THE EIGHT GEOMETRIES OF THE GEOMETRIZATION CONJECTJURE

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1. Introduction

The uniformization theorem tells us that every compact surface without boundary, or two-manifold, admits a geometric structure, and further, one of only three possible geometric structures. In 1982 William Thurston presented the geometrization conjecture, which suggested that all Riemannian three-manifolds can be classified similarly. However, in the case of three-manifolds a classification becomes more complicated. Thurston conjectured that a three-manifold can be uniquely decomposed via a two level decomposition into pieces such that each piece admits one of eight possible geometric structures, [6].

In this paper we explore the eight geometries of Thurston's geometrization conjecture. We begin in Section 2 with several preliminary definitions and theorems. We briefly discuss group actions, covering space topology, fiber bundles, and Seifert fiber spaces. Most importantly, we make precise the notion of a geometric structure.

In Section 3 we discuss the two dimensional geometries to provide a motivation for the three dimensional geometries. In this section we present a brief proof of the uniformization theorem, and explicitly construct several surfaces with a particular geometric structure. Having done this, we list each of the eight geometries of Thurston's geometrization conjecture, including the metric associated with each geometry. We then explore in depth the geometries of $S^2 \times \mathbb{R}$ and S^3.

With $S^2 \times \mathbb{R}$ we explicitly construct each of the seven three-manifolds with a geometric structure modeled on $S^2 \times \mathbb{R}$. We show that each of these seven manifolds is a fiber bundle. This geometry is simple, and thus makes a good introduction.

With S^3 we show that any three-manifold with a geometric structure modeled on S^3 is a Seifert fiber space. This is wonderful, as the Seifert fiber spaces are well understood and classified.

2. Preliminaries

We first specify which manifolds we are interested in. We wish to consider only Riemannian manifolds. As Riemannian manifolds are the objects of interest in this paper, we will assume that any manifold is Riemannian, unless otherwise stated.

We will discuss isometries throughout this paper, so we give some definitions here.

Definition 2.0.1 Let (M, g) and (M', g') be Riemannian manifolds. An **isometry** is a diffeomorphism $f : M \to M'$ such that $g = f^* g'$ where $f^* g'$ denotes the pullback of the metric tensor g' by f. If f is a local diffeomorphism then f is a **local isometry**. We say that M and M' are **isometric**, $M \simeq M'$, if there exists such a isometry between them. The set of isometries from M to itself forms a group under composition, and is denoted $\mathrm{Isom}(M)$.

We must discuss what we mean by a geometric structure in order to know what interesting aspects of three-manifolds to explore. We begin with group actions.

2.1. **Group Actions.**

We begin by defining a group action.

Definition 2.1.1 Let G be a group and M a set. A **left action** of G on M is a map from $G \times M$ to M, written $(g, m) \to g \cdot m$ such that $g_1 \cdot (g_2 \cdot m) = (g_1 g_2) \cdot m$ for all $g_1, g_2 \in G$ and $m \in M$, and $e \cdot m = m$ for e the identity element of G and all $m \in M$. A **right group action** can be defined similarly.

We can then define the **orbit** of an element $m \in M$.

Definition 2.1.2 Given a set M with a left action of a group G and $m \in M$, the **orbit** of m under the action of G is the set $orb(m) = \{g \cdot m : g \in G\}$, that is, the set of all images of m under the action of elements of the group G.

We can define the relation $q \backsim m$ if $q \in orb(m)$, and this relation is an equivalence relation. Thus the orbits of a group action partition the set it acts on into equivalence classes. We denote this the set of equivalence classes under the group action by M/G. We are interested in the case when M is a manifold and G is a group of isometries of M. Let M be a manifold and Γ the isometry group of M. We define a group action of Γ on M by $\gamma \cdot q = \gamma(q)$ where $\gamma \in \Gamma$ and $q \in M$, [4].

2.2. **Covering Spaces, Deck Groups, and Topological Tid Bits.**

Now we recall a few definitions from topology.

Definition 2.2.1 Let $p : E \to B$ be a continuous, surjective map between topological spaces E and B. An open set U of B is **evenly covered** by p if $p^{-1}(U) = \bigcup V_\alpha$ where the

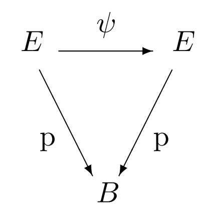

FIGURE 1. The diagram commutes

V_α are disjoint open sets in E such that for each α, $p|_{V_\alpha}$ is a homeomorphism onto U. If every point b of B has a neighborhood U that is evenly covered by p, then p is a **covering map** and E is a **covering space** of B. Given a covering map $p : E \to B$ the group of all automorphisms of the space E is called the **deck group** or **covering group** and is denoted $\mathcal{C}(E, p, B)$. That is, $\mathcal{C}(E, p, B)$ is the group of homeomorphisms $\psi : E \to E$ such that the digram in Figure 1 commutes.

When the spaces and covering map are understood, we will refer to the deck group simply as $\mathcal{C}$.

Now we review some basic covering space theory.

Definition 2.2.2 Let X and Y be topological spaces, and $h : X \to Y$ such that $h(x_0) = y_0$. Then we define the map $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by $h_*([f]) = [h \circ f]$, where $[f] \in \pi_1(X, x_0)$ is the equivalence class of the loop f in the space X centered at x_0. Let $p : E \to B$ be a covering transformation with $p(e_0) = b_0$. Define the group $H_0 = p_*(\pi_1(E, e_0))$. Note that H_0 is a subgroup of $\pi_1(B, b_0)$. Then p is a **regular covering map** if H_0 is a normal subgroup of $\pi_1(B, b_0)$.

Theorem 2.2.1 [7] If $p : E \to B$ is a regular covering map and $\mathcal{C}$ is its deck group, then there exists a homeomorphism $k : E/\mathcal{C} \to B$ such that $p = k \circ \pi$ where π is the projection map. That is, $E/\mathcal{C}$ is homeomorphic to B.

2.3. **Geometric Structures.** We begin with the definition of a geometric structure for two-manifolds.

Definition 2.3.1 Let X be one of E^2, S^2, or H^2, where E^2 is Euclidean two-space, S^2 is the two-sphere, and H^2 is hyperbolic two-space. Let Γ be a subgroup of Isom(X). If F is a two-manifold such that $F \simeq X/\Gamma$ and the projection $X \to X/\Gamma$ is a covering map, we say that F has **a geometric structure modeled on** X.

We will use Definition 2.3.1 to generalize the notion of geometric structures to three dimensions. However, there exists another definition of a geometric structure in any dimension.

Definition 2.3.2 A manifold M^n admits a geometric structure if it can be equipped with a complete, locally homogeneous metric.

In case the terms **homogeneous** or **complete** are unfamiliar, we provide the definitions below.

Definition 2.3.3 A metric on a manifold M is **locally homogeneous** if for all points x and y in M there exist neighborhoods U and V of x and y and an isometry $f : U \to V$. A metric on a manifold M is **homogeneous** if for all points $x, y \in M$ there exists an isometry of M sending x to y. A Riemannian manifold M is **complete** if it is complete as a metric space, that is, every Cauchy sequence in M converges.

Now we show that if the manifold in question is Riemannian, then Definition 2.3.2 implies Definition 2.3.1.

Lemma 2.3.1 If M^n is a Riemannian manifold which admits a geometric structure (Def. 2.3.2) and X is its universal covering space, then there exists a subgroup Γ of $\text{Isom}(X)$ such that M is isometric to X/Γ. Specifically, Γ is the deck group of X.

Proof. Consider a manifold M and a covering space for this manifold, $\tilde{M}$. Any covering space of M inherits a natural metric, the pull-back metric, such that the projection of $\tilde{M}$ onto M is a local isometry. Thus, if M admits a geometric structure in the sense of Def. 2.3.2, M has a complete, locally homogeneous metric, so the metric inherited by the universal covering space X of M is also complete and locally homogeneous.

We now use the fact that a locally homogeneous metric on any simply connected manifold is homogeneous[1]. The universal covering space is always simply connected, so the metric X inherits from M is in fact homogeneous, [8].

Let Γ be the isometry group of X. From Definition 2.3.3 we know that for any two points $x, y \in X$ there is a $\gamma \in \Gamma$ such that $\gamma(x) = y$. Viewing the action of γ on X as a group action, we see that every point of X is in the same orbit. We say that such a group action is **transitive**. That is, if a manifold M admits a geometric structure in the sense of Definition 2.3.2, then its universal covering space X has a transitive isometry group.

If $\psi \in \mathcal{C}$, then ψ must also be locally an isometry. However, because ψ is a diffeomorphism, it must be globally an isometry as well. Thus, the deck group of X is a subgroup of the isometry group of X, $\mathcal{C} < \text{Isom}(X)$.

[1] Scott in [8] cites I. M. SINGER, 'Infinitesimally homogeneous spaces', Comm. Pure Appl. Math., 13 (1960), 685-697 for this theorem

If $p : X \to M$ where X is a universal covering space, the group H_0 from Definition 2.2.2 is trivial, and thus normal, making p a regular covering map. Then Theorem 2.2.1 applies and we conclude that M is isometric to $X/\mathcal{C}$. Thus, given a Riemannian manifold M which admits a geometric structure in the sense of Definiyion 2.3.2, we can write M as the quotient of its universal covering space X by a subgroup of Isom(X). □

Thus we may make the following definition, without ambiguity.

Definition 2.3.4 A **geometry** is a simply connected, complete, homogeneous Riemannian manifold X together with its isometry group. A manifold M has a **geometric structure modeled on** X if $M \simeq X/\Gamma$ where Γ is a subgroup of the isometry group of X and $\simeq$ indicates that M is isometric to X/Γ.

We can also add some technical conditions to this definition which we will not be called upon to use in this paper, but which are worth mentioning for the sake of completeness.

Definition 2.3.5 Two geometries (X, G) and (X', G') are **equivalent** if G is isomorphic to G' and there exists an equivariant map $\phi : X \to X'$. That is, $\phi(g \cdot x) = g' \cdot \phi(x)$ where g' is the isomorphic image of g in G'.

