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Title: Optimization, Conservation and Valuation of Contingent Claims in Economic Resource Management under Uncertainty.

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Robert T. Smythe

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This dissertation consists of three papers studying optimization, conservation and valuation of contingent claims in economic resource management under uncertainty.

In the first paper the Markovian optimal policies are studied for resource management in a finite time horizon. Under some conditions, in particular, when the prices are stochastic and there is a positive fixed setup cost K, the existence of $\{S, s\}$-type Markovian optimal management policies is proved. When $K = 0$, the optimal policies are of $\{S\}$-type, in which case a comparison is made between the optimal policies under stochastic and deterministic prices. It turns out that under stochastic prices the optimal policies should be more conservative in order to maximize the present value of expected revenue.

The second paper investigates the issue of conservation under uncertainty of future benefits for both non-renewable and renewable resources. A framework is

introduced to demonstrate that under uncertainty of future benefits the optimal management policy will be more conservative than the optimal policy obtained when the uncertainty is ignored. This paper extends early results by Arrow and Fisher (1974) for the case of irreversible management policies for non-renewable resources. In particular it is shown that resource conservation can be a consequence of profit maximization or timing of investments when future benefits are uncertain and the process of developing or harvesting is irreversible.

In the third paper we discuss the valuation of contingent claims on economic resources. Contingent claims on certain economic resources may involve underlying assets with conversion costs or subsidies, in which case the price of underlying risky assets may be modeled by a geometric Brownian motion plus a deterministic conversion cost or subsidy. Under the circumstances, we derive the equation for valuing contingent claims on assets with conversion costs or subsidies and show that a unique arbitrage-free hedging strategy exists.

Optimization, Conservation and Valuation of Contingent Claims in Economic Resource Management under Uncertainty

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OPTIMIZATION, CONSERVATION AND VALUATION OF CONTINGENT CLAIMS IN ECONOMIC RESOURCE MANAGEMENT UNDER UNCERTAINTY

CHAPTER ONE

INTRODUCTION

This dissertation is on profit maximization under uncertainty and related methods and issues. The dissertation consists of three papers. The first paper studies optimal policies for resource management subject to market uncertainty. Under some conditions, the optimal management policies under market uncertainty are compared with the optimal policies for a deterministic market. It turns out that under price uncertainty the optimal policies should be more conservative in order to maximize the present value of expected revenue. The second paper introduces a general framework, in which the issue of conservation is investigated under uncertainty of future benefits for both non-renewable and renewable resources. It is demonstrated that under uncertainty of future benefits the optimal management policy will be more conservative than the optimal policy obtained when the uncertainty is ignored. This paper extends early results by Arrow and Fisher (1974) for the case of irreversible management policies for non-renewable resources. In particular it is shown that resource conservation can be a consequence of profit maximization or timing of investments when future benefits are uncertain and the process of developing or harvesting is irreversible. The third paper considers economic resources as risky assets in the market subject to uncertainty. Such risky assets may involve deterministic conversion costs or subsidies. In the paper the valuation problem is studied for contingent claims on assets with conversion costs or subsidies. A replicating portfolio that contains consumption or money infusion is used to determine the arbitrage-free hedging strategy.

The three studies have cost and benefit structures ranging from specific to general. Over a finite time horizon, the quantity of an economic resource, which is a risky asset in an uncertain market, is described by

$$x_i = Z_i f(x_{i-1}),$$

where x_i denotes the resource quantity at the i-th period, the Z_i's are independent with mean one and $Z_i > 0$ almost surely for all $i = 1, 2, \ldots, N$, and f is known as the recruitment/reproduction function. In this first-order growth equation, the expected resource population is determined by the recruitment/reproduction function f and the population standard deviation is determined by the distribution of Z_i. When $Z_i = 1$ almost surely for all i the growth is deterministic. The resource has no growth if $x_i = x_{i-1}$ for all i. When h_{i-1} units of the resource are harvested at the corresponding period,

$$x_i = Z_i f(x_{i-1} - h_{i-1}),$$

and a gross profit $h_{i-1}A_{i-1}$ is made, where A_{i-1} is the random market price of the resource at that time period. In the case that the resource as a risky asset involves a conversion cost or subsidy, the gross profit before subtracting harvesting costs is $h_{i-1}(A_{i-1} + c)$, where c represents a conversion cost for $c < 0$ or a subsidy for $c > 0$. While the harvested resource can be sold/bought directly in the market, there are many forms of contingent claim contracts on the resource, such as futures and options. These contingent claims bring about valuation problems.

The first paper deals with a rather specific cost and benefit structure. Under some conditions, in particular, when the prices are stochastic and there is a positive fixed setup cost K for each harvesting, the existence of $\{S, s\}$-type Markovian optimal management policies is proved in a finite time horizon. When $K = 0$, the optimal policies are of $\{S\}$-type, in which case a comparison is made between the optimal policies under stochastic and deterministic prices. It is shown that the optimal policies are more conservative under stochastic prices.

In the second paper there is a general framework of costs and benefits. The future is compressed into the second period in a two-period time horizon. In this framework, uncertainty in future benefits due to random fluctuations of market and/or resource inventory is considered. For renewable and non-renewable resources the relationship between resource conservation and expected profit maximization is illustrated. It is demonstrated that the goal of profit maximization leads to resource conservation. From the perspective of the timing of the investment, resource conservation is also implied by its positive intrinsic value, even though the future profits could be zero.

The third paper studies contingent claims on economic resources as underlying risky assets. In the case that there are deterministic conversion costs or subsidies involved, we derive the equation that determines the value of contingent claims. It is shown that a unique non-arbitrage hedging strategy can be obtained through a replicating portfolio containing a certain amount of consumption or money infusion.

CHAPTER TWO

A NOTE ON THE ECONOMIC MANAGEMENT OF INVENTORY OR RESOURCE UNDER STOCHASTIC PRICES

Siwei Jia

2.1 Abstract

The Markovian optimal policies are studied for the problem of economic inventory control or resource management in a finite time horizon. Under some conditions, in particular, when the prices are stochastic and there is a positive fixed setup cost K, the existence of $\{S, s\}$-type Markovian optimal management policies is proved. When $K = 0$, the optimal policies are of $\{S\}$-type, in which case a comparison is made between the optimal policies under stochastic and deterministic prices. It turns out that under stochastic prices the optimal policies should be more conservative in order to maximize the present value of expected revenue.

2.2 Introduction

As a problem in control theory, Clark (1971) studied the optimal management rule for renewable natural resources under the assumptions of a fixed unit selling price and a deterministic population level of the resources. In his model, the inventory level or resource population is described by

$$x_i = f(x_{i-1}),$$

where f is the "recruitment" or "reproduction" function. Reed (1974) extended the model, in which the unit selling price remained fixed, by adding random shocks in the growth of the resource population, i.e.,

$$x_i = Z_i f(x_{i-1}),$$

where the Z_i's are independent with mean one and $Z_i > 0$ with probability one for all i. This is a first-order growth equation. The expected resource population is decided by the recruitment/reproduction function f and the population standard deviation is determined by the distribution of Z_i. Under certain conditions, in particular when the unit selling price is constant and there is a positive fixed setup cost K, Reed (1974) has shown that the optimal harvesting polices are of $\{S, s\}$-type. More general conditions under which the Markovian optimal policies exist were studied by Bhattacharya and Majumdar (1989a, 1989b). On the other hand, if there is no fixed setup cost ($K = 0$), Reed (1974) found that the stochastic growth of the resource population does not effect the optimal harvesting rule obtained by Clark (1971). However, letting the price be random for a one-period time horizon model, Burnes (2001) has shown that the optimal harvesting rule for a renewable resource is more conservative than that obtained under a deterministic price. For non-renewable resources, such as environmental resources, Arrow and Fisher (1974) also pointed out in a one-period model that uncertainty would lead to preservation.

The objective of this article is two-fold:

1. extend the model to the case that the unit selling prices are stochastic and study the optimal management policies in finite time horizon under price uncertainty;

2. investigate the impact of price uncertainty when the optimal management policies under stochastic and deterministic prices are comparable.

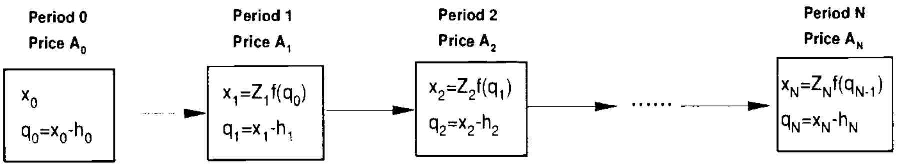
The first part of the problem is discussed in Section 2.3, in which the existence of Markovian optimal policies $\{S_i, s_i\}_{i=0}^{N}$ will be proved. Under the condition of zero set-up cost, $\{S, s\}$-type optimal policies become {S}-type. Then the optimal policies $\{S_{S,i}\}_{i=0}^{N}$ under stochastic prices and the policies $\{S_{D,i}\}_{i=0}^{N}$ under deterministic prices may be compared. In Section 2.4, by showing $S_{S,i} \geq S_{D,i}$ for all $i = 0, 1, \ldots, N$, we conclude that the Markovian optimal management policies need to be more conservative in order to maximize the present value of expected revenue.

2.3 The Model and Main Results

Let $A_0, A_1, \ldots, A_N$ be the stochastic prices for $N+1$ periods and the inventory level at the beginning of the ith period be denoted by x_i, $i = 0, 1, \ldots, N$. The growth of inventory satisfies

$$x_{i+1} = Z_{i+1} f(x_i - h_i),$$

where $Z_1, Z_2, \ldots, Z_N$ are positive and independent random variables with mean one. In addition, the A_i's and Z_j's are independent for all i, j. The quantity h_i is the amount harvested at the ith period. The recruitment function f defined on $[0, +\infty)$ with $f(0) = 0$ is assumed to be nondecreasing, concave and twice-differentiable. We also suppose that the marginal harvesting cost is a constant g and there is a (positive) fixed set-up cost K incurred every time a harvest is undertaken. The management policy at ith period can be defined by h_i or by $q_i = x_i - h_i$:



Generally, at the ith period a policy is decided by

$$q_i = q_i(x_0, A_0, q_0, \ldots, x_{i-1}, A_{i-1}, q_{i-1}, x_i, A_i).$$

The policy is Markovian if

$$q_i = q_i(x_i, A_i).$$

We will use the method of backward induction to show the Markovian optimal management policies are of $\{S, s\}$-type.

Notice in the final period,

$$V_0^*(x_N|A_N) = [x_N(A_N - g) - K]^+,$$

where $V_0^*(x_N|A_N)$ denotes the maximized revenue based on the price information A_N when there is *zero* period to go. The $\{S, s\}$-type optimal policy in the final period is given by $\{0, \frac{K}{A_N - g}\}$:

> harvesting is undertaken if and only if $A_N > g$ and $x_N > \frac{K}{A_N-g}$, in which case harvest down to zero.

When there are $k+1$ periods to go, the Bellman equation for the revenue function is of the form

$$\begin{aligned} & V_{k+1}(q_{N-(k+1)}; x_{N-(k+1)}|A_{N-(k+1)}) \qquad (1) \\ = \quad & (x_{N-(k+1)} - q_{N-(k+1)})(A_{N-(k+1)} - g) \\ & +\alpha \mathbf{E}_{Z_{N-k}, A_{N-k}} V_k^*(Z_{N-k} f(q_{N-(k+1)})) - K\mathbf{1}_{[q_{N-(k+1)} < x_{N-(k+1)}]}, \end{aligned}$$

where $\alpha \in (0, 1)$ is a constant discounting factor. We need to show that there is an $\{S, s\}$-type optimal policy to maximize (1) for each $k = 0, 1, \ldots, N-1$. To show this we use the method of backward induction and the properties of K-concave functions.

Let us recall the definition of K-concave functions. A continuous function $\Psi(y)$ is said to be (strictly) K*-concave* on an interval I if $\forall y_1, y_2, y_3 \in I$ with $y_1 < y_2 < y_3$,

$$\Psi(y_1) - \Psi(y_2) - (y_1 - y_2)\frac{\Psi(y_3) - \Psi(y_2)}{y_3 - y_2}(<) \leq K.$$

Some properties of K-concavity (see Bhattacharya and Waymire (1990), Reed (1974) and Scarf (1960)) which will be used are summarized below:

(i) 0-concavity is equivalent to ordinary concavity.

