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Title: The Sinking Fund Problem: Optimal Timing of Bond Purchases

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A sinking fund payment must be made each year to retire a bond issue. Instead of paying cash, the corporation will typically surrender bonds that it has purchased on the open market. The *sinking fund problem* is to determine an optimal strategy in the form of a stopping rule to decide between purchasing bonds now, or waiting for at least one more day. The problem is formulated and solved for two objective functions: (1) minimize expected unit cost, and (2) minimize maximum regret. Price movements are assumed to follow a conditional random walk. Using computer simulation, the performance and stopping characteristics of the two models are compared against a dollar averaging purchase strategy commonly used in practice. Both models outperform dollar averaging under most simulated conditions. A case study using actual price data is also developed to demonstrate the approach.

The Sinking Fund Problem:
Optimal Timing of Bond Purchases

by

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I. INTRODUCTION

Background

A *sinking fund* provision is often included in a bond indenture to guarantee the orderly retirement of a bond issue. Usually, the provision will require that a sinking fund "payment" be made by the corporation to a trustee on a specified date each year (called the *surrender date*). Instead of paying cash to the trustee, the corporation will frequently buy back its own bonds on the open market and surrender the bonds, themselves, to fulfill the sinking fund obligation. This tactic is always used if the bond is selling below par value on the open market. If the bond is not available on the open market at a discount, the company generally surrenders bonds it has "called" for redemption. If the call provision is used, the company pays par value for the bonds plus a premium of 1% - 8%.

Sinking fund payments are almost always required on both straight interest bonds and on convertible bonds.[1/]

1/ Personal communication with John Carberry, a bond specialist with Soloman Brothers. For convertible debentures, bonds converted into common stock during the year are generally applied as a credit against the sinking fund payment.

Sometimes bond indentures for utilities contain a "60% Provision" where, for example, a utility can offset a $1 million sinking fund payment by adding $1.6 million in plant and equipment. Presumably, the new assets provide the same element of protection tc the bondholder that a reduction in debt would provide. This provision is unique to utilities, however.

Moreover, the data given in Table 1 (next page) shows bond prices fluctuate considerably--even for the lowest risk issues--and annual SF payments involve millions of dollars.

TABLE 1. PRICE SPREADS AND SINKING FUND PAYMENTS ON STRAIGHT INTEREST BONDS

(SOURCE: MOODY'S INDUSTRIAL MANUAL, 1977)

	Fortune 500 a/ Rank	Coupon Rate ($)	Due (Year)	S & P Rating	1976 c/ Call Price ($)	Price Range c/ 1976	Price Range c/ 1975	Annual Sinking Fund Payment ($MM)
General Motors	1	8-5/8	2005	AAA	108	110-100	104-95	11.80
Proctor & Gamble	20	8-1/4	2005	AAA	107	107-94	101-92	15.00
General Foods	40	8-7/8	1990	AA	105	107-101	105-99	4.50
J. Deere & Co.	60	5.40	1992	AA	103	83-68	76-64	3.375
Allied Chemical	80	7-7/8	1996	A	105	102-84	92-81	5.00
United Brands	100	5-1/2	1994	B	103	58-38	44-29	4.00
Nabisco	120	7-3/4	2003	AA	105	98-78	93-85	6.25
Fruehauf	140	6	1987	BAA	103	84-73	74-60	2.50
NL Industries	160	7-1/2	1995	A	104	97-85	85-80	5.00
Martin-Murietta	180	6b/	1996	BA	104	100-69	78-65	2.50
Abbott Labs	200	7-5/8	1996	AA	105	94-88	95-85	6.00
Crane	220	7	1993	BA	101	90-75	82-68	1.00
Allegheny Ludlum	240	9	1995	A	106	98-90	94-89	2.00
Kennecott Copper	260	7-7/8	2001	A	104	98-88	90-83	10.00
Liggett Group	280	6	1992	A	103	85-76	80-71	3.00
Carborundum	300	6	1992	A	103	92-82	82-76	1.50
				Average Price Range:		1976 = 13 pts.	1975 = 9 pts.	

a/ "The 500 Largest Industrials (Ranked by Sales)", Fortune, May 8, 1978, pp. 240-259.
b/ Convertible Debenture
c/ Rounded to Nearest Dollar

The corporations in Table 1 were arbitrarily selected by taking every twentieth company in the Fortune 500 ranking (by sales) for 1977. This sample provides an interesting cross section of corporations ranging in size from the giant General Motors (1977 sales - $55 billion) to Carborundum, which ranked number 300 with sales of about $700 million. Studying the 16 bonds listed in Table 1 leads to the following generalizations about price fluctuations in the bond market:

- If there were such a thing as an "average" bond, its price would fluctuate about 10 full points during any 12 month period.
- The price fluctuations of bonds that are "under water" (deeply discounted) tend to be greater than for bonds priced in the mid 80's and 90's.
- Prices are frequently more volatile for convertible debentures than for straight interest bonds. Instead of a normal 10 point high-low price range, convertibles can jump 20-30 points in a year. This will occur when the price of the common stock nears (or falls away from) the bond conversion price. This situation occurred in 1976 for Martin-Murietta 6% convertibles (see Table 1) which posted a low price of 69 and a high of 100 (par).
- "Riskier" bonds (say, those rated below 'A' by Standard and Poors) tend to be more volatile price-wise.

Another point to make about Table 1 is that most of these companies have several debt issues outstanding. . .not just the issue listed. Allied Chemical, for example, has four separate debentures outstanding (see next page).

TABLE 2. SINKING FUND PAYMENT SCHEDULE FOR ALLIED CHEMICAL

(SOURCE: MOODY'S INDUSTRIAL MANUAL, 1977)

	Annual Sinking Fund Payment ($MM)						
Debenture	1977-81	1982-84	1985-86	1987-90	1991-92	1993-95	1996-99
5.20s, due 1991	6.00	6.75	6.75	7.50	-	-	-
6.60s, due 1993	5.00	5.00	5.00	5.00	5.00	-	-
7-7/8s, due 1996	5.00	5.00	5.00	5.00	5.00	5.00	-
9s, due 2000	-	-	4.60	4.60	4.60	4.60	4.60
	$16.00 MM	$16.75 MM	$21.35 MM	$22.10 MM	$14.60 MM	$ 9.60 MM	$ 4.60 MM

Total, 1977-96 = $337.75MM

Referring to Table 2, during 1977-1999 bonds totaling approximately $338 million par value must be surrendered to fulfill Allied Chemical's sinking fund obligation on its current debt. While the bonds could be called in for redemption, we might reasonably assume that most of the bonds will be purchased on the open market. This represents a sizable amount of market activity and with price fluctuations of about 10 points per year, trading gains or losses could be significant over the long term.

The Sinking Fund Problem

While debt covenants fix the amount of debt to be retired each year, the timing of SF purchases on the open market is purely discretionary. The situation facing the corporate treasurer is described below in terms of Allied Chemical 5.20s:

On January 1, 1978, $6 million of Allied Chemical's 5.20% debentures must be surrendered to meet the sinking fund requirement of this debt issue. Sometime during 1977 these debentures will be purchased on the open market because price is expected to fluctuate in the 70-80 price range.

Suppose no bonds have been purchased; that, say, 250 buying opportunities remain before the surrender date; and that the current market price is 75. The treasurer has the following sinking fund problem:

A. Should he purchase the entire $6 million of debentures at the current market price (paying roughly $4.5 million)?

B. Should he purchase part of the $6 million at this price and buy the remainder over the rest of the year to "average out" the price paid?

C. Should he not buy any, and wait for a better price?

The essence of the problem is deciding whether today's price is a "low" price relative to prices that will develop over the course of the year. Another facet of the problem is that the opportunity to save a few basis points by waiting for a lower price may be offset somewhat by the "continuation cost" incurred for every day the bonds remain outstanding. This continuation cost is defined precisely in Chapter VI, but may be positive or negative depending upon whether the coupon rate of the bond is higher than the opportunity cost associated with tying up funds that could be invested elsewhere. This decision situation will be hereafter referred to as the *sinking fund problem*.

Although the problem has been described in terms of one particular purchase decision, the problem is a recurring one because corporations usually maintain a perpetual care of long term debt for the financial leverage it provides. It is unfortunate, however, that even if the treasurer trades poorly, the bonds are generally purchased below their par value so a capital gain can be reported. That is, the corporation receives approximately 98 cents on every dollar when the debenture is sold and pays back, say, 70-80 cents on the dollar for bonds bought for sinking fund retirements. As a result, management may overlook the fact that gains could have been higher, still, by taking advantage of price fluctuations that occur during the year. This may lead to some complacency and may explain the widespread use of dollar averaging as a "solution" to the sinking fund problem.

Dollar Averaging

A common technique for coping with price fluctuations is called *dollar averaging*. Using this approach, if the annual sinking fund payment is $12 million, 12 units of $1 million each are purchased monthly for a year. Such a program guarantees that the whole fund is not purchased at the year's high.

However, as a trading strategy, dollar averaging is a passive, "do-nothing" type of approach.

- A fixed number of bonds are bought every month under dollar averaging so decisions as to timing and amount to buy at each price are eliminated. Price forecasting becomes unnecessary.
- The treasurer gives up the opportunity to profit from price fluctuations. Under dollar averaging,

there is no hope to ever attain the ideal least cost solution.

The challenge of developing a better approach provides the motivation for this study. 2/

Research Objective and Scope (Approach)

The objective of this research is to develop a new approach for making sinking fund purchase decisions which outperform the dollar averaging approach commonly used in practice. Instead of a heuristic approach, we decided to attack the problem in a way that draws upon as much relevant theory as possible. In particular, we decided to develop a decision model having a solid basis in the theory of optimal stopping of a stochastic process and in the theory of efficient capital markets.

2/ A frequently overlooked advantage of dollar averaging is that this method protects against a charge of utilizing "insider information" in timing bond purchases. However, this advantage fails to offset the shortcomings of using this approach as a sinking fund purchase strategy.

II. LITERATURE SEARCH OF RELATED RESEARCH

As a starting point for this research, a literature search was conducted having the following objectives:

1) Determine whether the sinking fund problem has been discussed (perhaps even been solved) in the literature.
2) If the solution to the sinking fund problem is still an open question, identify the techniques of operations research that might be relevant to effect a solution.

This section summarizes the results of this search.

Sinking Fund Problem: Has It Already Been Solved?

As far as can be determined, the sinking fund problem has not been treated specifically in the literature. The closest related research is the *bond refunding problem* which has been studied by Dyckman, et al (11), Bowlin (4), and Weingartner (28) to name just a few. In the bond refunding problem, a corporation having long term debt already outstanding periodically reviews its option to call the bonds and issue new bonds at a more advantageous interest rate. This section shows how the two problems are different and concludes that the sinking fund problem is still an open questions.

Bond Refunding

Early researchers on the subject of bond refunding concentrated on determining the breakeven point for the refunding decision. The choice of the interest rate to be used in discounting cash flows resulting from the refunding operation was the major issue for many years (see Bowlin (4) for an excellent discussion on this point). Today's consensus opinion is that debt refunding should join leasing as a special case where future interest savings should be discounted at the cost of debt rather than the (higher) cost of capital. The argument for this centers around the high degree of certainty associated with the savings (Weston and Brigham (29)).

Weingartner (28) and Bierman (3) were among the first to develop optimization models for the bond refunding decision. They used dynamic programming to establish a refunding strategy which maximized the interest savings from refunding.

The sinking fund problem differs from bond refunding in the following important ways:

- Both problems are similar to the extent that price is treated as a random variable and they are sequential decision problems. However, in the refunding problem both timing and the maturity of the refunding issue are optimized simultaneously. Unless price is treated in a simplified way, the problem becomes too unweildly to solve. Consequently, a major difference between the two problems is in the modeling of price, the sinking fund problem being able to accommodate a more complex price model.

- The planning horizon in the sinking fund problem is of finite duration (usually one year) where bond refunding is generally considered to be an infinite horizon problem. This greatly influences the approach one takes to solve the problem.
- For bond refunding, it is appropriate to use monetary expectations as the basis for the refunding decision. However, the corporate treasurer may be concerned more about safety and protection from second guessing than in achieving the lowest possible sinking fund cost. Consequently, the two problems may differ in the choice of objective function.

In summary, a literature search turned up nothing related closely enough to the problem at hand to jeopardize the originality of this research. The sinking fund problem is apparently still an open question.

During the literature search, a method of solution began to take shape which would utilize the theory of optimal stopping as the basic framework for optimization. This is in keeping with the importance of timing in this problem.

It also became apparent that a sophisticated price model would be required to be able to detect and act upon price aberrations or trends. Without this sophistication, the odds were against being able to outperform the dollar averaging approach. After surveying the literature, we came to the following conclusions about price modeling:

1) It is curious how little research has been published where price is treated as a random variable. In such instances where price is uncertain, assumptions about the underlying stochastic process are too simplistic for use in the sinking fund problem.

2) The price generating process should be consistent with research evidence compiled in conjunction with efficient markets theory.

To conclude this section on related research, a brief analysis of stopping theory, efficient markets research, and alternative price models is given with respect to the sinking fund problem.

Optimal Stopping Theory

The sinking fund problem is related to a class of mathematical problems called *stopping rule problems*. The essential features of these problems are as follows (5):

1) A probabilistic mechanism that moves the process from state to state under a known, partially known, or unknown probability law.

2) A payoff and decision structure such that, after observing the current state, we have our choice of at most two decisions:

   (a) Take our accumulated payoff to date and quit.

(b) Pay a fee for the privilege of watching one more transition.

If we define bond price as the "state" of the system and realize that a decision to wait for a better price implies paying some bondholder interest for one more period (i.e., a fee), then it is easy to imagine the sinking fund problem as being a stopping rule problem.

Optimal stopping theory has been treated extensively in the literature of statistics as it applies to sequential sampling schemes for quality control applications. Breiman (5) provides a good overall introduction to the subject.

Efficient Markets Theory

Efficient Markets Theory is concerned with successive changes in the price of common stocks, bonds, commodity futures, etc. The theory states that ". . .in highly competitive and organized markets (i.e., 'efficient' markets) price changes will be statistically independent of each other and will approximate a random variation that can be modeled as a random walk" (Fama (12)). A conclusion of this nature is profoundly important in the analysis and modeling of bond prices for the sinking fund problem. In view of the importance of this result, some of the research evidence pertaining to the random walk topic is reviewed below.

Random Walk Model: Price Changes Have No Memory

The most important conclusion from efficient markets theory is that changes in bond prices should be serially independent. If this is true, then forecasts based on historical price data have no theoretical justification. This includes forecasts based on moving averages, regression lines, and so forth. The theory says that price changes up or down are independent events which occur in response to new information formed in the market place. To quote Cootner (7): "Since there is no reason to expect such information to be non-random in appearance, the period-to-period price changes of a stock should be random movements, statistically independent of one another". A random walk model (Markov chain) can be used to describe the motion of this type of stochastic process.

Many empirical studies have been made to test the hypothesis that price changes are independent over time. The results of these studies are consistent. To quote Fama (12): "I know of no study in which standard statistical tools have produced evidence of important dependence in series of successive price changes. In general, these studies (and there are many of them) have tended to uphold the theory of random walks".

A recent study by Rogalski (24) examined bond prices from this point of view. Rogalski studied two different data sets:

1) Weekly Standard & Poor's yield indices for AAA, AA, A, and BBB rated bonds over an 11 year period, 1962-1972.

2) Weekly yields for two individual bonds selected from those making up each index for the same 11 year period.

Rogalski's study generally supported the random walk hypothesis although there was some evidence of low order serial correlation (implying price dependence). However, these irregularities were infrequent and unpredictable. So for practical purposes, the "lack of memory" property associated with a random walk can be assumed to hold for bonds.

3/ Another category of empirical research is concerned with attempts to fit particular probability distributions to price changes. In two pathbreaking papers, Bachelier (2) and Osborne (22) gave plausible theoretical arguments to support the conclusion that price changes are normally distributed. Their proposition is based on the following argument: If the price changes from transaction to transaction are independent, identically distributed random variables with finite variance, and if transactions are spaced uniformly through time, the central limit theorem says price changes across intervals of a day, a week, or a month will be normally distributed since they are simple sums of random variables.

We will use this information later to select an appropriate "step size" to use in a random walk model for bond prices. However, for our purposes, the evidence that price changes are serially independent is much more useful than the particular distribution form of price changes.

Is The Simple Random Walk Model Adequate?

In spite of all this research, there is a very disturbing problem associated with modeling bond prices as a simple random walk. The problem is this: If price level is indeed a random walk, then after, say t price changes have occurred the probability of price being with a certain range becomes absurdly small for large t. An example is given below to illustrate this point.

Suppose that at the end of each unit of time the change in price is either plus one or minus one with either outcome equally likely (i.e., the classical random walk). The possible outcomes (and their probability of occurring in parentheses) over successive units of time are plotted in Figure 1.

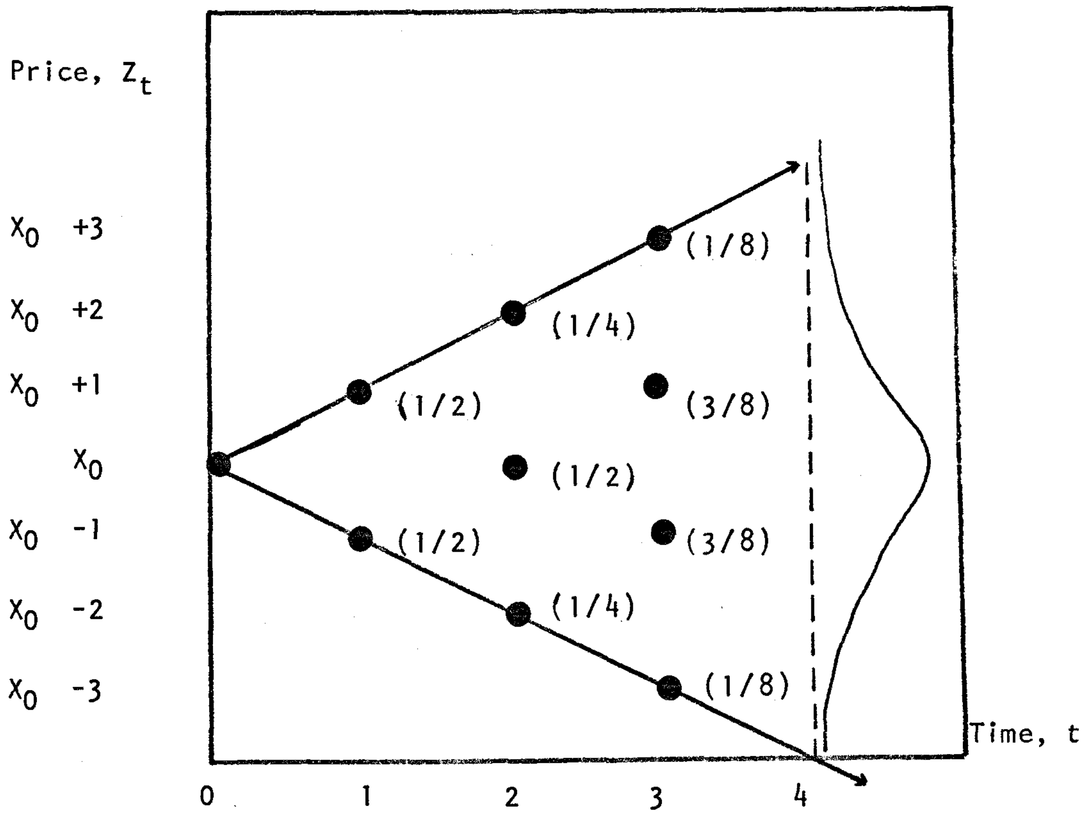

Figure 1. Sample Paths of a Simple Random Walk

This example illustrates the forecasting problem if prices actually behave according to this model. Since the variance of price increases with t, the probability distribution of price becomes unreasonably disbursed for large t.[4/]

The forecasting dilemma is the cause of the long standing controversy between professional security analysts on one side and economists on the other over the random walk hypothesis. To review the bidding:

- Economists believe that in ideal markets, price responds to new information in instantaneous jumps. While individual buyers or sellers may act in ignorance, taken as a whole the prices set in the market today reflect the best evaluation of currently available knowledge. Tomorrow, price may jump up or down from today's level depending on the random arrival of new information. However, at any given point in time, <u>the best forecast of the true, or intrinsic value of a security is the current market price</u>. If this position is correct, the random walk model should be adequate to describe bond price movements over time.

4/ Notice that it is possible to have a positive probability for negative bond prices after t has become sufficiently large. For this reason, Samuelson (26) would change the basic model assumptions by assuming price changes are log-normally distributed rather than normally distributed. This is still a random walk model, but the transformation confines price to positive values.

- The professional security analyst operates in the belief that certain trend generating facts exist, knowable today, that will enable one to forecast future prices if he can read the facts correctly. That is, the astute analyst will recognize intrinsic value before the market will, in general. A gradual spread of awareness of these facts throughout the market will be reflected in price eventually moving towards the intrinsic value. This, of course, implies that price series would show a dependence among successive price changes, a condition that would invalidate a random walk model. As mentioned previously, however, empirical results rarely uncover any dependence in speculative price changes.

There is another explanation that seems closer to reality because it encompasses random walk type of behavior (making economists and statisticians happy) plus restricts the range of the walk by incorporating a forecast (making security analysts happy). This notion will be developed in detail in Chapter III, "A Model to Describe Bond Price Movements". However, before developing this model, it is useful to review the other price models that have been treated in the operations research literature.

<u>Other Price Models in the Literature</u>

The simplest assumption for treating price as a random variable is to assume that prices are independent, identically distributed random variables. A famous illustration of this model is the "house selling problem" described in Ross (25). In this problem, an individual wants to sell his house at the highest price and an offer comes in each day. He must immediately decide whether or not to accept the offer. Once rejected, the offer is lost. The successive offers are assumed independent of each other and take on the value i with probability P_i. Ross uses this problem to illustrate an optimal stopping problem where the objective is to determine a policy which maximizes the expected selling price of the house. Ross shows that the optimal policy is of the form, "Accept any offer greater than or equal to i*, and reject any offer less than i*". The value of i* for each day is determined by dynamic programming.

We intend to solve the sinking fund problem using a similar approach. However, this price model, where prices are independent, identically distributed (IID) random variables, is not an adequate model for price movements in a speculative market. While price <u>changes</u> might be considered to be independent, identically distributed random variables, absolute price level can't be predicted by simply sampling from a distribution of possible prices. Whereas, in the house selling problem one might expect offers to be totally independent of one another, in a bond market knowing today's price tells us something about tomorrow's price, i.e., the conditional probability $Pr(P_{t+1}=j/P_t=i)$ should be quite different from the unconditional $Pr(P_{t+1}=j)$. Consequently, while the house selling problem's price model might be attractive mathematically (it does permit a closed form analytical solution to the problem), it is not appropriate for the sinking fund situation.

Markov Chain Price Models

One possible alternative to the independent, identically distributed (IID) price model just described is to view bond price levels as a Markov process. The states of the system may be thought of as the set of all possible prices that might be observed for a given bond. Since the number of states so defined is likely to be too large to manage, it would be necessary to group prices into a finite number of price ranges, or price classes. Each class could then be considered to be one of the states of the system.

Any Markov process can be completely described by its transition matrix, M, which gives the conditional probability, $(M)_{ij}$, of the jth state occurring next period given that the process is currently in the ith state. If, in addition to M, we know the initial state of the system (denoted by the vector P_o) one can compute the probabilities of the price level n periods hence by the expression

$$P_o M^n$$

where M^n is the nth power of the transition matrix, M.[5/]

While this Markov chain price level model is closer to reality than the previous model, it has several disadvantages from the point of view of implementation. Consider, for example, the problem of constructing the transition matrix, M. First, one must decide the range of possible bond prices over the planning horizon in order to establish the "states" of the system. Presumably, this would be done in conjunction with a price forecast. (For example, three standard deviations on either side of a trend line could be used to estimate the upper and lower limits of expected price.) However, one then must decide how to determine the conditional probabilities of moving from one price level to another.

Another drawback of the present Markov chain model is that the distribution of price n periods hence (from $P_o M^n$) converges rapidly to a unique steady state distribution, P, as n becomes large (say, exceeds n*). In this situation, the probability of being in price class K after n > n* periods is independent of the current price and n, and is given by the kth component

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5/ Knowing that the current price is in the ith class (price range), then the ith element of the vector P_o is set equal to one, and all other elements are set equal to zero.

of the vector P. This is essentially the same as the IID price model discussed earlier. The diagram in Figure 2, below, may make this clearer.

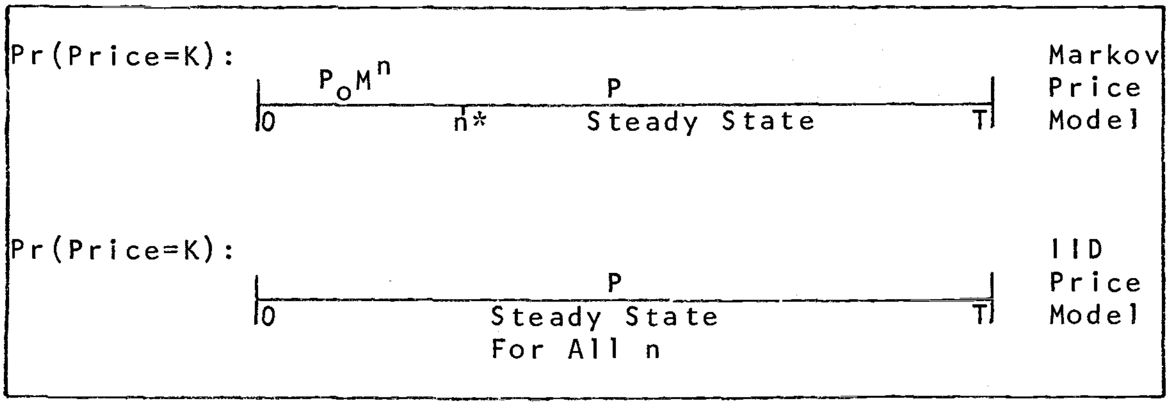

Figure 2. Steady State Markov Probabilities Approach IID Values

The Markov chain price level model just described is very attractive from a mathematical standpoint because it fits nicely into the framework of a Markov decision process.[6/] The bond refunding problem discussed earlier is often formulated using this type of model.

6/ A Markov decision process is a dynamic programming problem where the state of the system changes probabilistically at each stage, and state transition probabilities are defined by a Markov chain.

<u>Brownian Motion Model</u>

Empirical studies of speculative markets have generally found a close agreement between the empirical distribution of actual price changes, and the classical model of *Brownian Motion*. (This is not surprising because Brownian Motion is simply a continuous version of a random walk.)

A stochastic process $[P(t), t \geq 0]$ is said to be a Brownian Motion process with drift coefficient μ if:

(i) $P(0)=0$

(ii) $[P(t), t \geq 0]$ has stationary independent increments (i.e., has the "lack of memory property").

(iii) For every $t>0$, $P(t)$ is normally distributed with mean μt.

The above definition says that, starting from zero, the price level t time units from now is normally distributed with mean μt and (it can be shown) variance $\sigma^2 t$, where σ^2 is a function of the volatility of the price series. Note that with this price model, the further out one is looking in time, the broader is the range within which price might be expected to fall. Consequently, if one uses historical data to measure volatility, σ^2, and projects price at time t using, say, an econometric model[7/] having error S^2, then we may expect to have an inconsistency develop like the one shown in Figure 3, below.

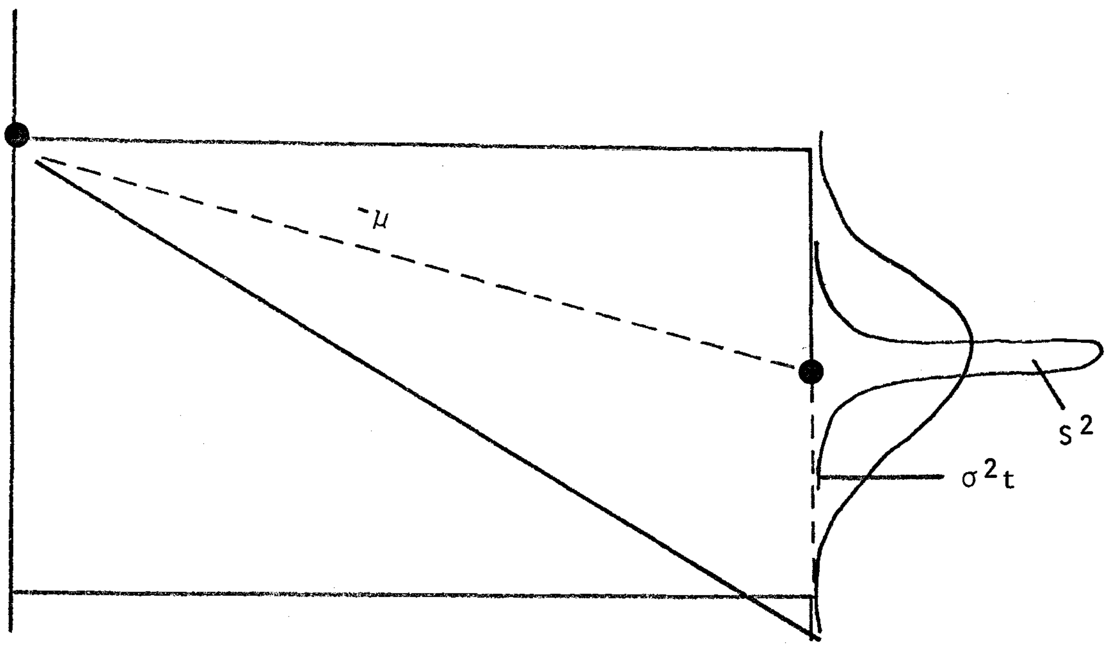

Figure 3. Inconsistency Between Forecast Error, S^2, and $\sigma^2 t$

The basic problem illustrated in Figure 3 is that S^2 might be a great deal less than $\sigma^2 t$; that is, the Brownian Motion Model is myopic in the sense that once σ^2 is defined and the process is set in motion it is assumed

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7/ A price projection based on the price series, itself, is not acceptable. . .empirical evidence has conclusively shown price "has no memory". However, an econometric forecast based on other exogenous variables (including price, if only lagged one period) could be allowed under a weaker type of random walk hypothesis.

free to wander anywhere within the domain defined by $\sigma^2 t$. Yet, the economic facts of the situation might predict price will lie in a much narrower range at time T, the neighborhood represented by S^2. Consequently, the brownian motion model for bond price movements suffers from the same deficiencies as the regular random walk model for bond prices discussed earlier. In particular, refer back to Figure 1 and note the similarity with Figure 3.

Constrained Brownian Motion Model

A logical way of increasing the validity of the brownian motion model might be to constrain the sample paths of the brownian motion process so that at time T, price is in the desired neighborhood predicted by the econometric model. It is possible, mathematically, to do this while still retaining the inherent price volatility reflected in σ^2.

Suppose, for example, that the current bond price (at t_1) is A and the projected price at the surrender date (t_2) is B. Assume price evolves in accordance with a brownian motion process P(t), with drift (i.e. trend) μ and volatility coefficient σ^2. Then it can be shown[8/] that the conditional distribution of P(t) given that $P(t_1)=A$ and $P(t_2)=B$, when $t_1<t<t_2$, is also a normal

8/ See Ross (25), pp. 180-181.

distribution with mean

$$A+\mu(t-t_1)+\frac{[B-A+\mu(t_2-t_1)]}{t_2-t_1}\cdot(t-t_1)$$

and variance

$$\frac{(t_2-t)(t-t_1)}{t_2-t_1}$$

These expressions for mean and variance assume that we have normalized σ^2 to one by employing the change of scale

$$P(t)=P'(t)/\sigma$$

where $P'(t)$ is the original time series of bond prices.

Price sample paths generated by the constrained brownian motion model would lie within an envelop similar to the one shown in Figure 4.