Definition 2.3.6 A geometry (X, G) is **maximal** if there is no geometry (X, G') with $G \subsetneqq G'$.

These two definitions are needed to identify all possible geometric structures in a given dimension, [1]. Thurston has identified the eight geometric structures in dimension three.

Theorem 2.3.1 (Thurston)[8] Any maximal, simply connected, three-dimensional geometry which admits a compact quotient is equivalent to one of the the geometries $(X, \text{Isom}(X))$ where X is one of E^3, H^3, S^3, $S^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$, $\widetilde{SL_2\mathbb{R}}$, Nil, or Sol.

In order to better understand the geometries of dimension three, we are interested in identifying the possible universal covering spaces, X and their isometry groups. Thurston has already done this for us in Theorem 2.3.1. In this paper we will not prove Thurston's theorem; a proof can be found in [8]. Having identified the eight geometric structures of dimension three, we are interested in exactly which manifolds possess a geometric structure modeled on one of these eight geometric structures. We begin by studying what subgroups of the isometry group will generate a Riemannian manifold with a geometry modeled on X. The answer is that if a subgroup Γ of Isom(X) acts freely and properly discontinuously then X/Γ is a Riemannian manifold. We will call such groups **discrete**. We now make this more precise:

Definition 2.3.7 A group G acts **properly discontinuously** on a space X if for any compact subset C of X the set $\{g \in G : gC \cap C \neq \emptyset\}$ is finite.

If G acts properly discontinuously on X and p is a point in X, then the stabilizer of p, stab(p), must be finite. This is because if C is a compact set in X containing p, and the stabilizer of p is not finite, then there are infinitely many $g \in G$ such that $p \in C$ and $p \in gC$. Then G would not act properly discontinuously on X.

Definition 2.3.8 Let G be a group acting on a space X. If stab(p) is trivial for all $p \in X$ then G **acts freely** on X.

The following theorem provides necessary conditions for the quotient X/Γ to be a smooth manifold, where Γ is a subgroup of Isom(X).

Theorem 2.3.2 [4] Suppose X is a connected smooth manifold and Γ is a finite or countably infinite group with the discrete topology acting smoothly, freely, and properly discontinuously on X. Then the quotient space X/Γ is a topological manifold and has a unique smooth structure such that $\pi : X \to X/\Gamma$ is a smooth normal covering map.

Now we explain exactly what is meant by a discrete group, and why it is called discrete. If G acts freely and properly discontinuously on a manifold X then the map $\pi : X \to X/G$ is a covering map with covering group G. Let $C(X)$ denote the space of continuous functions $f : X \to X$ considered with the compact open topology. If G acts properly discontinuously on a space X, then G is a a discrete subset of $C(X)$. However, the converse requires that X be a complete Riemannian manifold and G be a group of isometries of X. That is, if X is a complete Riemannian manifold, and G is a group of isometries of X that is a discrete subset of $C(X)$, then G acts properly discontinuously on X. So, when the manifold, X, under consideration is a complete Riemannian manifold, a subgroup of the isometry group of X is discrete if and only if it acts properly discontinuously. Because all the manifolds considered in this paper will be complete Riemannian manifolds, we will refer to such a group simply as **discrete**. We have the following definition.

Definition 2.3.9 [8] A **discrete group** G is a subgroup of the isometry group of a manifold X such that G acts freely and properly discontinuously on X.

Our goal in describing a geometry is to identify the universal covering space, X, and its isometry group, and then to identify some of the discrete subgroups of that isometry group. Many of the manifolds we encounter while describing X/Γ for some discrete group Γ are fiber spaces. Thus, it is useful to know something about fiber bundles and fiber spaces.

2.4. **Fiber Bundles and Seifert Fiber Spaces.** Fiber bundles are an important topic with many more applications than will be explored here. We will use the definitions given in this section in Section 5 to show that every manifold with a geometric structure modeled on $S^2 \times \mathbb{R}$ can be described as a fiber bundle. Then, in Section 6 we will show that every manifold with a geometric structure modeled on S^3 is a Seifert fiber space. Here we will discuss fiber bundles and Seifert fiber spaces in preparation for these sections.

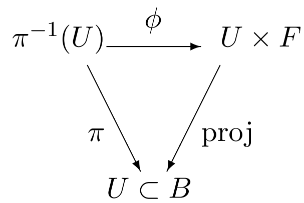

FIGURE 2. The diagram commutes

We begin with fiber bundles.

Definition 2.4.1 Let E, B, and F be topological spaces and $\pi : E \to B$ a continuous surjection. If π meets the triviality condition below, then we say that (E, B, π, F) is a fiber bundle. We call E the total space, B the base space, π the projection map, and F the fiber. The required triviality condition is as follows: for all $x \in B$ there exists an open neighborhood U of x and a homeomorphism $\phi : \pi^{-1}(U) \to U \times F$ such that $proj \circ \phi = \pi|_{\pi^{-1}(U)}$. That is, the diagram in Figure 2 commutes.

The idea is that the space E is "locally like" $B \times F$. When E is in fact globally homeomorphic to $B \times F$ we say that E is a trivial bundle over B.

There are some fiber bundles that are of special interest, and so get special names.

Definition 2.4.2 An **I-bundle** is a fiber bundle where the fiber is an interval.

The interval of an I-bundle can be any kind of interval, open, closed, half-open, half-closed, etc. If the interval is in fact all of $\mathbb{R}$ then such an I-bundle is called a **line bundle**.

Now we define Seifert fiber spaces which are a special case of a fiber bundle. First, however, we will need the following definitions.

Definition 2.4.3 Let I be the interval $[0, 1]$. Let $D^2 \times I$ be the solid fibered cylinder, with fibers $x \times I$ for $x \in D^2$. Then a **fibered solid torus** is obtained from the solid fibered cylinder as follows. Rotate $D^2 \times 1$ while holding $D^2 \times 0$ fixed, and then identify $D^2 \times 0$ with $D^2 \times 1$.

That is, starting with the solid fibered cylinder, rotate the top while leaving the bottom fixed, then glue the top the bottom. The result is a solid fibered torus. Notice that if the angle of rotation is 0 the result is just the solid torus. Also, note that the fiber corresponding to $(0, 0) \times I$ remains unchanged by this rotation. This is called the **middle fiber**, [9].

Definition 2.4.4 A **fiber preserving map** is a homeomorphism between two fiber bundles that maps fibers to fibers.

With these two definitions, we are ready to define a Seifert fiber space.

Definition 2.4.5 [9] A **Seifert fiber space** is a three-manifold M that is a disjoint union of fibers such that the following hold:

(1) Each fiber is a simple, closed curve.
(2) Each point of M lies in exactly one fiber.
(3) Each fiber H has a fiber neighborhood, that is, a subset of fibers containing H that can be mapped under a fiber preserving map onto the solid fibered torus, with H mapped to the middle fiber.

We can think of Seifert fiber spaces as fiber bundles over a base space where the fibers are circles.

An important Seifert fiber space is S^3 equipped with the Hopf fibration. To define this map we think of S^3 as sitting in $\mathbb{C}^2$. Then $S^3 = \{(z_1, z_2) : |z_1|^2 + |z_2|^2 = 1\}$. We can think of S^2 as the complex projective line $\mathbb{C}P^1$, that is, S^2 is described by pairs of complex numbers (z_1, z_2) with z_1 and z_2 not both zero, under the equivalence relation $[z_1, z_2] \sim [\lambda z_1, \lambda z_2]$ for $\lambda \in \mathbb{C}$ and nonzero. Notice that this is also the Riemann sphere, which is diffeomorphic to S^2, [10].

Definition 2.4.6 The Hopf map $h : S^3 \to S^2$ is defined by $h(z_1, z_2) \to [z_1, z_2]$.

We may also describe the Hopf map in an equivalent way which will be useful to us in Section 6. Still thinking of S^3 as pairs of complex numbers and S^2 as the extended complex plane, we can define $\tilde{h} : S^3 \to S^2$ by $\tilde{h}(z_1, z_2) = z_1/z_2$. Then $\tilde{h}^{-1}(\lambda)$ is the circle in S^3 described by $z_1 = \lambda z_2$, for $\lambda \in \mathbb{C} \cup \infty$.

Theorem 2.4.1 The set (S^3, S^2, h, S^1) is a fiber bundle.

Proof. First we show that h is a surjection. Suppose that $[z_1, z_2]$ is in S^2. We can normalize this point by taking $\lambda = (|z_1| + |z_2|)^{-1/2}$ and then choosing the representative $[\lambda z_1, \lambda z_2]$. Then $(\lambda z_1, \lambda z_2)$ is in S^3, and $h(\lambda z_1, \lambda z_2) = [z_1, z_2]$. Thus h is surjective.

Now we show that the required triviality condition holds. Let $x \in S^2$, and let $\bar{x}$ be the point antipodal to x. Then $U = S^2 \backslash \{\bar{x}\}$ is an open neighborhood of x in S^2. Define a map $\phi : h^{-1}(U) \to U \times S^1$ by

$$(z_1, z_2) \to \left(\left[1, \frac{z_2}{z_1}\right], \frac{z_2}{z_1}\right).$$

That is, the point $(z_1, z_2) \in S^3 \subset \mathbb{C}$ is sent to the equivalence class of $[z_1, z_2]$ in S^2, paired with the point

$$\frac{z_2}{z_1}$$

in S^1. Then $proj \circ \phi = h|_{h^{-1}(U)}$ and the triviality condition for a fiber bundle is met.

Consider $h^{-1}([z_1, z_2])$. This maps to all points $(\lambda z_1, \lambda z_2)$ in S^3 such that $|\lambda| = 1$. This is precisely a great circle in S^3. Thus, the fiber associated with this fiber bundle is indeed S^1. □

Corollary 2.4.1 The Hopf map makes S^3 a Seifert fiber space.