(ii) If $0 \leq K < M$, then K-concavity implies strict M-concavity.

(iii) If ϕ is K-concave and ψ is M-concave, then for $\alpha, \beta > 0$ $\alpha\phi + \beta\psi$ is $(\alpha K + \beta M)$-concave.

(iv) If f is nondecreasing and concave and ψ is nondecreasing and K-concave, then $\psi \circ f$ is K-concave.

(v) If $f_1(x), \ldots, f_n(x)$ are all K-concave then $\sum_1^n p_i f_i(x)$ is also K-concave if $p_i \geq 0$ and $\sum_1^n p_i = 1$. Alternatively if $f(x, \xi)$ is K-concave in x for each ξ then so is $\int f(x, \xi)\, d\Phi(\xi)$ for probability distribution functions $\Phi(\xi)$.

(vi) Suppose $\phi(x)$ is continuous and strictly K-concave on $[a, b]$, and let $M = \sup_{x \in [a,b]} \phi(x)$ and $S = inf\{t : \phi(t) = M, t \in [a, b]\}$. Then there exists at most one s, $S \le s \le b$, such that $\phi(s) = M - K$, and further if such an s exists, $\phi(x) < M - K$ for $x \in (s, b]$.

In addition we also require the following basic lemma which appears to be new.

Lemma 1 *Let the function $\psi(y)$ be of bounded variation on $[a, b]$. Define*

$$\Psi(y) = \Psi(a) + \int_a^y \psi(u)\, du.$$

Then $\Psi(y)$ is K-concave on $[a, b]$ if and only if for all $a \le y_1 < y_2 \le b$,

$$\int_{y_1}^{y_2} (u - y_1)\, d\psi(u) \le K. \tag{2}$$

Proof: First we prove sufficiency. Suppose (2) holds for all $a \le y_1 < y_2 \le b$.

Note for any $y_1 < y_2$,

$$\Psi(y_1) - \Psi(y_2) + (y_2 - y_1)\psi(y_2) = \int_{y_1}^{y_2} (u - y_1)\, d\psi(u) \le K.$$

Let

$$\lambda = \sup_{y_3 \in (y_2, b]} \{\frac{\Psi(y_3) - \Psi(y_2)}{y_3 - y_2}\}.$$

Since $\Psi(\cdot)$ is also of bounded variation on $[a, b]$, λ is finite and either there exists $y^* \in (y_2, b]$ such that

$$\lambda = \frac{\Psi(y^*) - \Psi(y_2)}{y^* - y_2}$$

or

$$\lambda = \Psi'_+(y_2) = \psi_+(y_2).$$

In the case that such y^* exists notice that

$$\frac{\Psi(y^*) - \Psi(w)}{y^* - w} \ge \lambda$$

for all $w \in (y_2, y^*)$. (Otherwise $\frac{\Psi(w)-\Psi(y_2)}{w-y_2} > \lambda$, which results in a contradiction.) Therefore, letting $w \uparrow y^*$, we have

$$\psi_-(y^*) \geq \lambda.$$

Now consider the following three cases:

(i) If $\lambda \leq \psi(y_2)$, then

$$\Psi(y_1) - \Psi(y_2) - (y_1 - y_2)\lambda \leq \int_{y_1}^{y_2} (u - y_1)\, d\psi(u) \leq K.$$

(ii) If $\lambda = \frac{\Psi(y^*)-\Psi(y_2)}{y^*-y_2} > \psi(y_2)$, then

$$\begin{aligned}
&\Psi(y_1) - \Psi(y_2) - (y_1 - y_2)\lambda \\
&= \int_{y_1}^{y_2} (u - y_1)\, d\psi(u) + (y_2 - y_1)(\lambda - \psi(y_2)) \\
&= \int_{y_1}^{y^*} (u - y_1)\, d\psi(u) - \int_{y_2}^{y^*} (u - y_1)\, d\psi(u) + (y_2 - y_1)(\lambda - \psi(y_2)) \\
&= \int_{y_1}^{y^*} (u - y_1)\, d\psi(u) + (y^* - y_1)(\lambda - \psi(y^*)) \\
&\leq \int_{y_1}^{y^*} (u - y_1)\, d\psi(u) + (y^* - y_1)(\psi_-(y^*) - \psi(y^*)) \\
&= \int_{y_1}^{y^*-} (u - y_1)\, d\psi(u) \leq K.
\end{aligned}$$

(iii) If $\lambda = \psi_+(y_2)$, then

$$\begin{aligned}
&\Psi(y_1) - \Psi(y_2) - (y_1 - y_2)\lambda \\
&= \int_{y_1}^{y_2} (u - y_1)\, d\psi(u) + (y_2 - y_1)(\psi_+(y_2) - \psi(y_2)) \\
&= \int_{y_1}^{y_2+} (u - y_1)\, d\psi(u) \leq K.
\end{aligned}$$

In all the three cases we have shown that

$$\Psi(y_1) - \Psi(y_2) - (y_1 - y_2)\frac{\Psi(y_3) - \Psi(y_2)}{y_3 - y_2} \leq \Psi(y_1) - \Psi(y_2) - (y_1 - y_2)\lambda \leq K.$$

Thus, sufficiency is proved.

To prove necessity we use the definition of K-concavity. Suppose $\Psi(\cdot)$ is K-concave on $[a, b]$. For arbitrary $a \leq y_1 < y_2 < y_3 \leq b$, let $y_3 \downarrow y_2$. Then

$$\begin{aligned} K &\geq \Psi(y_1) - \Psi(y_2) - (y_1 - y_2)\psi_+(y_2) \\ &= \int_{y_1}^{y_2} (u - y_1)\, d\psi(u) + (y_2 - y_1)(\psi_+(y_2) - \psi(y_2)) \\ &= \int_{y_1}^{y_2+} (u - y_1)\, d\psi(u). \end{aligned}$$

Because y_2 is arbitrary, we can conclude that $\int_{y_1}^{y_2} (u - y_1)\, d\psi(u) \leq K$. This completes the proof of the lemma. $\square$

Now we can show the existence of $\{S, s\}$-type optimal policies. We first show that $V_0^*(x|A_N)$ is K-concave for all A_N. Observe that

$$V_0^*(x|A_N) = [x(A_N - g) - K]^+ = \int_0^x \mathbf{1}_{[u > \frac{K}{A_N - g}, A_N > g]}(A_N - g)\, du.$$

It is easy to check that

$$\int_0^y u\, d(\mathbf{1}_{[u > \frac{K}{A_N - g}, A_N > g]}(A_N - g)) \leq K$$

for all $y > 0$. Therefore, by Lemma 1, $V_0^*(x|A_N)$ is K-concave. $V_0^*(x|A_N)$ is also nondecreasing in x.

Next assume that $V_k^*(x|A_{N-k})$ is K-concave and nondecreasing. By the properties (ii)–(v) of K-concavity,

$$\alpha \mathbf{E}_{A_{N-k}, Z_{N-k}} V_k^*(Z_{N-k} f(x)|A_{N-k})$$

is αK-concave and therefore is strictly K-concave, and so is

$$\xi_{k+1}(q) = (x_{N-(k+1)} - q)(A_{N-(k+1)} - g) + \alpha \mathbf{E}_{A_{N-k}, Z_{N-k}} V_k^*(Z_{N-k} f(q)|A_{N-k}).$$

Then by the property (vi) there exist $S_{N-(k+1)}$ at which $\xi_{k+1}(q)$ reaches its maximum and at most one $s_{N-(k+1)} \geq S_{N-(k+1)}$ such that $\xi_{k+1}(s_{N-(k+1)}) = \xi_{k+1}(S_{N-(k+1)}) - K$. To maximize

$$V_{k+1}(q_{N-(k+1)}; x_{N-(k+1)}|A_{N-(k+1)}) = \xi_{k+1}(q_{N-(k+1)}) - K\mathbf{1}_{[q_{N-(k+1)}<x_{N-(k+1)}]},$$

the $\{S, s\}$-type optimal policy is determined by $\{S_{N-(k+1)}, s_{N-(k+1)}\}$. In the case that $A_{N-(k+1)} \leq g$ or $\xi_{k+1}(q) > \xi_{k+1}(S_{N-(k+1)}) - K$ for all $q > S_{N-(k+1)}$, we wait until the next period, i.e., $s_{N-(k+1)} = x_{N-(k+1)}$.

To complete the backward induction we need to show that $V^*_{k+1}(x)$ is K-concave and nondecreasing in x for all $A_{N-(k+1)}$. Notice that

$$V^*_{k+1}(x) = \begin{cases} \alpha \mathbf{E} V^*_k(Z_{N-k} f(x)) & \text{if } x \leq s_{N-(k+1)} \text{ or } A_{N-(k+1)} \leq g; \\ (x - S_{N-(k+1)})(A_{N-(k+1)} - g) & \\ \quad + \alpha \mathbf{E} V^*_k(Z_{N-k} f(S_{N-(k+1)})) - K & \text{if } x > s_{N-(k+1)} \text{ and } A_{N-(k+1)} > g. \end{cases}$$

Clearly $V^*_{k+1}(x)$ is continuous and nondecreasing. Also it is K-concave when $A_{N-(k+1)} \leq g$. In the case that $A_{N-(k+1)} > g$, $V^*_{k+1}(x)$ is piecewise K-concave on $[0, s_{N-(k+1)}]$ and $(s_{N-(k+1)}, +\infty)$. Write

$$V^{*'}_{k+1}(x) = \frac{d}{dx} V^*_{k+1}(x) = \begin{cases} \frac{d}{dx} \alpha \mathbf{E} V^*_k(Z_{N-k} f(x)) & \text{if } x \leq s_{N-(k+1)}; \\ A_{N-(k+1)} - g & \text{if } x > s_{N-(k+1)}. \end{cases}$$

Note that $V^{*'}_{k+1}(s_{N-(k+1)}-) \leq A_{N-(k+1)} - g = V^{*'}_{k+1}(s_{N-(k+1)}+)$ since

$$\xi'_{k+1}(s_{N-(k+1)}) = -(A_{N-(k+1)} - g) + \frac{d}{dq} \alpha \mathbf{E} V^*_k(Z_{N-k} f(q))|_{q=s_{N-(k+1)}} \leq 0.$$

To see that $V^*_{k+1}(x)$ is K-concave on $[0, +\infty)$, we again apply Lemma 1. Let $0 \leq y_1 < s_{N-(k+1)}$ and $y_2 > s_{N-(k+1)}$.

$$\int_{y_1}^{y_2} (u - y_1)\, dV^{*'}_{k+1}(u)$$

$$
\begin{aligned}
&= \int_{y_1}^{s_{N-(k+1)}} (u - y_1)\, dV_{k+1}^{*'}(u) \\
&\qquad + (s_{N-(k+1)} - y_1)[V_{k+1}^{*'}(s_{N-(k+1)}+) - V_{k+1}^{*'}(s_{N-(k+1)}-)] \\
&= V_{k+1}^{*}(y_1) - V_{k+1}^{*}(s_{N-(k+1)}) + (s_{N-(k+1)} - y_1)(A_{N-(k+1)} - g) \\
&= \xi_{k+1}(y_1) - \xi_{k+1}(s_{N-(k+1)}) \le K.
\end{aligned}
$$

The last inequality is due to the property (vi) of ξ_{k+1}. This completes the backward induction. Consequently, optimal policies $\{S_i, s_i\}_{i=0}^{N}$ can be obtained to maximize the revenue over $N+1$ periods :

> at ith period, harvesting is undertaken if and only if $A_i > g$ and $x_i > s_i$, in which case harvest down to S_i.

The optimal management policies are Markovian since the policy at ith period depends only on x_i and A_i.

The above result may be generalized to the case of non-constant marginal harvesting cost g under some conditions. Suppose that a diminishing marginal harvesting cost $g(y)$ satisfies

$$
\lim_{y \to +\infty} g(y) = g = \inf_y g(y) \text{ and } \int_0^\infty [g(u) - g]\, du < rK,
$$

where r is the (fixed) interest rate. (The discount factor $\alpha = \frac{1}{r+1}$.) It can be shown that the optimal harvesting policies are also of $\{S, s\}$-type.

