AN ABSTRACT OF THE DISSERTATION OF

Gülden Karakök for the degree of Doctor of Philosophy in Mathematics Education presented on, March 11, 2009.

Title: Students' Transfer of Learning of Eigenvalues and Eigenvectors: Implementation of Actor-Oriented Transfer Framework.

Abstract approved:

Barbara S. Edwards

The purpose of this study was to investigate third year college students' transfer of learning of the concepts of eigenvalues and eigenvectors from the winter term physics courses to the interviews in which they participated before, during, and after these courses. Transfer of learning of each student was explored by implementing the Actor-Oriented Transfer (AOT) framework (Lobato, 1996). The research questions addressed by this study were

1. What characterizes upper-level physics students' emerging understanding of the concepts of eigenvalues and eigenvectors before, during, and after studying the concepts in an intensive linear algebra review week and implementing them during a series of three 3-week intensive physics courses?

2. What do students transfer about the concepts of eigenvalues and eigenvectors from this series of courses to an interview setting? More precisely, what kind of experiences and views related to matrices, methods of finding eigenvalues and eigenvectors, the interpretation and use of the eigenvalue equation, and the relationship between basis vectors and eigenvectors do upper-level physics students transfer from their coursework to the interview setting?
3. In what ways do the experiences students choose to transfer relate to their emerging understanding of the concept of eigenvalues and eigenvectors?

Seven junior level physics majors volunteered to participate in the study. Participants were enrolled in four physics courses during the winter term. The seven students participated in three in-depth interviews before, during, and after they were enrolled in these courses.

Four students were purposefully selected for in-depth case analysis and a cross-case analysis by actor-oriented transfer was conducted on the data from all seven students.

The results of this study suggest the importance of exploring the issue of transfer by implementing the actor-oriented transfer framework. The researcher found that the actor-oriented transfer analysis provided evidence of transfer from the winter term courses to the interviews. Six students transferred their experience from a small group activity to the second interviews. There were other experiences students seemed to transfer however the transfer from this small group activity was observed in six of the participants' data.

Students' Transfer of Learning of Eigenvalues and Eigenvectors: Implementation of Actor-Oriented Transfer Framework

by

Gülden Karakök

A DISSERTATION

submitted to

Oregon State University

in partial fulfillment of

the requirements for the

degree of

Doctor of Philosophy

Presented March 11, 2009

Commencement June 2009

Doctor of Philosophy dissertation of Gülden Karakök

presented on March 11, 2009.

APPROVED:

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I understand that my dissertation will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my dissertation to any reader upon request.

Gülden Karakök, Author

ACKNOWLEDGEMENTS

My deepest appreciation goes to Dr. Barbara Edwards, my advisor, mentor, and friend. Her continuous support and invaluable guidance throughout my research made this journey possible.

Along with my advisor this study was greatly improved by the contributions of my committee: Dr. Corinne Manogue, Dr. Tom Dick, Dr. Maggie Niess, and Dr. Lynda Ciueffetti.

I also want to share my appreciation for the participants of this study for letting me be a part of their learning process. A special thanks to the teaching team of the winter physics courses and Professor Clayton for having me in their classes and meetings.

I also would like to thank Dr. Shlomo Libeskind, Dr. Tevian Dray, and Dr. Liz Gere for their support and encouragement.

Most of all, I would like to acknowledge my friends and family—both near and far. Rachel and Ryan Harrington, Ching-Chia Ko and Tugrul Turan provided informal therapy sessions and constant encouragement that were priceless. Jessica and Joel Cohen, and Molly Phipps never stopped supporting me. Ekin Senturk, Murat Hamutcuoglu made the last six months of this journey a whole lot more fun! The biggest thanks must go to my parents, siblings, and in-laws who were a constant reminder of what is really important in life. I appreciate all of your support, and thankful for having all in my life.

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Students' Transfer of Learning of Eigenvalues and Eigenvectors: Implementation of Actor-Oriented Transfer Framework

CHAPTER ONE

INTRODUCTION

Transfer of learning has been discussed by researchers from various fields for over 100 years (Hoffding, 1892; Thorndike & Woodworth, 1901; Bransford & McCarrell, 1974; Anderson, 1976; Gick & Holyoak, 1983; Lave, 1988; Detterman, 1993; Greeno, Smith & Moore, 1993; Bransford & Schwartz, 1999; Lobato, 1996; Beach, 1999; Mestre, 2005). Early psychology researchers were interested in the issue of how information learned at one time influence the learners and their performance at a later time (Hoffding, 1892). Similar ideas have been observed in educational research studies (Greeno, Smith & Moore, 1993). Transfer of learning has been linked to one of the important goals of education: learners need to generalize their learning experiences from initial learning settings to other settings (Bransford, Brown, & Cocking, 1999).

To see if the goal was fulfilled, early researchers developed studies to measure the learners' ability to transfer knowledge and skills. These studies were focused on the question of whether a particular piece of knowledge or a skill transfers from the initial learning situation to the target situation. The results of these studies were very similar to each other indicating that learners *did not* transfer their knowledge and skills (Detterman, 1993). However, this conclusion did not help the educators understand the reasons behind the problem of transfer.

Some researchers suggested that the definition of transfer of learning was the source of the problem. Transfer of learning is defined as *the ability to apply knowledge learned in one context to new contexts* and according to a strict application of this definition learners mostly lacked the ability to transfer (Simon & Reed, 1976; Detterman, 1993). However, this definition of transfer suggested that knowledge could be separated from the situation in which it was constructed rather than contained in the overall connected experience (Lave, 1988; Greeno, Smith & Moore, 1993; Lobato & Siebert, 2002). It was not just the definition of transfer but the methodologies used that were problematic. The studies often used tasks (problems) developed by researchers and learners' answers were analyzed for evidence by researchers without further in-depth interviews with students (Gick & Holyoak, 1980; Ozimek et al., 2004).

To investigate the issues surrounding the research construct of transfer as well as to explore learner's transfer of learning, different approaches (definitions, theories and methodologies) have been suggested and are being developed by researchers (Bransford & Schwartz, 1999; Greeno, Smith & Moore, 1993; Lobato, 1996). The next section provides a brief discussion of different approaches to transfer of learning (a more detailed discussion can be found in Chapter 2).

Transfer of Learning

In the last few years, researchers have questioned the methodology and the experimental evidence for lack of transfer provided by earlier studies. Several researchers have claimed that the previous findings contradict the everyday experiences in which most learners do perform successfully in new situations by finding similarities from previous situations (Bransford & Schwartz, 1999; Hatano & Greeno, 1999; Mestre, 2005; Lobato, 2006). These researchers have observed that the methodology adopted in the earlier studies was problematic too. The methodology only focused on the researchers' and experts' perspectives, especially when developing the tasks (problems) and analyzing data.

Transfer studies often used tasks developed by researchers, and learners' answers were analyzed for evidence of transfer of learning of a predetermined piece of knowledge or a skill (Gick & Holyoak, 1980; Singley & Anderson, 1989). Researchers often tried to answer the question "Do students transfer?" and expected students to provide complete and correct answers. The target tasks in the earlier studies were conducted after students were introduced to the tested knowledge or skills (Gick & Holyoak, 1980; Bassok, 1990; Bassok & Holyoak, 1993). The context of the task, or surface feature, was designed to be different from the context in which the initial "learning" occurred, but the methods needed to solve the task, structural feature, was kept the same. For example, finding the components of a vector placed on a ramp would be considered to have the same structural feature as the initial problem students learn in their classrooms- finding the components of a vector on a Cartesian plane, while the problems would be deemed to have different surface features.

In the earlier paradigm the researcher would decide on the surface and structural features of the problems. These approaches to transfer assumed transfer and learning were passive processes; either they happened or did not happen. To complement these earlier perspectives on transfer of learning, new approaches were conceived to offer a dynamic process in which the learner continues to build knowledge even during the studies.

One of these contemporary approaches to transfer, Actor-Oriented Transfer (AOT) was developed by Lobato (1996). AOT views transfer as the "personal construction of similarities between the two situations- the initial learning situation and the target (new) situation" (Lobato & Siebert, 2002, p.89). The main focus of this framework is the learner (actor) and how he or she sees the target situation in relation to the initial learning situation. Obtaining evidence for actor-oriented transfer is done by *scrutinizing* (Lobato & Siebert, 2002, p.89) a given task together with data. Any indication of influence from previous tasks on the given task is considered to be evidence for actor-oriented transfer. Clearly, the examination of how individuals interpret situations as similar as well as the individual's definition of similarity play an important role in the formation of the evidence.

Most recent studies (Lobato & Siebert, 2002; Ozimek , 2004; Ozimek et.al, 2004; Cui, 2006) implement contemporary approaches, especially the AOT framework, as well as earlier approaches in the same study. The most common reason to use both approaches is if a lack of transfer is concluded by the traditional approach, researchers then can do an in-depth analysis of the topic with the contemporary approaches, namely deciding what students do transfer between the tasks. These studies have investigated transfer of learning at different educational levels from kindergarten through early college education, and within different subject areas (within mathematics, between calculus and physics, etc) (Gick & Holyoak, 1980, Elliot et al., 2001;Lobato & Siebert, 2002; Ozimek et.al, 2004; Cui, 2006). There are no studies of transfer of learning in post-calculus mathematics or physics courses.

Eigenvalues and Eigenvectors

One of the post-calculus mathematics courses required by mathematics and other disciplines is linear algebra. Most of the topics covered in a typical undergraduate linear algebra course- matrix algebra, determinants, systems of linear equations, linear transformations, eigenvalues and eigenvectors, and diagonalization, are often the prerequisite topics for many client disciplines- from physics, economics, statistics, computer science, and engineering (Carlson et al., 1997).

Eigenvalues and eigenvectors are among topics covered in linear algebra and revisited during study of quantum mechanics in physics. To form the consistent mathematical foundation for quantum mechanics, Jon von Neumann (1927, as cited in Dorier, 2000) analyzed two different viewpoints to quantum mechanics. The *viewpoint of matrices* using matrices and infinite sequences and the *viewpoint of waves* using wave functions, and both led to different models. However, the underlying *concept*[1] in both models was to analyze a functional equation to find eigenvalues and eigenvectors (Dorier, 2000). The eigenvectors of an operator only depend upon the operator and the vectors in question, not upon the particular basis in which they are expressed. The eigenvectors and the operator

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[1] The concept here refers to mathematical idea(s) which are known by the mathematics community.

with this property, invariant under change of basis, is one of the underlying mathematical features of the behaviors of quantum mechanical systems (cf. Dorier, 2000; Ismael, 2000). In other words, the concept of eigenvalues and eigenvectors plays an important role in quantum mechanics.

The application of eigenvalues and eigenvectors is not limited to physics. Another example of its application is seen in determining the reason for the Tacoma Narrows Bridge collapse[2]. One explanation of the incident is the resonance which requires the frequency of the driving force (in this case wind) to be close to a natural frequency of the structure (an eigenvalue of the structure) (Braun, 1983; pp. 171-173). In order to calculate the response of a structure under periodic loading, such as wind and earthquake, eigenvalues and eigenvectors are determined. Thus it becomes important for educators to know if students can transfer their learning of eigenvalue problems into real world situations.

The studies conducted on linear algebra topics indicate that students have difficulties understanding eigenvalues and eigenvectors in their mathematics courses (McWorter & Meyers, 1998; Hillel, 2000; Stewart & Thomas, 2006). Some of these difficulties include students lacking geometric views of eigenvalues and eigenvectors and students being unable to reason about the relationship between different representations of eigenvalues and eigenvectors (Stewart & Thomas, 2003 & 2006). Since this concept is a prerequisite to many other disciplines, educators may also benefit from an investigation of the transfer of learning of eigenvalues and eigenvectors.

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[2] The Tacoma Narrows Bridge in Washington which connected Tacoma to Kitsap Peninsula collapsed in November 7th, 1940.

Research Questions

The purpose of this study was to investigate students' transfer of learning of the concept of eigenvalues and eigenvectors from a 10-week physics courses to interviews in which seven students participated during and after the courses. Transfer of learning for each student was explored by implementing the Actor-Oriented Transfer (AOT) framework (Lobato, 1996).

To better understand what each student transferred to the interview setting from their coursework, each student's emerging understanding of the concept of eigenvalues and eigenvectors needed to be investigated before, during and after they are introduced to the concept in the course. More specifically, the question, "What ideas of eigenvalues and eigenvectors do students transfer to the interviews from the courses?" will be answered. The question can be best answered with a rich description of specific characteristics of students' emerging understanding, experiences students chose to transfer to the interviews and possible relationships between characteristics of students' emerging understanding and the experiences chosen by the students.

In particular, this study addresses the following research questions:

1. What characterizes upper-level physics students' emerging understanding of the concepts of eigenvalues and eigenvectors before, during and after studying the concepts in an intensive linear algebra review week and implementing them during a series of three 3-week intensive physics courses?

2. What do students transfer about the concepts of eigenvalues and eigenvectors from this series of courses to an interview setting? More precisely, what kind of experiences and views related to matrices, methods of finding eigenvalues and eigenvectors, the interpretation and use of the eigenvalue equation, and the relationship between basis vectors and eigenvectors do upper-level physics students transfer from their coursework to the interview setting?
3. In what ways do the experiences students choose to transfer relate to their emerging understanding of the concept of eigenvalues and eigenvectors?

Significance of the Study

The focus of this study is to investigate students' transfer of learning of eigenvalues and eigenvectors from coursework to interviews. There have been no studies conducted on transfer of learning of an upper-level mathematics course topic to a later setting. The studies conducted on transfer of learning to date focused on topics from lower-level mathematics courses. One of the significant features of this study is that it will provide results for transfer of learning of an upper-level mathematics course topic. Second, this study also investigates students' emerging understanding of an important linear algebra topic. The results can be used to understand students' difficulties in learning linear algebra and to improve the teaching of linear algebra.

CHAPTER TWO

REVIEW OF THE LITERATURE

The purpose of this chapter is to position this study in the context of relevant educational research and theory. The review first discusses the literature from two separate bodies of research: transfer of learning, and learning and teaching of linear algebra with a special focus on students' learning difficulties in linear algebra. A synthesis of these two bodies of literature supports the case for investigating students' transfer of learning of the concept of eigenvalues and eigenvectors from one context to another. The chapter concludes with the description of a theoretical framework that underlies the researcher's view on learning, transfer of learning, and understanding.

Transfer of Learning

There is a rich body of research on transfer of learning with a history of over 100 years. It has been studied by researchers from various fields including education and psychology (Hoffding, 1892; Thorndike & Woodworth, 1901; Bransford & McCarrell, 1974; Anderson, 1976; Gick & Holyoak, 1983; Lave, 1988; Detterman, 1993; Greeno, Smith & Moore, 1993; Bransford & Schwartz, 1999; Lobato, 1996; Beach, 1999; Mestre, 2005).Transfer has been traditionally defined as *the ability to apply knowledge learned in one context to new context* (Mestre, 2005).

One of the reasons that transfer of learning has been studied by researchers for over 100 years is its direct relation to a main goal of education: providing learning experiences that can be generalized and used by the learner outside the initial learning situation (Bransford, Brown, & Cocking, 1999). To see if the goal was fulfilled, educators and researchers have been developing studies to assess the learners' ability to transfer knowledge and skills.

The following review of literature is organized to outline the different views of transfer and their effect on the research studies.

Early Views of Transfer

Early psychological views of transfer were based on the mental abilities of a person. The intellectual performance of a person was believed to rely on the person's memory, attention, and judgment. The training of the basic mental functions (memory, attention and judgment) was thought to improve the person's ability to transfer ideas and skills to new situations. In other words, training was thought to have general effects that would transfer to new situations (Wolf, 1973). The educational application of this *general effects* view was that all students were required to take courses such as Latin and geometry, which were thought to improve students' minds by making them think more logically. These courses were believed to discipline students' minds and improve their abilities in other school subjects (Detterman, 1993).

After many experiments, Thorndike and his colleagues challenged the existing belief of *general effects* and proposed an alternative idea, *the theory of identical elements* (1901). Thorndike's work showed that even though people do well on a test of the specific content they had studied; this knowledge did not increase their learning in a new situation. They concluded that transfer from one task to another happened only when two tasks shared identical elements.

Thorndike's studies influenced educational practice in the sequencing of the curriculum activities. The curriculum activities were divided into certain behaviors and sequenced from lower level basic skills to higher level ones. As a consequence, drill and practice of skills became very important in the curriculum of each subject (Mayer & Wittrock, 1996). Such practices are still being discussed by researchers and educators (NCTM, 2000).

Researchers also pointed out that the theory of identical elements has limited applications to offer in situations beyond the identical ones. It did not help to explain the transfer between situations that do not share any identical elements. However, Thorndike's research paradigm dominated the investigation of transfer of learning during the 20th century.

Judd (1939) questioned Thorndike's view and proposed a different view on transfer. He claimed that transfer depended on "grasping the general principles or generalizations of the subject matter." (1939, as cited in Tuomi-Grohn & Engestrom, 2003) In his proposed view, transfer is determined by the extent to which the learner is aware of the underlying shared structure of tasks. In other words, the relationship between two tasks could be declared identical by the researchers; however, the learners might have a different opinion on the sameness and differences of two tasks. He also argued that the interaction between a learner and her environment should be an important consideration in transfer studies.

He also distinguished two types of learning- rote memorization with little meaning, and generalized knowledge with many intellectual associations (1936, as cited in Tuomi-Grohn & Engestrom, 2003). The essence of Judd's argument is formed by these different learning types. He claimed that transfer did not occur effortlessly and mindlessly as a consequence of rote memorization (as a reflex) but as a consequence of generalization. Even though the reports of Judd's experiments were found sketchy by other researchers (Detterman, 1993), his view of transfer and learning provided a new approach for investigation of transfer.

Thorndike's view represents the behaviorist paradigm—according to which learning is conceived as a reflex or a reinforcement of stimulus-response associations, where particular connections are strengthened by feedback from the environment. Judd's (1908) represents the cognitive paradigm—according to which learning is an active process of (mental) construction rather than a passive assimilation. The debate of looking at the issue of transfer did not stop after Thorndike and Judd.

Cognitive Views of Transfer

As cognitive theories develop, the theories of transfer of learning are also framed under the explanation of how the human cognitive system is structured and how it functions (Mestre, 2005). Schema theorists use the concept of schema as the theoretical basis for developing a new understanding of transfer. They define schema as "a cluster of knowledge that provides a skeleton structure for a concept that can be instantiated or filled out by the detailed properties of a particular instance. It is formed by induction from numerous previous experiences with various exemplars of general concept" (Thorndyke, 1984, p.167).

Reed (1993) indicates that algebra word problems explain the schema theory nicely. He claims that students usually categorize problems and use equations or formulas related with the problem categories to solve problems. The equations and formulas represent the symbolic schema and they are abstract representations of solutions in the problem solving schema. When a new problem is given, students sort the categories of problems to find the best fitting (similar) previously stored (solved) problem to get an appropriate equation or formula (symbolic representation) to solve the given problem. Transfer is said to occur if students can recognize the similarities between the given problem (target task) and the one used in initial learning (during sorting task).

From a cognitive perspective, transfer of learning then requires the ability of finding the correct mental representation of the problem and its solution in the problem solving schema. However, learners develop their own problem solving schema and they may not be fully developed yet or not have the proper categorization of problems with the correct solutions. As learners form their schema, it is possible that they categorize problems as they see them similar, not how researchers see them as similar.