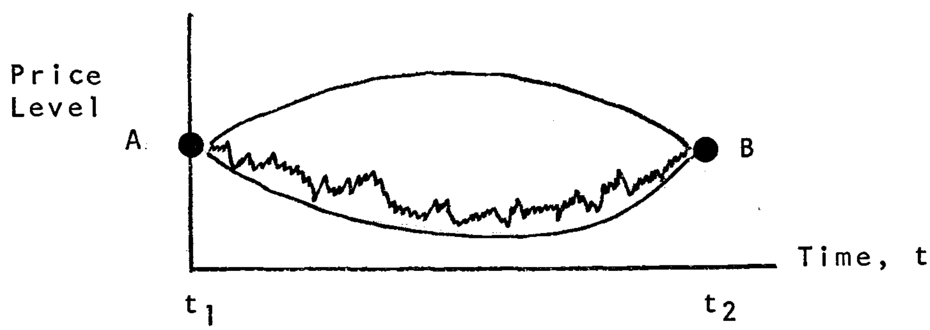

Figure 4. Sample Path of Constrained Brownian Motion

This model for bond prices is much better for our purposes than the other models we have discussed. It has two key attractive features:

1) The Markov chain type of behavior (i.e., transitions from state to state) is retained, and the difference in volatility of different bond issues can be reflected in the coefficient σ^2.

2) With this model, we can couple a forecasting mechanism (for example, an econometric model) which establishes the general level of future bond prices, to a mechanism which reproduces the type of fluctuations that actually occur as a real market moves prices in the direction of the forecast.

However, the constrained brownian motion model has a major disadvantage: the forecast, B, can only be incorporated in the form of a point estimate. A more satisfactory model would permit conditioning the sample paths to a neighborhood of B rather than through B, itself. In this way, we might expect that effort expended to develop a tighter forecast will result in a price model that behaves more realistically. . .a refinement that is needed if we are to succeed in outperforming the dollar averaging approach. The mathematical development and economic rationale for such a model is the topic of the next section, Chapter III: A Model to Describe Bond Price Movements.

III. A MODEL TO DESCRIBE BOND PRICE MOVEMENTS

In the previous chapter, empirical research was reviewed which shows rather conclusively that prices in speculative markets are serially independent, i.e., exhibit the "lack of memory" property of a Markov chain. Economists can explain this type of behavior in terms of an "efficient market" in which the current price reflects all known information - past, present and future - pertaining to the security in question, and fluctuations occur because new information, itself, arrives in a purely random manner.

Unfortunately, as we discussed earlier, a simple random walk-perfect market does not exist because the wandering of the walk (of prices) would be greater than what actually occurs in practice. Consequently, something must be done mathematically to restrict the walk to a more reasonable price range. This is the purpose of this chapter.

First, a market hypothesis proposed by Paul Cootner (8) is given which explains stock market price behavior in terms of a random walk with *reflecting barriers*, rather than as a simple random walk. Next, we argue that Cootner's model is conceptually appealing, but that it would be difficult to implement as a price model for the sinking fund problem. Finally, we extend Cootner's theory by developing his argument in such a way that it leads to a different type of model - a *constrained random* *walk*. This model will be used hereafter to describe bond prices in our solution to the sinking fund problem.

As background for an explanation of Cootner's hypothesis in terms of the bond market (Cootner's interest was in common stocks), it will be useful to define a term called *intrinsic value*.

Intrinsic Value

The intrinsic value, P_t*, of a bond at a particular point in time is a function of the coupon rate and term of the bond, plus premiums for:

- Liklihood of default on interest or principal payments.
- Marketability risk (liquidity).
- Possible shifts in interest levels.

The size of these premiums will vary depending on the individual outlook of each investor, so intrinsic value to the "market" exists only in the sense of a weighted average.

Unfortunately, this intrinsic value or equilibrium price (call it P_t*) is not observable. What is observable at any point in time is the market price, P_t. In general, P_t cannot be assumed to correspond exactly to the underlying equilibrium price, but will fluctuate around it. Cootner's hypothesis gives an explanation for random walk type of movement in the neighborhood of P_t^* and explains the existence of rubberlike barriers which limit the range of the walk.

Cootner's Market Hypothesis

Cootner's hypothesis is that the stock market is made up of two groups of participants. "One group, the non-professionals, are engaged in other occupations so it is very costly for them to devote time to detailed stock market research. As a result, they tend to accept prices as roughly representing fine differences in value and they choose between stocks largely on the grounds of their attitudes toward risk. Those non-professionals that do choose among stocks on the basis of information about future prospects are just as likely to be wrong as not."

"The other group of market participants are investors and speculators who specialize in the stock market. They know what to look for and where. Their profits are made by observing the random walk of the stock market produced by the non-professionals, and taking a position when the price wanders far enough away from the intrinsic value to give them the prospect of an adequate return. Prices will eventually move back toward their intrinsic value as non-professionals discover new information (to them) which the professionals have already discounted."

This market hypothesis differs from the pure random walk hypothesis because the latter assumes new information is instantaneously evaluated by all market participants so that current price always equals intrinsic value.

One possible model corresponding to Cootner's hypothesis is a random walk with "reflecting barriers". Conceptually, so called barriers would exist because whenever the discrepancy from intrinsic value is too large, the professionals will enter the market and this action will have a "restoring force" on price. The precise location of the barriers is fuzzy because buy/sell points are difficult for each individual.

A mathematical formulation of Cootner's hypothesis might take the following form:

> Suppose price level is a discrete random walk between two barriers of the reflecting type located (+a) and (-a) units to either side of the intrinsic value P*, where P* is constant during the time interval corresponding to n steps of the random walk. At each stage price can move one unit up or down, the probability of moving one unit toward P* being greater than the probability of moving one unit away from P* by an amount proportional to the deviation from P*. If price reaches one of the barriers, then it is certain to move to the adjacent interior positon (toward P*) at the next step. Figure 5 illustrates a typical sample path of this process.

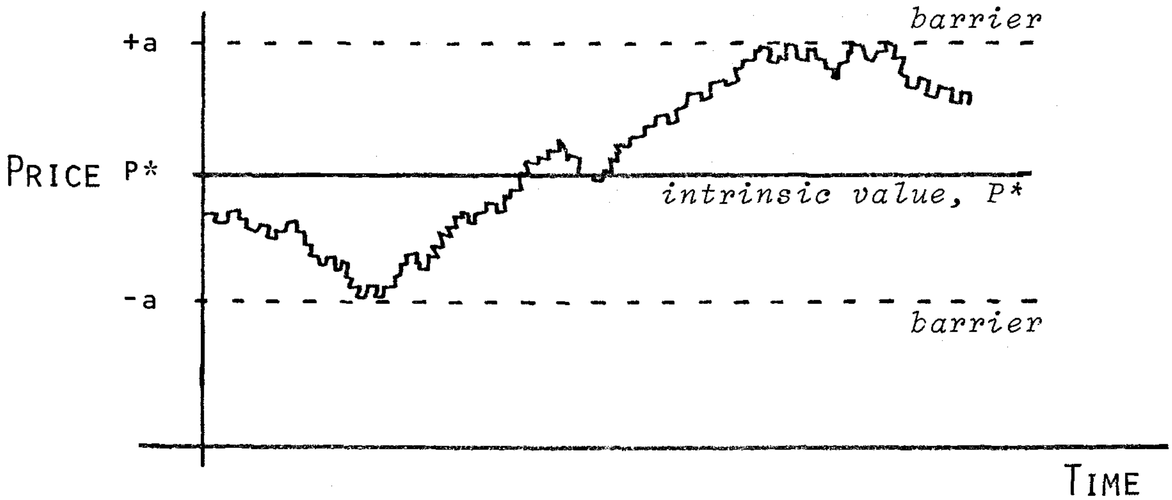

Figure 5. Sample Path of a Random Walk With Reflecting Barriers

Transition probabilities for this Markov chain are given below, where P_{ij}=probability (price changes to level j in the next stop of the walk given current price level is i.)

$$P_{j,j+1}=\frac{a-j}{2a}$$

$$P_{j,j-1}=\frac{a+1}{2a}$$

$$P_{a,a-1}=P_{-a,-a+1}=1$$

$$P_{j,k}=0, \text{ otherwise}$$

It is possible to derive the probability distribution of price after n steps (see Cox and Miller, (10), pp. 129-132) using well known properties of Markov chains.

Validity of Cootner's Model

One could argue that Cootner's model is not valid by taking the following position:

> There is no reason to expect that intrinsic value, P_t*, will not itself change over time. If it also changes randomly, then the "reflecting barriers" won't produce the desired constraining effect. In short, Cootner's reflecting barrier theory requires that a certain amount of stability in P_t* exist to restrict the range of the walk from wandering too much. The question is, does this stability actually occur?

Recognizing this problem, Cootner argued that the existence of the class of participants he called professionals gives P_t* the stability needed. His "professionals" do not adjust their expectations and/or react immediately to every piece of new information. They don't all look for the same things, evaluate data the same way, or take a new position simultaneously. The net result is what Cootner calls a "trend", that is a gradual (rather than instantaneous) spread of awareness and reaction among professionals to a significant change in intrinsic value. In such a world, "the path of prices over any substantial period of time would be composed of a random number of trends, each of which is a random walk with reflecting barriers." He goes on to say that "There is much random behavior in such a series, but it is substantially different from a random walk. . .the primary element being that the price series will simply not be free to wander as much as they would if the series were a pure random walk."

Cootner attempted to validate his model empirically, and his method and findings are reported in (9). His results support the reflecting barrier hypothesis in the statistical sense that his model cannot be rejected on the basis of the data sample analyzed. Unfortunately, the statistical techniques he used were not powerful enough to reject the basic random walk model for this data either. In comparing goodness of fit, Cootner's model did explain certain "irregularities" in his data set rather well compared to a basic random walk model.[9/]

The most worrysome aspect of Cootner's model - that intrinsic value is not stable enough - has been tested empirically by a student of Cootner's, William Steiger (27). He developed a test statistic specifically designed to test whether or not a time series was composed of "a few, relatively long trends each of which is a random walk with reflecting barriers". Based on stock prices from a 14-company sample, Steiger concluded that ". . .the

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9/ By irregularities, we are referring to such things as as a greater number of "reversals" than would be expected from a pure random walk, etc. . .things that could be explained nicely by Cootner's reflecting barrier model.

explanation offered by the Cootner model is a satisfactory description of stock price behavior."

As appealing as Cootner's market hypothesis is, it would be difficult to implement his barrier type of model in practice. The biggest drawback is establishing where the barriers should be. Obviously, they do not really exist except conceptually. Therefore, setting the boundaries is an artificial process, at best.

Another problem is how to handle the fact that intrinsic value is not observable. State transition probabilities depend on the size of the deviation from intrinsic value, so it must be estimated. This, itself, can be accomplished, but there is no practical way to account for confidence limits on this estimate within the framework of Cootner's Markov chain type of model.

An Alternative Model for Cootner's Hypothesis

Returning, now, to modeling sinking fund bond prices, we are going to accept the following concepts of Cootner as being true:

- That prices exhibit random walk type of behavior because of the activity of a class of market participants called non-professionals.
- That the range of the resulting random walk is restricted by other participants called professionals who enter the market whenever price wanders too far away from intrinsic value.

- That the arrival of new information of the sort that will cause professionals to change their estimates of intrinsic value occurs relatively infrequently (that is, P_t* is fairly stable).

We will assume that the corporate treasurer is a "professional" in Cootner's sense, and that he can estimate the intrinsic value at time t of the bond of interest. In particular, we require an estimate of intrinsic value at the surrender date. It is not necessary that this be a point estimate. Assume, in fact, that the forecast, F, is expressed in terms of a probability distribution on F.

Following Cootner's theory, the actions of market professionals will constrain the market price from being too far from F at the surrender date. Instead of using "barriers" to restrict the range of the walk, it is much cleaner mathematically to simply condition the walk to sample paths passing through a neighborhood of F at the surrender date. How this is accomplished is discussed next.

Conditional Random Walk

It is well known that a sample path of a random walk of length n can be modeled by recording the outcomes of n successive tosses of a coin. Suppose +1 is recorded for a head (H) and -1 for a tail (T), and suppose we keep track of the cumulative score S_k after the number of tosses, k. Then the final score S_n can take on integer values between -n and +n and the distribution of S_n is binomial. The probability of H heads occurring, and hence a score of H-(n-H)=2H-n is given by

$$Pr(S_n=2H-n)=\binom{n}{H}\left(\frac{1}{2}\right)^{H}\left(\frac{1}{2}\right)^{n-H}=\frac{n!}{H!(n-H)!}2^{-n}$$

and the final score has mean zero and standard deviation $\sqrt{n}$.

This coin tossing model results in the fan shaped domaine that has been discussed earlier, and which is illustrated in Figure 6a below.

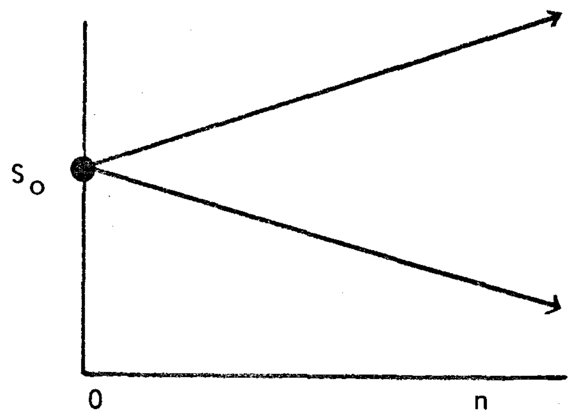

Figure 6a. Unrestricted Walk

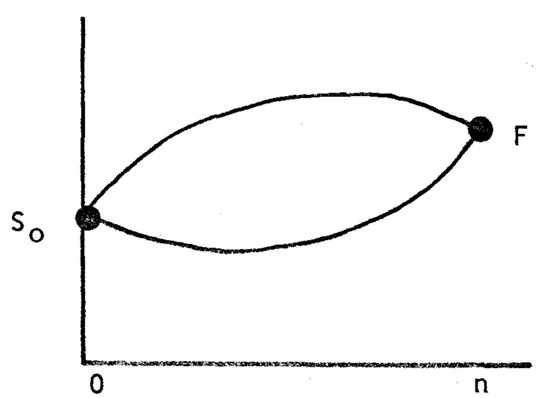

Figure 6b. Constrained Walk

If we have a forecast available, we would like to modify the coin tossing mechanism to generate only sample paths that terminate at F; that is, paths for which S_n=F (see Figure 6b). This can be done by making transition probabilities for the random walk conditioned on S_n=F, a process that is easily accomplished and which will be demonstrated below by an example. The important thing to note about this procedure, however, is that an individual sample path arising from the coin tosses conditioned on S_n will look identical to the paths arising from the unconstrained coin tossing situation (i.e., a sequence of unit steps either up or down). The conditioning process does not alter the fundamental vibration of the random walk, it only restricts its domaine. Conditioning is simply a way of rescaling the probabilities of each sample path of Figure 6a in such a way that all of the probability density is assigned to the subset of sample paths represented by Figure 6b.

Example 1

If we assume that the outcome of each coin toss is fixed before the sequence begins, then a final score of 2H-n must have resulted from a set of coins originally marked with H heads and (n-H) tails, since the number of heads minus number of tails equals the score. This knowledge can be incorporated into the coin tossing mechanism by allowing the probability of a head or tail to depend on the number of heads that have come up prior to the present toss. A numerical example should make this clear.

Suppose ten tosses are to be made, N=10, and the final score is projected to be 2, i.e., $F=s_{10}=2$. Then

$$2H-N=S_N \longrightarrow 2H-10=2$$

$$H+T=N \longrightarrow H+T=10$$

are the conditions used to specify the prior marking of heads (H) or tails (T) on the coins. Solving the equations, we find H=6 and T=4. Then, the state transition probabilities are no longer plus or minus 1/2 like they are in the unconstrained random walk, but instead depend on the current "state of the system" defined as follows:

> The state of the system is defined by the tuple (H,N), where H is the number of heads that have been observed before the Nth toss of a coin. That is, if four heads and two tails resulted (in any order) from six tosses, the current state is denoted by (4,7).

A table corresponding to a particular sequence of ten coin tosses (H H T H H H T H T T) is shown below, along with a diagram of the rescaled probability space defined by conditioning on the forecast F=2.

TABLE 3. TRANSITION PROBABILITIES OF A CONSTRAINED RANDOM WALK

TOSS NUMBER (N)	CURRENT STATE (H,N)	TRANSITION PROBABILITIES Pr(H)	Pr(T)	ACTUAL OUTCOME	SCORE (S_N)
1	(0,1)	6/10	4/10	H=+1	1
2	(1,2)	5/9	4/9	H=+1	2
3	(2,3)	4/8	4/8	T=-1	1
4	(2,4)	4/7	3/7	H=+1	2
5	(3,5)	3/6	3/6	H=+1	3
6	(4,6)	2/5	3/5	H=+1	4
7	(5,7)	1/4	3/4	T=-1	3
8	(5,8)	1/3	2/3	H=+1	4
9	(6,9)	0	1	T=-1	3
10	(6,10)	0	1	T=-1	2

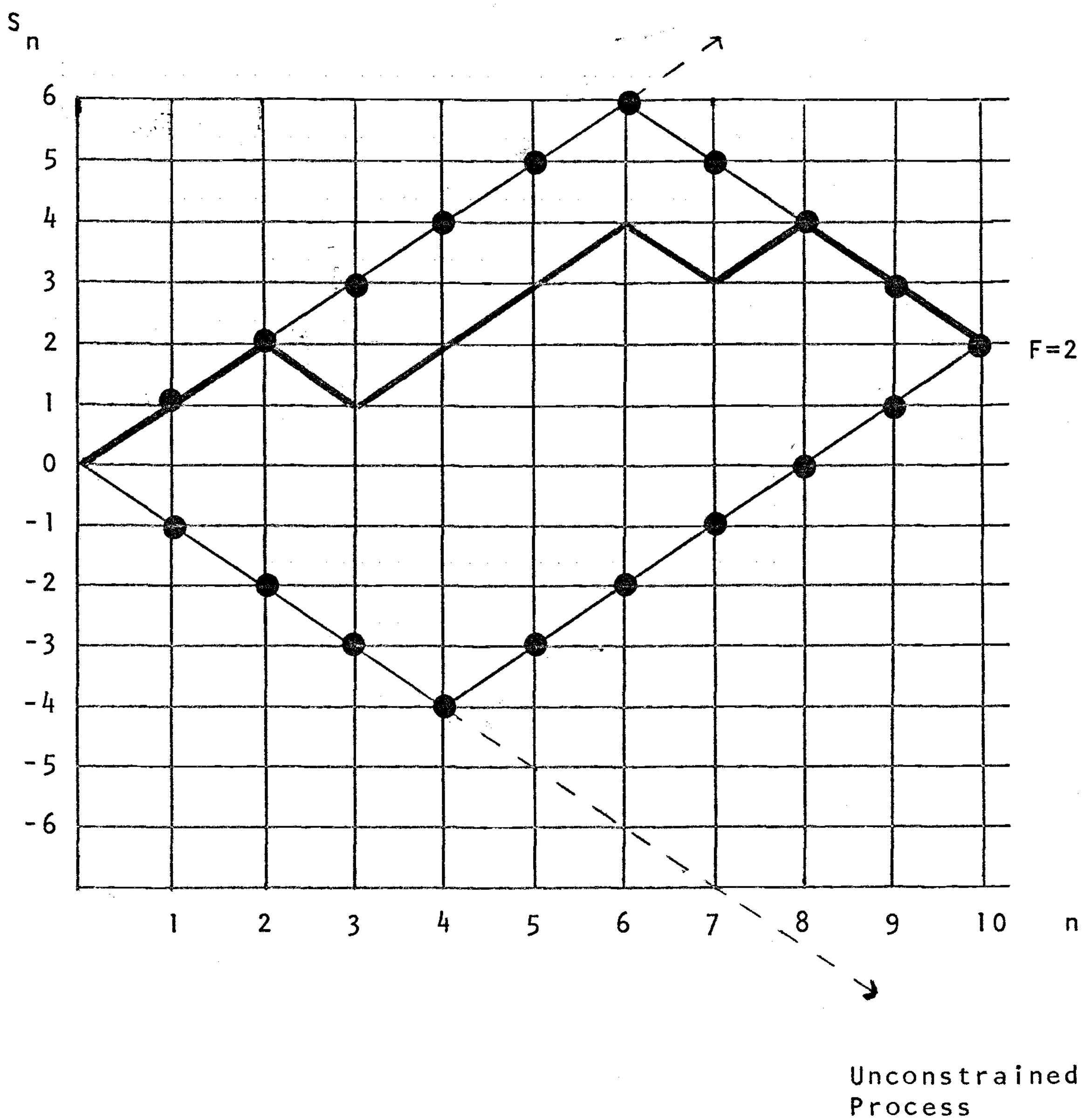

Figure 7. Rescaled State Space Caused By Conditioning the Random Walk

A random walk conditioned on F at the surrender date will be used as the standard price model in the remainder of this paper. However, we will make an extension to it which will make it even more useful. This change involves weakening the assumption we have made about the forecast, F. Instead of requiring F to be a point estimate of bond prices at the surrender date, we will generalize the procedure to allow F to be described by a probability distribution. This procedure is demonstrated in the next section.

Conditioning a Random Walk to a Neighborhood of F

A random walk is described by N tosses of a coin, with the outcome of the ith toss, Z_i, scored as +1 or -1 corresponding to a head (H) or tail (T), respectively. The basic probability space is the space of sequences $Z=Z_1,Z_2,\ldots Z_N)$, and the final score S_N is given by

$$S_N = \sum_{i=1}^{N} Z_i$$

If the walk is not constrained, the final score after N tosses can be any integer between -N and N. However, suppose that J is a subset of the integers [-N,N] and that we can predict that the final score will be one of the values of J. We are interested in how this changes the transition probabilities of the original walk.

Assume that a probability measure $F(\cdot)$ is available that gives the probability of a particular score. That is, $F(j)=Pr(S_N=j)$. Note that $F(\cdot)=0$ for all integers not in the prediction subset j.

Conditioning on $F(\cdot)$ is a process whereby the original unconstrained probability space is rescaled so that all the probability weight is assigned to sample paths that give scores in the prediction subset rather than over the entire interval $[-N,N]$. Furthermore, the sample paths are weighted so that the relative probability that a particular score will result is consistent with $F(\cdot)$.

An algorithm is developed below which will accomplish this rescaling process and calculate new transition probabilities. The algorithm relies on two simple Lemmas.

Lemma 1

Let $\underline{a}=(a_1,a_2,\ldots,a_n)$ and $\underline{b}=(b_1,b_2,\ldots,b_n)$ be any two sequences of n tosses from a N-step random walk conditioned on a subset J of the integers $[-N,N]$. Suppose that each sequence contains k heads and (n-k) tails, but in different orders. Then

(i) The probabilities of $\underline{a}$ and $\underline{b}$ are equal.

(ii) If $\underline{e}$ is any event that can occur on tosses (k+1) through N, the probability of $\underline{e}$ conditioned on $\underline{a}$ or $\underline{b}$ are equal.

Proof

Suppose an urn contains R red balls and W white balls (R+W=N). Suppose $\underline{a}$=(rrwr) and $\underline{b}$=(wrrr) are two sequences of 4 random draws from the urn, with each sequence having 3 red and 1 white balls drawn, but in a different order. Then, the assertion of Lemma 1 is that $Pr(\underline{a})=Pr(\underline{b})$.

Now

(Multiplicative law of conditional probability)

$$\begin{aligned} Pr(\underline{a}) &= Pr(rrwr) \\ &= Pr(r)\cdot pr(r/r)\cdot Pr(w/rr)\cdot Pr(r/rrw) \\ &= \frac{R}{N}\cdot\frac{(R-1)}{(N-1)}\cdot\frac{W}{(N-2)}\cdot\frac{(R-2)}{(N-3)} \\ &= \frac{W}{N}\cdot\frac{R}{(N-1)}\cdot\frac{(R-1)}{(N-2)}\cdot\frac{(R-2)}{(N-3)} \\ &= Pr(wrrr) \\ &= Pr(\underline{b}) \end{aligned}$$

This proves Part (i).

To prove Part (ii), let $\underline{e}$ be the event that the fifth ball drawn is red. Then, we want to show that

$$Pr(\underline{e}|\underline{a})=Pr(\underline{e}|\underline{b})$$

By definition

$$Pr(\underline{e}|\underline{a})=Pr(\underline{ea})/Pr(\underline{a})$$

and by Part (i):

$$\begin{aligned} &= Pr(rrwrr)/Pr(\underline{b}) \\ &= \frac{R}{N}\cdot\frac{(R-1)}{(N-1)}\cdot\frac{W}{(N-2)}\cdot\frac{(R-2)}{(N-3)}\cdot\frac{(R-3)}{(N-4)}/Pr(\underline{b}) \\ &= \frac{W}{N}\cdot\frac{R}{(N-1)}\cdot\frac{(R-1)}{(N-2)}\cdot\frac{(R-2)}{(N-3)}\cdot\frac{(R-3)}{(N-4)}/Pr(\underline{b}) \\ &= Pr(\underline{be})/Pr(\underline{b}) \\ &= Pr(\underline{e}/\underline{b}) \end{aligned}$$

This completes the proof of Lemma 1.

We'll now state and prove Lemma 2.

Lemma 2

Let p(n,k)=the probability of a particular sequence a of n tosses, k of which are heads.

(i) If n tosses have been made, k of them heads, then the probability that the (n+1)st toss is a head is

$$Pr(H)=p(n+1,k+1)/p(n,k)$$

and the probability of a tail is

$$Pr(T)=p(n+1,k)/p(n,k)$$

(ii) $p(n,k)=(n+1,k+1)+p(n+1,k)$

$$= p(n+1,k+1)$$

Proof

(i) Let $\underline{a}=(a_1,a_2,\ldots,a_n)$ be a sequence of n tosses, k of them heads with each a_k=+1 or -1. Then the probability of a is p(n,k). Let H be the probability that the (n+1)st toss is a head given a. Then the probability of the sequence $a'=(a_1,a_2,\ldots,a_n,+1)$ is p(i,j)H, from the definition of conditional probability P(AB)=P(A)p(B/A). Note, also, that H does not depend on the particular sequence a by Part (ii) of Lemma 1. However, a is a sequence of n+1 tosses with k+1 heads, so its probability is p(n+1,k+1). Thus, p(n+1,k+1)=p(n,k)H, so the transition probability H is given by Pr(H)=p(n+k,k+1)/p(n,k).

Similarly, the probability of a tail on the next toss given <u>a</u> is $Pr(T)=p(n+1,k)/p(n,k)$.

(ii) Since $Pr(H)+Pr(T)=1$, from Part (i):

$$\frac{p(n+1,k+1)}{p(n,k)}+\frac{p(n+1,k)}{p(n,k)}=1$$

or

$$p(n,k)=p(n+1,k)+p(n+1,k+1)$$

<u>This ends the proof of Lemma 2</u>. We can now show how to develop the transition probabilities.

<u>Algorithm 1: Calculation of Transition Probabilities</u>

Step (i):

$$p(N,j)=\begin{cases}\dfrac{F(2j-N)}{\binom{N}{j}}, & \text{for } j\varepsilon j\\ 0, & \text{for } j\notin j\end{cases}$$

Step (ii): For $n=1,2,\ldots,(N-1)$ and $j=0,1,\ldots,N$

$$p(n,j)=p(n+1,j)+p(n+1,j+1)$$

Step (iii): (Transition Probabilities)

(H) The probability that the next toss is a head given j out of the first n tosses have been heads is given by $H(n,j)=p(n+1,j+1)/p(n,j)$.

(T) The probability that the next toss is a tail given j out of the first n tosses have been heads is given by $T(n,j)=p(n+1,j)/p(n,j)$.

<u>Proof of Algorithm 1</u>

Step (i): Every sample path having j heads in N trials will have a score of $2j-N$. There are $\binom{N}{j}$

sample paths with this score and by Part (i) of Lemma 1 each is equally likely to occur. Therefore, the probability of a score of (2j-N). . .by definition F(2j-N). . .is given by

$$F(2j-N) = \binom{N}{j} p(N,j)$$

The probability of a particular sample path having j heads is, therefore, given by

$$p(N,j) = \frac{F(2n-N)}{\binom{N}{j}}$$

Step (ii): This is a consequence of Lemma 2 (ii).

Step (iii): Follows from Lemma 2 (i).

Example 2: Calculating Transition Probabilities Using Algorithm 1

Suppose ten tosses of a coin are to be made (N=10), and the final score is predicted to be 0, 2, or 4 with probabilities (1/4, 1/2, 1/4) respectively. This example is similar to the conditional random walk of page 42, except F is now allowed to have more than one value.

Knowing the number of tosses and the final score allows us to predetermine the number of heads and tails that will be tossed for each score to result. This is illustrated, below, for this example.

Predicted Score S_N	Can Only Occur for N=10 if: Number Heads	Number Tails	Estimated Probability of Score, S_N
0	5	5	1/4
2	6	4	1/2
4	7	3	1/4

Using the algorithm and working backwards in time:

(i) $p(10,5)=\frac{1/4}{\binom{10}{5}} = .0010$

$p(10,6)=.0024$

$p(10,7)=.0021$

$p(10,j)=0$ for $j\neq 5, 6, 7$

(ii) $p(9,4)=p(10,5)+p(10,4)$

$=.0010+0=.0010$

$p(9,5)=p(10,6)+p(10,5)$

$=.0024+.0010$

$=.0034$

$p(9,6)=p(10,7)+p(10,6)$

$=.0021+.0024$

$=.0045$

etc.

The complete set of p(i,j) for this example are tabulated on the following page in Table 4.

TABLE 4. p(i,j)=PROBABILITY (i TOSSES HAVE BEEN MADE, j OF THEM HEADS)

(S_N) j \ i	10	9	8	7	6	5	4	3	2	1	0
(-10) 0	-	-	-	-	-	.0010	.0084	.0399	.1400	.4032	1.00
(-8) 1	-	-	-	-	.0010	.0074	.0315	.1001	.2632	.6048	
(-6) 2	-	-	-	.0010	.0064	.0241	.0686	.1631	.3416		
(-4) 3	-	-	.0010	.0054	.0177	.0445	.0945	.1785			
(-2) 4	-	.0010	.0044	.0123	.0268	.0500	.0848				
(0) 5	.0010	.0034	.0079	.0145	.0232	.0340					
(2) 6	.0024	.0045	.0066	.0087	.0108						
(4) 7	.0021	.0021	.0021	.0021							
(6) 8	-	-	-								
(8) 9	-	-									
(10) 10	-	-									

Let $H(i,j)$=probability that the next toss is a head given j out of the first i tosses have been heads. From Part (iii) of the algorithm (using the values of Table 4) the rescaled state transition probabilities for the conditional random walk can be computed. Table 5 (next page) gives the values of $H(i,j)$ for this example.

TABLE 5. TRANSITION PROBABILITIES, H(i,j)=PROBABILITY THAT NEXT TOSS IS A HEAD GIVEN j OUT OF FIRST i TOSSES HAVE BEEN HEADS

j \ i	0	1	2	3	4	5	6	7	8	9
0	.60	.65	.72	.79	.88	1.00	-	-	-	-
1		.56	.62	.68	.77	.86	1.00	-	-	-
2			.52	.58	.65	.73	.84	1.00		
3				.47	.53	.60	.69	.81	1.00	
4					.40	.46	.54	.64	.77	1.00
5						.32	.38	.46	.57	.71
6							.19	.24	.32	.47
7								-	-	-

It is important to see the effect that the forecast F has on the transition probabilities. Table 6, below, compares the transition probabilities H(i,j) for example 1, for which F=2, and for example 2, for which [F=0,2,4].

TABLE 6. COMPARISON OF H(i,j) FOR EXAMPLE 1 VERSUS EXAMPLE 2

Toss Number	Cum Nbr. of Heads	Example 1 Probability (H)	Example 2 Probability (H)	Actual Outcome
1	0	.60	.60	H
2	1	.26	.56	H
3	2	.50	.52	T
4	2	.57	.58	H
5	3	.50	.53	H
6	4	.40	.46	H
7	5	.25	.38	T
8	5	.33	.46	H
9	6	0	.32	T
10	6	0	.47	T

Note, also, the significant effect conditioning has on the transition probabilities of the random walk. In the conventional unconstrained random walk, P(H)=P(T)=1/2 regardless of how many heads or tails have come up in the past.

