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Title COALESCENT SINGULAR POINTS OF DIFFERENTIAL SYSTEMS HAVING QUADRATIC RIGHT HAND SIDES

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In this paper we investigate the behavior of integral curves to the differential equation

\[ \frac{dy}{dx} = \frac{A_1 x^2 + 2A_2 xy + A_3 y^2 - 1}{B_1 x^2 + 2B_2 xy + B_3 y^2 - 1} \]

in a neighborhood of a point \((g, h)\) where both numerator and denominator vanish, i.e., a singular point. We specifically consider the coalescence of singular points of the type occurring in the bilinear case, and the integral curves behavior in a neighborhood of these coalescent points.
COALESCENT SINGULAR POINTS
OF DIFFERENTIAL SYSTEMS
HAVING QUADRATIC RIGHT HAND SIDES

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<table>
<thead>
<tr>
<th>TABLE OF CONTENTS</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>INTRODUCTION</strong></td>
</tr>
<tr>
<td><strong>CHAPTER ONE</strong></td>
</tr>
<tr>
<td>CLASSIFICATION OF CRITICAL POINTS</td>
</tr>
<tr>
<td><strong>CHAPTER TWO</strong></td>
</tr>
<tr>
<td>INTERSECTION POINTS AND CORRESPONDING EQUATION FORMS</td>
</tr>
<tr>
<td><strong>CHAPTER THREE</strong></td>
</tr>
<tr>
<td>COALESCENT POINTS</td>
</tr>
<tr>
<td><strong>CHAPTER FOUR</strong></td>
</tr>
<tr>
<td>INTEGRAL CURVES FOR $\mu$ NONZERO</td>
</tr>
<tr>
<td><strong>CHAPTER FIVE</strong></td>
</tr>
<tr>
<td>INTEGRAL CURVES FOR $\mu$ ZERO</td>
</tr>
<tr>
<td><strong>CHAPTER SIX</strong></td>
</tr>
<tr>
<td>SUMMARY OF RESULTS</td>
</tr>
<tr>
<td><strong>BIBLIOGRAPHY</strong></td>
</tr>
</tbody>
</table>
INTRODUCTION

In the study of ordinary differential equations authors such as Bendixson (1) and Lefschetz (4) treat the integral curves of the linear fractional equation

\[
\frac{dy}{dx} = \frac{T_1 x + T_2 y}{J_1 x + J_2 y} = \frac{Q(x, y)}{P(x, y)}
\]

in great detail. If the point \((g, h)\) is such that either the numerator or the denominator is nonzero then this point is termed regular. By the theorem on existence and uniqueness we know that there is only one solution through a regular point.

Furthermore if \((g, h)\) is such that both numerator and denominator vanish then \((g, h)\) is a critical point. Since (0-1) can be solved explicitly in terms of elementary functions, then the nature of its integral curves in the neighborhood of a critical point can be examined directly. This classifies these points into the following five groups: node, saddle, focus or spiral, center or vortex, and degenerate node.

The researches of Poincaré (5) and Bendixson extend the discussion to include the equation

\[
\frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)}
\]
where now \( Q(x, y) \), \( P(x, y) \) are allowed to be non-linear functions which are analytic in the neighborhood of the critical point. It should be noted that more than one critical point may arise for this differential equation.

However, this classical discussion is limited by the condition that the value of

\[
D = \frac{\partial P}{\partial x} \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial y} \frac{\partial Q}{\partial x}
\]  

(0-3)

is nonzero when evaluated at the critical point under consideration. With this restriction Bendixson was able to classify all critical points into the five groups previously noted. Such critical points at which \( D \) is nonzero are then called "simple".

In 1956 N. A. Gubar (3) presented a paper on the nature of the critical points of (0-2) when \( D \) is zero. Such critical points are called "compound". He arrived at the following seven groups in the classification of compound points: node, saddle, focus or spiral, center or vortex, point with closed nodal region, saddle-node, and degenerate point.

In this paper we wish to specialize \( Q(x, y) \), \( P(x, y) \) to the form:

\[
Q(x, y) = A_1 x^2 + 2A_2 xy + A_3 y^2 - 1 ,
\]

\[
P(x, y) = B_1 x^2 + 2B_2 xy + B_3 y^2 - 1 .
\]
Hence the critical points are the intersection points of the two central conics \( Q(x, y) = 0, \ P(x, y) = 0 \). By symmetry there will be an even number of critical points, 0, 2, or 4. We call the intersection points coalescent if the two conics have the same tangent line at the intersection and ordinary otherwise. These points may be either simple or compound.

We shall give no more than the results of classical theory for the simple points and Gubarí's results for the compound points. But for our special case we shall show that when critical points are coalescent the critical point is compound and this compound point is generally from one particular group. The detailed results are summarized in the theorem which concludes chapter six.
CHAPTER ONE

CLASSIFICATION OF CRITICAL POINTS

Let us first consider equation (0-1) and suppose $D$ is non-zero. We use the notation of Golomb and Shanks (2) to describe the nature of the integral curves in the neighborhood of the simple critical point which is $(0, 0)$.

Thus we have the following five conditions:

1. if $D$ is positive and if $\zeta$ is positive, where
   \[ \zeta = \left[ (\partial P/\partial x) + (\partial Q/\partial y) \right]^2 - 4D, \]
   then $(0, 0)$ is termed a node;

2. if $D$ is negative and $\zeta$ is positive then $(0, 0)$ is termed a saddle;

3. if $\zeta$ is zero then $(0, 0)$ is termed a degenerate node;

4. if $\zeta$ is negative and $\mu$ is nonzero, where
   \[ \mu = (\partial P/\partial x) + (\partial Q/\partial y), \]
   then $(0, 0)$ is termed a spiral or focus;

5. if $\zeta$ is negative and $\mu$ is zero then $(0, 0)$ is a center or vortex.

These conditions completely determine the nature of the integral curve in a neighborhood of the critical point $(0, 0)$ for the linear case.

We now consider equation (0-2) where $Q(x, y), P(x, y)$ are
analytic functions in a neighborhood of the critical point having
terms of at least the second power. Again we suppose \( D \) is non-
zero. Then the following theorem, Lefschetz (4), summarizes the
work of Poincaré and Bendixson.

