover,

$$\sum_{\omega \in \mathbb{Z}(p^\infty)} \left| \int_{|y| \leq R} |y| \, \omega^{-y} \, d\mu(y) \right|^2 \leq 2p^2 R^3. \tag{5.5}$$

Proof of Lemma 5.3. Let $R = 1/p^k$ and let $\|\omega\| = p^l$. We have

$$\begin{aligned} \int_{|y| \leq R} |y| \, \omega^{-y} \, d\mu(y) &= \sum_{j \geq k} \frac{1}{p^j} \int_{|y| = 1/p^j} \omega^{-y} \, d\mu(y) \\ &= \sum_{j \geq k} \frac{1}{p^j} \int_{\mathbb{Z}_p} \left(\mathcal{X}_{D(0,1/p^j)}(y) - \mathcal{X}_{D(0,1/p^{j+1})}(y) \right) \omega^{-y} \, d\mu(y) \end{aligned} \tag{5.6}$$

When $\|\omega\| \leq 1/R$, that is when $l \leq k$, using Lemma 3.5 and (5.6) we have

$$\begin{aligned} \int_{|y| \leq R} |y| \, \omega^{-y} \, d\mu(y) &= \sum_{j \geq k} \frac{1}{p^j} \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right) \\ &= \frac{p}{(p+1)(1/R)^2}. \end{aligned}$$

Thus (5.4) holds in this case. If $\|\omega\| > 1/R$, that is when $l \geq k+1$, again using Lemma 3.5 and (5.6) we have

$$\begin{aligned} \int_{|y| \leq R} |y| \, \omega^{-y} \, d\mu(y) &= \sum_{j \geq l} \frac{1}{p^j} \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right) - \frac{1}{p^{l-1} p^l} \\ &= -\frac{p^2}{(p+1)\|\omega\|^2}, \end{aligned}$$

and thus (5.4) holds in this case as well.

To check (5.5), we use the fact that for each $j \geq 1$, the group $\mathbb{Z}(p^\infty)$ contains p^j elements of order at most p^j, and $p^j - p^{j-1}$ elements of order exactly p^j. We then have

$$\begin{aligned}
\sum_{\omega \in \mathbb{Z}(p^\infty)} \left| \int_{|y| \leq R} |y|\, \omega^{-y}\, d\mu(y) \right|^2 &\leq p^2 \sum_{\omega \in \mathbb{Z}(p^\infty)} \frac{1}{\max(1/R, \|\omega\|)^4} \\
&= p^2 \left(\sum_{\|\omega\| \leq 1/R} R^4 + \sum_{\|\omega\| > 1/R} \frac{1}{\|\omega\|^4} \right) \\
&= p^2 \left(R^3 + \frac{p-1}{p(p^3-1)} R^3 \right) \\
&< 2p^2 R^3.
\end{aligned}$$

$\square$

Finally, we prove Lemma 5.2.

Proof of Lemma 5.2. Applying Theorem 3.4 to $f(x,y)$ and using (5.3) and (5.5), we conclude

$$\begin{aligned}
\|f\|^2 &= \sum_{\substack{\zeta, \omega \in \mathbb{Z}(p^\infty) \\ \zeta \neq 1}} |\hat{f}(\zeta, \omega)|^2 \\
&= \sum_{\substack{\zeta \in \mathbb{Z}(p^\infty) \\ \zeta \neq 1}} \left(\sum_{\omega \in \mathbb{Z}(p^\infty)} \left| \int_{|y| \leq 1/\|\zeta\|} |y| \omega^{-y}\, d\mu(y) \right|^2 \right) \left| \frac{1}{N} \sum_{n=1}^{N} \zeta^{x_n} \right|^2 \\
&\leq 2p^2 \sum_{\substack{\zeta \in \mathbb{Z}(p^\infty) \\ \zeta \neq 1}} \frac{1}{\|\zeta\|^3} \left| \frac{1}{N} \sum_{n=1}^{N} \zeta^{x_n} \right|^2 .
\end{aligned}$$

$\square$

Remark 5.2. The constants appearing in Lemmas 5.1 and 5.2 are $C_1(p) = \frac{p^9}{(p-1)^3}$ and $C_2(p) = 2p^2$. Therefore, the value for the constant $C(p)$ in 5.1 that we obtain from our proof of Theorem 1.10 is

$$C(p) = (C_1(p) C_2(p))^{1/4}$$

$$= \left(\frac{2p^{11}}{(p-1)^3}\right)^{1/4}.$$

We analyze the best possible value of the constant $C(p)$ appearing in Theorem 1.10. Consider the sequence $\{x_n\} = \{0, 0, 0, \ldots\}$, the zero sequence. Substituting this sequence into (5.1) we get

$$D_N \leq C(p) \left(\sum_{\zeta \in \mathbb{Z}(p^\infty) \setminus \{1\}} \frac{1}{\|\zeta\|^3} \right)^{\frac{1}{4}}. \tag{5.7}$$

We can evaluate the sum on the right hand side of (5.7) using the fact that there are exactly $p^r - p^{r-1}$ elements of $\mathbb{Z}(p^\infty)$ of order p^r. Hence,

$$\begin{aligned}
\sum_{\zeta \in \mathbb{Z}(p^\infty) \setminus \{1\}} \frac{1}{\|\zeta\|^3} &= \sum_{1 \leq r < \infty} \frac{(p^r - p^{r-1})}{p^{3r}} \\
&= \left(1 - \frac{1}{p}\right) \sum_{1 \leq r < \infty} \frac{1}{p^{2r}} \\
&= \left(1 - \frac{1}{p}\right) \left(\frac{p^2}{p^2 - 1} - 1\right) \\
&= \frac{1}{p(p+1)}.
\end{aligned}$$