For a proof of Corollary 2.4.1 see [9] in the section *Fiberings of* S^3.

2.5. **A Motivating Example.** In two dimensions all surfaces admit a geometric structure modeled on one of S^2, E^2, or H^2, as we will see in Section 3. Naturally, we wonder if we can do something similar with three-manifolds. There are three obvious geometries to consider, E^3, S^3, and H^3. Unfortunately, not all three-manifolds admit a geometric structure modeled on one of these. We can see this with an example.

Consider the three-manifold $S^2 \times S^1$. The two-sphere S^2 is its own universal cover, and the circle, S^1, has universal cover $\mathbb{R}$. Thus the universal covering space of $S^2 \times S^1$ is $S^2 \times \mathbb{R}$, which is not homeomorphic to E^3, S^3, or H^3.

This example shows us that we must at least consider more possible geometries than those modeled on E^3, S^3, and H^3. However, it is not even true that all three-manifolds admit a geometric structure at all. In Thurston's geometrization conjecture we must first decompose a three-manifold into pieces, each of which then admits a geometric structure. In this paper we will not explore the method of decomposing a three-manifold, but will instead focus on the potential geometric structures of these pieces.

3. Two Dimensional Geometries

In this section we present a proof that every connected surface (two-manifold) admits one of three geometries and explore each of these geometries. The claim is that any compact connected surface is locally isometric to one of E^2, S^2, or H^2. In fact, it is true that any *connected* surface admits a geometric structure modeled on one of E^2, S^2, or H^2; however, we present only the argument for compact surfaces. There are two methods to see this, one using complex analysis, and one using the topological classification of surfaces. While we present the arguments using topological classification of surfaces, it is interesting to note the use of complex analysis in a deep geometric result.

3.1. **Only Three Two Dimensional Geometries.**

Theorem 3.1.1 (Uniformization Theorem) Every compact, connected surface admits a geometric structure modeled on E^2, S^2, or H^2.

Proof. The topological classification of surfaces tells us that every compact surface is diffeomorphic to the sphere, a connected sum of n tori, or the connected sum of m projective planes, [7]. Every surface also has a well defined Euler characteristic, which is an invariant of the diffeomorphism type. We say that a surface S has Euler characteristic $\chi(S)$. If S is the two-sphere, then $\chi(S) = 2$. If S is the connected sum of n tori, then $\chi(S) = 2 - 2n$. If S is the connected sum of m projective planes, then $\chi(S) = 2 - m$. We may then use the Euler characteristic to break our argument into three cases: $\chi(S) > 0$, $\chi(S) = 0$, $\chi(S) < 0$. Within each of these three cases there are a few sub cases.

If $\chi(S) > 0$ then S is diffeomorphic to a two sphere or a projective plane. In the first case S is S^2 and so admits a geometric structure modeled on S^2. We know the projective plane is S^2 with antipodal points identified. Because the antipodal map is an isometry of S^2, the projective plane admits a geometric structure modeled on S^2 as well.

If $\chi(S) = 0$ then S is diffeomorphic to the torus or the connected sum of two projective planes (the Klein bottle). Each of these surfaces can be described as a quotient of E^2 by a subgroup of $\text{Isom}(E^2)$ and thus admits a geometric structure modeled on E^2, [1].

If $\chi(S) < 0$ then S is an n-torus with $n \geq 2$ or a sum of m projective planes with $m \geq 3$. Each of these admits a complete, locally homogeneous hyperbolic metric. By Lemma 2.3.1 we know that this implies that S has a geometric structure modeled on H^2. This aspect of the proof is more cumbersome, and we cite [1] as a reference for the interested reader. □

3.2. **Metrics on Manifolds.** To discuss these three geometries of two-manifolds we are interested in the group of isometries of the space (which requires knowledge about the metric), the discrete subgroups of the isometry group, and the corresponding quotient spaces.

A metric on a topological space X is a map from $X \times X$ to $\mathbb{R}$ with the properties of being positive, positive definite, symmetric, and having the triangle inequality hold. A **Riemannian metric** is rather different, as it is defined as an inner product on the tangent space of a smooth manifold X. However, a Riemannian metric induces a regular metric on the manifold. We will make this more precise below, but first introduce a motivating example. In Euclidean space, E^2, we can describe the metric as $ds^2 = dx^2 + dy^2$. This does not look like a map from $E^2 \times E^2$ to $\mathbb{R}$ but does seem to be related to the usual Pythagorean theorem that gives us a method for measuring distance in E^2. We can in fact calculate the length of any smooth path γ in E^2 by evaluating $\int_\gamma ds$. We can then define the distance between two points to be the infimum of the length of paths between them. The resulting metric is the usual Euclidean metric. In fact, given any Riemannian metric ds on the tangent space of X we can define a metric on X in this way. This approach makes it simpler to state the metric associated with a particular space.

We make this idea more precise by defining the first fundamental form. We denote the inner product on the tangent space of a manifold by $< w, v >_p$ for $w, v \in T_pM$.

Definition 3.2.1 Let M be a manifold. The **first fundamental form** is a map $I_p : T_pM \to \mathbb{R}$ such that $I_p(w) =< w, w >_p= |w|^2 \geq 0$. If $M = \mathbb{R}^n$ this is the usual dot product.

Suppose that $w \in T_pS$ where T_pS is the tangent space of a surface S at a point p. Then by definition w is a vector tangent to a curve $\alpha(t)$ with $\alpha(0) = p$. We parameterize α in the coordinate chart $x(u, v)$ of the surface S so $\alpha(t) = x(u(t), v(t))$ and $p = \alpha(0) = x(u_0, v_0)$ and $w = \alpha'(0)$. Then we can say the following, where x_u is the partial derivative of x with respect to u and x_v is the partial derivative of x with respect to v.

$$\begin{aligned} I_p(\alpha'(0)) &= < \alpha'(0), \alpha'(0)) >_p \\ &= < x_u u'(0) + x_v v'(0), x_u u'(0) + x_v v'(0) >_p \\ &= < x_u, x_u >_p (u'(0))^2 + 2 < x_u, x_v > (u'(0)v'(0)) + < x_v, x_v >_p ((v'(0))^2 \\ &= E(u'(0))^2 + 2F(u'(0)v'(0)) + G(v'(0))^2 \end{aligned}$$

where

$$\begin{aligned} E &= < x_u, x_u >_p \\ F &= < x_u, x_v >_p \\ G &= < x_v, x_v >_p . \end{aligned}$$

Hence E, F, and G are the coefficients of the first fundamental form in the basis $\{x_u, x_v\}$ of T_pS. We can then use the first fundamental form to determine distances on the surface without needing to refer to the ambient space. For instance, let $\alpha(t) = x(u(t), v(t))$ be a curve on the surface S. The arc length $s(t)$ of $\alpha : [0, 1] \to S$ is given by:

$$
\begin{aligned}
s(t) &= \int_0^t |\alpha'(t)|dt \\
&= \int_0^t \sqrt{I_p(\alpha'(t))}dt \\
&= \int_0^t \sqrt{E(u')^2 + 2F(u'v') + G(v')^2}dt
\end{aligned}
$$

However, instead of writing out this cumbersome notation whenever we want to talk about distance we rewrite the integrand in the notation of differentials. Than we can say that $ds^2 = Edu^2 + 2Fdudv + Gdv^2$ where $s(t) = \int_0^t ds$. For the standard Euclidean metric we obtain $E = G = 1$ and $F = 0$, [3].

3.3. **The Geometry of E^2.** The space E^2 is of course equipped with the usual Euclidean metric, $ds^2 = dx^2 + dy^2$. The isometries of E^2 are translations, reflections, rotations, and glide-reflections (a composition of a reflection over a line l with a translation along the line l). We can express any $\alpha \in \text{Isom}(E^2)$ as $\alpha(x) = A\mathbf{x} + \mathbf{b}$ where A is a 2×2 orthogonal matrix and $\mathbf{b}$ is a vector in E^2. Then the map sending the translation $\alpha(x) = A\mathbf{x} + \mathbf{b}$ to the matrix $A \in O(2)$ is surjective, and the kernel is the set of all translations.

We are interested in the discrete subgroups of $\text{Isom}(E^2)$. Let $G < \text{Isom}(E^2)$. If G acts freely on E^2 then for all $g \in G$, g has no fixed points. Thus, G can not include any rotation or reflection. We are left, then, with translations and glide reflections. There are four types of discrete subgroups of $\text{Isom}(E^2)$; these are subgroups generated by one or two translations, subgroups generated by one glide reflection, and subgroups generated by a translation and a glide reflection, [8]. We begin by considering the group $< f >$ generated by a single translation. We will show that this group is indeed discrete and discover the surface $E^2/ < f >$.

The function f is a translation, and thus has the coordinate form $f(x, y) = (a + x, b + y)$. Fix an arbitrary compact set C in E^2. Consider the set $D = \{g \in < f >: gC \cap C \neq \emptyset\}$. In E^2 a compact set is bounded, so there exits an $r \in \mathbb{R}$ such that the ball of radius r about the origin contains C, that is $B_r(0, 0) \supset C$. Then for any point $(x, y) \in C$ we have $|x| < r$ and $|y| < r$. Now, any $g \in < f >$ has the form f^n and so $g(x, y) = (x+na, y+nb)$. We can find an n so that $|na| > 2r$ and $|nb| > 2r$. Then, if $(x, y) \in C$ we have $|x + na| \geq |na| - |x| > r$ and $|y + nb| \geq |na| - |y| > r$ so the point $(x + na, y + nb)$ is not in C. Thus, for $N > n$, $f^N \notin D$. Thus D is finite, and $< f >$ acts freely and properly discontinuously on E^2.

Definition 3.3.1 A **fundamental region** for $G < \text{Isom}(S)$ is a set consisting of exactly one representative from each orbit of S under G.