**Theorem:** The nature of the integral curves for equation (0-2)
in the neighborhood of a critical point is the same as that of the
first approximation, except that a center for the first approxima-
tion may yield either a center or spiral for the equation being con-
sidered.

By the term **first approximation** Lefschetz means the linear
terms of \( Q(x, y), P(x, y) \) when \((0, 0)\) is the critical point under dis-
cussion. Hence the first approximation is an equation of the form
(0-1).

If \( D \) is nonzero, this disposes of the classification of critical
points for equation (0-2).

We are led to the general problem of the nature of the integral
curves in the neighborhood of a compound critical point, i.e., a
critical point when \( D \) is zero. For the remainder of this paper
we refer to the equation

\[
\frac{dy}{dx} = \frac{A_1 x^2 + 2A_2 xy + A_3 y^2 - 1}{B_1 x^2 + 2B_2 xy + B_3 y^2 - 1} = \frac{Q(x, y)}{P(x, y)}
\]  

(1-1)
as the ORIGINAL equation.

The following represents Gubar's findings, given without proof.

However we should first comment that Gubar studies the original equation by considering the related system

\[
\frac{dy}{dt} = Q(x, y), \quad \frac{dx}{dt} = P(x, y).
\]

Also the transformations he considers, and those that we do, are transformations linear in \( x, y \), and the parameter \( t \).

**Lemma I:** For

\[
\mu = (\partial P/\partial x) + (\partial Q/\partial y)
\]

nonzero there exists a non-singular transformation which converts the original equation into

\[
\frac{dy}{dx} = \frac{y + Q_2(x, y)}{P_2(x, y)}.
\]  

(1-2)

Here \( Q_2(x, y) \), \( P_2(x, y) \) are functions whose power series expansions, in a neighborhood of the critical point, have terms beginning with second degree.

**Lemma II:** There exists a function \( \Phi(x) \) which is the solution to

\[
y + Q_2(x, y) = 0
\]

and which vanishes at \( x = 0 \). We shall call this the explicit function.

Furthermore if the function \( \Gamma(x) \) is defined by
\( \Gamma(x) = P_2(x, \Phi(x)) \)

then in a neighborhood of the critical point \( \Gamma(x) \) has the form

\[ \Gamma(x) = \Delta_k x^k + \ldots, \quad k \geq 2, \Delta_k \neq 0. \]

**Theorem A:** For \( \mu \) nonzero the critical point \((0, 0)\) is a node or saddle when \( \Delta_k > 0 \) or \( \Delta_k < 0 \) respectively and \( k \) is odd. If \( k \) is even, then the critical point is termed a SIMPLER SADDLE-NODE.

Consider a compound point of the original equation. If perturbing functions \( \omega_1, \omega_2 \) can be added to the numerator and denominator so that the original compound splits into an equal number of simple nodes and simple saddle points, then we have a saddle node, or a simpler saddle-node if \( \mu \) is nonzero.

**Lemma III:** For \( \mu \) zero there exists a non-singular transformation which converts the original equation into

\[ \frac{dy}{dx} = \frac{C_2(x, y)}{y + P_2(x, y)} \]  

(1-3)

In this case \( \Gamma(x) = Q_2(x, \Phi(x)) \) where \( \Phi(x) \) is the solution to \( y + P_2(x, y) = 0 \).

**Lemma IV:** If \( \mu \) is zero and the function \( \sum (x, y) \) is defined by

\[ \sum (x, y) = (\partial P_2 / \partial x) + (\partial Q_2 / \partial y), \]
then in a neighborhood of the critical point the following representation holds:

\[ \sum (x, \Phi(x)) = \frac{\Delta}{n} x^n + \ldots, \quad n \geq 1, \; \Delta_n \neq 0, \]

\[ = 0 \quad , \quad \Delta_n = 0 \]

Theorem B: If \( \mu \) is zero, then the critical point \((0, 0)\) of the original equation is dependent upon the exponents \( k \) of lemma II and \( n \) of lemma IV. If \( k = 2m \) we have two cases possible:

a) \( \Delta_n \neq 0, \; n < m \).

The point is a SADDLE-NODE.

b) \( \Delta_n \neq 0, \; n \geq m \) or \( \Delta_n = 0 \).

The point is DEGENERATE.

We further note that if \( k = 2 \), then \( m = 1 \) and the point is always degenerate.

To define what is meant by a degenerate compound point we consider a compound point and two perturbing functions \( \omega_1, \omega_2 \). As before we add these functions to the numerator and denominator of the original equation. If only simple nodes for the altered equation arise from this one compound point, then this compound point is termed a degenerate point for the original equation.
If \( k = 2m + 1 \), then four cases are possible:

a) \( \Delta_k > 0. \)

The point is a SADDLE.

b) \( \Delta_k < 0, \Delta_n \neq 0, n \geq m, \) but

\[
\lambda = \left( \Delta \frac{2}{m} + 4 \{m + 1\} \Delta_k \right) < 0 \quad \text{or} \quad \Delta_k < 0, \Delta_n = 0.
\]

The point is the CENTER or SPIRAL.

c) \( \Delta_k < 0, n < m, \) \( n \) even or

\( n = m, \lambda \geq 0, n \) odd.

The point is a NODE.

d) \( \Delta_k < 0, n < m, \) \( n \) odd or

\( n = m, \lambda \geq 0, n \) odd.

The point is termed POINT WITH CLOSED NODAL REGION.

Thus theorems A and B divide the families of integral curves of the original equation into two broad classes; those where \( \mu \) is nonzero and those where \( \mu \) is zero. We intend to show that for the functions \( P(x, y), Q(x, y) \) of the original equation the exponent \( k \) of lemma II is always even. Therefore only saddle-node or degenerate points occur when \( P(x, y) = 0, Q(x, y) = 0 \) are central conics.
We consider the original equation

\[
\frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)} = \frac{N_1(x, y) - 1}{N_2(x, y) - 1}
\]

where

\[
N_1(x, y) = A_1x^2 + 2A_2xy + A_3y^2,
\]

\[
N_2(x, y) = B_1x^2 + 2B_2xy + B_3y^2.
\]

We assume \( N_1(x, y) \) is not a multiple of \( N_2(x, y) \) since if they were multiples then it is apparent that no critical points would exist. Note that this excludes the possibility that \( A_i = B_i \) for all \( i \).