On the other hand, the left hand side of (5.7) is equal to one. This could be seen as follows. Any disk $D(0, 1/p^k)$ centered at 0 of radius $1/p^k$ contains the zero sequence, and therefore

$$\frac{\left|\{x_1, x_2, ..., x_N\} \cap D(0, 1/p^k)\right|}{N} - 1/p^k = 1 - 1/p^k.$$

Letting k go to infinity we get $D_N = 1$, which is the maximum possible value of D_N. We conclude that in Theorem 1.10, the best possible value for $C(p)$ is no smaller than $(p(p+1))^{1/4}$.

6 Linear sequence in $\mathbb{Z}_p$

We have the following proposition on the equidistribution of linear sequences, a proof of which is given in [14] using elementary number theory. We present an alternate proof using Fourier analysis.

Proposition 6.1. *Let $a, b \in \mathbb{Z}_p$. The sequence $x_n = na + b$ is equidistributed in $\mathbb{Z}_p$ if and only if a is a unit in $\mathbb{Z}_p$.*

Proof. The forward implication follows from Weyl's criterion (Proposition 4.1). For suppose, a is not a unit. Then $a = p^k c$, where $k > 0$ and c is a unit. Now let $\zeta = e^{2\pi i/p^k}$. Then

$$\frac{1}{N}\sum_{n=1}^{N} \zeta^{x_n} = \frac{1}{N}\sum_{n=1}^{N} \zeta^{np^k c}\zeta^b = \zeta^b,$$

and Weyl's criterion will not hold.

For the reverse implication, let $\zeta \in \mathbb{Z}(p^\infty)$ with $\|\zeta\| = p^k$ for $k \geq 1$. There exists an m such that $1 \leq m < p^k$, with $p \nmid m$ and $\zeta = e^{2\pi i m/p^k}$. Suppose that a is a unit in $\mathbb{Z}_p$. Let $a = t_0 + t_1 p + t_2 p^2 + \ldots$ be the canonical expansion of a, with $t_0 \neq 0$. Then we let $a_k = t_0 + t_1 p + \ldots + t_{k-1}p^{k-1}$ be the truncation of this expansion to the first k terms. By our hypothesis that a is a unit in $\mathbb{Z}_p$, we know that p does not divide a_k. We have

$$\begin{aligned}
\frac{1}{N}\left|\sum_{n=1}^{N} \zeta^{na+b}\right| &= \frac{1}{N}\left|\sum_{n=1}^{N} \zeta^{na}\right| \\
&= \frac{1}{N}\left|\frac{1-\zeta^{(N+1)a_k}}{1-\zeta^{a_k}}\right| \\
&\leq \frac{1}{N}\frac{2}{|1-\zeta^{a_k}|} \\
&\leq \frac{1}{N}\left|\frac{1}{\sin(\pi m a_k/p^k)}\right|. \qquad (6.1)
\end{aligned}$$

Since $p \nmid m$, and $p \nmid a_k$, we have $\sin(\pi m a_k/p^k) \neq 0$ and hence $\frac{1}{N}\sum_{n=1}^{N} \zeta^{na+b} \to 0$ as $N \to \infty$; the proof of equidistribution now follows from the p-adic Weyl's criterion. $\square$

We now prove Corollary 1.2, that the discrepancy of the sequence $x_n = na + b$ is of the order $D_N = O\left(\frac{1}{\sqrt{N}}\right)$.

Proof of Corollary 1.2. Applying the bound given by Theorem 1.10 we get

$$\begin{aligned}
D_N^4 &\ll \sum_{\zeta \in \mathbb{Z}(p^\infty)\setminus\{1\}} \frac{1}{\|\zeta\|^3} \left| \frac{1}{N} \sum_{n=1}^{N} \zeta^{na+b} \right|^2 \\
&\leq \frac{1}{N^2} \sum_{k=1}^{\infty} \frac{1}{p^{3k}} \sum_{\substack{1 \leq m < p^k \\ p \nmid m}} \frac{1}{|\sin(\pi\, m a_k / p^k)|^2} \\
&\leq \frac{1}{N^2} \sum_{k=1}^{\infty} \frac{1}{p^{3k}} \sum_{1 \leq m < p^k} \frac{1}{|\sin(\pi\, m a_k / p^k)|^2} \\
&\leq \frac{1}{N^2} \sum_{k=1}^{\infty} \frac{1}{p^{3k}} \sum_{1 \leq l < p^k} \frac{1}{|\sin(\pi\, l / p^k)|^2} \\
&\leq \frac{2}{N^2} \sum_{k=1}^{\infty} \frac{1}{p^{3k}} \sum_{1 \leq l \leq p^k/2} \frac{1}{|\sin(\pi\, l / p^k)|^2}. \qquad (6.2)
\end{aligned}$$

Note that the second inequality in (6.2) comes from the last inequality in (6.1). For the fourth inequality, note that since a is a unit we have $p \nmid a_k$. Hence, $\gcd(a_k, p^k) = 1$ and so a_k generates $\mathbb{Z}/p^k\mathbb{Z}$. That is, $\mathbb{Z}/p^k\mathbb{Z} = \{ m a_k \mid m = 0, .., p^k - 1 \}$. The final inequality follows from the identities $|\sin(\theta)| = |\sin(-\theta)| = |\sin(\pi - \theta)|$, so that for $p^k/2 \leq l < p^k$ we have $|\sin(\pi l / p^k)| = |\sin(\pi(p^k - l)/p^k)|$. This allows us to double the sum over the first half of the interval.