In the final period, given any market price $A > g$, the revenue function is of the form

$$
V(q; x) = \int_q^x (A - g(u))\, du - K\mathbf{1}_{[q<x]}.
$$

The optimal $\{S, s\}$ policy is decided by S and s such that

$$g(S) = A \text{ and } \int_S^s (A - g(u))\, du = K,$$

and in the case that $A \leq g$, $s = x$. The optimized revenue

$$V_0^*(x) = \int_0^x \mathbf{1}_{[u>s]}(A - g(u))\, du.$$

Notice that $V_0^*(x)$ is convex and nondecreasing. Moreover, for arbitrarily large y,

$$\int_0^y u\, d[\mathbf{1}_{[u>s]}(A - g(u))] < \frac{K}{\alpha}.$$

To see this we notice that

$$\begin{aligned}
& \int_0^y u\, d[\mathbf{1}_{[u>s]}(A - g(u))] \\
= & \int_s^y u\, d(A - g(u)) + s(A - g(s)) \\
= & \ y(A - g(y)) - \int_s^y (A - g(u))\, du \\
= & \int_0^y (A - g(y))\, du - [\int_0^y (A - g(u))\, du - \int_0^S (A - g(u))\, du - K] \\
\leq & \int_0^y (g(u) - g)\, du + K < rK + K < \frac{K}{\alpha}.
\end{aligned}$$

Thus by Lemma 1 $V_0^*(x)$ is strictly $\frac{K}{\alpha}$-concave.

Now suppose that when there are k periods to go $V_k^*(x)$ is strictly $\frac{K}{\alpha}$-concave and nondecreasing. Then

$$\xi_{k+1}(q) = \int_q^x (A_{N-(k+1)} - g(u))\, du + \alpha \mathbf{E}_{Z_{N-k}, A_{N-k}} V_k^*(Z_{N-k} f(q))$$

is strictly K-concave since $\alpha \mathbf{E}_{Z_{N-k}, A_{N-k}} V_k^*(Z_{N-k} f(x))$ is strictly K-concave and the integral part in $\xi_{k+1}(q)$ is concave. Hence there exists $\{S, s\}$-type optimal policy to maximize

$$V_{k+1}(q;x) = \xi_{k+1}(q) - K\mathbf{1}_{[q<x]}.$$

The maximized revenue when there are $k+1$ periods to go is given by

$$\begin{aligned} & V^*_{k+1}(x|A_{N-(k+1)}) \\ = & \left\{ \begin{array}{ll} \alpha \mathbf{E}_{Z_{N-k},A_{N-k}} V^*_k(Z_{N-k}f(x)) & \text{if } x \le s \text{ or } A_{N-(k+1)} \le g; \\ \int_S^x (A_{N-(k+1)} - g(u))\,du & \\ \quad + \alpha \mathbf{E}_{Z_{N-k},A_{N-k}} V^*_k(Z_{N-k}f(S)) - K & \text{if } x > s \text{ and } A_{N-(k+1)} > g. \end{array} \right. \end{aligned}$$

For any $A_{N-(k+1)} = A$, $V^*_{k+1}(x)$ is continuous and nondecreasing. Also, $V^*_{k+1}(x)$ is K-concave on $[0,s]$ and convex on $(s,+\infty)$. Around the point s,

$$\frac{d}{dx}V^*_{k+1}(x)|_{s-} \le \frac{d}{dx}V^*_{k+1}(x)|_{s+}$$

since

$$\xi'_{k+1}(s) = -(A - g(s)) + \frac{d}{dq}\alpha \mathbf{E} V^*_k(Z_{N-k}f(q))|_s \le 0.$$

We need to show that $V^*_{k+1}(x)$ is strictly $\frac{K}{\alpha}$-concave. Write

$$\begin{aligned} & V^{*'}_{k+1}(x) = \frac{d}{dx}V^*_{k+1}(x) \\ = & \left\{ \begin{array}{ll} \frac{d}{dx}\alpha \mathbf{E}_{Z_{N-k},A_{N-k}} V^*_k(Z_{N-k}f(x)) & \text{if } x \le s\ ; \\ A - g(x) & \text{if } x > s\ . \end{array} \right. \end{aligned}$$

Let $y_1 \in [0,s]$ and $y_2 \in (s,+\infty)$. By Lemma 1, it suffices to show that

$$\int_{y_1}^{y_2} (u - y_1)\, d(V^{*'}_{k+1}(u)) < \frac{K}{\alpha}.$$

In fact,

$$\begin{aligned} & \int_{y_1}^{y_2} (u - y_1)\, d(V^{*'}_{k+1}(u)) \\ = & \int_{y_1}^{s} (u - y_1)\, d(V^{*'}_{k+1}(u)) + \int_{s}^{y_2} (u - y_1)\, d(V^{*'}_{k+1}(u)) \end{aligned}$$

$$
\begin{aligned}
&+(s-y_1)[V^{*'}_{k+1}(s^+)-V^{*'}_{k+1}(s^-)] \\
= \quad & V^*_{k+1}(y_1)-V^*_{k+1}(s)-\int_s^{y_2}(A-g(u))\,du+(y_2-y_1)(A-g(y_2)).
\end{aligned}
$$

Notice that

$$
V^*_{k+1}(y_1)-V^*_{k+1}(s)=\xi_{k+1}(y_1)-\xi_{k+1}(s)-\int_{y_1}^{s}(A-g(u))\,du
$$

since

$$
V^*_{k+1}(y_1)=\alpha\mathbf{E}_{Z_{N-k},A_{N-k}}V^*_k(Z_{N-k}f(y_1))=\xi_{k+1}(y_1)-\int_{y_1}^{x}(A-g(u))\,du
$$

and

$$
V^*_{k+1}(s)=\alpha\mathbf{E}_{Z_{N-k},A_{N-k}}V^*_k(Z_{N-k}f(s))=\xi_{k+1}(s)-\int_{s}^{x}(A-g(u))\,du.
$$

Therefore,

$$
\begin{aligned}
&\int_{y_1}^{y_2}(u-y_1)\,d(V^{*'}_{k+1}(u)) \\
= \quad & V^*_{k+1}(y_1)-V^*_{k+1}(s)-\int_s^{y_2}(A-g(u))\,du+(y_2-y_1)(A-g(y_2)) \\
= \quad & \xi_{k+1}(y_1)-\xi_{k+1}(s)-\int_{y_1}^{y_2}(A-g(u))\,du+(y_2-y_1)(A-g(y_2)) \\
= \quad & \xi_{k+1}(y_1)-\xi_{k+1}(s)+\int_{y_1}^{y_2}(g(u)-g(y_2))\,du \\
< \quad & K+rK=\frac{K}{\alpha}.
\end{aligned}
$$

The last inequality is due to property (vi) of K-concavity of $\xi_{k+1}(\cdot)$ and the conditions on $g(\cdot)$. This completes the backward induction. Thus we can conclude that if a diminishing marginal harvesting cost converges fast enough then the optimal policies are also of $\{S,s\}$-type. An example of such a marginal cost is given in Reed (1974).

2.4 A Comparison under Zero Set-up Cost

In the case that the fixed set-up cost $K = 0$, the revenue function V is 0-concave, which is concave. Consequently, the Markovian optimal management policies become S-type:

at ith period, harvesting is undertaken if and only if $A_i > g$ and $x_i > S_i$,

in which case harvest down to S_i.

In Reed's case the stochastic growth of the resource population does not effect the optimal harvesting rule that was obtained by Clark. Will the price uncertainty in our case influence the optimal management policies? Given the price information A_i at the ith period, $S_{S,i} = q^*_{S,i}$ decides the ith optimal rule when the future prices $\{A_j\}_{i+1}^N$ are stochastic; $S_{D,i} = q^*_{D,i}$ decides the optimal rule when the future prices $\{a_j\}_{i+1}^N$ are deterministic, where $a_j = \mathbf{E}A_j > g$ for $j = i+1, \ldots, N$. Is $S_{S,i}$ still equal to $S_{D,i}$ for all $i = 0, 1, \ldots, N$? In this section we will show that the uncertainty of the prices does effect the optimal management policies and $q^*_{S,i} \geq q^*_{D,i}$ for all $i = 0, 1, \ldots, N$. That tells us in order to maximize the present value of expected revenue the optimal management policies should be more conservative when the prices are stochastic.

We begin with the one-period model. When there is one period to go, given the price A_{N-1},

$$V_1(q_{N-1}; x_{N-1}|A_{N-1})$$

$$= (x_{N-1} - q_{N-1})(A_{N-1} - g) + \alpha \mathbf{E}_{Z_N, A_N} V_0^*(Z_N f(q_{N-1})|A_N). \quad (3)$$

Notice that when A_N is deterministically equal to a_N

$$\mathbf{E}_{Z_N,A_N} V_0^*(Z_N f(q_{N-1})|A_N) = f(q_{N-1})(a_N - g),$$

while in the case of stochastic price

$$\mathbf{E}_{Z_N,A_N} V_0^*(Z_N f(q_{N-1})|A_N) = f(q_{N-1})\mathbf{E}(A_N - g)^+.$$

(3) is clearly concave and the equation

$$-(A_{N-1} - g) + \alpha \frac{d}{dq_{N-1}} \mathbf{E}_{Z_N,A_N} V_0^*(Z_N f(q_{N-1})|A_N) = 0 \tag{4}$$

determines the maximizing q_{N-1}^*. In the case that $A_N = a_N$ with probability one, (4) is

$$f'(q_{D,N-1}^*) = \frac{1}{\alpha} \frac{A_{N-1} - g}{a_N - g}. \tag{4D}$$

Otherwise (4) is

$$f'(q_{S,N-1}^*) = \frac{1}{\alpha} \frac{A_{N-1} - g}{\mathbf{E}(A_N - g)^+}. \tag{4S}$$

Since $\mathbf{E}(A_N - g)^+ \geq a_N - g > 0$ (with strict inequality if $\mathbf{P}[A_N < g] > 0$) and f is nondecreasing and concave, We have

$$0 \leq q_{D,N-1}^* \leq q_{S,N-1}^* \leq +\infty. \tag{5}$$

Here are some remarks on (5).

1. $q_{D,N-1}^*$ and $q_{S,N-1}^*$ generally depend on A_{N-1}.

2. That $q^* = +\infty$ is practically the same as $q^* = x_{N-1}$ for any x_{N-1}, which means waiting until the next period.

3. If $f'(y) > \frac{1}{\alpha}\frac{A_{N-1}-g}{a_N-g}$ for all $y \geq 0$ then $q^*_{D,N-1} = q^*_{S,N-1} = +\infty$, which is also true when $A_{N-1} \leq g$.

4. If $f'(y) < \frac{1}{\alpha}\frac{A_{N-1}-g}{\mathbf{E}(A_N-g)^+}$ for all $y \geq 0$ then $q^*_{D,N-1} = q^*_{S,N-1} = 0$.

5. If (4D) and (4S) have distinct solutions then $q^*_{D,N-1} < q^*_{S,N-1}$.

(5) together with the above remarks imply that if the optimal rule is to preserve $q^*_{D,N-1}$ under deterministic prices, then under random prices the optimal rule is to preserve at least $q^*_{D,N-1}$, i.e., $q^*_{D,N-1} \leq q^*_{S,N-1}$. In other words, the optimal rule at the $(N-1)$th period should be more conservative under price uncertainty. Note that nondecreasing of f is necessary for us to have more conservative optimal management policies under price uncertainty. If f reaches its maximum at some m and becomes decreasing on $[m, +\infty)$, then in the case that $A_{N-1} < g$, (4D) and (4S) may have solutions $m < q^*_{S,N-1} < q^*_{D,N-1}$. This means that when the current population level is too high (well above m) to be salubrious for growth we will cut it down, even with a sacrifice when $A_{N-1} < g$, in order to raise the expected population in the next period. In the case of random prices we are willing to sacrifice more today ($m < q^*_{S,N-1} < q^*_{D,N-1}$) since the expected future profit per unit is higher, i.e., $\mathbf{E}(A_N - g)^+ \geq a_N - g$.

We proceed with showing that for $k = 1$,

(i) $V^*_{S,k}(y|A_{N-k})$ and $V^*_{D,k}(y; a_{N-k})$ are both nondecreasing and concave;

(ii) $\mathbf{E}_{A_{N-k}} \frac{d}{dy} V^*_{S,k}(y|A_{N-k}) \geq \frac{d}{dy} V^*_{D,k}(y; a_{N-k})$ for all $y \in [0, +\infty)$.