Gick and Holyoak (1980) investigated learners' ability to build analogies between similar problems (as perceived by the researcher) by series of experiments. The researchers posed problems that were structurally identical; meaning that they shared the same solution method, but the surface feature of the problems were different. In other words, the stories (context) of the problems were different.

In the experiments, students initially read a problem and its solution. The problem was about a small country that was ruled from a fortress by a dictator. The fortress was placed in the middle of the country and there were many paths and small roads leading to it. One of the generals decided to attack the fortress; however, he knew that all the roads leading to the fortress had mines on them. The mines were set to blow when there was a large force applied on them. So, small bodies of men could pass over them safely since the dictator needed to move his men in and out of fortress. The general looked for a plan to attack the fortress. He decided to divide his army into small groups where each used a different path to the fortress but all arrived at the same time to attack the fortress from all around. The *convergence solution* worked and the general overthrew the dictator.

After reading the problem students were presented with Duncker's (1945) *radiation problem* (a target task) which asked students to find the best method to destroy a tumor in the stomach by radiation without damaging healthy tissues. The best results would be gained if the *convergence solution* was applied to the problem. The use of rays of radiation converging onto the tumor from different sides with less intensity would cause less harm to the surrounding healthy tissues but at the same time provide the necessary intensity on the tumor.

The researchers investigated to what extent and under what conditions students made use of analogies to connect seemingly different but structurally identical problems. Gick and Holyoak (1980) first examined students' ability to access the relevant information to solve the radiation problem. In other words, they investigated if the students found the two situations similar, and in what ways they found them dissimilar. If students could not recognize the similarities, they were provided by hints. They found that students had a higher solution rate when they were provided a hint. The researchers interpreted the results as the initial story was being encoded in a useable fashion but students were unable to access it. The reason could be that students did not see two problems as being related to each other.

In the follow up experiment, Gick and Holyoak (1983) provided multiple structurally similar stories before a transfer problem. They reported that transfer rates were higher when subjects were exposed to multiple examples. They also provided a distinction between perceived and structural similarity of the initial and transfer (target) situations (Gick and Holyoak, 1983). They said perceived similarity of two tasks is "based on salient common features of their representations" and structural similarity is "based on the actual components determining appropriate response" (Gick and Holyoak, 1983, p.40). They claimed that transfer was affected by both of these types. Perceived similarity determined whether the students would attempt to transfer, structural similarity determined whether the transfer occurred or not.

Many other researchers followed the same research paradigm- initial learning task (or problem) followed by the target task (or problem). (e.g., Adams et al., 1988; Bassok, 1990; Brown & Kane, 1988; Novick, 1988; Perfetto, Bransford & Franks, 1983; Reed, Ernst, Banerji, 1974).   Most of the studies conducted under this paradigm reported failure of spontaneous transfer from one problem to the next.

One of the criticisms of this research paradigm comes from Bransford and his colleagues (1999). They note that the subjects during these experiments are like judges during a trial. They are *sequestered* during the final task (target task) in order to protect them from any exposure to information that could possibly "contaminate" their solution. The "sequestered problem solving" (SPS) experiments (Bransford & Schwartz, 1999) do not provide any opportunities for the subjects to learn to solve problems by interacting with others, receiving feedback, and revising the solution method. Therefore, the results from such experiments are binary: either transfer happens or not.

Researchers of situative cognition also criticize the cognitive theorists' approach to transfer (Greeno, 1997; Lave, 1988). They point out that most of the studies ask the same question, 'does transfer occur', and then similar results are reported, spontaneous transfer is negligible. These research studies indicate that providing the solution or more learning time for the initial problem (task) might increase the likelihood of having learners arrive at the correct solution to the transfer problem (also referred as target task). Additionally the problems have to be structurally identical and the similarities between them need to be explicitly pointed out to the learners. Even within the cognitive tradition researchers report similar criticisms. They are worried that transfer is too hard to find. After analyzing the results from many studies Detterman said: "There is no evidence to contradict Thorndike's general conclusions: Transfer is rare and its likelihood of occurrence is directly related to the similarity between two situations." (Detterman, 1993, p.6)

The cognitive views of transfer also bring up important concepts that they claim to influence transfer. They emphasize the role of metacognition in individuals' learning. Metacognition is "the awareness of the cognitive processes that one uses in monitoring and consequent regulation and orchestration of those processes" (Tuomi-Grohn & Engestrom, 2003, p.22). Under the metacognition view, successful transfer occurs when students recognize the requirements of the target situation (new problem) and are able to choose the appropriate skills among previously learned specific and general skills. A focus on metacognition during problem solving activities is reported to increase transfer.

The use of explicit structural schemas or concept maps by instructors is also advocated by several cognitively-oriented educational psychologists. The new information presented in this way is claimed to enhance its organization and relationships to other areas of knowledge domain (Ausubel, Hanesian, and Novak, 1978). This approach may indeed facilitate a better performance in school based learning; however, it remains unclear if it will improve the transfer of school knowledge to life situations.

Situated/Sociocultural Views of Transfer

In late 1970s, Broudy noted that people rapidly forgot the facts that they acquired in the school setting, probably as a result of *replicative knowing tests* (1977, as cited in Bransford & Schwartz, 1999, p.61). Tests of this sort would ask students to repeat the procedures and facts instead of asking students how and why these procedures work. Broudy's conclusions did not indicate that transfer was rare, but pointed out that transfer needed rethinking with respect to what it means to learn, the goal being a re-evaluation of educational practices.

Lave (1988) re-evaluates the cognitive view of transfer and her main concern is with the definition of transfer. The derivative of the definition "the ability to apply knowledge or procedures learned in one context to new contexts" (Mestre, 2003; p.3) suggests that transfer consists of measures of the proper use of previous learning in the new setting with the assumption that the settings (initial learning and the transfer) do not affect the learner's performance. In other words, the definition suggests that knowledge could be separated from the situations in which it is constructed instead of an overall connected experience (Lave, 1988; Lobato & Siebert, 2002). Also, learning between settings and during the target setting (for example, during an interview of an experiment) is mostly not included in the examination of transfer.

Lave (1988) also points out that most of the experimental studies done earlier were designed to have only one answer and/or method as evidence of transfer. In real life settings, most problems do not have one answer or only one method of solution. Thus, new views of transfer need to include the social and cultural mediating factors such as learning settings (classroom culture), language, tools, interactions, other learners, and more knowledgeable individuals (experts).

The critiques of earlier views are being addressed under these new perspectives of learning and knowing (Lave, 1988; Greeno et al., 1993; Cobb, 1994). The view of learning shifts from an "active process of (mental) construction to strengthening of the practices and participatory abilities that allow individuals to work within and contribute to communities." (Lave, 1988)

The situative view of knowing suggests an alternative way of practicing and analyzing instructional tasks. A student's whole school life becomes a connected unit of analysis. Knowing how to participate in different social practices in school becomes important in all aspects of a student's learning at school (Greeno et. al., 1993).

Researchers carefully examined previous studies of transfer of learning from a new perspective. The cognitive paradigm view of transfer states that transfer "depends upon acquiring an abstract mental representation in the form of a schema and designates relations of a structure that is invariant across situations" (Greeno et. al., 1993). For example, if a student develops a schema for multiplication based on experience can apply that schema to examples with feedback, use the examples to tune the procedures, and is aware of the application of a multiplication schema; she will be more likely to transfer that schema to novel problems.

However, transfer becomes a delicate issue under the situative perspective. Whether transfer is learning new practices within the community or learning to use learned practices outside the community is being questioned (Lave & Wenger, 1991). Different views of transfer are being proposed under the situative perspective (Greeno et. al., 1993; Bransford & Schwartz, 1999; Beach, 1999; Lobato 1996). These views include the social and cultural mediating factors. They also share the idea that what transferred is not only the knowledge from task to task but "patterns of participatory processes across situations" (Greeno, 1997).

Contemporary Approaches to Transfer

Several researchers of the situative perspective claim that the findings from the earlier studies of transfer contradict the everyday experiences in which most learners do perform successfully in new situations by finding similarities to previous situations (Bransford & Schwartz, 1999; Hatano & Greeno, 1999; Mestre, 2005; Lobato, 2006). The researchers suggest using different views of transfer and research methodologies to investigate transfer. They notice that earlier studies have only focused on the researchers' and experts' points of view, especially when developing the tasks (problems) and deciding on structural and surface features of tasks. Moreover, the contemporary approaches view transfer as an active process, rather than a passive one. They believe that students transfer during transfer experiments, and researchers' job is to figure out what they transfer in:

> *[…] assuming that participants in transfer studies were not daydreaming throughout the experiments, they were transferring something to reason and make sense of the tasks given to them to perform.* (Meste, 2005, p. xvii)

Contemporary approaches to transfer view it as a dynamic process in which the learner continues to build knowledge even in the target situations. Some of these contemporary approaches are *Affordances and Constraints Approach* (Greeno et al. 1993 &1996), *Preparation for Future Learning (*Bransford & Schwartz, 1999), and *the Actor-Oriented Transfer* (Lobato, 1996). These contemporary approaches are all formed under situative perspective and they use revised definitions of transfer together with modified research questions and methods to explore the issue of transfer.

The *Affordances and Constraints* approach examines the extent to which participation in an activity, when the learners are aware of the affordances and constraints of the activity, influences the learners' ability to participate in a different activity in a new situation (Greeno *et al*., 1993). Greeno and his colleagues' view of transfer represents a relation between situations and learners. They claim that transfer occurs when instructions in a situation shape learners' activities and their social interaction with each other. Learners in a learning activity need to receive instruction to influence them about the affordances and constraints of the activity that are invariant across situations. Such instructions are claimed to result in transfer between different situations of a learning activity.

Preparation for Future Learning (PFL) investigates participants' learning in transfer situations. It focuses on whether learners can *learn* to solve problems in new situations by utilizing similar methods to those (?) they initially learned the content (Bransford & Schwartz, 1999). The researchers did not evaluate if a learner could create a final product, but they rather looked for evidence of initial learning trajectories. For example, the questions that learners ask during the initial learning settings are claimed to shape their learning goals and future learning. The trace of such questions is investigated in the transfer situations (e.g., see Bereiter & Scardamalia, 1989; Hmelo, 1994).

The Actor-Oriented Transfer (AOT) approach conceives transfer as the "personal construction of similarities between activities where the 'actors,' i.e. learners, see situations as being similar" (Lobato, 1996 & 2003). She focuses on how the "actors" (or learners) see the two contexts as similar. The evidence for AOT is gathered by "scrutinizing a given activity by any indication of influence from previous activities and by examining how people construe situations as similar" (Lobato & Siebert, 2002, p.89).

Contemporary approaches to transfer (Bransford & Schwartz, 1999; Greeno *et al*., 1993; Lobato, 2003) account for the social and cultural aspects that the earlier studies did not consider. They view transfer from the learner's point of view rather than the researcher's point of view. Moreover, new approaches to transfer consider it as an active, dynamic process.

Actor-Oriented Transfer Framework

Actor-Oriented Transfer (AOT) was developed by Lobato (1996). AOT views transfer as the *personal construction of similarities between the two situations- the initial learning situation and the target (new) situation* (Lobato & Siebert, 2002).

The main foci of the framework are the learner (actor) and how the learners see the target situation (could be a given task or a problem) similar to the initial learning situation (could be an initial task or a problem). Obtaining evidence for actor-oriented transfer is done by *scrutinizing* (Lobato & Siebert, 2002, p. 89) a given task together with data. Any indication of influence from previous tasks on the given task is considered to be evidence for actor-oriented transfer. In other words, researcher should not decide or give a priority to what students should transfer *but rather adopting a student-centered perspective to find out what students do transfer and investigating the mediating factors* (Rebello et al., 2005, p.219).

Lobato and Siebert's (2002) case study demonstrated these main ideas of the AOT framework. In the case study, researchers focused on an eighth-grade student who participated in a 10-day (3 hr/day) teaching experiment. The goal of the teaching experiment was to improve quantitative reasoning of students. Students were introduced to the concept of slope and worked on activities. Throughout the teaching experiments students were also interviewed individually. There were three interviews conducted on the first, fifth, and last days of the teaching experiment. All the teaching and interview sessions were recorded.

The researchers focused on the last interview with one eighth-grade student. They were looking for possible occurrences of transfer between the events before the last interview and the students' problem solving activities during the last interview. The student chosen for the case study was working on measuring the slope of a wheelchair ramp when the length and height were given. The student had demonstrated difficulties distinguishing between height and the slope during the first interview. The student thought that if one walked upwards on a hill, then the path would get steeper and steeper as one got higher on the hill. The same confusion was seen when the student was asked to create wheelchair ramps with the same slope but different heights, one being higher than the first one. During the last interview the student seemed to have a sudden insight on how to create a one feet high wheelchair ramp which had the same steepness as a 15 foot long and 2 foot high wheelchair ramp. Students learned about slope formula (height over length), prior to the 10-day-teaching experiment. However this particular student did not use the slope formula. He found a different way to approach the problem. He calculated how much length went with 1ft of height first, then created the ratio 15:2 = 7.5:1. Then he tried to explain what the ratio meant to him. He said if he increased the height one height-unit (for example 1ft), then he needed to increase the length one length-unit (7.5 ft) as well. He calculated to find the length of 22.5 ft in order to have a 3ft wheelchair ramp.

The researchers were mainly investigating the possible occurrence of transfer. They did not pose a question of transfer prior to the student's insight. In other words, they did not ask the question: Did the students transfer previously learned slope formula to the new situation? However, with this interview data Lobato and Siebert (2002) asked the question: "Was the student's sudden insight a spontaneous event without any connection to what had taken place during the teaching experiment or can we find anything that has possibly foreshadowed and laid ground for the sudden insight?" The authors were looking for possible links between the teaching experiment and the sudden insight of the student during the last interview.

Researchers claimed that one of the teaching experiment sessions prior to the interview (session 8) could be linked to the student's approach in the interview. In session 8, the students were working on an activity in which they were trying to make an animated clown walk with the same speed. Students were using some computer software to make the animated clown walk across the computer screen. The clown initially was walking at a constant speed, covering 10 cm in 4 seconds. The students' task was to make a frog walk with the same speed but covering different distances and times. The researchers explored what the case study student had done during this activity. This particular student built his own reasoning to explain that the frog walked 20 cm in 8 seconds, 30 cm in 12 seconds which were respectively 2 and 3 units of the ratio 10 cm: 4 seconds.

Researchers aimed that the case study student probably used the same reasoning he created with the clown-frog task in the wheelchair problem. They postulated that the student demonstrated transfer between two situations by creating his own similarities between these two situations. The researchers reported that if they were investigating the student's transfer of learning of slope formula to new problems, then the results would be negative. The researchers also wanted to demonstrate that researchers should not initially identify the tasks of the experiments in terms of structural and surface similarities since students could have different ways of looking at these similarities.

Lobato's (2002, 2003, & 2006) *actor-oriented* transfer perspective complements the previous studies of performance measures of transfer. The previous studies informed us only with the *success or failure of individuals in modeling expert performance in transfer tasks* (Marton, 2006). Lobato's (2003) AOT framework, however, considers the social and learning environments and activities. The influence of social aspects on learner's construction of similarities helps the researchers to see the question of “what counts as transfer” from the learner’s perspective.

AOT and the contemporary approaches to transfer could be interpreted to suggest that transfer researchers need to acknowledge the participants’ internal knowledge as well as the learning situations in which the internal knowledge is constructed. Therefore, the concurrent consideration of the internal knowledge, the learning environments and the relationship between them need to be included in transfer studies (Lobato, 2003). Table 2.1 compares the early and actor-oriented perspectives of transfer of learning.

Table 2.1 Perspectives on Transfer of learning

	Perspectives on Transfer of Learning	
	Early Perspectives	**Actor-Oriented**
Definition	The application of knowledge learned in one situation to a new situation	The personal construction of relations of similarity across activities (i.e. seeing situations as the same).
Perspective	Observer's (expert's) perspective	Actor's (learner's) perspective
Research Method	Researchers look for improved performance between learning and transfer tasks	Researchers look for the influence of prior activity on current activity and how actors construe situations as similar
Research Questions	Was transfer obtained? (Can learners successfully apply knowledge previously acquired in the learning task to transfer task?) What conditions facilitate transfer?	What relations of similarities are created? How are they supported by the environment? (How do learners actively construct knowledge in the transfer task based on experiences in the learning task?)
Researcher's Role	The researcher pre-defines the structural similarities between the learning an transfer context.	The researcher investigates what the learner sees as similar between the two scenarios.
Metaphor (Dynamism)	Transfer is a static construct, i.e. students can either apply their knowledge in a transfer context or they cannot.	Transfer is dynamic, i.e. students can learn in the transfer context based on their prior experiences.

(Lobato, 2003)

Transfer of Learning Studies with Mathematics

Previous studies investigated the transfer of learning within mathematics and from various courses in mathematics to physics and chemistry courses. Bassok (1990) investigated transfer of learning from algebra to physics at the high school level and the results showed a “transfer asymmetry.” Most students were able to transfer their algebra knowledge to the “isomorphic” physics problems, however, only a few students were able to apply their physics knowledge to the “isomorphic” algebra problems. The researcher believed this asymmetry was due to the nature of algebra being context-free.

The study was designed under the cognitive view of transfer and no contemporary approaches were included. Students’ personal constructions of similarities between algebra and physics problems were not investigated thoroughly. The researcher decided in what ways the problems were similar. When students could not solve the algebra problems correctly after being exposed to “similar” physics problems, researchers did not conduct further examination of the solutions.

Tuminaro’s (2004) study focused on why algebra-based physics students perform poorly on mathematical problem-solving tasks in physics. The main assertion of his study was that students did not know how to apply the mathematical skills to particular problem situations in physics. The study did not focus on description of the students’ mathematical knowledge. Tuminaro’s framework was grounded in cognitive learning theories with traditional physics problems.

Most recent studies (Lobato& Siebert, 2002; Ozimek, 2004; Ozimek et.al, 2004; Cui, 2006) implement the contemporary approaches, especially the AOT framework as well as the early approaches in the same study. The most common reason to use both approaches is that if a lack of transfer is concluded with early perspectives, then researchers could provide an in-depth analysis of the topic with the contemporary approaches, for example by figuring out what students do transfer between the tasks. These studies investigate transfer of learning in different levels, such as K-12 and lower level undergraduate courses, and among different subjects (within mathematics, between calculus and physics, etc.) (Elliot et al., 2001;Lobato & Siebert, 2002; Ozimek, 2004; Ozimek et.al, 2004; Cui ,2006). New studies are needed to investigate the transfer of learning among upper division courses, for example, from mathematics to physics or from physics to engineering courses.

A recent study implemented both early and contemporary approaches to investigate students' transfer of learning from trigonometry to physics. In a quantitative study, Ozimek et al. (2004) examined retention and transfer at the introductory college level. They found no evidence of transfer when the cognitive view of transfer was implemented. Their questionnaires contained trigonometry problems and physics problems with the same trigonometric concepts. However, there was evidence that students did transfer what they learned in their trigonometry class to their physics class when contemporary perspectives of transfer were used. The frameworks of Preparation for Future Learning (Bransford and Schwartz, 1999) and Actor-Oriented Transfer (Lobato, 2003) were utilized to design the study and analyze the collected data. Ozimek et al's research provided a demonstration that early views of transfer are not enough to detect transfer, and contemporary perspectives provide more information on student's learning.