In order to apply the results of Gubar's paper we need to know three items, (1) the intersection points of \( Q(x, y) = 0 \) and \( P(x, y) = 0 \); (2) the form of the original equation after an intersection point is translated to the origin; (3) the forms which belong to the group where \( \mu \) is nonzero and which to the group where \( \mu \) is zero.

We first introduce the quantities \( S_i, D_i \) defined by \( S_i = A_i + B_i \), \( D_i = A_i - B_i \), \( i = 1, 2, 3 \). With this notation the sum and difference of the equations \( Q(x, y) = 0 \), \( P(x, y) = 0 \) can be written
In polar coordinates, these equations are

\[ S_1 \rho^2 \cos^2 \theta + 2S_2 \rho^2 \cos \theta \sin \theta + S_3 \rho^2 \sin^2 \theta = 2 \]  
\[ D_1 \rho^2 \cos^2 \theta + 2D_2 \rho^2 \cos \theta \sin \theta + D_3 \rho^2 \sin^2 \theta = 0 \]

Solving \((2-2a)\) for the angle \(\theta\), we find

\[ \theta = \arctan(-2D_2 \pm \Delta / 2D_3) \]  
where \(\Delta = \sqrt{(D_2^2 - D_1 D_3)}\),

and if \(D_3 = 0\), \(\theta = (n + 1)\pi / 2\).

We substitute these values back into \((2-1a)\) and we find for the values of \(\rho\),

\[ \rho_1 = \pm \sqrt{2/F_{13}}, \quad \rho_2 = \pm \sqrt{2/F_{23}}. \]

Here \(F_{13} = \sqrt{(S_1 D_3^2 - \{2S_3 D_2 - 2S_2 D_3\} \{\Delta - D_2\} - S_3 D_1 D_3)}\),

\[ F_{23} = \sqrt{(S_1 D_2^2 - \{2S_3 D_2 - 2S_2 D_3\} \{-\Delta - D_2\} - S_3 D_1 D_3)}. \]

Hence, if we let \((g, h)\) be the intersection point, we find these possibilities:

\[ (g, h) = (\pm \sqrt{2 D_3 / F_{13}}, \pm \sqrt{2 / \{\Delta - D_2\} / F_{13}}) \]

and

\[ (g, h) = (\pm \sqrt{2 D_3 / F_{23}}, \pm \sqrt{2 / \{-\Delta - D_2\} / F_{23}}). \]
Furthermore, if $D_3$ is zero, we see from the properties of symmetry that if $D_1$ is nonzero then equations (2-1a), (2-2a) yield points similar to (2-3), where only a simultaneous interchange of $g$ and $h$ and of the subscripts 1 and 3 is needed.

If it happens that $D_1$ is zero and $D_3$ is zero, then equation (2-2a) reduces to

$$\sin 2 \theta = 0,$$

$$\theta = (n + 1)\pi / 2, \ n = 0, 1, 2, 3.$$ 

Therefore (2-3) reduces to the simple form

$$(g, h) = (\pm \sqrt{2/S_1}, 0)$$

and

$$(2-4)$$

$$(g, h) = (0, \pm \sqrt{2/S_3}).$$

These sets of points (2-3), (2-4) are the only points of interest for this paper. We rule out the rest by noting that the other intersection points cannot coalesce. For if $D_3$ is zero and $S_3$ is zero then, in view of (2-1), (2-2), there are only two finite real points of intersection. These are given by

$$g = -2D_2 h / D_1,$$

$$h = \pm D_1 / (2\sqrt{-D_2 \Delta_{12}}),$$

where

$$\Delta_{ij} = \begin{vmatrix} A_i & A_j \\ B_i & B_j \end{vmatrix}.$$
Similarly if $D_1$ is zero and $S_1$ is zero we have by (2-1), (2-2) that
\[ h = -2D_2 g / D_3 \]
\[ g = \pm D_3 / (2\sqrt{\Delta_{23}}) \]  \hspace{1cm} (2-5a)

Thus item (1) has been found. We recall here that these intersection points are also the critical points for the original equation.

In the process of finding item (2) we shall also group the resultant equations and find item (3). Hence suppose at the critical point $(g, h)$ we have $\mu = (\partial P / \partial x) + (\partial Q / \partial y)$ nonzero. We translate points (2-3), (2-4) to the origin. Let $T_i$ and $J_i$ be given by:
\[ T_i = -2A_i g - 2A_{i+1} h , \]
\[ J_i = -2B_i g - 2B_{i+1} h , \]
and suppose the $x, y$ are the new coordinates and $N_1(x, y), N_2(x, y)$ are as previously described. The following types of equations arise:
\[ (A) \frac{dy}{dx} = \frac{T_1 x + T_2 y + N_1(x, y)}{J_1 x + J_2 y + N_2(x, y)} \]  \hspace{1cm} (2-6)

and (2-6) with these modifications;

1) $J_1 = J_2 = 0$
2) $T_2 = J_2 = 0$
3) $T_1 = J_1 = 0$
4) $T_1 = J_1 = J_2 = 0$.  \hspace{1cm} (2-6a)
\[ \frac{dy}{dx} = \frac{N_1(x, y)}{J_1 x + J_2 y + N_2(x, y)} \quad (2-7) \]

and the modified form

1) \( J_2 = 0 \).

Now suppose \( \mu \) is zero. Then we have equation (2-6) with one form

1) \( T_2 = J_2 = J_1 = 0 \) \quad (2-6b)

We see that the above forms of the differential equation are not generally in the form of (1-2), (1-3). Thus we need to know that linear transformations exist which will convert them into the form of (1-2) and (1-3).

Only three transformations are needed, two for the equations when \( \mu \) is nonzero and one for when \( \mu \) is zero. Let \( t' \) be the parameter of the equation to be transformed and \( t \) be the new parameter.

Then for equations (2-6), (2-6a) in the case where \( \mu \) is nonzero we use the transformation of the \( xy \)-plane which has the matrix

\[ \begin{bmatrix} -L & T_1 \\ H & T_2 \end{bmatrix} \quad (2-8) \]

and the parameter transformation

\[ t = (T_2 + T_1 / L) t' \quad (2-8), \]
where \( L \) is the ratio \( T_1/J_1 \) or \( T_2/J_2 \) if either exists and is 1 otherwise. \( H \) is to be 1 if either \( J_1 \) or \( J_2 \) is nonzero and \( H \) is to be zero if both \( J_1 \) and \( J_2 \) vanish.