Based on the given price A_{N-1}, the maximized revenue when there is one period to go is of the form

$$V^*_{S,1}(y|A_{N-1}) =$$
$$\begin{cases} \alpha \mathbf{E}_{Z_N,A_N} V^*_{S,0}(Z_N f(y)|A_N) & \text{if } A_{N-1} \le g \text{ or } y \le q^*_{S,N-1}, \\ (y - q^*_{S,N-1})(A_{N-1} - g) & \\ \quad + \alpha \mathbf{E}_{Z_N,A_N} V^*_{S,0}(Z_N f(q^*_{S,N-1})|A_N) & \text{if } A_{N-1} > g \text{ and } y > q^*_{S,N-1}; \end{cases}$$

Its derivative is

$$\begin{aligned} V^{*'}_{S,1}(y|A_{N-1}) &= \frac{d}{dy} V^*_{S,1}(y|A_{N-1}) \\ &= \begin{cases} \alpha f'(y) \mathbf{E}(A_N - g)^+ & \text{if } A_{N-1} \le g \text{ or } y \le q^*_{S,N-1}, \\ A_{N-1} - g & \text{if } A_{N-1} > g \text{ and } y > q^*_{S,N-1}, \end{cases} \end{aligned} \tag{6}$$

where $q^*_{S,N-1} = q^*_{S,N-1}(A_{N-1})$. On the other hand, when $A_{N-1} = a_{N-1}$ and $A_N = a_N$ deterministically,

$$V^*_{D,1}(y; a_{N-1}) = \begin{cases} \alpha f(y)(a_N - g) & \text{if } y \le q^*, \\ (y - q^*)(a_{N-1} - g) + \alpha f(q^*)(a_N - g) & \text{if } y > q^*; \end{cases}$$

and

$$V^{*'}_{D,1}(y; a_{N-1}) = \begin{cases} \alpha f'(y)(a_N - g) & \text{if } y \le q^*, \\ a_{N-1} - g & \text{if } y > q^*, \end{cases} \tag{7}$$

where q^* is determined by (4D) for $A_{N-1} = a_{N-1}$ with probability one. It is easy to verify (i). To show (ii) notice that $q^*_{S,N-1}(A_{N-1})$ is increasing as A_{N-1} decreases and $q^*_{S,N-1}(A_{N-1}) = +\infty$ when $A_{N-1} \le g$. This can be illustrated in the following figure.

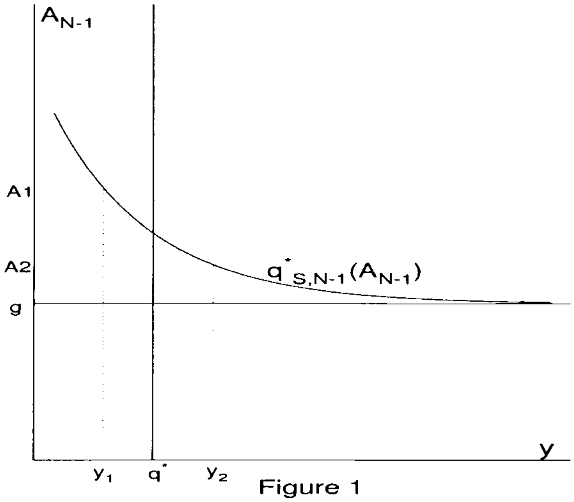


Figure 1

From Figure 1 we can see that if the inventory level y is at some $y_2 > q^*$, then

[a] in the case that $A_{N-1} \leq g$, $(6) = \alpha f'(y)\mathbf{E}(A_N - g)^+ > 0 \geq A_{N-1} - g$;

[b] in the case that $g < A_{N-1} \leq A2$, $f'(y) \geq f'(q^*_{S,N-1}(A_{N-1})) = \frac{A_{N-1}-g}{\alpha\mathbf{E}(A_N-g)^+}$

and therefore $(6) \geq A_{N-1} - g$;

[c] in the case that $A_{N-1} > A2$, $(6) = A_{N-1} - g$.

Thus for any $y > q^*$,

$$\mathbf{E}_{A_{N-1}} V_1^{*'}(y|A_{N-1}) \geq \mathbf{E}_{A_{N-1}}(A_{N-1} - g) = a_{N-1} - g = (7).$$

If the inventory level y is at some $y_1 \leq q^*$, then

[d] in the case that $A_{N-1} \leq A1$, $(6) = \alpha f'(y)\mathbf{E}(A_N - g)^+ > (7)$;

[e] in the case that $A_{N-1} > A1$, $f'(y) < f'(q^*_{S,N-1}(A_{N-1})) = \frac{A_{N-1}-g}{\alpha\mathbf{E}(A_N-g)^+}$

and therefore $(6) = A_{N-1} - g > \alpha f'(y)\mathbf{E}(A_N - g)^+ > (7)$.

So when $y \leq q^*$, we also have

$$\mathbf{E}_{A_{N-1}} V_1^{*'}(y|A_{N-1}) \geq \alpha f'(y)(a_N - g) = (7).$$

Hence, for all $y \in [0, +\infty)$, (ii) holds.

Now consider

$$V_{k+1}(x_{N-(k+1)}, q_{N-(k+1)}|A_{N-(k+1)})$$
$$= (x_{N-(k+1)} - q_{N-(k+1)})(A_{N-(k+1)} - g)$$
$$+\alpha \mathbf{E}_{Z_{N-k}, A_{N-k}} V_k^*(Z_{N-k} f(q_{N-(k+1)})|A_{N-k}). \qquad (8)$$

When the prices are stochastic, by adding subscript "S" to V and q in (8) we have (8S), while in the deterministic case we add subscript "D" and put $A_{N-k} = a_{N-k}$ with probability one to get (8D). The following backward induction will show that if (i) and (ii) hold for k then

[A] $q^*_{D,N-(k+1)} \leq q^*_{S,N-(k+1)}$;

[B] (i) also holds for $k+1$;

[C] (ii) also holds for $k+1$.

[A] That (i) holds for k implies concavity of (8). Observe

$$\frac{d}{dq_{N-(k+1)}} V_{k+1}(x_{N-(k+1)}, q_{N-(k+1)}|A_{N-(k+1)})$$
$$= -(A_{N-(k+1)} - g)$$
$$+\alpha f'(q_{N-(k+1)}) \mathbf{E}_{Z_{N-k}, A_{N-k}}[Z_{N-k} V_k^{*'}(Z_{N-k} f(q_{N-(k+1)})|A_{N-k})]. \qquad (9)$$

If $A_{N-(k+1)} \leq g$ then clearly $q^*_{S,N-(k+1)} = q^*_{D,N-(k+1)} = x_{N-(k+1)}$. For $A_{N-(k+1)} > g$, we can write (9D) for (9) in the deterministic case and similarly (9S) when the prices are stochastic in the way of writing (8D) and (8S). Then because of the induction assumption that (ii) holds for k,

$$\text{(9D)} > 0 \text{ for all } q_{N-(k+1)} \geq 0 \implies \text{(9S)} > 0 \text{ for all } q_{N-(k+1)} \geq 0,$$

in which case

$$q^*_{D,N-(k+1)} = x_{N-(k+1)} \Longrightarrow q^*_{S,N-(k+1)} = x_{N-(k+1)}.$$

For the same reason,

$$(9S) < 0 \text{ for all } q_{N-(k+1)} \geq 0 \Longrightarrow (9D) < 0 \text{ for all } q_{N-(k+1)} \geq 0,$$

in which case

$$q^*_{S,N-(k+1)} = 0 \Longrightarrow q^*_{D,N-(k+1)} = 0.$$

The following two cases are straightforward:

(9D) $= 0$ for some $q^*_{D,N-(k+1)}$ and (9S) > 0 for all $q_{N-(k+1)} \geq 0$;

(9S) $= 0$ for some $q^*_{S,N-(k+1)}$ and (9D) < 0 for all $q_{N-(k+1)} \geq 0$.

Now suppose that $q^*_{S,N-(k+1)}$ solves (9S) $= 0$ and $q^*_{D,N-(k+1)}$ solves (9D) $= 0$. Then (9S) evaluated at $q^*_{D,N-(k+1)}$ is

$$\begin{aligned}
&\alpha f'(q^*_{D,N-(k+1)})\mathbf{E}_{Z_{N-k},A_{N-k}}[Z_{N-k}V^{*'}_{S,k}(Z_{N-k}f(q^*_{D,N-(k+1)})|A_{N-k})] \\
&\qquad -(A_{N-(k+1)} - g) \\
&= (A_{N-(k+1)} - g)\left[\frac{\mathbf{E}_{Z_{N-k},A_{N-k}}[Z_{N-k}V^{*'}_{S,k}(Z_{N-k}f(q^*_{D,N-(k+1)})|A_{N-k})]}{\mathbf{E}_{Z_{N-k}}[Z_{N-k}V^{*'}_{D,k}(Z_{N-k}f(q^*_{D,N-(k+1)})|a_{N-k})]} - 1\right] \\
&\geq 0. \qquad (10)
\end{aligned}$$

In (10) the equality is due to the fact that

$$f'(q^*_{D,N-(k+1)}) = \frac{1}{\alpha}\frac{A_{N-(k+1)} - g}{\mathbf{E}_{Z_{N-k}}[Z_{N-k}V^{*'}_{D,k}(Z_{N-k}f(q^*_{D,N-(k+1)})|a_{N-k})]}$$

and the inequality is because (ii) holds for k. That $q^*_{D,N-(k+1)} \leq q^*_{S,N-(k+1)}$ follows from (10) and the concavity of $V_{k+1}(x_{N-(k+1)}, q_{N-(k+1)}|A_{N-(k+1)})$. Thus [A] is proved.

[**B**] Observe that

$$V^*_{S,k+1}(y|A_{N-(k+1)}) = \begin{cases} \alpha \mathbf{E}_{Z_{N-k},A_{N-k}} V^*_{S,k}(Z_{N-k}f(y)|A_{N-k}) \\ \qquad\qquad \text{if } A_{N-(k+1)} \leq g \text{ or } y \leq q^*_{S,N-(k+1)}, \\ (y - q^*_{S,N-(k+1)})(A_{N-(k+1)} - g) \\ \quad +\alpha \mathbf{E}_{Z_{N-k},A_{N-k}} V^*_{S,k}(Z_{N-k}f(q^*_{S,N-(k+1)})|A_{N-k}) \\ \qquad\qquad \text{if } A_{N-(k+1)} > g \text{ and } y > q^*_{S,N-(k+1)}; \end{cases}$$

and

$$V^*_{D,k+1}(y;a_{N-(k+1)}) = \begin{cases} \alpha \mathbf{E}_{Z_{N-k}} V^*_{D,k}(Z_{N-k}f(y);a_{N-k}) \\ \qquad\qquad \text{if } y \leq q^*_{D,N-(k+1)}, \\ (y - q^*_{D,N-(k+1)})(a_{N-(k+1)} - g) \\ \quad +\alpha \mathbf{E}_{Z_{N-k}} V^*_{D,k}(Z_{N-k}f(q^*_{D,N-(k+1)});a_{N-k}) \\ \qquad\qquad \text{if } y > q^*_{D,N-(k+1)}. \end{cases}$$

By applying (i) for k, it is easy to verify [B], i.e., (i) is true for $k+1$.

[**C**] From the above we have

$$V^{*'}_{S,k+1}(y|A_{N-(k+1)}) = \begin{cases} \alpha f'(y) \mathbf{E}_{Z_{N-k},A_{N-k}}[Z_{N-k} V^{*'}_{S,k}(Z_{N-k}f(y)|A_{N-k})] \\ \qquad\qquad \text{if } A_{N-(k+1)} \leq g \text{ or } y \leq q^*_{S,N-(k+1)}, \\ A_{N-(k+1)} - g \\ \qquad\qquad \text{if } A_{N-(k+1)} > g \text{ and } y > q^*_{S,N-(k+1)}; \end{cases} \tag{11}$$

and

$$V^{*'}_{D,k+1}(y;a_{N-(k+1)}) = \begin{cases} \alpha f'(y) \mathbf{E}_{Z_{N-k}}[Z_{N-k} V^{*'}_{D,k}(Z_{N-k}f(y);a_{N-k})] \\ \qquad\qquad \text{if } y \leq q^*_{D,N-(k+1)}, \\ a_{N-(k+1)} - g \qquad \text{if } y > q^*_{D,N-(k+1)}. \end{cases} \tag{12}$$

Notice that $q^*_{S,N-(k+1)} = q^*_{S,N-(k+1)}(A_{N-(k+1)})$ and $q^*_{D,N-(k+1)}$ is fixed based on $a_{N-(k+1)}$. Consider the following two cases.