Another recent study followed the same approach to transfer, first using early views of transfer and then implementing a contemporary framework to collect and analyze data. Cui (2006) conducted her study to investigate students' transfer of learning from calculus to physics at the college level. She suggested that transfer of learning from "*relatively abstract domains such as mathematics to relatively concrete domains such as physics must be examined from multiple perspectives of transfer"* (p.135). Her results from student interviews indicated that students were able to use their calculus schema when they were asked to solve problems on integration in physics. However, students had difficulties with variables in physics problems and with aligning those variables with the ones they had seen in calculus. In her study, students did not know when or what to activate from their calculus learning. Overall, students often implemented naïve problem-solving strategies when they could not figure out or match the problem with a previous one. Cui claimed that these behaviors suggested that students search for an appropriate schema to help them solve the problems, and when they are unsuccessful, they go back to naïve methods. Cui recommended that future research expanded on the content areas by investigating the transfer of learning from upper-division mathematics topics to upper-division physics settings or from physics courses to engineering courses.

Summary of Transfer Literature

The literature review on transfer of learning indicates that each transfer study reflects its own time and the influence of the learning theories of that time. These different views of transfer provide different ways to investigate the issue. Some researchers claim that such a variety of views creates conflicts and they even suggest that the topic of transfer should be avoided as a research construct

(Carraher & Schlieman, 2002). However, each perspective is valuable and addresses different theoretical and practical issues of transfer in a complementary way.

The early transfer studies focus on the one-shot performance of learners in the target situation after being exposed to an initial situation. Learners were expected to transfer previously learned knowledge and skills into the target situation. Such studies take researchers' perspectives and the interactions of learners within the learning environment are left out from the studies. To investigate the effects of such interactions, new transfer views are proposed. These views reflect the ideas from social learning theories. As views of learning start to include social interaction and environment as parts of the learning process, the views of transfer also start to take account of such constructs when looking for evidence of transfer.

All the previous research under different views of transfer reveal that transfer can happen in two planes- one in personal plane and one in social plane. The cognitive view of transfer provides insight on transfer on the personal plane; the situated views of transfer provide more information on transfer on the social plane. The literature review also shows that there is a need for more transfer studies in the higher level of undergraduate mathematics.

The view of transfer in this study aligns with the view outlined in Actor-Oriented Transfer Framework. The current study, however, will also address the researcher's perspective during the analysis of the data, even though AOT framework tries to focus only on the learner's perspective. It seems more realistic to provide both perspectives- those of the researcher and the learner- while looking for evidence for transfer. Therefore, in some cases the researcher will report her perspectives as well as interpreting the learner's.

Linear Algebra Literature Review

One of the upper-division mathematics course required for many majors is Linear Algebra. Applications of linear algebra topics are found within mathematics and also in other disciplines. For example, eigenvalues and eigenvectors form the building blocks of Quantum Mechanics in physics, and are also used analyzing population growth models in biology. Some research studies (Harel, 1989; Hillel & Sierpinska, 1994; Hillel, 2000; Sierpinska, 2000) as well as anecdotal evidence indicate that some students experience difficulties learning certain linear algebra topics. To address these difficulties, and to address an increasing demand for student understanding of linear algebra courses, Linear Algebra Curriculum Study Group (LACSG) was formed in 1990 to provide recommendations for the first course in linear algebra (Carlson et al, 1993).

The study group recommended that the syllabus and presentation of the first course in linear algebra should address the needs of client discipline and also the course should be matrix-oriented. They suggested that students should actively get involved in classes and faculty should consider students' needs and interests during the planning of and teaching the course. Utilizing technology starting from the first course of linear algebra was also suggested. Also, adding a second course, as a follow-up for the first linear algebra course was recommended for every mathematics curriculum.

The LACSG recommendations were mainly based on three sources. The first source was the research-based knowledge. Before the list of recommendation and the suggested syllabus were published, the LACSG reviewed the studies contemporary on *how students learn, how mathematics should be taught and what pedagogical and epistemological considerations are involved in the learning and teaching of linear algebra* (Harel, 1997). For example, the research finding that emphasizing geometric thinking in teaching first year linear algebra course improves students' understanding was considered in the LACSG recommendation. They suggested that there should be a strong emphasis on geometric interpretations when teaching the first year linear algebra course (Harel, 1989 & 1990).

The second source of the recommendation of the LACSG was the teaching experience of each member of the study group. Each member had taught some form of a linear algebra course many times and their experience helped them to form feasible curriculum suggestions (Carlson, 1997; Harel 1997).

The third resource used by the LACSG was the inputs of many client disciplines. The study group consulted with various client disciplines about the role of linear algebra in their courses. The LACSG asked them how each of those disciplines wanted the linear algebra curriculum to be improved. The inputs helped the LACSG to form the orientation of the recommended first course in linear algebra.

Even after recommendations were in place in curriculum activities, students still had difficulties with linear algebra topics (Stewart & Thomas, 2003; Dorrier 2000). Researchers continued to examine the root of these difficulties

(Stewart & Thomas, 2003 &2006; Hillel, 2000; Harel, 2006; Sierpinska, 2000). Dorier and Sierpinska claimed "the nature of linear algebra itself (conceptual difficulties) and the kind of thinking required for the understanding of linear algebra (cognitive difficulties)" were some of the sources of difficulties with the linear algebra course (Dorier, 2000; Sierpinska, 2000). Sierpinska noted that even after many improvements of the curriculum, the difficulties persisted. In her studies, she tried to introduce the theory to students after many innovations. Through the innovations the structural theory of linear algebra was unchanged but the presentation and the activities were improved. However, the results of her experiment showed that students still did not understand the theory. She claimed that the reason for students' lack of understanding was that students *wanted to grasp it [theory] with a practical rather than a theoretical mind (p.211).* She reported that students' practical thinking created an obstacle for understanding the theory of linear algebra (Sierpinska, 2000).

The issue of multiple representations used in linear algebra topics reported to be another source of difficulties for students (Hillel, 2000). Hillel noticed that students had problems translating between different representations. For example, representing a linear transformation using two different sets of basis was a major challenge for students in his study.

Dubinsky (1997) also underlined two main sources of difficulties in learning linear algebra. He stated that the first source was the pedagogical approach. He suggested that overall pedagogical approach in most linear algebra courses need to change from "telling and showing students how it works" (*p.93*) to promoting them to construct their own ideas about the concepts in linear algebra. He proposed that students need to interact with each other, their textbook and the instructor after constructing these concepts. The interactions should bring students to resolutions of possible conflicts and eventually help them to re-construct the understanding of the concepts.

The second source of difficulty came from students' lack of understanding of background concepts that were not part of linear algebra but very crucial for the understanding of it. For example, he stated that having a strong understanding of the function concept was important to understand linear transformations.

More in depth studies with various topics, such as understanding matrices and representations, vector spaces, eigenvalues and eigenvectors, have been and are being conducted by researchers (Dorier, 1998 & 2000; Uhlig, 2002; Hamdan, 2005; Stewart & Thomas, 2006).

The difficulties with eigenvalues and eigenvectors are still being investigated in linear algebra courses. Studies conducted by Stewart and Thomas (2003, 2006) have reported that students could not reason about relationships between a diagram and eigenvectors. Even though students seemed confident with algebraic and matrix procedures, a vast majority had difficulties with geometric reasoning of eigenvalues and eigenvectors (Hillel, 2000; Stewart & Thomas, 2006).

As suggested by the results of literature on transfer of learning and on student difficulties with linear algebra topics, further investigations are needed to understand students' understanding of linear algebra topics and transfer of learning of these topics. The focus of this study is to investigate students' emerging understanding of the concept of eigenvalues and eigenvectors and transfer of learning of this concept. In particular this study addresses the following research questions.

1. What characterizes upper-level physics students' emerging understanding of the concepts of eigenvalues and eigenvectors before, during and after studying the concepts in an intensive linear algebra review week, and how they implement them during a series of three 3-week intensive physics courses?
2. What do students transfer about the concepts of eigenvalues and eigenvectors from this series of courses to an interview setting? More precisely, what kind of experiences and views related to matrices, methods of finding eigenvalues and eigenvectors, the interpretation and use of the eigenvalue equation, and the relationship between basis vectors and eigenvectors do upper-level physics students transfer from their coursework to the interview setting?
3. In what ways do the experiences students choose to transfer relate to their emerging understanding of the concept of eigenvalues and eigenvectors?

The research questions and the research design of this study are determined by the theoretical framework of the researcher. The literature which has influenced the researcher's construction of her theoretical framework is discussed in the following section.

Theoretical Framework:

In order to describe student's emerging understanding and to investigate transfer of learning, two questions need to be answered:

1. What does emerging understanding mean?
2. How does the researcher view transfer of learning?

<u>1. What does emerging understanding mean?</u>

To define emerging understanding one needs first to investigate both learning and understanding. Many theories have offered explanations on how individuals learn and understand, one of them suggests that learning occurs through construction of one's own knowledge. The theoretical arguments for this theory were formed by Piaget (1977) and have been advanced by von Glasersfeld (von Glasersfeld, 1995). The empirical support was provided by numerous studies indicating that there are differences in students' understanding of a mathematical concept as students develop their understanding during instructional settings (Confrey, 1990; Cobb, 1994). With inspiration from Vygotsky and the activity theorists Davydov and Leont'ev, another approach to learning was proposed (Vygotsky, 1986; Cobb, 1994). This approach emphasized the social and cultural constructs in mathematics instructional settings. Empirical studies conducted Carraher et al. (1985), and Lave (1988) demonstrated that the participation in a community (socially and culturally) was an influential factor in learning. To investigate the relationship between psychological and social aspects of learning, a social constructivist perspective has been proposed. The emergent perspective (Cobb & Yackel, 1996) characterizes learning as a self-organization process; however, these organizations are claimed to happen during interactions. These interactions (with others and with self) are the key issues in construction of individual learning. The interaction between two planes, social and psychological, during reconstruction of knowledge defines the learning for the researcher. In other words, the researcher agrees with the emergent perspective that learning is a process which occurs within the individual's head and during social settings. Then understanding can be defined as an assimilation of a concept into a learner's existing network of concepts. This definition is incorporated from Skemp's definition to fit the researcher's perspective of learning (Skemp, 1987). In this definition 'concept' is defined to be an idea that's been accepted by the community of practice.

The individual categorization of concepts leads into conceptual structures in which concepts are connected with each other (Skemp, 1987). Skemp defines two particular connections; associative and conceptual. Associative connections are built by memorization or by meaningless associations. On the other hand, conceptual connections are built by relations. In this sense, a person may build associative connections between concepts and then assimilate these into an existing conceptual structure. According to the definition of understanding above, a person with associative connections is still considered to have an understanding of a particular concept. The researcher believes that learners sometimes demonstrate only their associative connections as their understanding of a concept.

Thus, it seems that an observer could not observe all the connections a learner makes during learning process and could only observe the ones that learner shares. For this reason, in this study the researcher distinguishes the understanding from "emerging understanding." The researcher defines the "emerging understanding" to be the partial understanding of a concept that the researcher could observe during the study.

2. How does researcher view transfer of learning?

In this study the researcher will implement the actor-oriented transfer (AOT) framework (Lobato, 1996) to investigate students' transfer of learning. One reason is that the AOT framework aligns with the researcher's perspective of learning. As previously mentioned, the AOT framework is one of the frameworks that conceives transfer as a dynamic construct and defines it as the *personal construction of similarities between activities where the 'actors,' i.e. learners, see situations as being similar* (Lobato, 1996 & 2003). Lobato's model considers the sociocultural influences on transfer and indicates that transfer is distributed across mental, material, social, and cultural planes. Lobato (2003) stated that her framework emphasizes the "learner's personal perceptions" in a similar way to Pigates' (1977) notion of "generalizing assimilation" but the framework considers the structuring roles of social artifacts. On the other hand, traditional transfer perspectives only measure a psychological phenomenon.

The framework focuses on how the "actors" (or learners) see the two contexts as similar and the evidence for the AOT is gathered by *scrutinizing a given activity by any indication of influence from previous activities and by* *examining how people construe situations as similar* (Lobato & Siebert, 2002, p.89). Even though the framework underlines that the researcher should take the student's perspective, during "scrutinizing a given activity" the researcher could possibly slide back to his/her own perspective. For this reason, in this study the evidence for the AOT is claimed when a student explicitly indicates an influence of a previous activity. An activity which seemed to have an indication of influence from previous activities but not explicitly stated by student is still considered as transfer, however is not listed under evidence for the AOT. In other words, the researcher takes both the learner's and the researcher's perspectives when analyzing data.

Summary

This section provided a description of the researcher's perspectives on learning and understanding, and transfer of learning, which is used to analyze the data. The researcher views learning as a process of construction of knowledge as it takes place in two planes, psychological and social. Thus, understanding is formed as a combination of psychological and social aspects and is an assimilation of a concept into a learner's existing network of concepts. However, the researcher believes that one's understanding of a concept can partially be observed by an outsider and for this reason she defines "emerging understanding" as part of the learner's understanding, which could be observed through ideas shared by the learners.

In this study the actor-oriented transfer framework is used to analyze data. The researcher focuses on what students transfer to the interviews instead of investigating whether students transfer. The evidence for the AOT is provided by focusing on activities as students explicitly indicate the influence of previous activity and also the researcher's perspective is included to provide possible cases of transfer that require further investigation.

CHAPTER 3

METHODS

The purpose of this study is to describe and analyze junior year physics students' emerging understanding of eigenvalues and eigenvectors through problems posed at three interviews. Students' transfer of learning of eigenvalues and eigenvectors from linear algebra courses or from their prior experiences to the second and third interviews is also explored by implementing the Actor-Oriented Transfer (AOT) framework developed by Lobato (1996, 2003 & 2006). More specifically, this study answers the question, "What ideas of eigenvalues and eigenvectors do students transfer to the interviews from the courses?" Furthermore, the relationship between students' emerging understandings and the experiences of transfer of learning of eigenvalues and eigenvectors is explored through a cross-case analysis. The detailed research questions are:

1. What characterizes upper-level physics students' emerging understanding of the concepts of eigenvalues and eigenvectors before, during, and after studying the concepts in an intensive linear algebra review week and implementing them during a series of three 3-week intensive physics courses?
2. What do students transfer about the concepts of eigenvalues and eigenvectors from this series of courses to an interview setting? More precisely, what kind of experiences and views related to

matrices, methods of finding eigenvalues and eigenvectors, the interpretation and use of the eigenvalue equation, and the relationship between basis vectors and eigenvectors do upper-level physics students transfer from their coursework to the interview setting?

3. In what ways do the experiences students choose to transfer relate to their emerging understanding of the concept of eigenvalues and eigenvectors?

These questions can be best answered with a rich description of characteristics of students' emerging understandings and the experiences that students choose to implement during the study. The relationships between them are described through a cross-case analysis.

In-depth interviews with students and interviews with experts (mathematics and physics professors), ethnographic field notes of classroom culture of the courses students took during the winter term (namely, during the linear algebra review week and subsequent 3 three-weeks physics courses) and observation of students' interactions, students' pre and post quiz data from the linear algebra week of the winter term, and students' final exam grades at the end of the winter term provide data for this study (Patton, 2002). The choice of the research methods has been influenced by the researcher's previous work in a pilot study and in classroom observations in a junior-level physics courses. The pilot study was conducted for the purpose of focusing the research questions of the study and refining the interview protocols. The researcher also hoped to gain additional experience conducting in-depth interviews and developing questioning strategies as well as information on students' apparent understanding of eigenvalues and eigenvectors. With the permission of the Physics faculty, she also conducted classroom observations of the linear algebra review week (LAW) which is offered every year prior to junior-level physics courses during the winter term.

This chapter is divided into five sections. The first section describes the pilot study and its influences on the current study. The second section provides information on the participants of the study, all of whom were enrolled in junior-level physics courses (four courses) during the winter term of the academic year of 2007-2008. The third section provides descriptions of each of the junior-level physics courses that are usually offered during the winter term which are followed after the linear algebra review week. The fourth section discusses the data which were collected during the study, and the fifth section describes the methods for analysis of those data.

Pilot Study

The pilot study was conducted during the winter term of the 2006-2007 school year. The purpose of the pilot study was to focus the research questions and to refine interview protocols that would be used for the final study. The pilot study involved nine volunteer students (eight males and one female), four of whom were enrolled in prerequisite linear algebra courses (either from a linear algebra course or a matrix and power series methods course) taught by the mathematics department. The remaining five students were from a junior-level physics course. All students had been introduced to eigenvalues and eigenvectors prior to the pilot study. The students from linear algebra courses were all from different majors including nuclear engineering (sophomore), mechanical engineering (sophomore), civil engineering (freshman) and forest and civil engineering (freshman).The students from the physics course were all physics majors and three of them were juniors. All nine students participated in a one-hour interview. The interviews were audio-taped and eight of them were also video-taped.

All students were asked to describe eigenvalues and eigenvectors, give examples of eigenvalues and eigenvectors, find eigenvalues and eigenvectors of a matrix and discuss what the matrix does to vectors in general, and some additional questions were posed if there was enough time. The focus was to explore what each student transferred to the interview setting after being introduced to eigenvalues and eigenvectors. The interview protocol was constructed using ideas from previous studies done on transfer of learning of mathematical concepts in algebra and calculus (Bransford & Schwartz, 1999; Ozimek, Engelhardt, Bennett & Rebello, 2004). Studies on students' difficulties with eigenvalues and eigenvectors guided the design of questions on eigenvalues and eigenvectors (Stewart & Thomas, 2003 & 2006). Some interview questions were adopted from studies that used the actor-oriented transfer framework (Cui, 2006). The physics students were included to investigate if they had transferred any ideas from physics to the interview setting.

During the interviews, it became evident that the students had different ideas related to eigenvalues and eigenvectors. Two students from one of the linear algebra courses had nothing to say about eigenvalues and eigenvectors. They said they have seen it in their mathematics course but they could not recall anything. One of the students requested to look at his notes to refresh his memory. Three students (one of them was from the physics course) described eigenvalues and eigenvectors by explaining the procedures to find them. They described how to find eigenvalues and eigenvectors but could not explain what they were. One of them said "I know how to do them but I don't remember how to, what they are. I think they represent something, [after some calculation, the researcher asks again, what this eigenvector tells us] I don't know what eigenvectors are used for" (Interview #4-6:19, Pilot Study). After talking about the procedure to find eigenvalues and eigenvectors, a physics student said that the class always found eigenvalues and eigenvectors of Hamiltonian operators but he was not sure what that meant.

The remaining four students, all from physics, gave similar descriptions of eigenvectors. They stated that an eigenvector was a vector which did not change. Two of these students specifically stated that eigenvectors did not change directions. These two students were also fluent in the procedure of finding eigenvalues and eigenvectors and they also seemed to know how the procedure of finding eigenvalues and eigenvectors related to the eigenvalue equation, $A\vec{v} = \lambda\vec{v}$. They were able to reason further about eigenvalues and eigenvectors by providing examples from physics. For example, when the question "How about if we have two distinct eigenvalues λ_1 and λ_2, will the sum of these be another eigenvalue?" was posed, one of the students said that it does not make sense to add two eigenvalues, and gave an example from his physics course. The analysis of interview data with these two students suggested that students were referring to their experiences in their physics courses to address the interview questions.

