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In this thesis, problems arising in pumping fluids from the ground are treated. The mathematical theory is derived for both the time dependent and independent cases. A numerical method is developed to solve these problems. Some particular examples are presented and computer plots are drawn to illustrate the movement of fluid near the pump. Listings of the computer programs and the plots are included in the appendix.

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LIST OF SYMBOLS

Symbol	Definition
D	Domain where fluid flow takes place and $D \subseteq R^3$
x	$x = (x_1, x_2, x_3)$, point in D
r	$r = \|x\|$, the distance of x from the origin, i.e. $r = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$
r_{xy}	$r_{xy} = \|x-y\|$, the distance between x and y
t, τ	Time variables
$d\sigma$	Element of surface
dx	$dx = dx_1 dx_2 dx_3$, volume element
∇	Gradient operator computed with respect to x_1, x_2, x_3
div	Divergence operator computed with respect to x_1, x_2, x_3
Δ	Laplacian operator computed with respect to x_1, x_2, x_3
Ω	A domain (open connected set) in R^3
$\partial\Omega$	Boundary of Ω
$\bar{\Omega}$	The closure of Ω, i.e. $\bar{\Omega} = \Omega \cup \partial\Omega$
ν	Exterior unit normal to $\partial\Omega$
$C^k(\Omega)$	The set of functions defined on Ω and k times continuously differentiable there
$C^0(\bar{\Omega})$	The set of functions which are defined and continuous on $\bar{\Omega}$.
$B_\rho(y)$	The ball centered at y with radius ρ, i.e. $B_\rho(y) = \{x \mid \|x-y\| < \rho\}$
$C^{k,\ell}(\Omega x(0,T])$	The set of functions which are k times continuously differentiable with respect to x in Ω and ℓ times with respect to t in $(0,T]$.

Symbol	Definition
$\vec{q}$	Volumetric flow rate or the volume of fluid passing over a cross-sectional area per unit time
k	Permeability
μ	Viscosity
p	Pressure
ρ	Density
$\vec{g}$	Acceleration due to gravity
ϕ	Porosity
S	Saturation

NUMERICAL COMPUTATIONS IN THE NEIGHBORHOOD OF ISOLATED SINGULARITIES WITH APPLICATIONS

I. INTRODUCTION

In this thesis, problems arising in pumping fluids from the ground are treated. Existence and uniqueness questions are discussed and numerical methods for solving some particular problems are given. Then contour maps from computer data are drawn which demonstrate the movement of the fluid near singular points.

The basic equations governing the movement of a fluid in a porous medium when variations in temperature are negligible are:

(1.1) $$\vec{q} = -\frac{k}{\mu}(\nabla p + \rho\vec{g}) \qquad \text{(Darcy's Law)}$$

$$(\rho\phi S)_t + \operatorname{div}(\rho\vec{q}) = 0 \qquad \text{(Continuity Equation)}$$

In these equations, $\vec{q}$ is the volume flow rate, i.e. the volume of fluid passing over a cross-sectional area per unit time, k is the permeability, μ is the viscosity, p is the pressure, ρ is the density, $\vec{g}$ is the acceleration due to gravity, ϕ is the porosity and S is the saturation. The gradient, ∇, and the divergence, div, operators are computed with respect to the spatial coordinates x_1, x_2, x_3. A precise definition and further elaboration of these concepts may be found in Corey [1], Fulks, Guenther, Roetman [3], Hubbert [9], Muskat [13], Polubarinova-Kochina [14], Scheidegger [16]. Furthermore, they discuss the applicability, meaning and possible generalizations of Darcy's law.

In the case where the fluid is compressible, an equation of state relating p and ρ must be included with (1.1) and (1.2). We assume one of the form

$$\rho = \rho(p) \tag{1.3}$$

The equations (1.1) and (1.2) are derived assuming there are neither sources nor sinks. Now let us assume that the origin, 0, is in D, the domain where fluid flow takes place, and $D \subseteq R^3$. A pump, i.e. a point sink, is located there and equations governing the movement of the fluid in the neighborhood of the origin will be derived.

Equations (1.1) and (1.2) still hold as long as $x \neq 0$. In order for the pump to produce continuously, fluid mass has to be present near the origin. In other words, the pump is a function of the fluid mass in some neighborhood of $x = 0$. The size of the neighborhood depends on the strength of the pump.

Let us assume that the mass, $m(t)$, upon which the pump depends is contained in a sphere of radius γ. Then

$$m(t) = \int_{|x| \leq \gamma} \rho\phi S dx . \tag{1.4}$$

Let f be the rate at which m is produced. Then f depends upon m and perhaps upon t (since the pump may be regulated manually or will wear) and what is pumped out is exactly what flows into the pump.

The net rate at which mass crosses the surface $|x| = \varepsilon$ is

$$-\int_{|x|=\varepsilon} \rho\vec{q} \cdot \nu d\sigma ,$$

where ν is the exterior unit normal, and $d\sigma$ is the element of surface of the sphere of radius ε.

Then the rate at which mass flows into the pump is the limit of this quantity as $\varepsilon \to 0$, i.e.

$$f(m(t), t) = \lim_{\varepsilon\to 0} \{-\int_{|x|=\varepsilon} \rho\vec{q} \cdot \nu d\sigma\} . \tag{1.5}$$

We shall make the following assumptions on the pump function $f = f(\lambda,t)$, assumed continuous in both variables: There exist positive constants λ_1 and c such that

(i) $f(\lambda,t) = 0$ for $\lambda \le 0$.

(ii) $\frac{\partial}{\partial\lambda} f(\lambda,t) = 0$ for $\lambda \ge \lambda_1$.

(iii) $|f(\lambda,t)| \le c$.

(iv) $|f(\lambda',t) - f(\lambda'',t)| \le c|\lambda'-\lambda''|$.

Assumption (i) refers to the fact that if the mass reaches a certain residual point, no more mass can be produced. We assume the problem has been normalized so that this residual value is 0.

Assumptions (ii) and (iii) refer to the fact that there are limits as to how much mass it is possible for the pump to produce.

Assumption (iv) is a mathematical assumption which pumps seem to satisfy.

The equations (1.1)-(1.5) govern the general pumping problem, where equations (1.1) and (1.2) hold for $x \neq 0$. If one desires to consider the problem when mass is pumped into the ground, one simply reverses the sign in (1.5) and modifies assumptions (i) and (ii) appropriately.

II. MATHEMATICAL THEORY FOR THE LINEAR PROBLEMS

1. Time-Independent Case

Assumptions on the Domain D

We shall always assume that D is a normal domain, i.e. it is a domain such that the Gauss-Divergence Theorem holds (see [11], [12]). We assume further that D is such that the Dirichlet problem for the Laplace equation is solvable. Then by our assumptions a Green's function exists for the Laplace equation in D.

Formulation of the Dirichlet Problem

For incompressible, steady, saturated flow, the basic equations are

$$\vec{q} = -\frac{k}{\mu}(\nabla p + \rho\vec{g}). \tag{1.1}$$

$$\text{div}\ \vec{q} = 0\ . \tag{2.1}$$

We assume k and μ are constant and we find that p must satisfy:

$$\Delta p = 0,\ x\in D,\ x \neq 0\ . \tag{2.2}$$

$$\lim_{\varepsilon\to 0} \frac{k\rho}{\mu} \int_{|x|=\varepsilon} \nabla p\cdot\nu d\sigma = f, \text{ since } \lim_{\varepsilon\to 0} \frac{k\rho}{\mu} \int_{|x|=\varepsilon} \rho\vec{g}\cdot\nu d\sigma = 0, \tag{2.3}$$

where f is a given constant.

Equations (2.2)-(2.3) must be solved subject to some boundary conditions. We shall here formulate the Dirichlet problem. We seek a function $p \in C^2(D-\{0\}) \cap C^0(\bar{D}-\{0\})$ such that

(1) p satisfies equations (2.2) and (2.3) for all $x \in D$, $x \neq 0$.

(2) $p(x) = \psi(x)$, x on ∂D, where $\psi(x)$ is a given continuous function defined on D.

(3) There exists a constant M such that $|p(x)| \leq \frac{M}{r}$ and $|\nabla p(x)| \leq \frac{M}{r^2}$ as $r \to 0$.

The motivation for (3) is as follows. Observe from equation (2.3) that $|\nabla p|$ is of the order $\frac{1}{r^2}$ since if $|\nabla p|$ were of the order $\frac{1}{r^{2-\varepsilon}}$, $0 < \varepsilon < 2$, the limit in (2.3) would be zero and if it were of the order of $\frac{1}{r^{2+\varepsilon}}$, it would be infinite. By integration we conclude that u must be of the order of $\frac{1}{r}$.

The explicit representation of the solution to the Dirichlet problem will be given in terms of the Green's function G in D. We summarize some of its properties in the following lemma. For the proof, see Gilbarg and Trudinger [4] or Kellogg [11].

<u>Lemma</u> Let G be a Green's function in D. Then G has the form

$$G(x,y) = \frac{1}{4\pi r_{xy}} - g(x,y) \quad ,$$

where g is a smooth function and $0 < g < \frac{1}{4\pi r_{xy}}$ for $x, y \in D$, and

(1) $\Delta G = 0$ for $x \neq y$,

(2) $G(x,y) = 0$ for $x \in D$, $y \in \partial D$,

(3) $G(x,y) > 0$, $x \neq y$, $x,y \in D$,

(4) $G(x,y) = G(y,x)$.

Theorem 1 Let $D \subseteq R^3$ be a bounded domain. Let ψ be continuous on ∂D and χ satisfy a uniform Hölder condition of order α, $0 < \alpha \leq 1$, on D. Then there exists a unique solution to the Dirichlet problem

(2.4)
$$\Delta p(x) = \chi(x),\ x \in D,\ x \neq 0,$$
$$p(x) = \psi(x),\ x \text{ on } \partial D,$$
$$\sup_{x\in\bar{D}} r|p(x)| < \infty,\quad \sup_{x\in\bar{D}} r^2|\nabla p(x)| < \infty,$$
$$\lim_{\varepsilon\to 0} \frac{k\rho}{\mu} \int_{|x|=\varepsilon} \nabla p\cdot\nu d\sigma = f, \text{ where } f \text{ is a constant.}$$

Proof

We first take up the question of the uniqueness of the solution. Let us assume p_1 and p_2 are two solutions to the problem and let $p = p_1 - p_2$. Let $x^0 \in D$ be arbitrary. Observe that for $y \neq x^0$

(2.5)
$$\operatorname{div}[p(y)\nabla G(x^0,y) - G(x^0,y)\nabla p(y)] = p(y)\Delta G(x^0,y) - G(x^0,y)\Delta p(y),$$

where the div and ∇ operators are computed with respect to the y variables.

Integrate equation (2.5) over $D - \bar{B}_\eta(x^0) - \bar{B}_\varepsilon(0)$ with respect to x, where ε and η are such that $\bar{B}_\eta \cap \bar{B}_\varepsilon = \phi$, $\bar{B}_\eta \subset D$ and $\bar{B}_\varepsilon \subset D$, and use the fact that $\Delta G(x^0,y) = 0$, $\Delta p(y) = 0$ there, to obtain

$$\int_{D-\bar{B}_\eta-\bar{B}_\varepsilon} \operatorname{div}[p(y)\nabla G(x^0,y) - G(x^0,y)\nabla p(y)]dx$$
$$= \int_{D-\bar{B}_\eta-\bar{B}_\varepsilon} [p(y)\Delta G(x^0,y) - G(x^0,y)\Delta p(y)]dx$$
$$= 0.$$

Applying the Gauss-Divergence theorem, we get

$$\int_{\partial[D-\bar{B}_\eta-\bar{B}_\varepsilon]} [p(y)\nabla G(x^0,y) - G(x^0,y)\nabla p(y)]\cdot\nu d\sigma = 0 .$$

Since p and G are zero on ∂D, we find

$$\int_{\partial\bar{B}_\eta \cup \partial\bar{B}_\varepsilon} [p(y)\nabla G(x^0,y) - G(x^0,y)\nabla p(y)]\cdot\nu d\sigma = 0 .$$

This equation can be rewritten as

$$(2.6)\quad \left\{\begin{aligned} &\int_{\partial\bar{B}_\eta} p(y)\nabla G(x^0,y)\cdot\nu d\sigma - \int_{\partial\bar{B}_\eta} G(x^0,y)\nabla p(y)\cdot\nu d\sigma \\ &+ \int_{\partial\bar{B}_\varepsilon} p(y)\nabla G(x^0,y)\cdot\nu d\sigma - \int_{\partial\bar{B}_\varepsilon} G(x^0,y)\nabla p(y)\cdot\nu d\sigma = 0 . \end{aligned}\right.$$

We now calculate the individual integrals in (2.6) and then let ε and η tend to zero. First

$$\int_{\partial\bar{B}_\eta} p(y)\nabla G(x^0,y)\cdot\nu d\sigma = \int_{\partial\bar{B}_\eta} p(y)\nabla\left(\frac{1}{4\pi\eta}\right)\cdot\nu d\sigma - \int_{\partial\bar{B}_\eta} p(y)\nabla g(x^0,y)\cdot\nu d\sigma ,$$

so that

$$\lim_{\eta\to 0} \int_{\partial\bar{B}_\eta} p(y)\nabla\left(\frac{1}{4\pi\eta}\right)\cdot\nu d\sigma = \lim_{\eta\to 0} \int_{|\omega|=1} p(x^0+\eta\omega)\left(-\frac{1}{4\pi\eta^2}\right)\eta^2 d\omega$$

$$= -p(x^0) ,$$

and since g is smooth,

$$\lim_{\eta\to 0} \int_{\partial\bar{B}_\eta} p(y)\nabla g(x^0,y)\cdot\nu d\sigma = 0 \ .$$

Thus,

$$\lim_{\eta\to 0} \int_{\partial\bar{B}_\eta} p(y)\nabla G(x^0,y)\cdot\nu d\sigma = -p(x^0) \ .$$

Next,

$$\left|\int_{\partial\bar{B}_\eta} G(x^0,y)\nabla p(y)\cdot\nu d\sigma\right| \le \{\max_{\bar{B}_\eta}|\nabla p(y)|\}\int_{\partial\bar{B}_\eta}|G(x^0,y)|d\sigma$$

$$\le \{\max_{\bar{B}_\eta}|\nabla p(y)|\}\int_{|\omega|=1}\frac{1}{4\pi\eta}\ \eta^2 d\omega$$

and we conclude that

$$\lim_{\eta\to 0} \int_{\partial\bar{B}_\eta} G(x^0,y)\nabla p(y)\cdot\nu d\sigma = 0 \quad .$$

The fourth integral can be treated as follows:

$$-\int_{\partial\bar{B}_\varepsilon} G(x^0,y)\nabla p(y)\cdot\nu d\sigma$$

$$= -\int_{\partial\bar{B}_\varepsilon} G(x^0,0)\nabla p(y)\cdot\nu d\sigma \quad \int_{\partial\bar{B}_\varepsilon} [G(x^0,y)-G(x^0,0)]\nabla p(y)\cdot\nu d\sigma,$$

$$\lim_{\varepsilon\to 0}\int_{\partial\bar{B}_\varepsilon} G(x^0,0)\nabla p(y)\cdot\nu d\sigma = \lim_{\varepsilon\to 0}\int_{\partial\bar{B}_\varepsilon} (G(x^0,0)\nabla(p_1 - p_2)\cdot\nu d\sigma$$

$$= \lim_{\varepsilon\to 0} G(x^0,0)\int_{\partial\bar{B}_\varepsilon}\nabla p_1\cdot\nu d\sigma$$

$$- \lim_{\varepsilon\to 0} G(x^0,0)\int_{\partial\bar{B}_\varepsilon}\nabla p_2\cdot\nu d\sigma$$

$$= G(x^0,0)\,\frac{\mu}{k\rho}\,(f-f)$$

$$= 0 \ .$$

Since $|G(x^0,y) - G(x^0,0)| \le M_1|y| = M_1\varepsilon$, by the Mean Value Theorem, where M_1 is some constant, we have

$$\left|\int_{\partial\bar{B}_\varepsilon} [G(x^0,y) - G(x^0,0)]\nabla p\cdot\nu d\sigma\right|$$

$$\le \int_{\partial\bar{B}_\varepsilon} |G(x^0,y) - G(x^0,0)|\,|\nabla p|d\sigma$$

$$\le \int_{|\omega|=1} M_1\varepsilon \ \frac{M}{\varepsilon^2}\ \varepsilon^2\ d\omega$$

so that

$$\lim_{\varepsilon\to 0}\int_{\partial\bar{B}_\varepsilon} [G(x^0,y) - G(x^0,0)]\nabla p\cdot\nu d\sigma = 0 \quad .$$

Hence

$$\lim_{\varepsilon\to 0}\int_{\partial\bar{B}_\varepsilon} G(x^0,y)\nabla p(y)\cdot\nu d\sigma = 0 \quad .$$

Combining all these integrals in (2.6) and letting η and ε tend to zero, we get

$$p(x^0) = 0 \quad .$$

Since x^0 is arbitrary, we conclude that $p \equiv 0$ in D, i.e., the solution is unique.

