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Edward Max Matzdorff for the M. S. in Mathematics
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Title COALESCENT SINGULAR POINTS OF DIFFERENTIAL SYSTEMS HAVING QUADRATIC RIGHT HAND SIDES

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In this paper we investigate the behavior of integral curves to the differential equation

$$\frac{dy}{dx} = \frac{A_1x^2 + 2A_2xy + A_3y^2 - 1}{B_1x^2 + 2B_2xy + B_3y^2 - 1}$$

in a neighborhood of a point (g, h) where both numerator and denominator vanish, i. e., a singular point. We specifically consider the coalescence of singular points of the type occurring in the bilinear case, and the integral curves behavior in a neighborhood of these coalescent points.

COALESCENT SINGULAR POINTS OF DIFFERENTIAL SYSTEMS HAVING QUADRATIC RIGHT HAND SIDES

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EDWARD MAX MATZDORFF

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In Charge of Major

[redacted]

Head of Department Mathematics

[redacted]

Dean of Graduate School

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INTRODUCTION

In the study of ordinary differential equations authors such as Bendixson (1) and Lefschetz (4) treat the integral curves of the linear fractional equation

$$\frac{dy}{dx} = \frac{T_1x + T_2y}{J_1x + J_2y} = \frac{Q(x, y)}{P(x, y)} \tag{0-1}$$

in great detail. If the point (g, h) is such that either the numerator or the denominator is nonzero then this point is termed regular. By the theorem on existence and uniqueness we know that there is only one solution through a regular point.

Furthermore if (g, h) is such that both numerator and denominator vanish then (g, h) is a critical point. Since (0-1) can be solved explicitly in terms of elementary functions, then the nature of its integral curves in the neighborhood of a critical point can be examined directly. This classifies these points into the following five groups: node, saddle, focus or spiral, center or vortex, and degenerate node.

The researches of Poincaré (5) and Bendixson extend the discussion to include the equation

$$\frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)} \tag{0-2}$$

where now $Q(x, y)$, $P(x, y)$ are allowed to be non-linear functions which are analytic in the neighborhood of the critical point. It should be noted that more than one critical point may arise for this differential equation.

However, this classical discussion is limited by the condition that the value of

$$\mathbf{D} = (\partial P/\partial x)(\partial Q/\partial y) - (\partial P/\partial y)(\partial Q/\partial x) \tag{0-3}$$

is nonzero when evaluated at the critical point under consideration. With this restriction Bendixson was able to classify all critical points into the five groups previously noted. Such critical points at which $\mathbf{D}$ is nonzero are then called "simple".

In 1956 N. A. Gubař (3) presented a paper on the nature of the critical points of (0-2) when $\mathbf{D}$ is zero. Such critical points are called "compound". He arrived at the following seven groups in the classification of compound points: node, saddle, focus or spiral, center or vortex, point with closed nodal region, saddle-node, and degenerate point.

In this paper we wish to specialize $Q(x, y)$, $P(x, y)$ to the form:

$$Q(x, y) = A_1x^2 + 2A_2xy + A_3y^2 - 1 ,$$

$$P(x, y) = B_1x^2 + 2B_2xy + B_3y^2 - 1 .$$

Hence the critical points are the intersection points of the two central conics $Q(x, y) = 0$, $P(x, y) = 0$. By symmetry there will be an even number of critical points, 0, 2, or 4. We call the intersection points coalescent if the two conics have the same tangent line at the intersection and ordinary otherwise. These points may be either simple or compound.

We shall give no more than the results of classical theory for the simple points and Gubař's results for the compound points. But for our special case we shall show that when critical points are coalescent the critical point is compound and this compound point is generally from one particular group. The detailed results are summarized in the theorem which concludes chapter six.

CHAPTER ONE

CLASSIFICATION OF CRITICAL POINTS

Let us first consider equation (0-1) and suppose $\mathbf{D}$ is nonzero. We use the notation of Golomb and Shanks (2) to describe the nature of the integral curves in the neighborhood of the simple critical point which is (0, 0).

Thus we have the following five conditions:

(1) if $\mathbf{D}$ is positive and if ζ is positive, where $\zeta = [(\partial P/\partial x) + (\partial Q/\partial y)]^2 - 4\mathbf{D}$, then (0, 0) is termed a <u>node</u>;

(2) if $\mathbf{D}$ is negative and ζ is positive then (0, 0) is termed a <u>saddle</u>;

(3) if ζ is zero then (0, 0) is termed a <u>degenerate node</u>;

(4) if ζ is negative and μ is nonzero, where $\mu = (\partial P/\partial x) + (\partial Q/\partial y)$, then (0, 0) is termed a <u>spiral</u> or <u>focus</u>;

(5) if ζ is negative and μ is zero then (0, 0) is a <u>center</u> or <u>vortex</u>.

These conditions completely determine the nature of the integral curve in a neighborhood of the critical point (0, 0) for the linear case.

We now consider equation (0-2) where $Q(x, y)$, $P(x, y)$ are analytic functions in a neighborhood of the critical point having terms of at least the second power. Again we suppose $\mathbf{D}$ is nonzero. Then the following theorem, Lefschetz (4), summarizes the work of Poincaré and Bendixson.

Theorem: The nature of the integral curves for equation (0-2) in the neighborhood of a critical point is the same as that of the first approximation, except that a center for the first approximation may yield either a center or spiral for the equation being considered.

By the term first approximation Lefschetz means the linear terms of $Q(x, y)$, $P(x, y)$ when $(0, 0)$ is the critical point under discussion. Hence the first approximation is an equation of the form (0-1).

If $\mathbf{D}$ is nonzero, this disposes of the classification of critical points for equation (0-2).

We are led to the general problem of the nature of the integral curves in the neighborhood of a compound critical point, i.e., a critical point when $\mathbf{D}$ is zero. For the remainder of this paper we refer to the equation

$$\frac{dy}{dx} = \frac{A_1x^2 + 2A_2xy + A_3y^2 - 1}{B_1x^2 + 2B_2xy + B_3y^2 - 1} = \frac{Q(x, y)}{P(x, y)} \tag{1-1}$$

as the ORIGINAL equation.