One of the goals of the pilot study was to see if students were able to transfer their learning of eigenvalue and eigenvectors to answer the interview questions. Utilizing a traditional transfer paradigms framework, the researcher concluded that seven students out of nine were not able to transfer their learning of eigenvalues and eigenvectors to the interview tasks because these students did not complete the interview questions correctly. They either gave partially correct or no answers at all to the questions[3]. On the other hand, when data was analyzed by implementing actor-oriented transfer framework it was observed that five of these seven students were explicitly referring to their prior experiences in either one of the linear algebra courses or physics courses. For example, one of the students could not describe eigenvalues and eigenvectors properly but he said that eigenvectors were multiples of a scalar and should somehow give the same thing as a matrix multiplied by the same eigenvector. This particular student could not remember what the eigenvalue equation was, however his description of eigenvectors resembled the equation. This specific episode could constitute evidence for actor-oriented transfer of the description of an eigenvector because the student seemed to use his prior experience with eigenvectors. He seemed to reorganize some of the information that is encapsulated in the eigenvalue equation as he provided his explanation. However, it was noticed that the pilot study did not have any data collected to document students' prior experiences with eigenvalues

[3] See Appendix A for Pilot Study Questions

and eigenvectors; unless students explicitly stated that they had answered a similar question or investigated a similar idea in one of the courses they took.

It was also observed that three of the students from the physics course seemed to reconstruct their experiences from physics courses to address the interview questions. They provided examples from their physics courses as they were answering the interview questions.

The pilot study results indicated that students' prior experiences with eigenvalues and eigenvectors are important but they need to be documented prior to the final interviews. To obtain information on students' prior experiences, two more interviews, observations of courses, and pre- and post- quizzes from the linear algebra week were to be collected to document students' experiences for the final study. To help to reduce the variety of prior experiences, only students who were enrolled in the junior-level physics courses during the winter term in the Physics Department were included in the final study. Thus, the design of the final study was modified to its current form. Moreover, the research questions were refined after the pilot study.

Participants

Undergraduate students who were enrolled in the junior-level physics course offered during the fall term of 2007 and planned to take all the junior-level physics courses during the winter term of 2008 were invited to participate in the study. On the recruitment day students were told that the study was about physics students' understanding of linear algebra topics. Fourteen students out of twenty volunteered to participate; however only eleven students were able to complete all three required interviews. Among those eleven students, four students' data were not usable. For example, one of the students spoke very softly so his responses were inaudible from both video and audio-recordings, and another student's third interview was not complete due to time constraints. The participants in the current study were seven undergraduate students who were enrolled in and completed the junior-level physics courses during the winter term of 2008. The researcher arranged to pay twenty-five dollars to each student who completed all three interviews.

The professor (who will be called by the pseudonym "Professor Clayton" throughout this study) who taught the linear algebra review week (LAW) and the professors of the other the winter term physics courses gave the researcher permission to observe, audio-tape and use the video-recordings of the class. The students also signed a permission form for audio and video recordings.

All seven students were required to take the junior-level physics courses offered during the winter term by their majors. On the recruitment day (the fall term of 2007) all twenty students were asked to fill out the informed consent form[4] and a background survey[5]. The purpose of the survey was to get more information on students' mathematics background, i.e., which courses they took and when they took them, if they planned to take more math courses, etc.. The results of the survey were initially intended for use as selection criteria for the final two interviews. However, in the case of a possible problem with the data, the

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[4] See Appendix B for the informed consent form

[5] See Appendix C for the background survey

researcher decided to continue with eleven students. Only seven of the students' data were used for the current study. The pseudonyms for the seven participants used in this study are Crosby, Deniz, Joey, Milo, Tom, Ozzy and Gus. Table 3.1 below lists the prerequisite mathematics courses for the junior-level physics courses of the winter term that were taken by each participating student, and their major and year in school.

Table 3.1 Participating Students and Their Mathematics Courses

Name	**Third Calculus Course/Year**	**Matrix and Power Series course/Year**	**Linear Algebra Course/Year**	**Major & Minor/Year at the Time of the Study**
Joey	-	Yes/Spring 07 (took earlier-fall 06 but withdrew)	-	Engineering Physics
Deniz	Yes/ Spring 05	-	Yes/ academic year of 2005/06	Physics/Junior
Crosby	-	Yes/Spring 07	Yes/ Fall 2007 (but also took Spring 07)	Physics (Naval Sciences Minor)/Junior
Milo	-	Yes /Summer 2007	Yes/ Fall 2007	Physics/Junior
Ozzy	-	Yes/ Fall 2007	-	Physics with Geophysics Option/Junior
Tom	-	Yes/ Spring 2007	-	Philosophy(Major)/Physics Minor /Senior
Gus	Yes/Spring 2006	-	-	Engineering Physics/Junior

Winter Term Junior-Level Physics Course Sequence

The participants of the current study were all enrolled in junior-level physics courses during the winter term of 2008. Students were introduced to eigenvalues and eigenvectors during the first week of the winter term and continued to use them throughout the term. The first week of this term was considered to be one of the experiences participants had with eigenvalues and eigenvectors. Participants were interviewed before and after this particular week and at the end of the winter term of 2008. For all these reasons the courses students usually take during the winter term at the junior level are described in this section. Further information and analysis of the observation of the first week of the winter term of 2008 is included in Section 3.4.

The Physics Department at which this study was conducted revised their junior and senior physics curricula in 1997. Prior to the revision, the physics curriculum was divided into separate sub-disciplines each of which was taught as a sequence of courses during the junior or senior year. For example, Electromagnetism, Classical Mechanics and Mathematics Methods were taught during the junior year and Quantum Mechanics and Thermodynamics/Statistical Mechanics were taught during the senior year. Some of the problems with the traditional curriculum were observed to be:

- Learning was assumed to be linear.
- Compared to the introductory physics courses taken in the freshman and sophomore years, the level of difficulty of topics in junior year seemed to increase suddenly.

- Students were not given enough opportunity to make connections between sub-disciplines of physics.
- Not enough time was given to students to develop their analytical and problem solving skills.

To address the problems in the traditional curriculum of the junior and senior years, a revision was proposed. The revised curriculum followed a spiral-approach which provided students opportunities to consider the main topics twice while building different but connected skills and ideas each time. In addition to the revision of the curriculum, the instruction times were also changed from ten weeks with three hours of lecture per week, to three 3-week sessions with seven hours of class time per week. The tenth week was either added to one of three courses or became a review week at the beginning of the term.

The junior-level physics courses during the winter term start with a week of review on linear algebra topics prerequisite for the courses. Students spend seven hours in class to review topics including, but not limited to, matrix manipulations, the determinant of a matrix, inverse matrices, symmetric matrices, linear transformations, vector spaces, eigenvalues and eigenvectors, and properties of Hermitian matrices. The review week of the winter term (which will be called Linear Algebra Week (LAW) in this study) is not a course by itself, but junior-level undergraduate physics students are required to participate in the week and their grades are included in one of the courses that follow.

During the rest of the winter term students take three junior-level physics courses each of which meets for seven hours a week for three weeks. The timeline of courses shown in Table 3.2 is followed by a brief description of topics and ideas covered in each course.

Table 3.2 Winter Term Junior-Level Physics Course Sequence

Week 1	Week 2 through 4	Week 5 through 7	Week 8 through 10
Linear Algebra Week (LAW)-review of linear algebra topics	Quantum Measurements and Spin	Waves	Central Forces

Linear Algebra Week (LAW)

This one-week review course is only offered during the first week of the winter term of junior-level physics courses. Students enrolled in any of the winter term junior-level physics courses are required to take this one-week course. The topics discussed in this course are pre-requisite for all the other winter term courses. Since it is a review week, it is not considered as a separate course and does not have an official catalogue description; however, the following topics are typically reviewed during this week.

- Matrix manipulations (addition, subtraction, multiplication of matrices)
- Determinant and some properties of the determinant of matrices
- Definition of trace, transpose, Hermitian adjoint of a matrix
- Finding inverses of matrices
- Definition of Hermitian matrix and properties of Hermitian matrix
- Definition of a symmetric matrix and properties of a symmetric matrix
- Definition of a linear transformation and using matrix representations of linear transformations
- Examples of linear transformations with a special focus on rotations as the main physical example
- Defining and finding eigenvalues and eigenvectors
- Diagonalization of a matrix
- Definition of vector spaces and basis vectors

Students are usually given a pre-quiz on the first day, two quizzes and two assignments during the week and their grades are included in the course that follows it (or the course students take after this week).

The goals of the linear algebra week are:

- Students are expected to be fluent in matrix multiplication and finding determinants, inverses, eigenvalues and eigenvectors of matrices.
- Students are expected learn the terminology and properties of Hermitian and symmetric matrices.
- Students are expected to learn and use Dirac Notation.
- Students are expected to learn what linear transformations are, think about them geometrically and determine what they do to a vector.
- Students are expected to view matrices as linear transformations.
- Students are expected to know what properties of matrices correspond to different transformations.
- Students are expected to think geometrically about what eigenvalues and eigenvectors are.
- Students are expected to know the eigenvalue equation and understand what it represents algebraically and geometrically.
- Students are expected to begin understanding basis vectors and that eigenvectors could be basis vectors.
- Students are expected to begin understanding what a vector space is.
- Students are expected to learn inner product and norms.
- Students are expected to begin hypothesis testing and going from examples to general statements or even proposals to theorems.
- Students are expected to become familiar with complex numbers as scalars (as entries in matrices) and manipulate them as needed.

The curriculum of the linear algebra week was developed earlier but always revised to meet the needs of students. However, Professor Clayton, who usually teaches the course, indicated that the needs of students each year change and the priority of meeting each goal is revised accordingly. So, it is possible that the list of goals and topics covered each year shows some differences year to year and some topics are left out. However, the topics and goals directly related to the concepts of eigenvalues and eigenvectors are always covered.

Spin and Quantum Measurements Course

The description of the course in the course catalogue states that *students will be introduced to a variety of quantum mechanical concepts and tools through their illustration via the Stern-Gerlach experiment.* Postulates of quantum mechanics are discussed through demonstrations for the simple spin ½ Stern-Gerlach experiments. Operators, eigenvalues probability, state reduction and time evolution are also discussed. Students are introduced to Schrödinger's Equation and then Schrödinger's time evolution is applied to the case of a spin ½ system. Students are expected to learn and improve the tools of use of Dirac notation, and matrix notation throughout the course.

Waves

The description of the course in the course catalogue states that *students will study waves in one dimension, in both classical and quantum systems.* Students are introduced to the terminology to describe waves as they focus on waves in electrical circuits, waves on ropes and the matter waves of quantum mechanics. Students also discuss the topics of barriers and wells, reflection and transmission, resonance and normal modes, and wave packets with and without dispersion.

Central Forces

The description of the course in the course catalogue states that *students will explore common features of central forces, as well as some special features of the most important examples in physics.* Students first discuss central forces in classical mechanics while the importance of the conservation of angular momentum is emphasized. The separation of variables in Schrödinger's equation and related equations is explored when there is a spherical symmetry. Students are also exposed to spherical harmonics and angular momentum operators. The course ends with an in-depth investigation of the quantum theory of the hydrogen atom.

Overall the linear algebra goals of these three courses are:

- Students are expected to understand that states in quantum mechanics form a vector space and eigenvectors of Hamiltonian operators are chosen to form a basis for this vector space.
- Students are expected to understand that any state of the vector space could be expanded as a linear combination of basis vectors.
- Students are expected to observe that the vector space mathematics underlying all three courses are the same.

Data

Data for this study come from different sources to address the research questions and to capture the different aspects of students' learning experiences. Table 3.3 details the data that were collected and when they were collected.

Table 3.3 Data Collection Matrix

Data	Date
Background Survey	10/18/07
Interview 1	10/27/07 to 11/23/07
Pre-quiz	First day of LAW on 01/07/08
Classroom observation of LAW	01/07/08 to 01/11/08
Quiz 1	01/09/08
Post-Quiz (or Quiz 2)	01/11/08
Homework Assignments from LAW (2 assignments)	01/09/08 and 01/11/08
Interview 2	01/14/08 to 01/25/08
Classroom observations of courses of winter term	01/14/08 to 3/14/08
Final Exam of Central Forces Course	03/17/2008
Interview 3	04/11/08 (second week) to 04/21/08 (forth week)

All students enrolled in the junior-level physics course during the fall term 2007 were asked to fill out a background survey. Students who volunteered for the study were scheduled for a ninety-minute interview a week after they completed the survey. Pre- and post- quizzes were conducted during the LAW of the winter term of 2008. The pre-quiz was given at the beginning of the first day of LAW and the post-quiz was given at the beginning of the last day of LAW. Two assignments were given during LAW and copies of both assignments were collected before they were graded. Only eleven students continued to take the winter term physics courses and they were again invited for a ninety-minute interview after the LAW was finalized. Classroom observations were conducted throughout the winter term. Copies of participants' final exams from the central forces course were also gathered before they were graded. The third interviews were conducted at the beginning of the spring term of 2008. Participants gave permission to gather copies of their homework assignments and tests from the physics course. Also, participants were given audio-recording devices while they were in groups during LAW.

As well as the participants, the teaching team of LAW and two mathematics professors were also interviewed during the current study. The teaching team (Professor Clayton and a post-doctoral researcher) was interviewed before LAW and their preparation for class was observed. Two mathematics professors from the Mathematics Department who usually teach the linear algebra courses were also interviewed before the third interview.

Background Survey

The background survey was given on the recruitment day in the fall term of 2007 and initially planned to use for selecting participants for the current study. Students were asked to list the mathematics courses they took and planned to take and the physics courses they planned to take. Since only 14 students volunteered, all were invited to participate in the study for the possible case of drop-outs and unforeseeable problems.

Pre- and Post- Quizzes[6]

All students enrolled in the junior-level physics course in the winter term of 2008 were given a pre-quiz at the beginning of the first day of linear algebra week. The pre-quiz took five minutes and it had four questions. Students were asked to multiply two three-by-three matrices, find the determinant of a three-by three matrix, find eigenvalues and eigenvectors of a two-by-two matrix and describe eigenvalues and eigenvectors and explain how they are used in physics. Professor Clayton stated that her goal was to see how much students knew or remembered about these ideas so that she could modify the content of the week accordingly.

The post-quiz was given at the beginning of the last day of the linear algebra week. Students were asked to find eigenvalues and eigenvectors of a two-by-two matrix, describe eigenvalues and eigenvectors and explain how eigenvalues and eigenvectors are used in physics.

On the third day of LAW, students were given another quiz asking them to multiply two three-by-three matrices and find the determinant of a three-by-three matrix. Copies of this quiz from participating students were also collected. However, this particular quiz data did not provide much information since the focus of the study is eigenvalues and eigenvectors.

The results from the pre-quiz helped the researcher to understand students' prior experience with eigenvalues and eigenvectors. The results from the post-quiz aided the researcher to explore the participants' apparent understanding of eigenvalues and eigenvectors after being introduced to the topic.

Classroom Observation of Linear Algebra Week (LAW)

[6] See Appendix D for copies of pre and post quizzes

The class met every day for one hour on Monday, Wednesday and Friday, and for two hours on Tuesday and Thursday during the first week of the winter term. The structure of each day varied from small group activities to whole group discussions and sometimes lectures were given. All class meetings were videotaped. Each small group activity was also separately audio and video-taped. The researcher collected field notes each day and marked the time of the episode which needed to be revisited during the data analysis. The topics covered each day during the LAW are outlined and examples of some marked episodes are described next.

Day 1: At the beginning of the class, students were given the pre-quiz and it was explained that the goal of the pre-quiz was to find out what to include in the linear algebra week. Professor Clayton introduced the course and explained the goals and the topics to be covered as well as the grading policy. Students were given a linear algebra review handout which included explanations and examples of matrix addition, scalar multiplication, matrix multiplication, transpose and trace of a matrix, symmetric matrices, the Hermitian adjoint of a matrix, how to find the determinant and inverse of a matrix, and Dirac notation. Students were given another handout on the Dirac (BraKet) Notation. On the first day the topics of matrix addition and subtraction, scalar multiplication, matrix multiplication, finding the transpose, the Hermitian adjoint, the trace and the determinant of a matrix were discussed. Instead of directly lecturing, Professor Clayton first asked a question and from the students' answers she developed an explanation and examples for each topic. She pointed out that students would be working with Hermitian matrices and matrices with complex entries and the complex number *i* appears in the Schrödinger Equation which will be used in all winter term courses. Professor Clayton also stated that the determinant is a geometric quantity that tends to be preserved by certain kind of transformations. Students were also introduced to Dirac Notation briefly. She stated that all this information was on the handouts and students should review the handouts.

Day 2: The second day started with a small group activity called Linear Transformations[7]. Students were divided into small groups and each group was given one of the matrices from the activity sheet. In this activity students were asked to operate on five vectors that were listed on the sheet with their assigned matrix. They were asked to graph the initial and the transformed vectors on their whiteboards. They were also asked to find the determinant of the matrix and make a note of any changes made to the initial vectors to get the transformed ones, specifically to look for rotations, inversion, length changes and so forth. Students were asked to report vectors that were unchanged by the transformations. Once all the groups were done (after 15 minutes), they were asked to present their results. Each group was asked to report what their group matrix was, what it did, what its determinant was, and the vectors that were unchanged by the transformation in their presentations. During the presentations Professor Clayton lead a whole group discussion on certain ideas. After the second group's presentation, she told the class to pay attention to the connection between the determinants of the matrices and what the matrices did. She explicitly stated that she wanted them to come out of this activity by understanding the geometric role of the determinant. After the fourth group presentation one of the students (Gus) made a conjecture that a matrix

[7] See Appendix E for all the activities of Linear Algebra week.

with the determinant one rotates the vectors and a matrix with the determinant negative one reflects the vectors and this became the running hypothesis. The fifth group's matrix had the determinant zero and it did not conserve the length of the vectors. Professor Clayton asked what happen to their hypothesis after this example and one student said that they had a new condition and another student stated that determinant zero implies linear dependence. Professor Clayton decided to investigate the idea of the linear dependence and asked further questions. During this investigation another student stated that a determinant of zero also meant that the matrix was inconsistent. This particular student was asked clarifying questions and then the idea of systems of equations was introduced. Professor Clayton discussed the meaning of being consistent and inconsistent of the system of equations. She stated that representing a system of linear equations with a matrix was one way that matrices were often used in linear algebra courses. She explicitly stated that she was using the wording of "operate on" on purpose to point out the different use of matrices. Students were told that matrices represent linear transformations such that a matrix multiplied by a vector gives another transformed vector. Professor Clayton said matrices are often called operators in Quantum courses and represent linear transformations. Then students discussed the case when the determinant was zero. It was noted that these matrices did not have inverses and they sometimes represented projection transformations. They then decided to call this type of matrix (matrix with determinant zero) a "scrinch" matrix. Since it was not always a projection, they referred to these types of matrices with "scrinch".

There was also a discussion on symmetric and anti-symmetric matrices. Towards the end of day two, Professor Clayton asked about the vectors that were unchanged, asking if students knew anything similar to this idea. One of the students said “eigen-something”. Then they had a brief discussion on eigenvalues and eigenvectors. One of the students wrote the eigenvalue equation $Av=\lambda v$ and they discussed the meaning of the equation. Professor Clayton wrapped up the discussion by stating that the equation states that if you multiply a vector by the matrix A, in other words if you put the vector in the black box and transformed it, you get a scalar multiple of the same vector. She pointed out that the unchanged vectors in the activity were the eigenvectors of the matrices. Students were told that they should think geometrically about eigenvectors, meaning that eigenvectors are the vectors that do not change direction, and negative direction is considered to be the same since it is obtained by a negative scalar multiplication.

Students concluded that matrices with positive determinants represent rotation transformations, negative determinants represent reflections and determinant zero matrices put all the vectors on a line. They also discussed the possible eigenvectors of each matrix in the activity.

<u>Day 3:</u> Students were given a quiz on matrix multiplication and finding the determinant of a matrix. They reviewed the ideas from the second day briefly. Students were introduced to the Dirac notation and found the length of a vector using this notation. During the rest of the day Professor Clayton lead a discussion on rotation transformations to identify the properties of rotation.