For equation (2-7) the corresponding transformation has matrix

\[
\begin{pmatrix}
0 & J_1 \\
1 & J_2 \\
\end{pmatrix}
\]

(2-9)

and

\[ t = (J_1 \cdot t') \]

(2-9).

If, however, \( \mu \) is zero the only transformation needed has the matrix

\[
\begin{pmatrix}
0 & 1 \\
1 & -T_2/T_1 \\
\end{pmatrix}
\]

(2-10)

and

\[ t = (T_1 \cdot t') \]

(2-10).
CHAPTER THREE

COALESCENT POINTS

Since the graphs of the equations

\[ N_1(x, y) = \beta, \quad N_2(x, y) = 1 \]

are symmetric with respect to the origin, so is the set of intersection points. Thus, if when \( \beta = 1 \) there is a coalescent point, then there is a second coalescent point. These two coalescent points are also symmetric with respect to the origin. Further, if there is coalescence for \( \beta = 1 \), then for suitable \( \beta \) near one, there must be four ordinary points of intersection.

Thus if for all \( \beta \) near 1 we have two or zero intersection points, then the intersection points for \( \beta = 1 \) cannot be coalescent. This rules out the point sets (2-5), (2-5a).

Furthermore suppose \( \phi_1 \) is the polar coordinate angle of one of the coalescent points and \( \phi_2 \) that of the other coalescent point. A necessary and sufficient condition for coalescence is, from the symmetry of the intersection set, \( \sin (\phi_1 - \phi_2) = 0 \).

We find it easiest to divide the conics into three groups:

1) \( A_i \neq B_i \), \( i = 1, 2, 3 \);

2) \( A_i = B_i \) for some, but not all, \( i \) and none of these equal coefficients vanish;
3) \( A_1 = B_1 = 0 \) for some, but not all, \( i \).

For the first group we have the points (2-3) to consider. Thus

\[
\sin \phi_1 = \frac{h_1}{r_1} = M_1(\Delta - D_2), \quad \cos \phi_1 = \frac{g_1}{r_1} = M_1(D_3)
\]

\[
\sin \phi_2 = \frac{h_2}{r_2} = M_2(-\Delta - D_2), \quad \cos \phi_2 = \frac{g_2}{r_2} = M_2(D_3),
\]

where \( M_1 \) is a nonzero factor.

Therefore

\[
\sin (\phi_1 - \phi_2) = 2M_1 M_2 \Delta D_3.
\]

Since \( A_3 \) is unequal to \( B_3 \), \( \sin (\phi_1 - \phi_2) \) vanishes if and only if \( \Delta \) is zero, i.e.,

\[
D_2^2 = D_1 D_3.
\]

For the second group we see that if \( A_1 \) equals \( B_1 \) and \( A_3 \) is unequal to \( B_3 \), then points (2-3) are again used and the result is the same as above. If, however, \( A_3 \) equals \( B_3 \) and \( A_1 \) is unequal to \( B_1 \), then by interchange of subscripts and of coordinates \( g \) and \( h \) we arrive again at the same result, i.e.,

\[
\sin (\phi_1 - \phi_2) = 2M_1 M_2 \Delta D_1 = 0.
\]

This implies

\[
D_2^2 = D_1 D_3.
\]

Now suppose \( A_1 = B_1 \), \( A_3 = B_3 \), and \( A_2 \neq B_2 \). In this case we have to use points (2-4) which yield
\[
\sin \phi_1 = 0 = \cos \phi_2 ,
\]
\[
\cos \phi_1 = 1 = \sin \phi_2 .
\]
Thus \(\sin (\phi_1 - \phi_2)\) is nonzero. Hence if coalescence of critical points is to occur for conics in group 2 we need \(D_2^2 = D_1 D_3\).

Suppose we have an equation belonging to the last group. We have already noted that either \(A_1 = B_1 = 0\) or \(A_3 = B_3 = 0\) implies coalescence cannot occur. Therefore we need only to consider \(A_2 = B_2 = 0\).

We have by the use of points (2-3):
\[
\sin \phi_1 = M_1 \sqrt{-D_1 D_3} , \quad \cos \phi_1 = M_1 (D_3) ,
\]
\[
\sin \phi_2 = M_2 (-\sqrt{-D_1 D_3}) , \quad \cos \phi_2 = M_2 (D_3) ,
\]
where \(M_i\) is as before and \(A_3\) is unequal to \(B_3\). Thus
\[
\sin (\phi_1 - \phi_2) = 2M_1 M_2 D_3 \sqrt{-D_1 D_3}
\]
and hence is zero if and only if \(D_1\) is zero, i.e., \(A_1\) equals \(B_1\).

In a similar manner we find that if \(A_1\) is unequal to \(B_1\) then \(\sin (\phi_1 - \phi_2)\) is zero if and only if \(D_3\) is zero, i.e., \(A_3\) equals \(B_3\). Again this means \(D_2^2 = D_1 D_3\).

Thus we have shown the following to be true;

**Lemma:** A necessary and sufficient condition for coalescence of intersection points of the two concentric conics \(N_1(x, y) = 1\), \(N_2(x, y) = 1\) is that
\[ D_2^2 = D_1 D_3, \]

where \( D_i = A_i - B_i, \quad i = 1, 2, 3. \)

Suppose we now consider a typical critical point \((g, h)\) of the original equation. When we translate this point to the origin we obtain the following differential equation:

\[
\frac{dy}{dx} = \frac{T_1 x + T_2 y + A_1 x^2 + 2A_2 xy + A_3 y^2}{J_1 x + J_2 y + B_1 x^2 + 2B_2 xy + B_3 y^2} \quad (2-6)
\]

where, as before,

\[ T_i = (-2A_i g - 2A_{i+1} h) \]

and

\[ J_i = (-2B_i g - 2B_{i+1} h) \quad \text{for} \quad i = 1, 2. \]

To this equation we apply the appropriate transformation which was previously found; it is changed into either \((1-2)\) which is explicitly

\[
\frac{dy}{dx} = \frac{y + K_1 x^2 + K_2 xy + K_3 y^2}{R_1 x^2 + R_2 xy + R_3 y^2} \quad (3-1)
\]

or \((1-3)\) which is explicitly

\[
\frac{dy}{dx} = \frac{R_1 x^2 + R_2 xy + R_3 y^2}{y + K_1 x^2 + K_2 xy + K_3 y^2} \quad (3-2)
\]

For us the \( R_i, K_i \) are expressible in terms of \( T_i, J_i \) and they will be dealt with later. Recall here that what we are heading
for is the evaluation of the smallest exponent in the equation

\[ R_1 x^2 + R_2 x \phi(x) + R_3 \phi^2(x) = 0, \]

where \( \phi(x) \) is the explicit function for

\[ y + K_1 x^2 + K_2 xy + K_3 y^2 = 0, \]

cf. lemma II, chapter one.