As a check on the validity of the algorithm, we will use it to calculate the transition probabilities for a walk constrained to F=2, i.e., the condition of example 1.

In this case

$$F(j)=\begin{cases} 1 & \text{for } j=6 \\ 0 & \text{for otherwise} \end{cases}$$

Then, p(i,j) is calculated recursively using Algorithm 1 as shown in Table 7.

Step (i): $p(10,6)=\frac{1}{\binom{10}{6}}=\frac{1}{210}=.0048$

Step (ii): $p(9,5)=p(10,6)+p(10,5)=.0048$

$p(9,6)=p)10,7)+p(10,6)=.0048$

etc.

TABLE 7. p(i,j) FOR EXAMPLE 1 ON PAGE 41

j \ i	10	9	8	7	6	5	4	3	2	1	0
0	-	-	-	-	-	-	.0048	.0336	.1344	.4032	1.00
1	-	-	-	-	-	.0048	.0288	.1008	.2688	.6048	
2	-	-	-	-	.0048	.0240	.0720	.1680	.3360		
3	-	-	-	.0048	.0192	.0480	.0960	.1680			
4	-	-	.0048	.0144	.0288	.0480	.0720				
5	-	.0048	.0096	.0144	.0192	.0240					
6	.0048	.0048	.0048	.0048	.0048						
7	-	-	-	-							
8	-	-	-								
9	-	-									
10	-										

For simplicity, only the transition probabilities corresponding to the sequence (H H T H H H T H T T) assumed for example 1 are calculated in Table 8. These are the same as those given on page 43, which shows that the algorithm works as it should (see next page).

TABLE 8. TRANSITION PROBABILITIES FOR EXAMPLE 1
CALCULATED BY ALGORITHM 1

j \ i	0	1	2	3	4	5	6	7	8	9
0	.60									
1		.56								
2			.50	.57						
3					.50					
4						.40				
5							.25	.33		
6									0	0
7										

IV. SOLUTION OF THE SINKING FUND PROBLEM

This chapter develops two algorithms for solving the sinking fund problem. While both use a dynamic programming approach, they differ with respect to the objective function to be minimized. The first section discusses why two different objective functions should be considered. The two are:

(1) Minimize expected unit cost, and

(2) Minimize maximum "regret".

The sinking fund problem is then formulated for both objective functions and a recursive type of algorithm is developed to solve each case. Both cases assume that realistic price sample paths can be generated by the constrained random walk model developed in Chapter III. Finally, each algorithm is demonstrated by means of a simple example. Later, in Chapter V, these two models will be shown to outperform dollar averaging as a sinking fund purchase strategy using both simulated and actual price data.

Objective Function

In a probabilistic situation, the most common objective function (choice criterion) is the expected monetary value (EMV) criterion which, for the sinking fund problem, would be expressed as "minimize expected unit cost". This objective function has a certain attractiveness in situations which occur repeatedly and for which the long run properties of mathematical expectation apply. We assume that this is the case, here, and consequently have elected to minimize the <u>expected cost per unit</u> in one approach.

Savage's "regret" criterion is also thought to be an appropriate choice criterion. Under this criterion, the corporate treasurer acts so as to minimize maximum regret (or opportunity cost), where regret is defined as the difference between

(1) the price actually paid for the sinking fund securities, and

(2) the lowest market price observed over the purchase horizon.

The principle argument for using this criterion has to do with "second guessing", and is as follows.

The corporate treasurer has no alternative but to buy bonds <u>sometime</u> during the year to surrender for the sinking fund. His performance can thus be measured (and second guessed) by comparing the actual sinking fund cost against the absolute minimum cost over the interval computed in retrospect, i.e., after prices become known. Therefore, decisions made using the minimax regret criterion as we have defined it may in fact be a better reflection of the treasurer's utility function than EMV. Optimal decision rules corresponding to both the EMV and regret choice criteria will be developed under the assumption that the constrained random walk model of Chapter III is a realistic model for bond price movements.

Minimize Expected Unit Cost Formulation

The sinking fund problem is modeled as a constrained random walk which is observed at time points (stages) corresponding to N successive tosses of a coin. Stages are labelled n=0,1,2,. . .,N, where N corresponds to the sinking fund surrender date.

If the process is in state j at time n, this means that j heads and (n-j) tails have resulted from n tosses. As usual, a head is scored as +1 and a tail as -1. The state space consists of non-negative integers 0, 1, 2,. . .,j*.

In order to apply the coin tossing probability model it is necessary to structure the problem in the following way.

X-axis: The time horizon is divided into N intervals.

Y-axis: The Y-axis represents the market price of the bond except that price is transformed by a change of scale. A transformation is required because the coin tossing model requires that the random walk have unit steps.

Step Size: There should be a correspondence between the step size and the number of steps, N; also, between the step size and the volatility of the price series. The following transformation accomplishes this.

Change of Scale:

Let

$$S_n = \frac{P_n - P_0}{\Delta}$$

where

S_n = the scaled (Y-axis) value for price (also referred to as "score" at time n.

P_n = the price at time n which is to be transformed.

P_0 = the price at the beginning of the horizon (assumed known).

Δ = the standard deviation of price changes measured over intervals corresponding to one step of the walk.

With this transformation, the score $S_n=2j-n$ represents a market price of

$$P_n=(2j-n)\Delta+P_0$$

and the random walk always starts at zero.

We will define a function F(h) which gives the probability that the number of heads that are observed in N tosses of a coin is h. F(h) is a probability mass function which associates probability weight to the integers in the range $j_* \leq h \leq j^*$. F(h) is determined from the price forecast that we assume is available by making use of the correspondence between score and the number of heads, S=2h-N. In other words,

$$Pr(h=\frac{S+N}{2})=Pr(\text{Score at } N=S)$$

The probability distribution of score at N is our price forecast.

After observing that the state is j at time n, one of two possible actions must be taken:

(i) Stop (action 1) with a score of j-(n-j)-2j-n

(ii) Continue (action 2) at a cost of C(j).[10/]

The objective is to act in such a way that the expected score is minimized.

If we let X_n denote the state of the process at time n, and if action 2 is chosen, then the next state is determined by the transition probabilities:

$$Pr(X_n+1=j+1 \mid X_n=j) = p(n+1,j+1/p(n,j)$$

$$Pr(X_n+1=j \mid X_n=j) = p(n+1,j)/p(n,j)$$

where the p(n,j)'s are defined by Algorithm 1 of Chapter III. After a transition at time (N-1), the system ceases to operate and action 1 is mandatory.

Solution Algorithm

For a given stage n, define $V_n(j)$ as follows:

$V_n(j)$=the minimum expected "score" that can be attained by following an optimal policy at stage n and thereafter, starting from state j.

10/ The cost C(j) is a function of two interest rates:
a) The interest cost of keeping the bonds outstanding for one more period - a constant depending on the coupon rate of the debenture.
b) The "savings" associated with being able to make use of the funds which would otherwise be employed to purchase the sinking fund bonds (a function of the current state j).
This subject is discussed in more detail in Chapter VI, but it should be noted that it is possible for C(j) to be a negative cost.

From the principal of optimality of dynamic programming, we know that an optimal policy (set of actions) can be found from the following recursion:

Algorithm 2

(i) $V_N(j)=\begin{cases} 2j-N, & \text{for } j_* \leq j \leq j^* \\ 0, & \text{for } j \text{ otherwise} \end{cases}$

since the only action possible at stage N is the stop action.

(ii) For $n=(N-1),(N-2),\ldots, 2, 1, 0$ and $j=0, 1, 2., \ldots, j^*$.

a) Set $A(1)=2j-n$

b) Set $A(2)=C(j)+[+1+V_{n+1}(j+1)]Pr(H)$
$+[-1+V_{n+1}(j)]Pr(T)$

where

$Pr(H)=p(n+1,j+1)/p(n,j)$

$Pr(T)=p(n+1,j)/p(n,j)$

c) Set $V_n(j)=\min[A(1),A(2)]$

and

If $A(1)\leq A(2)$, stop with score A(1)

If $A(2)<A(1)$, continue for at least one more period to receive the expected score A(2).

Algorithm 2 provides a recursive scheme for sequentially computing an optimal policy for the sinking fund problem. This procedure divides the state space into two sets - a continuation set and a stopping set. Bonds will be bought the first time a stopping state is encountered.

An example is given later to demonstrate the computations involved in the algorithm. However, before going to this example, we should consider the problem of how many bonds to buy when a stopping state is reached.

In the case where the objective function involves expected value, it turns out that the entire sinking fund requirement will either be bought all at once, or not at all in the optimal solution. This is proved in the following theorem.

Theorem 1

Suppose we have a total of \$Q par value of bonds to buy and that we are considering a purchase of some partial amount, q, when there are n trading days remaining. Then either q=Q, or q=0 is optimal at stage n under the EMV criterion.

Proof

Let the price for q units of bonds be P_n per unit at stage n. The minimum expected cost per unit for bonds purchased by buying q units now and (Q-q) units later is

$$V_n = \frac{qP_n + (Q-q)V_{n+1}}{Q}$$

where the right hand side of this expression is simply the total cost which results from buying q units at P_n and (Q-q) units at the lowest expected unit cost available over the remaining trading days in the horizon (i.e., $n=n+1,n+2,\ldots,N$), divided by the total quantity purchased.

This expression can be written as

$$V_n=\frac{q}{Q}[P_n-V_{n+1}]+V_{n+1}$$

Now, if P_n is greater than V_{n+1}, it is apparent that V_n will be more than V_{n+1} unless we put $q=0$, which means buy none. If P_n is less than V_{n+1}, we should attempt to increase q to the maximum possible amount, which means put $q=Q$. Therefore, in order to achieve the smallest possible cost per unit, we need not consider partial sales.

As a practical matter, it may not be possible to buy the entire sinking fund requirements at one time. Prices are negotiated prices for blocks of this size, and are likely to be quoted in the following manner:

Price at time n	Quantity Available at this Price
P_n^1	0 to q_1
P_n^2	q_1 to q_2
P_n^3	q_2 to Q

where

$$P_n^1<P_n^2<P_n^3$$

When the model signals "stop" at period n, the corporate treasurer should try to buy as much as he can at this price. Assuming that he can buy up to $q_1<Q$ at $\$P_n^1$, then he does so, and the new, reduced problem is simply one where there are (N-n) time

periods left in which to buy $\$(Q-q_1)$ of bonds according to decision rules established by the model. The initial state of the system is set at P_n^2, which will be a continuation state, and the process begins anew.

Example 3: Algorithm to Minimize Expected Unit Cost

Suppose ten tosses of a coin are to be made (N=10), and that the final score (price) is predicted to be 0, 2, or 4 with probabilities 1/4, 1/2, and 1/4, respectively. The process starts at zero, so prices are predicted to increase over the horizon. The relationship between score and the number of heads is given below.

Predicted Score S_N	Can Only Occur for N=10 if: Number Heads	Number Tails	Estimated Probability of Score, S_N
0	5	5	1/4
2	6	4	1/2
4	7	3	1/4

From the figures given above, the probability distribution for the number of heads in ten tosses is:

$$F(5)=1/4$$

$$F(6)=1/2$$

$$F(7)=1/4$$

and

$$F(j)=0 \text{ for } j=0,\ 1,\ 2,\ 3,\ 4$$

State transition probabilities, $p(n,j)$, for this example are those calculated by Algorithm 1 earlier in Example 2, and are given on page 54.[11/]

The objective is to minimize the expected score, and the optimal solution is found by Algorithm 2. To simplify the calculations, assume C(j)=0.

11/ Actually, Table 5 gives only the probability of a head being tossed. The complement is obtained by subtracting the Table 5 value from one.

Algorithm 2

N=10

j	$V_{10}(j)=2j-10$	$V_{10}(j)$	Optimal Action*
5	$V_{10}(5)=(2)(5)-10=0$	0	S
6	$V_{10}(6)=(2)(6)-10=2$	2	S
7	$V_{10}(7)=(2)(7)-10=4$	4	S

*Stop=S

Continue=C

N=9 $V_9(j)=\min[A(1),A(2)]$

j	A(1)=Stop	A(2)=Continue	$V_9(j)$	Optimal Action
4	-1	(+1+0)(1)+0=1	-1	S
5	1	(+1+2)(.71)+(-1+0)(.29)=1.84	1	S
6	3	(+1+4)(.47)+(-1+2)(.53)=2.88	2.88	C
7	5	0+(-1+4)(1)=3.00	3.00	C

N=8 $\quad V_8(j)=\min[A(1),A(2)]$

j	A(1)=Stop	A(2)=Continue	$V_8(j)$	Optimal Action
3	-2	(+1-1)(1)+0=0	-2	S
4	0	(+1+1)(.77)+(-1-1)(.23)=1.08	0	S
5	2	(+1+2.88)(.57)+(-1+1)(.43)=2.21	2	S
6	4	(+1+3)(.32)+(-1+2.88)(.68)=2.56	2.6	C
7	6	0+(-1+3)(1)=2	2.0	C

N=7 $\quad V_7(j)=\min[A(1),A(2)]$

j	A(1)=Stop	A(2)=Continue	$V_7(j)$	Optimal Action
1	-4	(1-3)(1)+0=-2	-4	S
2	-2	(1=1)(.84)+(-1-3)(.16)=-.64	-2	S
3	0	(1+1)(.69)+(-1-1)(.31)=.76	0	S
4	2	(1+2.2)(.54)+(-1+1)(.46)=1.73	1.7	C
5	4	(1+1.9)(.38)+(-1+2.2)(.62)=1.84	1.8	C
6	6	(1+1.0)(.19)(.81)=1.1	1.1	C

N=5 $V_5(j)=\min[A(1),A(2)]$

j	A(1)=Stop	A(2)=Continue	$V_5(j)$	Optimal Action
0	-5	(1-4)(1)+0=-3	-5	S
1	-3	(1-2)(.86)+(-1-4)(.14)=-1.6	-3	S
2	-1	(1+0)(.73)+(-1-2)(.27)=.08	-1	S
3	1	(1+1.7)(.6)+(-1+0)(.4)=1.22	1	S
4	3	(1+1.8)(.46)+(-1+1.7)(.54)=1.7	1.7	C
5	5	(1+1.1)(.32)+(-1+1.8)(.68)=1.2	1.2	C

N=4 $V_4(j)=\min[A(1,A(2)]$

j	A(1)=Stop	A(2)=Continue	$V_4(j)$	Optimal Action
0	-4	(1-3)(.88)+(-1-5)(.12)=2.5	-4	S
1	-2	(1-1)(.77)+(-1-3)(.23)=-.92	-2	S
2	0	(1+1)(.65)+(-1-1)(.35)=+.60	0	S
3	2	(1+1.7)(.53)+(-1+1)(.47)=1.43	1.43	C
4	4	(1+1.2)(.40)+)-1+1.7)(.60)=1.30	1.30	C

N=3 $V_3(j)=\min[A(1),A(2)]$

j	A(1)=Stop	A(2)=Continue	$V_3(j)$	Optimal Action
0	-3	(1-2)(.79)+(-1-4)(.21)=-1.8	-3	S
1	-1	(1+0)(.68)+(-1-2)(.32)=-.3	-1	S
2	1	(1+1.43)(.58)+(-1+0)(.42)=1.0	1	S,C
3	3	(1+1.3)(.47)+(-1+1.43)(.53)=1.31	1.3	C

N=2 $V_2(j)=\min[(A(1),A(2)]$

j	A(1)=Stop	A(2)=Continue	$V_2(j)$	Optimal Action
0	-2	(1-1)(.72)+)-1-3)(.28)=-1.12	-2	S
1	0	(1+1)(.62)+(-1-1)(.38)=.48	0	S
2	2	(1+1.3)(.52)+)-1+1)(.48)=1.2	1.2	C

N=1 $V_1(j)=\min[(A(1),A(2)]$

j	A(1)=Stop	A(2)=Continue	$V_1(j)$	Optimal Action
0	-1	(1+0)(.65)+(-1-2)(.35)=-.40	-1	S
1	1	(1+1.2)(.56)+(-1+0)(.44)=.79	.8	C

N=0 $V_0(j)=\min[(A(1),A(2)]$

j	$A(1)$=Stop	$A(2)$=Continue	$V_0(j)$	Optimal Action
0	0	(1+.8)(.6)+(-1-1)(.4)=.3	0	5

The solution to Example 3 is summarized below in Figure 8. For this particular example, the optimal solution is to buy bonds at the current price, paying a scaled value of zero. Waiting for a lower price is not optimal since the minimum expected cost of continuing is a scaled value of 0.3.

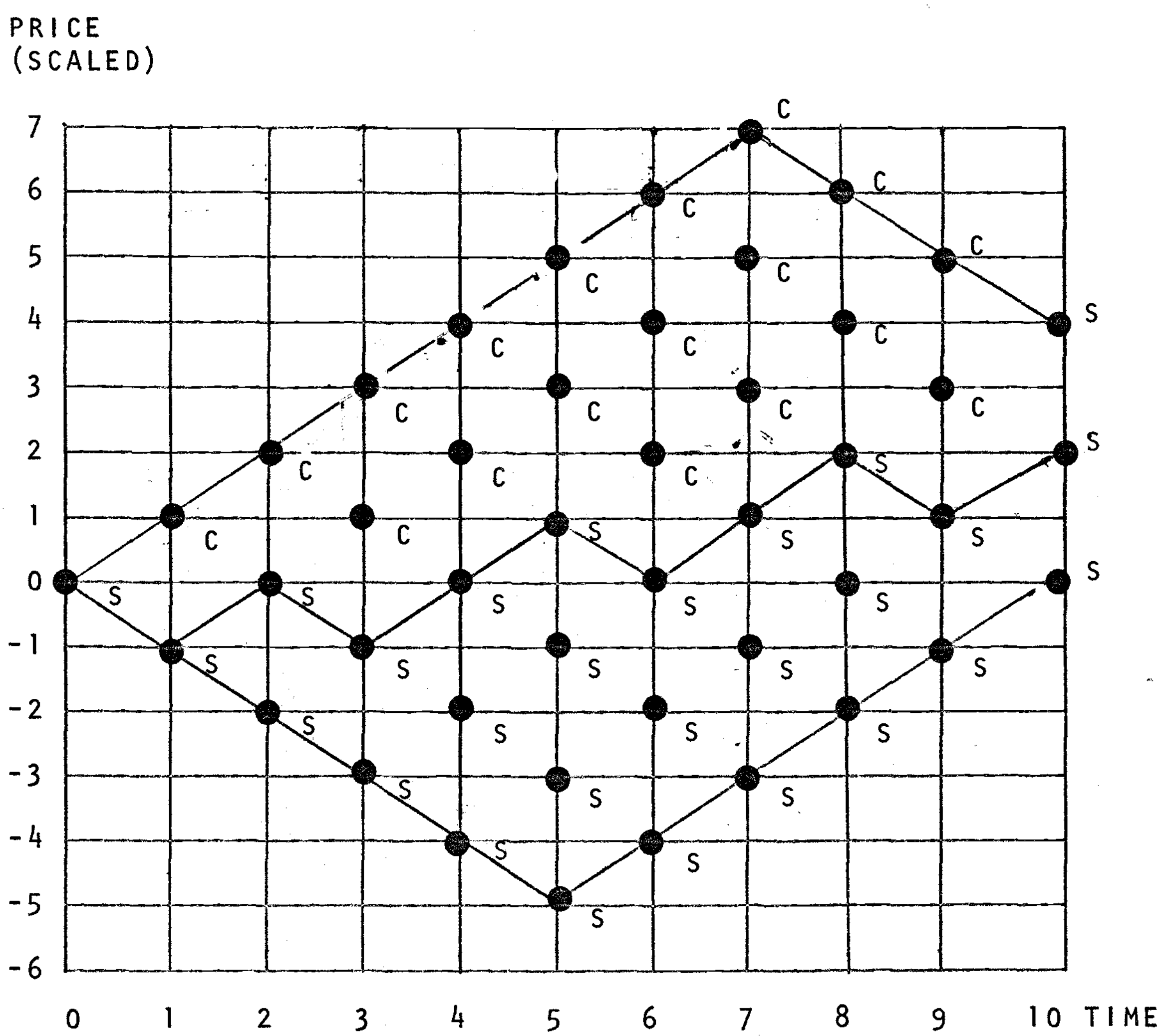

Figure 8. Solution to Example 3

Figure 8 illustrates the form a solution typically takes. A stopping boundary is defined which divides the solution space into a continuation region and a stopping region. In the case where $V_0(0)<0$, the optimal strategy is to continue until one of the stopping states is reached and then purchase the sinkers at this price.

In the next section, the sinking fund problem is formulated for a different objective function--minimize maximum regret. The example just solved using Algorithm 2 is then solved using the regret criterion. A different solution results, and the stopping characteristics of the two approaches are then compared.

Minimax Regret Formulation

The reason for considering a minimax policy was discussed earlier. What seems to be involved is a jittery feeling that a corporate treasurer experiences whenever he thinks back on a low price he passed up. The treasurer may be blamed (will blame himself) if the sinkers are bought too far from the year's low. Minimizing the maximum opportunity loss or regret, therefore, seems to be an appropriate objective function.

Definition of "Regret"

The regret a treasurer experiences is measured with respect to two prices. The first is the lowest price that is available over the entire horizon. Ideally, all of the bonds would be bought at this price. The second is the price at which the bonds are actually bought. Regret is defined as the difference between these two prices. This notion is made explicit in the mathematical formulation below.

Mathematical Formulation of the Regret Case 12/

Once again, the problem is modeled as a constrained random walk which is observed at 0, 1, 2, . . . ,N corresponding to N successive tosses of a coin. Everything is scaled so that the coin tossing model can be applied. However, in the regret formulation, the state of the system will be defined in terms of the number of tails that occur in n tosses rather than the number of heads.

If the number of tails observed in n tosses of a coin is i, then the score at stage n is given by

(1) $$S_n=(n-i)-i=n-2_i$$

Define M_n^* as the lowest score available through n periods, i.e.,

$$M_n^*=\underset{k\leq n}{\text{Min}}(S_k)$$

12/ The formulation of the regret case requires a good deal of notation, and the exposition is clearer if certain complications caused by the single period continuation cost, c(i), are discussed after the basic concept is understood. Consequently, certain refinements to the basic equations are deferred until we have formulated the problem in its easiest form and have worked through an example.

Then, the regret felt as of period n is given by $S_n - M_n^*$. This value (which will be zero if S_n is the low price up to now) is not exactly the regret we defined earlier because in the earlier definition, regret is measured against the minimum taken over the entire horizon. However, even if M_n^* is still unknown at n<N, we can still write an expression for it mathematically. This is done next.

Writing score in terms of the outcome of each individual toss, we get

$$S_n = S_o + \sum_{k=1}^{n} X_k$$

where

$$X_k = \begin{cases} 1, & \text{if the kth toss is a head} \\ -1, & \text{if the kth toss is a tail} \end{cases}$$

$S_o = 0$ due to the scaling procedure employed

Then, the lowest score over the entire horizon can be written in terms of the known minimum at an intermediate point n<N as follows:

$$M_N^* = M_n^* + \min(0, -M_n^* + S_n + \sum_{k=n+1}^{n^*} X_k) \qquad (2)$$

where n* is the state where the lowest score occurs if it does, in fact, occur after n.

With (2), we can write an expression for regret over the full horizon in terms of what is referred to as "current regret", $R = S_n - M_n^*$.

$$\text{Regret} = S_n - M_n^*$$

$$= S_n - M_n^* - \min(0, -M_n^* + S_n + \sum_{k=n+1}^{n^*} X_k)$$

$$(3) \qquad = R_n - \min(0, R_n + \sum_{k=n+1}^{n^*} X_k)$$

The expression for regret given in (3) provides the basis for solving the sinking fund problem. Note that only two terms are involved--R and a summation. The R term is fixed at stage n and reflects the regret felt by not having stopped the process sometime earlier. The summation term depends only on future coin tosses which are unknown, but which add to, or subtract from, the current regret depending on the particular outcomes.

We are now ready to define the "state" of the system as follows. If the process is in state (i,R) at stage n, then (1) n tosses have been made, i of them tails, and (2) the regret through stage n has the value R.

Figure 9 on the following page shows a typical situation where the process is in state (i,R) at stage $n<N$.

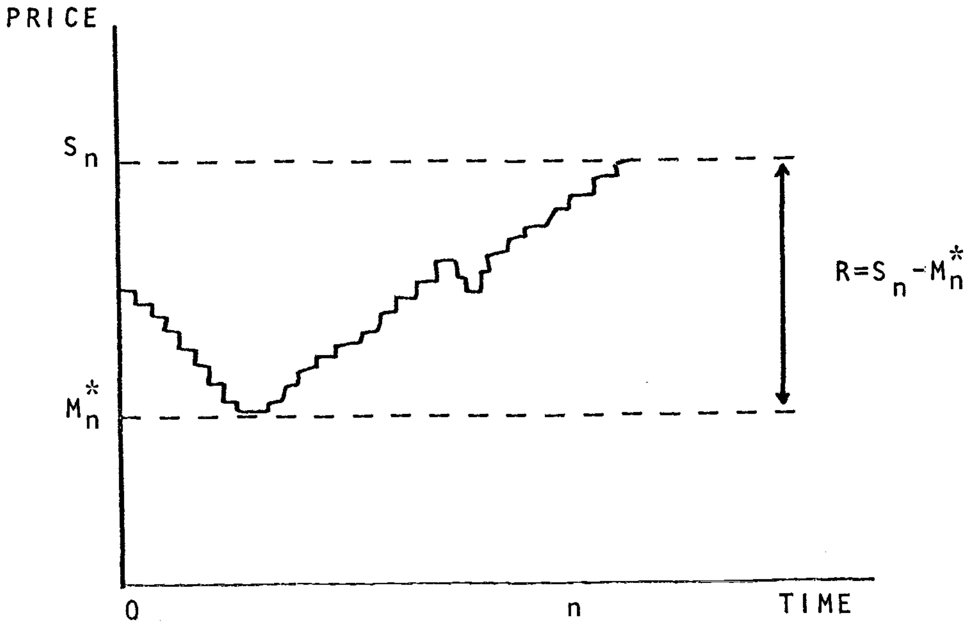

Figure 9. "Regret at Stage n<N

What we require in the regret formulation is a method for determining the maximum regret if the process is stopped (continued) at stage n<N with the process in state (i,R). Then, the optimal action under the minimax rule is to take whichever action minimizes the maximum regret.

Determining Maximum Regret: Action (1) - Stop

Let

R_n^s be the maximum regret possible if the process is stopped at state n in state (i,R).

A method for evaluating R_n^s is developed on the following page.

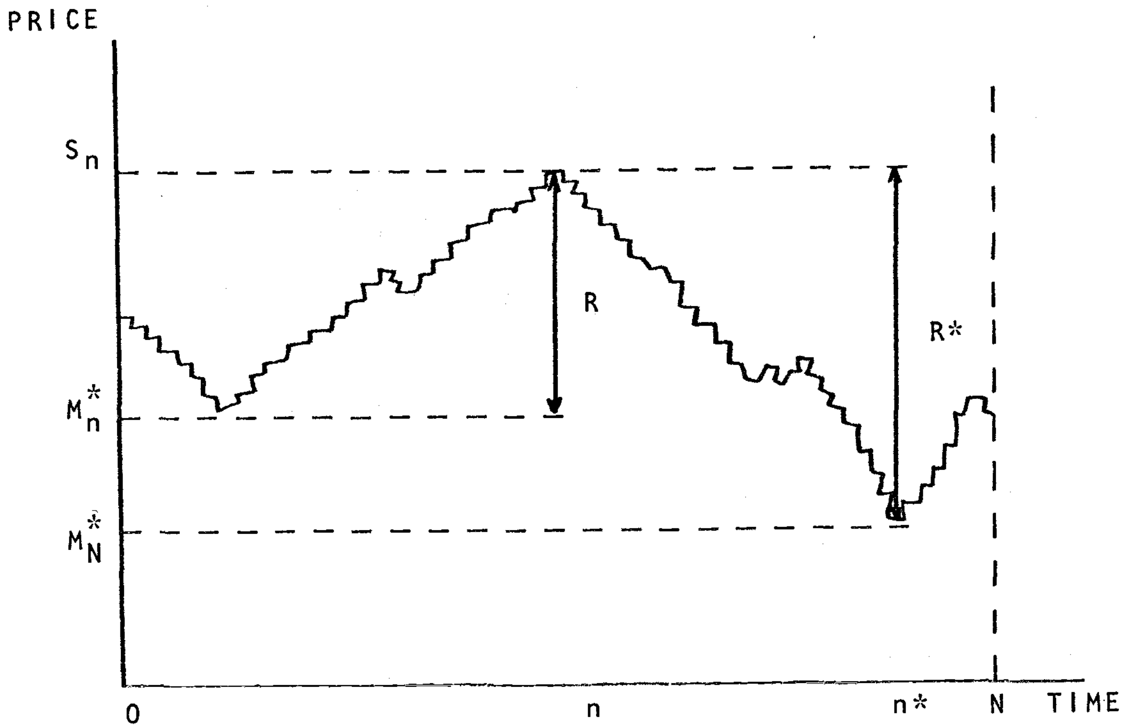

Figure 10. Maximum Regret if Process is Stopped at n<N

Referring to Figure 10 and equation (3), the current regret, R, is going to be the maximum regret R* if the sum of future coin tosses is less than, or equal to R; that is, if it is not possible to reach a new minimum in the time remaining starting from state (i,R). This can be evaluated if we can determine the maximum number of tails that can occur in the remaining (N-n) tosses given that i tails have already occurred. The assumption that price follows a constrained random walk enables us to accomplish this.

We assume that a forecast has been developed for the disbribution of price at the surrender date. Actually, we will only make use of the lowest price projected to occur at N because in terms of the coin tossing model, the lowest score corresponds to the largest number of tails. Let i* be the value of i corresponding to the lowest possible price forecast at N--say S_N--obtained by solving $S_N=N-2i^*$. Then, given the process is in state (i,R) at time n

- there are (N-n) coin tosses remaining.
- at most (i*-i) tails will occur in the remaining tosses (note that $(i^*-i)\leq(N-n)$).

Suppose (N-n)=8 and (i*-1)=3. Then there are $\binom{8}{3}=56$ possible sample paths the process can take, but the one that would produce the lowest possible score in the time remaining is the one that has tails in the next three tosses, i.e., (T T T H H H H H).

It follows from this logic that the maximum regret if we stop at stage n in state (i,R) is

(4) $$R_n^s=R-\min(0,R-D)$$

where

$$D=\min(i^*-i,N-n)$$

This is the relationship we were seeking.

<u>Determining the Maximum Regret: Action (2) - Continue</u>

Now, let

R_n^c be the maximum regret if the process is continued one more period, and a minimax strategy is followed thereafter.

$f_n(i,R)$ be the minimum (R_n^s, R_n^c) which is the value of regret at stage n if a minimax strategy is employed at stage n and thereafter.

One would expect R_n^c to be related to $f_{n+1}(i,R)$ in some fashion. To develop a solution by dynamic programming, we must define this relationship in a recursive form. This is our next step.

Suppose that the process is in state (i,R) at stage n and that we elect to continue for at least one more period by paying a cost C(i). Then, a transition will change the state to

[i+1,Max(0,R-1+C(i))] given the next toss is a tail.

or

[i,R+1+C(i)] given the next toss is a head.

If a minimax strategy is employed at stage n+1 and thereafter, then the maximum regret if action 2 is chosen at stage n is

$$(5)\quad R_n^c = \text{maximum}\begin{cases} f_{n+1}[i+1, \text{Max}(0, R-1+c(i))] \\ f_{n+1}[i, R+1+C(i)] \end{cases}$$

Since the state transition probabilities are available, it is tempting to substitute the expectation $E[R_n^c]$ for R_n^c. However, the expression given by (5) is conceptually the correct one for the minimax objective function, so the probability data will not be utilized at all in the regret formulation.