[f] $y \leq q^*_{D,N-(k+1)}$.

In this case

$$(12) = \alpha f'(y)\mathbf{E}_{Z_{N-k}}[Z_{N-k}V_{D,k}^{*'}(Z_{N-k}f(y); a_{N-k})].$$

For all $A_{N-(k+1)}$ such that $y \leq q_{S,N-(k+1)}^{*}(A_{N-(k+1)})$ (also including $A_{N-(k+1)} \leq g$), by the induction assumption,

$$(11) = \alpha f'(y)\mathbf{E}_{Z_{N-k},A_{N-k}}[Z_{N-k}V_{S,k}^{*'}(Z_{N-k}f(y)|A_{N-k})] > (12).$$

For all $A_{N-(k+1)}$ such that $y > q_{N-(k+1)}^{*}(A_{N-(k+1)})$, $(11) = A_{N-(k+1)} - g$. But in this case

$$\frac{d}{dq_{N-(k+1)}} V_{S,k+1}(x_{N-(k+1)}, q_{N-(k+1)}|A_{N-(k+1)}) \big|_{q_{N-(k+1)}=y} < 0,$$

which implies

$$A_{N-(k+1)} - g > \alpha f'(y)\mathbf{E}_{Z_{N-k},A_{N-k}}[Z_{N-k}V_{S,k}^{*'}(Z_{N-k}f(y)|A_{N-k})] > (12).$$

[g] $y > q_{D,N-(k+1)}^{*}$.

In this case $(12) = a_{N-(k+1)} - g$. Suppose the given price $A_{N-(k+1)}$ is such that $y \leq q_{S,N-(k+1)}^{*}(A_{N-(k+1)})$ but at the same time $A_{N-(k+1)} > g$. Then (11) evaluated at y is greater than zero, which implies that

$$(11) > A_{N-(k+1)} - g.$$

For any other realizations of $A_{N-(k+1)}$ that $(11) \geq A_{N-(k+1)} - g$ is clear. So the expectation of $(11) > a_{N-(k+1)} - g = (12)$.

Combining the arguments in [f] and [g], we obtain [C], i.e.,

$$\mathbf{E}_{A_{N-(k+1)}} \frac{d}{dy} V_{S,k+1}^{*}(y|A_{N-(k+1)}) \geq \frac{d}{dy} V_{D,k+1}^{*}(y; a_{N-(k+1)}) \text{ for all } y \in [0, +\infty).$$

This completes the backward induction.

A special case of nondecreasing and concave functions on $[0, +\infty)$ is

$$f(x) = bx$$

for $b > 0$. If this is the case, all the above arguments are valid. Moreover, for $k = 0, 1, \ldots, N$, it is easy to see that

(i') $V^*_{S,k}(y|A_{N-k})$ and $V^*_{D,k}(y; a_{N-k})$ are both linear and nondecreasing;

(ii') $\mathbf{E}_{A_{N-k}} \frac{d}{dy} V^*_{S,k}(y|A_{N-k}) \geq \frac{d}{dy} V^*_{D,k}(y; a_{N-k})$.

Therefore, (8) is linear and (9S) $\geq$ (9D), which implies that

the optimal policies are binary, i.e., $q^* = 0$ or x, and

$q^*_D = x \Longrightarrow q^*_S = x$.

Consequently, under our assumptions with $K = 0$, the comparison between $\{q^*_{S,i}\}_{i=0}^N$ and $\{q^*_{D,i}\}_{i=0}^N$ shows that $q^*_{S,i} \geq q^*_{D,i}$ for all $i = 0, 1, \ldots, N$. Hence under price uncertainty the Markovian optimal management policies need to be more conservative in order to maximize the present value of expected revenue.

Some characteristics, e.g., non-renewability, of a certain kind of inventory or resource may be defined by the pattern of its reproduction function f. If for a certain resource its population level grows deterministically and its reproduction function f is bounded by $u(x) = x$, then we may say that the resource is nonrenewable. The nonrenewable resources without decay have reproduction function $f(x) = x$, for which case the conservativeness of optimality holds under stochastic prices.

One example is the American option. Consider an American call option with executing price g and assume zero interest rate. The reproduction function here is $f(x) = x$. At any time the stock price A_i exceeds g, execution of the option will bring an earning of $A_i - g$. In the final period, execute the call option if $A_N > g$. The expected revenue is $\mathbf{E}(A_N - g)^+$. When there is one period left, the decision of executing or not depends on the comparison of $\mathbf{E}[(A_N - g)^+ | A_{N-1}]$ and the given value of $A_{N-1} - g$. Suppose the process $\{A_i\}_{i=0}^N$ is a martingale, i.e., $\mathbf{E}(A_i | A_{i-1}, \ldots, A_0) = A_{i-1}$. Then $\mathbf{E}[(A_N - g)^+ | A_{N-1}] > A_{N-1} - g$ and the decision is to wait. When there are two periods to go, given the price A_{N-2}, we can see

$$\mathbf{E}_{A_{N-1}}[\mathbf{E}((A_N - g)^+ | A_{N-1}) | A_{N-2}] > A_{N-2} - g.$$

So the decision is still to wait. Taking the backward iteration we have

$$\mathbf{E}_{A_1}\{\ldots \mathbf{E}_{A_{N-1}}[\mathbf{E}((A_N - g)^+ | A_{N-1}) | A_{N-2}] \ldots | A_0\} > A_0 - g.$$

Thus the optimal policy in dealing with this American call option is to wait until the final period.

We have compared the optimal policies under concavity of f together with the other conditions. It turns out that the comparison can also be made if f is strictly convex. In such a case

$$\begin{aligned} & V_1(x_{N-1}, q_{N-1} | A_{N-1}) \\ & = (x_{N-1} - q_{N-1})(A_{N-1} - g) + \alpha f(q_{N-1}) \mathbf{E}(A_N - g)^+ \end{aligned}$$

is also strictly convex, where in the deterministic case $A_N = a_N$ with probability one. Thus either $q_{N-1} = 0$ or $q_{N-1} = x_{N-1}$ maximizes $V_1(x_{N-1}, q_{N-1} | A_{N-1})$, i.e., the optimal policy is binary.

$$V_1^*(y|A_{N-1}) = max\{y(A_{N-1}-g),\ \alpha f(y)\mathbf{E}(A_N-g)^+\}$$
$$= \begin{cases} y(A_{N-1}-g) & \text{if } r_1 = \frac{y(A_{N-1}-g)}{f(y)\alpha\mathbf{E}(A_N-g)^+} \geq 1; \\ \alpha f(y)\mathbf{E}(A_N-g)^+ & \text{if } r_1 < 1. \end{cases} \quad (13)$$

The optimal rule is to harvest down to zero if and only if $r_1 \geq 1$. Clearly

$$r_{D,1} = \frac{x_{N-1}(A_{N-1}-g)}{f(x_{N-1})\alpha(a_N-g)} \geq \frac{x_{N-1}(A_{N-1}-g)}{f(x_{N-1})\alpha\mathbf{E}(A_N-g)^+} = r_{S,1}.$$

Thus $r_{D,1} < 1$ implies $r_{S,1} < 1$, i.e., $q_{D,1}^* = x_{N-1} \Longrightarrow q_{S,1}^* = x_{N-1}$. When the optimal rules are binary this implies conservativeness of optimality under price uncertainty. Notice that $V_{S,1}^*(y|A_{N-1}) = (13)$ with $r_{S,1} = r_1$ and

$$V_{D,1}^*(y|a_{N-1}) = max\{y(a_{N-1}-g),\ \alpha f(y)(a_N-g)\}$$
$$= \begin{cases} y(a_{N-1}-g) & \text{if } r_{D,1} = \frac{y(a_{N-1}-g)}{f(y)\alpha(a_N-g)} \geq 1; \\ \alpha f(y)(a_N-g) & \text{if } r_{D,1} < 1. \end{cases}$$

It is easy to verify that $V_{S,1}^*(y|A_{N-1})$ and $V_{D,1}^*(y|a_{N-1})$ are both nondecreasing, convex and

$$\mathbf{E}_{A_{N-1}}V_{S,1}^*(y|A_{N-1}) > V_{D,1}^*(y|a_{N-1}).$$

Now consider

$$V_{k+1}(x_{N-(k+1)}, q_{N-(k+1)}|A_{N-(k+1)})$$
$$= (x_{N-(k+1)} - q_{N-(k+1)})(A_{N-(k+1)} - g)$$
$$+\alpha\mathbf{E}_{Z_{N-k},A_{N-k}}V_k^*(Z_{N-k}f(q_{N-(k+1)})|A_{N-k}) \quad (14)$$

and apply backward induction. If $V_{S,k}^*(y|A_{N-k})$ and $V_{D,k}^*(y|a_{N-k})$ are both nondecreasing and convex, then (14) is convex. Therefore, given the price $A_{N-(k+1)}$, the optimal rules are decided by the ratio

$$r_{S,k+1} = \frac{x_{N-(k+1)}(A_{N-(k+1)} - g)}{\alpha \mathbf{E}_{Z_{N-k},A_{N-k}} V^*_{S,k}(Z_{N-k} f(x_{N-(k+1)})|A_{N-k})}$$

and

$$r_{D,k+1} = \frac{x_{N-(k+1)}(A_{N-(k+1)} - g)}{\alpha \mathbf{E}_{Z_{N-k}} V^*_{D,k}(Z_{N-k} f(x_{N-(k+1)})|a_{N-k})}.$$

By applying the induction assumption that

$$\mathbf{E}_{A_{N-k}} V^*_{S,k}(y|A_{N-k}) > V^*_{D,k}(y|a_{N-k}),$$

we can see that $r_{D,k+1} < 1$ implies $r_{S,k+1} < 1$. Therefore the optimal rule given the price at the $[N-(k+1)]th$ period is more conservative if the future prices are random. That both $V^*_{S,k+1}(y|A_{N-(k+1)})$ and $V^*_{D,k+1}(y|a_{N-(k+1)})$ are convex and nondecreasing with

$$\mathbf{E}_{A_{N-(k+1)}} V^*_{S,k+1}(y|A_{N-(k+1)}) > V^*_{D,k+1}(y|a_{N-(k+1)})$$

is straightforward. This completes the induction and finishes our comparison.

In this section we have shown that under certain conditions on f and under the assumed cost-price structure, the optimal management policies are more conservative when unit selling prices are random. It may also be possible to compare the $\{S, s\}$-type optimal policies under stochastic prices with those under deterministic prices. In a strict sense, saying that a policy $\{S_1, s_1\}$ is more conservative than another policy $\{S_2, s_2\}$, one has to show either $S_1 > S_2$ with $s_1 = s_2$ or $S_1 \geq S_2$ with $s_1 > s_2$. When $S_1 < S_2 < s_2 < s_1$, the two policies are not comparable. In this article we will not compare the different $\{S, s\}$-type optimal policies. If K is not so significantly large, we might use zero to approximate it. Then, as we have shown, the optimal harvesting policy should be more conservative under price uncertainty.

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CHAPTER THREE

RESOURCE CONSERVATION, UNCERTAINTY AND PROFIT MAXIMIZATION

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3.1 Abstract

The issue of conservation is investigated under uncertainty of future benefits for both non-renewable and renewable resources. A framework is introduced to demonstrate that under uncertainty of future benefits the optimal management policy will be more conservative than the optimal policy obtained when the uncertainty is ignored. This paper extends early results by Arrow and Fisher (1974) for the case of irreversible management policies for non-renewable resources. In particular it is shown that resource conservation can be a consequence of profit maximization or timing of investments when future benefits are uncertain and the process of developing or harvesting is irreversible.