Note that in the interval $[0, \pi/2]$, $\sin(\theta)$ is bounded from below by $2\theta/\pi$, so that

$$\frac{1}{|\sin(\theta)|} \leq \frac{\pi}{2\theta}.$$

This gives us

$$\begin{aligned}
\sum_{1 \leq l \leq p^k/2} \frac{1}{|\sin(\pi\, l / p^k)|^2} &\leq \sum_{1 \leq l \leq p^k/2} \frac{p^{2k}}{4 l^2} \\
&\leq \frac{p^{2k}}{4} \sum_{1 \leq l < \infty} \frac{1}{l^2}
\end{aligned}$$

$$\leq \frac{p^{2k}\pi^2}{24}. \tag{6.3}$$

Finally, applying the bound from (6.3) to (6.2) we get

$$D_N^4 \ll \frac{\pi^2}{12N^2}\sum_{k=1}^{\infty}\frac{1}{p^k}.$$

We conclude that $D_N = O\left(\frac{1}{\sqrt{N}}\right)$. □

Remark 6.1. Choosing $a = 1$ in Corollary 1.2, the natural numbers in the usual ordering are equidistributed in $\mathbb{Z}_p$ with discrepancy $D_N = O\left(\frac{1}{\sqrt{N}}\right)$.

7 Bounded variation and Koksma inequalities

The generalized Weyl's criterion Theorem 1.4 (and Theorem 4.1 for the analogous case in $\mathbb{Z}_p$) gives a connection between equidistribution and Riemann integration.

Theorem 7.1. *A sequence $\{x_n\}$ in $\mathbb{R}/\mathbb{Z}$ is equidistributed if and only if for every Riemann integrable function $f : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)\, dx.$$

The implication is that the arithmetic mean of any integrable function f evaluated over the N points of a sequence, converges to the integral of f if and only if the points are equidistributed. Evaluating the integral through such arithmetic means is important in numerical integration, they form the basis of quasi-Monte Carlo methods. Instead of pseudo random sequences, quasi-Monte Carlo methods use deterministic sequences with low discrepancy in the evaluation of integrals. Niederreiter in [19] provides an exposition of quasi-Monte Carlo methods. Morokoff and Caflisch in [17] do an extensive experimental study of the effectiveness of quasi-Monte Carlo methods in evaluating integrals in single and multi-dimensions. Such sequences are now being used in many engineering applications.

Given the discrepancy D_N of a sequence, it is then interesting to study bounds of the form

$$\left| \int_0^1 f(x)\, dx - \frac{1}{N} \sum_{n=1}^{N} f(x_n) \right| \leq C D_N, \tag{7.1}$$

where C is some constant dependent on the analytic properties of the function f. Of course, the analogue in $\mathbb{Z}_p$ is that the integral of interest be over $\mathbb{Z}_p$. The sharpness of the approximation depends on how well the sequence is distributed, which is measured by the discrepancy.

7.1 Koksma inequalities using bounded variation

The classical Koksma inequality involves the variation of a function f to produce a bound of the form (7.1). Let f be a complex-valued function on [0, 1], and recall that the variation of a function is defined to be

$$V(f) = \sup \left\{ \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \;\middle|\; n \geq 1,\, 0 = x_0 < x_1 < .... < x_n = 1 \right\}. \quad (7.2)$$

Thus the supremum ranges over all possible partitions of the interval [0, 1] into a finite collection of subintervals. The function f is said to be of bounded variation if $V(f) < \infty$ (see [10]).

We define the star discrepancy of a finite sequence $\{x_1, x_2, ..., x_N\}$ in $\mathbb{R}/\mathbb{Z}$ to be

$$D_N^* = \sup_{0 \leq b < 1} \left| \frac{|\{x_1, x_2, ..., x_N\} \cap [0,\, b]|}{N} - b \right|.$$

It can be shown that $D_N^* \leq D_N \leq 2D_N^*$, see for example page 2 of [16]. Koksma obtained the following bound on D_N^* for functions of bounded variation.

Theorem 7.2. *Let f be a function of bounded variation, and $\{x_1, .., x_N\}$ be N points on $\mathbb{R}/\mathbb{Z}$ with star discrepancy D_N^*. Then,*

$$\left| \int_0^1 f(x)\, dx - \frac{1}{N} \sum_{n=1}^{N} f(x_n) \right| \leq V(f) D_N^*.$$

Applying Theorem 7.2 to $e^{2\pi i x}$, gives us a bound on the Fourier sum appearing in Weyl's Criterion

$$\left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i x_n} \right| \leq 4D_N^*. \quad (7.3)$$

See [14] for both a proof of Theorem 7.2 and a proof of the inequality 7.3.

Similar notions of bounded variation can be defined for functions $f : \mathbb{Z}_p \to \mathbb{C}$. Beer in her article [1], uses the following definition of variation. The definition can be motivated by the fact that $\mathbb{Z}_p$ can be partitioned in to p^k disjoint disks of radius $1/p^k$ (see Proposition 2.10).