The following represents Gubař's findings, given without proof. However we should first comment that Gubař studies the original equation by considering the related system

$$dy/dt = Q(x, y) \ , \quad dx/dt = P(x, y) \ .$$

Also the transformations he considers, and those that we do, are transformations linear in x, y, and the parameter t.

Lemma I: For

$$\mu = (\partial P/\partial x) + (\partial Q/\partial y)$$

nonzero there exists a non-singular transformation which converts the original equation into

$$\frac{dy}{dx} = \frac{y + Q_2(x, y)}{P_2(x, y)} \ . \tag{1-2}$$

Here $Q_2(x, y)$, $P_2(x, y)$ are functions whose power series expansions, in a neighborhood of the critical point, have terms beginning with second degree.

Lemma II: There exists a function $\Phi(x)$ which is the solution to

$$y + Q_2(x, y) = 0$$

and which vanishes at $x = 0$. We shall call this the explicit function. Furthermore if the function $\Gamma(x)$ is defined by

$$\Gamma(x) = P_2(x, \Phi(x))$$

then in a neighborhood of the critical point $\Gamma(x)$ has the form

$$\Gamma(x) = \Delta_k x^k + \ldots \qquad , \qquad k \geq 2, \ \Delta_k \neq 0.$$

Theorem A: For μ nonzero the critical point (0, 0) is a node or saddle when $\Delta_k > 0$ or $\Delta_k < 0$ respectively and k is odd. If k is even, then the critical point is termed a SIMPLER SADDLE-NODE.

Consider a compound point of the original equation. If perturbing functions ω_1, ω_2 can be added to the numerator and denominator so that the original compound splits into an equal number of simple nodes and simple saddle points, then we have a saddle node, or a simpler saddle-node if μ is nonzero.

Lemma III: For μ zero there exists a non-singular transformation which converts the original equation into

$$\frac{dy}{dx} = \frac{Q_2(x, y)}{y + P_2(x, y)} \tag{1-3}$$

In this case $\Gamma(x) = Q_2(x, \Phi(x))$ where $\Phi(x)$ is the solution to $y + P_2(x, y) = 0$.

Lemma IV: If μ is zero and the function $\sum(x, y)$ is defined by

$$\sum(x, y) = (\partial P_2/\partial x) + (\partial Q_2/\partial y),$$

then in a neighborhood of the critical point the following representation holds:

$$\sum(x, \Phi(x)) = \bar{\Delta}_n x^n + \ldots \qquad , \quad n \geq 1, \ \bar{\Delta}_n \neq 0 ,$$
$$= 0 \qquad\qquad , \quad \bar{\Delta}_n = 0 \quad .$$

Theorem B: If μ is zero, then the critical point $(0, 0)$ of the original equation is dependent upon the exponents k of lemma II and n of lemma IV. If $k = 2m$ we have two cases possible:

a) $\bar{\Delta}_n \neq 0, \quad n < m.$

The point is a SADDLE-NODE.

b) $\bar{\Delta}_n \neq 0, \quad n \geq m \quad \text{or} \quad \bar{\Delta}_n = 0 .$

The point is DEGENERATE.

We further note that if $k = 2$, then $m = 1$ and the point is always degenerate.

To define what is meant by a degenerate compound point we consider a compound point and two perturbing functions ω_1 , ω_2 . As before we add these functions to the numerator and denominator of the original equation. If only simple nodes for the altered equation arise from this one compound point, then this compound point is termed a degenerate point for the original equation.

If $k = 2m + 1$, then four cases are possible:

a) $\Delta_k > 0$.

The point is a SADDLE.

b) $\Delta_k < 0$, $\bar{\Delta}_n \neq 0$, $n \geq m$, but

$\lambda = (\bar{\Delta}_m^{\ 2} + 4\{m + 1\}\,\Delta_k) < 0$ or

$\Delta_k < 0$, $\bar{\Delta}_n = 0$.

The point is the CENTER or SPIRAL.

c) $\Delta_k < 0$, $n < m$, n even or

$n = m$, $\lambda \geq 0$, n odd.

The point is a NODE.

d) $\Delta_k < 0$, $n < m$, n odd or

$n = m$, $\lambda \geq 0$, n odd.

The point is termed POINT WITH CLOSED NODAL REGION.

Thus theorems A and B divide the families of integral curves of the original equation into two broad classes; those where μ is nonzero and those where μ is zero. We intend to show that for the functions $P(x, y)$, $Q(x, y)$ of the original equation the exponent k of lemma II is always even. Therefore only saddle-node or degenerate points occur when $P(x, y) = 0$, $Q(x, y) = 0$ are central conics.

CHAPTER TWO

INTERSECTION POINTS AND CORRESPONDING EQUATION FORMS

We consider the original equation

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)} = \frac{N_1(x,y) - 1}{N_2(x,y) - 1}$$

where

$$N_1(x,y) = A_1x^2 + 2A_2xy + A_3y^2 ,$$

$$N_2(x,y) = B_1x^2 + 2B_2xy + B_3y^2 .$$

We assume $N_1(x,y)$ is not a multiple of $N_2(x,y)$ since if they were multiples then it is apparent that no critical points would exist. Note that this exludes the possibility that $A_i = B_i$ for all i. In order to apply the results of Gubaŕ's paper we need to know three items, (1) the intersection points of $Q(x,y) = 0$ and $P(x,y) = 0$; (2) the form of the original equation after an intersection point is translated to the origin; (3) the forms which belong to the group where μ is nonzero and which to the group where μ is zero.