3.2 Introduction

Improved scientific understanding of economic valuations for the management and conservation of ecosystems has been identified as a major problem confronting contemporary society, e.g. see Daily *et al* (2000), Vitousek *et al* (1997) and Polasky *et al* (1993). The seminal paper of Arrow and Fisher (1974) shows that ignoring uncertainty of future benefits will lead to an over-investment in the development of non-renewable environmental resources. For renewable animal resources, Reed (1974) extends the deterministic model developed by Clark (1971). Reed introduces uncertainty of future benefits through random shocks in the growth of resource inventory, but with constant prices. Assuming constant marginal cost and zero set-up cost for each harvest event, the results in Reed (1974) yield the identical optimal management policy as that obtained by

Clark (1971) where uncertainty in the resource size is ignored. This is a striking phenomenon in view of the Arrow-Fisher paradigm for the non-renewables. In particular, when random fluctuations in resource inventory are the only source of uncertainty in future benefits, no strict resource conservation is implied. However, Burnes (2001) and Jia (2003) consider models that incorporate uncertainty in future benefits due to stochastic prices and resource inventory. It is shown that profit maximization under uncertainty in future benefits caused by price fluctuations implies conservation for both stochastic and deterministic inventories.

In section 3.3 we present a general framework, from which each of the aforementioned models can be derived as a special case. The benefits that may be obtained from developing or harvesting one unit of resource are assumed to be uncertain and exogenously determined and independent of resource inventory level. For either renewable or non-renewable resources, the optimal management policy that considers uncertainty of future benefits is compared with the optimal policy that ignores uncertainty. In the case that the development or harvest is profitable under current conditions, the optimal policy under uncertainty conserves more resources than the policy that ignores uncertainty, and delivers a higher expected future resource inventory. On the other hand, when the current harvest is not profitable, there are stock related circumstances under which the optimal policy that considers uncertainty may consume more in order to yield a higher expected inventory in the future. However, in any case the profit maximizing management policy under uncertainty always leads to greater expected future resource inventory level than the optimal policy that ignores uncertainty. It is in this sense that the optimal policy under uncertainty is said to be more conservative. The question of why profit maximization results in conservation is considered further in section 3.4, where an alternative view based on the timing of development or harvesting is also shown to imply conservation. In particular, an interpretation of the intrinsic value of conservation is presented that explains why conservation is optimal even though it may forgo some current profits or suffer some current loss while the future profits could be zero.

3.3 The Model and Main Results

Consider development or harvest of a certain resource over a two-period time horizon, period zero and period one. Assume that at the beginning of period zero there are x_0 units of resource available. Let d_0 be the amount of the resource developed or harvested at period zero. In the notation of Reed (1974), the resource population x_1 at the beginning of period one is given by

$$x_1 = Zf(x_0 - d_0),$$

where f is the reproduction function and Z represents a random shock on the resource population growth. The reproduction function f is non-negative with $f(0) = 0$ and is assumed to be concave and differentiable over an interval 0 to m. Z is a non-negative random variable with mean one and independent of the information of benefits and costs. In the notation of Arrow and Fisher (1974), for one unit of the resource, let

b_p = benefits from preserving in period zero;

b_d = benefits from developing/harvesting in period zero;

β_p = benefits, conditional on b_p and b_d, from preserving in period one;

β_d = benefits, conditional on b_p and b_d, from developing/harvesting in period one;

c_0 = investment/harvesting costs in period zero;

c_1 = investment/harvesting costs in period one.

Benefits and costs are assumed to be independent of resource inventory level and β_p, β_d and c_1 represent present values. The process of development or harvesting is irreversible in the sense that

(a) developing/harvesting will disturb the growth of resource population and cannot be undone;

(b) c_0 and c_1 are sunk costs that cannot be recovered.

For any given information b_p, b_d and c_0 at period zero, the Bellman equation of total expected profits $V(d_0)$, $0 \leq d_0 \leq x_0$, to be maximized for the optimal policy d_0^*, is

$$V(d_0) = (b_d - c_0)d_0 + (x_0 - d_0)b_p + \mathbf{E}[\beta_d d_0 + x_1(\beta_p + (\beta_d - c_1 - \beta_p)^+)]. \quad (1)$$

Eqn(1) can be written as

$$V(d_0) = x_0 b_p + [(b_d - c_0 - b_p) + \mathbf{E}\beta_d]d_0 + f(x_0 - d_0)[\mathbf{E}\beta_p + \mathbf{E}(\beta_d - c_1 - \beta_p)^+].$$

$V(d_0)$ is concave and differentiable. Therefore the optimal policy, d_0^*, is decided by the following:

$$d_0^* = \begin{cases} x_0 & \text{if } f'(x) \leq \frac{(b_d - c_0 - b_p) + \mathbf{E}\beta_d}{\mathbf{E}\beta_p + \mathbf{E}(\beta_d - c_1 - \beta_p)^+} \text{ for all } 0 \leq x \leq x_0; \\ 0 & \text{if } f'(x) \geq \frac{(b_d - c_0 - b_p) + \mathbf{E}\beta_d}{\mathbf{E}\beta_p + \mathbf{E}(\beta_d - c_1 - \beta_p)^+} \text{ for all } 0 \leq x \leq x_0; \\ d_0^* & \text{if } 0 < d_0^* < x_0 \text{ and } f'(x_0 - d_0^*) = \frac{(b_d - c_0 - b_p) + \mathbf{E}\beta_d}{\mathbf{E}\beta_p + \mathbf{E}(\beta_d - c_1 - \beta_p)^+}. \end{cases} \tag{2}$$

If, on the other hand, the uncertainty of the future benefits is not taken into account, then the resulting optimal policy, δ_0^*, is decided by

$$\delta_0^* = \begin{cases} x_0 & \text{if } f'(x) \leq \frac{(b_d - c_0 - b_p) + \mathbf{E}\beta_d}{\mathbf{E}\beta_p + (\mathbf{E}\beta_d - \mathbf{E}c_1 - \mathbf{E}\beta_p)^+} \text{ for all } 0 \leq x \leq x_0; \\ 0 & \text{if } f'(x) \geq \frac{(b_d - c_0 - b_p) + \mathbf{E}\beta_d}{\mathbf{E}\beta_p + (\mathbf{E}\beta_d - \mathbf{E}c_1 - \mathbf{E}\beta_p)^+} \text{ for all } 0 \leq x \leq x_0; \\ \delta_0^* & \text{if } 0 < \delta_0^* < x_0 \text{ and } f'(x_0 - \delta_0^*) = \frac{(b_d - c_0 - b_p) + \mathbf{E}\beta_d}{\mathbf{E}\beta_p + (\mathbf{E}\beta_d - \mathbf{E}c_1 - \mathbf{E}\beta_p)^+}. \end{cases} \tag{3}$$

Observe that

$$\mathbf{E}(\beta_d - c_1 - \beta_p)^+ \geq (\mathbf{E}\beta_d - \mathbf{E}c_1 - \mathbf{E}\beta_p)^+. \tag{4}$$

By concavity of f and Eqns(2–4), the following can be shown:

(I) given the information that profits from developing/harvesting at period zero $(b_d - c_0 - b_p) + \mathbf{E}\beta_d \geq 0$,

$$0 \leq d_0^* \leq \delta_0^* \leq x_0 \text{ and } f(x_0 - d_0^*) \geq f(x_0 - \delta_0^*);$$

(II) given the information that profits from developing/harvesting at period zero $(b_d - c_0 - b_p) + \mathbf{E}\beta_d < 0$,

$$0 \leq \delta_0^* \leq d_0^* < x_0 \text{ and } f(x_0 - d_0^*) \geq f(x_0 - \delta_0^*).$$

Comparing the optimal management policy d_0^* that considers uncertainty of future benefits with the optimal policy δ_0^* that ignores uncertainty, case (I) implies that the optimal policy under uncertainty develops/harvests less amount of resources and leads to greater expected future resource inventory level. Case (II) gives the situation that the optimal policy under uncertainty develops/harvests more in order to raise the expected future inventory level. This can only happen if the current resource inventory level x_0 is in the interval at which the reproduction function is decreasing and current development/harvest is not profitable. In particular, case (II) can not occur for the resource that has an increasing reproduction function. Both cases (I) and (II) show that the optimal management policy under uncertainty of future benefits leads to greater expected future resource inventory level than the optimal policy that ignores uncertainty, and is therefore more conservative.

To appreciate the generality of this result, first consider the case of non-renewable resources. In this case the reproduction function is $f(x) = x$ and the inventory evolves according to $x_1 = x_0 - d_0$. For normalized $x_0 = 1$, and given the information at period zero, Eqn(1) can be reduced to

$$V(d_0) = [(b_d - b_p - c_0) + \mathbf{E}\min(\beta_d - \beta_p, c_1)]d_0 + b_p + \mathbf{E}\max(\beta_d - c_1, \beta_p),$$

as obtained by Arrow and Fisher (1974).

Moreover generalizations of the Clark-Reed models considered in Burnes (2001) and Jia (2003) can also be viewed as special case of Eqn(1). Specifically, when the term $\beta_d d_0$ in Eqn(1) is not considered, by taking

$x_1 = Zf(x_0 - d_0)$;

$b_p = \beta_p = 0$, i.e., no benefits from preservation;

$b_d = A_0$, the random market price at period zero;

$\beta_d = A_1$, the random market price at period one;

$c_0 = c_1 = g$, the constant marginal harvesting cost,

Eqn(1) reduces to the model in Clark-Reed framework with stochastic prices:

$$V(d_0|A_0) = (A_0 - g)d_0 + \mathbf{E}[Zf(x_0 - d_0)(A_1 - g)^+].$$

Notice that the term $\beta_d d_0$ in Eqn(1) represents the carry over benefits, that is the benefits accrued in the second period brought by the investment in the first period. Whether this term exists or not is determined by the nature of the investment and does not affect the results. That is, if the development is a building from which rents are obtained in each period, a flow of benefits will ensue. Alternatively, if development is a resource harvest, a one time marketing opportunity results.

While the present framework includes the above special cases as examples, the overall generality accommodates a much broader range of sure and/or uncertain benefits within both renewable and non-renewable resource environments.

3.4 Intrinsic Value of Conservation

The optimal management policy may be interpreted as a consequence of timing the investment to maximize profit under the interaction of irreversibility and uncertainty, e.g. see Dixit and Pindyck (1994). Here we demonstrate that resource conservation, which influences the time of development or harvest, results in greater expected profits.

Consider the case that

$$d_0^* \neq \delta_0^*.$$

Under uncertainty of future benefits the amount $\delta_0^* - d_0^*$, possibly negative, will be preserved at period zero for any given value of $(b_d - c_0 - b_p) + \mathbf{E}\beta_d$ not equal to zero. The cost of conserving, which is the forgone profit or loss suffered at period zero in order to raise the expected future inventory level, is

$$[\delta_0^* - d_0^*][(b_d - c_0 - b_p) + \mathbf{E}\beta_d].$$

The resulting expected profit at period one from conserving is

$$[f(x_0 - d_0^*) - f(x_0 - \delta_0^*)][\mathbf{E}\beta_p + \mathbf{E}(\beta_d - c_1 - \beta_p)^+].$$

Define the *intrinsic value of conservation* to be the difference between the expected profit and the cost of conserving:

$$[f(x_0 - d_0^*) - f(x_0 - \delta_0^*)][\mathbf{E}\beta_p + \mathbf{E}(\beta_d - c_1 - \beta_p)^+] - [\delta_0^* - d_0^*][(b_d - c_0 - b_p) + \mathbf{E}\beta_d].$$

Then it is worth conserving if and only if the intrinsic value is positive, i.e., the following basic inequality holds:

$$\frac{f(x_0 - d_0^*) - f(x_0 - \delta_0^*)}{\delta_0^* - d_0^*} \frac{\mathbf{E}\beta_p + \mathbf{E}(\beta_d - c_1 - \beta_p)^+}{(b_d - c_0 - b_p) + \mathbf{E}\beta_d} > 1. \tag{5}$$

Eqn(5) holds for strictly concave and differentiable f since, using Eqn(2) and Eqn(3), when $(b_d - c_0 - b_p) + \mathbf{E}\beta_d > 0$,

$$\frac{f(x_0 - d_0^*) - f(x_0 - \delta_0^*)}{\delta_0^* - d_0^*} > f'(x_0 - d_0^*) \geq \frac{(b_d - c_0 - b_p) + \mathbf{E}\beta_d}{\mathbf{E}\beta_p + \mathbf{E}(\beta_d - c_1 - \beta_p)^+};$$

and when $(b_d - c_0 - b_p) + \mathbf{E}\beta_d < 0$,

$$\frac{f(x_0 - d_0^*) - f(x_0 - \delta_0^*)}{\delta_0^* - d_0^*} < f'(x_0 - d_0^*) = \frac{(b_d - c_0 - b_p) + \mathbf{E}\beta_d}{\mathbf{E}\beta_p + \mathbf{E}(\beta_d - c_1 - \beta_p)^+}.$$

It can be shown that Eqn(5) is also valid when f is concave. As implied by Eqn(5), the optimal policy is willing to forgo some current profits or suffer some current loss in order to have higher expected future resource inventory level and achieve greater expected future profits — resulting in resource conservation.