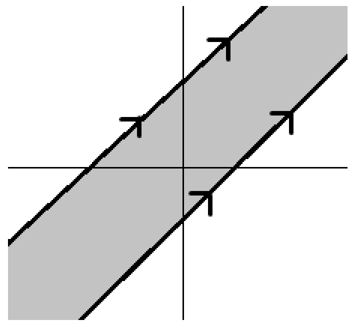

FIGURE 3. A fundamental region resulting in the infinite cylinder.

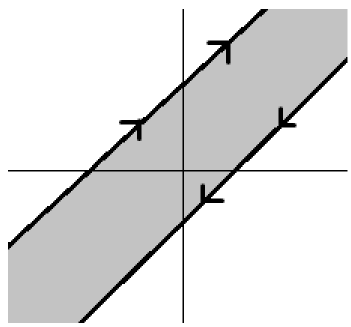

FIGURE 4. A fundamental region resulting in the infinite Möbius band

To discover what the manifold $E^2/ < f >$ is, we sketch the fundamental region for $< f >$. The result is an infinite strip with edges identified, resulting in an infinite cylinder. See Figure 3.

In a similar way we can show that if f is a glide reflection then $< f >$ acts discretely, and so $E^2/ < f >$ is a surface. In fact, the resulting surface is the infinite Möbius band. See Figure 4.

If $G =< f, g >$ where f and g are translations, then G is again a discrete subgroup, and the fundamental domain is a parallelogram with edges oriented as shown in Figure 5. Thus we see that E^2/G is the one-holed torus. Notice that the same fundamental region is obtained if f and g are distinct glide reflexions.

If $G =< f, g >$ where f is a translation and g is a glide reflection, then G is discrete, and the fundamental region of G is again a parallelogram, but with edges oriented as shown in Figure 6. The resulting surface is then a Klein bottle, or the connected sum of two projective planes.

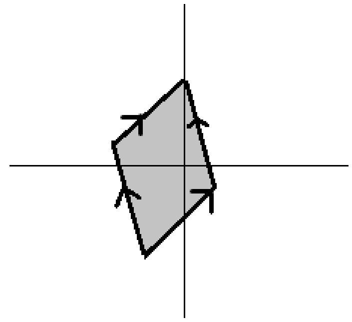

FIGURE 5. A fundamental region resulting in the torus.

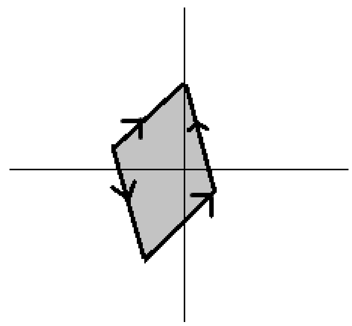

FIGURE 6. A fundamental region resulting in the Klein bottle.

Thus we have found the two compact topological surfaces with a geometry modeled on E^2, as discussed in the proof of Theorem 3.1.1. In addition, we have found two non-compact surfaces with a geometry modeled on E^2.

3.4. **The Geometry of S^2.** Next we consider the geometry of S^2. The sphere can be embedded in E^3 and so inherits the usual metric from this space. Thus we consider S^2 with the metric $ds^2 = dx^2 + dy^2 + dz^2$. Any isometry of E^3 which fixes the origin can be restricted to an isometry of S^2, and every isometry of S^2 can be extended to an isometry of E^3 which fixes the origin. The isometries of S^2 must then be exactly the isometries of E^3 which fix the origin. As with E^2, an isometry of E^3 which fixes the origin can be expressed as an orthogonal 3×3 matrix. Thus $\text{Isom}(S^2) \cong O(3)$. To find discrete subgroups of $\text{Isom}(S^2)$ we must find subgroups of $\text{Isom}(S^2)$ with no fixed points. Any orientation preserving isometry of S^2 must fix either a line through the origin or a great circle. Thus the only isometry which acts freely on S^2 is the antipodal map. The group G generated by this map has order two, as the antipodal map is its own inverse, and the surface S^2/G is the well known projective plane. Thus we have found the two compact surfaces with geometry modeled on S^2, and they are S^2 and $\mathbb{R}P^2$.

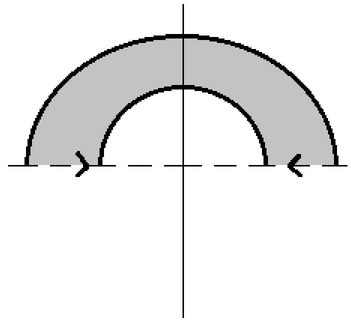

FIGURE 7. A fundamental region resulting in the annulus.

3.5. **The Geometry of H^2.** The geometry of H^2 is perhaps the most interesting because this class of surfaces is the largest. We define H^2 as the upper half of the complex plane, $\mathbb{C}^+ = \{x + iy : y > 0\}$ with the metric $ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$. The geodesics of H^2 are then vertical lines and semicircles with their center on the real axis. We know that any isometry must take a geodesic to a geodesic, so any isometry of H^2 takes the set of vertical lines and circles to itself. It is a standard result from complex analysis that transformations of the form $z \to \frac{az+b}{cz+d}$ for $a, b, c, d \in \mathbb{C}$ do just that. Such transformations are called Möbius transformations, and set of Möbius transformations contains Isom(H^2). We only want to consider the upper half plane, so the isometry group of H^2 will not be all Möbius transformations. It can be shown that the orientation preserving isometries of H^2 are exactly the set $\{z \to \frac{az+b}{cz+d} : a, b, c, d \in \mathbb{R} \text{ and } ad - bd > 0\}$, [8].

We will consider two examples of isometries that act discretely on H^2. The first is the group G generated by an isometry of the form $z \to \lambda z$ for $\lambda > 1$. The orbit of a single point w is $orb(w) = \{z | z = \lambda w\}$. Because multiplying a complex number by a constant only changes the radius of the point, this orbit is a set of points on a single ray from the origin, spaced a distance of λ apart. The fundamental region of G is as shown in Figure 7, and the resulting quotient space is an annulus.

If we take G to be instead the subgroup generated by an isometry of the form $z \to \lambda \bar{z}$ for $\lambda > 1$, we have the same sort of fundamental region, but the transformation is orientation reversing, so instead of the annulus we have the Möbius band. See Figure 8.

4. Three Dimensional Geometries

In this section we briefly describe each of the eight model geometries for three-manifolds identified by Thurston. We will not prove here that these are the only geometries; a proof can be found in [8].

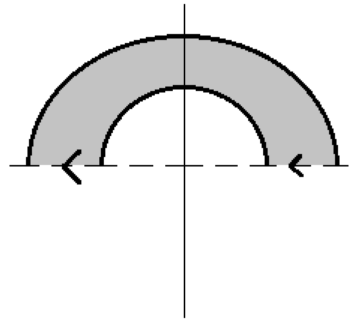

FIGURE 8. A fundamental region resulting in the Möbius band.

4.1. **Euclidean (E^3).** Euclidean 3-space, E^3, is the space $\mathbb{R}^3$ with the metric $ds^2 = dx^2 + dy^2 + dz^2$.

As in E^2 any isometry of E^3 can be written as $x \to Ax + b$, but now A is a real orthogonal 3×3 matrix and b is a translation vector in $\mathbb{R}^3$. Thus there is a group homomorphism $\phi : \mathrm{Isom}(E^3) \to O(3)$, and the kernel of ϕ is the translation subgroup of $\mathrm{Isom}(E^3)$. It is a theorem of Bieberbach that if G is a discrete subgroup of $\mathrm{Isom}(E^n)$, then G has a free Abelian subgroup of finite index in G of rank less than or equal to n. In the case where $n = 3$, it can be shown that this means that either the translation subgroup of G is of finite index in G, or G is a finite extension of $\mathbb{Z}$, where $\mathbb{Z} \cong \{g : g \text{ is a translation}\}$.

4.2. **Spherical (S^3).** The spherical geometry is the three-sphere and its isometry group. S^3 can be embedded in $\mathbb{R}^4$ and thus the metric on S^3 is the one induced from $\mathbb{R}^4$, that is, $ds^2 = dx^2 + dy^2 + dz^2 + dw^2$. The isometry group of S^3 is $O(3)$, the group of orthogonal 3×3 matrices. It is interesting to note that any orientation reversing isometry of S^3 has a fixed point, which limits the discrete subgroups of $\mathrm{Isom}(S^3)$ to subgroups of $SO(3)$. We explore S^3 more thoroughly in Section 6.

4.3. **Hyperbolic (H^3).** Hyperbolic three-space can be defined as the upper half of Euclidean three-space, $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$, with the metric $ds^2 = \frac{1}{z^2}(dx^2 + dy^2 + dz^2)$. Then geodesics will be vertical lines and circles with center on the xy-plane, similar to the geodesics in H^2. The isometry group of H^3 is generated by reflections, which are reflections across planes perpendicular to the xy-plane, and inversions in a sphere with center on the xy-plane. All orientation preserving isometries of H^3 can be identified with a Möbius transformation. We identify each point (x, y, z) in H^3 with the quaternion $w = x + yi + zj$. Then a Möbius transformation is

$$w \to \frac{aw + b}{cw + d}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

4.4. **$S^2 \times \mathbb{R}$.** The space $S^2 \times \mathbb{R}$ is precisely the Cartesian cross product of the unit two-sphere and the real line with the product metric. The isometry group of $S^2 \times \mathbb{R}$ is identified with the product of $\mathrm{Isom}(S^2)$ and $\mathrm{Isom}(\mathbb{R})$. That is, $\mathrm{Isom}(S^2 \times \mathbb{R}) \cong \mathrm{Isom}(S^2) \times \mathrm{Isom}(\mathbb{R})$. This geometry is relatively simple. In fact, there are exactly seven manifolds without boundary which have a geometric structure modeled on $S^2 \times \mathbb{R}$. We explore the geometry of $S^2 \times \mathbb{R}$ in more depth in Section 5.