We first introduce the quantities S_i, D_i defined by $S_i = A_i+B_i$, $i = 1, 2, 3$, $D_i = A_i - B_i$, $i = 1, 2, 3$. With this notation the sum and difference of the equations $Q(x,y) = 0$, $P(x,y) = 0$ can be written

$$S_1x^2 + 2S_2xy + S_3y^2 = 2 \qquad (2\text{-}1),$$

$$D_1x^2 + 2D_2xy + D_3y^2 = 0 \qquad (2\text{-}2).$$

In polar coordinates, these equations are

$$S_1\rho^2\cos^2\theta + 2S_2\rho^2\cos\theta\sin\theta + S_3\rho^2\sin^2\theta = 2 \qquad (2\text{-}1a),$$

$$D_1\rho^2\cos^2\theta + 2D_2\rho^2\cos\theta\sin\theta + D_3\rho^2\sin^2\theta = 0 \qquad (2\text{-}2a).$$

Solving (2-2a) for the angle θ, we find

$$\theta = \text{Arctan}\,(-2D_2 \pm \Delta / 2D_3) \text{ if } D_3 \neq 0,$$

where

$$\Delta = + \sqrt{(D_2^2 - D_1D_3)},$$

and if

$$D_3 = 0, \quad \theta = (n + 1)\pi/2.$$

We substitute these values back into (2-1a) and we find for the values of ρ,

$$\rho_1 = \pm\sqrt{2}/F_{13}, \quad \rho_2 = \pm\sqrt{2}/F_{23}.$$

Here $F_{13} = +\sqrt{(S_1D_3^2 - \{2S_3D_2 - 2S_2D_3\}\{\Delta - D_2\} - S_3D_1D_3)}$,

$$F_{23} = +\sqrt{(S_1D_2^2 - \{2S_3D_2 - 2S_2D_3\}\{-\Delta - D_2\} - S_3D_1D_3)}.$$

Hence, if we let (g, h) be the intersection point, we find these possibilities:

$$(g, h) = (\pm\sqrt{2}\,D_3/F_{13}, \pm\sqrt{2}\,\{\Delta - D_2\}/F_{13})$$

and $\qquad (2\text{-}3)$

$$(g, h) = (\pm\sqrt{2}\,D_3/F_{23}, \pm\sqrt{2}\,\{-\Delta - D_2\}/F_{23}).$$

Furthermore, if D_3 is zero, we see from the properties of symmetry that if D_1 is nonzero then equations (2-1a), (2-2a) yield points similar to (2-3), where only a simultaneous interchange of g and h and of the subscripts 1 and 3 is needed.

If it happens that D_1 is zero and D_3 is zero, then equation (2-2a) reduces to

$$\sin 2\theta = 0 \quad ,$$

$$\theta = (n+1)\pi/2, \ n = 0, 1, 2, 3.$$

Therefore (2-3) reduces to the simple form

$$(g, h) = (\pm\sqrt{\{2/S_1\}}, 0)$$

and (2-4)

$$(g, h) = (0, \pm\sqrt{\{2/S_3\}}) \ .$$

These sets of points (2-3), (2-4) are the only points of interest for this paper. We rule out the rest by noting that the other intersection points cannot coalesce. For if D_3 is zero and S_3 is zero then, in view of (2-1), (2-2), there are only <u>two</u> <u>finite</u> real points of intersection. These are given by

$$g = -2D_2h/D_1 \ ,$$

$$h = \pm D_1/(2\sqrt{\{-D_2\Delta_{12}\}}), \qquad (2\text{-}5)$$

where

$$\Delta_{ij} = \begin{vmatrix} A_i & A_j \\ B_i & B_j \end{vmatrix} \ .$$

Similarly if D_1 is zero and S_1 is zero we have by (2-1), (2-2) that

$$h = -2D_2\, g/D_3$$
$$g = \pm D_3/(2\sqrt{\{-D_2\Delta_{23}\}}) \,. \tag{2-5a}$$

Thus item (1) has been found. We recall here that these intersection points are also the critical points for the original equation.

In the process of finding item (2) we shall also group the resultant equations and find item (3). Hence suppose at the critical point (g, h) we have $\mu = (\partial P/\partial x) + (\partial Q/\partial y)$ nonzero. We translate points (2-3), (2-4) to the origin. Let T_i and J_i be given by:

$$T_i = -2A_i g - 2A_{i+1} h \,,$$

$$J_i = -2B_i g - 2B_{i+1} h \,,$$

and suppose the x, y are the new coordinates and $N_1(x, y), N_2(x, y)$ are as previously described. The following types of equations arise:

$$\text{(A)} \quad \frac{dy}{dx} = \frac{T_1 x + T_2 y + N_1(x, y)}{J_1 x + J_2 y + N_2(x, y)} \tag{2-6}$$

and (2-6) with these modifications;

$$\begin{aligned} &1) \quad J_1 = J_2 = 0 \\ &2) \quad T_2 = J_2 = 0 \\ &3) \quad T_1 = J_1 = 0 \\ &4) \quad T_1 = J_1 = J_2 = 0. \end{aligned} \tag{2-6a}$$

$$\text{(B)} \quad \frac{dy}{dx} = \frac{N_1(x, y)}{J_1 x + J_2 y + N_2(x, y)} \tag{2-7}$$

and the modified form

$$1) \quad J_2 = 0 \; .$$

Now suppose μ is zero. Then we have equation (2-6) with one form

$$1) \quad T_2 = J_2 = J_1 = 0 \tag{2-6b}$$

We see that the above forms of the differential equation are not generally in the form of (1-2), (1-3). Thus we need to know that linear transformations exist which will convert them into the form of (1-2) and (1-3).

Only three transformations are needed, two for the equations when μ is nonzero and one for when μ is zero. Let t' be the parameter of the equation to be transformed and t be the new parameter.

Then for equations (2-6), (2-6a) in the case where μ is nonzero we use the transformation of the xy- plane which has the matrix

$$\begin{bmatrix} -L & T_1 \\ H & T_2 \end{bmatrix} \tag{2-8}$$

and the parameter transformation

$$t = (T_2 + T_1/L)\, t' \qquad (2\text{-}8),$$

where L is the ratio T_1/J_1 or T_2/J_2 if either exists and is $\underline{1}$ otherwise. H is to be $\underline{1}$ if either J_1 or J_2 is nonzero and H is to be zero if both J_1 and J_2 vanish.