The final step is to select the action at stage n that minimizes the maximum regret. That is

If $R_n^s \leq R_n^c$ ⟶ stop at (i,R) with regret, R.

If $R_n^c < R_n^s$ ⟶ continue, with maximum regret R_n^c.

The minimax regret at stage n is therefore given by

$$f_n(i,R) = \text{minimum}[R_n^s, R_n^c]$$

A solution to this problem can be obtained by dynamic programming using the backtracking method with boundary condition

$$f_N(i,R) = R \text{ for } i = 0, 1, 2, \ldots, i^*; R \geq 0.$$

The example on page 50 will be solved using the regret criterion to demonstrate this approach.

Example 4: Minimize Maximum Regret Objective Function

In this example, ten tosses of a coin are made and the final score (price) is predicted to be 0, 2, or 4 with probabilities 1/4, 1/2, and 1/4, respectively. The number of heads and tails corresponding to these predicted scores are given below.

Predicted Score S_N	Can Only Occur for N=10 if: Number Heads	Number Tails	Estimated Probability of Score, S_n
0	5	5(i*)	1/4
2	6	4	1/2
4	7(j*)	3	1/4

The maximum number of tails corresponds to a score of zero, so i*=5. The maximum regret corresponds to the largest number of heads, so regret ranges from $0 \leq R \leq 7$. The required calculations, beginning backwards in time from state N=10, are tabulated in the following pages. To simplify the example, the continuation cost C(i) is assumed to be zero.

REGRET EXAMPLE

Stage:

n = 10

R \ i	0	1	2	3	4	5
0						
1						
2						
3						
4						
5						
6						
7						

$A = f_{11}[i+1, \text{Max}(0, R-1)]$

R \ i	0	1	2	3	4	5
0						
1						
2						
3						
4						
5						
6						
7						

$B = f_{11}(i, R+1)$

R \ i	0	1	2	3	4	5
0						
1						
2						
3						
4						
5						
6						
7						

$R^C_{10} = \text{Max}\ (A, B)$

R \ i	0	1	2	3	4	5
0				0	0	0
1				1	1	1
2				2	2	2
3				3	3	3
4				4	4	4
5				5	5	5
6				6	6	6
7				7	7	7

$R^S_{10} = R - \text{Min}[0, R - \text{Min}(N-n, i^* - i)]$

R \ i	0	1	2	3	4	5
0				0^s	0^s	0^s
1				1^s	1^s	1^s
2				2^s	2^s	2^s
3				3^s	3^s	3^s
4				4^s	4^s	4^s
5				5^s	5^s	5^s
6				6^s	6^s	
7				7^s		

$f_{10}(i, R) = \text{Min}(R^s_{10}, R^c_{10})$

REGRET EXAMPLE

Stage:

n = 9

R \ i	0	1	2	3	4	5
0			0	0	0	
1			0	0	0	
2			1	1	1	
3			2	2	2	
4			3	3	3	
5			4	4	4	
6			5	5	5	
7			6	6	6	

$A = f_{10}[i+1, \mathrm{Max}(0, R-1)]$

R \ i	0	1	2	3	4	5
0				1	1	1
1				2	2	2
2				3	3	3
3				4	4	4
4				5	5	5
5				6	6	6
6				7	7	7
7						

$B = f_{10}(i, R+1)$

R \ i	0	1	2	3	4	5
0			0	1	1	1
1			0	2	2	2
2			1	3	3	3
3			2	4	4	4
4			3	5	5	5
5			4	6	6	6
6			5	7	7	7
7			6	6	6	

$R^c_9 = \mathrm{Max}\ (A, B)$

R \ i	0	1	2	3	4	5
0			1	1	1	0
1			1	1	1	1
2			2	2	2	2
3			3	3	3	3
4			4	4	4	4
5			5	5	5	5
6			6	6	6	6
7			7	7	7	7

$R^s_9 = R - \mathrm{Min}[0, R - \mathrm{Min}(N-n, i^*-i)]$

R \ i	0	1	2	3	4	5
0			0^c	1^c	1^c	0^c
1			0^c	1^s	1^s	1^s
2			1^c	2^s	2^s	2^s
3			2^c	3^s	3^s	3^s
4			3^c	4^s	4^s	4^s
5			4^c	5^s	5^s	
6			5^c	6^s		
7			6^c			

$f_9\ (i, R) = \mathrm{Min}(R^s_9, R^c_9)$

REGRET EXAMPLE

Stage: n = 8

R \ i	0	1	2	3	4	5
0		0	1	1	0	
1		0	1	1	0	
2		0	1	1	1	
3		1	2	2	2	
4		2	3	3	3	
5		3	4	4	4	
6		4	5	5	5	
7		5	6	6	6	

$A = f_9\,[i+1, \mathrm{Max}(0, R-1)]$

R \ i	0	1	2	3	4	5
0			0	1	1	1
1			1	2	2	2
2			2	3	3	3
3			3	4	4	4
4			4	5	5	5
5			4	6	6	6
6			6	6	6	7
7						

$B = f_9\,(i, R+1)$

R \ i	0	1	2	3	4	5
0		0	1	1	1	1
1		0	1	2	2	2
2		0	2	3	3	3
3		1	3	4	4	4
4		2	4	5	5	5
5		3	5	6	6	6
6		4	6	6	6	7
7		5	6	6	6	

$R^C 8 = \mathrm{Max}\,(A, B)$

R \ i	0	1	2	3	4	5
0		2	2	2	1	0
1		2	2	2	1	1
2		2	2	2	2	2
3		3	3	3	3	3
4		4	4	4	4	4
5		5	5	5	5	5
6		6	6	6	6	6
7		7	7	7	7	7

$R^S 8 = R - \mathrm{Min}[0, R - \mathrm{Min}(N-n, i^* - i)]$

R \ i	0	1	2	3	4	5
0		0^c	1^c	1^c	1^c	0^s
1		0^c	1^c	2^c	1^s	1^s
2		0^c	2^c	2^s	2^s	2^s
3		1^c	3^c	3^s	3^s	3^s
4		2^c	4^c	4^s	4^s	
5		3^c	5^c	5^s		
6		4^c	6^c			
7		5^c				

$f_8\,(i, R) = \mathrm{Min}(R^S 8, R^C 8)$

REGRET EXAMPLE

Stage:

n = 7

R \ i	0	1	2	3	4	5
0	0	1	1	1	0	
1	0	1	1	1	0	
2	0	1	2	1	1	
3	0	2	2	2	2	
4	1	3	3	3	3	
5	2	4	4	4	4	
6	3	5	5	5	5	
7	4	6	6	6	6	

$A = f_8\,[i+1, \mathrm{Max}(0, R-1)]$

R \ i	0	1	2	3	4	5
0		0	1	2	1	1
1		0	2	2	2	2
2		1	3	3	3	3
3		2	4	4	4	4
4		3	5	5	5	5
5		4	6	6	6	6
6		5	6	6	6	7
7						

$B = f_8\,(i, R+1)$

R \ i	0	1	2	3	4	5
0	0	1	1	2	1	1
1	0	1	2	2	2	2
2	0	1	3	3	3	3
3	0	2	4	4	4	4
4	1	3	5	5	5	5
5	2	4	6	6	6	6
6	3	5	6	6	6	7
7	4	6	6	6	6	

$R^c_7 = \mathrm{Max}\,(A, B)$

R \ i	0	1	2	3	4	5
0	3	3	3	2	1	0
1	3	3	3	2	1	1
2	3	3	3	2	2	2
3	3	3	3	3	3	3
4	4	4	4	4	4	4
5	5	5	5	5	5	5
6	6	6	6	6	6	6
7	7	7	7	7	7	7

$R^s_7 = R - \mathrm{Min}[0, R - \mathrm{Min}(N-n, i^*-i)]$

R \ i	0	1	2	3	4	5
0	0^c	1^c	1^c	2^c	1^c	0^s
1	0^c	1^c	2^c	2^c	1^s	1^s
2	0^c	1^c	3^c	2^s	2^s	2^s
3	0^c	2^c	3^s	3^s	3^s	
4	1^c	3^c	4^s	4^s		
5	2^c	4^c	5^s			
6	3^c	5^c				
7	4^c					

$f_7\,(i, R) = \mathrm{Min}(R^s_7, R^c_7)$

REGRET EXAMPLE

Stage: n = 6

R \ i	0	1	2	3	4	5
0	1	1	2	1	0	
1	1	1	2	1	0	
2	1	2	2	1	1	
3	1	3	2	2	2	
4	2	3	3	3	3	
5	3	4	4	4	4	
6	4	5	5	5	5	
7						

$A=f_7\,[i+1, Max(0,R-1)]$

R \ i	0	1	2	3	4	5
0	0	1	2	2	1	1
1	0	1	3	2	2	2
2	0	2	3	3	3	3
3	1	3	4	4	4	4
4	2	4	5	5	5	5
5	3	5	6	6	6	6
6	4	6	6	6	6	7
7						

$B=f_7\,(i,R+1)$

R \ i	0	1	2	3	4	5
0	1	1	2	2	1	1
1	1	1	3	2	2	2
2	1	2	3	3	3	3
3	1	3	4	4	4	4
4	2	4	5	5	5	5
5	3	5	6	6	6	6
6	4	6	6	6	6	7
7						

$R^c{}_6 = Max\,(A,B)$

R \ i	0	1	2	3	4	5
0	4	4	3	2	1	0
1	4	4	3	2	1	1
2	4	4	3	2	2	2
3	4	4	3	3	3	3
4	4	4	4	4	4	4
5	5	5	5	5	5	5
6	6	6	6	6	6	6
7						

$R^s{}_6 = R - Min[0, R - Min(N-n, i^*-i)]$

R \ i	0	1	2	3	4	5
0	1^c	1^c	2^c	2^c	1^c	0^s
1	1^c	1^c	3^c	2^c	1^s	1^s
2	1^c	2^c	3^c	2^s	2^s	
3	1^c	3^c	3^s	3^s		
4	2^c	4^c	4^s			
5	3^c	5^c				
6	4^c					
7						

$f_6\,(i,R) = Min(R^s{}_6, R^c{}_6)$

REGRET EXAMPLE

Stage:

n = 5

R \ i	0	1	2	3	4	5
0	1	2	2	1	0	
1	1	2	2	1	0	
2	1	3	2	1	1	
3	2	3	2	2	2	
4	3	3	3	3	3	
5	4	4	4	4	4	
6						
7						

$A = f_6[i+1, \text{Max}(0, R-1)]$

R \ i	0	1	2	3	4	5
0	1	1	3	2	1	1
1	1	2	3	2	2	2
2	1	3	3	3	3	3
3	2	4	4	4	4	4
4	3	5	5	5	5	5
5	4	6	6	6	6	6
6						
7						

$B = f_6(i, R+1)$

R \ i	0	1	2	3	4	5
0	1	2	3	2	1	1
1	1	2	3	2	2	2
2	1	3	3	3	3	3
3	2	4	4	4	4	4
4	3	5	5	5	5	5
5	4	6	6	6	6	6
6						
7						

$R^c_5 = \text{Max}(A, B)$

R \ i	0	1	2	3	4	5
0	5	4	3	2	1	0
1	5	4	3	2	1	1
2	5	4	3	2	2	2
3	5	4	3	3	3	3
4	5	4	4	4	4	4
5	5	5	5	5	5	5
6						
7						

$R^s_5 = R - \text{Min}[0, R - \text{Min}(N-n, i^*-i)]$

R \ i	0	1	2	3	4	5
0	1^c	2^c	3^c	2^c	1^c	0^s
1	1^c	2^c	3^c	2^c	1^s	
2	1^c	3^c	3^c	2^s		
3	2^c	4^c	3^s			
4	3^c	4^s				
5	4^c					
6						
7						

$f_5(i, R) = \text{Min}(R^s_5, R^c_5)$

REGRET EXAMPLE

Stage:

n = 4

i

R	0	1	2	3	4	5
0	2	3	2	1	0	
1	2	3	2	1	0	
2	2	3	2	1	1	
3	3	3	2	2	2	
4	4	3	3	3	3	
5						
6						
7						

$A = f_5\ [i+1, \text{Max}(0, R-1)]$

i

R	0	1	2	3	4	5
0	1	2	3	2	1	1
1	1	3	3	2	2	2
2	2	4	3	3	3	3
3	3	4	4	4	4	4
4	4	5	5	5	5	5
5						
6						
7						

$B = f_5\ (i, R+1)$

i

R	0	1	2	3	4	5
0	2	3	3	2	1	1
1	2	3	3	2	2	2
2	2	4	3	3	3	3
3	3	4	4	4	4	4
4	4	5	5	5	5	5
5						
6						
7						

$R^C_4 = \text{Max}\ (A, B)$

i

R	0	1	2	3	4	5
0	5	4	3	2	1	
1	5	4	3	2	1	
2	5	4	3	2	2	
3	5	4	3	3	3	
4	5	4	4	4	4	
5						
6						
7						

$R^S_4 = R - \text{Min}[0, R - \text{Min}(N-n, i^* - i)]$

i

R	0	1	2	3	4	5
0	2^c	3^c	3^c	2^c	1^c	
1	2^c	3^c	3^c	2^c		
2	2^c	4^c	3^c			
3	3^c	4^c				
4	4^c					
5						
6						
7						

$f_4\ (i, R) = \text{Min}(R^S_4, R^C_4)$

REGRET EXAMPLE

Stage:

n = 3

R \ i	0	1	2	3	4	5
0	3	3	2	1		
1	3	3	2	1		
2	3	3	2	1		
3	4	3	2	2		
4						
5						
6						
7						

$A=f_4\,[i+1,\mathrm{Max}(0,R-1)]$

R \ i	0	1	2	3	4	5
0	2	3	3	2		
1	2	4	3	2		
2	3	4	3	3		
3	4	4	4	4		
4						
5						
6						
7						

$B=f_4\,(i,R+1)$

R \ i	0	1	2	3	4	5
0	3	3	3	2		
1	3	4	3	2		
2	3	4	3	3		
3	4	4	4	4		
4						
5						
6						
7						

$R^C{}_3 = \mathrm{Max}\,(A,B)$

R \ i	0	1	2	3	4	5
0	5	4	3	2		
1	5	4	3	2		
2	5	4	3	2		
3	5	4	3	3		
4						
5						
6						
7						

$R^S{}_3 = R-\mathrm{Min}[0,R-\mathrm{Min}(N-n,i*-i)]$

R \ i	0	1	2	3	4	5
0	3^c	3^c	3^c	2^c		
1	3^c	4^c	3^c			
2	3^c	4^c				
3	4^c					
4						
5						
6						
7						

$f_3\,(i,R)=\mathrm{Min}(R^S{}_3\,,R^C{}_3\,)$

REGRET EXAMPLE

Stage: n = 2

i

R	0	1	2	3	4	5
0	3	3	2			
1	3	3	2			
2	4	3	2			
3						
4						
5						
6						
7						

$A = f_3 [i+1, Max(0, R-1)]$

i

R	0	1	2	3	4	5
0	3	4	3			
1	3	4	3			
2	4	4	3			
3						
4						
5						
6						
7						

$B = f_3 (i, R+1)$

i

R	0	1	2	3	4	5
0	3	4	3			
1	3	4	3			
2	4	4	3			
3						
4						
5						
6						
7						

$R^c{}_2 = Max (A, B)$

i

R	0	1	2	3	4	5
0	5	4	3			
1	5	4	3			
2	5	4	3			
3						
4						
5						
6						
7						

$R^s{}_2 = R - Min[0, R - Min(N-n, i*-i)]$

i

R	0	1	2	3	4	5
0	3^c	4^c	3^c			
1	3^c	4^c				
2	4^c					
3						
4						
5						
6						
7						

$f_2 (i, R) = Min(R^s{}_2, R^c{}_2)$

REGRET EXAMPLE

Stage: n = 1

i

R	0	1	2	3	4	5
0	4	3				
1	4	3				
2						
3						
4						
5						
6						
7						

$A=f_2\,[i+1, \text{Max}(0, R-1)]$

i

R	0	1	2	3	4	5
0	3	4				
1	4	4				
2						
3						
4						
5						
6						
7						

$B=f_2\,(i, R+1)$

i

R	0	1	2	3	4	5
0	4	4				
1	4	4				
2						
3						
4						
5						
6						
7						

$R^C{}_1 = \text{Max}\ (A, B)$

i

R	0	1	2	3	4	5
0	5	4				
1	5	4				
2						
3						
4						
5						
6						
7						

$R^S{}_1 = R-\text{Min}[0, R-\text{Min}(N-n, i*-i)]$

i

R	0	1	2	3	4	5
0	4^c	4^c				
1	4^c					
2						
3						
4						
5						
6						
7						

$f_2\,(i, R) = \text{Min}(R^S{}_2, R^C{}_2)$

REGRET EXAMPLE

Stage:

n = 0

R \ i	0	1	2	3	4	5
0	4					
1						
2						
3						
4						
5						
6						
7						

$A = f_1 \, [i+1, Max(0, R-1)]$

R \ i	0	1	2	3	4	5
0	4					
1						
2						
3						
4						
5						
6						
7						

$B = f_1 \, (i, R+1)$

R \ i	0	1	2	3	4	5
0	4					
1						
2						
3						
4						
5						
6						
7						

$R^c_0 = Max \; (A, B)$

R \ i	0	1	2	3	4	5
0	5					
1						
2						
3						
4						
5						
6						
7						

$R^s_0 = R - Min[0, R - Min(N-n, i^* - i)]$

R \ i	0	1	2	3	4	5
0	4^c					
1						
2						
3						
4						
5						
6						
7						

$f_0 \; (i, R) = Min(R^s_0, R^c_0)$

The solution that has been developed over pages 86 - 96 is summarized below. It is in the form of a "stopping rule" that is interpreted as:

Stopping Rule: For stage N, stop in state (I,R) if regret, R, is greater than or equal to the value of R given in the matrix below. Continue if no value is given.

TABLE 9. SOLUTION TO EXAMPLE 4

I = Number of Tails in N Tosses

Stage N	0	1	2	3	4	5
0						
1						
2						
3						
4						
5		4	3	2	1	0
6			3	2	1	0
7			3	2	1	0
8				2	1	0
9				1	1	0
10				0	0	0

To illustrate how this stopping rule is applied, suppose that the price sample path is represented by the sequence (H H T H H H T H T T). Regret is calculated below for this case, and the process stops (bonds are bought) in time period 7.

TABLE 10. OPTIMAL STOPPING OF TYPICAL SAMPLE PATH

Stage N	Price Change X_i	State (I,R): No. Tails in n Tosses I	State (I,R): Regret R	Optimal Action (Table 8)
0	-	-	-	Continue
1	H=+1	0	1	Continue
2	H=+1	0	2	Continue
3	T=-1	1	1	Continue
4	H=+1	1	2	Continue
5	H=+1	1	3	Continue (R<4)
6	H=+1	1	4	Continue
7	T=-1	2	3	Stop (R=3, Score=3)
8	H=+1	2	4	
9	T=-1	3	3	
10	T=-1	4	2	

In solving the same example with the EMV criterion (page 75 bonds were bought immediately at a cost of zero. Using a dollar averaging strategy for this example would have resulted in a bond cost of 2.5. In summary, the three techniques would result in costs of

EMV	0
Dollar Averaging	2.5
Regret	3.0

respectively. For a different sample path, however, the regret model might outperform the other two. For example, the path T T T H T H H H H H would be stopped at -2 under regret -1.1 under dollar averaging, and 0 under EMV. Chapter VI will study the stopping characteristics of the three criteria in detail. To keep the regret formulation as simple as possible, we neglected certain aspects of dealing with continuation cost. This section clarifies these matters.

Complications Caused by C(i)

We want to take regret to be the difference between the lowest price that was available over [o,n] and the current price at n. Unfortunately, the single period cost C(i) complicates this a little. Suppose that the lowest score over [o,n] occurs at stage k. In the regret formulation developed earlier, regret was shown in Figure 10, page 81, to be

$$R=S_n-M_n^*$$
$$= S_n-S_k$$

when, in fact, it should be

$$R=S_n+(n-k)C(j)-S_k \qquad (6)$$

where the term (n-k)C(j) reflects the cost of continuing from period k to period n before stopping. Note, C(j) may be positive or negative depending upon whether the coupon rate of the bonds is greater or less than the rate that the firm can earn on the funds elsewhere. The index j is found from the relationship $S_k=n-2j$.

Using the same reasoning, the expression in (4) should read:

$$R_n^S=R-Min(0,R-D) \quad (7)$$

where

R is given by (6)

and

$$D=Min[(i*-i)(1-C(i)),N-n]$$

The expressions given by (6) and (7) are the ones used in Algorithm 3 to solve the regret formulation, and are coded into the computer program. Algorithm 3 (Regret Model) is summarized on the next page.

Algorithm 3

Minimax Regret Formulation

Step (i): If $S_N^1 < S_N^2 < \ldots < S_N^M$ are the scores predicted for the surrender date, set $i^* = \left[\frac{N - S_N^1}{2}\right]$ integer and $j^* = \left[\frac{S_N^M + N}{2}\right]$ integer.

Step (ii): For $i = 0, 1, 2, \ldots i^*$; and $0 < R < j^*$: Set

$$f_N(i,R) = R$$

$$A_N(i,R) = S$$

Step (iii): For $n = (N-1), (N-2), \ldots, 1, 0$; $i = 0, 1, 2, \ldots, i^*$; and $0 \leq R \leq j^*$:

Set: Stop

$$D = \mathrm{Min}[(i^* - i)(1 - C(i)), N - n]$$

$$R_n^s = R - \min(0, R - D)$$

Set: Continue

$$A = f_{n+1}[i+1, \mathrm{Max}(0, R - 1 + C(i))]$$

$$B = f_{n+1}[i, R + 1 + C(i)]$$

$$R_n^c = \mathrm{Max}[A, B]$$

Minimax

$$f_n(i,R) = \min(R_n^s, R_n^c)$$

$$A_n(i,R) = \begin{cases} S, & \text{if } R_n^s < R_n^c \\ C, & \text{if } R_n^c \leq R_n^s \end{cases}$$

V. COMPUTER MODEL VALIDATION

In the next chapter, a sensitivity analysis of continuation cost, forecast error, bond volatility and trend in interest rates is discussed in detail. These are the basic factors that should affect the average price obtainable with any trading strategy. In order to conduct this sensitivity analysis, computer programs were developed for each decision model, and a simulation was carried out using computer generated price sample paths. This chapter gives the computer solution to Examples 3 and 4 (pages 70 - 96) which were used to validate the computer programs. It also illustrates the form that the solution comes out of the computer.

Approach

For each set of assumptions corresponding to a single sensitivity, a "stopping rule" solution was developed using the EMV model (Algorithm 2) and the regret model (Algorithm 3). These stopping rules are applied against price sample paths generated as realizations of a constrained random walk process. The average price of bonds purchased is then compared to the dollar averaging price. In general, each sensitivity was evaluated as the average price obtained in 1,000 "trials".

Table 11, on the next page, shows a typical sample path for N=52 weeks. The initial price was set at par (100) and step size is one unit. This particular path is the fifth one simulated for this example. Normally, this output would be suppressed.

TABLE 11. SIMULATED SAMPLE PATH FOR A CONSTRAINED RANDOM WALK

				Scaled Price			Absolute Price
SIMULATION NUMBER		5					
TIME PERIOD	0		SCORE =	0.0000		PRICE =	100.0000
TIME PERIOD	1		SCORE =	1.0000		PRICE =	101.0000
TIME PERIOD	2		SCORE =	2.0000		PRICE =	102.0000
TIME PERIOD	3		SCORE =	3.0000		PRICE =	103.0000
TIME PERIOD	4		SCORE =	4.0000		PRICE =	104.0000
TIME PERIOD	5		SCORE =	3.0000		PRICE =	103.0000
TIME PERIOD	6		SCORE =	2.0000		PRICE =	102.0000
TIME PERIOD	7		SCORE =	1.0000		PRICE =	101.0000
TIME PERIOD	8		SCORE =	2.0000		PRICE =	102.0000
TIME PERIOD	9		SCORE =	1.0000		PRICE =	101.0000
TIME PERIOD	10		SCORE =	0.0000		PRICE =	100.0000
TIME PERIOD	11		SCORE =	1.0000		PRICE =	101.0000
TIME PERIOD	12		SCORE =	0.0000		PRICE =	100.0000
TIME PERIOD	13		SCORE =	-1.0000		PRICE =	99.0000
TIME PERIOD	14		SCORE =	-2.0000		PRICE =	98.0000
TIME PERIOD	15		SCORE =	-3.0000		PRICE =	97.0000
TIME PERIOD	16		SCORE =	-2.0000		PRICE =	98.0000
TIME PERIOD	17		SCORE =	-1.0000		PRICE =	99.0000
TIME PERIOD	18		SCORE =	0.0000		PRICE =	100.0000
TIME PERIOD	19		SCORE =	1.0000		PRICE =	101.0000
TIME PERIOD	20		SCORE =	0.0000		PRICE =	100.0000
TIME PERIOD	21		SCORE =	1.0000		PRICE =	101.0000
TIME PERIOD	22		SCORE =	0.0000		PRICE =	100.0000
TIME PERIOD	23		SCORE =	-1.0000		PRICE =	99.0000
TIME PERIOD	24		SCORE =	0.0000		PRICE =	100.0000
TIME PERIOD	25		SCORE =	1.0000		PRICE =	101.0000
TIME PERIOD	26		SCORE =	0.0000		PRICE =	100.0000
TIME PERIOD	27		SCORE =	1.0000		PRICE =	101.0000
TIME PERIOD	28		SCORE =	2.0000		PRICE =	102.0000
TIME PERIOD	29		SCORE =	3.0000		PRICE =	103.0000
TIME PERIOD	30		SCORE =	2.0000		PRICE =	102.0000
TIME PERIOD	31		SCORE =	1.0000		PRICE =	101.0000
TIME PERIOD	32		SCORE =	0.0000		PRICE =	100.0000
TIME PERIOD	33		SCORE =	-1.0000		PRICE =	99.0000
TIME PERIOD	34		SCORE =	-2.0000		PRICE =	98.0000
TIME PERIOD	35		SCORE =	-1.0000		PRICE =	99.0000
TIME PERIOD	36		SCORE =	0.0000		PRICE =	100.0000
TIME PERIOD	37		SCORE =	1.0000		PRICE =	101.0000
TIME PERIOD	38		SCORE =	0.0000		PRICE =	100.0000
TIME PERIOD	39		SCORE =	-1.0000		PRICE =	99.0000
TIME PERIOD	40		SCORE =	0.0000		PRICE =	100.0000
TIME PERIOD	41		SCORE =	1.0000		PRICE =	101.0000
TIME PERIOD	42		SCORE =	2.0000		PRICE =	102.0000
TIME PERIOD	43		SCORE =	1.0000		PRICE =	101.0000
TIME PERIOD	44		SCORE =	0.0000		PRICE =	100.0000
TIME PERIOD	45		SCORE =	1.0000		PRICE =	101.0000
TIME PERIOD	46		SCORE =	2.0000		PRICE =	102.0000
TIME PERIOD	47		SCORE =	1.0000		PRICE =	101.0000
TIME PERIOD	48		SCORE =	0.0000		PRICE =	100.0000
TIME PERIOD	49		SCORE =	1.0000		PRICE =	101.0000
TIME PERIOD	50		SCORE =	2.0000		PRICE =	102.0000
TIME PERIOD	51		SCORE =	1.0000		PRICE =	101.0000
TIME PERIOD	52		SCORE =	2.0000		PRICE =	102.0000

↑

CONSTRAINED RANDOM WALK

A simulation routine was developed that generates price sample paths according to transition probabilities computed by Algorithm 1 for a constrained random walk. The routine works very well, as is shown in Table 12 which compares the surrender price of 1,000 simulated sample paths against the forecast (input) probability distribution.

TABLE 12. SIMULATED SAMPLE PATHS FROM A CONSTRAINED RANDOM WALK

S I M U L A T I O N   S T A T I S T I C S

PRICE AT SURRENDER DATE	PROBABILITY PRICE	
	SIMULATED	INPUT
73.00	0.000	0.000
74.00	0.001	0.001
75.00	0.001	0.002
76.00	0.001	0.005
77.00	0.011	0.009
78.00	0.022	0.016
79.00	0.039	0.038
80.00	0.036	0.042
81.00	0.068	0.060
82.00	0.081	0.079
83.00	0.096	0.098
84.00	0.106	0.105
85.00	0.101	0.111
86.00	0.093	0.105
87.00	0.097	0.098
88.00	0.067	0.079
89.00	0.073	0.060
90.00	0.033	0.042
91.00	0.039	0.038
92.00	0.022	0.016
93.00	0.004	0.009
94.00	0.007	0.005
95.00	0.002	0.002
96.00	0.000	0.001
97.00	0.000	0.000

Validation of Computer Programs

In Chapter IV, both algorithms were demonstrated by working out a simplified example. The computer solution to the same example is given below. The output consists of:

- The optimal stopping rule for the EMV and regret models.
- A comparison of the forecast that was input and the prices actually simulated (they should be virtually the same).
- A comparison of average price paid using each strategy.
- Stopping time statistics for EMV and regret.

Comparing these results against the earlier solutions will verify that the computer programs work as they should.

Case 1. Computer Solution for the EMV Criterion

The computer solution to Example 3 pages 70 - 76 is developed on the following page. This example demonstrates the type of output obtained from the EMV computer program.

Part 1 (EMV Criterion): Calculation of Optimal Stopping Boundary

N = 10

J	A(1)	A(2)	VALUE	ACTION
7			4.0000	S
6			2.0000	S
5			0.0000	S

N = 9

J	A(1)	A(2)	VALUE	ACTION
7	5.0000	3.0000	3.0000	C
6	3.0000	2.8666	2.8666	C
5	1.0000	1.8235	1.0000	S
4	-1.0000	1.0000	-1.0000	S

STOPPING VALUE = 5

N = 8

J	A(1)	A(2)	VALUE	ACTION
7	6.0000	2.0000	2.0000	C
6	4.0000	2.5454	2.5454	C
5	2.0000	2.2025	2.0000	S
4	0.0000	1.0909	0.0000	S
3	-2.0000	0.0000	-2.0000	S

STOPPING VALUE = 5

N = 7

J	A(1)	A(2)	VALUE	ACTION
7	7.0000	1.0000	1.0000	C
6	5.0000	1.8965	1.8965	C
5	3.0000	2.1586	2.1586	C
4	1.0000	1.5691	1.0000	S
3	-1.0000	0.2592	-1.0000	S
2	-3.0000	-1.0000	-3.0000	S

STOPPING VALUE = 4

N = 6

J	A(1)	A(2)	VALUE	ACTION
6	6.0000	1.1111	1.1111	C
5	4.0000	1.8103	1.8103	C
4	2.0000	1.7089	1.7089	C
3	0.0000	0.7796	0.0000	S
2	-2.0000	-0.6250	-2.0000	S
1	-4.0000	-2.0000	-4.0000	S

STOPPING VALUE = 3

N = 5

J	A(1)	A(2)	VALUE	ACTION
5	5.0000	1.2235	1.2235	C
4	3.0000	1.6839	1.6839	C
3	1.0000	1.2337	1.0000	S
2	-1.0000	-0.0622	-1.0000	S
1	-3.0000	-1.5405	-3.0000	S
0	-5.0000	-3.0000	-5.0000	S

STOPPING VALUE = 3

Calculation of Stopping Boundary

For Example 3, Pages 70 - 76

(continued)

Stop Continue

N = 4

J	A(1)	A(2)	VALUE	ACTION
4	4.0000	1.3071	1.3071	C
3	2.0000	1.4201	1.4201	C
2	0.0000	0.5947	0.0000	S
1	-2.0000	-0.9396	-2.0000	S
0	-4.0000	-2.4761	-4.0000	S

STOPPING VALUE = 2

N = 3

J	A(1)	A(2)	VALUE	ACTION
3	3.0000	1.3081	1.3081	C
2	1.0000	0.9816	0.9816	C
1	-1.0000	-0.2587	-1.0000	S
0	-3.0000	-1.8421	-3.0000	S

STOPPING VALUE = 1

N = 2

J	A(1)	A(2)	VALUE	ACTION
2	2.0000	1.1973	1.1973	C
1	0.0000	0.4673	0.0000	S
0	-2.0000	-1.1400	-2.0000	S

STOPPING VALUE = 1

N = 1

J	A(1)	A(2)	VALUE	ACTION
1	1.0000	0.8058	0.8058	C
0	-1.0000	-0.3888	-1.0000	S

STOPPING VALUE = 0

N = 0

J	A(1)	A(2)	VALUE	ACTION
0	0.0000	0.2835	0.0000	S

STOPPING VALUE = 0

Part II (EMV Criterion): Results

S I M U L A T I O N R E S U L T S
FOR THE EMV OBJECTIVE FUNCTION

NUMBER OF SIMULATED
PRICE SAMPLE PATHS = 20

ASSUMPTIONS ---
STEP SIZE = 1.000
CONTINUATION COST,C(J)= 0.000
INITIAL PRICE = 80.000
DOLLAR AVERAGING
PURCHASE CYCLE = 1

BOND PURCHASE PRICE --	MEAN	STD.DEVIATION
- UNDER DOLLAR AVERAGING	81.16	0.96
- UNDER EMV STOPPING RULE	79.99	0.00
STOPPING TIME	0.00	0.00

S T O P P I N G B O U N D A R Y
(S O L U T I O N)

TIME PERIOD	SCORE (SCALED PRICE)	ABSOLUTE PRICE
0	0.00	80.00
1	-1.00	79.00
2	0.00	80.00
3	-1.00	79.00
4	0.00	80.00
5	1.00	81.00
6	0.00	80.00
7	1.00	81.00
8	2.00	82.00
9	1.00	81.00

S I M U L A T I O N S T A T I S T I C S

PRICE AT SURRENDER DATE	PROBABILITY PRICE SIMULATED	PROBABILITY PRICE INPUT
80.00	0.300	0.250
82.00	0.450	0.500
84.00	0.250	0.250

(Note that the simulated price distribution is not close enough to the input forecast with only 20 simulated sample paths. It generally requires a minimum of 200 "trials" to obtain an acceptable fit.)

