ON MULTIPLE POINT TAYLOR SERIES EXPANSIONS

by

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INTRODUCTION

The rapid advance in the electronic digital computer and its application has brought forth many advances in numerical and approximation methods. The literature published in this field has increased and work is being done in many areas.

One such area is the evaluation of a function in terms of its derivatives at one or more points. Such formulas have been developed and used by many authors although no general approach has been made. Boole (2, 107-109) used a semblance of this idea to solve a special problem. Milne (10, 537-542) appears to have priority with a method for the numerical integration of differential equations which depends on the values of the first three derivatives. Salzer (12, 103-106) has developed integration formulas using both the function and its first derivative. Luke (9, 298-307), Hammer and Hollingsworth (4, 92-96) and Fettis (3, 85-91) obtained various modified forms of the trapezoidal formula. Householder (5, 232-240) discussed several types of quadrature formulas. Hummel and Seebeck (6, 243-247) developed a treatment for the evaluation of a function without regard to quadrature methods.
Lotkin (7, 171-179) developed a specialized version of the first integral of the function. Andress (1, 394-396) developed essentially the same method on the complex plane. The bibliography here only indicates the numerous approaches that have been made to this problem.

The objective of this paper is to generalize Taylor's theorem of the expansion of a function about a single point to an expansion about two points. The uses of the multiple point expansion are: to obtain generalized formulas for mechanical quadrature; generalization of the Milne-Lotkin approach to the numerical integration of differential equations; a representation of functions by rational fractions which is of interest in the construction of tables.
THE TWO POINT THEOREM

The following assumptions will be made: $F(x)$ is continuous in an interval which contains 0 and $x$; all the derivatives up to and including the $(m+n+p)$th are continuous and the $(m+n+p+1)$th derivative is sectionally continuous in that interval; $F(x)$ must be of exponential order. The letters $m$ and $n$ are defined to be non-negative integers. The binomial coefficients will be denoted by $\binom{p}{q}$ with the understanding that the expression is zero for $q < 0$ or $q > p$. The symbol $\sum_{k=0}^{p} f(m,n,k)$ will mean to sum over $k$ from 0 to the larger of $m$ and $n$. Also $\sum_{k=0}^{p} f(k) = 0$ if $p < 0$.

A function $G(x,p)$ is defined by the definite integral

$$G(x,p) = \int_{0}^{x} F(t)(x-t)^{m} n t^{n} dt.$$  \hspace{1cm} (2.1)

Application of the standard one-sided Laplace transformation to both sides yields

$$g(s,p) = \frac{m!(-1)^{n}}{s^{m+p+1}} \frac{d^{n}}{ds^{n}} \left\{ s^{m+n+p+1} f(s) \right\}.$$  \hspace{1cm} (2.2)
Because of the Leibniz formula (2.2) becomes

\[- \sum_{k=0}^{m+n+p} \frac{\binom{m+n+p-k}{k}}{s^{m+n+p-k}} F(0) \]  

\[- \sum_{k=0}^{m+n+p} \binom{m+n+p-k}{k} \frac{F(k)}{s^{k+1}} \]

\[\frac{\Gamma(s)}{s} \frac{1}{\Gamma(1+q)} \]

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The inversion formulas

\[\mathcal{L}^{-1} \left\{ \frac{s^{-p+k}}{s} f(s) \right\} = (-1)^k \frac{d^{k+p}}{dx^{k+p}} F(x), \quad p \leq 0,\]
and

\[ \int \left\{ \frac{k!}{s^{k+1}} \right\} = \frac{x}{k} \]

(2.5)

lead to the form

\[ (-1)^n G(x, p) = m! \sum_{k=0}^{-p-1} \binom{n}{k} \frac{(m+n+p+1)!}{(m+p+1+k)!} \int \left\{ \frac{s^{p+k}(k)}{f(s)} \right\} \]

(2.6)

\[ + m! \sum_{k=0}^{n+p} \binom{n}{k-p} \frac{(m+n+p+1)!}{(m+k+1)!} \sum_{j=0}^{k} (-1)^{k-p} \binom{k}{j} \frac{(k-p)!}{(j-p)!} \frac{x^{j-p}(j)}{F(j)} \]

\[ - \sum_{k=0}^{m+p} \binom{m}{k-p} \frac{(m+n+p-k)!}{x^{k-p}(k)} \frac{k-p(k)}{F(0)} . \]