Thus for \( K_3 \) nonzero we can apply the quadratic formula to the last equation and obtain for the explicit function:

\[
\phi(x) = \frac{-(K_2 x + 1) \pm \sqrt{(K_2 x + 1)^2 - 4K_3 K_1 x^2}}{2K_3}
\]

We expand the radical in a Maclaurin series and obtain:

\[
\phi(x) = -K_1 x^2 + K_2 x^3 - \left\{2K_1 K_2 + 2K_1 K_3\right\} x^4 + \ldots \quad (3-3)
\]

If \( K_3 \) is zero we find the explicit function to be:

\[
\phi(x) = -K_1 x^2 + K_2 x^3 - K_1 K_2 x^4 + \ldots \quad (3-3a)
\]

by use of the binomial expansion.

We note, that if \( K_1 \) is zero then \( \phi(x) \) is identically zero, and that if \( x \) is zero then \( \phi(x) \) is again zero. Hence this explicit function has the desired properties.

Furthermore, with \( \phi(x) \) now determined, we may compute the function \( \Gamma(x) \) (as in lemmas II and III) to be
\[ \Gamma(x) = R_1 x^2 + R_2 x \{ -K_1 x^2 + K_1 K_2 x^3 + \ldots \} \]
\[ + R_3 \{ -K_1 x^2 + K_1 K_2 x^3 + \ldots \}^2. \]

\[ (3-4) \]

Hence if \( \mu \) is nonzero we only need to examine the numbers \( K_1, R_1, R_2 \). However when \( \mu \) is zero we need the representation of the function \( \sum \phi(x) \). Upon considering lemma IV we find it to be

\[ \sum [x, \phi(x)] = \{ 2K_1 + R_2 \} x + \{ K_2 + 2R_3 \} \phi(x) \]

\[ (3-5) \]

If \( \mu \) is zero we need to know the value of \( R_3 \) in addition to \( K_1, K_2, R_2 \). But we shall always examine \( R_1 \) first, since, from Theorem B, if \( R_1 \) is nonzero the critical point is always degenerate.
CHAPTER FOUR

INTEGRAL CURVES FOR $\mu$ NONZERO

In this chapter we consider the class of integral curves in the case

$$\mu = (\partial P/\partial x) + (\partial Q/\partial y)$$

is nonzero. We apply transformation (2-8) to equations of form (2-6), (2-6a) to reduce to form (3-1). We find

1) $R_1$ is a multiple of $\overline{R_1}$ where

$$\overline{R_1} = T_2^2 (A_1 - LB_1) + T_1 T_2 (2LB_2 - 2A_2)$$

$$+ T_1^2 (A_3 - LB_3);$$

2) $K_1$ is a multiple of $\overline{K_1}$ where

$$\overline{K_1} = T_2^3 A_1 + T_1^3 B_1 + T_2^2 T_1 (B_1 - 2A_2)$$

$$+ T_1^2 T_2 (A_3 - 2B_2);$$

3) $R_2$ is a multiple of $\overline{R_2}$ where

$$\overline{R_2} = T_2 L B_1 + T_1 L (A_1 + LA_2)$$

$$+ T_1 (A_2 + LA_3) - T_1 L (B_2 + LB_3).$$

In chapter three we have divided the conies into three groups; we first consider group 3), $A_i = B_i = 0$ for some, but not all, $i$; then group 2), $A_i = B_i$ for some $i$ and none of these equal coefficients vanish; and finally group 1), $A_i \neq B_i$ for all $i$. 
We suppose $A_2 = B_2 = 0$. Then we know, from the conditions for coalescence, that either $D_1$ is zero or $D_3$ is zero. Assume $D_1$ is zero. Then the critical points (2-3) become

$$ (g, h) = \left( \pm \sqrt{2} \frac{D_3}{F_{13}}, 0 \right). $$

Therefore $T_2 = 0$ and

$$ L = T_2/J_2 = (A_1 h)/(B_1 h) = 1. $$

Hence

$$ \overline{R}_1 = T_1^2 (A_3 - B_3) \neq 0. $$

Now we assume $D_3$ is zero. Then the critical points are found to be

$$ (g, h) = (0, \pm \sqrt{2} \frac{D_1}{F_{31}}). $$

Thus $T_1 = 0$ and

$$ L = T_2/J_2 = (A_3 h)/(B_3 h) = 1. $$

Again we have

$$ \overline{R}_1 = T_2^2 (A_1 - B_1) \neq 0. $$

If we now suppose that $A_2 = B_2 \neq 0$ then the only alteration needed in the above work would be: (a) for $D_1$ being zero then $T_2$ is no longer zero but all else holds; (b) for $D_1$ nonzero then $T_1$ is nonzero but everything else is valid.
Thus we can state that for groups 3) and 2), i.e., $A_i = B_i$ for some but not all $i$, if coalescence occurs and $\mu = (\partial P/\partial x) + (\partial Q/\partial y)$ is nonzero then the exponent $k$ of Theorem A is even.

Suppose we are now concerned with an equation in group 1), $A_i \neq B_i$ for all $i$. Then the critical points (2-3) become

$$(g, h) = (\pm \sqrt{2} D_3/F_{13}, \pm \sqrt{2} (-D_2)/F_{13}) .$$

We first note that when $A_2 \neq B_2$ then $T_2 = J_2$ and they are dependent upon the value of $\Delta_{23} = A_2 B_3 - B_2 A_3$. For if we consider the representation of $T_2$ and $J_2$ we find:

$$T_2 = -2 (A_2 g + A_3 h)$$
$$= -2 \left( \pm \sqrt{2}/F_{13} \right) (A_2 D_3 - D_2 A_3)$$
$$= + 2 \left( \pm \sqrt{2}/F_{13} \right) (\Delta_{23})$$

and

$$J_2 = -2 \left( \pm \sqrt{2}/F_{13} \right) (B_2 D_3 - D_2 B_3)$$
$$= + 2 \left( \pm \sqrt{2}/F_{13} \right) (\Delta_{23}) .$$

But we also have $T_1 J_2 = J_1 T_2$ since $D = T_1 J_2 - J_1 T_2 = 0$. Thus $T_1 = J_1$ when $A_2 \neq B_2$.