We obtain a representation of the solution by assuming that p is of the form

$$p(x) = \alpha G(x,0) + w(x) \ ,$$

where α is a constant which we choose below and w satisfies

$$(2.7) \quad \begin{cases} \Delta w(x) = \chi(x), \ x \in D, \ x \neq 0 \\ w(x) = \psi(x), \ x \text{ on } \partial D \ . \end{cases}$$

w is well defined by our assumptions on D, χ and ψ. α may be calculated as follows:

$$\lim_{\varepsilon\to 0} \int_{|x|=\varepsilon} \nabla p \cdot \nu d\sigma = \lim_{\varepsilon\to 0} \int_{|x|=\varepsilon} \nabla[\alpha G(x,0) + w(x)] \cdot \nu d\sigma$$

$$= \lim_{\varepsilon\to 0}\{\alpha \int_{|x|=\varepsilon} \nabla(\frac{1}{4\pi\varepsilon}) \cdot \nu d\sigma - \alpha \int_{|x|=\varepsilon} \nabla g(x,0) \cdot \nu d\sigma$$

$$+ \int_{|x|=\varepsilon} \nabla w(x) \cdot \nu d\sigma\}$$

Since g and w are smooth,

$$\lim_{\varepsilon\to 0} \alpha \int_{|x|=\varepsilon} \nabla g(x,0) \cdot \nu d\sigma = 0 \quad .$$

and

$$\lim_{\varepsilon\to 0} \int_{|x|=\varepsilon} \nabla w(x) \cdot \nu d\sigma = 0 \quad .$$

Further,

$$\lim_{\varepsilon\to 0} \alpha \int_{|x|=\varepsilon} \nabla\left(\frac{1}{4\pi\varepsilon}\right)\cdot\nu d\sigma = \lim_{\varepsilon\to 0} \alpha \int_{|\omega|=1} \left(-\frac{1}{4\pi\varepsilon^2}\right)\varepsilon^2 d\omega = -\alpha \ .$$

By equation (2.3),

$$\lim_{\varepsilon\to 0} \frac{k\rho}{\mu} \int_{|x|=\varepsilon} \nabla p\cdot\nu d\sigma = f,$$

hence

$$\alpha = -\frac{f\mu}{k\rho} \ .$$

Replacing α by $-\frac{f\mu}{k\rho}$, p can be written as

$$p(x) = -\frac{f\mu}{k\rho} G(x,0) + w(x),$$

and a simple calculation shows that this function is a solution to the problem which completes the proof of the theorem.

2. Time-Dependent Case for Gases

Assumptions on D are already made as in the time-independent case. By our assumptions a Green's function for the heat equation exists on D x (0,T].

One of the properties of gases is that $S \equiv 1$ and we can neglect gravitational effects.

Formulation of the First Initial Boundary Value Problem

Assume we have saturated flow and that ϕ, k and μ are constants. We further assume an equation of state for ρ of the form $\rho(p) = \alpha \exp(\beta p)$, where α and β are some constants (see [16]). Combining all these assumptions, using (1.1) and (1.2), and setting $u = \rho\phi$, we find that u must satisfy

(2.8) $$\frac{k}{\mu\phi\beta} \operatorname{div} [\nabla u] = u_t, \quad x \neq 0, \ x \in D, \ t > 0.$$

(2.9) $$\lim_{\varepsilon \to 0} \frac{k}{\mu\phi\beta} \int_{|x|=\varepsilon} \nabla u \cdot \nu d\sigma = f(m(t), t), \ t > 0.$$

where

(2.10) $$m(t) = \int_{|x| \leq \gamma} u(x,t) dx .$$

Equations (2.8)-(2.10) must be solved subject to some boundary and initial conditions. We will formulate and solve the first initial boundary value problem for $u = u(x,t)$.

We seek a function $u \in C^{2,1}(Dx(0,T]) \cap C^{0,0}(\bar{D}x[0,T])$ satisfying equations (2.8)-(2.10) and

(2.11) $u(x,0) = \chi(x)$, $x \neq 0$, where $\chi(x)$ is a given continuous function defined on D.

(2.12) $m(t) = \int_{|x| \leq \gamma} u(x,t) dx$ is continuous in t for $t > 0$ and $m(0+) = \int_{|x| \leq \gamma} \chi(x) dx$.

(2.13) There exists a constant M such that $|\nabla u(x,t)| \leq \frac{M}{r^2}$ and $|u(x,t)| \leq \frac{M}{r}$ as $r \to 0$.

(2.14) $u(x,t) = \psi(x,t)$ for $x \in \partial D$, $0 \leq t \leq T$.

The representation of the solution to the first initial boundary value problem is in terms of the Green's function G for $\bar{D} \times [0,T]$. We summarize some of the properties in the following lemma. For a proof see Friedman [2] or Tikhonov [17].

Lemma Let $G(x,y, t-\tau)$ be the Green's function for $\bar{D} \times [0,T]$. Then

(1) $G_\tau + \Delta_y G = 0$ for $(y,\tau) \neq (x,t)$.

(2) $G(x,y, t-\tau) = 0$ for $x \in D$, $y \in \partial D$.

(3) $G(x,y, t-\tau) > 0$ for $x \neq y$, $t \neq \tau$.

(4) $G(x,y, t-\tau) = G(y,x, t-\tau)$.

Theorem 2 Let D be a bounded domain. Then there exists a unique solution, u, satisfying the conditions (2.8)-(2.14) and the differential equation

$$\Delta u + F(x,t) = u_t, \ x \neq 0, \ x \in D, \ t > 0.$$

Here F represents the source density for internal volume sources and is known. We assume that F satisfies a Hölder condition in x and t on $\bar{D} \times [0,T]$. χ and ψ are assumed to be continuous and satisfy $\psi(x,0) = \chi(x)$ for $x \in \partial D$.

Proof Let $G(x,y, t-\tau)$ be the Green's function in $\bar{D} \times [0,T]$. Let x and t be fixed but arbitrary. For $(y,\tau) \neq (x,t)$,

$$G_\tau + \Delta_y G = 0 ,$$

$$u_\tau - \Delta_y u = F(y,\tau),$$

so that

$$(2.15)\quad \begin{cases} FG = (u_\tau - \Delta_y u)G + (G_\tau + \Delta_y G)u \\ \quad = (Gu)_\tau + \mathrm{div}_y[-G\nabla_y u + u\nabla_y G]. \end{cases}$$

Integrate (2.15) over $(D - \bar{B}_\varepsilon(0) - \bar{B}_\eta(x)) \times [\varepsilon_1, t-\varepsilon_1]$, where $\bar{B}_\varepsilon \subset D$, $\bar{B}_\eta \subset D$ and $\bar{B}_\varepsilon \cap \bar{B}_\eta = \phi$ and $0 < \varepsilon_1 < \frac{t}{4}$ is arbitrary:

$$\int_{\varepsilon_1}^{t-\varepsilon_1}\int_{D-\bar{B}_\varepsilon-\bar{B}_\eta} F(y,\tau)G(x,y,\ t-\tau)dy\ d\tau$$

$$=\int_{\varepsilon_1}^{t-\varepsilon_1}\int_{D-\bar{B}_\varepsilon-\bar{B}_\eta} (Gu)_\tau dy d\tau + \int_{\varepsilon_1}^{t-\varepsilon_1}\int_{D-\bar{B}_\varepsilon-\bar{B}_\eta} \mathrm{div}_y[-G\nabla_y u+u\nabla_y G]dy\ d\tau.$$

Apply the Gauss Divergence theorem to the second integral on the right side of the equation,

$$\int_{\varepsilon_1}^{t-\varepsilon_1}\int_{D-\bar{B}_\varepsilon-\bar{B}_\eta} F(y,\tau)G(x,y,\ t-\tau)dy\ d\tau$$

$$= \int_{\varepsilon_1}^{t-\varepsilon_1}\int_{D-\bar{B}_\varepsilon-\bar{B}_\eta} (Gu)_\tau dy\ d\tau$$

$$+ \int_{\varepsilon_1}^{t-\varepsilon_1}\int_{\partial[D-\bar{B}_\varepsilon-\bar{B}_\eta]} [-G\nabla_y u + u\nabla_y G]\cdot\nu d\sigma_y\ d\tau$$

$$= \int_{D-\bar{B}_\varepsilon-\bar{B}_\eta} u(y,\ t-\varepsilon_1)G(x,y,\ t-(t-\varepsilon_1))dy$$

$$- \int_{D-\bar{B}_\varepsilon-\bar{B}_\eta} u(y,\ \varepsilon_1)G(x,y,\ t-\varepsilon_1)dy$$

$$-\int_{\varepsilon_1}^{t-\varepsilon_1} \int_{\partial[D-\bar{B}_\varepsilon-\bar{B}_\eta]} [G(x,y,\ t-\tau)\nabla_y\ u(y,\tau)$$

$$-\ u(y,\tau)\nabla_y G(x,y,\ t-\tau)]\cdot\nu d\sigma_y\ d\tau.$$

But on ∂D, $G = 0$, $u = \psi$ so that on simplifying we get

$$(2.16)\quad \begin{cases} \displaystyle\int_{\varepsilon_1}^{t-\varepsilon_1} \int_{D-\bar{B}_\varepsilon-\bar{B}_\eta} F(y,\tau)G(x,y,\ t-\tau)dy\ d\tau \\ \displaystyle = \int_{D-\bar{B}_\varepsilon-\bar{B}_\eta} u(y,\ t-\varepsilon_1)G(x,y,\ \varepsilon_1)dy \\ \displaystyle - \int_{D-\bar{B}_\varepsilon-\bar{B}_\eta} u(y,\varepsilon_1)G(x,y,\ t-\varepsilon_1)dy \\ \displaystyle + \int_{\varepsilon_1}^{t-\varepsilon_1} \int_{\partial D} \psi(y,\tau)\nabla_y G(x,y,\ t-\tau)\cdot\nu d\sigma_y\ d\tau \\ \displaystyle - \int_{\varepsilon_1}^{t-\varepsilon_1} \int_{\partial\bar{B}_\varepsilon\cup\partial\bar{B}_\eta} [G(x,y,\ t-\tau)\nabla_y\ u(y,\tau) \\ \quad - u(y,\tau)\nabla_y G(x,y,\ t-\tau)]\cdot\nu d\sigma_y d\tau\ . \end{cases}$$

Now let $\varepsilon_1 \to 0$. Then (2.16) can be written as

$$\int_0^t \int_{D-\bar{B}_\varepsilon-\bar{B}_\eta} F(y,\tau)G(x,y,\ t-\tau)dy\ d\tau$$

$$= \int_{D-\bar{B}_\varepsilon-\bar{B}_\eta} u(y,t)G(x,y,0)dy$$

$$- \int_{D-\bar{B}_\varepsilon-\bar{B}_\eta} u(y,0)G(x,y,t)dy$$

$$+ \int_0^t \int_{\partial D} \Psi(y,\tau)\nabla_y G(x,y,\ t-\tau)\cdot\nu d\sigma_y\ d\tau$$

$$- \int_0^t \int_{\partial\bar{B}_\varepsilon\cup\partial\bar{B}_\eta} [G(x,y,\ t-\tau)\nabla_y u(y,\tau)$$

$$- u(y,\tau)\nabla_y G(x,y,\ t-\tau)]\cdot\nu d\sigma_y\ d\tau\ .$$

Rearranging,

$$(2.17)\quad \begin{cases} u(x,t) = \int_0^t \int_{D-\bar{B}_\varepsilon-\bar{B}_\eta} F(y,\tau)G(x,y,\ t-\tau)dy\ d\tau \\ \quad + \int_{D-\bar{B}_\varepsilon-\bar{B}_\eta} u(y,0)G(x,y,t)dy \\ \quad - \int_0^t \int_{\partial D} \psi(y,\tau)\nabla_y G(x,y,\ t-\tau)\cdot\nu d\sigma_y\ d\tau \\ \quad + \int_0^t \int_{\partial\bar{B}_\varepsilon} [G(x,y,\ t-\tau)\nabla_y u(y,\tau) \\ \qquad - u(y,\tau)\nabla_y G(x,y,\ t-\tau)]\cdot\nu d\sigma_y\ d\tau \\ \quad + \int_0^t \int_{\partial\bar{B}_\eta} [G(x,y,\ t-\tau)\nabla_y u(y,\tau) \\ \qquad - u(y,\tau)\nabla_y G(x,y,\ t-\tau)]\cdot\nu d\sigma_y\ d\tau. \end{cases}$$

We now let ε and η tend to zero in equation (2.17). To compute the limits, observe first that

$$|G(x,y,\ t-\tau) - G(x,0,t-\tau)| \le M_1|y| = M_1\varepsilon$$

for some constant M_1 so we have

$$\int_0^t \int_{\partial\bar{B}_\varepsilon} G(x,y,\ t-\tau)\nabla_y u(y,\tau)\cdot\nu d\sigma_y\ d\tau$$

$$= \int_0^t \int_{\partial\bar{B}_\varepsilon} [G(x,y,\ t-\tau) - G(x,0,\ t-\tau)]\nabla_y u(y,\tau)\cdot\nu d\sigma_y\ d\tau$$

$$+ \int_0^t \int_{\partial\bar{B}_\varepsilon} G(x,0,\ t-\tau)\nabla_y u(y,\tau)\cdot\nu d\sigma_y\ d\tau$$

Now, $\int_0^t \int_{\partial\bar{B}_\varepsilon} |G(x,y,\ t-\tau)-G(x,0,\ t-\tau)||\nabla_y u(y,\tau)|d\sigma_y d\tau$

$$\le \int_0^t \int_{|\omega|=1} M_1\varepsilon\ \frac{M}{\varepsilon^2}\ \varepsilon^2\ d\omega\ d\tau,$$

so that

$$\lim_{\varepsilon\to 0}\int_0^t\int_{\partial\bar{B}_\varepsilon}[G(x,y,\ t-\tau) - G(x,0,\ t-\tau)]\nabla_y u(y,\tau)\cdot\nu d\sigma_y d\tau$$
$$= 0\ .$$

Now,

$$\lim_{\varepsilon\to 0}\int_0^t\int_{\partial\bar{B}_\varepsilon} G(x,0,\ t-\tau)\nabla_y u(y,\tau)\cdot\nu d\sigma_y d\tau$$
$$=\int_0^t G(x,0,\ t-\tau)(\lim_{\varepsilon\to 0}\int_{\partial\bar{B}_\varepsilon}\nabla_y u(y,\tau)\cdot\nu d\sigma_y)d\tau$$
$$=\int_0^t \mu G(x,0,\ t-\tau)f(m(\tau),\tau)\ d\tau\ .$$

Since G is a smooth function of y when $x \neq y$, we have

$$\lim_{\varepsilon\to 0}\int_0^t\int_{\partial\bar{B}_\varepsilon} u(y,\tau)\nabla_y G(x,y,\ t-\tau)\cdot\nu d\sigma_y d\tau = 0\ .$$

We may conclude that

$$\lim_{\varepsilon\to 0}\int_0^t\int_{\partial\bar{B}_\varepsilon}[G(x,y,\ t-\tau)\nabla_y u(y,\tau)-u(y,\tau)\nabla_y G(x,y,\ t-\tau)]\cdot\nu d\sigma_y d\tau$$
$$=\int_0^t \mu G(x,0,\ t-\tau)f(m(\tau),\ \tau)\ d\tau\ .$$

In the same manner, the limit of the last integral on the right side of equation (2.17) can be calculated and we can conclude that

$$\lim_{\eta\to 0}\int_0^t\int_{\partial\bar{B}_\eta}[G(x,y,\ t-\tau)\nabla_y u(y,\tau)-u(y,\tau)\nabla_y G(x,y,\ t-\tau)]\cdot\nu d\sigma_y d\tau$$
$$= 0\ .$$

On substituting into equation (2.17), we obtain

$$(2.18)\quad \begin{cases} u(x,t) = \int_0^t \int_D F(y,\tau)G(x,y,\ t-\tau)dy\ d\tau \\ \quad + \int_D \chi(y)G(x,y,t)dy \\ \quad - \int_0^t \int_{\partial D} \psi(y,\tau)\nabla_y G(x,y,\ t-\tau)\cdot\nu d\sigma_y d\tau \\ \quad + \int_0^t \mu G(x,0,\ t-\tau)f(m(\tau),\ \tau)\ d\tau\ . \end{cases}$$

Integrate (2.18) over the sphere $|x| \leq \gamma$ to obtain the non-linear Volterra-type integral equation

$$(2.19)\qquad m(t) = \tilde{m}(t) - \int_0^t \mu \ell(t,\tau)f(m(\tau),\ \tau)d\tau\ ,$$

where

$$\tilde{m}(t) = \int_{|x|\leq\gamma} dx\{\int_0^t \int_D F(y,\tau)G(x,y,\ t-\tau)dy\ d\tau$$

$$+ \int_D \chi(y)G(x,y,\ t)dy - \int_0^t \int_{\partial D} \psi(y,\tau)\nabla_y G(x,y,\ t-\tau)\cdot\nu d\sigma_y d\tau\}$$

and

$$(2.20)\qquad \ell(t,\tau) = \int_{|x|\leq\gamma} G(x,0,\ t-\tau)dx\ .$$

Using standard estimates on the fundamental solution (see [2] or [6]) one can easily prove by iteration that (2.19) possesses a unique solution whence follow the assertions of the theorem.