The formula for interchange of order of summation,

\[ \sum_{k=0}^{n} \sum_{j=0}^{k} f(k, j) = \sum_{j=0}^{n} \sum_{k=j}^{n} f(k, j) , \]

(2.7)

yields

\[ (-1)^n G(x, p) = m! \sum_{k=0}^{-p-1} \binom{n}{k} \frac{(m+n+p+1)!}{(m+p+1+k)!} \int \left\{ \frac{s^{p+k}(k)}{f(s)} \right\} \]

(2.8)
\[
\sum_{j=0}^{n+p} \binom{n}{j-p} \frac{(m+n+p+1)!}{j!} x^{-p} f(j) x^{-j} \sum_{k=j}^{n+p} \binom{n-j+p}{k-j} (-1)^{k-j} \frac{k-p}{(m+k+1)!} \\
- \sum_{k=0}^{m+p} \binom{m}{k-p} (m+n+p-k)! x^{-p} f(k) x^{k-p} f(0),
\]

\[
m! \sum_{k=0}^{-p-1} \binom{n}{k} (m+n+p+1)! \sum_{s=0}^{p+k} \frac{(k)}{(m+p+1+k)!} f(s)
\]

\[
m! \sum_{j=0}^{n+p} \binom{n}{j-p} \frac{(m+n+p+1)!}{j!} (-x)^{j-p} f(j) x^{-j} \sum_{r=0}^{n-j+p} \binom{n-j+p}{r} (-1)^{r} \frac{r}{(m+r+j+1)!} \\
- \sum_{k=0}^{m+p} \binom{m}{k-p} (m+n+p-k)! x^{-p} f(k) x^{k-p} f(0).
\]

To evaluate the second summation (with respect to \( r \)) it is necessary to iterate the integration of both sides of the identity

\[
(2.9) \sum_{r=0}^{n-j+p} (-1)^{r} \binom{n-j+p}{r} x^{r+j} = x^{j-n-j+p} x (1-x) ^{n-j+p} 
\]

\( m+1 \) times, the last integration being between 0 and 1. That is,
\[
\sum_{r=0}^{n-j+p} (-1)^r \binom{n-j+p}{r} \frac{l}{(r+j+1) \ldots (m+r+j+1)}
\]

(2.10)

\[
= \frac{1}{m!} \int_0^1 x^{j} (1-x)^{m-n-j+p} \, dx
\]

\[
= \frac{j!(m-n-j+p)!}{m!(m+n+p+1)!}
\]

and (2.8) takes the form

\[
\frac{(-1)^n}{(m+n+p+1)!} G(x,p) = \sum_{k=0}^{p-1} \binom{n}{k} \frac{m!}{(m+p+1+k)!} \int_0^1 \{s^k f^{(k)}(s)\}
\]

(2.11)

\[- \sum_{k=0}^{p-k} \frac{(m+n+p-k)!}{(m+n+p+1)!} \{ \binom{m}{k} f^{(k)}(0) - (-1)^{k-p} \binom{n}{k-p} f^{(k-p)}(x) \} x^{k-p} \]

The theorem of the mean for integrals gives as an estimate of \( G(x,p) \),

\[
G(x,p) = x F^{(m+n+1)}(x) \int_0^1 (1-u) u \, du
\]

(2.12)

\[
= \frac{m! n! x^{m+n+1}}{(m+n+1)!} F^{(m+n+p+1)}(x)
\]

where \( 0 < X < x \). The final form of the two point Taylor
expansion is, therefore,

\[ \sum_{k=0}^{\frac{p-1}{p+k}} \binom{n}{k} \frac{m!}{(m+p+1+k)!} \mathcal{L}^{-1} \left\{ s \, f(s) \right\} \]

(2.13)

\[ = \sum_{k=0}^{\frac{m+n+p-k}{m+n+p+1}} \left\{ \frac{m}{(k-p)} f^{(k)}(0) - (-1)^{k-p} \frac{n}{(k-p)} f^{(k)}(x) \right\} x^{k-p} \]

\[ + R(x,p) , \]

where

(2.14) \[ R(x,p) = \frac{(-1)^{n} m! n! x}{(m+n+1)! (m+n+p+1)!} \]

The Boole (2, 107-109) development is quickly duplicated in modern notation by using the two-sided Laplace transformation. An obvious identity,

\[ (1 - e^{-sh})f(s) = \frac{1 - e^{-sh}}{1 + e^{-sh}} (1 + e^{-sh})f(s) \]