3.5 Conclusion

In this paper uncertainty in future benefits due to random fluctuations of market and/or resource inventory is considered. For renewable and non-renewable resources the relationship between resource conservation and expected profit maximization is illustrated. It is demonstrated that the goal of profit maximization leads to resource conservation. From the perspective of the timing of the investment, resource conservation is also implied by its positive intrinsic value, even though the future profits could be zero. In the framework, the market is independent of inventory level and random shocks on the growth of resource inventory play a less prominent role than uncertain market fluctuations. In the two-period time horizon framework considered in this paper the future is compressed into the second period. However, the framework can be generalized to a multi-period model. In the multi-period model a catastrophic loss can be defined by $\mathbf{P}(Z = 0) > 0$ in some period. While catastrophic loss is permitted in the two-period case developed here, the multi-period extension of the general conclusions of this paper requires the stronger hypothesis that $\mathbf{P}(Z > 0) = 1$.

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CHAPTER FOUR

VALUATION OF CONTINGENT CLAIMS ON ASSETS WITH CONVERSION COSTS OR SUBSIDIES

4.1 Abstract

Contingent claims on certain economic resources may involve underlying assets with conversion costs or subsidies, in which case the price of underlying risky assets may be modeled by a geometric Brownian motion plus a deterministic conversion cost or subsidy. Under the circumstances, we derive the equation for valuing contingent claims on assets with conversion costs or subsidies and show that a unique arbitrage-free hedging strategy exists.

4.2 Introduction

In classic Black-Scholes theory the valuation of a contingent claim on risky assets is through a replicating portfolio that contains the assets and the risk-free bond. It is assumed that risky assets do not involve conversion costs or subsidies, and the price of risky assets is lognormal. In many situations, however, the underlying risky assets exist in either a developed state or in an undeveloped state with conversion costs and/or subsidies, and the contingent claim contracts are on undeveloped assets, e.g. electric power options, contracts on natural resources. For the contingent claim contracts on assets with constant conversion costs, Thomann and Waymire (2003) developed a method of martingale localization in the framework of Harrison and Pliska (1981). A practical example of pricing a federal timber lease is given in Burnes *et al* (1999). With a constant conversion cost, it is shown that there exists a unique arbitrage-free hedging strategy.

In this paper we study the problem of valuing contingent claims on assets with deterministic conversion costs or subsidies. We derive the equation that determines the value of the contingent claims. From the point of view of a replicating portfolio, we show the existence of a unique arbitrage-free hedging strategy. To demonstrate this we first decompose the replicating portfolio under the framework of classic Black-Scholes theory. Then we generalize the decomposition to a portfolio that contains a certain amount of consumption or money infusion. It can be seen that if a replicating portfolio contains two deterministic parts (the risk-free bond and subsidy or conversion cost) with different accruing rates, then one may find arbitrage opportunities, and the arbitrage opportunities can be removed by including a certain amount of consumption or money infusion in the portfolio.

In Section 4.3 we introduce the models and give the main results. Discussions and conclusions are in Section 4.4. All the proofs are given in Section 4.5.

4.3 The Model and Main Results

We start with the standard Black-Scholes model. Assume that the risky asset price is given by the SDE

$$dS_t = \mu S_t dt + \sigma S_t dW_t \quad \text{with} \quad S(0) = S_0,$$

where $\mu > 0$ is the mean yield rate, $\sigma > 0$ the volatility, and $\{W_t : t \geq 0\}$ is a standard Brownian motion. The risk-free bond price satisfies

$$d\beta_t = r\beta_t dt \quad \text{with} \quad \beta_0 = 1,$$

where r is the interest rate. Then a contingent claim with payoff $V(S_T)$ at expiry T can be replicated by a self-financing portfolio

$$\pi_t = \pi(t, S_t) = \phi_t S_t + \psi_t \beta_t, \tag{1}$$

and $\pi(t, S_t)$ satisfies the Black-Scholes PDE

$$\frac{\partial \pi}{\partial t} + rS\frac{\partial \pi}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \pi}{\partial S^2} - r\pi = 0 \tag{2}$$

with $\pi_T = V(S_T)$. Write $\mathcal{Q}$ as the probability measure induced by the process

$$\tilde{W}_t = \frac{\mu - r}{\sigma} t + W_t, \quad 0 \leq t \leq T,$$

where $\tilde{W}_t$ is the standard Brownian motion under $\mathcal{Q}$. $\mathcal{Q}$ can be obtained by applying Girsanov theory, e.g. see Steele (2001), Karatzas and Shreve (1991). Then $\frac{S_t}{\beta_t}$ is a $\mathcal{Q}$-martingale. Applying the standard semigroup theory, e.g. see Oksendal (1995) and Friedman (1975), one can get the solution to Eqn(2):

$$\pi(t, s) = e^{-r(T-t)} \mathbf{E}_{\mathcal{Q}}[V(S_T) | S_t = s]$$

since Eqn(2) is driven by the process

$$S_0 \exp(rt - \frac{1}{2}\sigma^2 t + \sigma \tilde{W}_t). \tag{3}$$

It can be seen that $\frac{\pi_t}{\beta_t}$ is also a $\mathcal{Q}$-martingale. This implies the existence of a unique non-arbitrage hedging strategy (see Harrison and Pliska (1981)).

Now we consider contingent claims on a risky asset with a deterministic conversion cost or subsidy. In this case a replicating portfolio, which contains ϕ_t units of the risky asset and ψ_t units of the risk-free bond, is of the form

$$\pi_t = \phi_t (S_t + c_t) + \psi_t \beta_t, \tag{4}$$

where $S_t = S_0 \exp(\mu t - \frac{1}{2}\sigma^2 t + \sigma W_t)$ is the market price of the asset, $c_t < 0$ is the conversion cost ($c_t > 0$ when it represents the subsidy), and β_t denotes the risk-free bond. At expiry T, the value of the portfolio π_T matches the payoff $V(S_T)$ of the contingent claim. Under the self-financing condition,

$$\begin{aligned} d\pi_t &= \phi_t(dS_t + dc_t) + \psi_t r\beta_t dt \\ &= \phi_t \sigma S_t dW_t + [\phi_t \mu S_t + \psi_t r\beta_t + \phi_t \dot{c}_t]dt. \end{aligned} \tag{5}$$

Applying Itô's lemma, we also have

$$\begin{aligned} d\pi_t &= \frac{\partial \pi}{\partial t}dt + \frac{\partial \pi}{\partial S}dS_t + \frac{1}{2}\frac{\partial^2 \pi}{\partial S^2}(dS_t)^2 \\ &= \left[\frac{\partial \pi}{\partial t} + \mu S\frac{\partial \pi}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \pi}{\partial S^2}\right] dt + \sigma S\frac{\partial \pi}{\partial S}dW_t. \end{aligned} \tag{6}$$

Combining Eqn(5) and Eqn(6), we can obtain

$$\frac{\partial \pi}{\partial t} + rS\frac{\partial \pi}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \pi}{\partial S^2} - r\pi = \frac{\partial \pi}{\partial S}(\dot{c}_t - rc_t). \tag{7}$$

Eqn (7) together with $\pi_T = V(S_T)$ determines the value of the contingent claim.

We study the arbitrage-free hedging strategy through the replicating portfolio. Comparing portfolio (4) with portfolio (1), we see an extra term $\phi_t c_t$. Observe that the portfolio (1) can be written as

$$\pi_t = \phi_t S_t + [\psi_t - g(0)]\beta_t + g(t) + e^{rt}\int_0^t e^{-rs}[rg(s) - \dot{g}(s)]\, ds$$

since

$$g(t) + e^{rt}\int_0^t e^{-rs}[rg(s) - \dot{g}(s)]\, ds = g(0)e^{rt}.$$

Let $\tilde{\phi_t} = \phi_t$, $\tilde{\psi_t} = \psi_t - g(0)$, and $G(t) = \int_0^t e^{-rs}[rg(s) - \dot{g}(s)]\, ds$. We have

$$\pi_t = \tilde{\phi_t} S_t + \tilde{\psi_t}\beta_t + g(t) + e^{rt}G(t). \tag{1'}$$

Now suppose that we have a portfolio

$$\tilde{\pi}_t = \tilde{\pi}(t, S_t) = \tilde{\phi}_t S_t + \tilde{\psi}_t \beta_t + g(t), \tag{8}$$

which includes one unit of another deterministic financial instrument with value $g(t)$ at time t. Compare the portfolios (1') and (8). If we want to use portfolio (8) to hedge a contingent claim with payoff $V(S_T)$, there is a "hidden" consumption or money infusion $e^{rt}G(t)$. Define

$$G(t) = \int_0^t e^{-rs}[rg(s) - \dot{g}(s)]\, ds$$

to be the present value of consumption or money infusion up to time t. We have the following proposition.

Proposition 1 *To hedge the contingent claim with payoff $V(S_T)$, the following self-financing portfolios are equivalent and both satisfy the Black-Scholes equation:*

(i) $\pi_t = \phi_t S_t + \psi_t \beta_t$;

(ii) $\tilde{\pi}_t = \tilde{\phi}_t S_t + \tilde{\psi}_t \beta_t + g(t) + e^{rt}G(t)$,

where $G(t) = \int_0^t e^{-rs}[rg(s) - \dot{g}(s)]\, ds$ *is the present value of consumption or money infusion up to time t. Equivalence means that the arbitrage-free portfolios have the same uniquely determined delta, i.e.* $\phi_t = \tilde{\phi}_t$.

Notice that portfolio (8) coincides with portfolio (4) if $g(t) = \tilde{\phi}_t c_t$. For $g(t) = \tilde{\phi}_t c_t$, we have the following proposition.

Proposition 2 *To hedge the contingent claim with payoff $V(S_T)$, the following* *self-financing portfolios are equivalent and both satisfy the Black-Scholes equation:*

(i) $\pi_t = \phi_t S_t + \psi_t \beta_t$;

(ii) $\tilde{\pi}_t = \tilde{\phi}_t S_t + \tilde{\psi}_t \beta_t + \tilde{\phi}_t c_t + e^{rt} G(t)$,

where $G(t) = \int_0^t e^{-rs} \tilde{\phi}(s)[rc(s) - \dot{c}(s)]\, ds$ *is the present value of consumption or money infusion up to time* t. *The arbitrage-free replicating portfolios are equivalent in the sense that they have the same uniquely determined delta, i.e.* $\phi_t = \tilde{\phi}_t$.

Proposition 2 provides a rather straightforward way to deal with contingent claims on assets with deterministic conversion costs or subsidies. Since

$$\tilde{\pi}_t = \tilde{\phi}_t S_t + \tilde{\psi}_t \beta_t + \tilde{\phi}_t c_t + e^{rt} G(t)$$

satisfies the Black-Scholes equation (2), $e^{-rt}\tilde{\pi}_t$ is a martingale under the probability measure $\mathcal{Q}$ defined by the diffusion (3). Then a unique arbitrage-free hedging strategy can be found according to the theory of Harrison and Pliska (1981). As an example, we apply the proposition to the case of constant conversion cost, i.e. $c(t) = -c$, $c > 0$, and

$$\pi_t = \phi_t (S_t - c) + \psi_t \beta_t.$$

The present value of consumption or money infusion is

$$G(t) = -rc \int_0^t e^{-rs} \phi(s)\, ds.$$

Thus, by Proposition 2,

$$e^{-rt}\tilde{\pi}_t = e^{-rt}\pi_t - rc \int_0^t e^{-rs} \phi(s)\, ds$$

is a martingale under the equivalent martingale measure Q. Hence there exists a unique non-arbitrage hedging strategy. The result coincides with that in Thomann and Waymire (2003).