We can also obtain by the same technique an existence and uniqueness theorem for the second initial boundary value problem. In fact replace the condition (2.14) by

$$(2.14') \qquad \frac{\partial u}{\partial \nu} = \psi(x,t), \; x \in \partial D, \; 0 \leq t \leq T \quad .$$

Here ν represents the exterior unit normal to ∂D. We must assume that on ∂D, $\frac{\partial \chi}{\partial \nu} = \psi(x,0)$ and that ∂D is such that the second initial boundary value problem is solvable. We can then state

Theorem 3 There exists a unique solution $u(x,t)$ to the problem $\Delta u = u_t + F(x,t)$ in $D \times (0,T]$ satisfying (2.8)-(2.13) and (2.14').

The proof follows the same lines as above with the Green's function G for $\bar{D} \times [0,T]$ replaced by the Neumann function N.

III. NUMERICAL METHODS

1. Method for Approximating Solutions to Partial Differential Equations

The method of finite differences is used to approximate solutions to the partial differential equations. The basic idea is as follows.

Construct a rectilinear system of grid points. Let I, J and N be integers and set $h = \frac{1}{I}$, $k = \frac{1}{J}$, $\Delta t = \frac{1}{N}$. Here h, k and Δt are the x-step size, y-step size and the t-step size respectively. The lengths of the x, y and t intervals are taken to be 1 (the intervals could be any finite lengths, here they are taken to be 1 for simplicity). Define the grid points $(x_i, y_j, t_n) = (ih, jk, n\Delta t)$. Schematically, the points can be represented in a 3-dimensional array as in Figure 3.1.

The method is to determine a function U defined on the grid points which approximates the values of u, the solution to the partial differential equation, at the grid points by using some algorithms which will be derived in the next section. The values of U from the previous time step are used to calculate the values of U for the present time step.

In this thesis, numerical examples are done in two-dimensions to save computer time, but the method is the same in case of three or higher dimensions.

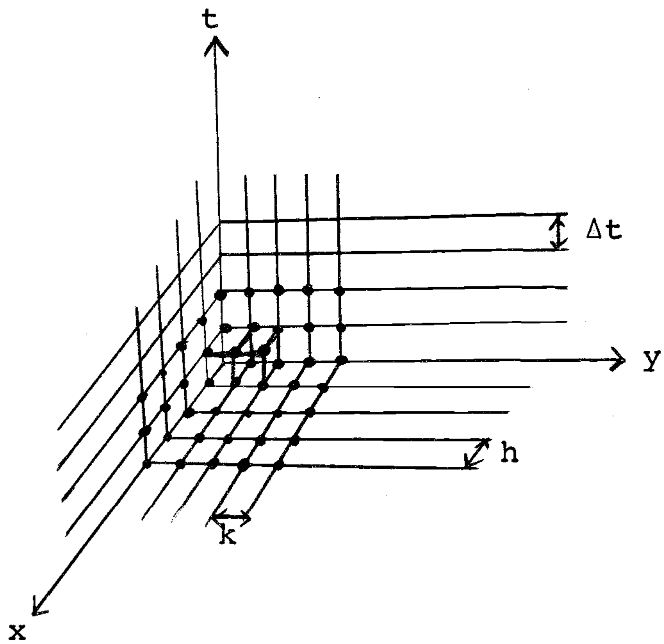

Figure 3.1. Representation of the grid points in 3-dimensional diagram.

All the examples shown will employ the forward time explicit scheme as it is easier to program than implicit schemes and reasonable accuracy can be obtained. The only restriction of the scheme is the limitation on the size of the mesh ratios.

2. Derivation of Algorithm for the Equation $u_{xx} + u_{yy} = u_t$

To derive the algorithm, Taylor's formula is used. It is found that:

$$\frac{u^n_{i+1,j} - 2u^n_{ij} + u^n_{i-1,j}}{h^2} = u_{xx} + \frac{h^2}{12} u_{xxxx} + O(h^4) \tag{3.1}$$

$$(3.2) \qquad \frac{u^n_{i,j+1} - 2u^n_{ij} + u^n_{i,j-1}}{k^2} = u_{yy} + \frac{k^2}{12} u_{yyyy} + O(k^4)$$

$$(3.3) \qquad \frac{u^{n+1}_{ij} - u^n_{ij}}{\Delta t} = u_t + \frac{\Delta t}{2} u_{tt} + O(\Delta t^2) \ ,$$

where $u^n_{ij} = u(x_i, y_j, t_n) = u(ih, jk, n\Delta t)$. Using (3.1), (3.2) and (3.3) in $u_{xx} + u_{yy} - u_t = 0$, we obtain

$$\frac{u^n_{i+1,j} - 2u^n_{ij} + u^n_{i-1,j}}{h^2} - \frac{h^2}{12} u_{xxxx} + O(h^4)$$

$$+ \frac{u^n_{i,j+1} - 2u^n_{ij} + u^n_{i,j-1}}{k^2} - \frac{k^2}{12} u_{yyyy} + O(k^4)$$

$$- \frac{u^{n+1}_{ij} - u^n_{ij}}{\Delta t} + \frac{\Delta t}{2} u_{tt} + O(\Delta t^2) = 0 \ .$$

Rearranging the terms,

$$\frac{u^n_{i+1,j} - 2u^n_{ij} + u^n_{i-1,j}}{h^2} + \frac{u^n_{i,j+1} - 2u^n_{ij} + u^n_{i,j-1}}{k^2} - \frac{u^{n+1}_{ij} - u^n_{ij}}{\Delta t}$$

$$= \frac{h^2}{12} u_{xxxx} + \frac{k^2}{12} u_{yyyy} - \frac{\Delta t}{2} u_{tt} + O(h^4+k^4+\Delta t^2).$$

Now let $k = h$,

$$(3.4) \begin{cases} \dfrac{u^n_{i+1,j} - 2u^n_{ij} + u^n_{i-1,j}}{h^2} + \dfrac{u^n_{i,j+1} - 2u^n_{ij} + u^n_{i,j-1}}{h^2} \\ - \dfrac{u^{n+1}_{ij} - u^n_{ij}}{\Delta t} = \dfrac{h^2}{12} [u_{xxxx} + u_{yyyy} - \dfrac{6\Delta t}{h^2} u_{tt}] + O(h^4 + \Delta t^2). \end{cases}$$

Equation (3.4) can be simplified by using the equation $u_{xx} + u_{yy} = u_t$.

On differentiating with respect to t, we get

$$(u_{xx})_t + (u_{yy})_t = u_{tt}$$

$$(u_t)_{xx} + (u_t)_{yy} = u_{tt}$$

$$(u_{xx} + u_{yy})_{xx} + (u_{xx} + u_{yy})_{yy} = u_{tt}$$

$$u_{xxxx} + u_{yyyy} + 2u_{xxyy} = u_{tt} .$$

Now choose $\frac{6\Delta t}{h^2} = 1$. Then equation (3.4) can be written as

$$(3.5)\quad \begin{cases} \dfrac{u^n_{i+1,j} - 2u^n_{ij} + u^n_{i-1,j}}{h^2} + \dfrac{u^n_{i,j+1} - 2u^n_{ij} + u^n_{i,j-1}}{h^2} \\ - \dfrac{u^{n+1}_{ij} - u^n_{ij}}{\Delta t} = \dfrac{h^2}{12}[u_{xxxx}+u_{yyyy}-(u_{xxxx}+u_{yyyy}+2u_{xxyy})] \\ + O(h^4 + \Delta t^2) \\ = -\dfrac{h^2}{6} u_{xxyy} + O(h^4 + \Delta t^2) \end{cases}$$

U_{xxyy} in equation (3.5) can be written in the form

$$(u_{xx})_{yy} = \left(\frac{u^n_{i+1,j} - 2u^n_{ij} + u^n_{i-1,j}}{h^2}\right)_{yy}$$

$$= \frac{1}{h^2}\left\{\frac{u^n_{i+1,j+1} - 2u^n_{i+1,j} + u^n_{i+1,j-1}}{h^2}\right.$$

$$- 2\,\frac{u^n_{i,j+1} - 2u^n_{ij} + u^n_{i,j-1}}{h^2}$$

$$\left. + \frac{u^n_{i-1,j+1} - 2u^n_{i-1,j} + u^n_{i-1,j-1}}{h^2}\right\} + O(h^4).$$

Hence equation (3.5) can be written as

$$\frac{u^n_{i+1,j} - 2u^n_{ij} + u^n_{i-1,j}}{h^2} + \frac{u^n_{i,j+1} - 2u^n_{ij} + u^n_{i,j-1}}{h^2}$$

$$- \frac{u^{n+1}_{ij} - u^n_{ij}}{\Delta t}$$

$$= -\frac{1}{6h^2}\{u^n_{i-1,j+1} - 2u^n_{i-1,j} + u^n_{i-1,j-1} - 2u^n_{i,j+1} + 4u^n_{ij}$$

$$- 2u^n_{i,j-1} + u^n_{i+1,j+1} - 2u^n_{i+1,j} + u^n_{i+1,j-1}\}$$

$$+ 0(h^4 + \Delta t^2).$$

Thus, the solution u satisfies

$$(3.6) \quad \left\{ \begin{aligned} u^{n+1}_{ij} &= \{\tfrac{1}{36} \; 16u^n_{ij} + 4u^n_{i+1,j} + 4u^n_{i-1,j} + 4u^n_{i,j+1} \\ &\quad + 4u^n_{i,j-1} + u^n_{i-1,j+1} + u^n_{i-1,j-1} + u^n_{i+1,j+1} \\ &\quad + u^n_{i+1,j-1}\} + 0(h^4 + \Delta t^2) \end{aligned} \right.$$

at the grid points. Schematically, this can be represented by Figure 3.2.

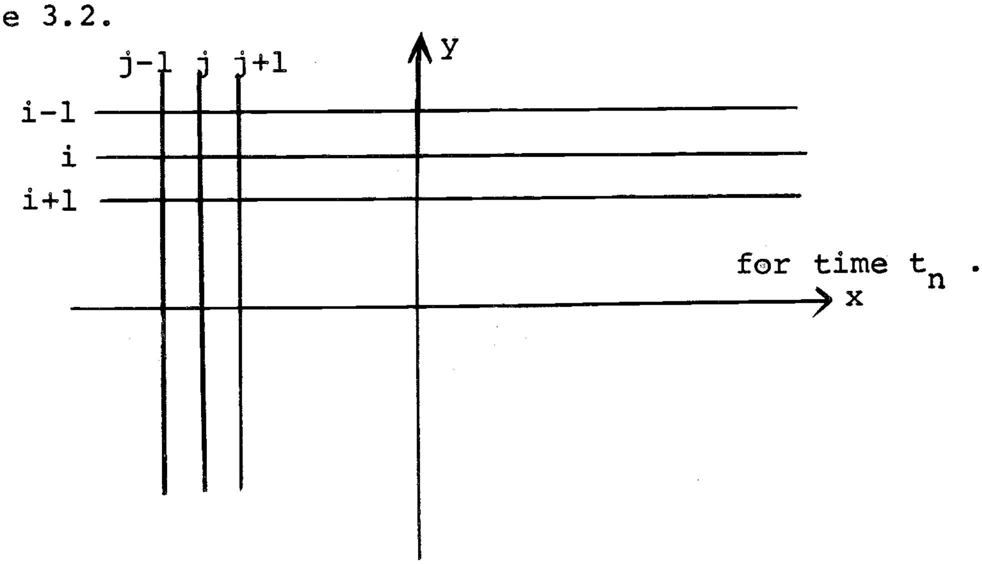

Figure 3.2. Schematic representation of the grid points.

To calculate U at the point (x_i, y_j) for the time step t_{n+1}, 8 points around (x_i, y_j) from the time step t_n are used.

3. Derivation of Algorithm to Solve $m(t) = \iint_{\Sigma} u(x,y,t)\,dxdy$, where $\Sigma = \{(x,y)\,|\,x^2+y^2 \leq .2^2\}$

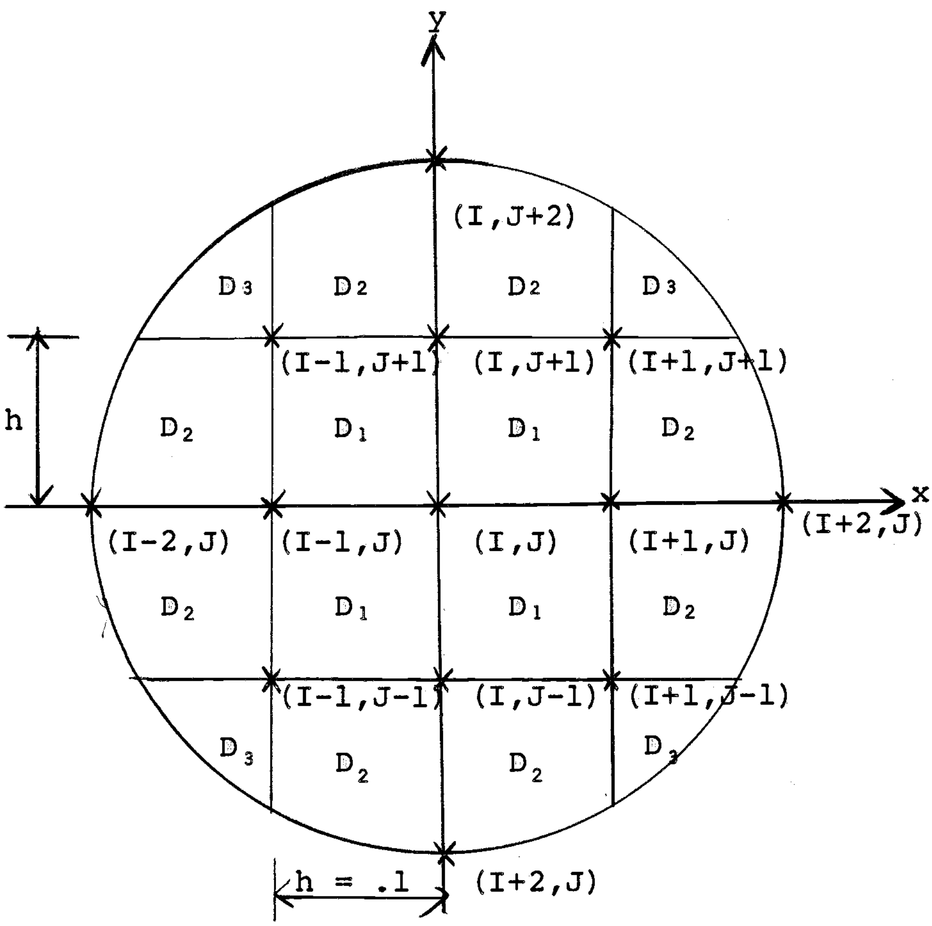

Figure 3.3. Region Σ.