(2.15)

\[ = \left\{ \frac{sh}{2} - \frac{(sh)}{24} \ldots \right\} (1 + e^{-sh})f(s) \]

\[ = \sum_{n=0}^{\infty} (-1)^{n} E_n (sh) (1 + e^{-sh})f(s) , \]

where the \( E_n \) are the Bernoulli numbers, is written down.
The inversion yields

\[ F(x) - F(x-h) = \sum_{n=0}^{\infty} \frac{2n+1}{(2n+1)(2n+1)} (-1)^n \frac{1}{n} \frac{1}{h} \{ F(x) + F(x-h) \} \]

which is equivalent to the original form of Boole.
SPECIAL CASES

The replacement of \( p \) by \( 0 \) in the general formula of (2.13) yields

\[
0 = \sum_{k=0}^{\infty} \frac{(m+n-k)!}{(m+n+1)!} \left\{ \binom{m}{k} F^{(k)}(0) - (-1)^{k} \binom{n}{k} F^{(k)}(x) \right\} x^{k}
\]

\[+ R(x,0) \]

or

\[
F(x) - F(0)
\]

\[
= \sum_{k=1}^{\infty} \frac{(m+n-k)!}{(m+n)!} \left\{ \binom{m}{k} F^{(k)}(0) - (-1)^{k} \binom{n}{k} F^{(k)}(x) \right\} x^{k}
\]

\[+ (m+n+1) R(x,0) , \]

where

\[
R(x,0) = \frac{(-1)^{n} m! n! x^{m+n+1}}{(m+n+1)!} F^{(m+n+1)}(x) .
\]

This formula is that of Hummel and Seebeck (6, 243-247), who state that many special cases arise from variations of \( m \) and \( n \). If \( n = 0 \) equation (3.2) becomes the Taylor series with remainder.

Since

\[
(3.4) \quad \mathcal{L}^{-1} \left\{ \frac{f(s)}{s} \right\} = \int_{0}^{x} F(t) \, dt
\]
the choice of $p = -1$ reduces (2.13) to

$$\int_0^x F(t) \, dt = R(x, -1)$$

(3.5)

$$+ \sum_{k=0}^{m+n-1-k} \frac{(m+n-1-k)!}{(m+n)!} \left\{ \binom{m}{k} F(0) - (-1)^k \binom{n}{k} F(x) \right\} x,$$

where

(3.6) \quad R(x, -1) = \left\{ \frac{n}{m! n!} \frac{m+n+1}{(m+n)!} \chi \right\}.$$

Here again varied formulas are obtained by adjusting $m$ and $n$. For $m = n$,

$$\int_0^x F(t) \, dt = \sum_{k=0}^{n-1} \frac{(2n-1-k)!}{(2n)!} \left\{ \binom{n}{k} F(0) - (-1)^k \binom{2n}{k} F(x) \right\} x$$

(3.7)

$$+ R(x, -1),$$

where

(3.8) \quad R(x, -1) = \frac{(-1)^{n!} (2n+1) x}{(2n+1)! (2n)!}.$$

Thus (3.5) is a generalization of Lotkin's (8, 29-34) formula since he limited his study to the case $m = n$.

In their papers on the numerical solution of differential equations Milne (10, 537-542) and Lotkin (8, 29-34) have used the two point expansion formula, $n = m = 3$, as
a "corrector", with the Taylor formula used as a "predictor". Lotkin's proof of convergence for such a method may be generalized without difficulty.

The transform pairs

\[
(3.9) \quad \mathcal{L}^{-1} \left\{ \frac{f(s)}{s^2} \right\} = \int_0^x \int_0^{t_1} F(t_2) dt_2 dt_1 = \int_0^x (x-t)F(t) \, dt
\]

and

\[
(3.10) \quad \mathcal{L}^{-1} \left\{ \frac{f^{(1)}(s)}{s} \right\} = -\int_0^x tF(t) \, dt
\]

reduce (2.13) for \( p = -2 \) to

\[
m x \int_0^x F(t) \, dt - (m+n) \int_0^x tF(t) \, dt = R(x,-2)
\]

\[
(3.11)
\sum_{k=0}^{(m+n-2)} \frac{(m-n-2-k)!}{(m+n-1)!} \left\{ \left( \frac{m}{k+2} \right)^{\frac{k}{k+2}} \left( \frac{n}{k+2} \right)^{(k)} \right\} x^{k+2}.
\]

Probably a better approach to finding a formula for

\[
\int_0^x t^{-p+1} F(t) \, dt
\]

would be to replace \( F(x) \) by \( x^{-p+1}F(x) \) in (3.5).
NUMERICAL EXAMPLES