Definition 7.1 (Variation of Beer)**.** Fix an integer $\lambda \geq 1$. Then consider the following dictionary ordering of the integers from 0 to $p^{\lambda}-1$, where we order the points according to their p-adic canonical expansion $a_0 + a_1 p + ... + a_{\lambda-1}p^{\lambda-1}$ as follows

$$\begin{array}{lcl} m_0 & = & 0+0p^1+0p^2+\cdots+0p^{\lambda-2}+0p^{\lambda-1} \\ m_1 & = & 0+0p^1+0p^2+\cdots+0p^{\lambda-2}+1p^{\lambda-1} \\ m_2 & = & 0+0p^1+0p^2+\cdots+0p^{\lambda-2}+2p^{\lambda-1} \\ \vdots & & \\ m_{p-1} & = & 0+0p^1+0p^2+\cdots+0p^{\lambda-2}+(p-1)p^{\lambda-1} \\ & & \\ m_p & = & 0+0p^1+0p^2+\cdots+1p^{\lambda-2}+0p^{\lambda-1} \\ m_{p+1} & = & 0+0p^1+0p^2+\cdots+1p^{\lambda-2}+1p^{\lambda-1} \\ \vdots & & \\ m_{2p-1} & = & 0+0p^1+0p^2+\cdots+1p^{\lambda-2}+(p-1)p^{\lambda-1} \\ & & \\ m_{2p} & = & 0+0p^1+0p^2+\cdots+2p^{\lambda-2}+0p^{\lambda-1} \\ \vdots & & \\ \vdots & & \\ m_{p^{\lambda}-1} & = & (p-1)+(p-1)p+....+(p-1)p^{\lambda-1}=p^{\lambda}-1. \end{array} \tag{7.4}$$

For each $i \in \{0,1,2,...,p^{\lambda}-1\}$, let $E_i = D(m_i, 1/p^{\lambda})$, and for $f : \mathbb{Z}_p \to \mathbb{R}$ define

$$V_{\lambda}(f) = \sup \sum_{i=1}^{p^{\lambda}-1} |f(x_i) - f(x_{i-1})|, \tag{7.5}$$

where the supremum is taken over all choices of points $x_{i-1} \in E_{i-1}, x_i \in E_i$. Then define the Beer variation of f to be

$$V_B(f) = \sup\nolimits_{\lambda} V_{\lambda}(f). \tag{7.6}$$

Remark 7.1. The motivation for the dictionary ordering of Beer can be explained as follows. Let $k \geq 0$ and let $a_0, a_1, ..., a_{k-1}$ be elements of $\{0,1,2,....,p-1\}$. The disk with center $a = a_0 + a_1p + a_2p^2 + ... + a_{k-1}p^{k-1}$ and radius $1/p^k$ in $\mathbb{Z}_p$ can be written as

$$D(a, 1/p^k) = a + p^k\mathbb{Z}_p.$$

That is, $D(a, 1/p^k)$ is the set of all elements $x \in \mathbb{Z}_p$ whose first k terms in the canonical expansion begin with $a_0 + a_1p + a_2p^2 + ... + a_{k-1}p^{k-1}$. Therefore, for $0 \le k \le \lambda$, any disk of radius $1/p^k$ in $\mathbb{Z}_p$ is the union of a block of **consecutive** disks E_i in the dictionary ordered partition of Beer.

As an example of Definition 7.1, consider the case when $p = 3$ and $\lambda = 2$. Then we have the sequence of integers

$$\{m_0, m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8\} = \{0, 3, 6, 1, 4, 7, 2, 5, 8\}.$$

Hence, the corresponding disks are $E_i = D(m_i, 1/9)$. Now consider a finer partition with $p = 3$ and $\lambda = 3$. Then,

$$\begin{aligned}\{m_0, m_1, m_2, ..., m_{26}\} = \{&0, 9, 18,\ \ 3, 12, 21,\ \ 6, 15, 24,\\ &1, 10, 19,\ \ 4, 13, 22,\ \ 7, 16, 25,\\ &2, 11, 20,\ \ 5, 14, 23,\ \ 8, 17, 26\},\end{aligned}$$

with corresponding disks $E_i = D(m_i, 1/27)$.

Example 7.1. Consider the characteristic function $f(x) = \mathcal{X}_{D(a,1/p^k)}(x)$ of a disk $D(a, 1/p^k)$ with $k \ge 1$. Then,

$$V_B(f) = \begin{cases} 1 & \text{if } 0 \text{ or } -1 \text{ is in } D(a, 1/p^k) \\ 2 & \text{else.} \end{cases} \tag{7.7}$$

For each $\lambda \ge 1$, 0 is always contained in the first disk $E_0 = E(0, 1/p^\lambda)$ of the partition, and -1 is always contained in the last disk $E_{p^\lambda-1} = E(p^\lambda - 1, 1/p^\lambda)$. If 0 is in $D(a, 1/p^k)$, then $D(a, 1/p^k)$ is the union of an initial block of disks E_i in the ordered Beer partition. It is then easy to that $V_\lambda(f) = 1$. Similarly, if -1 is in $D(a, 1/p^k)$, then $D(a, 1/p^k)$ is the union of the final block of disks in the Beer partition and

$V_\lambda(f) = 1$. If neither 0 or -1 is in $D(a, 1/p^k)$, then for $\lambda \geq k$, $D(a, 1/p^k)$ is the union of block of disks E_i containing neither E_0 nor $E_{p^\lambda - 1}$. Therefore, $V_\lambda(f) = 2$, and (7.7) follows.

This is very similar to the case of the classical variation of the characteristic function of a proper subinterval $[a, b]$ of $[0, 1]$, which is 1 if $a = 0$ or $b = 1$, and 2 if $0 < a < b < 1$.