The region Σ is represented in Figure 3.3 with grid points as shown. The location of the pump is at (x_i, y_j). The x- and y-step sizes are chosen to be of the same length h, where h is chosen to be 0.1 in all the examples presented in this thesis. Subregions marked D_1 have area h^2, subregions D_2 have area $h^2[\frac{\pi}{3} + \frac{\sqrt{3}}{2} - 1]$ and subregions D_3 have area $h^2[\frac{\pi}{3} - \sqrt{3} + 1]$. The average value of u in each subregion is multiplied by the area of that subregion. This gives the approximate mass in that subregion. Summing up all the masses in all the subregions will give the value of m at the corresponding time step t_n. Hence

$$
\begin{aligned}
m(t_n) \approx {} & \Big\{\frac{u^n_{i-1,j} + u^n_{ij} + u^n_{i,j+1} + u^n_{i-1,j+1}}{4} \\
& + \frac{u^n_{ij} + u^n_{i,j+1} + u^n_{i+1,j} + u^n_{i+1,j+1}}{4} \\
& + \frac{u^n_{i-1,j} + u^n_{ij} + u^n_{i-1,j-1} + u^n_{i,j-1}}{4} \\
& + \frac{u^n_{ij} + u^n_{i+1,j} + u^n_{i,j-1} + u^n_{i+1,j-1}}{4}\Big\}h^2 \\
& + \Big\{\frac{u^n_{i-2,j} + u^n_{i-1,j} + u^n_{i-1,j+1}}{3} + \frac{u^n_{i-2,j} + u^n_{i-1,j} + u^n_{i-1,j-1}}{3} \\
& + \frac{u^n_{i-1,j-1} + u^n_{i,j-1} + u^n_{i,j-2}}{3} + \frac{u^n_{i,j-1} + u^n_{i+1,j-1} + u^n_{i,j-2}}{3} \\
& + \frac{u^n_{i+1,j-1} + u^n_{i+1,j} + u^n_{i+2,j}}{3} + \frac{u^n_{i+1,j} + u^n_{i+2,j} + u^n_{i+1,j+1}}{3} \\
& + \frac{u^n_{i,j+2} + u^n_{i,j+1} + u^n_{i+1,j+1}}{3} \\
& + \frac{u^n_{i-1,j+1} + u^n_{i,j+1} + u^n_{i,j+2}}{3}\Big\}h^2\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2} - 1\right)
\end{aligned}
$$

$$+ \{u^n_{i-1,j+1} + u^n_{i-1,j-1} + u^n_{i+1,j-1} + u^n_{i+1,j+1}\}h^2(\frac{\pi}{3} - \sqrt{3} + 1).$$

On simplifying,

$$(3.7) \quad m(t_n) \approx \{u^n_{ij} + \frac{1}{2}(u^n_{i,j+1} + u^n_{i-1,j} + u^n_{i,j-1} + u^n_{i+1,j})$$

$$+ \frac{1}{4}(u^n_{i-1,j+1} + u^n_{i+1,j+1} + u^n_{i-1,j-1} + u^n_{i+1,j-1})\}h^2$$

$$+ \{u^n_{i-2,j} + u^n_{i-1,j} + u^n_{i-1,j-1} + u^n_{i,j-1}$$

$$+ u^n_{i,j-2} + u^n_{i+1,j-1} + u^n_{i+1,j} + u^n_{i+2,j}$$

$$+ u^n_{i+1,j+1} + u^n_{i,j+1} + u^n_{i,j+2}$$

$$+ u^n_{i-1,j+1}\}\frac{2h^2}{3}(\frac{\pi}{3} + \frac{\sqrt{3}}{2} - 1)$$

$$+ \{u^n_{i-1,j+1} + u^n_{i-1,j-1} + u^n_{i+1,j-1}$$

$$+ u^n_{i+1,j+1}\}h^2(\frac{\pi}{3} - \sqrt{3} + 1) .$$

IV. EXAMPLES

In this chapter, some examples are solved using the explicit finite difference method. A complete listing of the programs and the plots is shown in the appendix. In drawing the plots, a subroutine CONTUR in the user library PLOTLIB stored on UN=LIBRARY is used. The subroutine is a computer program designed by the computer center at this university and its purpose is to generate contour plots. It accepts an arbitrary sized two-dimensional array of floating point numbers and produces the contour map on a one-inch square grid (the actual grid lines may be suppressed). Options are included to label some of the contours. The accuracy of the program is limited by the linear interpolation process used in finding the values of the points along contour segments. It is also limited by the resolution of the off-line plotter. A more detailed description about the application of the subroutine is included in the appendix.

In all the examples shown here, we take, without loss of generality, the region under consideration to be a square of length 2 with center at the origin. The origin is where the pump is located. This is shown in Figure 4.1.

We will treat three different types of problems here. Each problem is solved twice by applying a different pump function. Problems 1, 3, and 5 are three different types

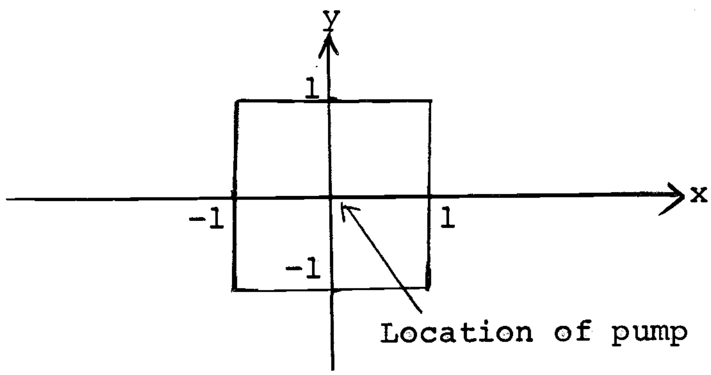

Figure 4.1. The region to be considered and the location of the pump.

of problems with the pump function defined by

$$(4.1)\quad \begin{cases} f(m(t),t) = \begin{cases} 0, & m(t) < 0 \\ m(t), & m(t) \geq 0 \end{cases} \\ \text{where } m(t) = \int_{\Sigma} u(x,y,t)\,dxdy \\ \text{and} \quad \Sigma = \{(x,y,)\,|\,x^2+y^2 \leq .2^2\}. \end{cases}$$

Actually, we tacitly assume for the purposes of computation that for values of m sufficiently large, i.e. larger, say, than the total mass of the system, f becomes constant. Problems 2, 4 and 6 are the same problems as problems 1, 3 and 5 except with a different pump function which is given by

$$(4.2)\qquad f(m(t),t) = \begin{cases} 0, & m < 0 \\ [m(t)]^2, & 0 \leq m \leq p \\ p^2, & m > p. \end{cases}$$

Here m(t) is the same as in (4.1). In all the examples, p is taken to be 3. The graphs of f in (4.1) and (4.2) are represented as follows.

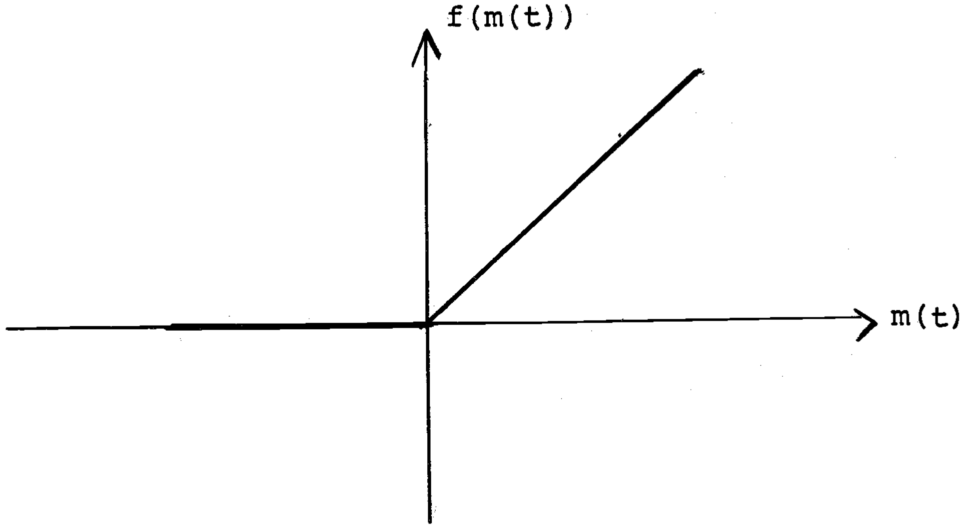

Figure 4.2. Graph of the pump function f defined in (4.1).

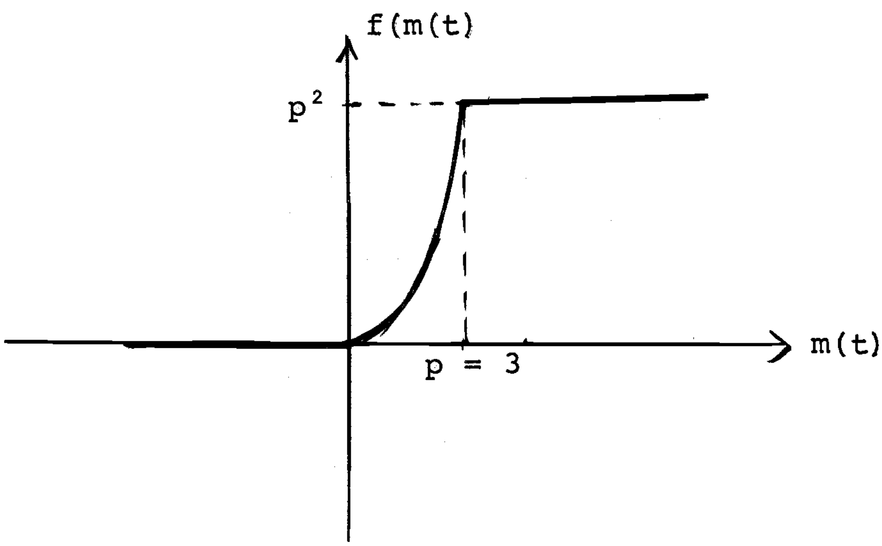

Figure 4.3. Graph of the pump function f defined in (4.2).

The six problems are as follows.

<u>Problem 1</u> Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$, $-1 \leq x,\ y \leq 1,\ t > 0,$

$$u(x,y,0) = 100, \quad -1 \leq x \leq 1, \quad -1 \leq y \leq 1,$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=-1} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=1} = 0,$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=-1} = 0, \quad \left.\frac{\partial u}{\partial y}\right|_{y=1} = 0,$$

$$u(0,0,t) = -f(m(t)),$$

where f(m(t)) is defined in (4.1).

The problem is represented in Figure 4.4.

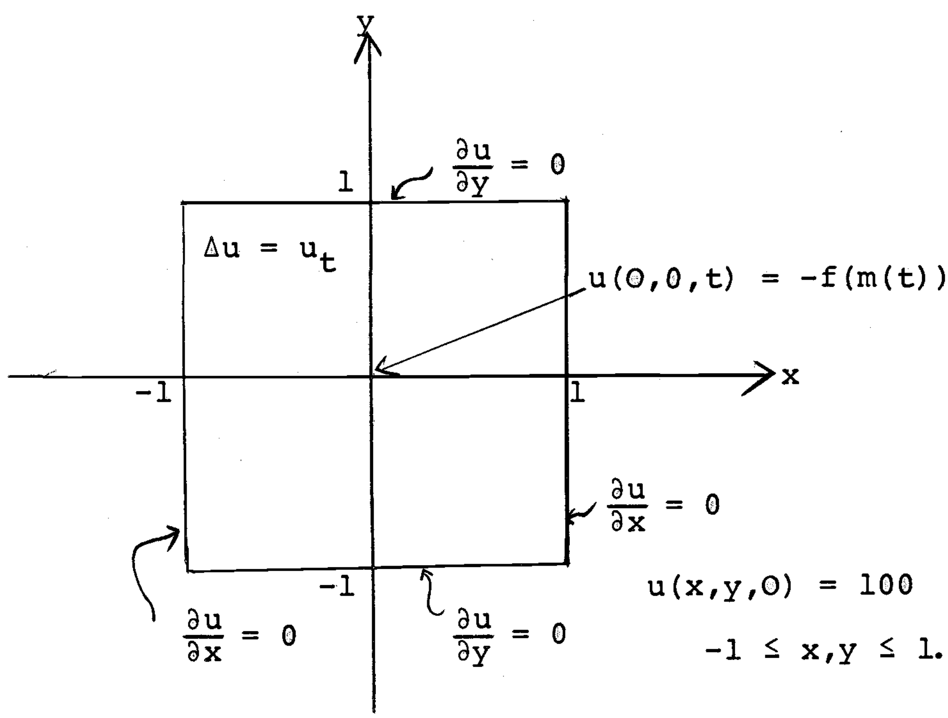

Figure 4.4. Representation of Problem 1.

To solve the problem numerically, (3.6) in Chapter 3 is used to calculate the value of u at the interior points. To calculate the value of u at the boundaries, let us consider first the boundary x=1, where $u_x = 0$. It is found that

(4.3)
$$u(1,y,t) = \frac{18u(1-h,y,t)-9u(1-2h,y,t)+2u(1-3h,y,t)}{11} + O(h^4).$$

This is obtained by using the approximation

$$u_x \approx \alpha_1 u(1,y,t) + \alpha_2 u(1-h,y,t) + \alpha_3 u(1-2h,y,t) + \alpha_4 u(1-3h,y,t)$$

With the condition $u_x = 0$, α_1, α_2, α_3 and α_4 are found to be equal to $\frac{11}{6h}$, $\frac{-3}{h}$, $\frac{3}{2h}$, $\frac{-1}{3h}$ respectively. A similar approach can be applied to the other three boundaries, it is found that at the boundary x = -1,

$$u(-1,y,t) = \frac{18u(-1+h,y,t)-9u(-1+2h,y,t)+2u(-1+3h,y,t)}{11} + O(h^4).$$

At the boundary y=1,

(4.4)
$$u(x,1,t) = \frac{18u(x,1-h,t)-9u(x,1-2h,t)+2u(x,1-3h,t)}{11} + O(h^4).$$

At the boundary y = -1,

$$u(x-1,t) = \frac{18u(x,-1+h,t)-9u(x,-1+2h,t)+2u(x,-1+3h,t)}{11} + O(h^4).$$

Hence all the boundary values can be obtained by using these approximations. To calculate the value of u at the origin, (3.7) and (4.1) are used. When all the data points are obtained, subroutine CONTUR is called to plot contour graphs.

Problem 2 Same as Problem 1 with f defined by (4.2).

Problem 3 Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$, $-1 \le x, y \le 1, \ t > 0,$

$$u(x,y,0) = 100, \quad -1 \le x \le 1, \quad -1 \le y \le 1,$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=1} = 0, \quad \left.\frac{\partial u}{\partial y}\right|_{y=1} = 0,$$

$$u(-1,y,t) = 50, \quad -1 \le y \le 1,$$

$$(\vec{q} \cdot \nu)\Big|_{y=-1} = 10,$$

$$u(0,0,t) = -f(m(t)),$$

where f(m(t)) is defined by (4.1).

The problem is represented in Figure 4.5.