Day 4: During the first part of day four, Professor Clayton wrapped up the ideas related to rotation transformations. Then Professor Clayton started to explain how to find eigenvalues and eigenvectors of a matrix. She started with the eigenvalue equation and showed how the characteristic equation was derived. Then she worked through an example of finding eigenvalues and eigenvectors of a complex two-by-two matrix on the board. After her example, students were again divided into small groups and each group was assigned a matrix. They were asked to find eigenvalues and eigenvectors of their matrix. Students presented their results to the class and Professor Clayton asked questions during the presentations. She asked each group to tell what their matrix did to vectors and what happened to eigenvectors. Professor Clayton's goal was to show students how the geometric interpretation of eigenvectors relates to algebraically calculated eigenvectors.

Day 5: Students were given the second quiz (post-quiz) and asked to find eigenvalues and eigenvectors of a complex two-by-two matrix, to describe eigenvalues and eigenvectors and to explain how eigenvalues and eigenvectors are used in Physics. The quiz was followed a lecture on the definition of a linear transformation. Professor Clayton stated that if a linear transformation operated on the standard basis vectors, the results would demonstrate what happens to the whole vector space under the transformation. Students were shown three different languages for representing vectors, one of which was the Dirac notation. A discussion on the inner product operator was followed by the introduction of Hermitian matrices. Professor Clayton proved two theorems related to Hermitian matrices. She showed that the eigenvalues of Hermitian matrices were real and eigenvectors belonged to different eigenvalues of a Hermitian matrix were orthogonal to each other.

Overall, it was observed that students were exposed to almost all the listed topics except for the definition of a vector space and diagonalization of matrices. Professor Clayton stated that some years they do not have enough time to cover everything and each year students' needs and in-class discussions are different. She modifies the curriculum to meet the needs of the students as well as to address the questions that surface during in-class discussions. She stated that she wanted to spend more time on the idea of using eigenvectors as basis vectors during LAW. She did not have enough time to discuss this idea during LAW of 2008.

The purpose of observing LAW was to document students' experiences as they were exposed to linear algebra topics and specifically the concept of eigenvalues and eigenvectors. Also, the class observation helped the researcher to document particular phraseology or ideas that might be adopted by the students because of their experience in class.

Homework Assignments from LAW

Students were given two assignments during LAW and the copies of the participants' assignments were collected before they were graded. The assignments were collected to document part of the participants' experiences in the linear algebra week. Homework assignments also helped the researcher to be alerted to ideas that might be adopted by the students because of the assignment questions.

Final Exam from Central Forces Course

The final exam of Central Forces course was given at the end of the winter term of 2008. One of the questions asked students to describe eigenvalues and eigenvectors and explain how they are used in physics. Since the same question was part of the pre- and post- quizzes during linear algebra week, copies of the participating students' final exams were collected as part of the data. The exam was administered between the second and third interviews of the study. Participants' answers were included in the data analysis. The final exams served to document the participating students' apparent understanding of eigenvalues and eigenvectors at that point in their learning of eigenvalues and eigenvectors.

Classroom Observations of Winter Courses

The junior-level physics courses students took after LAW of the winter term of 2008 were also observed. Each class was video-taped and researcher took field notes during each class. The date and the time of the episodes that were directly related to the second and third interviews were marked and revisited again during the data analysis. The classroom observation informed the researcher on the students' experiences with eigenvalues and eigenvectors in physics settings. Also, by observing these courses the researcher became aware of the particular examples and ideas that might be implemented by students.

Interviews

Three ninety-minute, in-depth interviews were conducted during three different terms of the academic year of 2007-2008. The first interview was conducted toward the end of fall term, the second one took place after LAW and the third interviews were at the beginning of spring term. The overall goal of the interviews was to explore each participating students' emerging understanding and determine how students transfer their experiences with eigenvalues and eigenvectors from LAW and the subsequent physics courses to the interview tasks. The reasons to have more than one interview were to make students comfortable working on problems in the researcher's presence and to document the students experiences with eigenvalues and eigenvectors as they were exposed to them in each course.

All interviews were audio and video taped. The video camera was facing the participant and running on its own. The participant and the researcher were the only people in the room. All interview questions and possible follow-up questions were prepared in advanced and a certain interview protocol was followed in each interview. In addition to prepared follow-up questions, the interviewer asked questions when she did not understand what was said or to clarify her understanding of the interviewee's response.

The researcher also conducted interviews with two mathematics professors who usually teach linear algebra courses in the mathematics department prior to the third interviews with participants. Also, Professor Clayton and the post-doctoral researcher who co-taught the linear algebra week and the central forces course were interviewed by the researcher. The post-doctoral researcher also co-taught the other two junior-level physics courses with two other professors from the Physics Department. Each interview helped this study in different specific ways, as discussed below.

<u>Interviews with Physics and Mathematics Faculty:</u>

Math Experts: Two professors from the Mathematics Department were interviewed before the third interviews with the participating students. Each interview took approximately thirty minutes with the focus of the interviews to explore how the experts solve some of the questions from the second and third interviews of this study. The professors were asked to comment on the interview questions and how they would expect a student who took a linear algebra course to solve them. Since these professors usually teach the linear algebra courses in the math department, their experiences with the content of the course and with the students of the course were important.

Physics Experts: Professor Clayton and the post doctoral researcher were interviewed twice during the study. They were first interviewed before they taught the linear algebra week and asked what they expected students to gain from this week. Their preparation for LAW was also observed. They were contacted later to discuss the third interview questions (before the third interviews were conducted with the participants). The purpose of these interviews was to document their perspectives in choosing and engaging in the observed curriculum and instructional practices.

<u>Interviews with Participants</u>

Three ninety-minute, in depth interviews were conducted with eleven students during the fall, winter and spring terms of the academic year 2007-2008.

Interview 1: The first interviews with the participants were conducted after the sixth week of the fall term. There were two reasons for this timing: to maintain student interest in the study by scheduling interviews close to the winter term, and to coincide with the linear algebra instruction through the mathematics department. The matrix and power series methods courses usually cover eigenvalues and eigenvectors toward the end of the fifth week and eigenvalues and eigenvectors are the last topic covered in the linear algebra course. To capture students' experiences with these topics prior to LAW, the researcher attempted to schedule the first interviews after students have seen these topics. However, there were scheduling problems with some participants. Both Milo and Crosby were busy with their final exams, so their first interviews could not be scheduled until after they had been exposed to eigenvalues and eigenvectors in a linear algebra course.

One purpose of the first interview was to document students' experiences with eigenvalues and eigenvectors prior to the linear algebra week in the winter term. Students were asked to describe eigenvalues and eigenvectors and to give examples. If students could not recall or did not know anything about eigenvalues and eigenvectors, they were asked to explain what matrices represent. The researcher then asked questions related to matrix multiplication, vectors and vector multiplications. The purpose of these questions was to capture the participants' experiences with basic linear algebra topics. If students could talk about eigenvalues and eigenvectors, then they were given a linear transformation and asked to determine its eigenvalues and eigenvectors.

Interview 2: The second interviews with participants were conducted after the linear algebra week, but before the end of the winter term. One purpose of this interview was to investigate the students' emerging understanding and to characterize each student's emerging understanding in terms of eigenvalues and eigenvectors after his/her LAW experience.

Another purpose of the second interview was to address the research question on transfer of learning of eigenvalues and eigenvectors by exploring whether students' emerging understanding observed in the second interview had roots in their prior experiences or if they were spontaneously constructed at the interview.

Students were first asked to describe the most interesting thing they learned that they did not know before LAW. The purpose of this question was to direct students' attention to their LAW experiences. If students did not mention eigenvalues and eigenvectors, they were asked to describe eigenvalues and eigenvectors and then asked to find eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ after explaining what A did to vectors. The matrix A was chosen purposely to be different than the matrices that students were exposed to in LAW. The matrices used in LAW had integer entries, mostly -1, 0 and 1. This question was chosen because of its similarity to the questions explored in LAW, yet it would allow students to demonstrate their experience from LAW if they chose to.

Two math professors were also asked the same questions about the matrix $A = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$. One of the professors conjectured that the matrix rotates the vectors and after checking her conjecture she said it was a reflection. She found eigenvalues and eigenvectors and stated that the first eigenvector found laid on the reflection axis. One of the ideas she said she emphasized in her linear algebra courses was that eigenvectors were not necessarily perpendicular to each other and in this example they happened to be perpendicular to each other. The second professor said she usually used basis vectors (0, 1) and (1,0) to figure out what A takes the vectors to and she did not think geometrically about matrices but she made sure to spend time on geometric ideas in her linear algebra courses.

If time permitted students were asked to explain their experiences with eigenvalues and eigenvectors in LAW and if they had learned anything new during LAW about eigenvalues and eigenvectors. Participants who were able to describe eigenvalues and eigenvectors at the first interview were asked to watch and comment on their answers from the first interview. The purpose of this question was to get students' perspectives on their earlier experiences with eigenvalues and eigenvectors after being introduced to them in LAW.

Interview 3: The third interviews were conducted at the beginning of the spring term. The researcher wanted to make sure that participants were all exposed to the same ideas before the third interview, so she waited until all courses were completed. The purpose of the third interview was similar to the second one; however, this time participants had worked more with ideas related to eigenvalues and eigenvectors, where in the second interview students were still in the process of being introduced to eigenvalues and eigenvectors. One of the purposes of the interview was to describe the students' emerging understanding after they had worked with eigenvalues and eigenvectors throughout the winter term. Another purpose of the third interview was to investigate students' transfer of learning of eigenvalues and eigenvectors by exploring whether students' emerging understanding observed in the third interview had roots in their experiences during the winter term or whether they were spontaneously constructed at the time of the third interview.

In the first question students were told that a square matrix or an operator had eigenvectors e_1 and e_2 and they were asked what they could say about the sum of the eigenvectors, in particular if it was an eigenvector or not. One of the goals of the winter term physics courses was to expand any vector as a linear combination of eigenvectors. Students used this idea in both waves and central forces courses. The purpose of this question was to explore if students would find the interview question similar to what they had been exposed to in their courses.

Students were told in the second interview question that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ was an eigenvector that associated with the eigenvalue of 1 and the second eigenvector was $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ with the eigenvalue of -1 an operator M. They were asked what they could say about the matrix M with the given information. A similar question was discussed in the spin and quantum measurements course and the purpose of the interview question was to see whether students would find this question similar to their class experience.

Students were again asked to describe eigenvalues and eigenvectors and if there was enough time, another question was asked. The final question of the third interview was to give the general solution to the differential equation $i\frac{d}{d\varphi}f(\varphi) - af(\varphi) = 0$ subject to the condition $f(\varphi) = f(\varphi + 2\pi)$. Once students had an answer, they were asked if it was possible to change this question into an eigenvalue problem. The purpose of the question was to see if students could use their experiences from the winter term courses. Even though they had not worked on a similar question, they had worked on differential operators during the winter term.

These interviews were transcribed in their entirety. Table 3.4 outlines the relationship between interviews and research questions.

Table 3.4 Interviews and Research Questions Matrix

Interview Number	***Corresponding Research question***	***Interview Time***
Interview #1	*Research Question 1 (Prior Experience)*	*Fall Term 2007*
Interview #2	*Research Questions 1, 2 and 3*	*Winter Term 2008*
Interview #3	*Research Questions 1. 2 and 3*	*Spring Term 2008*

Qualitative Data Analysis

Data analysis was an ongoing process beginning with the information collected in the background survey and the analysis of the first interviews collected during the fall term of 2007. Data from the surveys were compiled into an Excel spreadsheet and used to create Table 3.1 after checking the accuracy of the information with the participants at the first interview. The first interview transcriptions were also checked for accuracy and the eigenvalues and eigenvectors sections of the transcriptions were marked. Data from pre- and post-quizzes and the final examination were also compiled into an Excel spreadsheet once they were collected. Table 3.5 presents an example from this Excel spreadsheet.

Table 3.5 Pre Quiz, Post Quiz and Final Exam Excel Spreadsheet Example

Student Name	**Pre Quiz (Day 1-LAW, winter term) Eigenvalue Question**	**Post Quiz (Day 5-LAW, winter term) Eigenvalue Question**	**Final (End of winter term) Eigenvalue Question**
Crosby	It's a value that you multiply a vector by which does the same thing as if you had multiplied that specific vector by a specific matrix. $Ax = \lambda x$	It is a value that has the same affect when multiplied on a specific vector as when a certain matrix operates on that specific vector. $Av = \lambda v$	An eigenvalue is a particular number (scalar) which can replace a matrix when operating on a particular eigenvector. In quantum, eigenvalues represent possible measurements.

After the second interviews were conducted, they were transcribed and transcriptions were checked for accuracy before the third interview. A similar process was followed after the third interviews; verbatim transcripts of the audio tapes were made and checked for accuracy while the researcher watched the video tapes of the third interviews.

As the second and third interviews were watched to check the accuracy of transcriptions, the parts in which students seemed to use an example or an idea from the winter term courses were marked. The purpose of marking these episodes was to alert the researcher to these examples and ideas when the videos of the winter term courses were watched.

There was a significant amount of video data from the winter term courses. Everyday class meetings were video taped and videos of small groups working during class were collected.

The analysis of these data followed modified versions of two established qualitative methods for video analysis, specifically the *whole-to-part (inductive) approach* and the *part-to-whole (deductive) approach* (Erickson, 2006). The *whole-to-part approach* suggested that the videos were watched without stopping, so videos of LAW were watched without stopping and notes were taken. These notes were checked against the field notes taken in class. Then, the researcher reviewed the videos again by stopping and re-watching the parts that seemed related to students' responses in the second and third interviews. A similar, but less intensive, approach was followed for the videos of the other three winter term courses.

The *part-to-whole approach* was implemented when the researcher was watching the participants' interactions during LAW. According to this approach, the researcher watched and took notes during a single participant's interactions during small group activities of LAW. For example, notes were made regarding the participant's actions- if he/she took notes during the small group activities, asked questions, did all the talking, or did not talk or participate.

Once all the data were collected, checks for accuracy, and notes were prepared from classroom videos; a case folder for each participant was compiled consisting of the printouts of transcriptions of their three interviews with marks, copies of pre- and post- quizzes, final exam and two homework assignments from

LAW and notes from the winter term courses related to the individual participant obtained through course videos. The purpose of the case folders was to do a case-by-case analysis of each participant's data to present the results based upon the repeating ideas related to the concept of eigenvectors and eigenvalues. After this analysis the data from second and third interviews were analyzed by implementing the actor-oriented transfer (AOT) framework (Lobato, 2002). The purpose of this analysis was to explore and address the second research question:

> What do students transfer about the concepts of eigenvalues and eigenvectors from this series of courses to an interview setting? More precisely, what kind of experiences and views related to matrices, methods of finding eigenvalues and eigenvectors, the interpretation and use of the eigenvalue equation, and the relationship between basis vectors and eigenvectors do upper-level physics students transfer from their coursework to the interview setting?

The AOT analysis was followed by a cross-case analysis of all data.

Case Analysis

Analysis of the case folder of each participant started with an initial reading of the transcripts of the first, second and third interviews. The researcher took notes and highlighted parts that were directly related to eigenvalues and eigenvectors. For example, the part in which the participant was talking about matrix multiplication did not get marked; however the parts in which the participant explicitly said, or referred to, eigenvalues and eigenvectors were highlighted. After the initial reading of the data a modified version of the Qualitative Hypothesis-Generating analysis (Auerbach & Sliverstein, 2003) was implemented.

For each participant, the researcher first identified the relevant texts in the transcriptions of the interviews and pre- and post- quizzes and final exam data. Auerbach and Sliverstein (2003, p.46) defined the relevant text as an idea that could be relevant to the research questions. To identify the relevant texts, the researcher asked the following questions

- Does this text (sentence, paragraph, or a whole episode from transcripts or other pieces of data) help to understand the idea (or answer) presented by the participant?
- Does this text (sentence, paragraph, or a whole episode from transcripts or other pieces of data) relate to the concept of eigenvalues and eigenvectors?
- Does this text (sentence, paragraph, or a whole episode from transcripts or other pieces of data) help me to understand how the participant thinks about eigenvalues and eigenvectors?
- Does this text (sentence, paragraph, or a whole episode from transcripts or other pieces of data) seem important and why?

After identifying the relevant texts for each participant, the relevant texts were compiled into an Excel spreadsheet. Table 3.6 illustrates examples of relevant texts from one of the participant's data.

Table 3.6 Examples of Relevant Text

Student Name	**Data Source**	**Relevant Text**
Crosby	Interview 1 Lines from transcripts:	Interviewer: Can you give me an example of how to find them? Crosby: Yes, it has been awhile, since Spring. I remember when you have like a matrix of some numbers, A, B, C, I take the diagonal pieces and you subtract lambda. Then you take the determinant and – I should probably do it two by two – but you take the determinant and solve for lambda, which you will have more than one answer for. It would give you quadrilateral for a two by two. That is as far as I remember what they are.
Crosby	Interview 2 Lines from transcripts:	Crosby: OK. So [writing]. So we have to use this formula, determinant of the identity matrix times lambda minus the matrix equals 0, which is lambda minus 1, lambda plus 1.
Crosby	Interview 3 Lines from transcripts:	Crosby: Yes. We have to take, let's see, it is the matrix. It is the determinant of the matrix minus the identity matrix times lambda, or the eigenvalues, equal to 0. Then you solve for lambda. That is how you find the eigenvalues. Then use eigenvalues to find the eigenvectors, using the eigen-statement. Interviewer: What is the eigen-statement? Crosby: That is this thing here. [pointing to $Av=\lambda v$]

All relevant texts were organized chronologically to document the participant's experiences with eigenvalues and eigenvector. After the relevant texts of the data of a participant were identified, they were organized into repeating ideas. A repeating idea is defined to be an idea stated in relevant texts more than once by the participant (or repeated by two or more participants). Examples of repeating ideas for Crosby included: the Eigenvalue Equation (algebraic interpretation), Procedures, Basis Vectors, Perpendicular, Proof, Matrix Representation, and Confidence. Examples of relevant texts that were identified as the Eigenvalue Equation (algebraic interpretation) repeating idea were listed in Table 3.7.

Table 3.7 Example of Crosby's Repeating Ideas

Relevant Text	**Data Source**	**Repeating Idea**
Interviewer: I am really interested in eigenvalues and eigenvectors, so what can you tell me about eigenvalues and eigenvectors? Anything that comes to mind? Crosby: I seem to remember not much about them. I know they are important. Let's see. Something about some multiple of some sort. I kind of remember how to find them and determinant, using a determinant. I don't remember what they do, why they are so great.	I. 1	Eigenvalue Equation (algebraic interpretation)
It's a value that you multiply a vector by which does the same thing as if you had multiplied that specific vector by a specific matrix. $Ax = \lambda x$. And then the vector stated above [answer for an eigenvector]	Pre-quiz	Eigenvalue Equation (algebraic interpretation)
It is a value that has the same affect when multiplied on a specific vector as when a certain matrix operates on that specific vector. $Av=\lambda v$. It is a vector that when operated on by a certain matrix yields the same result as if operated on by its eigenvalue.	Post-quiz	Eigenvalue Equation (algebraic interpretation)
Interviewer: Going back and thinking about that class, I remember you guys discussed eigenvalues and eigenvectors. What can you tell me about eigenvalues and eigenvectors? Crosby: Eigenvalue is a number that when acting on a certain vector does the same thing as if you used a matrix, a specific matrix. You start with a matrix and you add a vector that you use the matrix on. Eigenvalue does the same thing. It is just easier. Interviewer: Can you show me how on a piece of paper? Crosby: Sure. OK. So with a specific eigenvector, you use the scalar eigenvalue and it replaces what this matrix does. [writes $Ax=\lambda x$] Interviewer: These are eigenvalues and eigenvectors? Crosby: Yeah.	I.2	Eigenvalue Equation (algebraic interpretation)
Crosby: OK, what I know about eigenvectors – a matrix can have an eigenvector, which is a very specific vector that when the matrix is multiplied by it, it is the same as if the matrix were just a scalar.	I. 3	Eigenvalue Equation (algebraic interpretation)

After organizing the relevant texts into repeating ideas, these ideas were then organized under the four chosen goals of LAW and the linear algebra goals of the winter term courses to form the common themes. These four goals were

- Students are expected to view matrices as linear transformations
- Students are expected to learn algebraic and geometric interpretations of the eigenvalue equation $Av=\lambda v$
- Students are expected to understand the expansion of any vector with eigenvectors. (Superposition idea)
- Students are expected to be fluent in finding eigenvalues and eigenvectors of a matrix or an operator.