Example 7.2. Let $f(x) = |x|_p$. Then $V_B(f) = 1$. To show this, first note that $f(x)$ is constant on every disk in $\mathbb{Z}_p$ that is not centered at 0 (or equivalently does not contain 0).

Fix $\lambda \geq 1$. For the ordered Beer partition associated to λ, we can group the points m_i and the corresponding disks E_i into p blocks, where $|x|_p$ is constant on each block, except the first block which contains only E_0. For example, with $\lambda = 2$, the disk $E_0 = D(0, 1/9)$ is the first block with $|x|_p \leq 1/9$, $E_1 = D(3, 1/9), E_2 = D(6, 1/9)$ form the second block whose elements have constant p-adic norm $|x|_p = 1/3$, and the rest of the E_i form a block with elements of norm 1.

Hence, in calculating $V_B(f)$ the sum occurring in the definition of $V_B(f)$ as given by (7.5) is maximized when $x_0 = 0$, and the choices of $x_1, x_2, ..., x_{p^\lambda - 1}$ in their respective disks E_i are arbitrary. It follows that,

$$\begin{aligned}
V_\lambda(f) &= \sum_{i=1}^{p^\lambda - 1} |f(x_i) - f(x_{i-1})| \\
&= (1/p^{\lambda-1} - 0) + (1/p^{\lambda-2} - 1/p^{\lambda-1}) + ... + (1/p - 1/p^2) + (1 - 1/p) \\
&= 1.
\end{aligned}$$

Taking the sup over all λ, we conclude that $V_B(f) = 1$.

Beer also proves a Koksma inequality in [1] using Definition 7.1, as stated in the next theorem.

Theorem 7.3 (p-adic Koksma inequality of Beer)**.** *Let $f : \mathbb{Z}_p \to \mathbb{R}$ be an integrable* *function. Then,*

$$\left| \int_{\mathbb{Z}_p} f(x)\, d\mu(x) - \frac{1}{N} \sum_{n=1}^{N} f(x_n) \right| \leq 2pV_B(f)D_N.$$

Taibleson in [23], defines an order free notion of variation for functions in a local field. He works with the specific example of the dyadic group 2^ω. Extending this definition to $\mathbb{Z}_p$ would be as follows.

Definition 7.2 (Taibleson variation)**.** Let $f : \mathbb{Z}_p \to \mathbb{R}$, and $\Pi = \{S_k\}_{k=1}^{n}$ be any partition of $\mathbb{Z}_p$ into disks S_k. That is, $S_k \cap S_l = \{\emptyset\}$ for $k \neq l$, and $\mathbb{Z}_p = \cup_{k=1}^{n} S_k$. Let

$$V_k(\Pi) = \sup_{x \in S_k} f(x) - \inf_{x \in S_k} f(x),$$

and

$$V(\Pi) = \sum_k V_k(\Pi).$$

Define the Taibleson variation by $V_T(f) = \sup_\Pi V(\Pi)$. Then f is of bounded Taibleson variation if $V_T(f) < \infty$.

For $f : \mathbb{Z}_p \to \mathbb{C}$, we consider $f(x) = \Re(f)(x) + i\Im(f)(x)$, and let $V_\Re(f) = V_T(\Re(f)(x))$, $V_\Im(f) = V_T(\Im(f)(x))$. Define the Taibleson variation by $V_T(f) = V_\Re(f) + V_\Im(f)$. Then f is of bounded Taibleson variation if $V_T(f) < \infty$.

Example 7.3. The characteristic function of any disk $D(a, 1/p^r)$ with $r > 0$ has Taibleson variation 1. By taking the partition to be just $\Pi = \{\mathbb{Z}_p\}$, we see that $V(\Pi) = 1$. For any other partition $\Pi = \{S_k\}$, if $D(a, 1/p^r)$ is strictly contained in some S_k then $V(\Pi) = 1$. On the other hand, if any S_k is contained in $D(a, 1/p^r)$ then $V(\Pi) = 0$. Therefore, $V_T(D(a, 1/p^r)) = 1$.

Example 7.4. The norm function $f(x) = |x|_p$ also has Taibleson variation 1. By taking the partition to be just $\Pi = \{\mathbb{Z}_p\}$, we see that $V(\Pi) = 1$. For any other partition $\Pi = \{S_k\}$, suppose that S_{k_0} contains 0 and has radius $1/p^r$ (such a disk must exist since 0 must be in some disk). Then f is constant on each $S_k, k \neq k_0$, and it follows that $V(\Pi) = 1/p^r$. We conclude that $V_T(f) = 1$.

Proposition 7.1. *For any $f : \mathbb{Z}_p \to \mathbb{R}$, we have*

$$V_T(f) \leq V_B(f).$$

Proof. Let $\epsilon > 0$ and $\{S_k\}_{k=1}^n$ be a partition into disks as in the Taibleson variation. For each k define α_k and β_k such that $\alpha_k \neq \beta_k$, and

$$\begin{array}{rcl} f(\alpha_k) & \geq & \sup_{x \in S_k} f(x) - \frac{\epsilon}{2^{k+1}} \\ f(\beta_k) & \leq & \inf_{y \in S_k} f(y) + \frac{\epsilon}{2^{k+1}}. \end{array}$$

(Note that the ϵ factors are needed since we cannot guarantee that the extrema are achieved). We produce a corresponding Beer partition as follows. Take a $\lambda \geq 1$ large enough such that all of the disks S_k in the Taibleson partition have radius at least $1/p^\lambda$, and all of the α_k and β_k are distinct modulo p^λ. We can now enumerate the integers $0, 1, 2, ..., p^{\lambda-1}$ in the dictionary ordering of Beer, the ordered set $\{m_0, m_1, m_2...., m_{p^\lambda-1}\}$ along with the corresponding disks $E_i = D(m_i, 1/p^\lambda)$. By our choice of λ, each disk S_k in the Taibleson partition is a union of consecutive disks $E_t, E_{t+1}, ..., E_{t+r}$ in the Beer partition.