For equation (2-7) the corresponding transformation has matrix

$$\begin{bmatrix} 0 & J_1 \\ 1 & J_2 \end{bmatrix} \tag{2-9}$$

and

$$t = (J_1)\, t' \qquad (2\text{-}9).$$

If, however, μ is zero the only transformation needed has the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & -T_2/T_1 \end{bmatrix} \tag{2-10}$$

and

$$t = (T_1)\, t' \qquad (2\text{-}10).$$

CHAPTER THREE

COALESCENT POINTS

Since the graphs of the equations

$$N_1(x, y) = \beta \quad , \quad N_2(x, y) = 1$$

are symmetric with respect to the origin, so is the set of intersection points. Thus, if when $\beta = 1$ there is a coalescent point, then there is a second coalescent point. These two coalescent points are also symmetric with respect to the origin. Further, if there is coalescence for $\beta = 1$, then for suitable β near one, there must be four ordinary points of intersection.

Thus if for <u>all</u> β near 1 we have two or zero intersection points, then the intersection points for $\beta = 1$ cannot be coalescent. This rules out the point sets (2-5), (2-5a).

Furthermore suppose ϕ_1 is the polar coordinate angle of one of the coalescent points and ϕ_2 that of the other coalescent point. A necessary and sufficient condition for coalescence is, from the symmetry of the intersection set, $\sin(\phi_1 - \phi_2) = 0$.

We find it easiest to divide the conics into three groups:

1) $A_i \neq B_i$, $i = 1, 2, 3$;

2) $A_i = B_i$ for some, but not all, i and none of these equal coefficients vanish;

3) $A_i = B_i = 0$ for some, but not all, i.

For the first group we have the points (2-3) to consider. Thus

$$\sin \phi_1 = h_1/r_1 = M_1(\Delta - D_2), \quad \cos \phi_1 = g_1/r_1 = M_1(D_3)$$

$$\sin \phi_2 = h_2/r_2 = M_2(-\Delta - D_2), \quad \cos \phi_2 = g_2/r_2 = M_2(D_3),$$

where M_i is a nonzero factor.

Therefore

$$\sin(\phi_1 - \phi_2) = 2M_1M_2\Delta D_3.$$

Since A_3 is unequal to B_3, $\sin(\phi_1 - \phi_2)$ vanishes <u>if</u> and <u>only if</u> Δ is zero, i.e.,

$$D_2^{\,2} = D_1D_3.$$

For the second group we see that if A_1 equals B_1 and A_3 is unequal to B_3, then points (2-3) are again used and the result is the same as above. If, however, A_3 equals B_3 and A_1 is unequal to B_1 then by interchange of subscripts and of coordinates g and h we arrive again at the same result, i.e.,

$$\sin(\phi_1 - \phi_2) = 2M_1M_2\Delta D_1 = 0.$$

This implies

$$D_2^{\,2} = D_1D_3.$$

Now suppose $A_1 = B_1$, $A_3 = B_3$, and $A_2 \neq B_2$. In this case we have to use points (2-4) which yield

$$\sin \phi_1 = 0 = \cos \phi_2 ,$$

$$\cos\phi_1 = 1 = \sin\phi_2 .$$

Thus $\sin(\phi_1 - \phi_2)$ is nonzero. Hence if coalescence of critical points is to occur for conics in group 2 we need $D_2^2 = D_1D_3$.

Suppose we have an equation belonging to the last group. We have already noted that either $A_1 = B_1 = 0$ or $A_3 = B_3 = 0$ implies coalescence cannot occur. Therefore we need only to consider $A_2 = B_2 = 0$.

We have by the use of points (2-3):

$$\sin \phi_1 = M_1\sqrt{(-D_1D_3)} , \ \cos \phi_1 = M_1(D_3),$$

$$\sin \phi_2 = M_2(-\sqrt{\{-D_1D_3\}}), \ \cos \phi_2 = M_2(D_3),$$

where M_i is as before and A_3 is unequal to B_3. Thus

$$\sin(\phi_1-\phi_2) = 2M_1M_2D_3\sqrt{(-D_1D_3)}$$

and hence is zero if and only if D_1 is zero, i.e., A_1 equals B_1.

In a similar manner we find that if A_1 is unequal to B_1 then $\sin(\phi_1 - \phi_2)$ is zero if and only if D_3 is zero, i.e., A_3 equals B_3. Again this means $D_2^2 = D_1D_3$.

Thus we have shown the following to be true;

Lemma: A necessary and sufficient condition for coalescence of intersection points of the two concentric conics $N_1(x, y) = 1$, $N_2(x, y) = 1$ is that

$$D_2^2 = D_1 D_3 ,$$

<u>where</u> $D_i = A_i - B_i$, $i = 1, 2, 3$.

Suppose we now consider a typical critical point (g, h) of the original equation. When we translate this point to the origin we obtain the following differential equation:

$$\frac{dy}{dx} = \frac{T_1 x + T_2 y + A_1 x^2 + 2A_2 xy + A_3 y^2}{J_1 x + J_2 y + B_1 x^2 + 2B_2 xy + B_3 y^2} \tag{2-6}$$

where, as before,

$$T_i = (-2A_i g - 2A_{i+1} h)$$

and

$$J_i = (-2B_i g - 2B_{i+1} h) \qquad \text{for } i = 1, 2 .$$

To this equation we apply the appropriate transformation which was previously found; it is changed into either (1-2) which is explicitly

$$\frac{dy}{dx} = \frac{y + K_1 x^2 + K_2 xy + K_3 y^2}{R_1 x^2 + R_2 xy + R_3 y^2} \tag{3-1}$$

or (1-3) which is explicitly

$$\frac{dy}{dx} = \frac{R_1 x^2 + R_2 xy + R_3 y^2}{y + K_1 x^2 + K_2 xy + K_3 y^2} . \tag{3-2}$$

For us the R_i , K_i are expressible in terms of T_i , J_i and they will be dealt with later. Recall here that what we are heading for is the evaluation of the smallest exponent in the equation

$$R_1 x^2 + R_2 x \; \Phi(x) + R_3 \, \Phi^2(x) = 0 ,$$

where $\Phi(x)$ is the explicit function for

$$y + K_1 x^2 + K_2 xy + K_3 y^2 = 0 ,$$

cf. lemma II, chapter one.