4.5. **$H^2 \times \mathbb{R}$.** The space $H^2 \times \mathbb{R}$ is the Cartesian cross product of hyperbolic two-space and the real line with the product metric. It has isometry group $\mathrm{Isom}(H^2 \times \mathbb{R}) \cong \mathrm{Isom}(H^2) \times \mathrm{Isom}(\mathbb{R})$. There are infinitely many manifolds with a geometric structure modeled on $H^2 \times \mathbb{R}$ because, if H is a hyperbolic surface, then $H \times S^1$ and $H \times \mathbb{R}$ both have a geometric structure modeled on $H^2 \times \mathbb{R}$. Because there are infinitely many hyperbolic surfaces, they give rise to infinitely many manifolds with a geometric structure modeled on $H^2 \times \mathbb{R}$.

4.6. **$\widetilde{SL_2\mathbb{R}}$.** The group $SL_2\mathbb{R}$ is the group of 2×2 real matrices with determinant one, and is in fact a Lie group. The space $\widetilde{SL_2\mathbb{R}}$ is the universal covering space of the Lie group $SL_2\mathbb{R}$. The metric on $\widetilde{SL_2\mathbb{R}}$ can be derived as follows. The unit tangent bundle of H^2 can be identified with $PSL_2\mathbb{R}$, which is covered by $SL_2\mathbb{R}$. The metric on H^2 can then be pulled back to induce a metric on $\widetilde{SL_2\mathbb{R}}$.

4.7. **Nil.** The geometry of Nil is the three dimensional Lie group of all real 3×3 upper triangular matrices of the form

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$$

under multiplication together with its isometry group. This geometry is called Nil because the Lie group is nilpotent. Nil can be identified with $\mathbb{R}^3$, but with the metric $ds^2 = dx^2 + dy^2 + (dz - xdy)^2$.

4.8. **Sol.** Sol is also a Lie group. We can identify Sol with $\mathbb{R}^3$ with the multiplication given by $(x, y, z)(x', y', z') = (x + e^{-z}x', y + e^{z}y', z + z')$. Then the metric on Sol is $ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$. This group is called Sol because it is a solvable group.

5. The Geometry of $S^2 \times \mathbb{R}$

In this section we will expand on the description of $S^2 \times \mathbb{R}$. Our discussion of the manifolds with a geometric structure modeled on $S^2 \times \mathbb{R}$ will be helped by the discussion about fiber bundles in Section 2.4.

There are only seven three-manifolds without boundary with geometric structure modeled on $S^2 \times \mathbb{R}$. We will discover these from several perspectives. First, we consider a few subgroups of the isometry group which clearly act discretely and the associated quotient spaces. Then we consider each of these spaces instead as fiber bundles.

The isometry group of $S^2 \times \mathbb{R}$ is isomorphic to $\text{Isom}(S^2) \times \text{Isom}(\mathbb{R})$. We know $\text{Isom}(S^2)$ is generated by the identity, the antipodal map, rotations, and reflections. $\text{Isom}(\mathbb{R})$ consists only of identity, translations, and reflections. There are only a few ways elements of these two groups can be paired to generate a discrete subgroup of $\text{Isom}(S^2) \times \text{Isom}(\mathbb{R})$.

Let $(\alpha, \beta) \in \text{Isom}(S^2) \times \text{Isom}(\mathbb{R})$ and let G be the subgroup generated by (α, β). If one of α or β is identity, then $(S^2 \times \mathbb{R})/G$ will be determined entirely by the action of the other element. There is only one isometry which acts freely on $\mathbb{R}$, and that is translation. Because the quotient of $\mathbb{R}$ by the group generated by a translation is a circle, we know that if α is identity and β is a translation then $(S^2 \times \mathbb{R})/G$ is $S^2 \times S^1$. There is also only one freely acting isometry in $\text{Isom}(S^2)$, and that is the antipodal map. Because the quotient of S^2 by the antipodal map is $\mathbb{R}P^2$, we know that if α is the antipodal map and β is identity, then $(S^2 \times \mathbb{R})/G$ is $\mathbb{R}P^2 \times \mathbb{R}$. Of course, we also have the trivial case, where both α and β are identity, in which case $(S^2 \times \mathbb{R})/G$ is just itself. Notice that these are all the possible combinations (α, β) with one the identity such that $\langle(\alpha, \beta)\rangle$ acts freely.

Let $G = \langle(\alpha_1, \beta)\rangle$. When α_1 is the identity and β is a translation, then $(S^2 \times \mathbb{R})/G$ is $S^2 \times S^1$. This is a fiber bundle, where $S^2 \times S^1$ is the total space, S^1 is the base space, and S^2 is the fiber. That is, $(S^2 \times \mathbb{R})/G$ is the *trivial S^2 bundle over the circle.* Now, let α_2 be the antipodal map, and let $H = \langle(\alpha_2, \beta)\rangle$. Then $(S^2 \times \mathbb{R})/H$ can also be described as a fiber bundle. We still have the circle as the base space and S^2 as the fiber. The only difference is the map.

To describe $(S^2 \times \mathbb{R})/G$ and $(S^2 \times \mathbb{R})/H$ as fiber bundles we must first define the map π. Recall that $G = \langle(\alpha_1, \beta)\rangle$ where α_1 is identity on S^2 and β is a translation of $\mathbb{R}$, and $H = \langle(\alpha_2, \beta)\rangle$ where α_2 is the antipodal map on S^2 and β is still a translation of $\mathbb{R}$. Let $x \in S^1$ and U be an open neighborhood of x. Then we have the map $\pi_G : (S^2 \times \mathbb{R})/G \to U$ by π_G of the coset $(a, b)G$ is the equivalence class of b under the translation map, that is $(a, b)G \to [b]$. Then $\phi_G : (S^2 \times \mathbb{R})/G \to U \times S^2$ by mapping the coset $(a, b)G$ to $[b] \times S^2$. Then $((S^2 \times \mathbb{R})/G, S^1, \pi_G, S^2)$ meets the definition of a fiber bundle. Because ϕ_G is in fact a diffeomorphism from $(S^2 \times \mathbb{R})/G$ to $S^2 \times S^1$ this is the trivial S^2 fiber bundle over S^1. Now we consider $(S^2 \times \mathbb{R})/H$. The map π_H is defined in the same way as π_G, and ϕ_H is defined in the same way as ϕ_G, except that the domain space of these functions is different. In $(S^2 \times \mathbb{R})/H$ the antipodal points of the sphere are also identified, so the map ϕ_H is not a bijection, and thus the fiber bundle $((S^2 \times \mathbb{R})/H, S^1, \pi_H, S^2)$ is no longer trivial. Thus we can say that $(S^2 \times \mathbb{R})/H$ is a nontrivial S^2 bundle over S^1.

Now let α_1 be the antipodal map, β_1 be the identity, α_2 be the antipodal map, and β_2 be a reflection of $\mathbb{R}$. Let $G = \langle(\alpha_1, \beta_1)\rangle$ and $H = \langle(\alpha_2, \beta_2)\rangle$. We have already seen that $(S^2 \times \mathbb{R})/G$ is $\mathbb{R}P^2 \times \mathbb{R}$. We could also describe this as the trivial line bundle over $\mathbb{R}P^2$. We will see that $(S^2 \times \mathbb{R})/H$ is a nontrivial line bundle over $\mathbb{R}P^2$.

Define a map $\pi : (S^2 \times \mathbb{R})/H \to \mathbb{R}P^2$ by $\pi((a, b)H) = [a] \in \mathbb{R}P^2$. That is, π takes the coset $(a, b)H$ to the equivalence class of the point a in $\mathbb{R}P^2$. If x is a point in $\mathbb{R}P^2$ and U is an open neighborhood of x, define $\phi : (S^2 \times \mathbb{R})/H \to U \times \mathbb{R}$ by $\phi((a, b)H) = ([a], [b])$ where $[a]$ is the equivalence class of $a \in S^2$ under the action of α_2 and $[b]$ is the equivalence class of $b \in \mathbb{R}$ under the action of β_2 Then $proj \circ \phi = \pi$ and we can say that $((S^2 \times \mathbb{R})/H\mathbb{R}, P^2, \pi, \mathbb{R})$ is a fiber bundle. Because ϕ is not in fact a global bijection, we know that this is a nontrivial fiber bundle. We say that $(S^2 \times \mathbb{R}/)H$ is a nontrivial line bundle over $\mathbb{R}P^2$.

Now consider a subgroup G of $\text{Isom}(S^2 \times \mathbb{R})$ which is generated by two elements. Let α_1 be the antipodal map, β_1 be identity, α_2 be identity, and β_2 be a translation of $\mathbb{R}$. Let $G = \langle(\alpha_1, \beta_1), (\alpha_2, \beta_2)\rangle$. Then $(S^2 \times \mathbb{R})/G$ is just $\mathbb{R}P^2 \times S^1$. We could also call this the trivial circle bundle over $\mathbb{R}P^2$, or the trivial $\mathbb{R}P^2$ bundle over S^1.

Now let $\alpha_1 = \alpha_2$ be the antipodal map, and β_1 and β_2 be distinct reflections in $\mathbb{R}$. Let $H = \langle(\alpha_1, \beta_1), (\alpha_2, \beta_2)\rangle$. Then $(S^2 \times \mathbb{R})/H$ can be described as the union of two nontrivial I-bundles over $\mathbb{R}P^2$.

This gives us a description of all seven surfaces as fiber bundles.

It can be shown that if M is a manifold which is isomorphic to $(S^2 \times \mathbb{R})/G$ where G is a subgroup of $\text{Isom}(S^2 \times \mathbb{R})$ which acts discretely on $S^2 \times \mathbb{R}$, then M must be one of the seven manifolds found above. Thus there are exactly seven manifolds with a geometric structure modeled on $S^2 \times \mathbb{R}$, [8].

6. The Geometry of S^3

In this section we explore the geometry of S^3. We view S^3 in three different ways depending on which is most convenient. First, we may think of S^3 as the unit sphere in $\mathbb{R}^4$. Because $\mathbb{R}^4$ can be identified with $\mathbb{C}^2$ we may also think of S^3 as ordered pairs of complex numbers. In this case, $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$. Then again we can identify $\mathbb{C}^2$ with the quaternions, denoted $\mathbb{H}$, by sending $(z_1, z_2) \in \mathbb{C}^2$ to $z_1 + z_2 j \in \mathbb{H}$. Then S^3 becomes the group of unit quaternions.