To show the value of the two point formula it is first applied to the Bessel differential equation,

\[ xy'(x) + y(x) + xy(x) = 0, \]  
\[ y(0) = 1, y'(0) = 0, \]

and \( y(x) = J_0(x) \). Taking \( y'(x) = z(x) \) the solution of the two equations defining \( y(x) \) and \( z(x) \) in terms of their values at zero for \( n = m = 1, 2 \) and \( 3 \) are

\[ y(x,1) = \frac{12 - x}{2(6 + x^2)}, \]

\[ y(x,2) = \frac{(6 - x)(80 - x)}{2(240 + 17x + x^2)}, \]

\[ y(x,3) = \frac{201,600 - 41,280x + 1134x^2 - 3x}{8(25,200 + 1140x^2 + 33x^4 + x^6)} , \]

with appropriate remainder terms. The error in \( y(1,3) \) is less than \( 2 \cdot 10^{-6} \).

By taking \( m = n = 2 \) and setting up the equations for a stepwise procedure the representations of \( y(x) \) and \( z(x) \) in terms of values at an arbitrary \( x_o \) become
(4.3) \[ y = y_0 + \frac{(x-x_0)}{2} (y_0' + y') + \frac{(x-x_0)}{12} (y_0'' - y'') \]

and

(4.4) \[ z = z_0 + \frac{(x-x_0)}{2} (z_0' + z') + \frac{(x-x_0)}{12} (z_0'' - z'') \]

To reduce these to solvable equations in terms of \( y(x) \) and \( z(x) \) the values of \( z' \) and \( z'' \) are obtained by differentiating (4.1). Finally

\[
y \left[ 12 - (x-x_0)^2 \right] - z \left( \frac{(x-x_0)(7x-x_0)}{x} \right) \]

(4.5) \[
= 12y_0 + 6(x-x_0)y_0' + (x-x_0)^2 y_0''
\]

and

\[
y \left\{ \frac{(x-x_0)(7x-x_0)}{x} \right\} + z \left\{ 12 + \frac{6(x-x_0)}{x} + \frac{(x-x_0)(2-x)^2}{x^2} \right\} \]

(4.6) \[
= 12y_0' + 6(x-x_0)y_0' + (x-x_0)^2 y_0'''
\]

and successive applications for \( x_0 = 0, 1, 2, \ldots \) leads to the table
Table 1.

<table>
<thead>
<tr>
<th>x</th>
<th>y(x)</th>
<th>J_0(x)</th>
<th>z(x)</th>
<th>J'_0(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.7655</td>
<td>.7652</td>
<td>-.4399</td>
<td>-.4401</td>
</tr>
<tr>
<td>2</td>
<td>.2246</td>
<td>.2239</td>
<td>-.5770</td>
<td>-.5767</td>
</tr>
<tr>
<td>3</td>
<td>-.2596</td>
<td>-.2601</td>
<td>-.3400</td>
<td>-.3391</td>
</tr>
<tr>
<td>4</td>
<td>-.3978</td>
<td>-.3971</td>
<td>.06503</td>
<td>.06604</td>
</tr>
<tr>
<td>5</td>
<td>-.1791</td>
<td>-.1776</td>
<td>.3277</td>
<td>.5276</td>
</tr>
<tr>
<td>6</td>
<td>.1496</td>
<td>.1506</td>
<td>.2781</td>
<td>.2767</td>
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<td>.00643</td>
<td>.00468</td>
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<td>.1613</td>
<td>.1717</td>
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<td>-.2487</td>
<td>-.2459</td>
<td>-.02758</td>
<td>-.04347</td>
</tr>
</tbody>
</table>

That is, $J_0(x)$ has been rather quickly estimated from 0 to 10 with sufficient accuracy for drawing curves. The values of $J_0(x)$ and $J'_0(x)$ have been taken from Watson (13, 886-885).

A form of the error function serves as a second numerical example. For

$$(4.7) \quad y(x) = e^x \int_x^\infty e^{-t^2} dt$$

the two point formula for $m = n = 4$ and $x_0 = 0$ reads


11. Rosser, J. Barkley. Theory and application of
\[ \int_0^z e^{-x^2} \, dx \quad \text{and} \quad \int_0^z e^{-p^2y^2} \, dy \quad \int_0^y e^{-x^2} \, dx. \]

12. Salzer, Herbert E. Osculatory quadrature formulas.