Thus for K_3 nonzero we can apply the quadratic formula to the last equation and obtain for the explicit function:

$$\Phi(x) = \frac{-(K_2 x + 1) \pm \sqrt{(K_2 x + 1)^2 - 4K_3 K_1 x^2}}{2 K_3}$$

We expand the radical in a Maclaurin series and obtain:

$$\Phi(x) = -K_1 x^2 + K_1 K_2 x^3 - \{2K_1 K_2^2 + 2K_1^2 K_3\} x^4 + . . . \qquad (3\text{-}3)$$

If K_3 is zero we find the explicit function to be:

$$\Phi(x) = -K_1 x^2 + K_1 K_2 x^3 - K_1 K_2 x^4 + . . . \qquad (3\text{-}3a)$$

by use of the binomial expansion.

We note, that if K_1 is zero then $\Phi(x)$ is identically zero, and that if x is zero then $\Phi(x)$ is again zero. Hence this explicit function has the desired properties.

Furthermore, with $\Phi(x)$ now determined, we may compute the function $\Gamma(x)$ (as in lemmas II and III) to be

$$\Gamma(x) = R_1 x^2 + R_2 x\{-K_1 x^2 + K_1 K_2 x^3 + \ldots\}$$
$$+ R_3\{-K_1 x^2 + K_1 K_2 x^3 + \ldots\}^2. \tag{3-4}$$

Hence if μ is nonzero we only need to examine the numbers K_1, R_1, R_2. However when μ is zero we need the representation of the function $\sum(x)$. Upon considering lemma IV we find it to be

$$\sum[x, \Phi(x)] = \{2K_1 + R_2\} x + \{K_2 + 2R_3\} \Phi(x) \tag{3-5}$$

If μ is zero we need to know the value of R_3 in addition to K_1, K_2, R_2. But we shall always examine R_1 first, since, from Theorem B, if R_1 is nonzero the critical point is always degenerate.

CHAPTER FOUR

INTEGRAL CURVES FOR μ NONZERO

In this chapter we consider the class of integral curves in the case

$$\mu = (\partial P/\partial x) + (\partial Q/\partial y)$$

is nonzero. We apply transformation (2-8) to equations of form (2-6), (2-6a) to reduce to form (3-1). We find

1) R_1 is a multiple of $\overline{R}_1$ where

$$\overline{R}_1 = T_2^2(A_1 - LB_1) + T_1T_2(2LB_2 - 2A_2) + T_1^2(A_3 - LB_3);$$

2) K_1 is a multiple of $\overline{K}_1$ where

$$\overline{K}_1 = T_2^3 A_1 + T_1^3 B_1 + T_2^2 T_1 (B_1 - 2A_2) + T_1^2 T_2 (A_3 - 2B_2);$$

3) R_2 is a multiple of $\overline{R}_2$ where

$$\overline{R}_2 = T_2 L(B_1 + LB_2) - T_2(A_1 + LA_2) + T_1(A_2 + LA_3) - T_1 L (B_2 + LB_3).$$

In chapter three we have divided the conics into three groups; we first consider group 3), $A_i = B_i = 0$ for some, but not all, i; then group 2), $A_i = B_i$ for some i and none of these equal coefficients vanish; and finally group 1), $A_i \neq B_i$ for all i.

We suppose $A_2 = B_2 = 0$. Then we know, from the conditions for coalescence, that either D_1 is zero or D_3 is zero. Assume D_1 is zero. Then the critical points (2-3) become

$$(g, h) = (\pm \sqrt{2 D_3 / F_{13}}\, ,\ 0\ ) .$$

Therefore $\qquad T_2 = 0 \quad$ and

$$L = T_1 / J_1 = (A_1 h)/(B_1 h) = 1.$$

Hence

$$\overline{R}_1 = T_1^{\ 2} (A_3 - B_3) \neq 0.$$

Now we assume D_3 is zero. Then the critical points are found to be

$$(g, h) = (0, \pm \sqrt{2 D_1 / F_{31}}) .$$

Thus $\qquad T_1 = 0 \quad$ and

$$L = T_2 / J_2 = (A_3 h)/(B_3 h) = 1.$$

Again we have

$$\overline{R}_1 = T_2^{\ 2} (A_1 - B_1) \neq 0.$$

If we now suppose that $A_2 = B_2 \neq 0$ then the only alteration needed in the above work would be: (a) for D_1 being zero then T_2 is no longer zero but all else holds; (b) for D_1 nonzero then T_1 is nonzero but everything else is valid.

Thus we can state that for groups 3) and 2), i.e., $A_i = B_i$ for some but not all i, if coalescence occurs and $\mu = (\partial P/\partial x) + (\partial Q/\partial y)$ is nonzero then the exponent k of Theorem A is <u>even</u>.

Suppose we are now concerned with an equation in group 1), $A_i \not\equiv B_i$ for all i. Then the critical points (2-3) become

$$(g, h) = (\pm\sqrt{2}\, D_3/F_{13}\, ,\ \pm\sqrt{2}\,(-D_2)/F_{13})\ .$$

We first note that when $A_2 \not\equiv B_2$ then $T_2 = J_2$ and they are dependent upon the value of $\Delta_{23} = A_2B_3 - B_2A_3$. For if we consider the representation of T_2 and J_2 we find:

$$\begin{aligned} T_2 &= -2\,(A_2 g + A_3 h) \\ &= -2\,(\pm\sqrt{2}/F_{13})\,(A_2D_3 - D_2A_3) \\ &= +2\,(\pm\sqrt{2}/F_{13})\,(\Delta_{23}) \end{aligned}$$

and

$$\begin{aligned} J_2 &= -2\,(\pm\sqrt{2}/F_{13})\,(B_2D_3 - D_2B_3) \\ &= +\ 2\,(\pm\sqrt{2}/F_{13})\,(\Delta_{23})\ . \end{aligned}$$

But we also have $T_1J_2 = J_1T_2$ since $\mathbf{D} = T_1J_2 - J_1T_2 = 0$. Thus $T_1 = J_1$ when $A_2 \not\equiv B_2$.