A general mathematical result can be obtained by studying the following replicating portfolio:

$$\hat{\pi}_t = \hat{\pi}(t, S_t) = \hat{\phi}_t S_t + \hat{\psi}_t \beta_t + \xi_t g(t),$$

where $g(t)$ is deterministic and ξ_t is taken to be a function of $\hat{\phi}_t$ (e.g. $\xi_t = (\hat{\phi}_t)^2$) or a function of the history of $\hat{\phi}_t$ (e.g. $\xi_t = \int_0^t \hat{\phi}_u \, du$). Under the self-financing condition, we have $\hat{\phi} = \frac{\partial \hat{\pi}}{\partial S}$ and

$$\frac{\partial \hat{\pi}}{\partial t} + rS\frac{\partial \hat{\pi}}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \hat{\pi}}{\partial S^2} - r\hat{\pi} = \xi_t(\dot{g}(t) - rg(t)). \tag{9}$$

It can be shown that the following portfolio satisfies the Black-Scholes equation (2):

$$\hat{\pi}_t = \hat{\phi}_t S_t + \hat{\psi}_t \beta_t + \xi_t g(t) + e^{rt} G(t),$$

where the present value of consumption or money infusion up to time t is given by

$$G(t) = \int_0^t e^{-rs} \xi(s)[rg(s) - \dot{g}(s)] \, ds.$$

The above results are summarized in the following proposition, which may be used to obtain the solution to the non-linear equation (9).

Proposition 3 *To hedge the contingent claim with payoff $V(S_T)$, the following self-financing portfolios are equivalent and both satisfy the Black-Scholes equation:*

(i) $\pi_t = \phi_t S_t + \psi_t \beta_t$;

(ii) $\hat{\pi}_t = \hat{\phi}_t S_t + \hat{\psi}_t \beta_t + \xi_t g(t) + e^{rt} G(t)$,

where ξ_t *is a function of* $\hat{\phi}_t$ *or a function of the history of* $\hat{\phi}_t$*, and* $G(t) = \int_0^t e^{-rs} \xi(s)[rg(s) - \dot{g}(s)]\, ds$ *is the present value of consumption or money infusion up to time t. The arbitrage-free replicating portfolios are equivalent in the sense that they have the same uniquely determined delta, i.e.* $\phi_t = \hat{\phi}_t$.

4.4 Conclusion

In this paper we study the valuation of contingent claims on assets with deterministic conversion costs or subsidies. When an underlying risky asset involves a deterministic conversion cost or subsidy $c(t)$, the value of the contingent claim is given by the equation

$$\frac{\partial \pi}{\partial t} + rS\frac{\partial \pi}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \pi}{\partial S^2} - r\pi = \frac{\partial \pi}{\partial S}(\dot{c}(t) - rc(t))$$

with π_T matching $V(S_T)$, the payoff of the contingent claim at expiry T. It is shown that the replicating portfolio

$$\tilde{\pi}_t = \tilde{\phi}_t S_t + \tilde{\psi}_t \beta_t + \tilde{\phi}_t c(t) + e^{rt} G(t),$$

where $G(t) = \int_0^t e^{-rs} \tilde{\phi}(s)[rc(s) - \dot{c}(s)]\, ds$ is the present value of consumption or money infusion up to time t, determines a unique arbitrage-free hedging strategy. Notice that if $c(t)$ and β_t accrue at the same rate, i.e. $rc(t) - \dot{c}(t) = 0$, then the consumption or money infusion in the portfolio is zero. From this we can see that if a replicating portfolio contains two or more deterministic financial instruments with different growth rates then arbitrage opportunities appear, and such arbitrage opportunities can be removed by including consumption or money infusion $e^{rt}G(t)$ in the portfolio.

The method we used in this paper essentially provides a rather general way to value contingent claims and find arbitrage-free hedging strategies. If the price of an underlying risky asset can be modeled by

$$S_t = S_0 \exp(\mu t - \frac{1}{2}\sigma^2 t + \sigma W_t) + g(t), \tag{10}$$

then we can value the contingent claim on that asset and find the arbitrage-free hedging strategy. There is more freedom in modeling price processes statistically by using (10) instead of a simple geometric Brownian motion. Also, the replicating portfolio

$$\hat{\pi}_t = \hat{\phi}_t S_t + \hat{\psi}_t \beta_t + \xi_t g(t) + e^{rt}G(t)$$

allows us to deal with some complicated situations.

4.5 Proof of the Propositions

The proof of Proposition 1 is straightforward. Proposition 2 can be obtained from Proposition 3 by taking $\xi_t = \hat{\phi}_t$. We only need to prove Proposition 3.

Proof of Proposition 3:

We start with the replicating portfolio

$$\hat{\pi}_t = \hat{\pi}(t, S_t) = \hat{\phi}_t S_t + \hat{\psi}_t \beta_t + \xi_t g(t) + H(t),$$

with $\hat{\pi}_T = V(S_T)$. Assuming the self-financing condition,

$$\begin{aligned} d\hat{\pi}_t &= \hat{\phi}_t dS_t + \hat{\psi}_t r\beta_t dt + \xi_t \dot{g}(t)dt + \dot{H}(t)dt \\ &= \hat{\phi}_t \sigma S_t dW_t + [\hat{\phi}_t \mu S_t + \hat{\psi}_t r\beta_t + \xi_t \dot{g}(t) + \dot{H}(t)]dt. \end{aligned} \tag{11}$$

By Itô's lemma,

$$\begin{aligned} d\hat{\pi}_t &= \frac{\partial \hat{\pi}}{\partial t}dt + \frac{\partial \hat{\pi}}{\partial S}dS_t + \frac{1}{2}\frac{\partial^2 \hat{\pi}}{\partial S^2}(dS_t)^2 \\ &= \left[\frac{\partial \hat{\pi}}{\partial t} + \mu S_t \frac{\partial \hat{\pi}}{\partial S} + \sigma^2 S_t^2 \frac{1}{2}\frac{\partial^2 \hat{\pi}}{\partial S^2}\right] dt + \sigma S_t \frac{\partial \hat{\pi}}{\partial S} dW_t. \end{aligned} \tag{12}$$

Equate Eqn(11) and Eqn(12). We obtain

$$\hat{\phi}_t = \frac{\partial \hat{\pi}}{\partial S},$$

and

$$\frac{\partial \hat{\pi}}{\partial t} + rS\frac{\partial \hat{\pi}}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \hat{\pi}}{\partial S^2} - r\hat{\pi} = \xi_t \dot{g}(t) - r\xi_t g(t) + \dot{H}(t) - rH(t). \tag{13}$$

Eqn(13) coincides with the standard Black-Scholes equation if and only if

$$H(t) = e^{rt}G(t),$$

where

$$G(t) = \int_0^t e^{-rs}\xi(s)[rg(s) - \dot{g}(s)]\, ds$$

for $G(0) = 0$. Thus both replicating portfolios

$$\hat{\pi}_t = \hat{\phi}_t S_t + \hat{\psi}_t \beta_t + \xi_t g(t) + e^{rt}G(t)$$

and

$$\pi_t = \pi(t, S_t) = \phi_t S_t + \psi_t \beta_t$$

satisfy the standard Black-Scholes equation and have the same solution

$$\hat{\pi}(t,s) = e^{-r(T-t)}\mathbf{E}_Q[V(S_T)|S_t = s] = \pi(t,s).$$

Now we show that $\hat{\phi}_t = \phi_t$ for all $t \in [0,T]$. Notice that both $\frac{\hat{\pi}}{\beta_t}$ and $\frac{\pi}{\beta_t}$ are Q-martingales. Therefore by the martingale representation theorem there exists a unique α_t such that

$$d(\frac{\hat{\pi}}{\beta_t}) = \alpha_t dW_t = d(\frac{\pi}{\beta_t}),$$

which yields

$$\alpha_t \beta_t dW_t = d\pi - r\pi dt = \phi_t dS_t + \psi_t r \beta_t dt - r\pi dt \tag{14}$$

and

$$\alpha_t \beta_t dW_t = d\hat{\pi} - r\hat{\pi} dt = \hat{\phi}_t dS_t + \hat{\psi}_t r \beta_t dt + \xi_t \dot{g}(t) dt + d(e^{rt} G(t)) - r\hat{\pi} dt. \tag{15}$$

Since $\frac{S_t}{\beta_t}$ is also a Q-martingale, we have a unique representation $\gamma_t dW_t = d(\frac{S_t}{\beta_t})$ and therefore

$$dS_t = \beta_t \gamma_t dW_t + rS_t dt. \tag{16}$$

Combine Eqns(14–16). We have

$$\hat{\phi}_t = \phi_t = \frac{\alpha_t}{\gamma_t}.$$

Once $\phi_t = \frac{\alpha_t}{\gamma_t}$ is uniquely determined then so is ψ_t. Also, for uniquely determined $\hat{\phi}_t = \frac{\alpha_t}{\gamma_t}$, ξ_t is a function of $\hat{\phi}_t$ or a function of the history of $\hat{\phi}_t$. Therefore, $\hat{\phi}_t$, ξ_t and $\hat{\psi}_t$ are all uniquely determined. □

4.6 References

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CHAPTER FIVE
CONCLUSION

In the three studies included in this dissertation we have developed tools, methods and frameworks for economic resource management and valuation of contingent claims. We study optimal management strategies for profit maximization and provide insight into uncertainty, conservation, timing of investment, and their connections. We develop a method for valuing contingent claims on assets with deterministic conversion costs or subsidies.

In Chapter Two we have a rather specific cost structure, from which we derive Markovian $\{S, s\}$-type optimal management policies for resource control under uncertainty. A comparison is made under some conditions, which shows that under price uncertainty the optimal policies are more conservative than in the deterministic case.

Such conservation is investigated in Chapter Three in a framework with a general cost and benefit structure. It is shown that under market uncertainty, conservation is a consequence of profit maximization. Moreover, conservation also reflects the strategy of timing of investment. Under market uncertainty, there is an intrinsic value to conserve. The framework introduced in Chapter Three has general results for both renewable and non-renewable resources. It extends the early results of Arrow and Fisher (1974) for non-renewable environmental resources.

Chapter Four treats economic resources as risky assets in the market subject to uncertainty. In many situations the underlying risky assets involve conversion costs or subsidies, and the actual price of the risky assets is modeled by a geometric Brownian motion plus a deterministic conversion cost or subsidy. The equation that determines the value of the contingent claims is derived. It is shown that a unique arbitrage-free hedging strategy exists.

The method developed in Chapter Four essentially provides a rather general way to value contingent claims and find arbitrage-free hedging strategies. If the price of an underlying risky asset can be modeled by

$$S_t = S_0 \exp(\mu t - \frac{1}{2}\sigma^2 t + \sigma W_t) + g(t),$$

then we can value the contingent claim on that asset and find the arbitrage-free hedging strategy. Also, a general replicating portfolio which contains a certain amount of consumption or money infusion may allow us to deal with some complicated situations.

Notice that in Chapters Two and Three the conditions on prices (benefits) are very general. In Chapter Four, however, a strong condition is imposed on the price of underlying risky assets. We assume that the price of the underlying risky asset follows a geometric Brownian motion, which is one of the major assumptions in the Black-Scholes model (see Black and Scholes (1973)). One of the main problems with the Black-Scholes model is that the log returns of most financial assets do not follow a Normal distribution, e.g. see Schoutens (2003). A general way to solve this problem is to model price behaviour by more sophisticated stochastic processes than geometric Brownian motion. Nualart and Schoutens (2001) use geometric Lévy processes to model underlying assets and have an integro-differential option pricing equation. A probabilistic method of computing and analyzing this type of integro-differential equations is introduced by Bhattacharya el al (2003).

According to the development by Harrison and Pliska (1981), non-arbitrage hedging strategy rests on the existence of an equivalent martingale measure. There exists a unique equivalent martingale measure in the case of lognormal underlying asset prices (e.g. as in Chapter Four). However, an equivalent martingale measure is generally not unique when prices of underlying assets are driven by Lévy processes, in which case we have an incomplete financial market. Under the circumstances, an equivalent martingale measure may be selected from a variety of perspectives, e.g. see Gerber and Shiu (1996). Deep study of incomplete market models is a huge research area and we will not discuss the details in this dissertation.

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