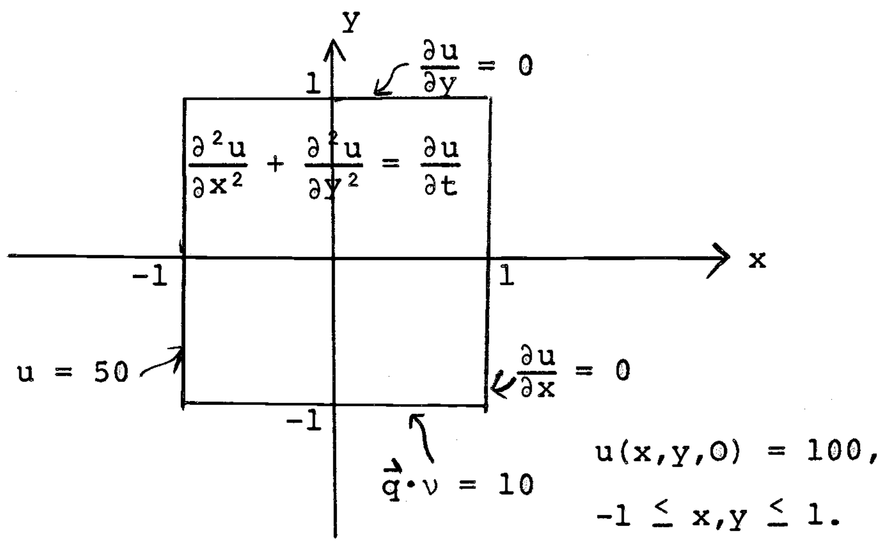

Figure 4.5. Representation of Problem 3.

Using (1.1) in Chapter 1 and taking $\vec{q}$ to be zero in the case of gaseous flow,

$$\vec{q} = -\frac{k}{\mu} \nabla p .$$

By a simple calculation,

$$\vec{q} = -\frac{1}{\mu\rho} \nabla u ,$$

$$(\vec{q} \cdot \nu) \Big|_{y=-1} = \frac{1}{\mu\rho} u_y .$$

After using the boundary condition $\vec{q} \cdot \nu = 10$ and taking $\mu\rho$ to be one, we get

$$u_y = 10.$$

u_y is approximated by an expression similar to (4.3). As before we find that

$$u(x,-1,t) = \frac{18u(x,-1+h,t)-9u(x,-1+2h,t)+2u(x,-1+3h,t)-60h}{11} + O(h^4),$$

$u(1,y,t)$ and $u(x,1,t)$ are given by (4.3) and (4.4) respectively.

<u>Problem 4</u> Same as Problem 3 with f defined by (4.2).

<u>Problem 5</u> Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$, $-1 \le x,\ y \le 1,\ t > 0,$

$$u(x,y,0) = 100, \quad -1 \le x \le 1, \quad -1 \le y \le 1,$$

$$u(x,1,t) = 0, \quad -1 \le x \le 1, \quad t > 0,$$

$$u(x,-1,t) = 10(1-x^2), \quad -1 \le x \le 1, \quad t > 0,$$

$$u(1,y,t) = 0 , \quad -1 \le y \le 1, \quad t > 0,$$

$$u(-1,y,t) = 20(1-y^4), \quad -1 \le y \le 1, \quad t > 0 .$$

The problem is represented in Figure 4.6.

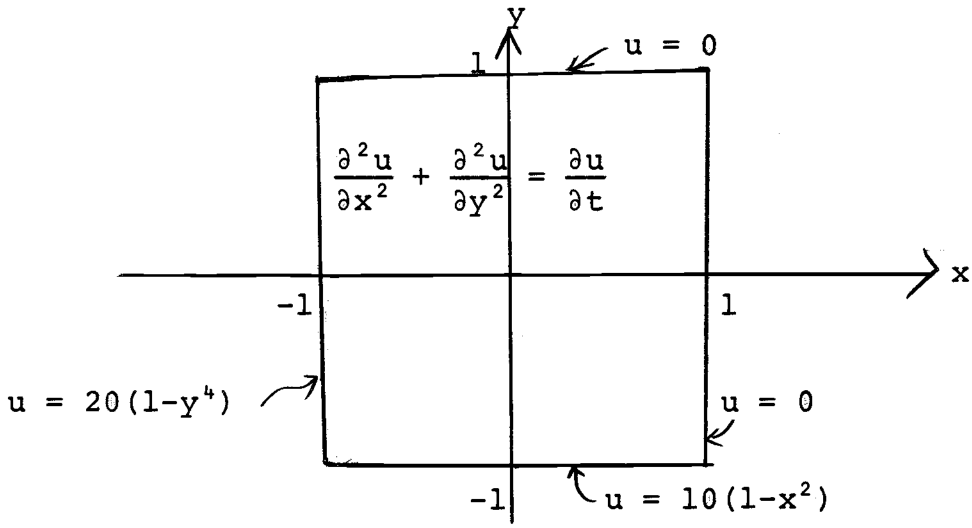

Figure 4.6. Representation of Problem 5.

Problem 6 Same as Problem 5 with f defined by (4.2).

V. CONCLUSIONS

Contour plots at two different instants of time for each problem are examined. We notice there is a change in the density which indicates the movement of fluid near the pump and in the region. By comparing plots of the same problem with different pump functions, once again we observe a change in the density but the patterns of the movement of fluid are similar in both cases. The change in the density is due to the amount of fluid being pumped. The results of each problem are discussed below.

1. Problem 1

There is symmetry about the origin in this problem. Initial pressure is set at 100, at the five-hundredth time step, there is a decrease in the pressure. At the thousandth time step, there is a further decrease. This is due to the fact that there is no flow across the boundaries, the initial fluid present in the region flows into the pump while no additional fluid is allowed to enter the region. Eventually there will be no fluid left in the region as all of it is drawn into the pump.

2. Problem 2

A very similar interpretation as that of Problem 1 except that the pump is operated at a different rate.

Problems 1 and 2

At the five-hundredth and the thousandth time steps, the mass of fluid in the region is greater than p (in all the examples presented in this thesis, p is taken to be 3). Hence the pump in Problem 2 is stronger than that in Problem 1 in the flow regime under consideration. A stronger pump means more fluid is drawn into the pump from the region and this is brought about by causing a bigger reduction in the density in Problem 2 compared to Problem 1. In Problem 1, fluid is drawn towards the pump at a slower rate but the flow patterns in both problems are very similar at the corresponding time steps.

3. Problem 3

The two boundaries at $x = 1$ and $y = 1$ are insulated, that means no fluid is allowed to cross the boundaries there. At the boundary where $x = -1$ some fluid is allowed to enter the region, but the amount of fluid entering is governed by the fact that the density at the boundary always has to be maintained at 50. At the boundary where $y = -1$, fluid is brought in at a rate of 10 mass units per time. By comparing the plots at the five-hundredth and the thousandth time steps, a movement of fluid is observed. Fluid enters the region at the boundaries where $x = -1$ and $y = -1$ and is being drawn into the pump. A change in the flow pattern is observed over time.

4. Problem 4

A similar interpretation as to that of Problem 3 except with a different pump function.

Problems 3 and 4

In both problems, the mass of fluid in the region is greater than p, hence Problem 4 has a stronger pump in the flow regime under consideration. The flow patterns in both problems are very similar at the corresponding time steps.

5. Problem 5

At the boundaries where $x = 1$ and $y = 1$, pressure is kept at zero. That means for example there is a ditch at those boundaries. At the boundaries where $x = -1$ and $y = -1$, fluid is brought into the region at a given rate. A flow pattern is observed over time while fluid is drawn into the pump.

6. Problem 6

A similar interpretation as to that of Problem 5 except with a different pump function.

Problems 5 and 6

In both problems, mass of fluid is less than 0.5 at both the time steps considered, hence the pump Problem 5 is slightly stronger in the flow regime under consideration.

As a result, more fluid is drawn into the pump and the density is reduced. The flow patterns are nearly the same in both problems. The change in density is very insignificant and it is due to the fact that the two pumps are almost of the same strength here.

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APPENDIX

Description of the Computer Program for Problem 1

The x- and y-step sizes are taken to be 0.1 and the t-step size is chosen to be [x-step size]^2/6. In the computer program, U(I,J,1), U(I,J,2) represent the values of U at the grid point (I,J) at the present and the next time steps respectively. The labelling convention of the grid points is the same as that in a matrix. The first index I represents the row, the second index J represents the column. The labelling convention is described by Figure A-1.

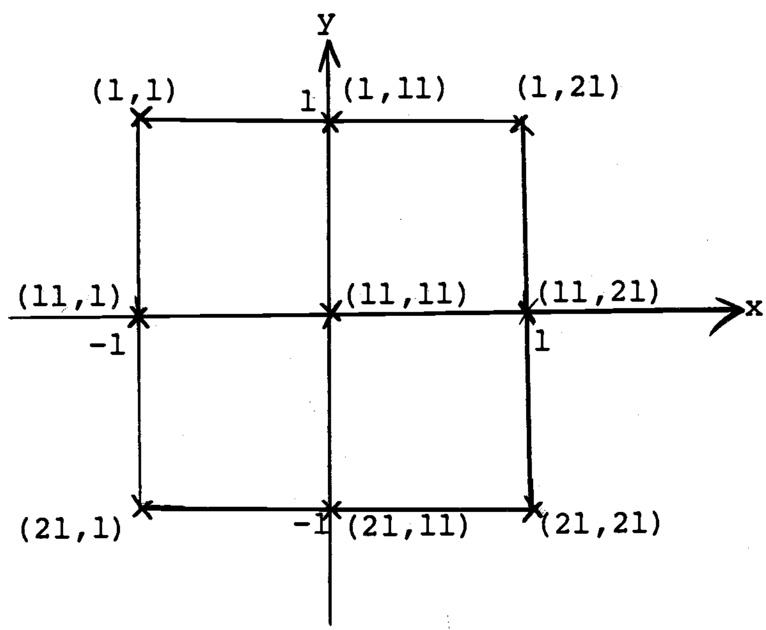


Figure A-1. Labelling convention of the grid points.

In the program, RM(N) represents the quantity m(t). The dimension of the array, RM, is 1,000.

Description of the Subroutine CONTUR

The purpose of the subroutine is to draw contour graphs. The calling statement is

CALL CONTUR (AM, M, N, CL, NL, LABMIN, LABBDR, IGRID)

The symbols are defined as follows:

AM The floating point array to be contoured. The whole array will be contoured. Its dimensions are passed as M and N.

M Integer variable which is the number of rows in AM.

N Integer variable which is the number of columns in AM.

CL A floating point array containing the contour levels to be plotted.

NL An integer variable whose absolute value is the number of contour levels to be found for this graph.

LABMIN An integer flag. LABMIN = 1 indicates that exterior contour segments and interior contour segments which represent a local maximum are to be labelled. If LABMIN = 0, contour levels are not labelled.

LABBDR An integer flag. LABBDR = 1 requests tick marks and corresponding scale values one inch apart on the frame of the contour graph. If LABBDR = 0, tick marks are drawn one inch apart, but they are not labelled.

IGRID An integer flag. If IGRID=1, a one-inch by one-inch grid is superimposed on the contour graph. If IGRID = 0, no grid is drawn.

The physical aspects of the plot are defined in labelled common as follows:

COMMON/PLT/ICODE, WIDTH, HEIGHT, NCHAR, LABELS (8), MODEL, IRATE, LUN

Here the symbols are defined as follows:

ICODE A code number indicating the plot device. In all the examples presented in this thesis, ICODE is set equal to three. This represents a Gerber plot device.

WIDTH The width of the plot in inches.

HEIGHT The height of the plot in inches.

NCHAR The number of characters in the user supplied label.

LABELS (8) An array dimensioned to eight containing the user label.

MODEL Indicates the type of Tek Terminal.

IRATE The transmission BAUD rate.

LUN The logical unit number or tape number assigned to the hard copy plotter. Usually 10 is used.

Listing of Programs and Plots

Here we present only the computer program of Problem 1. The other problems use the same basic computer program except with minor changes due to the boundary conditions and the pump functions.

Setting up the Job Deck

```
USER, <USER NUMBER>, <PASSWORD> .
CHARGE, <CHARGE NUMBER>, <PROJECT NAME>.
SETTL, <TIME IN SECONDS>.
TITLE. <YOUR NAME> .
ATTACH, PLOTLIB/UN = LIBRARY.
FTN, PD = 6.
LIBRARY, COMPLOT, PLOTLIB.
TITLE, FN = TAPE10, DC = GL.
LGO.
ROUTE, TAPE10, DC = GL.
7/8/9
```

```
PROGRAM ONE (INPUT=/150, OUTPUT=/150, TAPE10=0)
.
.
END
```

```
7/8/9
<YOUR NAME>, <CHARGE NUMBER>, 1 PLOT, 13 BY 13
6/7/8/9
```

```
      PROGRAM ONE (INPUT=/150,OUTPUT=/150,TAPE10=0)
      COMMON/PLT/ICODE,WIDTH,HEIGHT,NCHAR,LABELS(8),MODEL,IRATE,LUN
      DIMENSION U(21,21,2),CL(9),AM(21,21),RM(1000)
      DATA CL/20.0,25.0,30.0,40.0,45.0,50.0,60.0,65.0,70.0/
      DATA LABELS/10HPRESSURE-L,10HEVEL GRAPH,5HPLOT1/
      LUN=10
      WIDTH=10.
      HEIGHT=10.
      NCHAR=25
      LABMIN=1
      LABBDR=1
      IGRID=1
      ICODE=3
      MODEL=0
      IRATE=0
      H=DY=DX=0.1
      DT=DX**2/6.
      N4=IFIX(2./DX+0.5)
      N1=N4+1
      N3=N1+1
***** LET (IP,JP) BE THE LOCATION O F THE PUMP
      IP=JP=N4/2+1
      PI=4.*ATAN(1.)
      B=2.*(PI/3.+3.**0.5/2.-1.)/3.*H**2
      C=(PI/3.-3.**0.5+1.)*H**2
      WRITE 900
  900 FORMAT (1H1)
      PRINT 5,DX,DT
    5 FORMAT (5X,≠DX=≠,G20.10,5X,≠DT=≠,G20.10)
      DO 11 J=1,N1
      DO 10 I=1,N1
      U(I,J,1)=100.
   10 CONTINUE
   11 CONTINUE
      DO 25 N=1,1000
      M1=N/500
      N6=N-(500*M1)
      IF (N6 .EQ.0) 21,22
   21 PRINT 30,N
   30 FORMAT (5X,≠FOR THE≠,I5,≠TIME STEP≠,/,5X,≠FOR I,J=1 TO 21, U(I,J)
     C=≠)
***** FIND U FOR THE INTERIOR POINTS
   22 DO 35 J=2,N4
      DO 40 I=2,N4
      IF (I.EQ.IP.AND.J.EQ.JP)GO TO 45
      U(I,J,2)=(16.*U(I,J,1)+4.*U(I+1,J,1)+4.*U(I-1,J,1)+4.*U(I,J+1,1)
     C+4.*U(I,J-1,1)+U(I-1,J+1,1)+U(I-1,J-1,1)+U(I+1,J+1,1)+U(I+1,J-1,1)
     C)/36.
   45 U(IP,JP,2)=0
   40 CONTINUE
   35 CONTINUE
***** FIND U FOR THE BOUNDARY POINTS
      DO 70 J=1,N1
      U(1,J,2)=(18.*U(2,J,2)-9.*U(3,J,2)+2.*U(4,J,2))/11.
   70 CONTINUE
      DO 75 J=1,N1
      U(N1,J,2)=(18.*U(N1-1,J,2)-9.*U(N1-2,J,2)+2.*U(N1-3,J,2))/11.
   75 CONTINUE
```

```
      DO 80 I=1,N1
      U(I,N1,2)=(18.*U(I,N1-1,2)-9.*U(I,N1-2,2)+2.*U(I,N1-3,2))/11.
   80 CONTINUE
      DO 85 I=1,N1
      U(I,1,2)=(18.*U(I,2,2)-9.*U(I,3,2)+2.*U(I,4,2))/11.
   85 CONTINUE
      U(IP,JP,2)=-((0.5*(U(IP,JP+1,2)+U(IP-1,JP,2)+U(IP,JP-1,2)
     C+U(IP+1,JP,2))+0.25*(U(IP-1,JP+1,2)+U(IP+1,JP+1,2)+U(IP-1,JP-1,2)
     C+U(IP+1,JP-1,2)))*H**2+(U(IP-2,JP,2)+U(IP-1,JP,2)+U(IP-1,JP-1,2)
     C+U(IP,JP-1,2)+U(IP,JP-2,2)+U(IP+1,JP-1,2)+U(IP+1,JP,2)+U(IP+2,JP,
     C2)+U(IP+1,JP+1,2)+U(IP,JP+1,2)+U(IP,JP+2,2)+U(IP-1,JP+1,2))*3
     C+(U(IP-1,JP+1,2)+U(IP-1,JP-1,2)+U(IP+1,JP-1,2)+U(IP+1,JP+1,2))*C)
     C/(1.+H**2)
      IF (U(IP,JP,2).GT.0)12,14
   12 U(IP,JP,2)=0
   14 M=N/500
      N5=N-(500*M)
      IF (N5 .EQ. 0)16,51
   16 RM(N)=-U(IP,JP,2)
      DO 50 I=1,N1
      PRINT 55,(U(I,J,2),J=1,N1)
   55 FORMAT (+(/,5(5X,G20.10)),/,5X,G20.10)
   50 CONTINUE
      PRINT 52, RM(N)
   52 FORMAT (5X,≠RM=≠,G20.10)
      DO 101 I=1,21
      DO 102 J=1,21
      AM(I,J)=U(I,J,2)
  102 CONTINUE
  101 CONTINUE
      CALL CONTUR (AM,21,21,CL,9,LABMIN,LABBDR,IGRID)
   51 DO 60 J=1,N1
      DO 65 I=1,N1
      U(I,J,1)=U(I,J,2)
   65 CONTINUE
   60 CONTINUE
   25 CONTINUE
      END
```