Case 2. Computer Solution for the Regret Criterion

The tables on the following pages show the N-stage computer calculations made to develop an optimal stopping rule for Example 4 under the minimax regret objective function. These computer generated tables are identical to the ones given on pages 84 - 96 which were calculated by hand.

Part 1 (Regret Criterion): Calculating Optimal Stopping Rule

STAGE= 10

F(I,R) TABLE FOR STAGE 10

	I= 0	1	2	3	4	5
R= 0				0.0	0.0	0.0
R= 1				1.0	1.0	1.0
R= 2				2.0	2.0	2.0
R= 3				3.0	3.0	3.0
R= 4				4.0	4.0	4.0
R= 5				5.0	5.0	5.0
R= 6				6.0	6.0	
R= 7				7.0		

REGRET PROGRAM VALIDATION

(CONTINUED)

STOP TABLE FOR STAGE 9

	I= 0	1	2	3	4	5
R= 0			1.0	1.0	1.0	0.0
R= 1			1.0	1.0	1.0	1.0
R= 2			2.0	2.0	2.0	2.0
R= 3			3.0	3.0	3.0	3.0
R= 4			4.0	4.0	4.0	4.0
R= 5			5.0	5.0	5.0	
R= 6			6.0	6.0		
R= 7			7.0			

CONTINUE TABLE FOR STAGE 9

	I= 0	1	2	3	4	5
R= 0			0.0	1.0	1.0	1.0
R= 1			0.0	2.0	2.0	2.0
R= 2			1.0	3.0	3.0	3.0
R= 3			2.0	4.0	4.0	4.0
R= 4			3.0	5.0	5.0	5.0
R= 5			4.0	6.0	6.0	
R= 6			5.0	7.0		
R= 7			6.0			

STAGE= 9

F(I,R) TABLE FOR STAGE 9

	I= 0	1	2	3	4	5
R= 0			0.0	1.0	1.0	0.0
R= 1			0.0	1.0	1.0	1.0
R= 2			1.0	2.0	2.0	2.0
R= 3			2.0	3.0	3.0	3.0
R= 4			3.0	4.0	4.0	4.0
R= 5			4.0	5.0	5.0	
R= 6			5.0	6.0		
R= 7			6.0			

REGRET PROGRAM VALIDATION

(CONTINUED)

STOP TABLE FOR STAGE 8

	I= 0	1	2	3	4	5
R= 0		2.0	2.0	2.0	1.0	0.0
R= 1		2.0	2.0	2.0	1.0	1.0
R= 2		2.0	2.0	2.0	2.0	2.0
R= 3		3.0	3.0	3.0	3.0	3.0
R= 4		4.0	4.0	4.0	4.0	4.0
R= 5		5.0	5.0	5.0	5.0	
R= 6		6.0	6.0	6.0		
R= 7		7.0	7.0			

CONTINUE TABLE FOR STAGE 8

	I= 0	1	2	3	4	5
R= 0		0.0	1.0	1.0	1.0	1.0
R= 1		0.0	1.0	2.0	2.0	2.0
R= 2		0.0	2.0	3.0	3.0	3.0
R= 3		1.0	3.0	4.0	4.0	4.0
R= 4		2.0	4.0	5.0	5.0	5.0
R= 5		3.0	5.0	6.0	6.0	
R= 6		4.0	6.0	7.0	0.0	
R= 7		5.0	6.0			

STAGE= 8

F(I,R) TABLE FOR STAGE 8

	I= 0	1	2	3	4	5
R= 0		[illegible]	1.0	1.0	1.0	0.0
R= 1		0.0	1.0	2.0	1.0	1.0
R= 2		0.0	2.0	2.0	2.0	2.0
R= 3		1.0	3.0	3.0	3.0	3.0
R= 4		2.0	4.0	4.0	4.0	4.0
R= 5		3.0	5.0	5.0	5.0	
R= 6		4.0	6.0	6.0		
R= 7		5.0	6.0			

REGRET PROGRAM VALIDATION

(CONTINUED)

STOP TABLE FOR STAGE 7

	I= 0	1	2	3	4	5
R= 0	3.0	3.0	3.0	2.0	1.0	0.0
R= 1	3.0	3.0	3.0	2.0	1.0	1.0
R= 2	3.0	3.0	3.0	2.0	2.0	2.0
R= 3	3.0	3.0	3.0	3.0	3.0	
R= 4	4.0	4.0	4.0	4.0		
R= 5	5.0	5.0	5.0			
R= 6	6.0	6.0				
R= 7	7.0					

CONTINUE TABLE FOR STAGE 7

	I= 0	1	2	3	4	5
R= 0	0.0	1.0	1.0	2.0	1.0	1.0
R= 1	0.0	1.0	2.0	2.0	2.0	2.0
R= 2	0.0	1.0	3.0	3.0	3.0	3.0
R= 3	0.0	2.0	4.0	4.0	4.0	
R= 4	1.0	3.0	5.0	5.0		
R= 5	2.0	4.0	6.0			
R= 6	3.0	5.0				
R= 7	4.0					

STAGE= 7

F(I,R) TABLE FOR STAGE 7

	I= 0	1	2	3	4	5
R= 0	0.0	1.0	1.0	2.0	1.0	0.0
R= 1	0.0	1.0	2.0	2.0	1.0	1.0
R= 2	0.0	1.0	3.0	2.0	2.0	2.0
R= 3	0.0	2.0	3.0	3.0	3.0	
R= 4	1.0	3.0	4.0	4.0		
R= 5	2.0	4.0	5.0			
R= 6	3.0	5.0				
R= 7	4.0					

REGRET PROGRAM VALIDATION

(CONTINUED)

STOP TABLE FOR STAGE 6

	I= 0	1	2	3	4	5
R= 0	4.0	4.0	3.0	2.0	1.0	0.0
R= 1	4.0	4.0	3.0	2.0	1.0	1.0
R= 2	4.0	4.0	3.0	2.0	2.0	
R= 3	4.0	4.0	3.0	3.0		
R= 4	4.0	4.0	4.0			
R= 5	5.0	5.0				
R= 6	6.0					

CONTINUE TABLE FOR STAGE 6

	I= 0	1	2	3	4	5
R= 0	1.0	1.0	2.0	2.0	1.0	1.0
R= 1	1.0	1.0	3.0	2.0	2.0	2.0
R= 2	1.0	2.0	3.0	3.0	3.0	
R= 3	1.0	3.0	4.0	4.0		
R= 4	2.0	4.0	5.0			
R= 5	3.0	5.0				
R= 6	4.0					

STAGE= 6

F(I,R) TABLE FOR STAGE 6

	I= 0	1	2	3	4	5
R= 0	1.0	1.0	2.0	2.0	1.0	0.0
R= 1	1.0	1.0	3.0	2.0	1.0	1.0
R= 2	1.0	2.0	3.0	2.0	2.0	
R= 3	1.0	3.0	3.0	3.0		
R= 4	2.0	4.0	4.0			
R= 5	3.0	5.0				
R= 6	4.0					

REGRET PROGRAM VALIDATION

(CONTINUED)

STOP TABLE FOR STAGE 5

	I= 0	1	2	3	4	5
R= 0	5.0	4.0	3.0	2.0	1.0	0.0
R= 1	5.0	4.0	3.0	2.0	1.0	
R= 2	5.0	4.0	3.0	2.0		
R= 3	5.0	4.0	3.0			
R= 4	5.0	4.0				
R= 5	5.0					

CONTINUE TABLE FOR STAGE 5

	I= 0	1	2	3	4	5
R= 0	1.0	2.0	3.0	2.0	1.0	1.0
R= 1	1.0	2.0	3.0	2.0	2.0	
R= 2	1.0	3.0	3.0	3.0		
R= 3	2.0	4.0	4.0			
R= 4	3.0	5.0				
R= 5	4.0					

STAGE= 5

F(I,R) TABLE FOR STAGE 5

	I= 0	1	2	3	4	5
R= 0	1.0	2.0	3.0	2.0	1.0	0.0
R= 1	1.0	2.0	3.0	2.0	1.0	
R= 2	1.0	3.0	3.0	2.0		
R= 3	2.0	4.0	3.0			
R= 4	3.0	4.0				
R= 5	4.0					

REGRET PROGRAM VALIDATION

(CONTINUED)

STOP TABLE FOR STAGE 4

	I= 0	1	2	3	4
R= 0	5.0	4.0	3.0	2.0	1.0
R= 1	5.0	4.0	3.0	2.0	
R= 2	5.0	4.0	3.0		
R= 3	5.0	4.0			
R= 4	5.0				

CONTINUE TABLE FOR STAGE 4

	I= 0	1	2	3	4
R= 0	2.0	3.0	3.0	2.0	1.0
R= 1	2.0	3.0	3.0	2.0	
R= 2	2.0	4.0	3.0		
R= 3	3.0	4.0			
R= 4	4.0				

STAGE= 4

F(I,R) TABLE FOR STAGE 4

	I= 0	1	2	3	4
R= 0	2.0	3.0	3.0	2.0	1.0
R= 1	2.0	3.0	3.0	2.0	
R= 2	2.0	4.0	3.0		
R= 3	3.0	4.0			
R= 4	4.0				

REGRET PROGRAM VALIDATION

(CONTINUED)

STOP TABLF FOR STAGE 3

	I= 0	1	2	3
R= 0	5.0	4.0	3.0	2.0
R= 1	5.0	4.0	3.0	
R= 2	5.0	4.0		
R= 3	5.0			

CONTINUE TABLE FOR STAGE 3

	I= 0	1	2	3
R= 0	3.0	3.0	3.0	2.0
R= 1	3.0	4.0	3.0	
R= 2	3.0	4.0		
R= 3	4.0			

STAGE= 3

F(I,R) TABLE FOR STAGE 3

	I= 0	1	2	3
R= 0	3.0	3.0	3.0	2.0
R= 1	3.0	4.0	3.0	
R= 2	3.0	4.0		
R= 3	4.0			

REGRET PROGRAM VALIDATION

(CONTINUED)

STOP TABLE FOR STAGE 2

	I= 0	1	2
R= 0	5.0	4.0	3.0
R= 1	5.0	4.0	
R= 2	5.0		

CONTINUE TABLE FOR STAGE 2

	I= 0	1	2
R= 0	3.0	4.0	3.0
R= 1	3.0	4.0	
R= 2	4.0		

STAGE= 2

F(I,R) TABLE FOR STAGE 2

	I= 0	1	2
R= 0	3.0	4.0	3.0
R= 1	3.0	4.0	
R= 2	4.0		

REGRET PROGRAM VALIDATION

(CONTINUED)

STOP TABLE FOR STAGE 1

	I= 0	1
R= 0	5.0	4.0
R= 1	5.0	

CONTINUE TABLE FOR STAGE 1

	I= 0	1
R= 0	4.0	4.0
R= 1	4.0	

STAGE= 1

F(I,R) TABLE FOR STAGE 1

	I= 0	1
R= 0	4.0	4.0
R= 1	4.0	

REGRET PROGRAM VALIDATION

(CONTINUED)

```
STOP TABLE FOR STAGE    0

                          I=      0
       R=   0                   5.0


CONTINUE TABLE FOR STAGE    0

                          I=      0
       R=   0                   4.0


STAGE=    0
                          I=      0
F(I,R) TABLE FOR STAGE    0
       R=   0                   4.0
```

Part II (Regret Criterion): Results

STOPPING RULE FOR REGRET CRITERION

	I= 0	1	2	3	4	5
N= 0						
N= 1						
N= 2						
N= 3						
N= 4						
N= 5		4	3	2	1	0
N= 6			3	2	1	0
N= 7			3	2	1	0
N= 8				2	1	0
N= 9				1	1	0
N= 10				0	0	0

Decision Rule: For stage N, stop in state (I,R) if R is greater than or equal to the value in the table above. Continue if no value is given. ("I" is the number of tails in N tosses, "R" is regret.)

S I M U L A T I O N   R E S U L T S

NUMBER OF SIMULATED
PRICE SAMPLE PATHS = 10

ASSUMPTIONS ---
STEP SIZE = 1.000
CONTINUATION COST,C(J)= 0.000
INITIAL PRICE = 100.000
DOLLAR AVERAGING
PURCHASE CYCLE = 1

BOND PURCHASE PRICE --	MEAN	STD.DEVIATION
- UNDER DOLLAR AVERAGING	101.14	1.02
- UNDER REGRET CRITERION	101.70	1.09
STOPPING TIME STATISTICS	7.10	1.13

S I M U L A T I O N   S T A T I S T I C S

PRICE AT SURRENDER DATE	PROBABILITY PRICE SIMULATED	PROBABILITY PRICE INPUT
100.00	0.300	0.250
102.00	0.500	0.500
104.00	0.200	0.250

Summary

This chapter shows that both computer models are working properly and that the simulation routine generates sample paths that behave according to the constrained random walk model of Chapter III. The models are used to conduct a sensitivity analysis of the parameters of the EMV and regret models in the next chapter.

VI. SENSITIVITY ANALYSIS OF MODEL ASSUMPTIONS

This chapter is a sensitivity analysis of the following assumptions:

(1) Continuation cost (C)

(2) Step size (Δ)

(3) Tightness of forecast (T)

(4) Trend (μ)

As a general approach, in sensitivity analysis we try to vary these factors one at a time and observe how they influence the performance of each trading strategy. However, this is very difficult, here, because the four factors are very interdependent. To appreciate this fact, consider the following.

(1) When the continuation cost, C, is the same magnitude as the step size, Δ, then C has a much greater effect on stopping time than it would have if the ratio C/Δ is small.

(2) A forecast is considered to be a tight forecast when it constrains price at the surrender date to a narrower range than would be "predicted" from a simple unconstrained random walk. Since the domaine of the unrestricted walk depends on its step size, we see that the tightness of forecast sensitivity is also intertwined with step size.

(3) Trend, or drift, in price levels is tied to tightness of forecast. A tight forecast in a down-trending market will encourage buying late in the horizon where a tight forecast of an uptrend has the opposite effect.

Because the four factors are so interrelated, a great many sensitivity combinations are possible and this makes the results difficult to summarize. In order to minimize confusion, we will organize the sensitivity analysis in the following manner:

Part I: Sensitivity analysis objectives

Part II: Discussion of meaningful ranges and combinations for each factor

Part III: Summary of major results

We will begin by stating our objectives.

Sensitivity Analysis Objectives

The objectives of this sensitivity analysis are as follows:

(1) Compare the performance of the EMV and regret models against dollar averaging under a wide variation in conditions.

- Does one technique dominate the other two?
- If not, under what conditions is one technique preferred to another?
- In general, is the payoff from using one of the new models worth the bother of changing from the traditional dollar averaging approach?

(2) Sensitize the assumptions of the models.

- Continuation cost.
- Step size (price volatility).
- Tightness of forecast.
- Trend in interest rates.

The next four sections develop meaningful ranges and combinations of each factor.

Sensitivity of Step Size

The volatility of a bond refers to the amount of price movement that it has exhibited in the past, and is measured by calculating the variance or price sample paths after adjusting for trend. We want to evalute how sensitive the EMV, regret and dollar averaging methods are to the amount of price fluctuation going on.

In the EMV and regret models, volatility is reflected in the step size of the random walk. The more volatile the bond, the larger the step size, and vice versa. Step size was sensitized with respect to trend, μ, forecast error, S^2, and continuation cost, C.

From a sample of bonds picked at random, average weekly price changes of a 52 week period ranged from about 25 - 50 basis points. The market, itself, fluctuated by about 36 basis points per week over the same period (see Table 13). Therefore, step size was sensitized at 25 and 50 basis points.

TABLE 13. WEEKLY PRICE FLUCTUATIONS OF THE MARKET

S & P Weekly Bond Index
High Grade Corporate Price (AAA)

1976	Price	Basis Points Change	1976	Price	Basis Points Change
Jan.	56.88	-	July	56.63	-52
	57.30	42		56.88	25
	57.14	- 16		57.28	40
	56.81	- 33		56.95	-33
Feb.	57.06	25		57.24	29
	57.70	64	Aug.	57.76	52
	57.14	- 56		57.66	-10
	57.44	30		57.89	23
Mar.	57.02	- 42		58.28	39
	56.93	- 9	Sept.	58.48	20
	56.95	2		58.79	31
	57.60	65		58.79	0
	57.90	30		59.21	50
Apr.	57.96	6		58.54	-67
	58.66	70	Oct.	59.05	51
	58.63	- 3		59.38	33
May	57.57	-106		58.99	-49
	57.35	- 22		58.97	- 2
	56.64	- 71	Nov.	58.88	- 9
	56.16	- 48		58.71	17
June	55.97	- 19		58.96	25
	56.23	26		60.41	145
	56.70	47	Dec.	60.75	34
	57.28	58		61.27	52
	57.15	- 13		61.31	4
				61.38	7
				61.97	59

AVG = 36.5
(+ or -) Basis
points

Sensitivity of Continuation Cost

A corporate treasurer's decision to purchase bonds may be inordinately influenced by how cheap they presently are in relation to historical price levels. What may be overlooked is their cheapness relative to other short term investment opportunities. That is, if bonds are purchased before the surrender date ("prepurchased"), funds that could be invested elsewhere are tied up before it is mandatory to do so. Consequently, one component of continuation cost is an opportunity cost of interest foregone by employing the funds in the sinking fund purchase rather than on a Treasury Bill, Commercial Paper, Certificate of Deposit, etc. The appropriate rate to use is discussed later.

An offsetting factor is a saving of coupon payments on the principal purchased, and the availability of these unpaid coupon payments for investment or for reduction of short term debt. Consequently, if the coupon rate is less than the use of funds rate the continuation "cost" is negative, which has the effect of penalizing prepayment. A sensitivity analysis of continuation cost should, therefore, consider both the magnitude and the "sign", plus or minus, of the cost.

Appropriate Use of Funds Rate

Typically, a sinking fund prepurchase is made with working capital funds which would otherwise be invested in a short term money market instrument. Therefore, money market rates are more appropriate than the normally higher reinvestment rate (hurdle rate) to value these funds. Another possibility is to use the cost of capital as the use of funds rate. The cost of capital is a composite rate approximating the cost of all forms of long term capital used in funding corporate operations. However, the cost of capital would seem to be inappropriate because sinking fund investments are considerably smaller and bear less risk than long term capital investments. In the event the corporation has to borrow money to make the prepurchase, the appropriate rate would be the prime rate, adjusted for compensating balances, or the issuer's rate on Commercial Paper.

In actual practice, a corporation with cash available to fund an annual sinking fund payment would probably roll over short term investments several times throughout the year. Consequently, we shall use the average rate forecast for the year, say, a 90-day T-bill in our calculations.

To calculate continuation cost, C(j), at stage n:

(i) Calculate the current price

$$P_n=(2j-n)(\Delta)+P_o$$

(ii)

$$C'(j)=\left[\frac{(CU\%)(100)}{N}-\frac{(UF\%)(P_n)}{N}\right]$$

where

Δ = step size

CU% = coupon rate of the bond (decimal)

UF% = use of funds rate (decimal)

N = number of time periods

100 = par value

(iii) Scale $C'(j)$ to get $C(j)$:

$$C(j)=C'(j)/\Delta$$

Notice that continuation cost is state dependent--increasing during downtrends and decreasing during uptrends because of the influence of the alternative use of funds, rate, UF.

Coupon rates on most NYSE bonds range from about 4% to 10%. In the last few years, CD's and T-bill rates available have been in the 5% to 9% range. Consequently, a reasonable range of rates for sensitivity analysis is -5(≃4%-9%) to 5%(≃10%-5%). This corresponds to a continuation cost range of about -10 to 10 basis points per week and we sensitized at both extremes.

Table 14 shows the combinations that were evaluated under different trend and tightness of forecast combinations.

TABLE 14. CONTINUATION COST SENSITIVITY ALTERNATIVES

		Continuation Cost, C (Basis Points) -10	10
Step Size, Δ (Basis Points)	25	C/Δ=-40%	C/Δ=40%
	50	C/Δ=-20%	C/Δ=20%

Sensitivity Analysis of Tight vs. Loose Forecasts

The point was made in Chapter III that the simple random walk model has the problem of wandering too far afield, hence the constraining technique of making transition probabilities conditional on a forecast was preferred. A "tight" forecast is, therefore, one that pins down price at the surrender date more precisely than would be predicted under an unrestricted random walk assumption. This section develops a tightness index, T, and gives the values of t which were sensitized.

Tightness Index

Suppose than an unrestricted random walk having unit steps X_i starts at the origin and makes N transitions. The possible positions of the walk at time N are $K=0,\pm 1,\ldots,\pm N$. Let

$$Pr(X_i=+1)=p$$

$$Pr(X_i=-1)=q,\ p+q=1$$

If μ and σ_x^2 denote the mean and variance of a single step, then

(1) $$\mu=p-q$$

(2) $$\sigma_x^2=1-(p-q)^2$$

Now, suppose $Y_i=\Delta\cdot X_i$, where Δ is a scale factor. Then (1) and (2) become

(3) $$\mu=\Delta(p-q)$$
$$\sigma_y^2=\Delta^2[1-(p-q)^2]$$

The mean and variance of the position of the unrestricted walk at the surrender date N, is given by

$$E[S_N]= N\cdot\mu$$
(4) $$Var[S_N]= N\sigma_y^2$$

We assume that the corporate treasurer has developed a forecasting technique[13/] that also assigns probability weights to the integers $R=0,\pm1,...,\pm N$. Let $\overline{X}$ and S^2 denote the price forecast mean and variance. Then, a relative measure of how "tight" the forecast is with respect to the accuracy of the naive random walk model is given by the tightness index

(5) $$T=\frac{S^2}{N\cdot\sigma_y^2}$$

where σ_y^2 is determined in the following manner.

(i) Starting at the origin, X_0, the walk is forecast to terminate at $\overline{X}$. . .requiring a "drift" of $=(\overline{X}-X_0)/N$ units per step.

(ii) From (10),

$$\Delta(p-q)=(\overline{X}-X_0)/N$$
$$p+q=1$$

are solved to yield p and q to accomplish this drift.

(iii) From (3) and (4)

$$(N)\cdot(\Delta^2)\cdot[1-(p-q)^2]$$

is the unrestricted variance of price at the surrender date used in the denominator of the tightness index.

Table 15 on the following page gives six different forecasts for a particular bond. The average "score" is zero in each case, but the forecast accuracy as measured by standard deviation, S, varies. The forecast error is assumed to be normally distributed, and the values in Table 15 are calculated using the normal approximation to a binomial distribution, that is

$$Pr(j \leq S_N \leq k) \simeq N\left(\frac{k+1-\bar{X}}{S}\right) - N\left(\frac{j-1-\bar{X}}{S}\right)$$

13/ The particular forecasting approach used is immaterial. All that we require is that the forecast be translated into a probability mass function on the integers [-N,N]. One forecasting approach used in industry is discussed later in Chapter VII.

TABLE 15. FORECAST ERROR DISTRIBUTION USED IN SENSITIVITY ANALYSES

Forecast Error, S

Forecast of "Score" at Surrender Date (S_N)	0	$\frac{\sqrt{N}}{8}$	$\frac{\sqrt{N}}{4}$	$\frac{\sqrt{N}}{2}$	$\frac{\sqrt{N}}{4}$	$\sqrt{N}$
24						.000
22					.000	.001
20					.000	.002
18					.001	.005
16					.002	.009
14					.005	.016
12				.000	.013	.038
10				.005	.027	.042
8			.000	.019	.050	.060
6		.000	.005	.055	.079	.079
4		.001	.055	.119	.112	.098
2		.133	.242	.190	.137	.105
0	1.000	.733	.396	.222	.147	.111
- 2		.133	.242	.190	.137	.105
- 4		.001	.055	.119	.112	.098
- 6		.000	.005	.055	.079	.079
- 8			.000	.019	.050	.060
-10				.005	.027	.042
-12				.000	.013	.038
-14					.005	.016
-16					.002	.009
-18					.001	.005
-20					.000	.005
-22					.000	.001
-24						.000

Table 16 shows the tightness index for appropriate combinations of forecast error (measured there by standard deviation, S,) and step size.

TABLE 16. TIGHTNESS INDEX COMBINATIONS

Forecast Error, S

Step Size, Δ (Δ↓ / S→)	0	$\frac{\sqrt{N}}{8}$	$\frac{\sqrt{N}}{4}$	$\frac{\sqrt{N}}{2}$	$\frac{\sqrt{N}}{4}$	$\sqrt{N}$
1.00	0	1/64	1/16	1/4	9/16	1
.50	0	1/16	1/4	1	9/4	4
.25	0	1/4	1	4	9	16

Some values of T given in Table 16 are greater than one which may seem inconsistent with our assumption that forecasting puts a tighter constraint on the domaine of a simple random walk. However, it is not improbable that T values greater than unity will occur because of uncertainty in trend. Therefore, it is worthwhile to sensitize over T values greater than one.

Sensitivity Analysis of Trend

The trend of bond prices reflects the general level of interest rates, with prices tending to increase as interest rates fall and tending to decrease as interest rates go up. The trend effect is reflected in our analysis by making the mean of the price forecast, $\overline{X}$, higher or lower than the initial price X_0. Price drift toward $\overline{X}$ is accomplished through the effect of the transition probabilities developed by conditioning on the forecast.

From Table 1 on page 3, bond price savings of ten full points during a 52 week period were not uncommon. A swing of this magnitude would require a drift of 20 basis points per week for 50 weeks. Superimposed on this trend component would be the typical 25 - 50 basis points of random walk type of vibration.

We have sensitized trend at five levels and studied its interactions with price volatility (step size). Table 17 shows the combinations that were sensitized.

TABLE 17. TREND SENSITIVITY COMBINATIONS

Price Trend, μ
(Basis Points Per Week)

		Downtrend		Level	Uptrend	
		-20	-8	0	+8	+20
Step Size, Δ (Basis Points)	25	μ/Δ=-80%	μ/Δ= -35%	μ/Δ= 0	μ/Δ= 35%	μ/Δ=80%
	50	μ/Δ= -40%	μ/Δ=-16%	μ/Δ= 0	μ/Δ= 16%	μ/Δ=40%

Summary of Major Sensitivity Results

The major results from the sensitivity analysis are as follows:

Performance vs. Dollar Averaging

(1) Under most conditions, both the EMV and regret models outperform dollar averaging. Tables 18a and 18b show the savings realized under the conditions of interest. On the average over many simulated price sample paths, the EMV model dominates the other two as far as achieving a lower expected bond cost. Perhaps, the most interesting observation about the EMV model is that it has the ability to evaluate the available data (i.e., the price forecast, continuation "cost", price volatility, etc.) and synthesize this data into a buy/wait recommendation. Because of this aspect of EMV, the model recognizes losing or winning situations before the first toss of the coin, so to speak. The result is that stopping often occurs at the endpoints of the horizon--either immediately, or else near the surrender date depending on whether prices are increasing or decreasing. Over the long run, the EMV strategy is a good one for the corporate treasurer whose objective function is cost minimization oriented.

(2) The regret model does very well against dollar averaging when prices are drifting downward, although bond cost is slightly higher, on average, than with the EMV model. However, it should be remembered that this criterion minimizes opportunity loss (regret) rather than expected cost, so comparing average price paid may not be entirely appropriate. In any given year, the regret experienced by following an EMV strategy can be quite large. Recall the solution to Example 3, page 75, for example. The stopping rule is such that the treasurer would buy too early roughly half the time. The EMV cost minimization strategy can be very inappropriate for the corporate treasurer who is more concerned about looking bad in a given year than with long run average cost.

(3) As another point of difference, the standard deviation of price paid in a random sample of 1,000 simulated price sample paths is significantly greater for the regret criterion than for the other two. This is apparently a characteristic of "regret" as we have defined it. Under regret, the price path leading to the current price is very important because it defines regret at that stage. On the other hand,

with EMV the current price is all that matters. Therefore, points on the stopping boundary under EMV are always stopping points, whereas with regret a price may trigger a buy action in one instance, and a continue action in another. This would cause the higher variance in price paid under regret.