These goals were chosen among the list of the goals formed by Professor Clayton. They were chosen because they were the ones directly related to the concept of eigenvalues and eigenvectors. For example, one of the chosen goals was that "Students are expected to view matrices as linear transformations" and it was chosen to be one of the four goals because the geometric interpretation of the eigenvalue equation was developed on viewing the matrices as a linear transformation, rather than a system of equations.

The repeating ideas *the Eigenvalue Equation (algebraic interpretation)*, and *the Eigenvalue Equation (geometric interpretation)*, for example, were combined under the goal of "Students are expected to know the eigenvalue equation and understand what it represent algebraically and geometrically." Not all participating students had the same repeating ideas under the same goals. For example, Crosby repeated many times that eigenvectors were perpendicular to each other and used this particular idea to reason with the idea of superposition, which is part of the goal that states “Students are expected to understand that any state of the vector space could be expanded as a linear combination of basis vectors”. The repeating idea of *perpendicular* was not observed in every participant’s data.

Some of the repeating ideas that were observed in Crosby’s data did not directly fit under any of the goals. For example, the repeating ideas of *proof* and *not my kind of math* did not help the researcher to characterize Crosby’s eigenvalues and eigenvectors image. Even though the repeating ideas did not directly answer the research questions, they helped the researcher answer two questions, “How do these repeating ideas help understanding the participant in general and how do they relate to the participant’s way of thinking about linear algebra concepts?”

After repeating ideas were categorized under goals, case studies of seven students were written and four were chosen to be presented in Chapter Four: CASES. These students’ case studies were chosen because these students had “different” backgrounds.

Actor-Oriented Transfer (AOT) Analysis

In order to analyze the data by using the AOT framework the researcher assumed the perspective of *epoche*. *Epoche* is defined to be a process in which the researcher engaged to eliminate or at least be attentive of his or her viewpoint or assumptions regarding the phenomenon (Patton 2002, p.485). This process helped the researcher look at the data with an open mind, without any prejudgments.

The transcripts of the second and third interviews of each participant were read again. The participant's answers to each interview question and follow-up questions were analyzed individually and then divided into episodes. For example, a participant was asked to find eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ and what this matrix did to vectors in the second interview. A participant's answer was divided into two episodes; one addressing eigenvalues and eigenvectors; and the second one was the part of the answer that focused on what the matrix did. Then each episode was analyzed by searching for evidence of prior experiences documented in any earlier parts of the study.

Once the AOT episodes were developed, they were again categorized under the goals which they seemed to fit.

Cross-Case Analysis

After each case analysis was performed, some repeating ideas in each participant data under the goals of LAW and the winter term physics courses noticed to be common among all participants and some of them were characteristic of only one participant. Similarly, some prior experiences of participants were observed to be similar when conducting the actor-oriented transfer analysis of the data. These commonalities and differences among repeating ideas and prior experiences then decided to be analyzed across all cases by a cross-case analysis.

The cross-case analysis began by reading and studying each participant's interview transcripts, repeating ideas and prior experiences. However this time each repeating idea and prior experiences were studied to compare and contrast the data among participants according to how they fit into four chosen goals of the linear algebra week and the junior-level winter term physics courses. These four goals were:

- Students are expected to view matrices as linear transformations
- Students are expected to learn algebraic and geometric interpretations of the eigenvalue equation $Av=\lambda v$
- Students are expected to understand the expansion of any vector with eigenvectors. (Superposition idea)
- Students are expected to be fluent in finding eigenvalues and eigenvectors of a matrix or an operator.

The four chosen cases will be discussed in the next chapter.

CHAPTER FOUR

CASES

The purpose of this chapter is first to characterize the participating students' emerging understandings of the concept of eigenvalues and eigenvectors as they worked on problems during three interviews. Data were also analyzed to describe the participants' transfer of learning of eigenvalues and eigenvectors from a reconceived transfer perspective, namely the Actor-Oriented Transfer (AOT) framework (Lobato, 1996). Specifically, the question of whether a participant's emerging understanding of the concept of eigenvalues and eigenvectors has roots in his/her prior experiences is explored.

Chapter Three, Methods, contains a description of the selection of seven students as the participants of this study. Four of the students' case analyses are chosen to be presented in this chapter. Three of these students (Deniz, Crosby, and Gus) were chosen because they had different backgrounds in terms of which prerequisite mathematics courses they took prior to the winter term of 2008. Even though the fourth student (Milo) took the same courses as Crosby, Milo's data analysis revealed different ideas. Milo's case analysis was shared to show that students with the same backgrounds do not necessarily have the same experiences.

Each case starts with some background information about the participant including the mathematics courses taken prior to the current study, the courses taken concurrently with the study, and notes from in-class observations. The analyzed data is categorized according to four goals of the linear algebra review week (LAW) and the physics courses of the winter term. These goals were chosen because they were directly related to the concept of eigenvalues and eigenvectors.

These four goals are described briefly:

Goal 1: Matrix Representation

Professor Clayton stated in an interview with the researcher that she wanted students to understand that matrices represent linear transformations (as operators). She said that she uses an activity titled Linear Transformations on day two to introduce the topic. In this activity students are asked to operate on an assortment of vectors with a given matrix to find the transformed vectors. The activity also introduces terminology that will be used during the term. Students are asked to draw both the initial and the transformed vectors on the same graph. It is hoped that with this approach students will be able to see the vectors that are unchanged by the transformations. Professor Clayton then helps her students to realize that eigenvectors of linear transformations are the special vectors that do not change direction, assuming the reverse direction to be the same. (In other words, vectors pointing south and north are assumed to have the same direction since their direction could be changed through a negative scalar multiplication). Professor Clayton feels that viewing matrices as linear transformations will help students understand the geometric meaning of eigenvectors.

Goal 2: Finding eigenvalues and eigenvectors

Professor Clayton expected students to be fluent in finding eigenvalues and eigenvectors of matrices. Students are usually introduced to finding eigenvalues and eigenvectors on the fourth day of LAW. Professor Clayton first explains how and why the characteristic equation is obtained from the eigenvalue equation. She then works on an example to demonstrate how eigenvalues and eigenvectors are found using the characteristic and the eigenvalue equations. Professor Clayton emphasized that students should not try to be efficient when finding eigenvectors, and they should do each case separately by writing out the eigenvalue equation with each eigenvalue. She demonstrates how each eigenvector is found using the eigenvalue equation. Students then practice finding eigenvalues and eigenvectors in a small group activity called Eigenvalues and Eigenvectors. Each group is assigned a matrix to find its eigenvalues and eigenvectors, and later, groups present their results. At the beginning of each presentation, Professor Clayton asks students to explain what kind of transformation the matrix represents and which vectors are unchanged by it before the students present the results of the calculations. She suggests that students should figure out what the matrix represented and the vectors that are unchanged by it each time prior to finding eigenvectors. She says it would help students to avoid algebra mistakes. Students are asked to practice finding eigenvalues and eigenvectors after class. On the fifth day, students are given a quiz on finding eigenvalues and eigenvectors of a two-by-two complex matrix.

Goal 3: The Eigenvalue Equation and Its Interpretations

Professor Clayton expected her students to understand the algebraic and geometric interpretations of the eigenvalue equation, $A\vec{v} = \lambda\vec{v}$ and use these interpretations when solving problems. Students were expected to understand that the equation

$A\vec{v} = \lambda\vec{v}$ algebraically indicates that when the matrix A acts on its eigenvector v, the result is a scalar multiple of the same eigenvector and the scalar is the eigenvalue λ of the matrix A. Also, the equation could be interpreted geometrically to mean that eigenvectors do not change direction but only change length when the matrix acts on them, and the length change is determined by the eigenvalues.

Professor Clayton emphasized many times during the interview with the researcher that she expected students to understand the geometric meaning of eigenvalues and eigenvectors. It was observed that this expectation was related to the geometric interpretation of the eigenvalue equation. Moreover, the algebraic meanings of eigenvalues and eigenvectors were part of the algebraic interpretation of the eigenvalue equation. The eigenvalue then could be interpreted algebraically as a scalar value replacing what matrix do. Its geometric interpretation could be stated as a scalar value which changes the length of the eigenvector with which it is associated. Similarly, eigenvectors could be interpreted geometrically as special vectors of the matrix that do not change direction, and algebraically, they are the vectors that become a scalar multiple of themselves under matrix multiplication.

Goal 4: Using Eigenvectors as Basis Vectors

Students were also expected to understand that the eigenvectors could be chosen as the basis vectors of a vector space and any vector in the vector space could be expanded linearly with the eigenvectors (the later idea is called “superposition”). Professor Clayton said in the interview with the researcher that students could develop this understanding after completing the winter term physics courses. She stated that all three winter term physics courses try to convey this goal by using different vector spaces. In all three courses students were given a vector in superposition of eigenvectors and asked to figure out if it was an eigenvector of the operator.

It was also observed that there was not a discussion of eigenvalues with the multiplicities (which was called “degeneracy”). This idea arose once on the fourth day of LAW. Professor Clayton stated each year discussion on the multiplicities changed depending on the questions from students. For example, they had a long discussion on the topic during LAW of 2009. She also stated that the degeneracy examples in physics were not discussed until the senior year courses.

As described in Chapter 3, each student’s data were first analyzed into repeating ideas, and these repeating ideas were categorized under each relating goal. The data from students were also analyzed by implementing the actor-oriented transfer framework. Episodes constituting evidence of the actor-oriented transfer were described under each relating goal. The ideas and episodes which were not directly related to the concept of eigenvalues and eigenvectors were left out of the case analysis. In addition, some students do not have data relating to every goal.

In this chapter and in Chapter 5, whenever portions from a student’s interview transcripts are shared the student’s initial are used and the letter “I” will indicate the interviewer. Also, the abbreviations I.1, I.2, and I.3 are used for interview 1, interview 2, and interview 3, respectively.

Milo

Milo was a junior physics major and expected to graduate at the end of spring term 2009. He was enrolled in a linear algebra course when the first interview was conducted during the fall term. At the time of the interview the concepts of eigenvalues and eigenvectors had not been introduced in the course. Milo had taken the matrix and power series methods course during the summer term of 2007; he had taken a calculus sequence (two courses), and a vector calculus sequence (two courses) during his sophomore year, and differential equations and matrix and power series methods courses during the summer term of 2007, prior to his junior year. He had completed all of his mathematics course requirements at the end of the fall term when he finished the Linear Algebra 1 course. He did not plan to take any further mathematics courses.

Table 4.1 shows the physics related courses in which he was enrolled during this study.

Table 4.1 Courses Milo took during the study

Course Name	**Term Taken**
Linear Algebra	Fall Term 2007
3 Modules of Physics Courses	Fall Term 2007
Analog and Digital Electronics in Physics	Winter Term 2008
3 Modules of Physics Courses	Winter Term 2008
Computer Interfacing in Physics	Spring Term 2008
3 Modules of Physics Courses	Spring Term 2008
Classical Mechanics in Physics	Spring Term 2008

At the first interview, Milo mentioned that the topics of matrices and determinants were being discussed in the linear algebra course, and he said it was “really boring.” He did not explain what he meant by “boring.”

When he could not perform matrix multiplication as fluently as he wanted; he said, “It was irritating to me that I am not better at that. I should not have to think about that.” He also said twice that he could not think in front of the camera during the first interview. For example, when finding the determinant of a two-by-two matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, he first stated that the determinant was $a_{21}a_{12} - a_{11}a_{22}$ and then he said, “Of course, now that I am on camera, I think I am doing it backwards. I am doing it backwards, $a_{22}a_{11} - a_{21}a_{12}$. Cause I remember now seeing it in just variables a, b, c, d and it was *ad-bc.*” He was asked how he decided that $a_{21}a_{12} - a_{11}a_{22}$ was not correct, and he was still not sure which one was correct. He, however, did not mention any discomfort about the camera during the second interview.

In the third interview, he did not explicitly state any discomfort, but he seemed to be very cautious when talking about some of the materials from the spin and quantum measurements course. For example, he recognized that the matrix on which he was working was very similar to the “Poly-Matrices” discussed in the spin and quantum measurements course but he said, “I shouldn’t have said anything (laughter)-I have to talk about it.[…] I shouldn’t talk about that” [I.3].

In the first interview Milo explained how to find the magnitude of a vector first using the Pythagorean Theorem and then by the dot product. He was also asked to explain what a dot product represented geometrically. After he explained that it represented the length of the projection of the vector *a* onto vector *b* or *b* onto *a* after drawing the vectors *a* and *b,* he was asked if he usually thought of the dot product geometrically.

> **M:** *Yes, I actually don't think about this [looking at the whiteboard containing the drawing of vectors.]*
> **I:** *What do you think about?*
> **M:** *I just remember, honestly I remember the formula for it and how to do a dot product in the sense of you adding the first terms and add, or add the multiplication of the first terms and second terms. I don't think in pictures.*
> [I.1, lines 117-123]

In the second interview he was asked about his experience in the linear algebra course and if eigenvalues and eigenvectors had been introduced before the end of the course. Milo stated that eigenvalues and eigenvectors were the last topic covered in the course but he said he did not attend the classes because he did not like the course. He stated that it was not the subject matter but the presentation of the topics that did not fit his learning style. He preferred more active engagement and interaction as in LAW.

During LAW, Milo was active in both whole class discussions and small group activities. He answered the questions and posed ideas during the whole-group discussions and helped his group members when working on activities (Classroom Observation Notes, LAW). During the second day of LAW, the students were given a matrix and asked to use it as an operator on a group of five vectors. Milo's group began to discuss why Professor Clayton might have chosen the wording "to operate on the vectors with the matrix" instead of "multiplying the vectors with the matrix." Milo decided to ask Professor Clayton, and she said that she wanted them to think of the matrices as an operator that transforms the vector to another one. The researcher observed that Milo's group was one of the few groups who had noticed the wording of this particular question.

The following section describes the repeating ideas observed in Milo's data as they fit under the four chosen goals of LAW and winter term physics courses. Then, Milo's attempts to use his experience from LAW and courses he took after LAW at the second and third interviews are also analyzed and described by implementing the Actor-Oriented Transfer (AOT) framework as they fit under the four chosen goals of LAW and winter term physics courses.

Milo was one of the few students who took both the matrix and power series methods course and a linear algebra course prior to the winter term. He and Crosby had a similar background, and the researcher wanted to analyze the data from both of them as well as from the other students with different backgrounds.

Goal 1: Matrix representation

As reported previously, Professor Clayton expected her students to view matrices as linear transformations during the winter term courses, and for this reason, she usually used the activity, linear transformations, on the second day of LAW. In this section Milo's view of matrices before and after LAW as they repeated during the interviews are described and the analysis of whether Milo uses his experience from LAW and winter term physics courses at the second and third interviews is reported.

<u>Ideas on Matrix Representation:</u> Only one idea related to matrix representation seemed to be repeated in Milo's data, and it was that a matrix represents a linear transformation. Even though Milo stated that matrices represented vectors and a

system of equations, these two ideas were only observed in the first interview and were not repeated again in other interviews.

At the first interview, Milo was asked to tell everything he knew about matrices. He said a matrix was a way of representing multiple vectors. He, however, decided to change his description and said, “Um, oh, actually it is better described as a system of equations, I think” [I.1, line 171]. He gave an example by using variables. He created a system of equations and represented it with a matrix. He concluded that through some matrix operation, one could figure out a solution for the system and matrices made it easier to solve the system. He was then asked what the solution represented. He stated that the answer found would satisfy the equations simultaneously and it was a set of solutions, rather than only one. He was then asked what the equations and solution represented geometrically. He said with his example, a three-by-three matrix, the equations represented planes and the answer would represent the point where all three planes come together. Milo was asked if he had learned about these ideas in the matrix and power series methods course he had taken over the summer term or in some other course. However, he did not answer the question, and he kept talking about the geometric meaning of a matrix representation. He seemed uncomfortable talking about the matrices representing vectors. He also seemed to connect the idea of a system of equations and vectors represented by a matrix.

I: *Ok, so this is an example of a matrix. You were mentioning that it represents a system of equations or vectors. So, is that something you learned from the matrix and power series course, or linear algebra course, or is that how you remember it?*

M: *Well, I remember the rows and columns being referred to as row vectors and column vectors. Um, and I suppose you can think of the vectors as living on the plane represented by the equations. Maybe, so if you can get those three vectors to cross at one point, that is your solution, but I am not really comfortable with calling them vectors because I do not have a good way to visualize them, at least not yet.*

[I.1, lines 188-195]

At the first interview, he was asked one more time if he had any drawing or pictures or a geometric way of thinking of a matrix while he was trying to find eigenvalues and eigenvectors of a matrix. He said that he had thought about a matrix as either a system of equations or vectors and had no pictures in his mind. Since these ideas were only observed in the first interview, they did not become repeating ideas.

The only repeating idea that was observed in Milo's data was the "matrix as a linear transformation" idea. Milo was asked to talk about linear transformations at the first interview. He said, "If there is some kind of matrix that is a linear transformation, you can multiply the vector by that matrix and it will transform the vector" [I.1, lines 281-282]. He did not mention a matrix representing a linear transformation prior to this moment, and it came up when he was asked to talk about linear transformations. He was asked to find eigenvalues and eigenvectors of a linear transformation L defined from R^2 to R^2 which reflected vectors over the x-axis. Milo started to draw some vectors and their images under this transformation and proposed that the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ represented this linear transformation. He later checked to see if he was correct by multiplying the matrix with the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ and decided that his proposed matrix was correct.

Similar ideas were observed at the second and third interviews. At the second interview, he was given the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ and first asked what he could tell about this matrix. He immediately referred to the LAW activity which was done on the second day and calculated the determinant of the matrix to decide what the matrix did. He could not decide if it was a rotation or a reflection matrix. Then to check to see which one it was, he multiplied the basis vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ with the matrix and then drew the initial and transformed vectors. He concluded that it was not a rotation matrix, but was not sure if it was a reflection. However, overall it did seem that he treated the matrix as a linear transformation rather than a system of equations.

At the third interview when he was given eigenvalues and eigenvectors of a matrix M and asked to talk about this matrix M, he proposed that M was not a rotation matrix and provided reasons relating to what would happen to vectors if M was a rotation. He then used the eigenvalue equation $A\vec{x} = \lambda\vec{x}$ to figure out the entries of the matrix M. He was asked if the calculated matrix M was a reflection.