Viewing S^3 as the unit sphere in $\mathbb{R}^4$ we see that S^3 inherits a metric from the Euclidean metric on $\mathbb{R}^4$. In this metric the geodesics of S^3 are exactly the great circles of S^3. This view of S^3 also tells us about the isometry group of S^3. In analogy to $S^2 \subset \mathbb{R}^3$, $\text{Isom}(S^3) \cong O(4)$, [8].

In order to discover manifolds with geometric structure modeled on S^3 we need to find subgroups of $\text{Isom}(S^3)$ which act discretely on S^3. It is known that any orientation reversing isometry of S^3 has a fixed point, [8]. Further, we can describe any orientation preserving isometry of S^3 as an element of $SO(4)$. Thus, only subgroups of $SO(4)$ have any chance of acting freely, and therefore discretely, on S^3.

Recall that a group G acts properly discontinuously on a space X if for all compact subsets C of X the set $\{g \in G : gC \cap C \neq \emptyset\}$ is finite. We are considering the case where $X = S^3$. We know that S^3 is itself compact. Any infinite subgroup of $SO(4)$ fails to act properly discontinuously, and thus discretely, on S^3. Thus our search reduces to considering finite subgroups of $SO(4)$ which act freely on S^3. These subgroups have been classified, and a complete classification can be found in [8]. We will not prove the classification, but we will show that every such subgroup of $SO(4)$ preserves the Hopf fibration on S^3, and thus conclude that every three-manifold with geometric structure modeled on S^3 is a Seifert fiber space. We begin by finding an explicit description of orientation preserving isometries of S^3.

Thinking of S^3 as the group of unit quaternions, consider the action of S^3 on itself by $x \to q_1 x q_2^{-1}$. This is an isometry.

Lemma 6.0.1 For $x, q_1, q_2 \in S^3$, the map $x \to q_1 x q_2^{-1}$ is an isometry of S^3.

A proof that left translation is an isometry of S^3 may be found in [2] on page 60. A similar argument shows that right translation is also an isometry of S^3. These two together constitute a proof of Lemma 6.0.1.

Theorem 6.0.1 The map $\phi : S^3 \times S^3 \to SO(4)$ defined by $(q_1, q_2) \to (x \to q_1 x q_2^{-1})$ is a surjective homomorphism with kernel $\{(1,1), (-1,-1)\}$.

Corollary 6.0.1 Every orientation preserving isometry of S^3 may be described as a map of the form $x \to q_1 x q_2^{-1}$.

Once we know Theorem 6.0.1, Corollary 6.0.1 follows because $SO(4)$ is exactly the orientation preserving isometries of S^3.

Proof of Theorem 6.0.1. We may describe any isometry of the form $x \to q_1 x q_2^{-1}$ as an ordered pair $(q_1, q_2) \in S^3 \times S^3$. Define a map $\phi : S^3 \times S^3 \to O(4)$ by
$(q_1, q_2) \to (x \to q_1 x q_2^{-1})$. We first show that ϕ is a homomorphism.

We wish to show that $\phi((q_1, q_2) \cdot (w_1, w_2)) = \phi((q_1, q_2)) \circ \phi((w_1, w_2))$ where (q_1, q_2) and (w_1, w_2) are in $S^3 \times S^3$. Let x be an arbitrary element of S^3. We will show that

$$\phi((q_1, q_2) \cdot (w_1, w_2))(x) = \phi((q_1, q_2)) \circ \phi((w_1, w_2))(x).$$

$$\begin{aligned}
\phi((q_1, q_2) \cdot (w_1, w_2))(x) &= \phi((q_1 w_1, q_2 w_2)(x) \\
&= q_1 w_1 x (q_2 w_2)^{-1} \\
&= q_1 w_1 x w_2^{-1} q_2^{-1} \\
&= \phi(q_1, q_2)(w_1 x w_2^{-1}) \\
&= \phi(q_1, q_2) \circ \phi(w_1, w_2)(x)
\end{aligned}$$

Thus ϕ is a homomorphism.

Because $O(4)$ is the isometry group of S^3, and $(x \to q_1 x q_2^{-1})$ is an isometry, the image of ϕ is contained in $O(4)$. Now, S^3 is a connected space, so $S^3 \times S^3$ is also connected. The image of a connected space under a homomorphism is again connected, so the image of ϕ must be contained in one of the connected components of $O(4)$. Because $S^3 \times S^3$ contains the identity isometry, we know that the image of ϕ lies in the connected component containing identity, that is, $SO(4)$. Thus we may restate the codomain of ϕ as $SO(4)$. In fact, we know that $S^3 \times S^3$ is six dimensional as a group, and so the image of ϕ must also be a six dimensional subgroup of $SO(4)$. However, $SO(4)$ is itself six dimensional, so ϕ has image $SO(4)$. Thus ϕ is surjective.

We now describe the kernel of ϕ. If $(q_1, q_2) \in \ker(\phi)$ then $q_1 x q_2^{-1} = x$ for all $x \in S^3$. Choose $x = 1$. Then this relationship says that $q_1 = q_2$. That means that for this element to be in $\ker(\phi)$ we must have $q_1 x q_1^{-1} = x$, or $q_1 x = x q_1$. This says that $\ker(\phi) = \{(q, q) : q \in C(S^3)\}$, where $C(S^3)$ is the center of S^3 as a group. There are only two elements in $C(S^3)$, namely 1 and -1. Thus $\ker(\phi) = \{(1, 1), (-1, -1)\}$. By the first isomorphism theorem we know $S^3 \times S^3 / \{(1, 1), (-1, -1)\} \cong SO(4)$. □

Theorem 6.0.1 gives us a useful map between $S^3 \times S^3$ and $SO(4)$. In fact, $S^3 \times S^3$ is a double cover of $SO(4)$. We will use this map later.

Corollary 6.0.1 tells us that any manifold with a geometric structure modeled on S^3 has the form S^3/G where G is a finite group of isometries of the form $x \to q_1 x q_2^{-1}$. Our goal is to show that every such manifold is a Seifert fiber space. To do this, we want to show that every finite subgroup of $SO(4)$ which acts freely on S^3 preserves the Hopf fibration. We start by showing that certain isometries preserve the Hopf fibration. Then we show that if a group acts freely on S^3 it contains only these types of isometries.

Lemma 6.0.2 Any isometry $\alpha : S^3 \to S^3$ of the form $\alpha(x) = xq$ where $q \in S^3$ preserves the Hopf fibration.

Lemma 6.0.3 Any isometry $\alpha : S^3 \to S^3$ of the form $\alpha(x) = qx$ where q has the form $(w_1, 0)$ or $(0, w_2 j)$ for $w_1, w_2 \in S^1$ preserves the Hopf fibration.

Proof of Lemma 6.0.2. Let $w = (w_1, w_2)$ be a fixed point in S^3. If $z = (z_1, z_2)$ lies in a fiber of the Hopf fibration such that $z_1/z_2 = \lambda$, consider the action of right multiplication by w on this fiber.

$$(z_1, z_2)(w_1, w_2) = (z_1 w_1 - z_2 \bar{w_2}, z_2 \bar{w_1} + z_1 w_2)$$

Now consider the ratio of the first component to the second. If this ratio is constant, then the fiber $z_1/z_2 = \lambda$ will have been transformed into another fiber of the Hopf fibration by the action of w.

$$\frac{z_1 w_1 - z_2 \bar{w_2}}{z_2 \bar{w_1} + z_1 w_2} = \frac{\lambda w_1 - \bar{w_2}}{\bar{w_1} + \lambda w_2}$$

Because w was a fixed point of S^3, this ratio is indeed constant, and thus right multiplication by an element of S^3 preserves the Hopf fibration. □

Proof of Lemma 6.0.3. Let $w = (w_1, 0)$ be a fixed point of S^3, and consider the action of left multiplication by w on a fiber of the Hopf fibration, $z_1/z_2 = \lambda$. Then $(w_1, 0)(z_1, z_2) = (w_1 z_1, w_1 z_2)$, and the ratio of the first component to the second is simply $z_1/z_2 = \lambda$. In this case, not only does the action by w preserve the Hopf fibration, but the fiber remains fixed under the action.

Let $w = (0, w_2)$ be a fixed point of S^3, and consider the action of left multiplication by w on a fiber of the Hopf fibration, $z_1/z_2 = \lambda$. In this case, $(0, w_2)(z_1, z_2) = (-w_2 \bar{z_2}, w_2 \bar{z_1})$. The ratio of the first component to the second is then

$$-\frac{\bar{z_2}}{\bar{z_1}} = -\frac{1}{\bar{\lambda}}.$$

Because this ratio is constant, we see that the fiber $z_1/z_2 = \lambda$ is transformed into another fiber of the Hopf fibration.

Thus left multiplication of S^3 by elements of the form $(w_1, 0)$ or $(0, w_2)$ preserve the Hopf fibration.

□

Now we show that every finite subgroup of $SO(4)$ which acts freely on S^3 is conjugate in $SO(4)$ to a subgroup which preserves the Hopf fibration.

Lemma 6.0.4 Let α be an isometry of S^3 such that $\alpha(x) = q_1 x q_2^{-1}$, $q_1, q_2 \in S^3$. Then α has a fixed point if and only if q_1 and q_2 are conjugate in S^3.

Proof. Suppose α has a fixed point x. Then

$$\alpha(x) = q_1 x q_2^{-1} = x \leftrightarrow q_1 x = x q_2 \leftrightarrow q_1 = x q_2 x^{-1}$$

Thus q_1 and q_2 are conjugate in S^3.