TABLE 18A. SAVINGS OVER DOLLAR AVERAGING IN A VOLATILE MARKET

(Δ=50 BASIS PTS/WEEK)

CONTINUATION COST/STEP SIZE RATIO, C/Δ:

		C/Δ=-.4				C/Δ=-.2				C/Δ=+.2				C/Δ=+.4			
Tightness Index, T→		0	.25	1	4	0	.25	1	4	0	.25	1	4	0	.25	1	4
Sharp Uptrend μ=20%	EMV	3.00	3.16	3.20	3.45	3.21	3.23	3.24	3.29	2.84	2.86	2.88	2.92	2.73	2.75	2.78	2.81
	R	.21	.02	(.20)	(.33)	.04	(.25)	(.40)	(.55)	(.15)	(.40)	(.58)	(.75)	(.31)	(.75)	(.93)	(.95)
Moderate Uptrend μ=8%	EMV	.74	.66	.49	.15	.52	.42	.19	.01	.84	.99	.93	.85	.70	.72	.66	.61
	R	.41	.26	.15	.03	.32	.19	.05	(.11)	.29	.14	(.01)	(.21)	.28	.01	(.29)	(.33)
Level	EMV	.95	.63	.55	.46	.98	.61	.44	.31	.76	.57	.19	(.04)	.34	.29	.11	(.16)
μ=0	R	.97	.73	.48	.39	.88	.60	.43	.22	.69	.53	.32	.15	.55	.40	.23	.09
Moderate Decline μ=-8%	EMV	1.82	1.70	1.53	1.30	1.66	1.50	1.31	1.18	1.43	1.25	.92	.74	1.31	1.18	.83	.61
	R	1.73	1.36	.90	.79	1.50	1.25	.84	.62	1.40	1.11	.70	.51	1.35	1.22	.95	.78
Sharp Decline μ=-20%	EMV	3.51	3.41	3.36	3.32	3.42	3.26	3.24	3.22	3.14	2.86	2.85	2.82	3.03	2.76	2.73	2.70
	R	3.47	3.10	3.02	2.96	3.32	2.90	2.85	2.83	2.95	2.87	2.81	2.80	2.88	2.64	2.59	2.53

TABLE 18B. SAVINGS OVER DOLLAR AVERAGING IN A QUIET MARKET

(Δ = 25 BASIS PTS/WEEK)

CONTINUATION COST/STEP SIZE RATIO, C/Δ:

		C/Δ=-.4				C/Δ=-.2				C/Δ=+.2				C/Δ=+.4			
Tightness Index, T→		0	1	4	16	0	1	4	16	0	1	4	16	0	1	4	16
Sharp Uptrend μ=20%	EMV	1.51	1.58	1.61	1.63	1.60	1.63	1.72	1.76	1.47	1.50	1.51	1.53	1.37	1.40	1.43	1.46
	R	(.10)	(.30)	(.36)	(.48)	(1.5)	(.47)	(.56)	(.70)	(.19)	(.59)	(.67)	(.81)	(.23)	(.66)	(.87)	(.96)
Moderate Uptrend μ=8%	EMV	.40	.38	.34	.20	.27	.22	.14	(.05)	.44	.44	.43	.42	.36	.36	.33	.30
	R	.21	.15	.08	(.05)	.19	.10	(.02)	(.19)	.14	.03	(.11)	(.30)	.14	(.10)	(.27)	(.58)
Level μ=0	EMV	.50	.37	.27	.22	.58	.33	.25	.17	.37	.28	.10	(.06)	.17	.13	.01	(.27)
	R	.50	.35	.25	.20	.46	.31	.22	.13	.36	.26	.16	.07	.28	.20	.11	.05
Moderate Decline μ=-8%	EMV	.96	.85	.77	.70	.90	.74	.68	.57	.86	.63	.48	.40	.69	.57	.46	.30
	R	.87	.70	.51	.40	.75	.63	.44	.31	.70	.58	.37	.26	.70	.63	.46	.40
Sharp Decline μ=-20%	EMV	1.75	1.70	1.65	1.63	1.71	1.63	1.62	1.60	1.69	1.51	1.48	1.41	1.53	1.40	1.36	1.30
	R	1.76	1.73	1.69	1.65	1.75	1.69	1.65	1.61	1.69	1.61	1.56	1.50	1.61	1.53	1.44	1.39

Stopping Characteristics of EMV vs. Regret

(4) Under the EMV criterion, it is usually optimal to either stop quickly, or else continue almost to the surrender date. This is less true when actual price data is used because price jumps are often larger than the step size of the random walk model.

(5) With the regret model, the process is never stopped in the first half of the horizon. Stopping usually occurs in the third quarter or early forth quarter. Under regret, the process was always stopped before the surrender date was reached.

(6) For both EMV and regret, stopping always occurred late in the horizon with price declines. For the cases where prices are increasing, the EMV model can recognize when the odds against doing better than the initial price are unfavorable and will stop early (usually immediately). As mentioned before, the regret model will ride a trend at least half the horizon before stopping. . .so when prices are increasing the performance falls down compared to EMV and dollar averaging.

(7) Stopping times for both criteria are summarized in Tables 19a and 19b for the sensitivity alternatives of interest.

TABLE 19A. STOPPING TIMES IN A VOLATILE MARKET

(Δ=50 BASIS PTS/WEEK)

CONTINUATION COST/STEP SIZE RATIO, C/Δ:

		C/Δ=-.4				C/Δ=-.2				C/Δ=+.2				C/Δ=+.4			
Tightness Index, T→		0	.25	1	4	0	.25	1	4	0	.25	1	4	0	.25	1	4
Sharp Uptrend μ=20%	EMV	20	21	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	R	31	36	39	41	29	30	30	34	25	25	26	26	25	25	25	25
Moderate Uptrend μ=8%	EMV	44	45	45	49	41	41	42	0	0	0	0	0	0	0	0	0
	R	36	38	39	41	34	38	38	38	30	30	31	31	25	25	27	27
Level	EMV	49	49	50	51	46	46	50	51	33	31	24	23	20	20	0	0
μ=0	R	43	43	44	44	41	41	41	42	30	31	31	32	30	30	32	32
Moderate Decline μ=-8%	EMV	49	50	50	51	49	49	50	51	44	45	47	50	39	40	40	42
	R	44	44	44	45	41	41	42	42	38	38	39	39	36	37	37	37
Sharp Decline μ=-20%	EMV	52	52	52	52	51	51	52	52	50	50	51	51	49	49	50	50
		44	44	44	45	42	42	42	43	40	40	41	41	36	37	37	37

TABLE 19B. STOPPING TIMES IN A QUIET MARKET

(Δ=25 BASIS PTS/WEEK)

CONTINUATION COST/STEP SIZE RATIO C/Δ:

		C/Δ=-.4				C/Δ=-.2				C/Δ=+.2				C/Δ=+.4			
Tightness Index, T→		0	1	4	16	0	1	4	16	0	1	4	16	0	1	4	16
Sharp Uptrend	EMV	25	23	19	0	0	0	0	0	0	0	0	0	0	0	0	0
μ=20%	R	30	32	33	35	29	29	30	30	25	26	26	27	25	26	27	27
Moderate Uptrend	EMV	40	43	43	44	38	40	40	41	0	0	0	0	0	0	0	0
μ=8%	R	36	37	37	38	35	35	35	36	33	34	34	35	28	29	29	29
Level	EMV	49	50	51	51	46	46	47	47	37	37	38	38	30	30	31	31
μ=0	R	43	43	44	44	41	41	41	41	32	32	33	33	29	29	30	31
Moderate Decline	EMV	49	51	51	51	50	50	51	51	43	44	45	47	38	40	40	41
μ=-8%	R	41	43	43	43	41	41	42	43	39	39	40	40	36	37	37	37
Sharp Decline	EMV	52	52	52	52	51	51	52	52	50	50	51	51	49	49	50	50
μ=-20%	R	42	42	43	43	40	42	42	43	38	40	40	41	36	37	37	38

Sensitivity of Major Assumptions

(8) Step Size--The ability of both new models to outperform dollar averaging is tied to the volatility of the price series and hence to the step size assumption. For step sizes between 25 and 50 basis points per week (the range of practical interest), savings appear to be proportional to step size. Consequently, as one might expect, the simulation shows that savings from the new models are more impressive in more volatile markets. From our analysis, savings in a 50 basis point issue are roughly twice as large as for a 25 basis point issue.

(9) Continuation Cost--The inclusion of continuation cost into the model turns out to be significant. Both the sign and the magnitude of continuation cost are important. Referring to Tables 19a and 19b, a consistent pattern is evident: Savings decrease significantly as the ratio C/Δ is ranged from -.4 to +.4, no matter what the trend or step size, Δ. Notice that the penalities for positive continuation costs appear to affect regret more than EMV during uptrends. This is due to EMV stopping immediately, whereas regret stops the process near the midpoint of the horizon.

(10) Accuracy of Price Forecast--The sensitivity results clearly indicate that savings from both new models are improved by tightening the forecast. In particular, an accurate prediction of trend has a high payoff. Forecast error, as measured by the tightness index, is not as sensitive to performance as trend measurement. However, the value of perfect information is clearly high.

(11) Trend--The effect of trend in prices is particularly important. In a downtrending market, one can profit considerably using either new model and it is less important which one is used. On the basis of the simulation results, we would recommend that the regret model be used in both level and declining markets. This allows the treasurer to do well in sharp downturns with relatively low cost penalties. In rising markets, the advantage is clearly in favor of EMV because it can recognize when it is prudent to stop immediately.

A final point to make is that these "theoretical" conclusions would be expected to hold in the real world provided market prices actually conform to the constrained random walk model. Insofar as savings are concerned, "imperfections" in the real market place will tend to effect dollar averaging as well as the two new models, so the improved performance over dollar averaging should hold up. However, the stopping characteristics of the two models in actual practice may be different from those simulated because actual price increases can be much larger than the step sizes assumed in developing optimal stopping rules. When this "imperfection" is present, "buy" signals would tend to occur earlier.

The next chapter attempts to show how these models would have performed in an actual historical case study situation.

VII. CASE STUDY OF AN ACTUAL SINKING FUND SITUATION

The purpose of this chapter is to provide a richer, more realistic example of how the models that have been developed might be applied in practice. We have shown in Chapter VI that both the EMV and regret models outperform dollar averaging using simulated price sample paths. It remains to show how the new techniques perform when actual market prices are used.

Situation

Table 20 shows the price fluctuations that occurred during the 1968 - 1975 time period for six sinking fund debentures of Anheuser-Busch.

TABLE 20. HIGH-LOW PRICE SPREAD ON ANHEUSER-BUSCH DEBENTURES

(SOURCE: STANDARD & POOR'S BOND GUIDE)

(BONDS RATED 'A' 1968 - 73; 'AA' 1974 - 77)

	3.375s (1977)	4.5s (1989)	5.45s (1991)	6s (1992)	7.95s (1999)	9.2s (2005)
1968	6.75	5.50	8.50	12.50		
1969	2.50	11.50	13.25	17.75		
1970	1.00	6.25	10.00	14.00		
1971	4.13	17.87	7.50	12.00		
1972	7.75	8.25	3.00	5.25		
1973	2.50	7.25	8.50	11.50		
1974	7.00	4.00	13.38	14.75	14.00	
1975	No Sales	3.00	8.00	6.50	8.50	5.38
Average	4.52*	7.95*	9.02	11.78	11.25	5.38

*Closely held issues

During that period of time, all of these bonds were discount bonds with market prices in the 70's and 80's, primarily. Two issues, the 3-3/8s and 4.5s were being accumulated by Mellon Bank (accumulation is discussed in Chapter VIII.) Consequently, there was very little liquidity in these issues after 1974.

On the other hand, the 5.45s, 6s, 7.95s and 9.2s were widely held and actively traded; market prices tended to reflect the prevailing market rate of return for A and AA rated bonds. Of these four, the 7.95s and 9.2s were newly issued and sinking fund payments had not begun. Consequently, we will focus our attention on the Anheuser-Busch 5.45s and 6s.

Table 21 gives the par value of Anheuser-Busch bonds likely to be purchased on the open market in approximately the year and amount shown. The importance to Anheuser-Busch of making rational market decisions is obvious from the size of these transactions.

TABLE 21. FUTURE ANHEUSER-BUSCH SINKING FUND PURCHASES

($ THOUSANDS)

Year	Amount	Year	Amount	Year	Amount
1976	$ 3,200	1986	$20,500	1996	$13,700
1977	6,800	1987	20,600	1997	16,200
1978	6,800	1988	18,600	1998	7,200
1979	6,800	1989	17,400	1999	7,200
1980	6,800	1990	17,800	2000	7,200
1981	6,800	1991	13,700	2001	7,200
1982	6,800	1992	13,700	2002	7,200
1983	13,300	1993	13,700	2003	7,200
1984	20,500	1994	13,700		
1985	20,500	1995	13,700		

Total ($ thousands) = $342,100

Suppose that our models had been used during the early 1970's instead of dollar averaging, which was the strategy actually employed. How well would the models have performed? This is an intriging question, and we will attempt to simulate it as realistically as possible subject to data limitations.

Developing a Forecast

The most difficult aspect of an attempt to simulate the past is recreating a "price at the surrender date" forecast. In actual practice, the corporate treasurer would utilize many sources of information to develop this forecast and would probably not rely on any single source. However, we are forced to resort to a more mechanical approach because there is no better alternative. The approach we will use makes use of a widely respected forecasting service from the Wharton Econometric Forecasting Associates, Inc. (WEFA), a not for profit corporation owned by the University of Pennsylvania. Anheuser-Busch has subscribed to the "Wharton Model" for many years, including the 1971 - 1978 time period, and they use it to shape management's expections about future economic factors--among them, general interest rate levels. The method we will employ is:

(1) to develop a relationship between Anheuser-Busch bond prices and Moody's composite average of corporate bonds.

(2) obtain the Wharton model forecast for Moody's composite average of corporate bonds.

(3) translate that back into a forecast of Anheuser-Busch bond price at the surrender date.

Correlation Between Anheuser-Busch Bonds and Moody's Index

Data for Anheuser-Busch 5.45s and 6s, and for Moody's composite index of corporate bonds is given in Table 22.

TABLE 22. BOND VS. INDEX CORRELATION DATA

Yield to Maturity

Date	A-B 5.45s	A-B 6s	Moody's
7/71	7.70	7.90	8.14
8/71	7.35	7.50	8.12
9/71	7.50	7.60	7.97
10/71	7.20	7.20	7.88
11/71	7.25	7.30	7.77
12/71	7.00	7.10	7.79
1972			
1/72	7.15	7.30	7.66
2/72	7.10	7.10	7.68
3/72	7.00	7.12	7.66
4/72	7.20	7.30	7.71
5/72	7.00	7.10	7.71
6/72	7.20	7.25	7.66
7/72	7.20	7.35	7.66
8/72	6.95	7.15	7.61
9/72	7.10	7.20	7.59
10/72	7.05	7.10	7.59
11/72	6.95	7.05	7.52
12/72	7.00	7.05	7.47

This data is plotted against Moody's composite index in Figures 11 and 12. From this data, we develop the rule of thumb that Anheuser-Busch 5.45s and 5s are priced to yield 50 basis points less than the composite index.

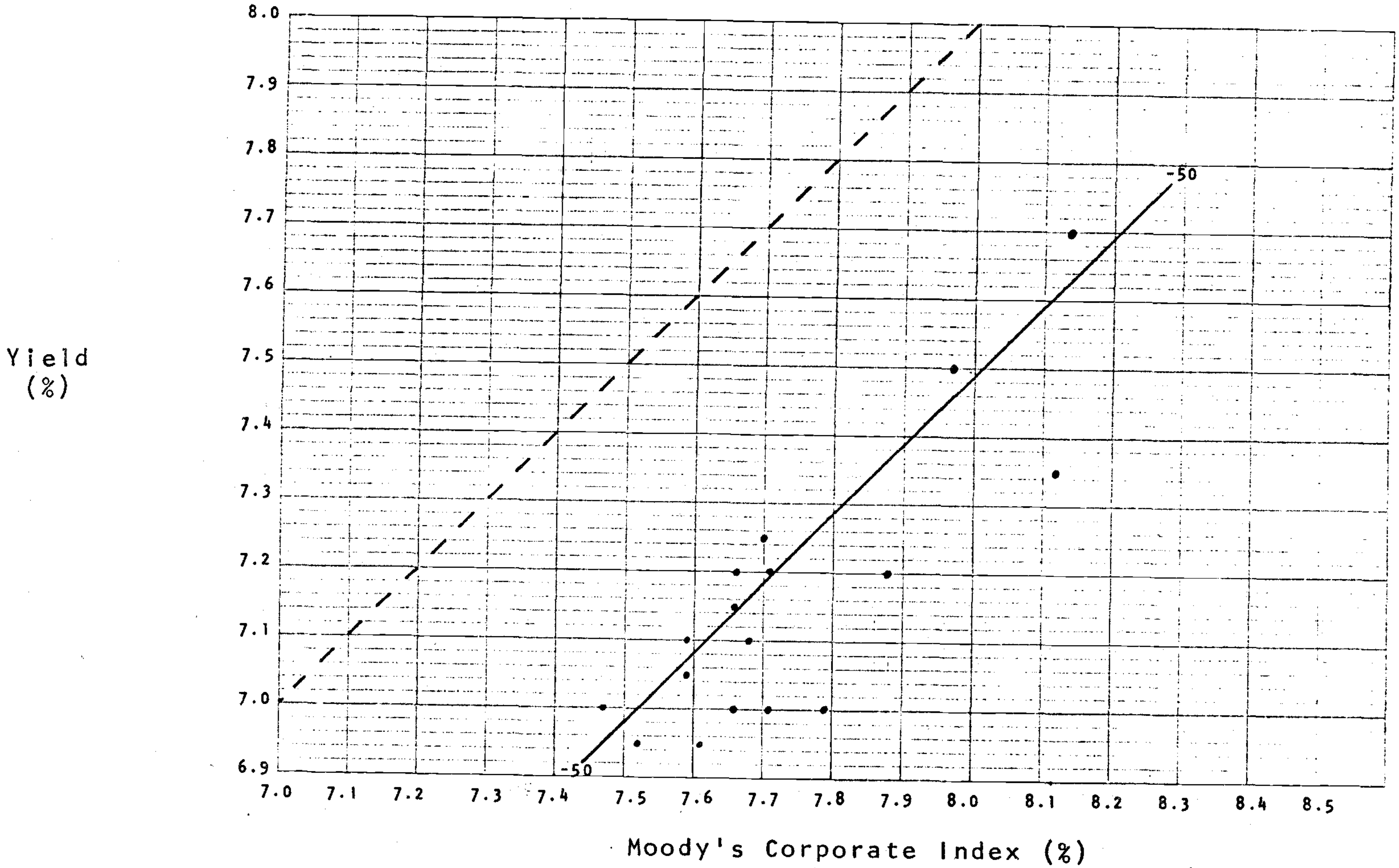

Figure 11. Anheuser-Busch 5.45s Vs. Moody's Index

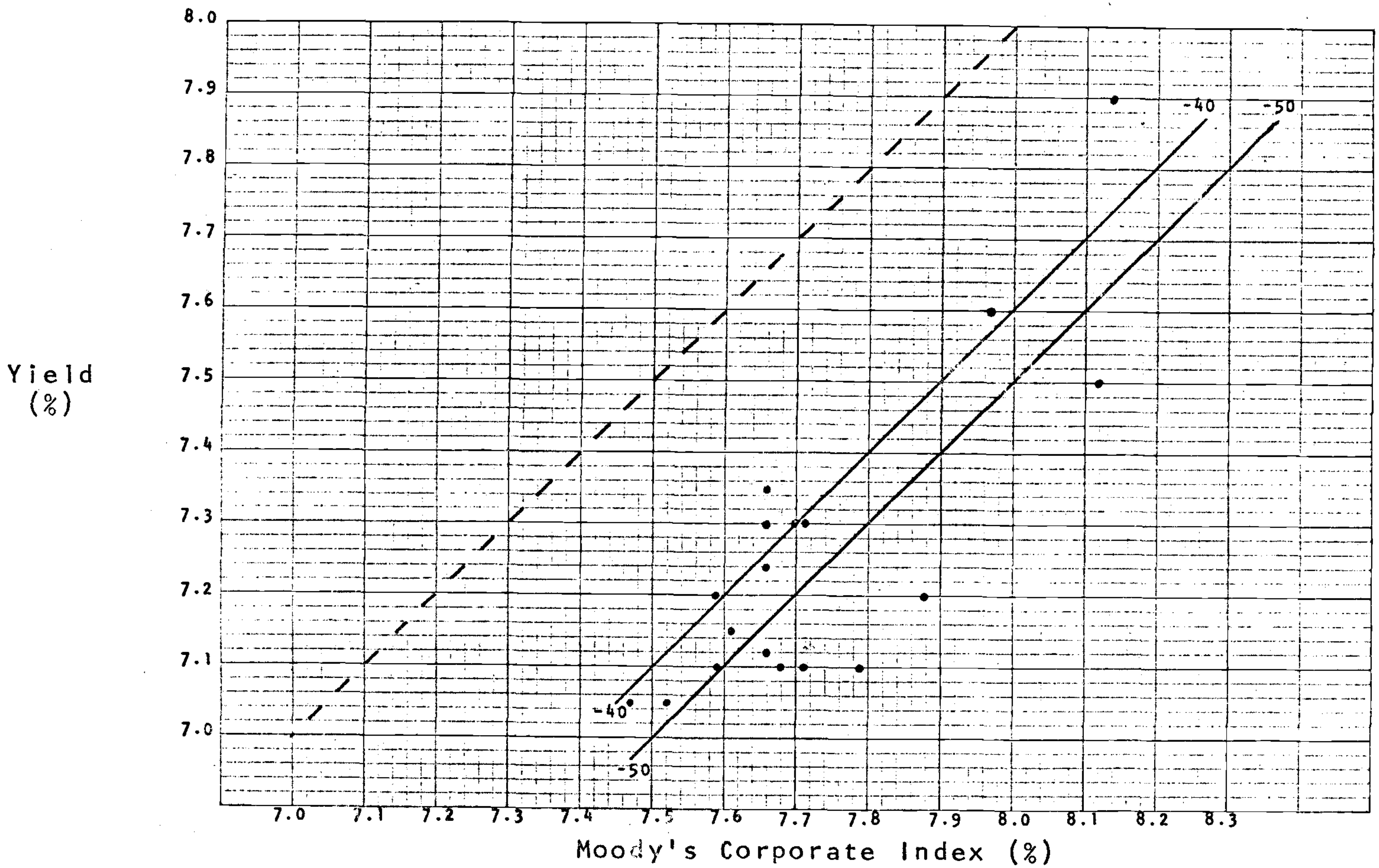

Figure 12. Anheuser-Busch 6s Vs. Moody's Index

The data used to develop the relationship between Anheuser-Busch bond yield and market yield will be used in the next step, which involves forecasting Anheuser-Busch bond prices using the Wharton model.

Wharton Model Forecast

Historical interest rate forecasts were obtained from WEFA's computer data base. The forecasts made during 1972 - 1975 period are shown in Table 23. To read the table, the actual data for each period appears in the left hand margin to the right of the date. Under the Q0 heading is the lagged value of the data for the quarter before the forecast. The forecast for any quarter can be read by going across the line to the Q1 column and then down the diagonal. For example, the forecast available in October, 1972 (1972,4) is:

Forecast for	Value	
1972,4	7.6	("Within the Quarter" Forecast)
1973,1	7.6	
1973,2	7.6	
1973,3	7.6	
1973,4	7.7	

This enables one to examine the forecast values for any quarter by reading across the line. Thus, the actual corporate bond yield for 1973,4 was 8.0. Reading across the line we have 8.0 as the value for 1973,3, 8.2 as the forecast within the quarter, 8.7 as the forecast for the 4th quarter made in the 2nd quarter, and so on.

TABLE 23. WHARTON FORECAST, MOODY'S CORPORATE BONDS

LAST AVAILABLE	ACTUAL	FORECAST HORIZON Q0	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8
1972,1	7,7	7,8	7,8							
1972,2	7,7	7,7	7,7	7,7						
1972,3	7,6	7,7	7,7	7,7	7,7					
1972,4	7,5	7,6	7,6	7,7	7,8	7,7				
1973,1	7,6	7,5	7,6	7,6	7,7	7,8	7,7			
1973,2	7,6	7,6	7,7	7,7	7,6	7,8	7,9	7,8		
1973,3	8,0	7,6	8,2	7,9	7,7	7,6	7,9	8,0	7,8	
1973,4	8,0	8,0	8,2	8,7	8,0	7,7	7,7	8,0	8,1	7,9
1974,1	8,2	8,0	8,0	8,3	9,0	8,1	7,7	7,7	8,1	8,2
1974,2	8,7	8,2	8,5	8,0	8,3	9,1	8,1	7,7	7,6	8,2
1974,3	9,5	8,7	9,0	8,6	7,9	8,2	9,1	8,1	7,7	7,6
1974,4	9,7	9,4	9,2	9,2	8,7	7,9	8,1	9,0	8,1	7,6
1975,1	9,5	9,6	9,4	9,5	9,4	8,7	7,9	7,9	8,8	8,0
1975,2	9,6	9,4	9,6	9,2	9,4	9,4	8,7	7,9	7,8	8,6
1975,3	9,6	9,5	9,7	9,4	9,0	9,1	9,4	8,7	7,9	7,7
1975,4	9,6	9,5	9,7	9,9	9,4	8,9	9,0	9,4	8,6	7,9

The average forecast error for interest rate level from the Wharton model is as follows (over the 1972,1 - 1978,2 time period):

Forecast Length	RMSE
4 quarters ahead	0.30
3 quarters ahead	0.28
2 quarters ahead	0.28
1 quarter ahead	0.29
Within quarter	0.28

We will use this as the forecast error component in our forecast.

Conversion to Anheuser-Busch Bond Forecast

We will develop our forecast using the following method:

Step (i) At the beginning of the horizon (say 1972,4), look up the Wharton model forecast for bond yield in the quarter of the surrender date (4 quarters ahead, or 1973,4=7.7%). Make this the expected interest rate, $\overline{X}$.

Step (ii) Assume that the error in forecasting interest rates at the surrender date is normally distributed with mean $\overline{X}$ from Step (i), and standard deviation equal to the RMSE value given above (i.e., 0.30 for a forecast 4 quarters ahead).

Step (iii) Subtract the 0.50 yield differential between Moody's index and Anheuser-Busch bonds, and convert this yield to maturity value (YTM) to price level. This price distribution and associated probabilities from Step (ii) are used as the price forecast.

Results

The details of developing inputs for the simulation are omitted because they are lengthy and are straightforward data manipulations. The salient factors that affect the EMV and regret models are tabulated on the following page, however.

TABLE 24. ASSUMPTIONS FOR 1973 - 74 SIMULATION

	5.45s		6s	
1. Step Size, Δ	33 Basis Pts. Per Week		38 Basis Pts. Per Week	
2. Continuation Cost, C				
- Coupon Rate	1973-75=5.45%		6.00%	
- Use of Funds Rate	1973=8.18%		8.18%	
(From Wharton Model, Avg.)	1974=9.83%		9.83%	
4-6 Mo. Comc'l Paper)	1975=6.33%		6.33%	
3. Forecast	Mean	Std. Dev.	Mean	Std. Dev.
- Yield to Maturity	1973= 7.2%	.30%	7.2%	.30%
	1974= 7.6%	.30%	7.6%	.30%
	1975= 8.5%	.30%	8.5%	.30%
- Absolute Price Level	1973=82.85	2.25	87.68	2.71
	1974=79.60	2.22	84.29	2.68
	1975=74.23	2.21	77.90	2.70
4. Trend				
- Initial Price	1973=83.78		88.50	
	1974=80.43		84.71	
	1975=74.20		79.78	
- Weekly Drift, μ	1973=-2 basis pts.		-2 basis pts.	
	1974=-2 basis pts.		-1 basis pts.	
	1975=-6 basis pts.		-4 basis pts.	

The important ratios corresponding to these assumptions are:

		5.45s	6s
C/Δ	1973	- 8%	- 6%
	1974	-14%	-12%
	1975	+ 1%	- 8%
μ/Δ	1973	- 2%	- 2%
	1974	- 2%	0
	1975	- 6%	- 4%
$T=\frac{s^2}{N\cdot\sigma_y^2}$	1973	85%	98%
	1974	83%	96%
	1975	82%	90%

The exhibits on the next two pages show how the two new models would have performed on Anheuser-Busch 5.45s and 6s during 1973 - 1975. The optimal stopping boundary for the EMV model is plotted on the graph. Calculations are also given for the 5.45s to illustrate how the regret stopping point is determined for each year (calculations for the 6s are similar so are omitted).

Anheuser-Busch 5.45s

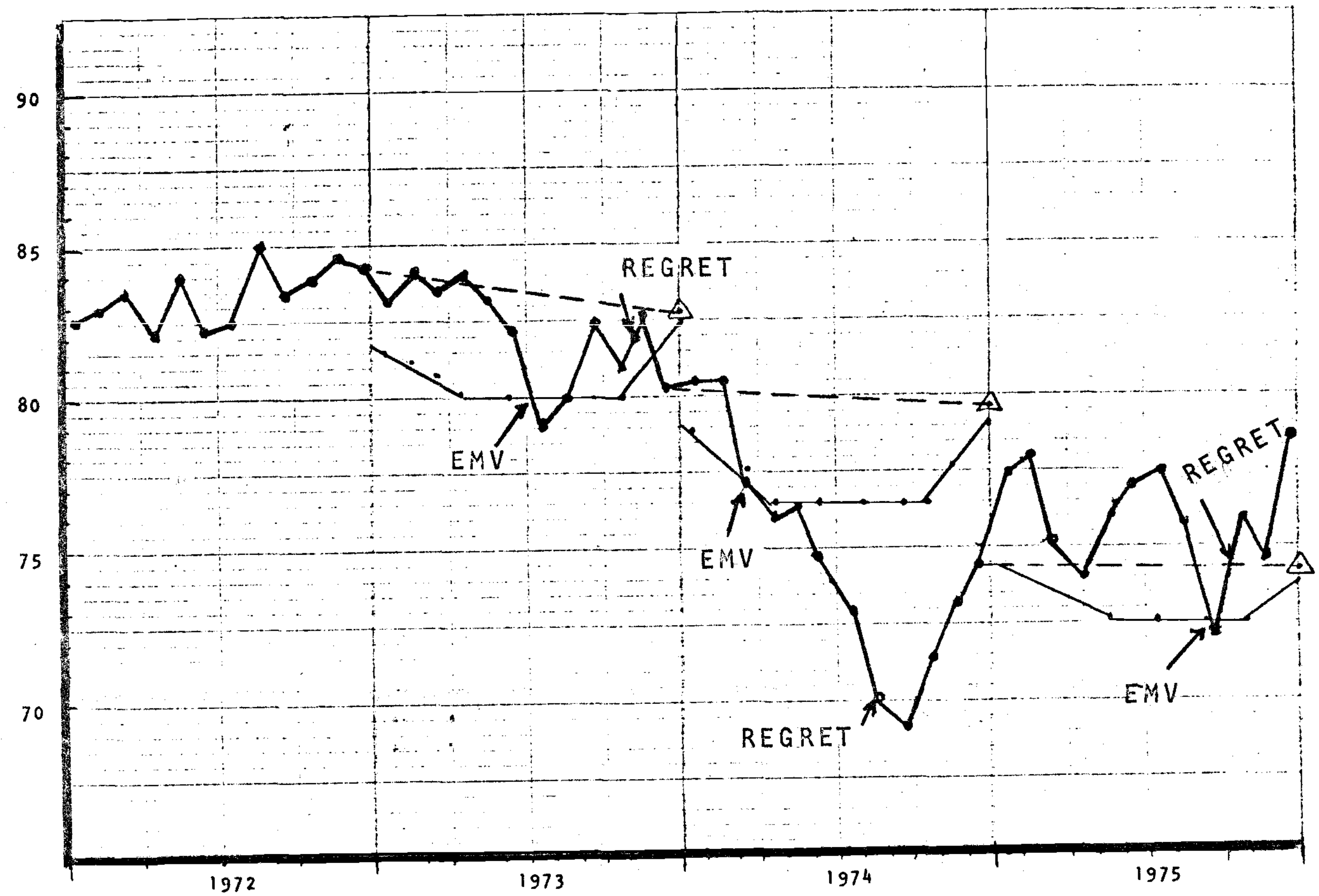

Figure 13. Bond Purchase Points Using the New Models
Case 1--Anheuser-Busch 5.45s

Anheuser-Busch 6s

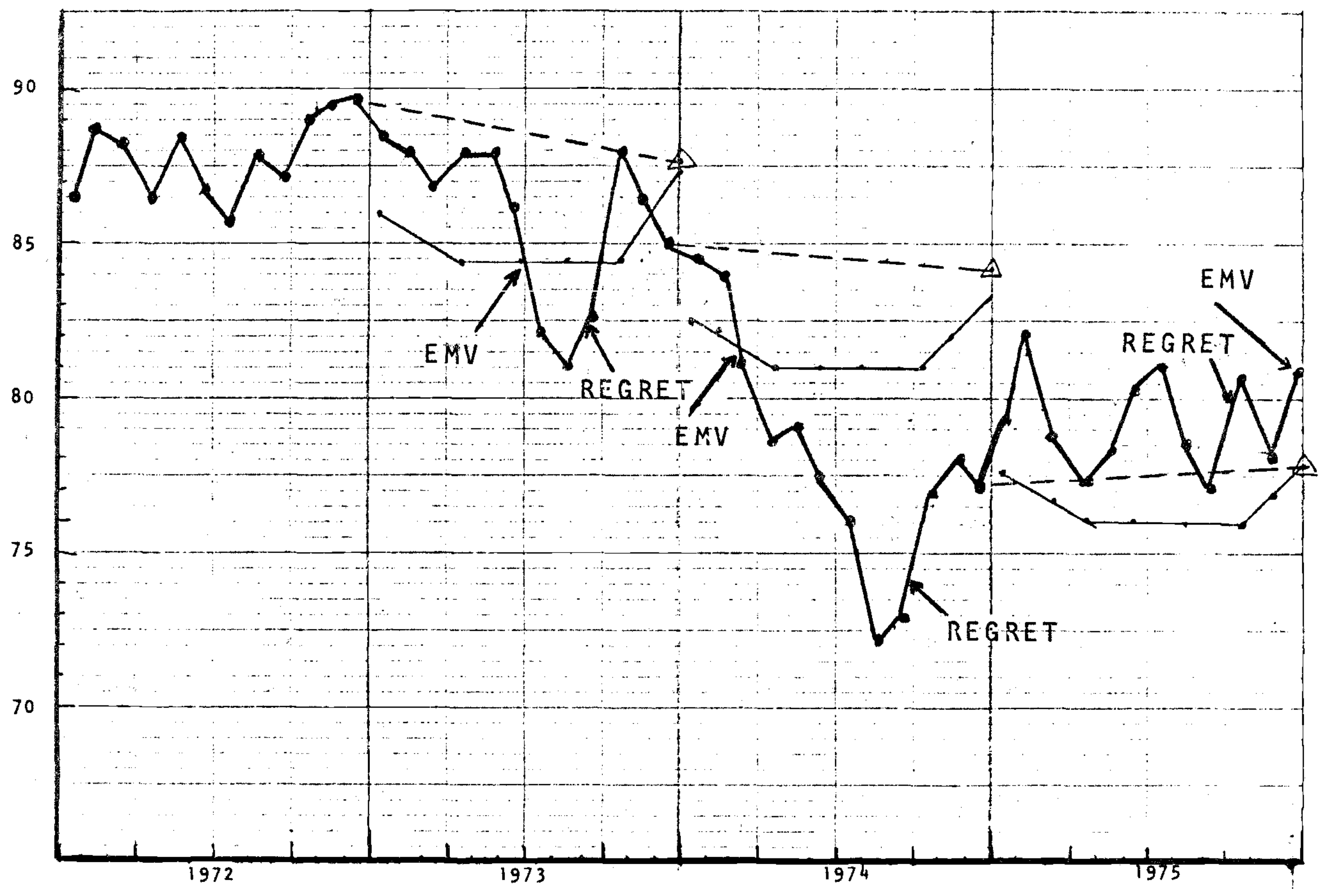

Figure 14. Bond Purchase Points Using the New Models
Case 2--Anheuser-Busch 6s

Anheuser-Busch 5.45s in 1973

Regret Calculations

Initial Price: 83.80

Step Size: 125 basis pts/month

1973	Stage N	Actual Price	Regret 1/	Calculated Number of Tails 2/	Critical Regret Value	Optimal Action
J	1	83.20	0	3	-	Continue
F	2	84.10	.72	1	-	Continue
M	3	83.50	.24	2	-	Continue
A	4	84.00	.64	2	-	Continue
M	5	83.05	0	3	-	Continue
J	6	82.05	0	4	-	Continue
J	7	79.00	0	5	-	Continue
A	8	80.00	.80	5	3	Continue
S	9	82.25	2.60	5	3	Continue
O	10	80.95	1.56	6	2	Continue
N	11	82.70	3.00	6	1	Stop (at 82.20= 80.95+1.25)
D	12	80.40				
	AVG =	82.10				

1/ Regret is scaled by dividing the difference between current price and the lowest price to date by step size.