I: *OK. You said something about it being a reflection, right here prior to the calculation. How can you check if that idea is correct?*

M: *OK, well, I can act this matrix on a vector and see what happens to it. So let's take 1 0. So, oops, 1 0 acting on and that would give me mirrored 1. So I have -- this vector turns into this vector. So.*

[I.3, lines 163-168]

It is interesting to note that Milo was using the linear transformation language, "act the matrix on a vector" as he explained his thinking. It seemed that Milo treated the matrices as a linear transformation before and after the LAW. He was aware of the system of equations representation prior to the LAW; however, he did not mention this representation again during the second and third interviews. He seemed to be aware of matrices representing linear transformations prior to LAW; however, he did not prefer to use this description as his first choice at the first interview.

<u>Actor-Oriented Transfer Episode Related to Matrix Representation:</u> The second and third interviews data were analyzed by implementing the actor-oriented transfer framework. The episodes which have evidence of AOT were then categorized according to these four chosen goals of the courses. The following episode describes the AOT of Milo that seemed to be related to the first goal. The episode is described briefly; then, the evidence of AOT is discussed.

<u>*Episode 1.*</u> This episode was observed close to the beginning of the second interview. Milo described what eigenvalues and eigenvectors were and gave an example. Then he was asked to talk about the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$. Milo looked at the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ and immediately said he was thinking about the things they did with "these." He was asked to explain further what he meant.

I: *What do you mean by that?*
M: *Well, I can tell you the determinant. Um, which is negative one minus three, so negative four. And we did talk about*

what it means when the determinant was negative. I can't remember if it was a reflection or a rotation.

I: *How can you check that?*

M: *Well, I can just plug in the vector one zero, or any vector or multiple vectors and see what happens.*

[I.2, lines 73-79]

He multiplied the basis vectors of R^2 to see what happened to them, and he also drew the initial and transformed vectors. He was not sure what the matrix did after having the sketches of the vectors. He decided that it was not a rotation matrix because the angle the vectors were rotated about seemed different in both situations, and a matrix could not rotate one vector by pi/2 radians and another one by pi/4. He conjectured then that it was a reflection matrix.

<u>AOT Analysis of Episode 1.</u> It was observed that when Milo looked at the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$, he seemed to have made a connection between the interview question and his experience with the activity called linear transformations that was done on the second day of LAW. Even though he did not explicitly say which activity or which day of the LAW, he said, "I am thinking about the things we did with "these," and then he calculated the determinant of the matrix to decide what kind of transformation it represented. According to the researcher's class observation notes, the determinant idea was from the second day of LAW during the linear transformations activity.

One of the goals of this activity was to show students that determinants help to decide what a matrix represents as a linear transformation. Milo seemed to connect to this activity when he saw the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$. He calculated the determinant to determine what the matrix did; however, he could not recall what a negative determinant represented, a rotation or a reflection. He did not explicitly state that the matrix represented a linear transformation, but his decision to multiply the matrix with some vectors to see what happened to the vectors indicated that he was aware of this representation.

His approach to this problem seemed to be developed from the ideas done in LAW. He seemed to reorganize his experience of this particular activity to answer the interview question, and for this reason the episode constitutes evidence of actor-oriented transfer.

Goal 2: Finding Eigenvalues and Eigenvectors

Another goal of LAW was that students are expected to be fluent in finding eigenvalues and eigenvectors of a matrix or an operator. In this section the repeating ideas that describe Milo's fluency in finding eigenvalues and eigenvectors are discussed. An episode exploring actor-oriented transfer is also described after the repeating ideas.

<u>Ideas on Finding Eigenvalues and Eigenvectors:</u> Two repeating ideas seemed to describe Milo's approach to finding eigenvalues and eigenvectors. These repeating ideas were referred to as the characteristic equation and the eigenvalue equation since Milo kept mentioning these words as he used them in his calculations of finding eigenvalues and eigenvectors.

At the first interview, Milo was asked to describe eigenvalues and eigenvectors. He stated that he could "remember a couple of general things" and he wrote the eigenvalue equation $Ax = \lambda x$; he explicitly said that A was a matrix and λ was the eigenvalue. He said there was an eigenvector that associated with "that," but it was not clear to what "that" referred.

He was asked to give an example after his description, and he then started to talk about the characteristic equation. He said there was something called the characteristic equation, but he could not recall it. He said that he had never gotten a "concrete feel" for what it really meant, and it was not attached to anything in his head. It was just "this thing that we had written down on a little formula," and he had used it on the test of the matrix and power series methods course. He said that he had forgotten all about it.

The characteristic equation idea came up again towards the end of the first interview when he was talking about finding the eigenvalues and eigenvectors. He recalled that the characteristic equation made the calculations really simple.

On the pre-quiz, Milo was able to find eigenvalues and eigenvectors of a two-by-two matrix correctly. Even though he did not explicitly state which equation was the characteristic equation, he used it to find eigenvalues. It was also observed that he used the modified version of the eigenvalue equation $(A - \lambda I)\vec{v} = 0$ to find eigenvectors.

The post-quiz was given on the last day of LAW, and Milo found eigenvalues and eigenvectors of a complex two-by-two matrix correctly. He used the eigenvalue equation $A\vec{v} = \lambda\vec{v}$ to find the eigenvectors this time.

At the second interview, he seemed to recall more about finding eigenvalues and eigenvectors, and again he found them correctly. He again mentioned both the characteristic and the eigenvalue equation. He wrote the eigenvalue equation when he started to find eigenvectors.

> ***I:*** *One thing that I realize here that you have this equation [pointing at $A\vec{v} = \lambda\vec{v}$]. Is this what you use all the time in your calculation of eigenvectors?*
>
> ***M:*** *No, there is a characteristic equation, which is derived from that I think. Yeah, because there is some kind of funny multiplying something by the identity matrix, and I don't remember how it was derived. I think it was pretty simple, but [wrote det(A-λI)=0], solving this for lambda gives you the eigenvalues. I use this [pointing at $A\vec{v} = \lambda\vec{v}$] just to remind myself how to solve for the eigenvector. Really, I should do that every time because I always end up staring at the ceiling and realized I should just write that down and not have to think so hard.*
>
> [I.2, lines 185-193]

He said he knew the characteristic equation was derived from the eigenvalue equation. He also explicitly stated that the characteristic equation was used finding eigenvalues and the eigenvalue equation helped him to find eigenvectors.

Similar ideas were observed at the third interview. He again found the eigenvalues and eigenvectors of the given matrices correctly. However, he was very sick at the time of the third interview, and he seemed to be having problems recalling some of the ideas.

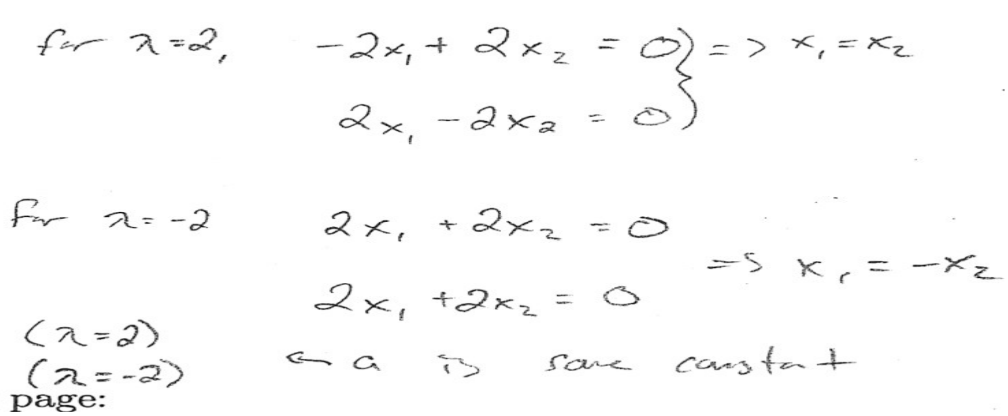
> ***I:*** *How about the eigenvectors in this situation? What are the eigenvectors associated with each eigenvalue? How many eigenvectors are there?*
>
> ***M:*** *I guess I'll have to… hang on a second. I'm sick. My brain is not working as well as I would like it to. Even the characteristic equation is kind of escaping me. A minus lambda I and determinant of that equals zero. OK. I am*

> *kind of on track. That is just for finding eigenvalues. A times the vector gives me lambda times the vector. So [wrote Av =λv and calculated eigenvalues and eigenvectors]*
>
> [I. 3, lines 41-46]

Milo seemed to be fluent in finding eigenvalues and eigenvectors after the first interview. On the pre-quiz he used the modified version of the eigenvalue equation to find the eigenvectors, but after LAW he used the eigenvalue equation to find eigenvectors. It seemed that he was aware of the connection between the characteristic equation and the eigenvalue equation.

<u>Actor-Oriented Transfer Episode Related to Finding Eigenvalues and Eigenvectors</u>*:* The second and third interviews' data were analyzed by implementing the actor-oriented transfer framework. The episodes which have evidence of AOT were then categorized according to the goals of the courses. The following episode describes the AOT of Milo that seemed to be related to the second goal. The episode is described briefly; then, the evidence of AOT is discussed.

<u>*Episode 2.*</u> On the pre-quiz, Milo used a modified eigenvalue equation $(A-\lambda I)\vec{v}=0$ to find eigenvectors as seen in Figure 4.1.



for $\lambda = 2$, $\left.\begin{array}{l} -2x_1 + 2x_2 = 0 \\ 2x_1 - 2x_2 = 0 \end{array}\right\} \Rightarrow x_1 = x_2$

For $\lambda = -2$ $\begin{array}{l} 2x_1 + 2x_2 = 0 \\ 2x_1 + 2x_2 = 0 \end{array} \Rightarrow x_1 = -x_2$

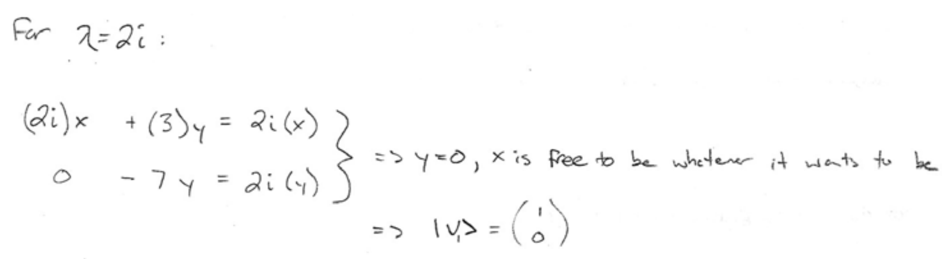
$(\lambda = 2)$

$(\lambda = -2)$ ← a is some constant

page:

Figure 4.1 Milo's pre-quiz

On the post-quiz, Milo used the eigenvalue equation $A\vec{v} = \lambda\vec{v}$ to find eigenvectors as seen in Figure 4.2.



For $\lambda = 2i$:

$\left.\begin{array}{l} (2i)x + (3)y = 2i(x) \\ 0 - 7y = 2i(y) \end{array}\right\} \Rightarrow y = 0$, x is free to be whatever it wants to be

$\Rightarrow |v_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Figure 4.2 Milo's post-quiz

In both the second and third interviews, he kept writing the eigenvalue equation when he was finding eigenvectors, and he explicitly used it.

<u>*AOT Analysis of Episode 2.*</u> The researcher noticed on the pre-quiz that Milo was familiar with a modified eigenvalue equation to find eigenvectors prior to LAW, and he seemed to change it to the eigenvalue equation after LAW. He also kept writing the eigenvalue equation when he was asked to find eigenvectors, and he used the equation consistently during the second and third interviews.

He did not explicitly state why he decided to use the eigenvalue equation instead of *(A-λI)=0* after LAW, but it was observed that Professor Clayton had used the eigenvalue equation as she worked on an example showing how to find eigenvalues and eigenvectors of a matrix on the fourth day of LAW. He began to use this approach on the post-quiz (which was conducted on the fifth day of LAW) and consistently used it throughout the interviews. He seemed to adapt this approach after its introduction on the fourth day. Milo did use his experience of finding eigenvectors from LAW in the second and third interviews, so this episode does constitute evidence of actor-oriented transfer.

Goal 3: The Eigenvalue Equation and Its Interpretations

One of the goals of LAW was to introduce the eigenvalue equation, $A\vec{v} = \lambda\vec{v}$ and its algebraic and geometric interpretations to students. Students were expected to learn both interpretations of the equation and apply them when appropriate.

In this section, Milo's apparent understanding of eigenvalues and eigenvectors and his reasoning while he worked with them before and after LAW are described through repeating ideas. Then, episodes exploring actor-oriented transfer are discussed.

<u>Repeating Ideas: The Eigenvalue Equation and Its interpretations</u>: Two repeating ideas were observed in Milo's data from interviews, pre- and post-quizzes, and the final examination. Milo described eigenvalues using an algebraic interpretation,

and he used a geometric interpretation for eigenvectors; these ideas were repeated in his data, as well.

The algebraic interpretation of eigenvalues was present at the first interview. Milo was asked to describe eigenvalues and eigenvectors. He stated that the eigenvalue was λ in the equation $Ax = \lambda x$ and recalled that eigenvectors were associated with the eigenvalues; in addition, he could not provide any further explanation. He did not mention the words "the eigenvalue equation" when he had written the equation $Ax = \lambda x$ and also throughout the first interview. When he was asked how the equation $Ax = \lambda x$ was used, he recalled the words "characteristic equation." When he was asked if the equation $Ax = \lambda x$ was the characteristic equation, he said, "No, there is something else" [I.1, line 217].

The researcher wanted to investigate further what else Milo had to say about the equation $Ax = \lambda x$. It was observed that he initially did not recall that the variable x was representing a vector. The researcher asked him what each variable represented, and then Milo said that he thought it was "maybe" a vector. Milo decided to look at an example to check the presented idea where A was a two-by-two matrix and x was a two-by-one vector.

> ***M:*** *Well you can multiply (pause 9 seconds) a matrix times a vector and if you do a two by one times, or a two by two times a two by one it gives you a two by one. So, something there.*
>
> ***I:*** *Ok.*
>
> ***M:*** *So, if you multiply a constant by a two by one, you still get a two by one. (whispers something to himself) Huh, I never put that together during class, I just now taught myself that. Unless I am completely wrong then I taught myself something wrong.*
>
> [I.1, lines 246-251]

After his "discovery," he did not know what to do next. It was still unclear if he knew that x was an eigenvector of A. To investigate further, he was then given a linear transformation that reflected the vectors over x-axis and was asked to find its eigenvalues and eigenvectors. He first drew some vectors and applied the linear transformation; then, he drew the transformed vectors. He also found the matrix representation of the linear transformation and checked if it was the correct one by operating it on some vectors. Once he was convinced that the matrix representation was correct, he started to think about how to find eigenvalues and eigenvectors. He tried to implement his "discovery."

> ***M:*** *So that would have to be some type of constant that I could multiple with this vector to get this outcome, I don't think there is one. I don't think I can multiple a constant by that and get this.*
>
> ***I:*** *Constant?*
>
> ***M:*** *Well, an eigenvalue… oh wait. I am trying to remember what, what exactly you know from the eigenvectors. They are vectors made up from eigenvalues. There is usually more than one. I am trying to remember if the eigenvalues themselves become the components of a vector or…I don't remember. From what I was trying to reason before, oh on that, an eigenvalue was just some constant, and so you would have to have a constant multiplied by the vector, sorry the vector multiplied by the constant would be the same effect as multiplying with matrix. Just a constant.*
>
> [I.1, lines 248-258]

The researcher tried to make him focus on the vectors in the equation $Ax = \lambda x$. She asked if he could find a vector such that when it was reflected over the x-axis it would only be a scalar multiple of itself. He looked at the vectors that he had worked on earlier and said none of them would satisfy the proposed idea. He was asked if he could find any other ones.

> ***M:*** *Oh, you are saying what I have written here won't work. But you're saying find a vector that will work with that.*

I: *Yeah, ok.*
M: *Well, if there was no x component, so if you have the vector zero one this transformation would be the same thing as multiplying this by negative one…*
[I.1, lines 370-374]

It was still unclear if he recognized that (0, 1) was an eigenvector associated with the eigenvalue -1, so the researcher decided to tell Milo that (0,1) was an eigenvector associated by the eigenvalue -1 and asked if there were any other eigenvalues and eigenvectors. He said that the vectors along the x-axis would not be affected by this linear transformation, and we could have any vector along the x-axis. After working on this problem, he was asked to describe eigenvalues and eigenvectors.

M: *Um, well I guess you can say that an eigenvalue is somehow a condensed version of a matrix, or a transformation, I suppose. So it is finding a simpler way to come up with things that only have specific vectors that apply to it. But it, the eigenvalue itself accomplishes the same thing that the transformation does. So you are finding a way to transform or alter a vector using a constant instead of a vector or a 1 by 1 matrix instead of an n by n. I can't come up with anything more intelligent than that. Sorry.*
[I.1, lines 413-418]

Milo stated the algebraic interpretation of the eigenvalue equation by only focusing on the eigenvalue. He did not include eigenvectors in his description at the first interview. A similar idea was repeated on the pre-quiz. He described eigenvalues as "Values of λ that satisfy det(A-λI)=0; they take the place of the a matrix operation by turning it into a scalar multiplication for a given eigenvector." For an eigenvector description, he wrote: "Vectors for which a matrix operation may be replaced by scalar multiplication."

On the post-quiz, he included a more comprehensive description of eigenvectors by using a geometric interpretation of eigenvectors. This was the first appearance of the repeating idea, geometric interpretation of eigenvectors. At the beginning of the second interview, he stated that the geometric interpretation of eigenvectors was one of the interesting things that he had learned which he had not known before LAW. He was asked to describe eigenvalues and eigenvectors.

> ***M:*** *Um, well based on last week an eigenvector is, well for a given matrix, we only talk about square matrices. So I will say for given for a square matrix an eigen vector is a vector who, whose direction is unchanged by the matrix operation. Um, and the matrix operation can be accomplished on an eigenvector also by simple scalar multiplication with the eigenvalue. Yes (both laugh).*
>
> [I.2, lines 49-53]

He later stated that an eigenvalue was the scalar, and he could reverse his definition of eigenvectors to describe eigenvalues. He used both algebraic and geometric interpretations of the eigenvalue equation in his description of eigenvalues and eigenvectors.

He repeated the geometric interpretation of eigenvectors again when he was asked if nonzero scalar multiples of an eigenvector were eigenvectors of the same operator. He said they would be eigenvectors and explained using the reflection transformation on which he was working earlier in the second interview.

> ***M:*** *Well, it is based on the understanding that the eigenvector lies on the line around the reflection, and it doesn't matter how long you make that vector it is still be on that line. Or you could reverse its direction for a negative scalar; it's still on the same line of reflection, so its direction does not change.*
>
> [Interview 2, lines 148-151]

At the end of the second interview he watched a clip from his first interview in which he commented that he did not have a "concrete feel" for what eigenvalues and eigenvectors meant, and he was asked what he thought about that clip.

> ***M:*** *Well, definitely the discussion in class that showed us the geometric interpretation of the eigenvector made it a lot easier to understand this little formula here [pointing to* $A\vec{v} = \lambda\vec{v}$ *] and there is certainly something concrete now that I have attached in my head. It is, I remember that interview and thinking that this is something I should know, and I know I have been taught this. But I don't remember, and I am pretty sure I am going to remember at least the very basic concept of what an eigenvector is, as far as I have been taught it. Maybe there is a lot more to it; I am sure there is, but, yes, I have something to hold on to now.*
> [I.2, lines 204-211]

He was again referring to the geometric interpretation of eigenvectors using the eigenvalue equation. The same ideas were repeated on the final exam. He stated that eigenvalues were scalars such that when eigenvectors were multiplied by them, they result in the same thing as when the operator was performed on the eigenvectors. He described eigenvectors as the vectors whose directions were unchanged under an operator; instead, they were stretched, shrunken, and/or reversed.