At this point we divide the work into two cases according to whether $\Delta_{23}$ is nonzero or zero.

If $\Delta_{23}$ is nonzero we have

$$L = T_1 / J_1 = T_2 / J_2 = 1.$$
Hence
\[ R_1 = T_2^2D_1 - 2T_1T_2D_2 + T_1^2D_3 \]
\[ = D_1(T_2 - D_2T_1/D_1)^2 \]

upon completing the square. Inserting the values of \( T_1, T_2 \), we have, upon arrangement,
\[ R_1 = 8D_1(D_2D_3 + D_2^2) - A_3D_2D_1 - A_1D_2D_3)^2/F_13^2 \]
\[ = 8D_1(A_2^2D_3 - D_2^2) - A_3D_2D_1 - A_1D_2D_3 + 2A_2D_2^2)^2/F_13^2. \]

We substitute the values for \( D_i \) to get
\[ R_1 = 8D_1D_2^2(2A_2^2 - A_1A_3) + A_3B_1 + A_1B_3 - 2A_2B_2)^2/F_13^2. \]

However since \( D_2^2 = D_1D_3 \) we have
\[ 2A_2B_2 - A_3B_1 - A_1B_3 = (A_2^2 - A_1A_3) + (B_2^2 - B_1B_3). \] (4-1)

Insert this in \( R_1 \); this yields
\[ R_1 = 8D_1D_2^2((A_2^2 - A_1A_3) - (B_2^2 - B_1B_3))^2/F_13^2. \]

At this stage we have determined the following: when \( A_i \neq B_i \) for all \( i \) and \( \Delta_{23} \) is nonzero, \( R_1 \) is zero if and only if
\[ A_2^2 - A_1A_3 = B_2^2 - B_1B_3. \]
Note that when $\Delta_{23}$ is nonzero, so is $T_2$.

But we next show that if $\Delta_{23}$ is nonzero then

$$A_2^2 - A_1A_3 \neq B_2^2 - B_1B_3.$$  

Consequently we shall conclude that when $A_i \neq B_i$ for all $i$ and $T_2$ is nonzero, then $R_1$ is nonzero.

Now if we consider the values of $T_1$ and $J_1$ we find

$$T_1 = -2(\pm \sqrt{2}/F_{13}) (D_2A_2 - D_2A_1)$$

$$= -2(\pm \sqrt{2}/F_{13}) (A_2B_2 - A_1B_3 - \{A_2^2 - A_1A_3\})$$

and

$$J_1 = -2(\pm \sqrt{2}/F_{13}) (B_2A_2 - B_1A_3 - \{B_2^2 - B_1B_3\}).$$

Since $T_1 = J_1$ we have:

$$(A_2^2 - A_1A_3) - (B_2^2 - B_1B_3) = A_3B_1 - A_1B_3$$

$$= \Delta_{31}.$$

Thus our problem is reduced to showing that $\Delta_{23}$ nonzero implies $\Delta_{31}$ is nonzero. We assume $\Delta_{23}$ is nonzero but $\Delta_{31}$ is zero and show a contradiction.

Since $\Delta_{31}$ is zero, equation (4-1) implies that

$$A_2^2 - A_1A_3 = A_2B_2 - A_1B_3,$$

or

$$A_2D_2 - A_1D_3 = 0.$$
Multiplying by $D_1$ and setting $D_1D_3 = D_2^2$ we have,

$$A_2D_2D_1 - A_1D_2^2 = 0.$$

Hence $D_2\Delta_{21}$ is zero. Since $D_2$ is nonzero, then $\Delta_{21}$ is zero. Obviously if this is so then $\Delta_{23}$ is zero because $\Delta_{31}$ is zero. This contradicts our assumption. Thus if $\Delta_{23}$ is nonzero then $R_1$ is nonzero also. This completes the discussion in case $\Delta_{23}$ is nonzero.

We now suppose $\Delta_{23}$ is zero. Then $T_2$ and $J_2$ are zero. We first assume that $T_1$ is nonzero. From $\Delta_{23}$ being zero we have

$$A_2/B_2 = A_3/B_3.$$

Then because $L = T_1/J_1$ we find

$$R_1 = T_1^2(A_3 - B_3T_1/J_1)$$

$$= T_1^2B_3(A_3/B_3 - T_1/J_1).$$

Therefore $R_1$ is zero if

$$A_3/B_3 = T_1/J_1 = A_2/B_2.$$

But then we have

$$R_2 = T_1(A_2 + A_3T_1/J_1) - (T_1^2/J_1)(B_2 + B_3T_1/J_1)$$

$$= (T_1^2/J_1)(A_2B_2/A_2 + A_3) - (T_1^2/J_1)(B_2 + B_3A_3/B_3))$$

$$= 0.$$
Since $R_1$ and $R_2$ are zero and $P_2(x, y)$ is nonzero, we conclude that $R_3$ is nonzero. Also we see that our explicit function $\Phi(x)$ is not identically zero because

$$\bar{K}_1 = T_1^3 B_3 \neq 0 \text{ as } B_3 \neq 0.$$ 

We note that if $B_3$ were zero then

$$\bar{R}_1 = A_3 J_1 \neq 0.$$ 

Thus whether $B_3$ vanishes or not we find the exponent $k$ to be even under the assumption that $T_1$ is nonzero.

We now assume $T_1$ is zero and still assume $\Delta_{23}$ is zero. Then we use the appropriate transformation on (2-7), i.e., the equation where $T_1 = T_2 = 0$. We find the altered equation to be

$$\frac{dy}{dx} = \frac{y + x^2 B_3 / J_1 + 2xy B_2 / J_1 + y^2 B_1 / J_1}{x^2 A_3 / J_1 + 2xy A_2 / J_1 + y^2 A_1 / J_1}$$

$$= \frac{y + x^2 K_1 + 2xy K_2 + y^2 K_3}{x^2 R_1 + 2xy R_2 + y^2 R_3}.$$ 

Thus we have new descriptions for $R_1, R_2, K_1$ obtained by identifying coefficients in the two forms above.