Now α_k is in one of these disks, say E_a and β_k is in another, say E_b. Select $x_a = \alpha_k$, $x_b = \beta_k$, and make arbitrary choices for the remaining $x_i \in E_i$, for $i \neq a$ and $i \neq b$. Presuming $a < b$, we have

$$\begin{array}{rcl} f(\alpha_k) - f(\beta_k) & = & f(x_a) - f(x_b) \\ & = & \displaystyle\sum_{a<i\leq b} f(x_{i-1}) - f(x_i) \\ & \leq & \displaystyle\sum_{a<i\leq b} |f(x_{i-1}) - f(x_i)| \\ & \leq & \displaystyle\sum_{t<i\leq t+r} |f(x_{i-1}) - f(x_i)|. \end{array}$$

If instead $b < a$, we still have

$$f(\alpha_k) - f(\beta_k) \leq \sum_{t<i\leq t+r} |f(x_{i-1}) - f(x_i)|,$$

by a similar argument. Now, summing over all $1 \leq k \leq n$ we obtain

$$\sum_k f(\alpha_k) - f(\beta_k) \leq \sum_{1<i\leq p^\lambda-1} |f(x_{i-1}) - f(x_i)|.$$

On the other hand, we have

$$\begin{aligned}\sum_{1\leq k\leq n} (f(\alpha_k) - f(\beta_k)) &\geq \textstyle\sum_{1\leq k\leq n} \left(\sup_{x\in S_k} f(x) - \inf_{x\in S_k} f(x) - \frac{\epsilon}{2^k}\right) \\ &\geq \textstyle\sum_{1\leq k\leq n} \left(\sup_{x\in S_k} f(x) - \inf_{x\in S_k} f(x)\right) - \epsilon.\end{aligned}$$

As ϵ was arbitrary, we obtain

$$\begin{aligned}\textstyle\sum_{1\leq k\leq n} \sup_{x\in S_k} f(x) - \inf_{x\in S_k} f(x)) &\leq \sum_{1<i\leq p^\lambda-1} |f(x_{i-1}) - f(x_i)| \\ &\leq V_\lambda(f) \\ &\leq V_B(f).\end{aligned}$$

Taking the supremum over all Taibleson partitions $\{S_k\}$, we obtain $V_T(f) \leq V_B(f)$.

□

The two types of variations have their pros and cons. On the one hand, the variation as defined by Beer is well suited to derive a Koksma inequality as given by Theorem 7.3. However, the variation of Taibleson is easier to calculate, and is not sensitive to 0 and -1 as the variation of Beer. It would be interesting to derive a Koksma inequality using Taibleson variation, although it is not straightforward.

7.2 A Fourier analytic Koksma inequality

Kuipers and Niederreiter have obtained a Fourier analytic Koksma inequality on $\mathbb{R}/\mathbb{Z}$, where the constant C in (7.1) depends upon the Fourier coefficients of the function f. See Exercise 5.21 in [14]. We state the result below as a theorem.

Theorem 7.4 (Fourier analytic Koksma inequality)**.** *Let $f : [0, 1] \to \mathbb{R}$ be a continuous function, with Fourier coefficients $\hat{f}(k)$, $k \in \mathbb{Z}$. Let $\{x_1, ..., x_N\}$ be N points in*

[0, 1) *with star discrepancy* D_N^*. *Then,*

$$\left| \int_0^1 f(x)\, dx - \frac{1}{N} \sum_{n=1}^{N} f(x_n) \right| \leq 8 D_N^* \sum_{k=1}^{\infty} k |\hat{f}(k)|. \tag{7.8}$$

Motivated by Theorem 7.4, we derive an analogous Fourier analytic Koksma inequality in $\mathbb{Z}_p$, for any continuous $f : \mathbb{Z}_p \to \mathbb{C}$.

Theorem 7.5 (Fourier analytic Koksma inequality in $\mathbb{Z}_p$)**.** *Let* $f : \mathbb{Z}_p \to \mathbb{C}$ *be a continuous function with Fourier coefficients* $\hat{f}(\zeta)$. *Then, for a set of* N *points* $\{x_1, ..., x_N\}$ *in* $\mathbb{Z}_p$ *with discrepancy* D_N *we have*

$$\left| \int_{\mathbb{Z}_p} f\, d\mu(x) - \frac{1}{N} \sum_{n=1}^{N} f(x_n) \right| \leq D_N \sum_{\substack{\zeta \in \mathbb{Z}(p^\infty) \\ \zeta \neq 1}} \|\zeta\| \left| \hat{f}(\zeta) \right|. \tag{7.9}$$

Note that the discrepancy D_N here is as defined in Definition 1.4.