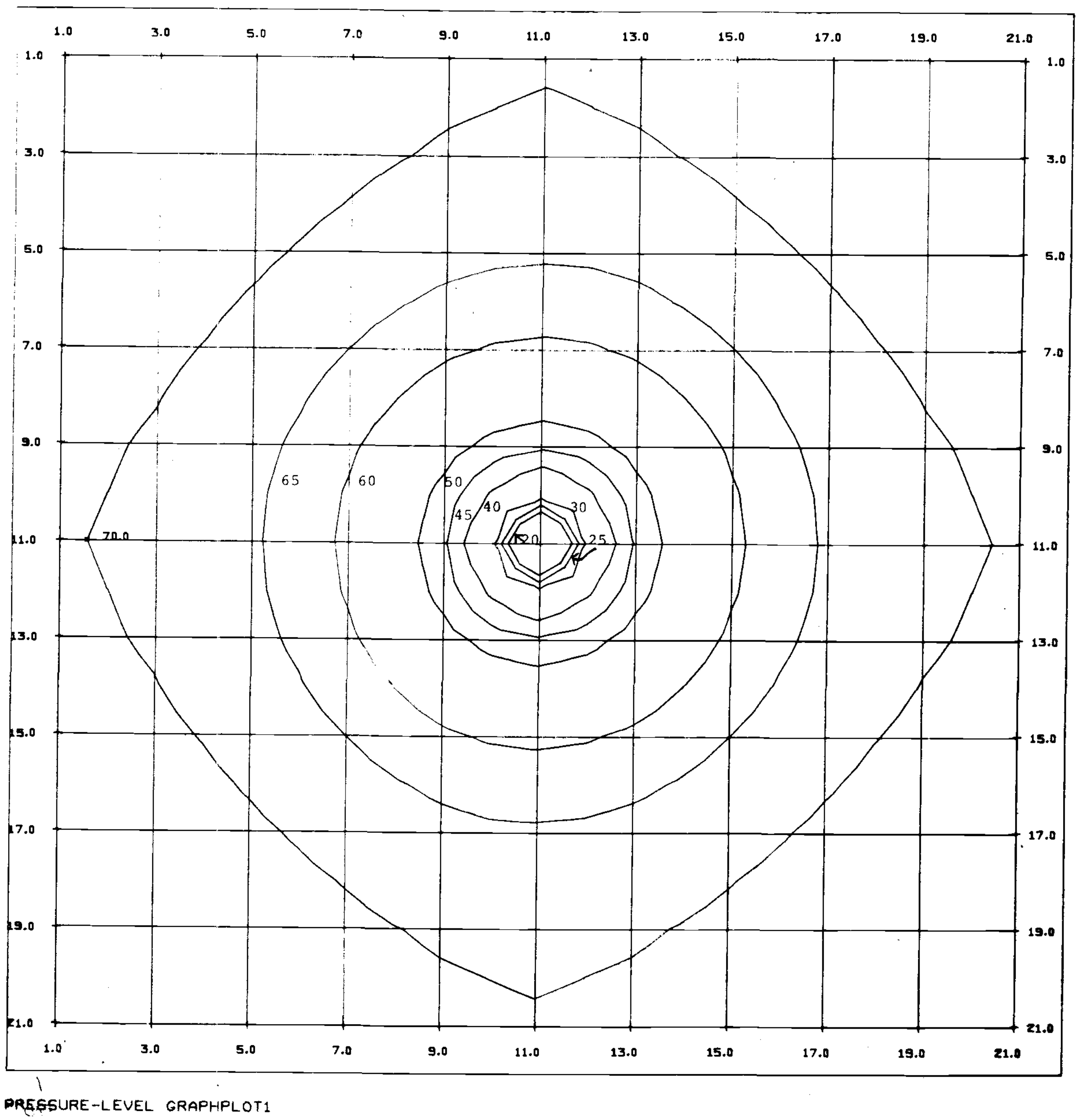


Figure A.2. Pressure-level graph at the 500th time step for Problem 1.

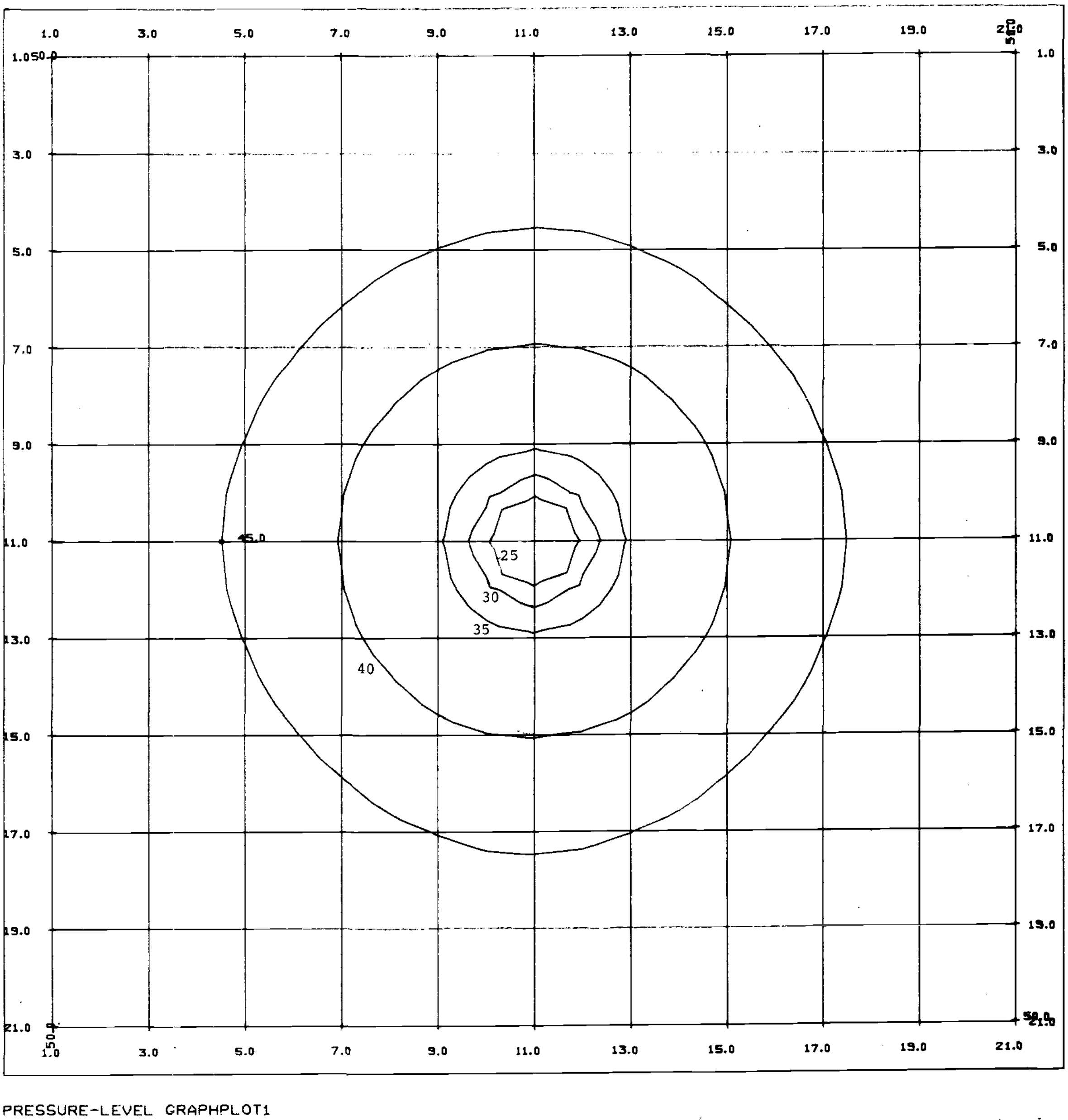


Figure A.3. Pressure-level graph at the 1000th time step for Problem 1.

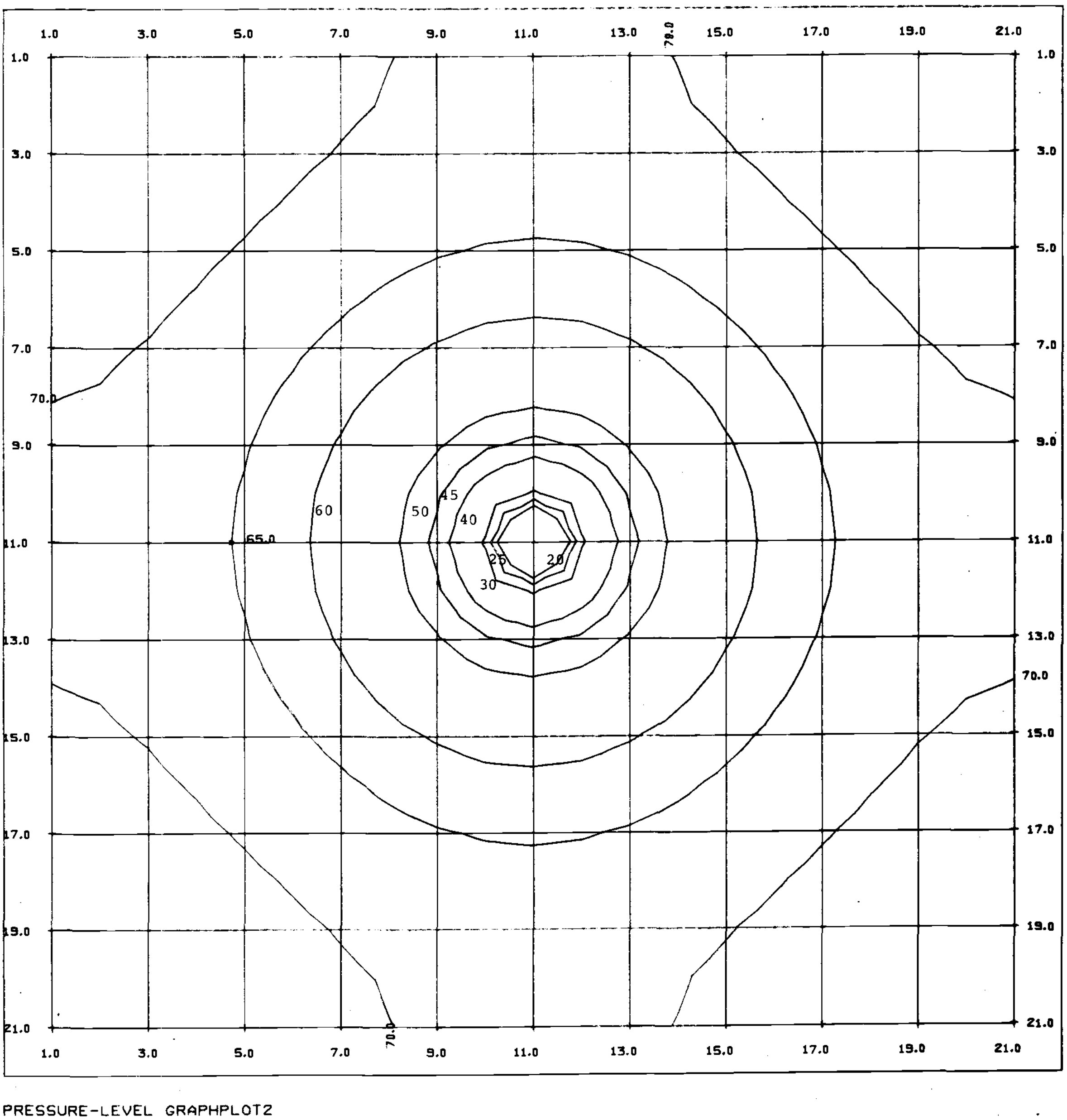
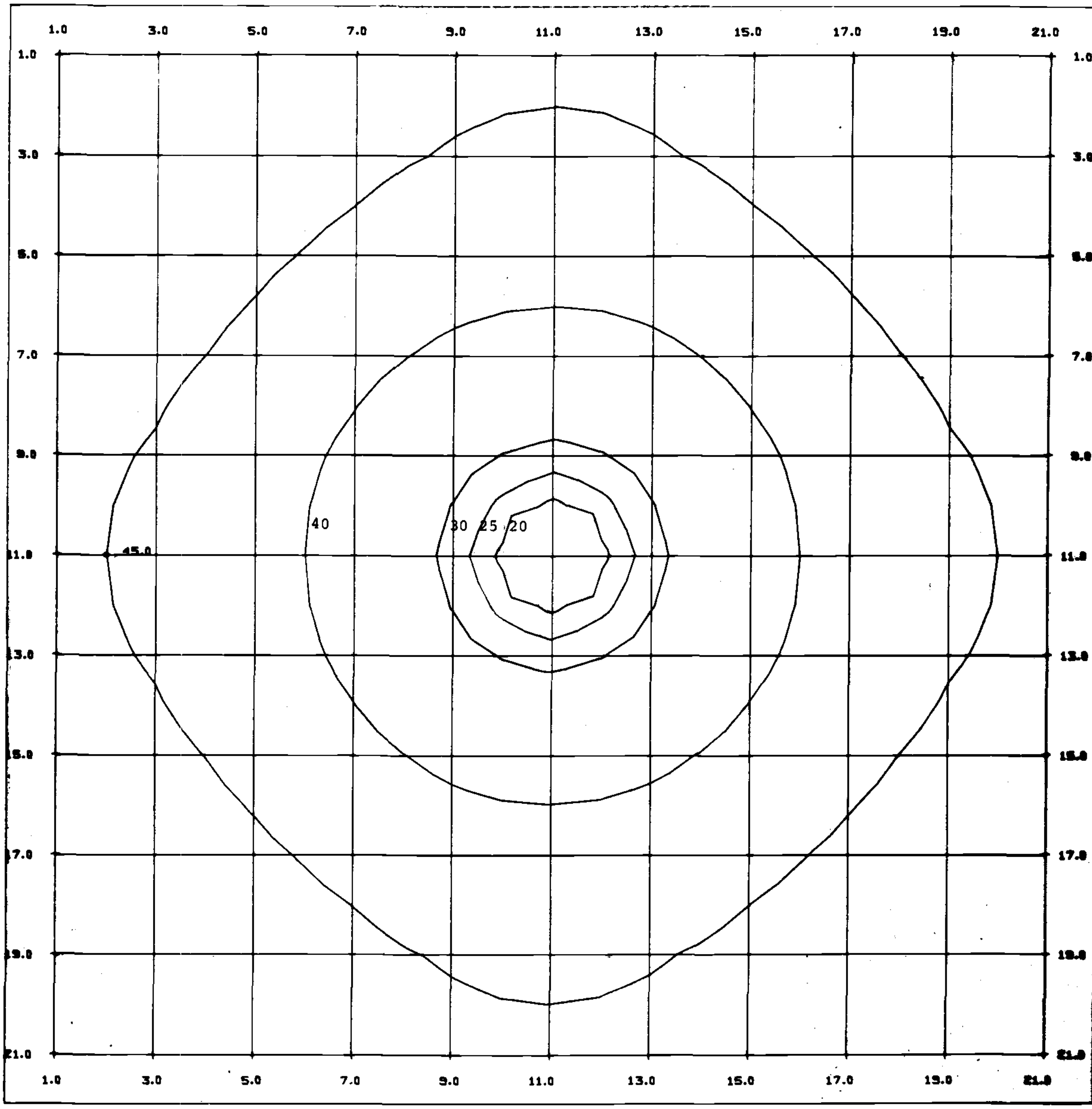
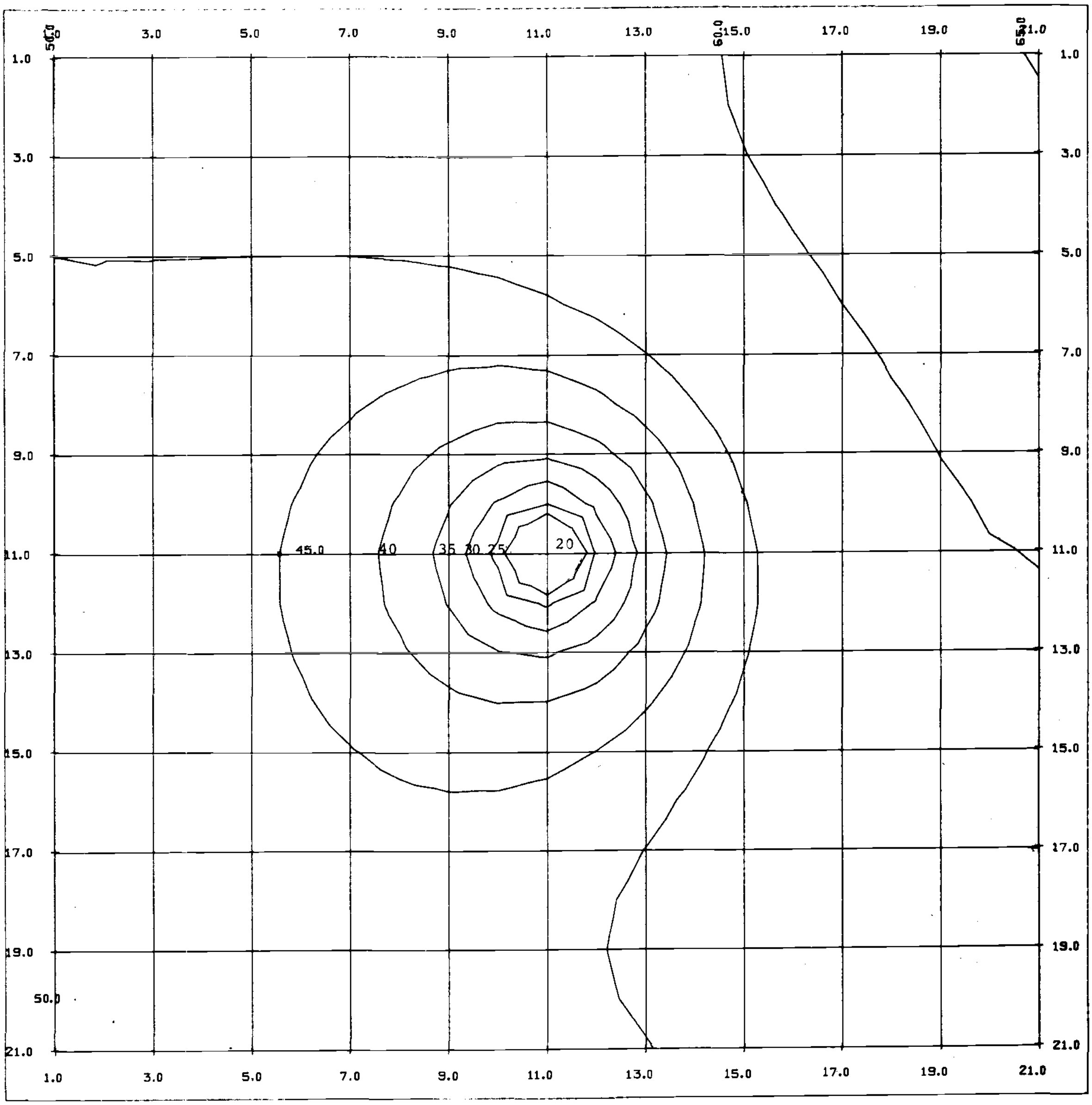