Because $\Delta_{23}$ is still zero, we have the three possibilities:

1) $B_2 = B_3 = 0$, 2) $A_2 = A_3 = 0$, 3) $A_2 B_3 - A_3 B_2 = 0$,

$A_i \neq 0$, $B_i \neq 0$. 
Suppose we have the first possibility. Then, as we are in the case \( A_i \) not equal to \( B_i \), \( A_2 \) is nonzero and \( A_3 \) is nonzero. Therefore

\[
R_1 = \frac{A_3}{J_1} \neq 0.
\]

Suppose now that we consider the second possibility. Then \( B_2 \) is nonzero and \( B_3 \) is nonzero. Hence we immediately obtain:

\[ R_1 \text{ and } R_2 \text{ are zero and } R_3 \text{ is nonzero. Also } \]

\[
K_1 = \frac{B_3}{J_1} \neq 0.
\]

For the last case we suppose \( A_3 \) nonzero and \( B_2 A_3 = A_2 B_3 \). Then we again have

\[
R_1 = \frac{A_3}{J_1} \neq 0.
\]

Therefore we find that if \( T_1 = T_2 = J_2 = 0 \) the exponent \( k \) is again even.

In summary, we have so far shown that if \( \mu \) is nonzero, the exponent \( k \) of Theorem A is even. Hence when \( \mu \) is not zero, the critical point is a simpler saddle-node.
CHAPTER FIVE

INTEGRAL CURVES FOR $\mu$ ZERO

We again consider the original equation after a translation which sends a coalescent critical point to the origin. However we now have the condition that

$$\mu = (\partial P/\partial x) + (\partial Q/\partial y) = 0.$$ 

We use transformation (2-10) to convert it into an equation of form (1-3).

We find that under this transformation the values of $R_1$, $R_2$, $R_3$, $K_1$, and $K_2$ are as follows:

$$R_1 = \left( [T_2^2(B_1T_1 + T_2A_1)/T_1^4] - [2T_2(B_2T_1 + A_2T_2)/T_1^3] \right. + \left. [B_3T_1 + T_2A_3]/T_1^2 \right);$$

$$R_2 = 2\left[ (B_2T_1 + A_2T_2)/T_1^2 \right] - 2\left[ T_2(B_1T_1 + T_2A_1)/T_1^3 \right];$$

$$R_3 = (B_1T_1 + T_2A_1)/T_1^2;$$

$$K_1 = ([A_1T_2^2/T_1^2] - 2A_2(T_2/T_1) + A_3)/T_1;$$

$$K_2 = ([ -2A_1T_2/T_1] + 2A_2)/T_1.$$ 

Again we divide the conics into three groups, 1) $A_i \neq B_i$ for all $i$; 2) $A_i = B_i$ for some $i$ and none of these coefficients
vanish; and 3) \( A_1 = B_1 = 0 \) for some, but not all, \( i \).

We first suppose that \( A_2 = B_2 = 0 \), i.e., \( D_2 \) is zero. Then we have, cf. chapter four, either \( D_1 \) is zero or \( D_3 \) is zero. Assume \( D_1 \) is zero. As in the case where \( \mu \) was nonzero, and from the dependence of \( T_2 \) on \( A_{23} \), we have \( T_2 \) is zero. But \( \mu \) being zero implies that \( T_2 = J_1 = 0 \). However we have the representation of \( J_1 \) as \((-2B_1g)\). As we know that the \( g \) coordinate is nonzero we find that

\[
B_1 = 0 = A_1 \quad \text{since} \quad D_1 \text{ is zero.}
\]

But we have already seen that \( A_2 = B_2 = A_1 = B_1 = 0 \) does not yield a coalescent point.

Now we assume \( D_1 \) is nonzero. The \( g \) coordinate of the critical point is zero and this yields the fact that \( J_1 \) is zero. This and the vanishing of \( \mu \) implies that \( T_2 \) is zero. But \( T_2 \) equal zero implies that \( A_3 = B_3 = 0 = A_2 = B_2 \) and again coalescence cannot occur.

Thus we have shown that if \( A_2 = B_2 = 0 \) and \( \mu = 0 \) then coalescence does not occur.

We now suppose that \( A_2 = B_2 \neq 0 \), i.e., we suppose we have an equation in group 2. If we assume that \( D_1 \) is zero this will yield the same critical points as above. Since the \( h \) coordinate is now zero and since \( A_1 \) equals \( B_1 \) we find that \( T_1 \) equals \( J_1 \).
But \( \mu = T_2 + J_1 = 0 \) implies \( T_1 = J_1 = -T_2 \).

Substituting these facts into our formula for \( R_1 \) we find

\[
R_1 = (-D_3/T_1) \not= 0.
\]

On the other hand if we assume that \( D_1 \) is nonzero then our critical point has the \( g \) coordinate zero. Then, because \( \mu \) is zero, we find that

\[
J_1 = -2B_2 h = -T_2 = 2A_3 h.
\]

Thus \( -B_2 = A_3 \) and hence \( A_2 = B_2 = -A_3 = -B_3 \).

Therefore we see that \( T_1 = -2A_2 h = -T_2 \).

Again, substitution into our formula for \( R_1 \) reveals

\[
R_1 = (-D_1/T_1) \not= 0.
\]

Therefore we see, in light of Theorem B, that if \( D_i = 0 \) for some \( i \) and if coalescence of critical points occurs under the condition \( \mu = 0 \), then the critical point \((g, h)\) formed by this coalescence is a degenerate point, i.e., the exponent \( k \) of Theorem B is even.

Suppose we now are in the last group where \( A_i \) is not equal to \( B_i \) for all \( i \). In chapter four we found that \( T_2 = J_2, \ T_1 = J_1 \) when \( A_2 \) does not equal \( B_2 \). Hence if we use the condition \( \mu \) is zero we have the equality

\[
T_2 = J_2 = -T_1 = -J_1 \not= 0.
\]
Therefore upon substitution and rearrangement we see that

\[ R_1 = \left( T_2 D_1 + 2T_2 D_2 + T_2 D_3 \right) / T_1^2 \]

\[ = -\left( \frac{D_2^2}{D_3} + 2D_2 + D_3 \right) / T_1 \]

\[ = -\left( D_2 + D_3 \right)^2 / T_1 D_3 . \]

Hence we have \( R_1 \) is zero if and only if \( D_2 = -D_3 \). But since
\[ D_2^2 = D_1 D_3 \] this implies \( D_2 = -D_3 = -D_1 \) if \( R_1 \) is to be zero.