At this point we divide the work into two cases according to whether Δ_{23} is nonzero or zero.

If Δ_{23} is nonzero we have

$$L = T_1/J_1 = T_2/J_2 = 1.$$

Hence

$$\overline{R}_1 = T_2^{\,2}D_1 - 2T_1T_2D_2 + T_1^{\,2}D_3$$
$$= D_1\,(T_2 - D_2T_1/D_1)^2$$

upon completing the square. Inserting the values of T_1, T_2, we have, upon arrangement,

$$\overline{R}_1 = 8D_1(A_2\{D_3D_1 + D_2^{\,2}\} - A_3D_2D_1 - A_1D_2D_3)^2/F_{13}^{\,2}$$

$$= 8D_1(A_2\{D_3D_1 - D_2^{\,2}\} - A_3D_2D_1 - A_1D_2D_3 + 2A_2D_2^{\,2})^2/F_{13}^{\,2}.$$

We substitute the values for the D_i to get

$$\overline{R}_1 = 8D_1D_2^{\,2}(2\{A_2^{\,2} - A_1A_3\} + A_3B_1 + A_1B_3 - 2A_2B_2)^2/F_{13}^{\,2}.$$

However since $D_2^{\,2} = D_1D_3$ we have

$$2A_2B_2 - A_3B_1 - A_1B_3 = (A_2^{\,2} - A_1A_3) + (B_2^{\,2} - B_1B_3). \qquad (4\text{-}1)$$

Insert this in $\overline{R}_1$; this yields

$$\overline{R}_1 = 8D_1D_2^{\,2}(\{A_2^{\,2} - A_1A_3\} - \{B_2^{\,2} - B_1B_3\})^2/F_{13}^{\,2}.$$

At this stage we have determined the following: when $A_i \neq B_i$ for all i and Δ_{23} is nonzero, R_1 is zero if and only if

$$A_2^{\,2} - A_1A_3 = B_2^{\,2} - B_1B_3.$$

Note that when Δ_{23} is nonzero, so is T_2.

But we next show that if Δ_{23} is nonzero then

$$A_2^{\,2} - A_1A_3 \neq B_2^{\,2} - B_1B_3 .$$

Consequently we shall conclude that when $A_i \neq B_i$ for all i and T_2 is nonzero, then R_1 is nonzero.

Now if we consider the values of T_1 and J_1 we find

$$T_1 = -2(\pm\sqrt{2}/F_{13})\,(D_3A_1 - D_2A_2)$$

$$= -2(\pm\sqrt{2}/F_{13})\,(A_2B_2 - A_1B_3 - \{A_2^{\,2} - A_1A_3\})$$

and

$$J_1 = -2(\pm\sqrt{2}/F_{13})\,(B_2A_2 - B_1A_3 - \{B_2^{\,2} - B_1B_3\}).$$

Since $T_1 = J_1$ we have:

$$(A_2^{\,2} - A_1A_3) - (B_2^{\,2} - B_1B_3) = A_3B_1 - A_1B_3$$

$$= \Delta_{31} .$$

Thus our problem is reduced to showing that Δ_{23} nonzero implies Δ_{31} is nonzero. We assume Δ_{23} is nonzero but Δ_{31} is zero and show a contradiction.

Since Δ_{31} is zero, equation (4-1) implies that

$$A_2^{\,2} - A_1A_3 = A_2B_2 - A_1B_3 ,$$

or

$$A_2D_2 - A_1D_3 = 0.$$

Multiplying by D_1 and setting $D_1D_3 = D_2^2$ we have,

$$A_2D_2D_1 - A_1D_2^2 = 0 .$$

Hence $D_2\Delta_{21}$ is zero. Since D_2 is nonzero, then Δ_{21} is zero. Obviously if this is so then Δ_{23} is zero because Δ_{31} is zero. This contradicts our assumption. Thus if Δ_{23} is nonzero then R_1 is nonzero also. This completes the discussion in case Δ_{23} is nonzero.

We now suppose Δ_{23} is zero. Then T_2 and J_2 are zero. We first assume that T_1 is nonzero. From Δ_{23} being zero we have

$$A_2/B_2 = A_3/B_3 .$$

Then because $L = T_1/J_1$ we find

$$\begin{aligned}\overline{R}_1 &= T_1^2(A_3 - B_3T_1/J_1)\\ &= T_1^2B_3(A_3/B_3 - T_1/J_1) .\end{aligned}$$

Therefore R_1 is zero if

$$A_3/B_3 = T_1/J_1 = A_2/B_2 .$$

But then we have

$$\begin{aligned}\overline{R}_2 &= T_1(A_2 + A_3T_1/J_1) - (T_1^2/J_1)(B_2 + B_3T_1/J_1)\\ &= (T_1^2/J_1)(A_2\{B_2/A_2\}+A_3) - (T_1^2/J_1)(B_2 + B_3\{A_3/B_3\})\\ &= 0 .\end{aligned}$$

Since R_1 and R_2 are zero and $P_2(x, y)$ is nonzero, we conclude that R_3 is nonzero. Also we see that our explicit function $\Phi(x)$ is not identically zero because

$$\overline{K}_1 = T_1^{\,3} B_3 \neq 0 \quad \text{as} \quad B_3 \neq 0.$$

We note that if B_3 were zero then

$$\overline{R}_1 = A_3 J_1 \neq 0 .$$

Thus whether B_3 vanishes or not we find the exponent k to be <u>even</u> under the assumption that T_1 is nonzero.