Figure A.4. Pressure-level graph at the 500th time step for Problem 2.



PRESSURE-LEVEL GRAPHPLOT2

Figure A.5. Pressure-level graph at the 1000th time step for Problem 2.



PRESSURE-LEVEL GRAPHPLOT3

Figure A.6. Pressure-level graph at the 500th time step for Problem 3.

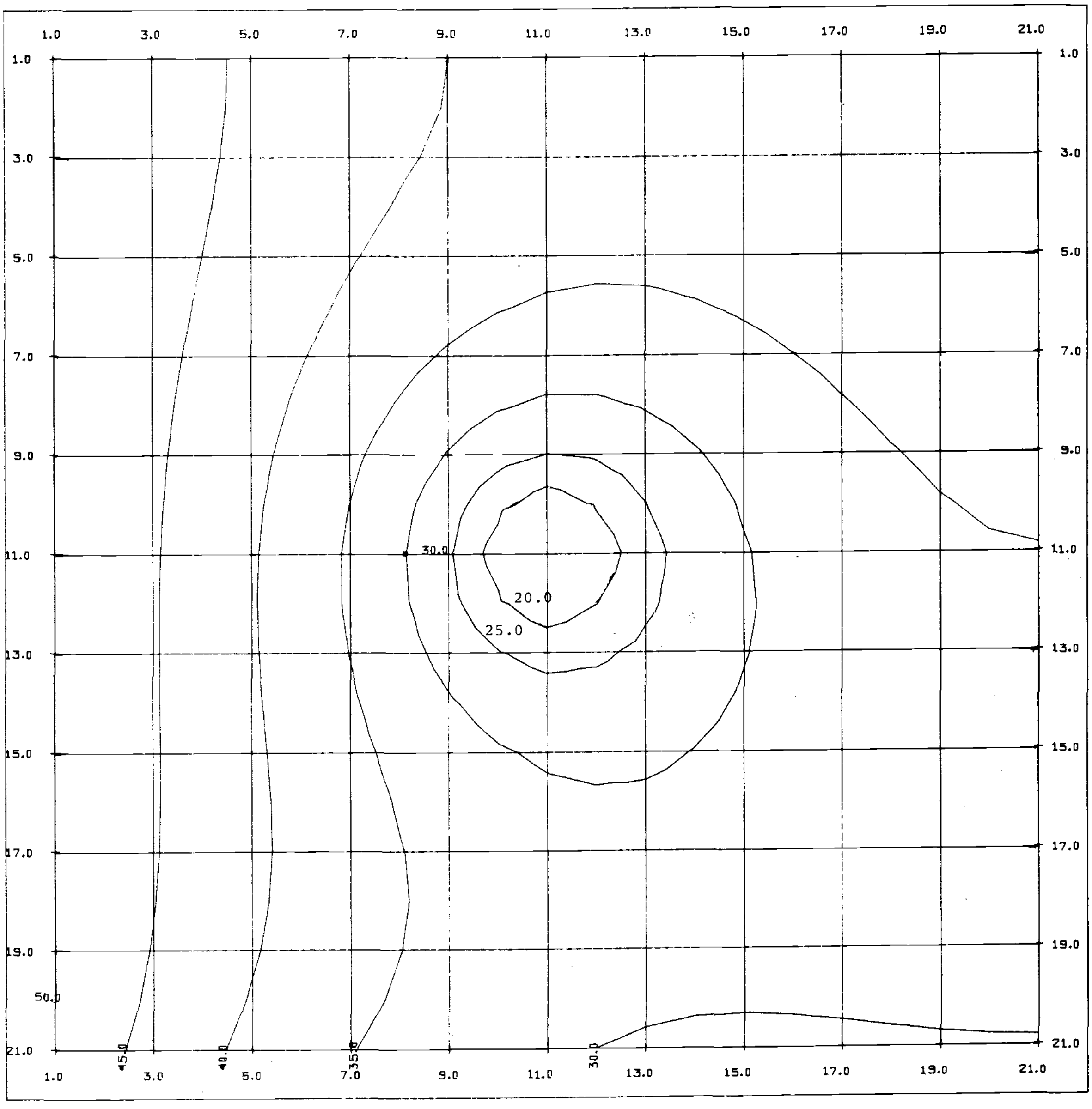


Figure A.7. Pressure-level graph at the 1000th time step for Problem 3.

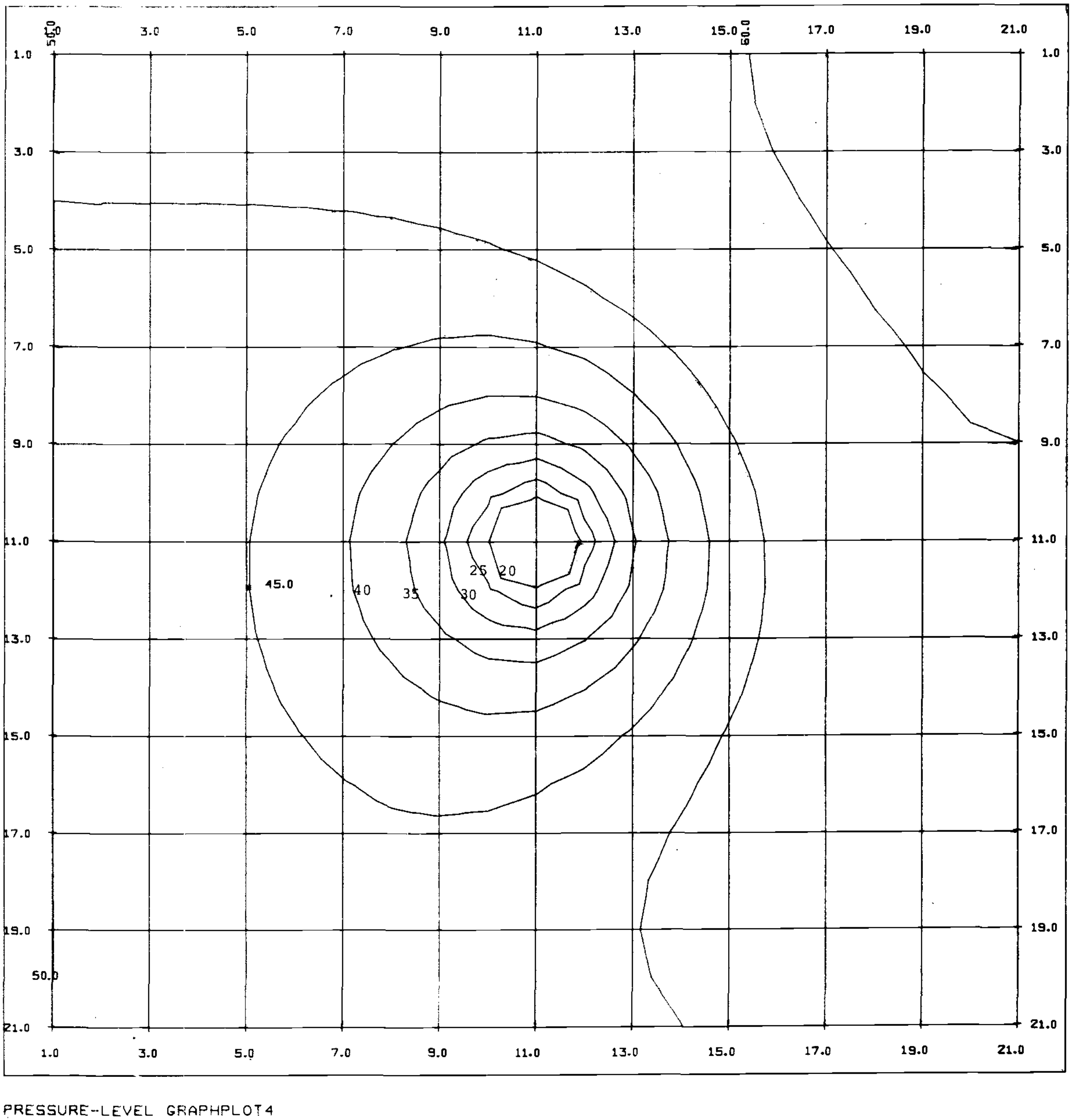
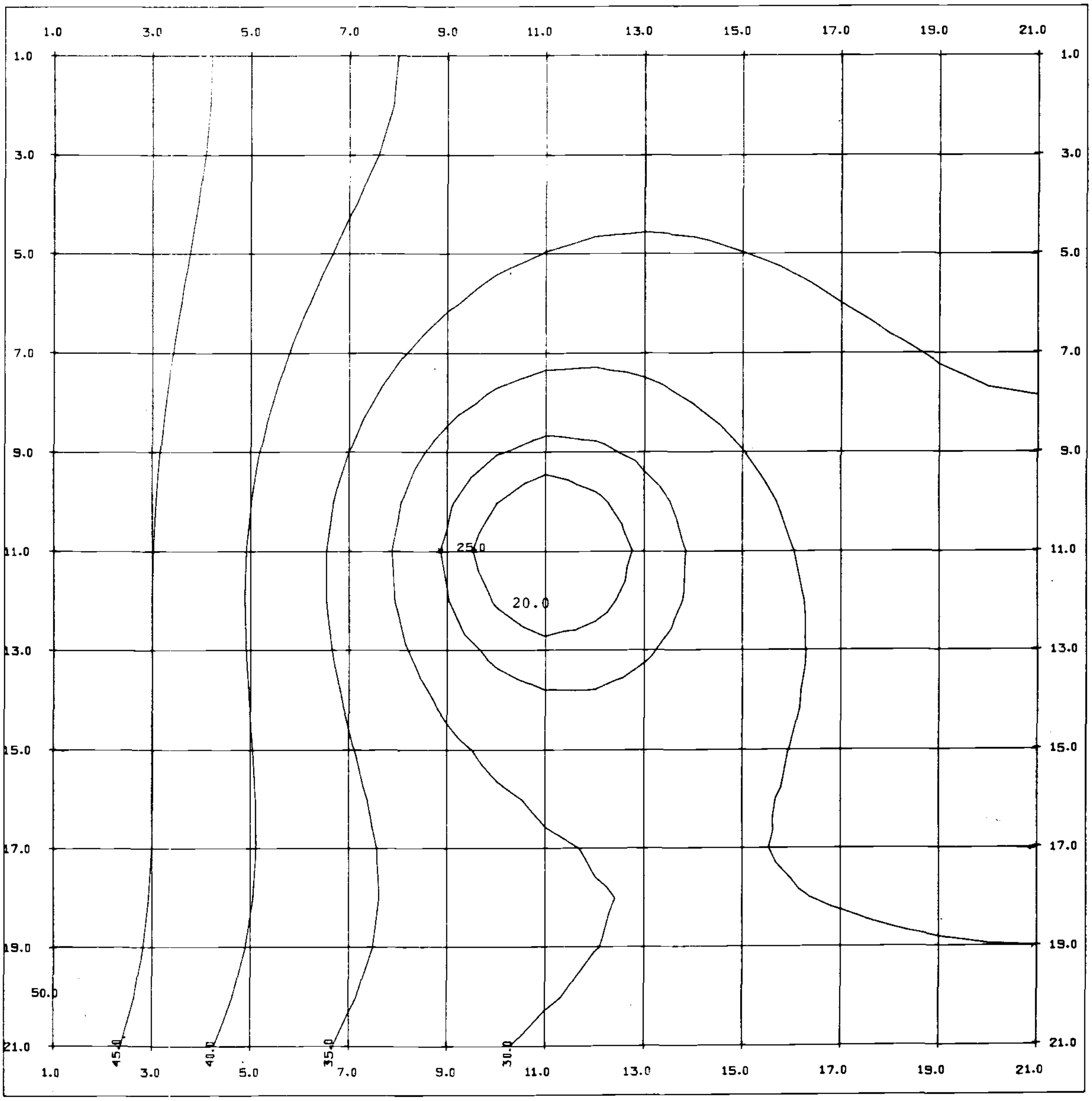


Figure A.8. Pressure-level graph at the 500th time step for Problem 4.