Suppose that q_1 and q_2 are conjugate in S^3. Then by the above argument there exists some x so that $q_1 = x q_2 x^{-1}$, and $x = q_1 x q_2^{-1}$. Thus α has a fixed point. □

Lemma 6.0.5 If G is a subgroup of $SO(4)$ which acts freely on S^3 and has order two, then $G = \{I, -I\}$ where I is the identity matrix. This implies that a subgroup of $SO(4)$ which acts freely either has odd order or one element of order two, namely $-I$.

Proof. If G is a subgroup of order two it must have one nontrivial element; let α be this element. We know from Lemma 6.0.1 that α has the form $\alpha(x) = q_1 x q_2^{-1}$ for $q_1, q_2 \in S^3 \subset \mathbb{H}$. We also know that α must have order two in G, and so $x = q_1^2 x q_2^{-2}$ for all x. Choosing $x = 1$ we see that $q_1^2 = q_2^2$. Then, letting x be arbitrary, we now have $x = q_1^2 x q_1^{-2}$ and $x = q_2^2 x q_2^{-2}$. Thus q_1^2 and q_2^2 commute with every element of S^3, which means they are central in S^3. Thus $q_1^2 = q_2^2$ and their shared value must be ± 1 since the center of $\mathbb{H}$ is the subgroup $\{\pm 1\}$.

If the shared value is -1, then q_1 and q_2 have order 4, and it can be shown that all unit quaternions of order 4 are conjugate, and thus by Lemma 6.0.4 α has a fixed point. We assumed that G acts freely on S^3, so this is a contradiction. Thus the shared value is 1, and q_1 and q_2 are either ± 1.

If $q_1 = q_2$ for either ± 1 then $\alpha = I$, but we assumed that α was nontrivial. Thus one of q_1 and q_2 is 1 and the other is -1. Then $\alpha(x) = -x$ and $\alpha = -I$. □

To continue, we need to define another map of S^3. We have already seen the surjective homomorphism $\phi : S^3 \times S^3 \to SO(4)$ in Theorem 6.0.1. Now we define a classic homomorphism between S^3 and $SO(3)$.

Proposition 6.0.1 Let $q \in S^3$ and define $\psi(q) = \alpha_q$ where $\alpha_q(x) = q x q^{-1}$. Then $\psi : S^3 \to SO(3)$ and ψ is a surjective homomorphism with $\ker(\psi) = \{1, -1\}$.

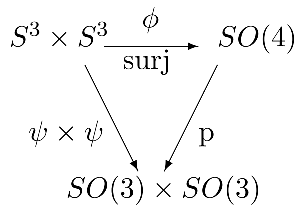

FIGURE 9. The function $p(x) = (\psi \times \psi)(\phi^{-1}(x))$

Proof. First we must show that the codomain of ψ is in fact $SO(3)$. We know from the proof of Theorem 6.0.1 that ψ maps into $SO(4)$. Now, the isometry α_q fixes 1 for all q, so ψ maps into the subgroup of $SO(4)$ which fixes 1. This subgroup can be naturally identified with $SO(3)$. Further, as a manifold $SO(3)$ has dimension three, and is connected. Because ψ maps from a three dimensional manifold and has finite kernel, its image must be a three dimensional connected manifold. Thus the image of ψ is in fact all of $SO(3)$, and ψ is a surjection.

Now we will show that ψ is a homomorphism. Let $q_1, q_2 \in S^3$.

$$\begin{aligned}
\psi(q_1 q_2) &= \alpha_{q_1 q_2} \\
\alpha_{q_1 q_2}(x) &= q_1 q_2 x (q_1 q_2)^{-1} \\
&= q_1 q_2 x q_2^{-1} q_1^{-1} \\
&= \alpha_{q_1} \circ \alpha_{q_2}(x)
\end{aligned}$$

Thus, $\psi(q_1 q_2) = \psi(q_1)\psi(q_2)$ and so ψ is a homomorphism.

To complete the proof we need only show that $\ker(\psi) = \{1, -1\}$. Suppose that $q \in \ker(\psi)$. Then $\alpha_q(x) = qxq^{-1} = x$. This means that $qx = xq$, so q is central in S^3. There are only two elements in the center of S^3, and these are 1 and -1. A quick check shows that in fact both of these elements are in $\ker(\psi)$, and thus $\ker(\psi) = \{1, -1\}$. For further details, see [8] and [2]. □

With this homomorphism in hand we can now describe the following map and prove some of its properties.

Corollary 6.0.2 The map $p : SO(4) \to SO(3) \times SO(3)$ such that $\psi \times \psi = p \circ \phi$ has kernel $\{I, -I\}$, where I is the identity matrix. See Figure 9.

Proof. We have already defined ψ in Proposition 6.0.1 and ϕ in Theorem 6.0.1, and noted their respective kernels. Recall that $\psi : S^3 \to SO(3)$ and $\ker(\psi) = \{1, -1\}$, while $\phi : S^3 \times S^3 \to SO(4)$ and $\ker(\phi) = \{(1,1), (-1,-1)\}$. We can then define $\psi \times \psi : S^3 \times S^3 \to SO(3) \times SO(3)$ by $\psi \times \psi(q) \to \psi(q) \times \psi(q)$. Notice that $\psi \times \psi$ must have kernel $\ker(\psi) \times \ker(\psi)$. We can certainly define a map $p : SO(4) \to SO(3) \times SO(3)$ such that $\psi \times \psi = p \circ \phi$. Now we will find the kernel of this map.

We know $\ker(\psi \times \psi) = \{(1,1), (1,-1), (-1,1), (-1,-1)\}$. This must also be the kernel of $p \circ \phi$. Thus $\phi(\ker(\psi \times \psi)) = \ker(p)$. We can see by inspection that $\phi(\ker(\psi \times \psi)) = \{I, -I\}$. Thus $\ker(p) = \{I, -I\}$. □

Lemma 6.0.6 If G is a finite subgroup of $SO(4)$ which acts freely on S^3, then $p(G)$ is a subgroup of $SO(3) \times SO(3)$ which acts freely on $\mathbb{R}P^3$.

Proof. First we need to show that $p(G)$ is in fact a subgroup of $\text{Isom}(\mathbb{R}P^3)$. By definition, $p(G)$ is a subgroup of $SO(3) \times SO(3)$. We will show that $SO(3) \times SO(3)$ is contained in $\text{Isom}(\mathbb{R}P^3)$. We know that $S^3/\{I, -I\} \simeq S^3/\{1, -1\} \simeq \mathbb{R}P^3$, as this is the classic topological construction of $\mathbb{R}P^3$. However, we also know from Proposition 6.0.1 and the first isomorphism theorem, that $S^3/\{1, -1\} \cong SO(3)$. Thus $\mathbb{R}P^3$ is diffeomorphic to $SO(3)$. Then the map $\alpha : \mathbb{R}P^3 \to \mathbb{R}P^3$ by $x \to u_1 x u_2^{-1}$ where $u_1, u_2 \in \mathbb{R}P^3$ is an isometry of $\mathbb{R}P^3$, as it is already an isometry of S^3. Thus every element of $SO(3) \times SO(3)$ is an isometry of $\mathbb{R}P^3$, and $p(G)$ is a subset of $\text{Isom}(\mathbb{R}P^3)$.

We can add further details to our knowledge of isometries of $\mathbb{R}P^3$ by using the relationship between $\mathbb{R}P^3$ and S^3. The argument that α has a fixed point if and only if u_1 and u_2 are conjugate in $\mathbb{R}P^3$ follows from the same argument as for S^3. Because conjugacy is respected by isomorphisms, u_1 conjugate to u_2 in $\mathbb{R}P^3$ means that u_1 and u_2 are also conjugate in $SO(3)$. We can think of $SO(3)$ as the group of rotations of E^3, and then the conjugacy classes of $SO(3)$ are well understood. In fact, they are exactly determined by their rotation angles. In particular, two elements of order two in $SO(3)$ must be conjugate because are both rotations through π.

Now, assume $G < SO(4)$ acts freely on S^3. If G has even order, then by Lemma 6.0.5 we know that G contains the subgroup $\{I, -I\}$. In this case, we obtain S^3/G by first considering $S^3/\{I, -I\} = \mathbb{R}P^3$ and then letting $p(G)$ act on this group, so that we have $S^3/G = \mathbb{R}P^3/p(G)$. If G acts freely on S^3 then $p(G)$ acts freely on $\mathbb{R}P^3$.

If G has odd order and acts freely on S^3 then $p(G)$ will also act freely on $\mathbb{R}P^3$. Let $\bar{G} =< G, -I >$, that is, the group generated by G and $-I$. Then elements of $\bar{G}$ are isometries of S^3, $x \to g(x)$ where $g \in G$, together with the isometries $x \to -g(x)$. Suppose that G acts freely on S^3, but $\bar{G}$ does not. Then there exists an isometry $g \in G$ and an $x \in S^3$ such that x is a fixed point for $-g$, that is $x = -g(x)$. Then $g^2(x) = x$. Because we are assuming that G acts freely on S^3, this means that $g^2 = I$. However, then g has order two, which is a contradiction since we assumed that G had odd order.

So, if G has odd order and acts freely on S^3, then $\bar{G}$ also acts freely on S^3 and has even order. Further, $p(G) = p(\bar{G})$. We have already shown that if $\bar{G}$ has even order and acts freely on S^3, then $p(\bar{G})$ acts freely on $\mathbb{R}P^3$. Thus, $p(\bar{G})$ acts freely on $\mathbb{R}P^3$, and thus $p(G)$ also acts freely on $\mathbb{R}P^3$. □

Corollary 6.0.3 If $p(G) \subseteq SO(3) \times SO(3) \simeq \mathbb{R}P^3 \times \mathbb{R}P^3$ acts freely on $\mathbb{R}P^3$ then $p(G)$ cannot contain a non-trivial element $(u_1, u_2) \in \mathbb{R}P^3 \times \mathbb{R}P^3$ with u_1, u_2 conjugate in $SO(3)$. Moreover, this means that $p(G)$ can not contain an element (u_1, u_2) where u_1 and u_2 have order two in $SO(3)$.