We now assume T_1 is zero and still assume Δ_{23} is zero. Then we use the appropriate transformation on (2-7), i. e., the equation where $T_1 = T_2 = 0$. We find the altered equation to be

$$\frac{dy}{dx} = \frac{y + x^2 B_3/J_1 + 2xy\, B_2/J_1 + y^2 B_1/J_1}{x^2 A_3/J_1 + 2xy\, A_2/J_1 + y^2 A_1/J_1}$$

$$= \frac{y + x^2 K_1 + 2xy\, K_2 + y^2 K_3}{x^2 R_1 + 2xy\, R_2 + y^2 R_3} .$$

Thus we have new descriptions for R_1, R_2, K_1 obtained by identifying coefficients in the two forms above.

Because Δ_{23} is still zero, we have the three possibilities:

1) $B_2 = B_3 = 0$, 2) $A_2 = A_3 = 0$, 3) $A_2B_3 - A_3B_2 = 0$,

$A_i \neq 0$, $B_i \neq 0$.

Suppose we have the first possibility. Then, as we are in the case A_i not equal to B_i, A_2 is nonzero and A_3 is nonzero. Therefore

$$R_1 = A_3 / J_1 \neq 0 .$$

Suppose now that we consider the second possibility. Then B_2 is nonzero and B_3 is nonzero. Hence we immediately obtain: R_1 and R_2 are zero and R_3 is nonzero. Also

$$K_1 = B_3 / J_1 \neq 0.$$

For the last case we suppose A_3 nonzero and $B_2 A_3 = A_2 B_3$. Then we again have

$$R_1 = A_3 / J_1 \neq 0.$$

Therefore we find that if $T_1 = T_2 = J_2 = 0$ the exponent k is again <u>even.</u>

In summary, we have so far shown that if μ is nonzero, the exponent k of Theorem A is <u>even.</u> Hence when μ is not zero, the critical point is a <u>simpler</u> <u>saddle-node.</u>

CHAPTER FIVE

INTEGRAL CURVES FOR μ ZERO

We again consider the original equation after a translation which sends a coalescent critical point to the origin. However we now have the condition that

$$\mu = (\partial P/\partial x) + (\partial Q/\partial y) = 0 .$$

We use transformation (2-10) to convert it into an equation of form (1-3).

We find that under this transformation the values of R_1, R_2, R_3, K_1, and K_2 are as follows:

$$R_1 = ([T_2^2\{B_1T_1 + T_2A_1\}/T_1^4] - [2T_2\{B_2T_1 + A_2T_2\}/T_1^3]$$

$$+ [\{B_3T_1 + T_2A_3\}/T_1^2]) ;$$

$$R_2 = 2[\{B_2T_1 + A_2T_2\}/T_1^2] - 2[T_2\{B_1T_1 + T_2A_1\}/T_1^3] ;$$

$$R_3 = (B_1T_1 + T_2A_1)/T_1^2 ;$$

$$K_1 = (\{A_1T_2^2/T_1^2\} - 2A_2\{T_2/T_1\} + A_3)/T_1 ;$$

$$K_2 = (\{-2A_1T_2/T_1\} + 2A_2)/T_1 .$$

Again we divide the conics into three groups, 1) $A_i \neq B_i$ for all i ; 2) $A_i = B_i$ for some i and none of these coefficients vanish; and 3) $A_i = B_i = 0$ for some, but not all, i.

We first suppose that $A_2 = B_2 = 0$, i.e., D_2 is zero. Then we have, cf. chapter four, either D_1 is zero or D_3 is zero. Assume D_1 is zero. As in the case where μ was nonzero, and from the dependence of T_2 on Δ_{23}, we have T_2 is zero. But μ being zero implies that $T_2 = J_1 = 0$. However we have the representation of J_1 as $(-2B_1 g)$. As we know that the g coordinate is nonzero we find that

$$B_1 = 0 = A_1 \qquad \text{since } D_1 \text{ is zero.}$$

But we have already seen that $A_2 = B_2 = A_1 = B_1 = 0$ does not yield a coalescent point.

Now we assume D_1 is nonzero. The g coordinate of the critical point is zero and this yields the fact that J_1 is zero. This and the vanishing of μ implies that T_2 is zero. But T_2 equal zero implies that $A_3 = B_3 = 0 = A_2 = B_2$ and again coalescence cannot occur.

Thus we have shown that if $A_2 = B_2 = 0$ and $\mu = 0$ then coalescence does not occur.

We now suppose that $A_2 = B_2 \neq 0$, i.e., we suppose we have an equation in group 2. If we assume that D_1 is zero this will yield the same critical points as above. Since the h coordinate is now zero and since A_1 equals B_1 we find that T_1 equals J_1.

But $\mu = T_2 + J_1 = 0$ implies $T_1 = J_1 = -T_2$.

Substituting these facts into our formula for R_1 we find

$$R_1 = (-D_3/T_1) \neq 0.$$

On the other hand if we assume that D_1 is nonzero then our critical point has the g coordinate zero. Then, because μ is zero, we find that

$$J_1 = -2B_2h = -T_2 = 2A_3h .$$

Thus $\qquad -B_2 = A_3 \quad$ and hence $\quad A_2 = B_2 = -A_3 = -B_3$.

Therefore we see that $T_1 = -2A_2h = -T_2$.

Again, substitution into our formula for R_1 reveals

$$R_1 = (-D_1/T_1) \neq 0.$$

Therefore we see, in light of Theorem B, that if $D_i = 0$ for some i and if coalescence of critical points occurs under the condition $\mu = 0$, then the critical point (g, h) formed by this coalescence is a degenerate point, i. e., the exponent k of Theorem B is even.

Suppose we now are in the last group where A_i is not equal to B_i for all i. In chapter four we found that $T_2 = J_2$, $T_1 = J_1$ when A_2 does not equal B_2. Hence if we use the condition μ is zero we have the equality

$$T_2 = J_2 = -T_1 = -J_1 \neq 0.$$

Therefore upon substitution and rearrangement we see that

$$R_1 = (T_2D_1 + 2T_2D_2 + T_2D_3)/T_1^2$$

$$= -(\{D_2^2/D_3\} + 2D_2 + D_3)/T_1$$

$$= -(D_2 + D_3)^2/T_1D_3 \quad .$$

Hence we have R_1 is zero if and only if $D_2 = -D_3$. But since $D_2^2 = D_1D_3$ this implies $D_2 = -D_3 = -D_1$ if R_1 is to be zero.