2/ Number of tails is calculated using the relationship:

$$\text{Price} = (n-2i)\cdot\Delta + P_o$$

where i = number of tails
Δ = step size
P_o = initial price

Anheuser-Busch 5.45s in 1973

Regret Calculations

Stage 1: $(83.20-83.80)=(1-2i)(1.25)$

$i=2(1.48)\doteq 3$

Stage 2: $(84.10-83.80)=(2-2i)(1.25)$

$i\doteq 1$

Stage 3: $(83.50-83.80)=(3-2i)(1.25)$

$i=1.4\doteq 2$

Stage 4: $(84.00-83.80)=(4-2i)(.125)$

$i\doteq 2$

Stage 5: $(83.05-83.80)=(5-2i)(1.25)$

$i\doteq 3$

Stage 6: $(82.05-83.80)=(6-2i)(1.25)$

$i\doteq 4$

Stage 7: $(79.00-83.80)=(7-2i)(.125)$

$i\doteq 5$

Stage 8: $(80.00-83.70)=(8-2i)(1.25)$

$i\doteq 5$

Stage 9: $(82.25=83.70)=(9-2i)(1.25)$

$i\doteq 5$

Stage 10: $(80.95-83.70)=10-2i)(1.25)$

$i\doteq 6$

Stage 11: $(82.70-83.70)=(11-2i)(1.25)$

$i\doteq 6$

Anheuser-Busch 5.45s in 1974

Regret Calculations

Initial Price: 80.40

Step Size: 125 basis pts/month

1974	Stage N	Actual Price	Regret 1/	Calculated Number of Tails 2/	Critical Regret Value	Optimal Action
J	1	80.50	.08	0	-	Continue
F	2	80.50	.08	1	-	Continue
M	3	77.20	0	3	-	Continue
A	4	76.10	0	3	-	Continue
M	5	76.40	.24	4	-	Continue
J	6	74.70	0	5	-	Continue
J	7	73.00	0	6	-	Continue
A	8	70.00	0	8	0	Stop (price=70.00)
S	9	69.00	0			
O	10	71.40	1.92			
N	11	73.00	3.20			
D	12	74.20				
	AVG =	74.67				

1/ Regret is scaled by dividing the difference between current price and the lowest price to date by step size.

2/ Number of tails is calculated using the relationship:

$$\text{Price} = (n-2i)\cdot\Delta + P_o$$

where i = number of tails

Δ = step size

P_o = initial price

Anheuser-Busch 5.45s in 1974

Regret Calculations

Stage 1: $(80.50-80.40)=(1-2i)(1.25)$

$$i=.46\doteq 0$$

Stage 2: $(80.50-80.40)=(2-2i)(1.25)$

$$i\doteq 1$$

Stage 3: $(77.20-80.40)=(3-2i)(1.25)$

$$i\doteq 3$$

Stage 4: $(76.10-80.40)=(4-2i)(1.25)$

$$i\doteq 3$$

Stage 5: $(76.40-80.40)=(5-2i)(1.25)$

$$i\doteq 4$$

Stage 6: $(74.70-80.40)=(6-2i)(1.25)$

$$i=5$$

Stage 7: $(73.00-80.40)=(7-2i)(1.25)$

$$i\doteq 6$$

Stage 8: $(70.00-80.40)=(8-2i)(1.25)$

$$i\doteq 8$$

Anheuser-Busch 5.45s in 1975

Regret Calculations

Initial Price: 74.20

Step Size: 125 basis pts/month

1975	Stage N	Actual Price	Regret 1/	Calculated Number of Tails 2/	Critical Regret Value	Optimal Action
J	1	77.50	2.64	1	-	Continue
F	2	78.00	3.04	1	-	Continue
M	3	75.10	.72	1	-	Continue
A	4	73.90	0	2	-	Continue
M	5	76.00	1.68	2	-	Continue
J	6	77.00	2.48	2	-	Continue
J	7	77.50	2.88	2	5	Continue
A	8	75.75	1.48	3	4	Continue
S	9	72.10	0	5	3	Continue
O	10	76.00	3.12	4	2	Stop (at 74.60= 72.10+2(1.25))
N	11	74.70				
D	12	78.60				
	AVG =	76.00				

1/ Regret is scaled by dividing the difference between current price and the lowest price to date by step size.

2/ Number of tails is calculated using the relationship:

$$\text{Price} = (n=2i)\cdot\Delta+P_o$$

where i = number of tails
Δ = step size
P_o = initial price

Anheuser-Busch 5.45s in 1975

Regret Calculations

Stage 1: $(77.50-74.20)=(1-2i)(1.25)$

$$i \doteq 1$$

Stage 2: $(78.00-74.20)=(2-2i)(1.25)$

$$i \doteq 1$$

Stage 3: $(75.10-74.20)=(3-2i)(1.25)$

$$i \doteq 1$$

Stage 4: $(73.90-74.20)-(4-2i)(1.25)$

$$i \doteq 2$$

Stage 5: $(76.00-74.20)=(5-2i)(1.25)$

$$i \doteq 2$$

Stage 6: $(77.00-74.20)-(6-2i)(1.25)$

$$i \doteq 2$$

Stage 7: $(77.50-74.20)-(7-2i)(1.25)$

$$i \doteq 2$$

Stage 8: $(75.75-74.20)=(8-2i)(1.25)$

$$i \doteq 3$$

Stage 9: $(72.10-74.20)=(9-2i)(1.25)$

$$i \doteq 5$$

Stage 10: $(76.00-74.20)=(10-2i)(1.25)$

$$i \doteq 4$$

Summary

The price that would have been paid for the bonds under dollar averaging, EMV, and regret are summarized below:

	1973		1974		1975	
	5.45s	6s	5.45s	6s	5.45s	6s
Dollar Averaging	82.10	86.00	74.67	78.50	76.00	78.95
EMV	80.00	84.20	77.20	81.40	72.50	81.00
Regret	82.20	82.40	70.00	73.50	74.60	80.00

Overall, the two new models did very well compared to a dollar averaging strategy over the same period. While this historical look back proves nothing statistically, the Anheuser-Busch case study nevertheless demonstrates how the new methods can be used in a practical decision situation. This example would in all likelihood be sufficient to get the Anheuser-Busch corporate treasurer to try the new approaches.

The next, and final chapter summarizes the conclusions of this study and discusses a special case of the sinking fund problem that requires a slightly different approach.

VIII. SUMMARY AND CONCLUSIONS

Recapitulation

This study was motivated by a conversation with a corporate treasurer who was wrestling with his own version of the sinking fund problem. His educational background and experience were probably typical of most corporate treasurers, and he was thoroughly exasperated with his attempt to develop a systematic sinking fund purchasing strategy that promised to do better than the averaging approach he had been using. He had not been doing badly with dollar averaging, but it seemed to him that with all the sources of information and analysis at his disposal, he could do better. Perhaps as important a consideration was that he had been receiving a great deal of pressure from his management to put his decision process on a more scientific basis. Consequently, the two new models developed in this study are in response to a need that is <u>real</u>, and probably also is prevalent.

Furthermore (from our earlier examples), if Anheuser-Busch and Allied Chemical are typical of most large corporations, then the potential savings that could be realized by improving bond purchase decisions is significant: Allied Chemical will probably purchase \$338 million and Anheuser-Busch \$342 million on the open market by the year 2000.

In researching this problem, it became apparent to the author that cost minimization was not necessarily the most appropriate objective function for the sinking fund problem. Because of the growing amount of debt to be retired each year, the performance of the corporate treasurer in this area becomes increasingly visible. His performance is easily measured after the fact, and his own personal well being may cause him to try to minimize "regret" rather than cost. For this reason, a decision was made early to try to define and solve the sinking fund problem under a "regret" formulation as well as a cost minimization formulation. As far as can be determined, the definition of regret used in this study and the solution under this criterion are original. The EMV criterion (cost minimization) is certainly appropriate here, also, and is the usual criterion of choice in optimal stopping problems of this type.

While there is still a considerable amount of discussion in the financial literature about the random walk hypothesis, it is generally recognized as the foremost model to represent price movements in speculative markets. As pointed out in Chapter II, however, it suffers from a basic flaw: the price sample paths generated from a simple random walk are too free to wander. Consequently, we attempted to justify conceptually, based on economic rationale, a model that would constrain the sample paths of a random walk to a more limited neighborhood. The development of the mathematics of a constrained random walk model, its economic justification in terms of the bond market, and its use in the sinking fund problem is original research.

Conclusions

The major results of this study are as follows:

(1) Under most conditions, both the EMV and regret models can be expected to outperform dollar averaging. This was conclusively demonstrated by simulating price sample paths generated from the constrained random walk model assumed to reflect market price movements. While not proof as such, a case study of Anheuser-Busch 5.45% and 6% sinking fund bonds shows that during the 1973 - 75 period analyzed, both models performed well against the dollar averaging appraoch actually employed by the firm.

(2) One of the most useful features of the EMV model, in particular, is that it provides a mechanism for synthesizing several pieces of information that are interrelated and which affect the purchase decision. That is, the treasurer can "plug into the model" his price forecast, the coupon rate of the bond in question, and "alternate use of funds" rate

and the volatility of the bond. The model will then consolidate this information into an actionable, simple decision rule. By sensitizing the inputs the treasurer can get a good fix on how safe it is if he buys today, or decides to wait. With this tool available to him, his decisions are made in a much more scientific and justifiable manner. If he turns out to be wrong, at least he can now explain why he made the wrong decision.

(3) The regret model does particularly well when prices are forecast to decline. The steeper the downturn, the more likely the regret model will turn in the best performance of the three alternative approaches. On the other hand, we would recommend that the EMV model be used when the forecast indicates prices will rise. In a level situation, we would recommend using the regret model.

(4) The potential savings that both new models can achieve with respect to dollar averaging is directly proportional to the volatility of price (reflected by the step size of the random walk). The greater the volatility, the more potential there is to realize a significant savings.

(5) The effort spent in developing a good price forecasting method will pay off using either model. The tighter the forecast, the higher the savings relative to dollar averaging. In particular, it pays to be able to forecast price trends.

(6) The "continuation cost" aspect of the sinking fund problem is significant--in sign as well as magnitude. It is possible to have a negative "cost" in the case where the ability to put off committing funds to retire a sinking fund offsets the coupon interest payments that must be paid while the bonds are outstanding. In sensitivity analysis of this factor, it was shown that this "cost" can have a big effect on the savings realized over dollar averaging.

Possible Extension of This Research: The Special Case of a Sinking Fund Accumulator

In recent years, a few large banks (Mellon Bank and First National Bank of Atlanta, for example) have been astute enough to recognize an opportunity to corner the market, so to speak, in sinking fund issues nearing maturity.[14/] For example, First National Bank

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14/ Details of this strategy, called accumulation, are given in (18).

of Atlanta has cornered 95% of the market in Anchor Hocking 5.125% sinkers due 4/15/91. While today the bond shows a yield to maturity of 5.95%, compared to 8.50% for most industrials, the actual yield will be much higher. The reason is that in less than 12 months, the company will be forced to turn to First National to fulfill its sinking fund requirment. Obviously, the First National is in a position to demand a substantial premium for its bonds.

This "lock up" strategy is not as effective now as it once was because corporations that were burned by this tactic have learned that they can preempt an accumulator's interest in an issue by prepurchasing several sinking fund payments in advance of their due dates. This strategy is being employed by Anheuser-Busch, for example, after Mellon Bank worked a corner on their 4.5 issue due 3/1/89. By "anticipating" several sinking fund payments, a company can remain absent from the market for a number of years. The accumulator's rationale for holding the sinking fund issue requires selling into the fund at a substantial capital gain. Anticipation deprives the collector of the opportunity to realize a capital gain for several years, thus reducing the yield to what is attainable on alternative investments.

The problem with anticipation from the companies point of view is that a prepurchase means tying up funds before it becomes mandatory to do so on the surrender date. Bonds repurchased by a corporation cannot be resold without again being registered. Consequently, anticipation is usually limited to sinking fund payments for two or three years foward.

From the above it is clear that a logical extension of our basic approach is to lengthen the time horizon from one year up to, say, three years. This is an even more interesting problem in some ways because forecasts are obviously going to be more widely disbursed and continuation costs become more of a factor.

However, when bonds are in the hands of an accumulator, there is very little liquidity in the issue and prices are set primarily as a result of direct negotiation between the company and the accumulator. Consequently, the price model developed earlier will not be valid. However, in these situations the accumulator will generally not sell a portion of his holdings to the corporation; usually, the corporation must buy out the accumulator's entire position. This, then, is the essence of a special sinking fund problem (assumulation):

Usually, a firm has to float a special debt issue to raise enough cash ($40 - 60 million, possibly) to buy out the accumulator. Suppose a firm must buy out an accumulator anytime within the next three years. The problem is to time this debt issue most economically, i.e., when interest rates are lowest.

Interest rates (or equivalently bond prices) for a new issue are modeled as a constrained random walk and either the EMV or regret model can be applied to determine the optimal reissue point.

In this manner, the special situation where a corporation's bonds are in the hands of one or more accumulators and are not priced "at market" can be treated as a multi-period extension of the basic sinking fund problem we have solved.

BIBLIOGRAPHY

(1) Alexander, Sidney S., "Price Movements in Speculative Markets: Trends or Random Walks", Industrial Management Review, Vol. 27, May, 1961, pp. 7-26.

(2) Bachelier, Louis, "Theory of Speculation", appeared in (9), pp. 17-78.

(3) Bierman, Harold, "The Bond Refunding Decision as a Markov Process", Management Science, Vol. 12, Aug. 1966, pp. 545-551.

(4) Bowlin, Oswald D., "The Refunding Decision: Another Special Case in Capital Budgeting", J. of Finance, Mar. 1966, pp. 55-68.

(5) Breiman, Leo, "Stopping Rule Problems", in Applied Combinatorial Mathematics, Edwin F. Bechenback, ed., John Wiley & Sons, New York, 1964, pp. 224-319.

(6) Cargill, Thomas F., and Rausser, Gordon C., "Time and Frequence Domain Representations of Future Prices as a Stochastic Process", J. of the Am. Statistical Assn., Vol. 67, No. 337, March, 1972, pp. 23-30.

(7) Cootner, Paul H., "Comments on the Variation of Certain Speculative Prices, The M.I.T. Press, Cambridge, Mass., 1964, pp. 333-337.

(8) Cootner, Paul H., "Stock Prices: Random vs. Systematic Changes", Industrial Management Review, Vol. 3, Spring, 1972, pp. 24-45.

(9) Cootner, Paul H., The Random Character of Stock Market Prices, The M.I.T. Press, Cambridge, Mass., 1964, 536 pages.

(10) Cox, D. R., and Miller, H. D., The Theory of Stochastic Processes, Methen, Landon, 1965.

(11) Dyckman, Thomas R., Thomas, L. Joseph, and Magee, Robert P., "Dynamic Models of Bond Refunding", Decisions Sciences, Vol. 6, No. 4., Oct. 1955, pp. 614-630.

(12) Fama, Eugene F., "Efficient Capital Markets: A Review of Theory and Empirical Work", J. of Finance, Vol. 26, No. 1, May, 1970, pp. 383-417.

(13) Fama, Eugene F., and Blume, Marshall E., "Filter Rules and Stock Market Trading", J. of Business, Vol. 39, Jan., 1966, pp. 226-241.

(14) Fama, Eugene F., "Mandelbrot and the Stable Paretian Hypothesis", J. of Business, Vol. 36, No. 4, Oct. 1963, pp. 420-429.

(15) Fama, Eugene F., "Tomorrow on the New York Stock Exchange", J. of Business, July, 1965, pp. 285-299.

(16) Granger, C. W. J. and Morgenstern, O., "Spectral Analysis of New York Stock Market Prices", Kyklos, Vol. 16, 1963, pp. 1-27.

(17) Granger, C. W. J., and Labys, Walter C., Speculation, Hedging and Commodity Price Forecasts, D. C. Heath & Co., Lexington, Mass., 1970, 320 pages.

(18) Leader, Susan, "Playing the Cat-and-Mouse Game in Sinking Fund Bonds", Institutional Investor, April, 1977, pp. 89-96.

(19) Levy, Robert A., "Relative Strength as a Criterion for Investment Selection", J. of Finance, Vol. 22, Dec., 1967, pp. 595-610.

(20) Mandelbrot, Benoit, "On the Distribution of Stock Price Differences", Operations Research, Nov.-Dec., 1967, pp. 120-130.

(21) Mandelbrot, Benoit, "The Variation of Certain Speculative Prices", J. of Business, Vol. 36, No. 4, Oct. 1963, pp. 394-419.

(22) Osborne, M. F. M., "Brownian Motion in the Stock Market", Operations Research, Vol. 7, March-April, 1959, pp. 145-173.

(23) Osborne, M. F. M., "Periodic Structure in the Brownian Motion of Stock Prices", Operations Research, Vol. 10, May-June, 1962, pp. 345-379.

(24) Rogalski, C. A., "Bond Prices: Trends or Random Walks?", J. of Finance, Vol. 27, No. 1, May, 1973, pp. 380-400.

(25) Ross, Sheldon, "Applied Probability Models with Optimization Applications", Holden Day, 1970, p. 138.

(26) Samuelson, Paul A., "Proof That Properly Anticipated Prices Fluctuate Randomly", Industrial Management Review, Vol. 6, Spring 1965, pp. 41-43.

(27) Steiger, William, "Nonrandomness in Stock Price Changes", appeared in (9), pp. 253-261.

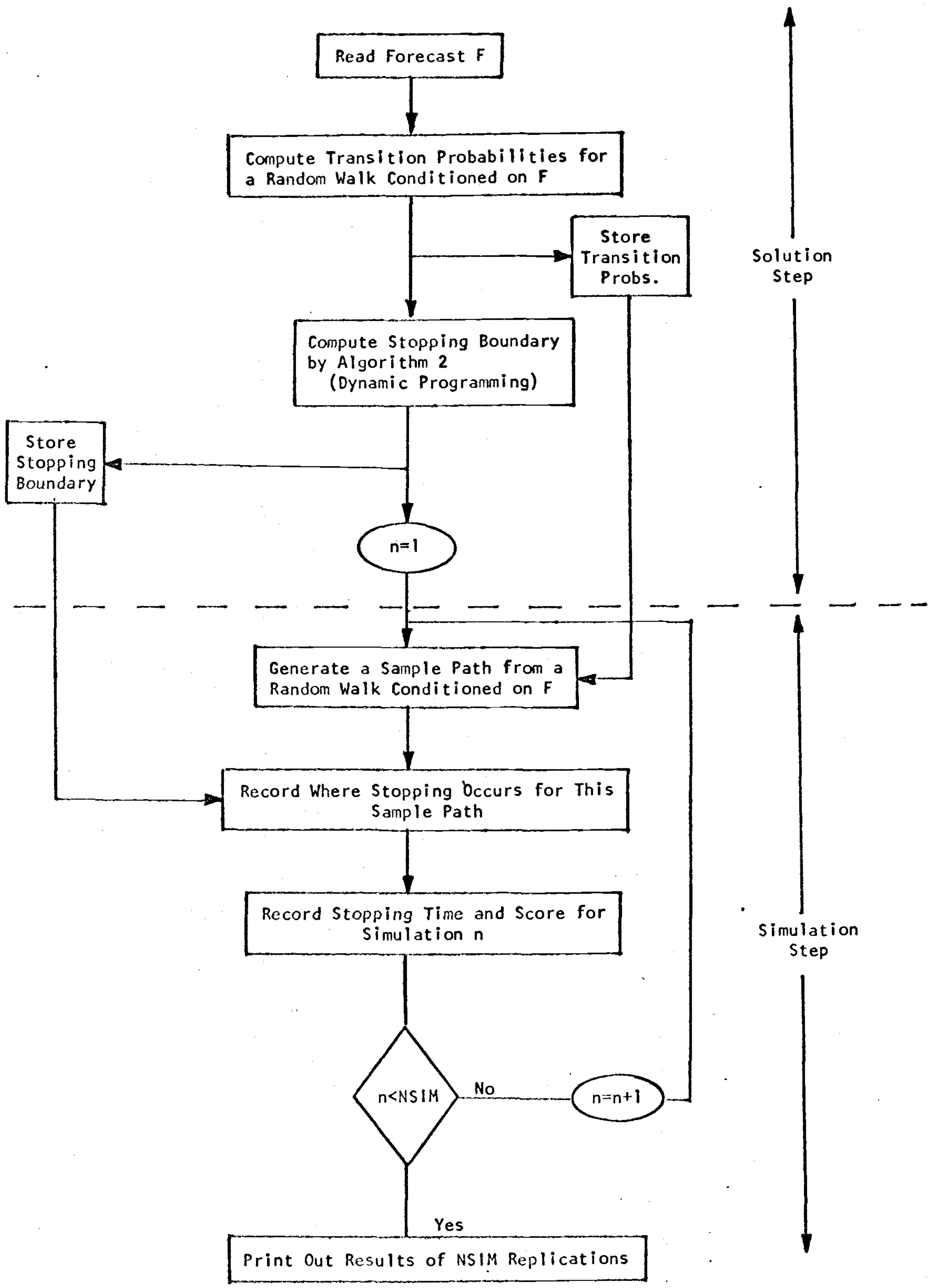
(28) Weingartner, Harvey M., "Optimal Timing of Bond Refunding", Management Science, Vol. 13, No. 7, Mar. 1967, pp. 511-524.

(29) Weston, J. Fred and Brigham, Eugene F., Managerial Finance, 4th Ed. Holt, Rinehart, and Winston, New York, 1972, pp. 441-444.

(30) Working, Holbrock, "A Theory of Anticipating Prices", American Economic Review, Papers and Proceedings, Vol. 48, May, 1958, pp. 188-199.

APPENDIX: COMPUTER PRINTOUTS

FLOW DIAGRAM FOR EMV SIMULATION



```
      INTEGER FCAST(55),HEADS(55),HMIN,HMAX,ACT(2)
      DIMENSION A22(55),QZ(55)
      DIMENSION FPROB(55),H(55,55),P(55,55),V(55),ICOFF(55),SCORE(55),
     *PSUR(55),NHEAD(55),PRICE(55),SIMU(55),AVG(3),VAR(3)
      EQUIVALENCE(H(1,1),P(1,1))
      DATA PSUR/55 * 0.0/
      DATA SIMU/55 * 0.0/
      DATA AVG/3 * 0.0/
      DATA VAR/3 * 0.0/
      DATA ACT,HEADS/'S','C',10*0/
C     INITIALIZATION
      DO 5 I = 1,55
      DO 5 J = 1,55
      H(I,J)=0.
      P(I,J)=0.
      V(I)=0.
      ICOFF(I)=0
      SCORE(I)=0.
      NHEAD(I)=0
      PRICE(I)=0.
    5 CONTINUE
C     READ BASIC INPUT DATA
      READ(2,1) N,I,NSIM,IDELT
    1 FORMAT(4I5)
      READ(2,2) PRICE(1),SIGMA
    2 FORMAT(2F10.3)
C  INTEREST RATES ARE DECIMALS
      READ(2,2) CU,UF
      WRITE(5,3) N,I,NSIM,IDELT,PRICE(1),SIGMA,CU,UF
    3 FORMAT(5X,'INPUT DATA FOLLOWS'//1X,'CARD 1',3X,'N=',I7,3X,'I=',
     1I7,3X,'NSIM=',I7,3X,'IDELT=',I7,3X//1X,'CARD 2',3X,
     2'PRICE(1)=',F10.3,3X,'SIGMA=',F10.3//1X,'CARD 3',3X,'COUPON RATE='
     3,F10.3,3X,'FUNDS RATE=',F10.3)
      I9 = I
      DO 45 K=1,I
      READ(2,15) FCAST(K),FPROB(K)
   15 FORMAT(I4,F5.0)
C FCASTS SHOULD BE ORDERED FROM HIGHEST FIRST TO LOWEST AND SHOULD BE SCALED.
      XF = FCAST(K)
      PSUR(K) = XF * SIGMA + PRICE(1)
      HEADS(K)=(FCAST(K)+N)/2
C     COMPUTE HMAX AND HMIN
      HMIN=HEADS(1)
      HMAX=HEADS(1)
      DO 40 L=1,I
      IF(HEADS(L)-HMAX) 30,25,25
   25 HMAX=HEADS(L)
      GO TO 40
   30 IF(HEADS(L)-HMIN) 35,35,40
   35 HMIN=HEADS(L)
      IF(HMIN) 36,40,40
   36 HMIN=0
   40 CONTINUE
      IHMIN=HMIN
      IHMAX=HMAX
C     COMPUTE COMBINATORIALS
      M=HEADS(K)
      AL=XCOMB(N,M)
C     COMPUTE P(N,J)'S
      IF(AL-1.E5) 1660,1661,1661
 1660 P(N+1,M+1)=(FPROB(K)*1.E3)/ABS(AL)
      GO TO 1559
```

```
 1661 P(N+1,M+1)=(FPROB(K)*1.E15)/ABS(AL)
 1559 CONTINUE
   45 CONTINUE
C     COMPUTE REMAINING P(I,J)'S
      DO 50 I=1,N
      DO 50 J=1,N
      L=N-I+1
      P(L,J)=P(L+1,J+1)+P(L+1,J)
   50 CONTINUE
      I=0
      J=0
      NN=N+1
   55 I=I+1
   56 J=J+1
      P(I,J+1)=0.
      IF(J-NN) 56,60,60
   60 IF(I-NN) 65,70,70
   65 J=I
      GO TO 55
   70 CONTINUE
C     COMPUTE TRANSITIONAL PROBABILITIES
      DO 75 I=1,N
      DO 75 J=1,N
      H(I,J)=P(I+1,J+1)/P(I,J)
   75 CONTINUE
C     WRITE(5,80)
   80 FORMAT('1')
C     WRITE(5,85) N
   85 FORMAT(5X,'N =  ',I3/9X,'J',13X,'A(1)',11X,'A(2)',11X,'VALUE',11X,
     *'ACTION'//)
C     COMPUTE V(N,J)'S
      HMIN=HMIN+1
      HMAX=HMAX+1
      DO 95 KK=HMIN,HMAX
      JJ=HMAX-KK+HMIN
      J=JJ-1
      V(JJ)=2*J-N
C     WRITE(5,90) J,V(JJ),ACT(1)
   90 FORMAT(8X,I2,38X,F9.4,14X,A1)
   95 CONTINUE
C     ITERATE FROM N-1 TO 0
      DO 155 ITER=1,N
      DO 1130 IQ=1,N
 1130 QZ(IQ)=V(IQ)
      IC=0
      NN=N-ITER+1
      NNN=NN-1
C     WRITE(5,85) NNN
      HMIN=HMIN-1
      IF(HMIN-1) 100,105,105
  100 HMIN=1
  105 IF(NN-HMAX) 110,115,115
  110 HMAX=NN
  115 DO 130 KK=HMIN,HMAX
      JJ=HMAX-KK+HMIN
      J=JJ-1
      A1=2*J-NNN
      AT  =1. + QZ(JJ+1)
      BT= QZ(JJ) - 1.
      CT = 1. - H(NN,JJ)
      DT = H(NN,JJ)
C CALCULATE CONTINUATION COST,C
C  CU IS COUPON RATE(DECIMAL), UF IS USE OF FUNDS RATE(DECIMAL
      PP=SIGMA*A1 + PRICE(1)
      PP=(CU*100. - UF*PP)/ 52.
      C = PP/SIGMA
```

```
      A2 = C + (AT * DT) + (BT * CT)
      IF(A1-A2) 120,120,125
  120 V(JJ)=A1
      IF(IC) 1111,2222,1111
 2222 ICOFF(NN)=J
      IC=1
 1111 CONTINUE
C     WRITE(5,121) J,A1,A2,V(JJ),ACT(1)
  121 FORMAT(8X,I2,5X,F12.4,2(3X,F12.4),14X,A1)
      GO TO 130
  125 V(JJ)=A2
C     WRITE(5,121) J,A1,A2,V(JJ),ACT(2)
      IF(IC) 130,988,130
  988 ICOFF(NN)=-99
      A22(NN)=A2
  130 CONTINUE
C     WRITE(5,151) ICOFF(NN)
  151 FORMAT(9X,'STOPPING VALUE =',I3/)
  155 CONTINUE
C     WRITE(5,160)
  160 FORMAT('1')
      TADV=0.
C     SIMULATION ROUTINE
      ASQ = 0.
      BSQ = 0.
      XMU = 0.
      XSQ = 0.
      IX4=11
      XND=N
      X3 = NSIM
      DO 215 ISIM =1,NSIM
C     WRITE(5,165) ISIM
  165 FORMAT(/3X,'SIMULATION NUMBER ',I5/)
C     GENERATE SAMPLE PATH
      NEND=N+1
      K=0
      SCORE(1)=0.
C     WRITE(5,176) K,SCORE(1),PRICE(1)
      DO 185 L=1,N
      CALL RANDU(IX4,IY,Y)
      IX4=IY
      NH=NHEAD(L)+1
C     WRITE(5,676) L,NH,Y,H(L,NH)
  676 FORMAT(5X,I4,I6,2F10.6)
      IF(Y-H(L,NH)) 180,180,175
  175 ICHK=N-L+NHEAD(L)
C     WRITE(5,813) ICHK,IHMIN,IHMAX,NHEAD(L)
  813 FORMAT(5X,4I10)
      IF(ICHK-IHMIN) 180,177,177
  177 SCORE(L+1)=2*NHEAD(L)-L
      PRICE(L+1)=SIGMA*SCORE(L+1)+PRICE(1)
C     WRITE(5,176) L,SCORE(L+1),PRICE(L+1)
  176 FORMAT(3X,'TIME PERIOD ',I6,6X,'SCORE =',F10.4,6X,'PRICE =',F10.4)
      GO TO 184
  180 NHEAD(L)=NHEAD(L)+1
      IF(NHEAD(L)-IHMAX) 181,181,175
  181 SCORE(L+1)=2*NHEAD(L)-L
      PRICE(L+1)=SIGMA*SCORE(L+1)+PRICE(1)
C     WRITE(5,176) L,SCORE(L+1),PRICE(L+1)
  184 NHEAD(L+1)=NHEAD(L)
      SCORE(L+2)=SCORE(L+1)
  185 CONTINUE
C     FIND ALGORITHM STOPPING POINT
      DO 195 I=1,N
      IF(ICOFF(1))1190,190,1190
 1190 IF(ICOFF(I)+99) 1191,195,1191
```

```
1191 IF(NHEAD(I)-ICOFF(I)) 190,190,195
 190 COST=PRICE(I)
     II=I-1
C STOPPING TIME IS GIVEN BY II.  TAB FOR MEAN AND VARIANCE
     XI=FLOAT(II)
     XMU = XMU + XI
     XSQ = XSQ + XI**2.
C    WRITE(5,191) II,SCORE(I),PRICE(I)
 191 FORMAT(3X,'STOP IN PERIOD',I4,3X,'SCORE =',F10.5,3X,'PRICE =',F10.
    *4)
     GO TO 200
 195 CONTINUE
     XMU=XMU+XND
     XSQ=XSQ+XND**2.
C    WRITE(5,191) N,SCORE(NEND),PRICE(NEND)
     COST=PRICE(NEND)
C    COMPUTE DOLLAR AVERAGING COST
 200 DACST=0.
     ADV=0.
     J=0
     TCNT=0.
     PR=0.
     DO 777 K5 = 1,I9
     ACAS = FCAST(K5)
C    WRITE(5,9000) NEND,SCORE(NEND),ACAS
9000 FORMAT(I5,2F10.4)
     ABC = SCORE(NEND) - ACAS
     IF(ABS(ABC) - .001) 775,775,777
 775 SIMU(K5) = SIMU(K5) + 1./X3
 777 CONTINUE
     DO 205 L=1,NEND,IDELT
     J=J+1
     TCNT=TCNT+FLOAT(IDELT  )*(UF*PRICE(L) -CU*100.)/52.
     PR=PR+PRICE(L)
 205 CONTINUE
     DACST=(PR+TCNT)/FLOAT(J)
C    COMPUTE ALGORITHM ADVANTAGE
     AVG(1) = AVG(1) + 1./X3 * DACST
     AVG(2) = AVG(2) + 1. / X3 * COST
     ASQ = ASQ + DACST**2.
     BSQ = BSQ + COST**2.
     ADV=DACST-COST
     TADV=TADV+ADV
     AVGG = TADV/ISIM
C    WRITE(5,210)       PRICE(I),DACST,ADV,     AVGG
 210 FORMAT(                              /3X,'EMV BOND PRICE ='  ,F7.
    *3/3X,'DOLLAR AVERAGING UNIT PRICE =',F7.3/3X,'ALGORITHM ADVANTAGE
    *=',F7.3/3X,'AVERAGE ADVANTAGE=',F7.3)
     DO 212 I=2,NEND
     PRICE(I)=0.
     SCORE(I)=0.
     NHEAD(I-1)=0
 212 CONTINUE
 215 CONTINUE
C CALCULATE STOPPING TIME STATISTICS AVG(3) = MEAN, VAR(3) = VARIANCE
     X3 = NSIM
     AVG(3) = XMU/X3
     VAR(3) = (XSQ - X3 * AVG(3)**2.)/(X3-1.)
     VAR(1) = (ASQ - X3 * AVG(1)**2.)/(X3-1.)
     VAR(2) = (BSQ - X3 * AVG(2)**2.) /(X3-1.)
     DO 570 J = 1,3
 570 VAR(J) = SQRT(VAR(J))
     WRITE(5,500)
 500 FORMAT('1' 13X,'S T O P P I N G   B O U N D A R Y' 12X,'I' 10X,
    1 'S I M U L A T I O N   S T A T I S T I C S'/20X,
    2 '( S O L U T I O N)' 21X,'I')
```