Proof. We have already seen that if u_1 and u_2 are conjugate in $\mathbb{R}P^3$ then the isometry $x \to u_1 x u_2^{-1}$ has a fixed point. Because u_1 and u_2 may be regarded as points in $SO(3)$ we can look for conjugacy classes in this well understood group of rotations of E^3. In particular, conjugacy classes are determined by the angle of rotation. If an element of $SO(3)$ has order two, then it is a rotation through π, and thus all elements of order two are conjugate. Thus if u_1 and u_2 each have order two, then they are conjugate, and the isometry $x \to u_1 x u_2^{-1}$ has a fixed point, and thus does not act freely. □

Lemma 6.0.7 Let H be a finite subgroup of $SO(3) \times SO(3)$ and let H_1 and H_2 be the projection of H onto each of the two factors. If H acts freely on $\mathbb{R}P^3$ then at least one of H_1 and H_2 is cyclic.

Proof. The finite subgroups of $SO(3)$ are well known. A finite subgroup of $SO(3)$ is cyclic, dihedral, the tetrahedral symmetry group, the octahedral symmetry group, or the icosahedral symmetry group. Each of these groups has even order, except the odd ordered cyclic groups.

Let $H_1 = \{(x, I) : (x, y) \in H \subseteq SO(3) \times SO(3)\}$ where I is the identity matrix in $SO(3)$. Similarly, let $H_2 = \{(I, y) : (x, y) \in H \subseteq SO(3) \times SO(3)\}$. That is H_i is the projection of H onto the ith component of $SO(3) \times SO(3)$. Let $H_i' = H \cap H_i$.

We claim that H_i' is a normal subgroup of H_i. We prove this for H_1'. Suppose that $x \in H_1'$, and $g \in H_1$. Then x has the form $(x, I) \in SO(3) \times SO(3)$, and g has the form $(g_1, I) \in SO(3) \times SO(3)$. We wish to show that $gxg^{-1} = (g_1 x g_1^{-1}, I) \in H_1'$. By the definition of H_1, there exists some g_2 so that $(g_1, g_2) \in H$. By the definition of H_1' $(x, I) \in H$ as well. Then $(g_1, g_2)(x, I)(g_1, g_2)^{-1} = (g_1 x g_1^{-1}, I) \in H$. Then $(g_1 x g_1^{-1}, I)$ is in H as well as H_1 and so is in H_1'. Thus H_1' is normal in H_1. A similar argument shows that H_2' is normal in H_2. Thus we can talk about H_i/H_i'. In fact, $H_1/H_1' \cong H_2/H_2'$. To see this, let $\varphi : H_1/H_1' \to H_2/H_2'$ by $\varphi((h_1, I)H_1') = (I, h_2)H_2'$ where $(h_1, h_2) \in H$. Then it can be checked that φ is an isomorphism.

Because H contains $H_1' \times H_2'$ Corollary 6.0.3 tells us that either H_1' or H_2' must have odd order and thus be cyclic.

Suppose that H_1' is cyclic with odd order, and that $b \in H_2$ is an element of order two. Then there is some $a \in H_1$ so that $(a, b) \in H$, by the definition of H_1 and H_2. Then $(a, b)^2 = (a^2, 1) \in SO(3) \times I$, and so $a^2 \in H_1'$. Thus a^2 has odd order, say r. That is $(a^2)^r = 1$, or $(a^r)^2 = 1$. Thus a^r is either the identity or has order two. We also know that $(a, b)^r = (a^r, b) \in H$ so Corollary 6.0.3 tells us that a^r can not have order two. Thus $a^r = 1$, and $(1, b) \in H$, so $b \in H_2'$. Thus H_2' contains every element of order two in H_2.

If $H_2' = H_2$ then it follows that $H_1' = H_1$, so that H_1 is cyclic.

If $H_2' \neq H_2$ then H_2 can not be generated by an element of order two, as H_2' contains every element of order two in H_2. This means that H_2 can not be equal to the dihedral group, the octahedral symmetry group, or the icosahedral symmetry group, as each of these is generated by an element of order two. Thus H_2 is either cyclic, or isomorphic to the tetrahedral symmetry group, T. We know that $T \cong A_4$. If H_2 is isomorphic to A_4, then $H_2' \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then H_2/H_2' has order three, because H_2 has order 12 and H_2' has order 4. By the isomorphism between H_1/H_1' and H_2/H_2' we know that H_2/H_2' also has order three. However, we assumed that H_1' had odd order, and so the order of H_1 must also be odd in order for their quotient to have order 3. Thus, H_1 must be cyclic.

Reversing the roles of H_1 and H_2, we see that at least one of H_1 and H_2 is cyclic when H acts freely on $\mathbb{R}P^3$. □

Theorem 6.0.2 Let $\Gamma_1 = \phi(S^1 \times S^3)$ and $\Gamma_2 = \phi(S^3 \times S^1)$. If G is a finite subgroup of $SO(4)$ acting freely on S^3, G is conjugate in $SO(4)$ to a subgroup of Γ_1 or Γ_2.

Proof. Let G be a finite subgroup of $SO(4)$ acting freely on S^3. Then $p(G)$ is a subgroup of $H_1 \times H_2$. By Lemma 6.0.7 we know that one of H_1 or H_2 is cyclic. Suppose that H_1 is cyclic. Let $\tilde{G} = \phi^{-1}(p^{-1}(H_1 \times H_2))$, which is a subgroup of $S^3 \times S^3$. Then $\tilde{G}$ must also equal $\psi^{-1}(H_1) \times \psi^{-1}(H_2)$, where $\psi : S^3 \to SO(3)$ is the map defined in Proposition 6.0.1. It is sufficient to show that $\tilde{G}$ is conjugate to a subgroup of $S^1 \times S^3$. For this we need only show that $\psi^{-1}(H_1)$ is conjugate in S^3 to a subgroup of the unit complex numbers, S^1.

We know that H_1 is cyclic and thus has a single generator. Let $q \in S^3$ be such that $\psi(q)$ generates H_1. By conjugation in S^3, we can put q in S^1. Then $\psi^{-1}(H_1)$ lies in the subgroup of S^3 generated by q and $-q$, because $\ker(\psi) = \{1, -1\}$. This is certainly a subgroup of S^1. Thus $\psi^{-1}(H_1)$ is conjugate in S^3 to a subgroup of S^1, through the conjugation of q. Then $\tilde{G}$ is conjugate to a subgroup of $S^1 \times S^3$, and so $p(G)$ is as well. Thus G is conjugate in $SO(4)$ to a subgroup of Γ_1.

A similar argument shows that if H_2 is cyclic then G is conjugate to a subgroup of Γ_2. □

This proof, together with Lemmas 6.0.2 and 6.0.3 tell us that every finite subgroup of $S0(4)$ which acts freely on S^3 preserves the Hopf fibration. If G preserves any particular Seifert fibration, then the quotient space S^3/G naturally inherits a Seifert bundle structure from S^3. Thus, if G is a subgroup which preserves the Hopf fibration, S^3/G is a Seifert fiber space. This tells us that every manifold with a geometric structure modeled on S^3 is a Seifert fiber space. Remarkable!

7. Conlcusion

We have considered the eight geometric structures of Thurston's geometrization conjecture. In order to do so, we have made precise the notion of a geometric structure modeled on X where X is a simply connected, complete, homogeneous Riemannian manifold. We have considered the geometric structures of $S^2 \times \mathbb{R}$ and S^3 in detail. We have identified each of the seven three-manifolds with geometric structure modeled on $S^2 \times \mathbb{R}$ as a fiber bundle. In S^3 we have shown that every three-manifold with a geometric structure modeled on S^3 is in fact a Seifert fiber space.

Thurston's geometrization conjecture entails much more than these eight geometries. It states that after a three-manifold has been split into its connected sum and Jaco-Shalen-Johannson torus decomposition, each of the remaining pieces admits one of the eight geometries discussed in this paper. Further understanding of this geometric classification of three-manifolds requires an understanding of this two level decomposition, which includes knowledge about the Ricci flow.

The geometrization conjecture implies the Poincaré conjecture. The Poincaré conjecture states that every simply connected, closed three-manifold is homeomorphic to the three-sphere. Because the geometrization conjecture has now been proven, the Poincaré conjecture is also true. There are, however, recent shorter proofs of the Poincaré conjecture which do not involve the full proof of the geometrization conjecture.

Thurston's geometrization conjecture has had a long road to being proven. In 2003 Grigori Perelman presented a proof of the geometrization conjecture in his paper "Ricci Flow with Surgery on Three-Manifolds." Since then, details in Perelman's arguments have been filled in, and the conjecture is now considered true some 21 years after its original presentation.

References

[1] Bonahon, Francis. "Geometric Structures on 3-Manifolds." Handbook of Geometric Topology, edited by R.J. Daverman, R.B. Sher. Elsevier B.V. Amsterdam, 2002
[2] Curtis, Morton L. *Matrix Groups*, Springer-Verlag, New York, 1984.
[3] DoCarmo, Manfredo. *Differential Geometry of Curves and Surfaces*, Prentice-Hall, Englewood Cliffs New Jeresy, 1976.
[4] Lee, John M. *Introduction to Smooth Manifolds*, Springer, New York, 2003.
[5] McCleary, John. *Geometry from a Differential Viewpoint.*
[6] Milnor, John. "Towards the Poincare Conjecture and the Classification of 3-Manifolds," Notices of the AMS, Vol. 50 Number 10, pages 1226-1233, Nov 2003.
[7] Munkres, James R. *Topology.* Prentice -Hall Inc, New Delhi, 2000.
[8] Scott, Peter, "The Geometries of 3-Manifolds," Bull. London Math. Soc, 15 (1983), 401-487.
[9] Seifert, Herbert. *Topology of 3-Dimensional Fibered Spaces.* Academic Press Inc. New York 1980.
[10] Steenrod, Norman. *The Topology of Fibre Bundles.* Princeton University Press, London, 1951.