We suppose R_1 is nonzero; hence $D_2 + D_3$ is nonzero. Then we conclude that the exponent k equals <u>two</u> and therefore the critical point is a <u>degenerate point.</u>

If, however, we suppose $D_2 = -D_3 = -D_1$ then R_1 is zero. But when we substitute into our formula for R_2 we get

$$R_2 = 2(T_2D_1 + T_2D_2)/T_1^2 = 0 \ .$$

Thus we deduce, since $P_2(x, y)$ is nonzero, that R_3 is nonzero. Since we are in the case where $\mu = 0$ we need to examine the value of K_1. We find that

$$K_1 = (A_1 + 2A_2 + A_3)/T_1 \quad .$$

Let us consider the consequence of T_1 being nonzero. Since

$$T_1 = -2(\pm\sqrt{2}/F_{13})(A_1D_3 - A_2D_2)$$

$$= -2(\pm\sqrt{2}/F_{13})(D_3\{A_1 + A_2\})$$

$$\neq 0$$

we see that $(A_1 + A_2)$ is nonzero.

Likewise since

$$T_1 + T_2 = -2(\pm\sqrt{2}/F_{13})(D_3\{A_1 + A_2\} + D_2\{A_2 + A_3\})$$

is zero we see that

$$A_1 = A_3 \ .$$

Hence

$$A_1 + 2A_2 + A_3 = 2(A_1 + A_2)$$

is nonzero and so K_1 is nonzero also.

Then we have the functions $\Gamma(x)$ and $\sum(x)$ represented by the expansion (3-4) and (3-5)

$$\Gamma(x) = R_3K_1^2x^4 + \ldots$$

$$= \Delta_4 x^4 + \ldots \qquad \text{and}$$

$$\sum(x, \Phi(x)) = x(2K_1) + \ldots$$

$$= \bar{\Delta}_1 x + \ldots \qquad .$$

Thus we have $k = 4$, $n = 1$, $\bar{\Delta}_n$ nonzero and by Theorem B these are the conditions for a saddle-node.

So we see that if coalescence of critical points occurs and μ is zero then except for <u>one</u> case a <u>degenerate point</u> arises. In the exceptional case, where $D_2 = -D_3 = -D_1$, we have a <u>saddle-node</u> at the critical point.

CHAPTER SIX

SUMMARY OF RESULTS

Before we summarize it would be advisable to show that a coalescent point is indeed a compound point. To do this we consider our original equation

$$\frac{dy}{dx} = \frac{A_1x^2 + 2A_2xy + A_3y^2 - 1}{B_1x^2 + 2B_2xy + B_3y^2 - 1} = \frac{Q(x, y)}{P(x, y)} .$$

Then since $\mathbf{D} = (\partial P/\partial x)(\partial Q/\partial y) - (\partial Q/\partial x)(\partial P/\partial y)$ we have upon substitution and rearrangement

$$\begin{aligned}\mathbf{D} &= (2B_1x + 2B_2y)(2A_2x + 2A_3y) \\ &\quad - (2A_1x + 2A_2y)(2B_2x + 2B_3y) \\ &= 4\{x^2(A_2B_1 - B_2A_1) + xy(A_3B_1 - B_3A_1) + y^2(A_3B_2 - B_3A_2)\}\end{aligned}$$

where (x, y) is the critical point.

Now in the study of our conics we found that for coalescence we needed $D_2^2 = D_1D_3$. Also for simplicity we divided the conics into the three groups, 1) $A_2 = B_2 = 0$, 2) $A_2 = B_2 \neq 0$, 3) $A_2 \neq B_2$.

We first suppose $A_2 = B_2 = 0$. Then we see that

$$\mathbf{D} = 4xy\,(A_3B_1 - B_3A_1) .$$

But the critical point (x, y) is of the form $(x, 0)$ if D_1 is zero or of the form $(0, y)$ if D_1 is nonzero. In either case the value of $\mathbf{D}$ is zero and hence the coalescent critical point is compound.

We now suppose we have an equation corresponding to the second group, in which $A_2 = B_2 \neq 0$. Again the critical point has the form $(x, 0)$ or $(0, y)$. If we use $(x, 0)$ then we have $D_1 = 0$, i.e., $A_1 = B_1$. But the value of $\mathbf{D}$ is

$$\mathbf{D} = 4x^2 (A_2B_1 - B_2A_1)$$

$$= 0 \qquad\qquad \text{since } A_1 = B_1,\ A_2 = B_2 .$$

Likewise D_1 is nonzero if $(0, y)$ is used and then D_3 is zero at coalescence and hence

$$\mathbf{D} = 4y^2 (A_3B_2 - B_3A_2) = 0.$$

So finally suppose A_2 does not equal B_2. Then

$$(x, y) = (\pm D_3 \sqrt{2}/F_{13} , \mp D_2\sqrt{2}/F_{13}) .$$

Hence

$$\mathbf{D} = 8\{D_3^2\Delta_{21} - D_3D_2\Delta_{31} + D_2^2\Delta_{32}\}/F_{13}^2$$

$$= -8D_3 \begin{vmatrix} D_1 & D_2 & D_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} /F_{13}^2 .$$

Since the first row is the difference of the second and third rows, $\mathbf{D} = 0$. Thus coalescence of critical points does indeed yield a compound point.

We summarize:

Theorem: Given a differential equation

$$\frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)}$$

in which

$$Q(x, y) = A_1 x^2 + 2A_2 xy + A_3 y^2 - 1$$

$$P(x, y) = B_1 x^2 + 2B_2 xy + B_3 y^2 - 1 .$$

Set $\quad D_i = A_i - B_i \,, \quad i = 1, 2, 3 .$

There is coalescence of simple critical points into compound critical points if and only if

$$D_2^2 = D_1 D_3 \quad .$$

If $D_1 \neq D_2$ and if at the critical point in question we have

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0 ,$$

then the compound critical point is degenerate. In all other cases the compound critical point is a saddle-node.

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