We suppose \( R_1 \) is nonzero; hence \( D_2 + D_3 \) is nonzero. Then we conclude that the exponent \( k \) equals two and therefore the critical point is a degenerate point.

If, however, we suppose \( D_2 = -D_3 = -D_1 \) then \( R_1 \) is zero. But when we substitute into our formula for \( R_2 \) we get

\[ R_2 = 2\left( T_2 D_1 + T_2 D_2 \right) / T_1^2 = 0 . \]

Thus we deduce, since \( P_2(x, y) \) is nonzero, that \( R_3 \) is nonzero.

Since we are in the case where \( \mu = 0 \) we need to examine the value of \( K_1 \). We find that

\[ K_1 = \left( A_1 + 2A_2 + A_3 \right) / T_1 . \]

Let us consider the consequence of \( T_1 \) being nonzero. Since
\[ T_1 = -2(\pm \sqrt{2/F_{13}}) (A_1 D_3 - A_2 D_2) \]
\[ = -2(\pm \sqrt{2/F_{13}}) (D_3 \{A_1 + A_2\}) \]

is zero we see that \((A_1 + A_2)\) is nonzero.

Likewise since

\[ T_1 + T_2 = -2(\pm \sqrt{2/F_{13}}) (D_3 \{A_1 + A_2\} + D_2 \{A_2 + A_3\}) \]

is zero we see that

\[ A_1 = A_3. \]

Hence

\[ A_1 + 2A_2 + A_3 = 2(A_1 + A_2) \]

is nonzero and so \(K_1\) is nonzero also.

Then we have the functions \(\Gamma(x)\) and \(\sum (x)\) represented by

the expansion (3-4) and (3-5)

\[ \Gamma(x) = R_3 K_1 \frac{2}{4} x^4 + \ldots \]

\[ = A_4 x^4 + \ldots \]

and

\[ \sum (x, \Phi(x)) = x(2K_1) + \ldots \]

\[ = A_1 x + \ldots \]

Thus we have \(k = 4, \ n = 1, \ \frac{A}{n}\) nonzero and by Theorem B these are the conditions for a saddle-node.
So we see that if coalescence of critical points occurs and $\mu$ is zero then except for one case a degenerate point arises. In the exceptional case, where $D_2 = -D_3 = -D_1$, we have a saddle-node at the critical point.
CHAPTER SIX

SUMMARY OF RESULTS

Before we summarize it would be advisable to show that a coalescent point is indeed a compound point. To do this we consider our original equation

\[ \frac{dy}{dx} = \frac{A_1x^2 + 2A_2xy + A_3y^2 - 1}{B_1x^2 + 2B_2xy + B_3y^2 - 1} = \frac{Q(x, y)}{P(x, y)}. \]

Then since \( D = (\partial P/\partial x)(\partial Q/\partial y) - (\partial Q/\partial x)(\partial P/\partial y) \) we have upon substitution and rearrangement

\[ D = (2B_1x + 2B_2y)(2A_2x + 2A_3y) \]

\[ - (2A_1x + 2A_2y)(2B_2x + 2B_3y) \]

\[ = 4\{x^2(A_2B_1 - B_2A_1) + xy(A_3B_1 - B_3A_1) + y^2(A_3B_2 - B_3A_2)\} \]

where \((x, y)\) is the critical point.

Now in the study of our conics we found that for coalescence we needed \( D_2^2 = D_1 D_3 \). Also for simplicity we divided the conics into the three groups, 1) \( A_2 = B_2 = 0 \), 2) \( A_2 = B_2 \neq 0 \), 3) \( A_2 \neq B_2 \).

We first suppose \( A_2 = B_2 = 0 \). Then we see that

\[ D = 4xy (A_3B_1 - B_3A_1) \].
But the critical point \((x, y)\) is of the form \((x, 0)\) if \(D_1\) is zero or of the form \((0, y)\) if \(D_1\) is nonzero. In either case the value of \(D\) is zero and hence the coalescent critical point is compound.

We now suppose we have an equation corresponding to the second group, in which \(A_2 = B_2 \neq 0\). Again the critical point has the form \((x, 0)\) or \((0, y)\). If we use \((x, 0)\) then we have \(D_1 = 0\), i.e., \(A_1 = B_1\). But the value of \(D\) is

\[
D = 4x^2 (A_2 B_1 - B_2 A_1)
\]

\[= 0\quad\text{since } A_1 = B_1, \ A_2 = B_2.\]

Likewise \(D_1\) is nonzero if \((0, y)\) is used and then \(D_3\) is zero at coalescence and hence

\[
D = 4y^2 (A_3 B_2 - B_3 A_2) = 0.
\]

So finally suppose \(A_2\) does not equal \(B_2\). Then

\[
(x, y) = (\pm D_3 \sqrt{2}/F_{13}, \mp D_2 \sqrt{2}/F_{13}).
\]

Hence

\[
D = 8\{D_3^2 A_{21} - D_3 D_2 A_{31} + D_2^2 A_{32}\}/F_{13}^2
\]

\[
= -8D_3 \begin{vmatrix} D_1 & D_2 & D_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}/F_{13}^2.
\]
Since the first row is the difference of the second and third rows, \( \mathbf{D} = 0 \). Thus coalescence of critical points does indeed yield a compound point.

We summarize:

**Theorem:** Given a differential equation

\[
\frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)}
\]

in which

\[
Q(x, y) = A_1 x^2 + 2A_2 xy + A_3 y^2 - 1
\]

\[
P(x, y) = B_1 x^2 + 2B_2 xy + B_3 y^2 - 1.
\]

Set

\[
D_i = A_i - B_i, \quad i = 1, 2, 3.
\]

There is coalescence of simple critical points into compound critical points if and only if

\[
D_2^2 = D_1 D_3.
\]

If \( D_1 \neq D_2 \) and if at the critical point in question we have

\[
\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0,
\]

then the compound critical point is degenerate. In all other cases the compound critical point is a saddle-node.
BIBLIOGRAPHY