Proof. Without loss of generality, we may assume that $\sum_{\zeta \neq 1} \|\zeta\| \left| \hat{f}(\zeta) \right| < \infty$, otherwise the theorem would hold vacuously. Note that since f is continuous on $\mathbb{Z}_p$, its Fourier series converges uniformly to f by Corollary 3.2. Hence

$$f(x) = \sum_{\zeta \in \mathbb{Z}(p^\infty)} \hat{f}(\zeta) \zeta^x,$$

for all x in $\mathbb{Z}_p$. Using this, we get

$$\begin{aligned}
\left| \int_{\mathbb{Z}_p} f(x)\, d\mu(x) - \frac{1}{N} \sum_{n=1}^{N} f(x_n) \right| &= \left| \int_{\mathbb{Z}_p} f(x)\, d\mu(x) - \frac{1}{N} \sum_{n=1}^{N} \sum_{\zeta \in \mathbb{Z}(p^\infty)} \hat{f}(\zeta) \zeta^{x_n} \right| \\
&= \left| \frac{1}{N} \sum_{\zeta \neq 1} \hat{f}(\zeta) \sum_{n=1}^{N} \zeta^{x_n} \right| \\
&\leq \frac{1}{N} \sum_{\zeta \neq 1} \left| \hat{f}(\zeta) \right| \left| \sum_{n=1}^{N} \zeta^{x_n} \right|.
\end{aligned}$$

Recall the fact that each character ζ^x is constant on each of the $\|\zeta\|$ disks of radius $1/\|\zeta\|$ in $\mathbb{Z}_p$, as described in the proof of Proposition 4.1. Using this, and the orthogonality condition as given in (4.2) we have

$$\begin{aligned}
\left|\frac{1}{N}\sum_{n=1}^{N}\zeta^{x_n}\right| &\leq \left|\sum_{0\leq i<\|\zeta\|}\zeta^{i}\left(\frac{1}{N}\sum_{n=1}^{N}\mathcal{X}_{D(i,1/\|\zeta\|)}(x_n)-\frac{1}{\|\zeta\|}\right)\right| \\
&\leq \sum_{0\leq i<\|\zeta\|}\left|\frac{1}{N}\sum_{n=1}^{N}\mathcal{X}_{D(i,1/\|\zeta\|)}(x_n)-\frac{1}{\|\zeta\|}\right| \\
&\leq D_N\|\zeta\|,
\end{aligned}$$

so that

$$\left|\int_{\mathbb{Z}_p} f(x)\,d\mu(x)-\frac{1}{N}\sum_{n=1}^{N}f(x_n)\right|\leq D_N\sum_{\zeta\neq 1}\|\zeta\|\left|\hat{f}(\zeta)\right|.$$

□

Remark 7.2. Note that in order for Theorem 7.5 to be non-vacuous, we would require sufficient smoothness conditions on f so that the expression on the right hand side of (7.9) converges.

Corollary 7.1. *Let $\zeta \in \mathbb{Z}(p^{\infty})$ and $\{x_1, ..., x_N\}$ be N points in $\mathbb{Z}_p$ with discrepancy D_N, then*

$$\left|\frac{1}{N}\sum_{n=1}^{N}\zeta^{x_n}\right|\leq\|\zeta\|D_N.$$

Proof. Take $f(x) = \zeta^{x}$ in Theorem 7.5. □

Niederreiter in his article [18], derives a general Koksma inequality on compact abelian groups. Here, he also presents a Fourier analytic Koksma inequality. The inequality being over any general compact group G, does not involve any notion of discrepancy defined on G. Instead, it considers the distribution of points on the unit circle under the character maps from G to the unit circle. In contrast, our derivation as given in Theorem 7.5 is directly related to the structure of $\mathbb{Z}_p$.

The following result gives a relationship between the Taibleson variation and the Fourier-analytic constant in our Koksma inequality in Theorem 7.5. In particular, it shows that any real valued function with rapidly decaying Fourier coefficients must have bounded Taibleson variation.

Proposition 7.2. *Let $f : \mathbb{Z}_p \to \mathbb{R}$. Then,*

$$V_T(f) \leq 2 \sum_{\zeta \neq 1} \|\zeta\| \left| \hat{f}(\zeta) \right|,$$

where $V_T(f)$ is the Taibleson variation of f.

Proof. Let D be a disk of radius $\delta = p^{-r}$ in $\mathbb{Z}_p$ and let x, y be in D such that $f(x) > f(y)$. Then,

$$\begin{aligned} |f(x) - f(y)| &= \left| \sum_{\zeta \neq 1} \hat{f}(\zeta) \left(\zeta^x - \zeta^y \right) \right| \\ &\leq \sum_{\zeta \neq 1} |\hat{f}(\zeta)| \left| \zeta^{x-y} - 1 \right| \\ &\leq \sum_{\zeta \neq 1} |\hat{f}(\zeta)| \left(\delta 2 \|\zeta\| \right) \\ &= 2\delta \sum_{\zeta \neq 1} \|\zeta\| |\hat{f}(\zeta)|. \end{aligned} \tag{7.10}$$

The third line in 7.10 can be seen as follows. If $\|\zeta\| = p^n \leq p^r = 1/\delta$, then since $|x - y|_p \leq 1/p^r$ we have $\zeta^{x-y} = 1$, or $|\zeta^{x-y} - 1| = 0$.

If $\|\zeta\| = p^n > p^r = 1/\delta$, then $1 < \delta\|\zeta\|$ so that $|\zeta^{x-y} - 1| \leq 2 \leq 2\delta\|\zeta\|$.