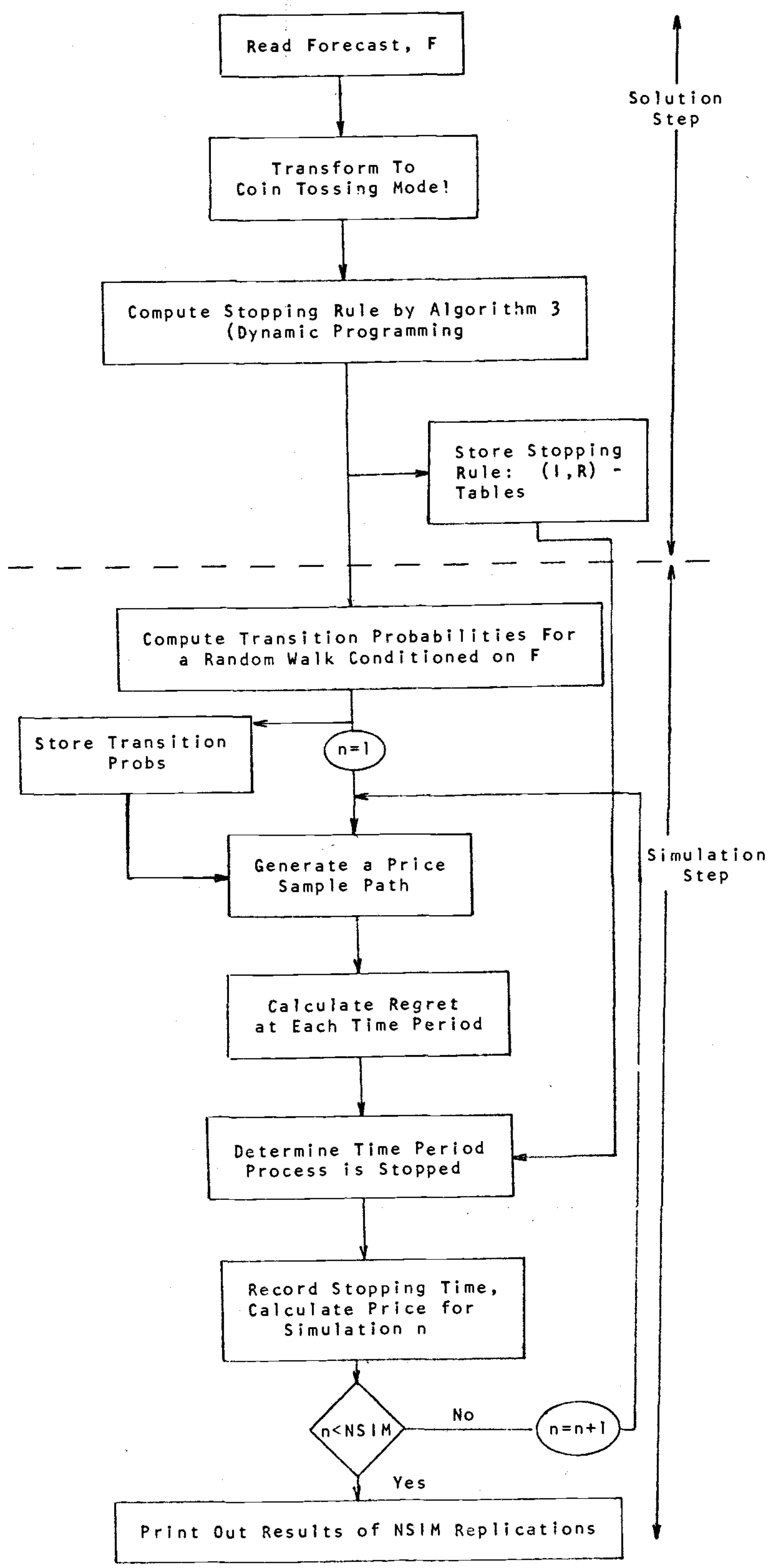
```
      WRITE(5,501)
  501 FORMAT(59X,'I' 9X,'PRICE AT' 15X,'PROBABILITY PRICE'/10X,'TIME',
     111X,'SCORE'12X,'ABSOLUTE'9X,'I',8X,'SURRENDER'13X,22('-')/
     19X,'PERIOD'5X,'(SCALED PRICE)'10X,'PRICE'10X,'I',12X,'DATE'15X,
     3'SIMULATED'7X,'INPUT'/59X'I')
      NND = N
      DO 400 I = 1,NND
      NI = I - 1
      SC = 2 * ICOFF(I) - NI
      IF(ICOFF(I)+99) 398,399,398
  399 SC=A22(I)
  398 AP = SC * SIGMA + PRICE(1)
      SC = SC + (SC/ABS(SC)) * .005
      AP = AP + (AP / ABS(AP)) * .005
      PSUR(I) = PSUR(I) + (PSUR(I) / ABS(PSUR(I))) * .005
      SIMU(I) = SIMU(I) +(SIMU(I) / ABS(SIMU(I))) * .0005
      FPROB(I) = FPROB(I) + (FPROB(I) / ABS(FPROB(I))) * .0005
      IF(I - I9) 405,405,410
  405 WRITE(5,510) NI,SC,AP,PSUR(I),SIMU(I),FPROB(I)
      GO TO 400
  410 WRITE(5,550) NI,SC,AP
  550 FORMAT(10X,I3,8X,F10.2,9X,F10.2,9X,'I')
  400 CONTINUE
  510 FORMAT(10X,I3,8X,F10.2,9X,F10.2,9X,'I',6X,F10.2,13X,F10.3,3X,
     1 F10.3)
      WRITE(5,520)
  520 FORMAT('1' 12X,'S I M U L A T I O N   R E S U L T S'/)
      WRITE(5,525) NSIM
  525 FORMAT('0NUMBER OF SIMULATED'/'  PRICE SAMPLE PATHS = 'I5//)
      WRITE(5,530) SIGMA,C,PRICE(1),IDELT
  530 FORMAT('0ASSUMPTIONS ---'/4X,'STEP SIZE'13X,'=',F10.3/
     1 4X,'CONTINUATION COST,C(J)='F10.3/4X,'INITIAL PRICE'9X,'=' F10.3/
     24X,'DOLLAR AVERAGING'/6X,'PURCHASE CYCLE'6X,'=',2X,I5///)
      WRITE(5,535) (AVG(K),VAR(K),K=1,3)
  535 FORMAT('0BOND PURCHASE PRICE --'9X,'MEAN'7X,'STD.DEVIATION'/
     1'0- UNDER DOLLAR AVERAGING'F11.2,5X,F10.2/
     2'0- UNDER EMV STOPPING RULE'F10.2,5X,F10.2/
     3'0STOPPING TIME STATISTICS 'F10.2,5X,F10.2)

      CALL EXIT
      END
```

```
      SUBROUTINE RANDU(IX,IY,YFL)
C******   RANDOM NUMBER GENERATOR FOR IBM 1130    ******
      IY=IX*899
      IF(IY) 5,6,6
    5 IY=IY+32767+1
    6 YFL=IY
      YFL=YFL/32767.
      RETURN
      END
```

```
      FUNCTION XCOMB(N,M)
C     SUBROUTINE TO COMPUTE THE NUMBER OF WAYS TO SELECT N
C     OBJECTS M AT A TIME
      XCOMB=1.
      IF(2*M-N) 20,20,10
   10 K=N-M
      GO TO 30
   20 K=M
   30 DO 40 I=1,K
      XI=FLOAT(I)-1.
      XCOMB=(FLOAT(N)-XI)*XCOMB/(XI+1.)
   40 CONTINUE
      RETURN
      END
```

FLOW DIAGRAM FOR REGRET SIMULATION



```
      INTEGER ACT(32)
      INTEGER HMIN,HMAX, RMAX,R,HEADS(55),FCAST(55) ,RS,RC,RF,R7
      DIMENSION RR(55) ,SOLN(55,55)
      DIMENSION AVG(3),VAR(3)
      DIMENSION P(55,55),H( 55,55),SCORE(55),NHEAD(55)
      DIMENSION FPROB(55),PSUR(55),PRICE(55)     ,SIMU(55)
      DIMENSION S(55,55),CC(55,55),F(2,55,2000 ), IPNT(11)
      EQUIVALENCE (S(1,1),P(1,1)),(CC(1,1),H(1,1))
      DATA ACT/'0','1','2','3','4','5','6','7','8','9','10','11','12',
     1'13','14','15','16','17','18','19','20','21','22','23',
     2'24','25','26','27','28','29','30',' '/
      DATA IPNT/0,1,2,3,4,5,6,7,8,9,10/
C  ZERO OUT ARRAYS
      DO 3210 NTIM=1,50
      READ(2,2) JTT
      IF(JTT) 3215,3211,3215
 3211 CONTINUE
      LMAX=55
      DO 700 L=1,3
      DO 700 I=1,2
      DO 700 J=1,LMAX
      DO 700 K=1,LMAX
      AVG(L)=0.
      VAR(L)=0.
      SOLN(J,K)=0.
      NHEAD(K)=0
      RR(K)=0.
      SIMU(K)=0.
      P(J,K)=0.
      H(J,K)=0.
      PRICE(K)=0.
      F(I,J,K)=0.
      S(J,K)=0.
      CC(J,K)=0.
      FPROB(K)=0.
      SCORE(K)=0.
      PSUR(K)=0.
      HEADS(K)=0
      FCAST(K)=0
  700 CONTINUE
C  READ INPUT DATA
      READ(2,1) NN,II,NSIM,IDELT
    1 FORMAT(4I5)
      READ(2,2) PRICE(1),SIGMA
    2 FORMAT(2F10.3)
      READ(2,2) CU,UF
      IQ=ACT(1)
      Z=IDELT
C  READ FORECAST DATA
      DO 45 K=1,II
      READ(2,15) FCAST(K),FPROB(K)
   15 FORMAT(I4,F5.3)
C  FIND MAXIMUM NBR HEADS AND TAILS
      XF= FCAST(K)
      PSUR(K)=XF*SIGMA+PRICE(1)
      HEADS(K)=(FCAST(K)+NN)/2
      HMIN=HEADS(1)
      HMAX=HEADS(1)
      DO 40 L=1,II
      IF(HEADS(L)-HMAX) 30,25,25
   25 HMAX=HEADS(L)
```

```
      GO TO 40
   30 IF(HEADS(L)-HMIN) 35,35,40
   35 HMIN=HEADS(L)
      IF(HMIN) 36,40,40
   36 HMIN=0
   40 CONTINUE
   45 CONTINUE
      XN=NN
      FI=NN-HMIN
      PC=SIGMA*(XN-2.*FI) + PRICE(1)
      PP=(CU*100.-UF*PC)/52.
      CI=PP/SIGMA
      IF(ABS(CI)-.0001) 91,91,90
C  NEGLECT CONTINUATION COST
   91 STP=1
      CI=0.
      RMAX=HMAX+1
      GO TO 391
C NORMAT SITUATION CI NOT ZERO
C  RMAX=HMAX)(1)+NN.(CI/SIGMA). HOVEVER,RMAX MUST BE INTEGER SO DIVIDE
C   BY (CI/SIGMA)=CI
   90 STP=1./CI + .005
      RMAX = HMAX*STP+NN
C 391 WRITE(5,3) RMAX, STP
    3 FORMAT(5X,'RMAX=',I9,'CI/SIGMA=',I5)
  391 CONTINUE
      I9=II
      NMAX=NN+1
      IMAX=NN-HMIN+1
      IMIN=NN-HMAX+1
      RMAX=HMAX+1
C  CHECK IF ARRAYS ARE EXCEEDED
      IF(NMAX-LMAX)400,400,4000
  400 IF(IMAX-LMAX)401,401,4000
  401 IF(RMAX-LMAX) 402,402,4000
 4000 WRITE(5,405) LMAX,NMAX,IMAX,RMAX
  405 FORMAT(1X,'ERROR,ARRAY TOO SMALL',4I4)
      CALL EXIT
  402 CONTINUE
C$$$$$$$  STAGE N  *******
      DO 100 I=IMIN,IMAX
      I2=I-1
      DO 100 IR=1,RMAX
      R=IR-1
      ICT=I2+R
      IF(ICT-NN) 1000,1000,100
 1000 SOLN(NMAX,I)=0.
C1000 SOLN(NMAX,I)=ACT(1)
      F(1,I,IR)=R
  100 CONTINUE
C     WRITE(5,167) NN
C     WRITE(5,1101) (IPNT(M),M=1,IMAX)
 1101 FORMAT(23X,'I=',11I5)
C     WRITE(5,1102) (SOLN(NMAX,M),M=   1,IMAX)
 1102 FORMAT(1X,'SOLUTION=',15X,11F5.1)
C1102 FORMAT(1X,'SOLUTION=',15X,11A5)
C     WRITE(5,1103) NN
 1103 FORMAT(1X,'F(I,R) TABLE FOR STAGE',I4)
      DO 103 IR=1,RMAX
      R=IR-1
C 103 WRITE(5,1104) R, (F(1,I,IR),I=   1,IMAX)
  103 CONTINUE
C  ******      END OF STAGE N  *******
C
C *****    MAIN LOOP FOR STAGES N-1 TO 0  *****
      N=NN
```

```
      DO 222 ITER=1,N
      N1=N-ITER
      NK=N1+1
      XN=FLOAT(N1)
      IMIN=N1-HMAX+1
      IF(IMIN) 3000,3000,300
 3000 IMIN=1
C  ******    I LOOP   ******
  300 DO 212 II=IMIN,IMAX
      I2=II-1
      FI=FLOAT(I2)
C  CONTINUATION COST AT STATE(I,R)
      PC=SIGMA*(XN-2.*FI) + PRICE(1)
      PP=.(CU*100.-UF*PC)/52.
      CI=PP/SIGMA
      IF(ABS(CI)-.0001) 891,891,890
  891 STP=1
      CI=0.
      GO TO 896
  890 STP=1./CI  + .0005
C*****   R LOOP   ******
      CI=0.
      ICUT=0
      DO 200 IR=1,RMAX
      R=IR-1
C  MAXIMUM REGRET IF STOPPED
C  I+R CANNOT EXCEED N
      ICT=I2+R
      IF(ICT-N1) 2000,2000,200
 2000 D1=FLOAT(NN-HMIN-I2)*(1.-CI)
      D2=FLOAT(ITER)
      IF(D1-D2) 911,911,912
  911 D=D1
      GO TO 33
  912 D=D2
   33 IF(R-D) 4,55,55
    4 RNS=D
      GO TO 6
   55 RNS=R
    6 CONTINUE
      S(II,IR)=RNS
      IA=NK-HMAX
C  MAXIMUM REGRET IF CONTINUED
      IF(CI) 893,892,893
  892 IR1=IR-1
      IR2=IR+1
      GO TO 999
  893 IR1=IR-STP+1
      IR2=IR+STP+1
  999 IF(IR1) 7,7,8
    7 IF(II-IMAX) 59,9,9
   59 A = F(1,II+1,1)
C    B IS NOT DEFINED FOR II .LE. IA
      IF(II-IA) 11,11,9
    8 IF(II-IMAX) 218,9,9
  218 IF(IR1-RMAX) 88,200,200
   88 IF(IXX-NMAX) 488,488,  9
  488 A=F(1,II+1,IR1+1)
      IF(II-IA) 11,11,9
    9 IF(IR2-RMAX)89,89,81
   81 IF(II-IMAX) 11,13,13
   89 B=F(1,II,IR2)
      IF(II-IMAX) 69,10,10
   69 IF(A-B) 10,10,11
   10 RNC=B
      GO TO 12
```

```
   11 RNC=A
   12 CC(II,IR)= RNC
C  MINIMAX DECISION
      IF(RNS-RNC) 13,14,14
   13 F(2,II,IR)=RNS
      IF(ICUT) 16,152,16
C 152 SOLN(NK,II)=ACT(IR)
  152 SOLN(NK,II)=R
      ICUT=1
      GO TO 16
   14 F(2,II,IR)=RNC
   16 CONTINUE
  200 CONTINUE
C  ICUT=0 IMPLIES CONTINUE FOR ALL R AT THIS STAGE
      IF(ICUT) 420,420,212
C 420 SOLN(NK,II)=ACT(32)
  420 SOLN(NK,II)=NK

C******       END OF R LOOP******
  212 CONTINUE
C *******     END OF I LOOP   *******
C  WRITE OUT STOP AND CONTINUE TABLES FOR STAGE N1
C     WRITE(5,1105) N1
 1105 FORMAT('1','STOP TABLE FOR STAGE',I4/)
C     WRITE(5,1101) (IPNT(M),M=1,IMAX)
      DO 1106 RS=1,RMAX
      R=RS-1
C1106 WRITE(5,1104) R, (S(IS,RS),IS=   1,IMAX)
 1106 CONTINUE
 1104 FORMAT(10X,'R=',I3,10X,11F5.1/)
C     WRITE(5,1107) N1
 1107 FORMAT(///1X,'CONTINUE TABLE FOR STAGE',I4/)
C     WRITE(5,1101) (IPNT(M),M=1,IMAX)
      DO 1108 RC=1,RMAX
      R=RC-1
C1108 WRITE(5,1104) R,(CC(IC,RC),IC=   1,IMAX)
 1108 CONTINUE
C  WRITE OUT F(I,R) TABLE FOR STAGE N1
  167 FORMAT('1'1X,'STAGE=',I4)
C     WRITE(5,167) N1
C     WRITE(5,1101)(IPNT(M),M=1,IMAX)
C     WRITE(5,1102) (SOLN(NK,M),M=   1,IMAX)
C     WRITE(5,1103) N1
      DO 1110 RF=1,RMAX
      R=RF-1
C1110 WRITE(5,1104) R,(F(2,M,RF),M=   1,IMAX)
 1110 CONTINUE
C END OF OUTPUT
C COPY F(N) TO F(N+1), THAT IS F(2) BECOMES F(1)
      DO 1111 I7=IMIN,IMAX
      DO 1111 R7=1,RMAX
 1111 F(1,I7,R7)=F(2,I7,R7)
  222 CONTINUE
C  ******       END OF MAIN N LOOP    *******
C  PRINT RESULTS
C     WRITE(5,1112)(IPNT(M),M=1,IMAX)
 1112 FORMAT('1'25X,'S T O P P I N G   R U L E   F O R   R E G R E T   C
     1 R I T E R I O N    '//23X,'I=',11I5//)
      DO 1114 NK=1,NMAX
      N1=NK-1
 1114 CONTINUE
C1114 WRITE(5,1113) N1,(SOLN(NK,I),I=   1,IMAX)
C1113 FORMAT(10X,'N=',I3,10X,11A5/)
 1113 FORMAT(10X,'N=',I3,10X,11F5.1/)
      IHMAX=HMAX
      IHMIN=HMIN
```

```
      DO 6345 K=1,19
C     COMPUTE COMBINATORIALS
      M=HEADS(K)
      IF(M) 4141,6345,4141
 4141 AL=XCOMB(N,M)
      IF(AL) 1663,1666,1663
 1666 WRITE(5,1664) K,N,M,AL
 1664 FORMAT(5X,'COMBINATORIAL ERROR',3I5,5X,F10.1)
      CALL EXIT
 1663 P(N+1,M+1)=FPROB(K)/ABS(AL)
 6345 CONTINUE
C     COMPUTE REMAINING P(I,J)'S
      DO 50 I=1,N
      DO 50 J=1,N
      L=N-I+1
      P(L,J)=P(L+1,J+1)+P(L+1,J)
   50 CONTINUE
      I=0
      J=0
      NN=N+1
 5588 I=I+1
   56 J=J+1
      P(I,J+1)=0.
      IF(J-NN) 56,60,60
   60 IF(I-NN) 65,70,70
   65 J=I
      GO TO 5588
   70 CONTINUE
C     COMPUTE TRANSITIONAL PROBABILITIES
      DO 75 I=1,N
      DO 75 J=1,N
      IF(P(I,J)) 75,75,7759
 7759 H(I,J)=P(I+1,J+1)/P(I,J)
   75 CONTINUE
C     WRITE(5,80)
   80 FORMAT('1')
C     SIMULATION ROUTINE
      N=NN -1
      IYY=11
      XND=N
      X3 = NSIM
      SASQ=0.
      SA=0.
      CJ=0.
      DAP=0.
      VAP=0.
      SUMC=0.
      SUM=0.
      XSQ=0.
      XCSQ=0.
      DO 215 ISIM =1,NSIM
C     WRITE(5,165) ISIM
  165 FORMAT(/3X,'SIMULATION NUMBER ',I5/)
C     GENERATE SAMPLE PATH
      NEND=N+1
      K=0
      SCORE(1)=0.
C     WRITE(5,176) K,SCORE(1),PRICE(1)
      DO 185 L=1,N
      CALL RANDU(IYY,IYX,Y)
      IYY=IYX
      NH=NHEAD(L)+1
      IF(Y-H(L,NH)) 180,180,175
  175 ICHK=N-L+NHEAD(L)
      IF(ICHK-IHMIN) 180,177,177
  177 SCORE(L+1)=2*NHEAD(L)-L
      [illegible]
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```
      PRICE(L+1)=SIGMA*SCORE(L+1)+PRICE(1)
C     WRITE(5,176) L,SCORE(L+1),PRICE(L+1)
  176 FORMAT(3X,'TIME PERIOD ',I6,6X,'SCORE =',F10.4,6X,'PRICE =',F10.4)
      GO TO 184
  180 NHEAD(L)=NHEAD(L)+1
      IF(NHEAD(L)-IHMAX) 181,181,175
  181 SCORE(L+1)=2*NHEAD(L)-L
      PRICE(L+1)=SIGMA*SCORE(L+1)+PRICE(1)
C     WRITE(5,176) L,SCORE(L+1),PRICE(L+1)
  184 NHEAD(L+1)=NHEAD(L)
      SCORE(L+2)=SCORE(L+1)
  185 CONTINUE
      RR(1)=0.
      ZMIN=0.
      DO 195 I=2,NEND
      IF(SCORE(I)-ZMIN) 710,710,711
  710 ZMIN=SCORE(I)
      RR(I)=0.
      GO TO 195
  711 RR(I)=SCORE(I)-ZMIN
  195 CONTINUE
      DO 720 K9=2,NEND
      KN=K9-1
      I=KN-NHEAD(KN)
      I5=I+1
C  FIND STOPPING POINT
      IF(SOLN(K9,I5)-KN) 723,723,720
  723 IF(RR(KN+1)-SOLN(K9,I5))720,726,726
  726 XKN=KN
      SUM=SUM+XKN
      XSQ=XSQ+XKN**2.
C  STOPPING TIME CALCULATED ABOVE
      COST=PRICE(K9)
      SUMC=SUMC+COST
      XCSQ=XCSQ+COST**2.
C     WRITE(5,9000) KN,COST
 9000 FORMAT(5X,'STOPPED PERIOD',I5,5X,'COST=',F10.2)
      GO TO 7747
  720 CONTINUE
C  COMPUTE DOLLAR AVERAGING COST
 7747 CJ=0.
      SASQ=0.
      SA=0.
      DO 747 I=1,NEND,IDELT
      CJ=CJ+1.
      SP=SCORE(I)*SIGMA+PRICE(1)
      SA=SA+SP
  747 SASQ=SASQ+SP**2.
      AVG(1)=SA/CJ
      VAR(1)=(SASQ-CJ*AVG(1)**2.)/CJ
C     WRITE(5,9050) AVG(1),VAR(1)
 9050 FORMAT(5X,'D.A. COST=',F10.2,'VAR=',F10.2)
      DAP=DAP+AVG(1)*1./X3
      VAP=VAP+VAR(1)*1./X3
      DO 777 K5 = 1,19
      ACAS = FCAST(K5)
      ABC = SCORE(NEND) - ACAS
      IF(ABS(ABC) - .001) 775,775,777
  775 SIMU(K5) = SIMU(K5) + 1./X3
  777 CONTINUE
      DO 3212 I=2,NEND
      PRICE(I)=0.
      SCORE(I)=0.
      NHEAD(I-1)=0
 3212 CONTINUE
  215 CONTINUE
C  AVERAGES AND VARIANCES
```

```
C    AVERAGES AND VARIANCES
      AVG(1)=DAP
      VAR(1)=SQRT(VAP)
      AVG(2)=SUMC/FLOAT(NSIM)
      AVG(3)=SUM/FLOAT(NSIM)
      VAR(2)=(XCSQ-X3*AVG(2)**2.)/X3
      VAR(3)=(XSQ-X3*AVG(3)**2.)/X3
      VAR(2)=SQRT(VAR(2))
      VAR(3)=SQRT(VAR(3))
      C=(CU*100.-UF*AVG(1))/52.
      WRITE(5,500)
  500 FORMAT('1' 13X,'S T O P P I N G   B O U N D A R Y' 12X,'I' 10X,
     1 'S I M U L A T I O N   S T A T I S T I C S'/20X,
     2 '( S O L U T I O N)' 21X,'I')
      WRITE(5,501)
  501 FORMAT(59X,'I' 9X,'PRICE AT' 15X,'PROBABILITY PRICE'/10X,'TIME',
     111X,'SCORE'12X,'ABSOLUTE'9X,'I',8X,'SURRENDER'13X,22('-')/
     19X,'PERIOD'5X,'(SCALED PRICE)'10X,'PRICE'10X,'I',12X,'DATE'15X,
     3'SIMULATED'7X,'INPUT'/59X'I')
      DO 4400 I=1,I9
      WRITE(5,510)  PSUR(I),SIMU(I),FPROB(I)
 4400 CONTINUE
  510 FORMAT(66X,F10.2,13X,F10.3,3X,
     1 F10.3)
      WRITE(5,520)
  520 FORMAT(     12X,'S I M U L A T I O N   R E S U L T S'/)
      WRITE(5,525) NSIM
  525 FORMAT('ONUMBER OF SIMULATED'/'  PRICE SAMPLE PATHS = 'I5//)
      WRITE(5,530) SIGMA,C,PRICE(1),IDELT
  530 FORMAT('OASSUMPTIONS ---'/4X,'STEP SIZE'13X,'=',F10.3/
     1 4X,'CONTINUATION COST,C(J)='F10.3/4X,'INITIAL PRICE'9X,'=' F10.3/
     24X,'DOLLAR AVERAGING'/6X,'PURCHASE CYCLE'6X,'=',2X,I5///)
      WRITE(5,535) (AVG(K),VAR(K),K=1,3)
  535 FORMAT('OBOND PURCHASE PRICE --'9X,'MEAN'7X,'STD.DEVIATION'/
     1'0- UNDER DOLLAR AVERAGING'F11.2,5X,F10.2/
     2'0- UNDER REGRET CRITERION 'F10.2,5X,F10.2/
     3'OSTOPPING TIME STATISTICS 'F10.2,5X,F10.2)
 3210 CONTINUE
 3215 CALL EXIT
      END
```