PRESSURE-LEVEL GRAPHPLOT4

Figure A.9 Pressure-level graph at the 1000th time step for Problem 4

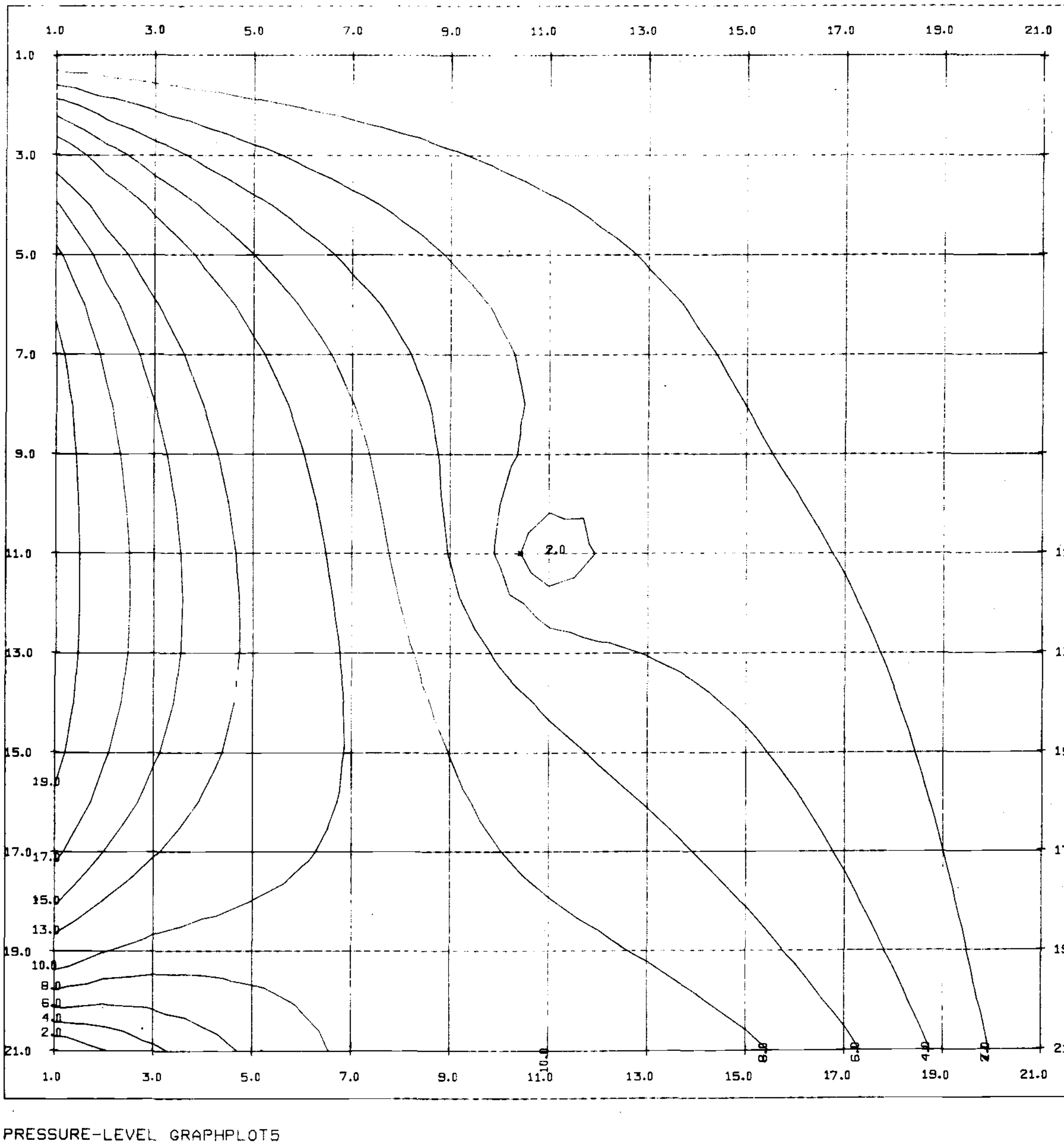
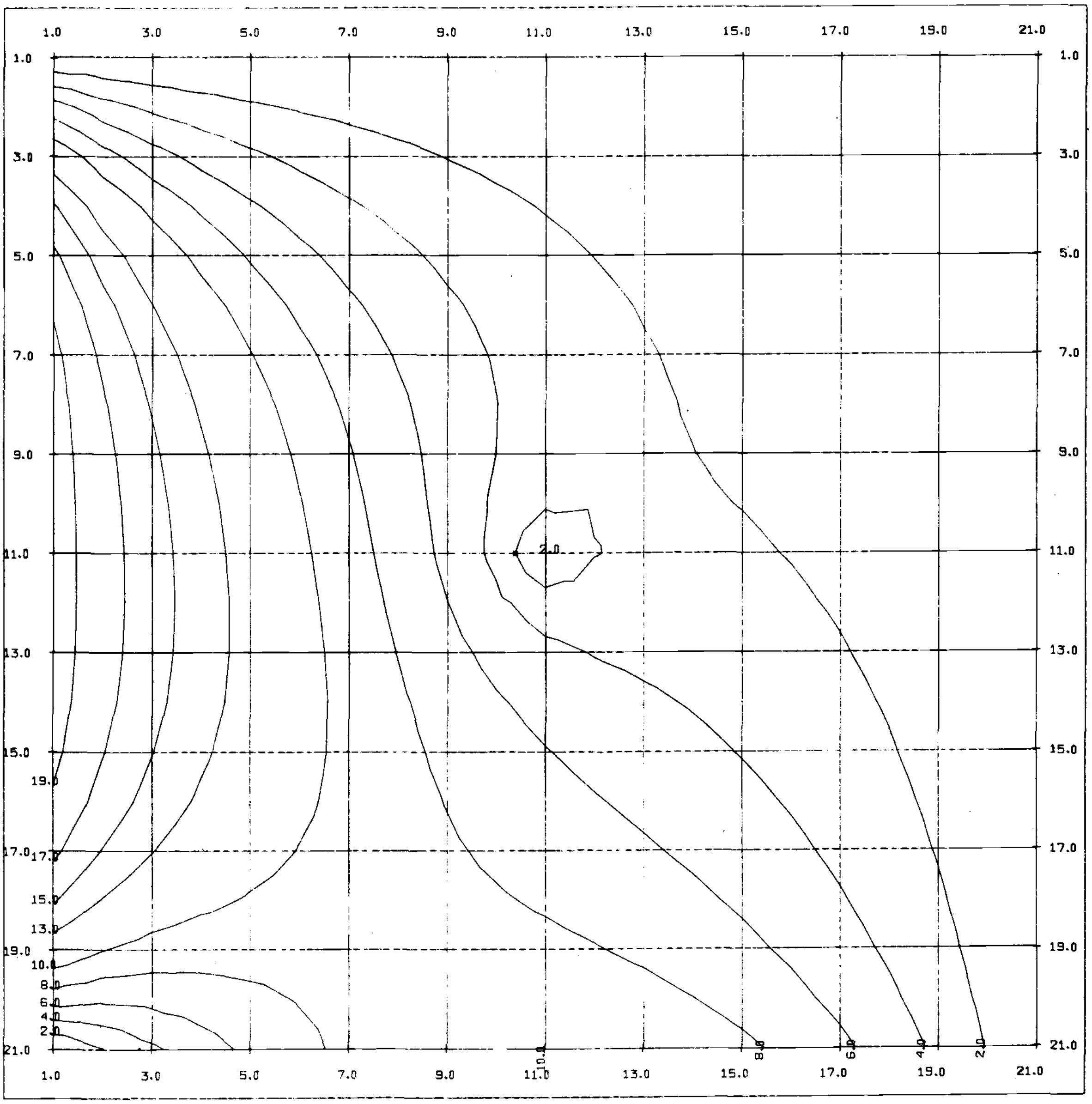
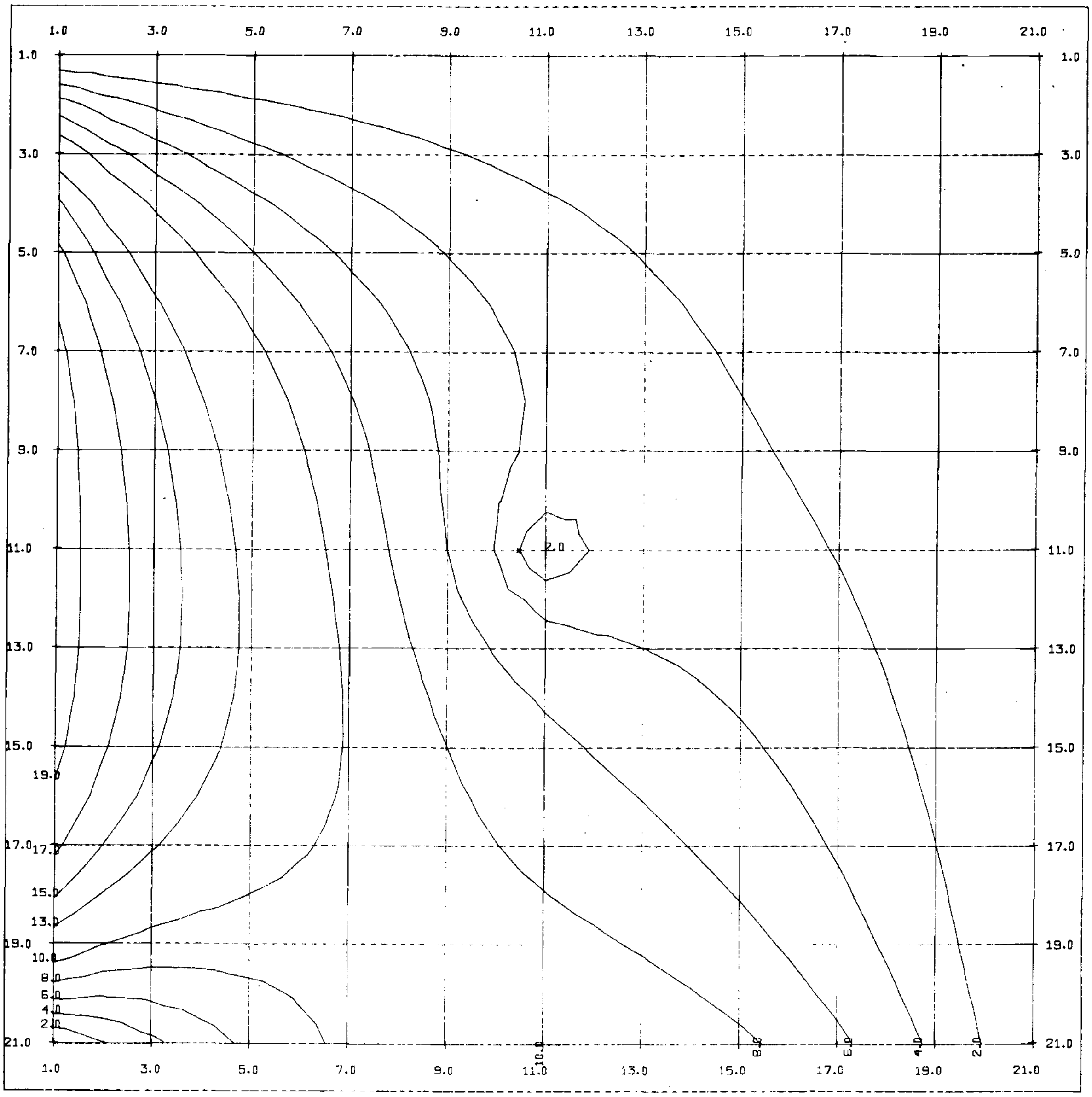


Figure A.10 Pressure-level graph at the 500th time step for Problem 5 .



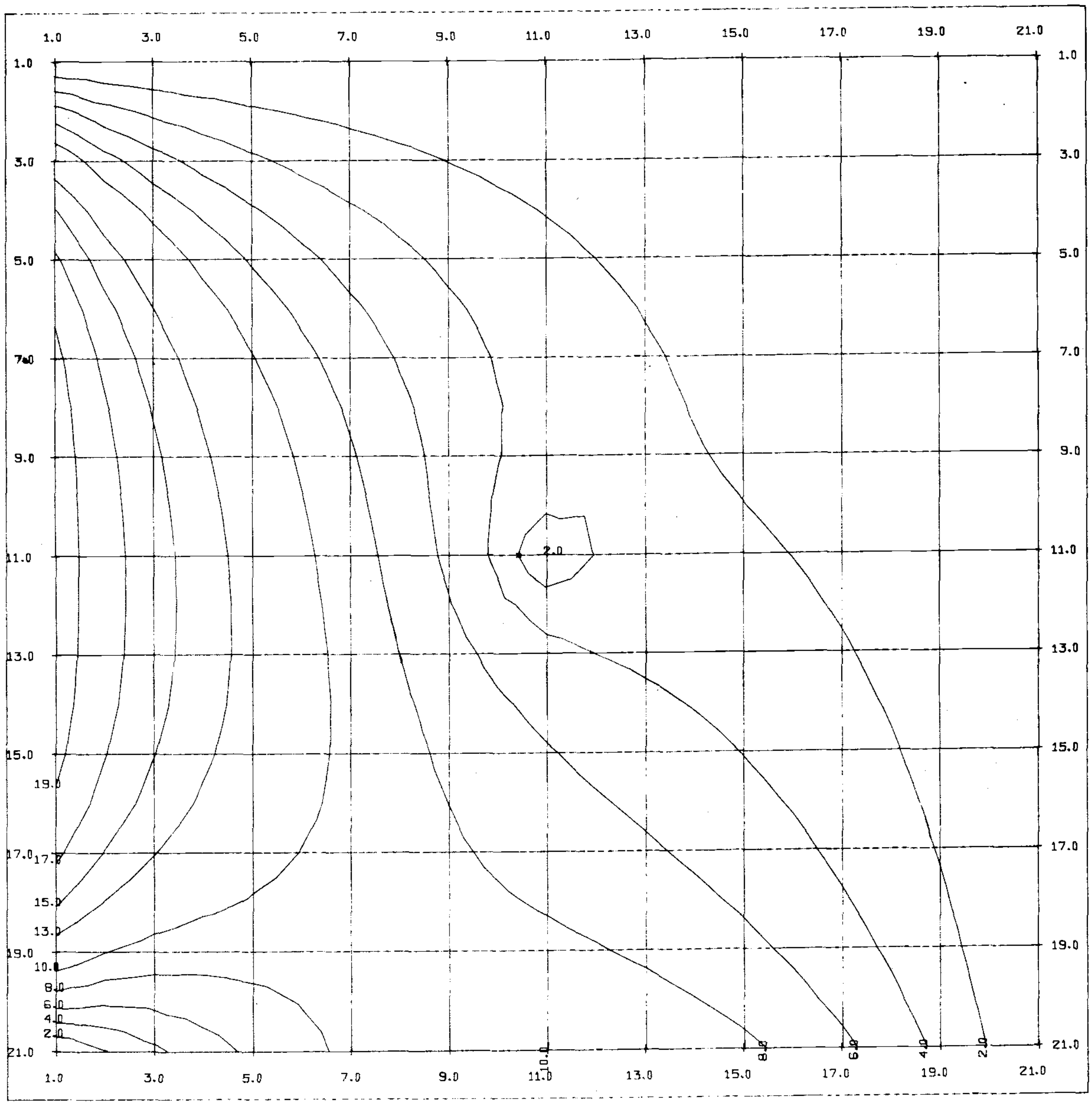
PRESSURE-LEVEL GRAPHPLOT5

Figure A.11 Pressure-level graph at the 1000th time step for Problem 5



PRESSURE-LEVEL GRAPHPLOT6

Figure A.12 Pressure-level graph at the 500th time step for Problem 6



PRESSURE-LEVEL GRAPHPLOT6

<u>Figure</u> A.13 Pressure-level graph at the 1000th time step for Problem 6