Now the result follows by applying the upper bound on $|f(x) - f(y)|$ given by (7.10) to the sum over any partition in Definition 7.2, and taking the supremum over all partitions. □

Example 7.5. Consider the function $f(x) = |x|_p^2$. Since $f(x)$ is constant on circles centered at 0, we can write it as

$$f(x) = \sum_{j=0}^{\infty} \frac{1}{p^{2j}} \mathcal{X}_{S(0,1/p^j)}(x),$$

where $S(0, 1/p^j) = \{x \in \mathbb{Z}_p \,|\, |x|_p = 1/p^j\}$ is a circle of radius $1/p^j$ centered at zero. For $\zeta \neq 1$, the Fourier coefficients are given by

$$\hat{f}(\zeta) = \int_{\mathbb{Z}_p} f(x) \zeta^{-x} \, d\mu(x)$$

$$= \sum_{j=0}^{\infty} \frac{1}{p^{2j}} \int_{\mathbb{Z}_p} \mathcal{X}_{S(0,1/p^j)}(x)\zeta^{-x}\, d\mu(x).$$

Using Lemma 3.5 we have,

$$\begin{aligned}
\int_{\mathbb{Z}_p} \mathcal{X}_{S(0,1/p^j)}(x)\zeta^{-x}\, d\mu(x) &= \int_{|x|\le 1/p^j} \zeta^{-x}\, d\mu(x) - \int_{|x|\le 1/p^{j+1}} \zeta^{-x}\, d\mu(x), \\
&= \begin{cases} \frac{1}{p^j}(1-1/p) & 1 \le \|\zeta\| \le p^j, \\ \frac{-1}{p^{j+1}} & \|\zeta\| = p^{j+1}, \\ 0 & \|\zeta\| > p^{j+1}. \end{cases}
\end{aligned}$$

Therefore, with $\|\zeta\| = p^t$ we have

$$\begin{aligned}
\hat{f}(\zeta) &= \sum_{t-1\le j<\infty} \frac{1}{p^{2j}} \int \mathcal{X}_{S(0,1/p^j)}(x)\zeta^{-x}\, d\mu \\
&= \frac{1}{p^{2(t-1)}}\left(\frac{-1}{p^t}\right) + \left(1 - \frac{1}{p}\right) \sum_{j=t}^{\infty} \frac{1}{p^{3j}} \\
&= \frac{-p^2}{p^{3t}} + \frac{1}{p^{3t}}\left(\frac{p-1}{p}\right)\left(\frac{p^3}{p^3-1}\right) \\
&= \frac{1}{\|\zeta\|^3}\left(-p^2 + \frac{p^2(p-1)}{p^3-1}\right),
\end{aligned}$$

so that

$$|\hat{f}(\zeta)| = p^2\left(\frac{p^3-p}{p^3-1}\right)\frac{1}{\|\zeta\|^3}.$$

Now using the Koksma bound of Theorem 7.5 we have

$$\begin{aligned}
\left|\int_{\mathbb{Z}_p} f\, d\mu - \frac{1}{N}\sum_{n=1}^{N} f(x_n)\right| &\le D_N \sum_{\zeta\ne 1} \|\zeta\| \left|\widehat{f}(\zeta)\right| \\
&= p^2\left(\frac{p^3-p}{p^3-1}\right) D_N \sum_{\zeta\ne 1} \frac{1}{\|\zeta\|^2} \\
&= p^2\left(\frac{p^3-p}{p^3-1}\right) D_N \sum_{1\le r<\infty} \frac{p^r - p^{r-1}}{p^{2r}}
\end{aligned}$$

$$
\begin{aligned}
&= p\left(\frac{p^3-p}{p^3-1}\right) D_N \\
&\leq pD_N, \qquad (7.11)
\end{aligned}
$$

where we used the fact that there are precisely $p^r - p^{r-1}$ elements of $\mathbb{Z}(p^\infty)$ of order $\|\zeta\| = p^r$ in line 3 of (7.11).

Remark 7.3. An easy calculation shows that $f(x) = |x|_p^2$ has Beer variation 1, so the Koksma inequality given by our Theorem 7.5 is sharper than Beer's Theorem 7.3 in this particular case. The Fourier analytic p-adic Koksma inequality we prove in Theorem 7.5 only requires continuity and a certain condition on the convergence of the Fourier coefficients in order for the right hand side of (7.9) to be convergent. In contrast, Beer's bound requires bounded variation which may be a more restrictive condition in some cases. Moreover, the Fourier coefficients of a function f may in some cases be easier to calculate than the variation $V_B(f)$ or $V_T(f)$. For example, consider the function expressed by its Fourier series

$$
f(x) = \sum_{\zeta \neq 1} \frac{(\zeta + \zeta^{-1})}{\|\zeta\|^4} \zeta^x.
$$

Then,

$$
\begin{aligned}
\sum_{\zeta \in \mathbb{Z}(p^\infty)} \|\zeta\| |\hat{f}(\zeta)| &= \sum_{\zeta \neq 1} \frac{|\zeta + \zeta^{-1}|}{\|\zeta\|^3} \\
&= 2\sum_{\zeta \neq 1} \frac{|\Re(\zeta)|}{\|\zeta\|^3} \\
&\leq 2\sum_{\zeta \neq 1} \frac{1}{\|\zeta\|^3} \\
&= 2 \sum_{1 \leq r < \infty} \frac{p^r - p^{r-1}}{p^{3r}} \\
&= \frac{2p}{p+1}.
\end{aligned}
$$

So, we have

$$
\left| \int_{\mathbb{Z}_p} f \, d\mu - \frac{1}{N} \sum_{n=1}^{N} f(x_n) \right| \leq \frac{2p}{p+1} D_N.
$$

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