At the third interview, he was again asked to describe eigenvalues and eigenvectors. He provided the geometric interpretation of eigenvector, but this time he included an algebraic interpretation.

> ***M:*** *An eigenvector is a special vector that when operated on by our operator changes only by a scalar multiple. It can't change direction. It can become negative, or it can just stretch or [inaudible] along the same direction, but the direction essentially is unchanged. The scalar multiple by*

> *which is it stretched or shrunk is the eigenvalue. That is where the eigenvalue equation, the interpretation of that equation, in other words. Eigenvalues are special because in physics they correspond to physical values. They are measurables.*
>
> [I.3, lines 211-216]

He, however, this time described eigenvalues slightly different from the previous interviews. His description of eigenvalues seemed more geometric. He also stated some ideas from the physics courses he had taken.

Overall, it was observed that two ideas, algebraic interpretation of eigenvalues and geometric interpretation of eigenvectors were repeated by Milo as he described eigenvalues and eigenvectors. The first idea only focused on the eigenvalue, and it was a part of the algebraic interpretation of the eigenvalue equation. Similarly, the second idea was focused on the eigenvector and provided the geometric interpretation of the eigenvalue equation for eigenvectors. At the second and third interview, Milo included both geometric and algebraic interpretations of eigenvalues and eigenvectors. Finally, at the third interview, he stated that his descriptions of eigenvalues and eigenvectors were from the interpretation of the eigenvalue equation.

<u>Actor-Oriented Transfer Episodes Related to the Eigenvalue Equation and Its Interpretations:</u> Data from the second and third interview were analyzed by implementing the Actor-Oriented Transfer framework. The episodes constituting evidence of AOT were then categorized according to these goals, and the following episodes describe the AOT of Milo that seemed to be related to the third goal. The episodes are described briefly; then, evidence of AOT is discussed.

<u>*Episode 3:*</u> This episode occurred at the second interview, and some parts of it were discussed separately in Episode 1. Milo was given the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ and asked first what he could say about it. The researcher planned to follow up this initial question by asking him to describe what the matrix did and to find its eigenvalues and eigenvectors.

Milo calculated the determinant of the matrix as -4 and said, "I can't remember if it was a reflection or a rotation." He then implemented ideas from the second day of LAW (see Episode 1 for detailed discussion) and multiplied the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ with the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ and then drew the initial vectors and the transformed ones. After he completed this process, he said that it did not seem to be either a rotation or a reflection but that he was "pretty sure" that it was not a rotation. After stating his reason (see Episode 1), he thought that it could be a reflection matrix.

> **M:** *I can't think of any line about which it could be reflected that would, cause, to get this[pointing to one of the transformed vectors] out of this with a reflection there would have to be some line right in there[drawing a line in between vectors]. Oh well wait; that might actually be… maybe it is a reflection.*
> **I:** *OK*
> **M:** *I could see reflection based on my dotted lines with non scientific counts*
> [I.2, lines 85-96]

The researcher asked him how he could check that idea, and Milo said that he could find out the eigenvectors of the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$. He continued his explanation by saying, “Because if there is an eigenvector, and that is this guy [pointing at the dotted line on his drawing] then it would, I would say, it would prove to me anyways that there is a reflection with some kind of scalar multiple also” [I.2, lines 98-101]. He found the eigenvalues and eigenvectors of $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ and drew the found eigenvectors.

Before he found the second eigenvector, he was asked if he had any intuition about where the second eigenvector would be.

> ***M:*** *I think it will be orthogonal to this vector. Because if it is a reflection then an vector along the line of reflection will be a eigenvector, and any line orthogonal to the line of reflection will be an eigenvector because its direction isn’t changed, just the way its pointing. But it is said in the physics course; well, the north and south are facing the same direction.*
>
> [I.2, lines 122-126]

He concluded that his calculations and drawing supported his idea of reflection over the “dotted line” because one of the eigenvectors was on the “dotted line”.

<u>AOT analysis of Episode 3</u>. It was observed that when Milo looked at the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ he made a connection between the interview question and his experience with the activity called Linear Transformations that was done on the second day of LAW. Even though he did not explicitly say which activity or which day of LAW, he said, “I am thinking about the things we did with these,” and then he calculated the determinant of the matrix to decide what kind of a transformation

it represented. According to the researcher's notes, these ideas came from the linear transformations activity from the second day.

He could not recall if a matrix with a negative determinant represented a rotation or a reflection. However, he did not quit trying and continued to implement other ideas from the same activity. After some calculations and drawings, he conjectured that it could be a reflection over a line that he drew between the original vectors and the transformed ones. To check his idea, he seemed to reconstruct his ideas related to eigenvectors by focusing on the geometric interpretation of eigenvectors. He did not explicitly state how finding eigenvectors of the matrix would help him to figure out the reflection line. However, his later explanation about the second eigenvector indicated that he was indeed reconstructing the geometric interpretation of eigenvectors to answer the interview problem.

He seemed to create a similarity between the interview question and the activity done on the second day of LAW. Then, he seemed to reorganize the ideas from the experience with this activity to reason through the interview problem. In other words, his ideas of finding eigenvectors did not seem to be constructed spontaneously at the time of the interview. Because of these reasons, Episode 3 constitutes evidence of actor-oriented transfer. Since he implemented the geometric interpretation of eigenvectors when solving the problem, this episode is presented under the third goal.

Goal 4: Basis Vectors

One of the ideas that Professor Clayton expected students to understand after completing the winter term physics courses was that any vector could be linearly expanded with eigenvectors; in other words, she wanted them to realize that eigenvectors could be used as the basis vectors of the vector space.

It was noted that Milo did not mention eigenvectors being related to basis vectors during the first two interviews. He only mentioned something related to this idea on the final exam of the central forces course and the third interview. So, no repeating ideas were formed. However, this does not mean that he was not aware of this idea. For example, at the beginning of the third interview he was asked if the sum of two eigenvectors was again an eigenvector. He immediately said it was not and continued explaining that eigenvectors were linearly independent of each other, and so it was not possible to have an eigenvector which was linearly dependent of the other two. His reason seemed to imply that he was aware of the idea that eigenvectors could be linearly independent. To investigate his reasoning further he was asked to give an example. At first, he said he could not think of an example, but later he seemed to recall some ideas from the spin and quantum measurements course.

> ***M:*** *OK, we talk about spin, and if the two eigenvectors are eigenstates, then if you add them together, then all you have is a super-position of eigenstates. You don't have a whole new eigenstate.*
>
> [I.3, lines 17-19]

In his example, he only used the word "spin" and did not explicitly state how this idea was used in this course. However, he talked more about the superposition idea later in the interview when he was asked to give a mini-lecture on ideas related to eigenvalues and eigenvectors. He described them and then talked about how to find them. At the end of his lecture, he brought up the idea that any state of a system could be represented as the super-position of the eigenstates. He repeated twice that this was a very important concept. He was asked how this was possible, and he then started to talk about ideas related to basis vectors.

> ***I:*** *How is that possible?*
> ***M:*** *Well, the eigenvectors – this probably isn't always true – but the eigenvectors make up a spanning set for this base you are working in. It is analogous to – we just did Cartesian coordinate system. If you have i, j, k – any point in space can be represented as xi+yj + z k. So any point is a super-position of the three eigenvectors, in a sense. That is really important. I am not doing it justice. It is really important.*
> [Interview 3, lines 257-262]

He could not provide any further explanation or an example. It was possible that due to his illness, he could not think of an example.

On the final examination of the central forces course, Milo wrote "Eigenvalues represent possible measurements or allowed values. Eigenvectors or eigenstates are the basis of [orthonormal states] from which all states may be expressed" as his answer to how eigenvalues and eigenvectors are used in physics. He seemed to be aware that eigenvectors could be used as basis vectors prior to the third interview. He was asked to comment on his answer at the third interview. He said that the part where he wrote "the basis of [orthonormal states]" was redundant, and he probably did not mean that. Since the eigenvectors could be used as basis vectors, the idea was not repeated at more than one interview. It did not become a repeating idea for Milo. However, this does not imply that he was not aware of this idea. Also, there were no actor-oriented transfer episodes related to this particular goal.

Summary

Milo initially stated that matrices represented a system of equations or vectors; however, this idea was only seen in the first interview. At the first interview, he also mentioned that linear transformations could be represented with matrices. He continued to view matrices as linear transformations in the second and third interviews, as well. Since this type of representation of matrix was used consistently during LAW and the winter term physics courses, he seemed to adapt this idea while he was working on matrices in the interviews.

Milo seemed to be fluent in finding eigenvalues and eigenvectors. He initially could not recall how to find them, but he mentioned that there existed an equation called the characteristic equation that helped him to find eigenvalues. He could not recall the equation but stated that it was different than the equation $Ax=\lambda x$ he had written at the first interview. He was not sure how the equation $Ax=\lambda x$ was used to find eigenvalues and eigenvectors. He correctly found eigenvalues and eigenvectors at the pre-test, prior to LAW. At the second and third interviews, he used both the characteristic and the eigenvalue equation, and he was aware of the idea that the characteristic equation was derived from the eigenvalue equation. It was also noticed that he adapted Professor Clayton's way of finding eigenvectors by using the eigenvalue equation after being introduced to it on the fourth day of LAW. Throughout the remainder of the data gathering portion of the study, Milo consistently used this approach.

He seemed to develop an algebraic interpretation of eigenvalues at the first interview. He continued to describe eigenvalues algebraically, and at the third interview he included a geometric interpretation of eigenvalues. He stated that the geometric interpretation of eigenvectors was very interesting, and he consistently used this idea after being introduced to it in LAW. He also mentioned its algebraic interpretation using the eigenvalue equation. Since the algebraic and geometric interpretations of eigenvalues and eigenvectors are derived from the eigenvalue equation and its interpretations, Milo seemed to develop these ideas. It was also observed that he implemented the geometric interpretation of eigenvectors while he was trying to figure out what a matrix represented. Thus, matrices representing linear transformations and geometric interpretation of eigenvectors seemed to be connected for Milo.

Milo did not mention that eigenvectors could be used as basis vectors until the third interview. He was aware of this idea and mentioned that it was very important to know that any states could be represented as a superposition of eigenstates. However, he did not provide any examples from his physics courses when he was asked to do so.

Overall Milo seemed to develop a comprehensive understanding of the geometric interpretation of eigenvectors, and he was fluent in finding eigenvalues and eigenvectors after the LAW and winter term physics courses.

Crosby

Crosby was a junior physics major with a minor in Naval Sciences who expected to graduate spring term of 2009. During his first two years in college, he took a calculus sequence (two courses), a vector calculus sequence (two courses), differential equations, and matrix and power series methods courses. He was also enrolled in a linear algebra course during the spring term of his sophomore year, simultaneously with the matrix and power series course. However, he struggled with the linear algebra course and withdrew, retaking the linear algebra course during the fall term of his junior year. Table 4.2 shows the physics-related courses in which he was enrolled during this study.

Table 4.2 Courses Crosby Took During the Study

Course Name	**Term Taken**
Linear Algebra Course	Fall Term 2007
3 Modules of Physics Courses	Fall Term 2007
Analog and Digital Electronics in Physics	Winter Term 2008
3 Modules of Physics Courses	Winter Term 2008
Naval Sciences	Winter Term 2008
3 Modules of Physics Courses	Spring Term 2008
Computer Interfacing in Physics	Spring Term 2008
Classical Mechanics in Physics	Spring 2008
Energy Alternatives	Spring 2008
Naval science	Spring 2008

Crosby mentioned during the first interview that matrices were hard for him and “it wasn’t his math.” In the second interview, he said that linear algebra was “pointless math” because according to him, he never used linear algebra in any of his courses. He also mentioned in that interview that he had received a B- in the linear algebra course.

Crosby worked well with others in class. He contributed to group discussions by asking questions and trying to help the group members whenever he could (Classroom Observation Notes of Linear Algebra Week (LAW)). He, however, did not contribute to whole-class discussions. He did not volunteer to present the group work on the board during the LAW week even though he was the only student who had completed two courses on linear algebra topics in his group.

During one of the group activities, the students were asked to find eigenvalues and eigenvectors of a given matrix after being introduced to the topic. One of Crosby's group members commented on the steps of finding eigenvectors saying he did not understand the reason of getting a free variable when finding eigenvectors. It was this particular student's first time learning about eigenvalues and eigenvectors. He asked Crosby for his input. Crosby said he did not know why they were doing that and stated that he just followed the steps (Classroom Observation Notes, Day 4, LAW).

Crosby showed a similar trend in the interviews. He repeatedly said that he did not know. He was not afraid to talk, but he did not seem to be confident in his answers. He seemed to be cautious with his statements and used phrases such as "I don't know", "I have no idea", and "I guess" frequently. After stating answers, he asked if he was correct. For example, during the first interview the idea of dot products arose when he was trying to explain matrix multiplication, and he called it "dotting." Then, he was asked to explain what dotting represented geometrically. He was able to give a formula which was presented as the geometric representation of the dot product in one of the physics courses he took earlier during the fall term.

C: *It is really, dotting it, isn't it [cut by the interviewer]*
I: *I am not familiar with dotting? What do you mean by dotting?*

C: *It is four and two times five, one and one. So the first entry of the row is times by the first entry of the column, so two times one, plus four times one. I think it is dotting. I don't know.*
I: *Dotting. Do you use that dotting idea somewhere else?*
C: *Yeah, it is how you deal with, like, multiplying vectors. You can dot vectors to get a scalar value. That is just where you are multiplying all of their components, like all i hat components together and adding them to j hat components that are multiplied together.*
I: *What does that multiplication represent geometrically, do you know?*
C: *You dot two – it is their, I want to say it is the A dotted with B equals the magnitude of A times the magnitude of B times the cosine between them, the cosine of the angle between them and that is – I can't remember – projection. So it is this distance right here, I think. I think.*
I: *OK, not confident?*
C: *Not confident.*
I: *Why not?*
C: *Well, there are just a lot of little things to remember and it is really easy to get them confused, so I might be thinking of something completely different.*
[I.1, lines 173-191]

Even though he stated the correct formula and the conceptual idea (projection), he was not confident in his answer. He did not check the formula he had suggested. He thought that he might be confusing it with other ideas. He stated that "there were a lot of little things to remember" and this seemed to suggest that he may have memorized the formula as the geometric representation of the dot product and had forgotten why it represented projection. It is also possible that he may have not understood the connection between the formula and the geometric representation.

The following sections first describe the repeating ideas related to the goals of the linear algebra review week (LAW) and winter term physics courses as they occur in Crosby's data. Then, Crosby's attempts to use his experience from LAW and courses he took after LAW at the second and third interviews are analyzed and described by implementing the Actor-Oriented Transfer (AOT) framework.

Crosby was one of the few students who took both the matrix and power series method course and the linear algebra course prior to the winter term physics courses. His background makes him one of the unique students in the study. For this reason, Crosby's experience is described as one of the case studies of this study.

Goal 1: Matrix representation

As reported previously, Professor Clayton expected students to view matrices as linear transformations during the winter term courses, and she said she usually used an activity called linear transformations on the second day of LAW. In this section Crosby's view of matrices before and after LAW through repeated ideas during the interviews are described and also an investigation of whether Crosby used his experience from LAW and winter term physics courses at the second and third interviews are discussed.

<u>Ideas on Matrix Representation:</u> Two repeating ideas were observed in Crosby's data. First, during the first and second interviews, he seemed to view matrices as representing systems of equations. Second, especially after LAW, Crosby talked about matrices operating or acting on vectors, which could imply that he was aware that matrices represented linear transformations.

At the first interview when Crosby was asked if he knew what a linear transformation was he said, "It is like a matrix you can use, like elementary matrix, that you use to change or transform the matrix that you have, which is a system of linear equations” (Interview 1). Crosby seemed to assume that linear transformations were not the same as matrices, but they were something that changed the matrices which represented systems of equations. He later stated explicitly that one “could think of matrices as a way of expressing systems of equations” (Interview 1). He seemed very comfortable with the idea and created an example. In his example two lines that intersected at the point (0,1) (See Figure 4.3).

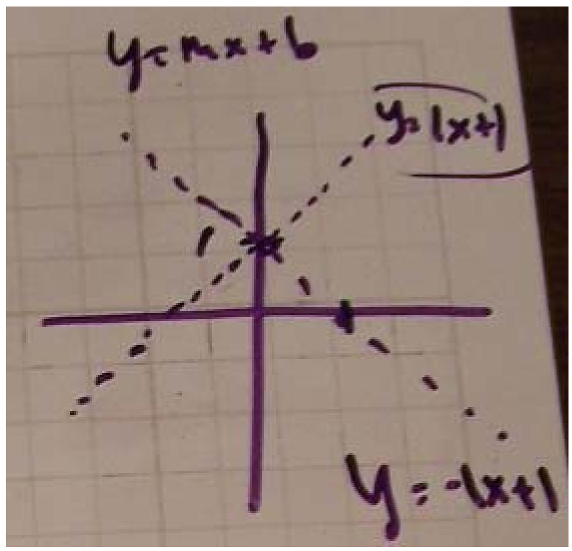


Figure 4.3 Crosby’s Example of a System of Equations-1

He then made a system of equations from these lines and represented the system with a matrix. He applied elementary row operations to reduce the matrix, and at the end he got x=0 and y= 1.

During the first interview he was also given the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and asked what it represented. He first said that the matrix was equal to the identity matrix multiplied by some elementary matrices. He stated that his linear algebra course they had talked about the elementary matrices. Then without being asked, he wanted to figure out what the matrix represented geometrically. While he was working on the problem, he said they have done similar questions in his matrix and power series methods course, but he could not remember how it was done. To decide what the matrix did geometrically, he multiplied the matrix with the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ and stated that geometrically the matrix transformed the vectors by making all new x values into –y values and the new y values into x values, and hence, he got the transformed vector $\begin{bmatrix} -y \\ x \end{bmatrix}$. It seemed that he was also aware that matrices transformed vectors.

Similar ideas were observed at the second interview. The second interview started with Crosby answering the question, "Tell me one of the interesting things you have learned during the linear algebra week that you did not know before." He said that the activity in which students related the determinants of matrices to what they did was very interesting. He never knew that before, and prior to that, he had thought determinants were only used to decide if matrices were singular.

The interview continued with the investigation of what the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ did and what were its eigenvalues and eigenvectors. It was observed that he was referring to the matrix as an operator.

C: *What does this [pointing to the matrix] do?*
I: *Yeah.*
C: *Well, it is a matrix that operates on it, it could operate on a vector and change it somehow.*
I: *Do you know how it will change?*
C: *No.*
I: *Any guesses? Can you [cut off by Crosby]*

C: *I think if I just multiply it like just in this one, I just say [writing] by some random vector,* $\begin{bmatrix} x \\ y \end{bmatrix}$. *Then I get* $\begin{bmatrix} 1x+\sqrt{3}y \\ \sqrt{3}x-1y \end{bmatrix}$ *so that would be just how it changed it.*

[I.2, lines 93-102]

He did not find the determinant of the matrix to decide what it did, but algebraically calculated how a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ would change. He did not state what happened to this vector geometrically either. He then continued to find eigenvalues and eigenvectors. After finding the eigenvectors, he proposed that eigenvectors formed lines perpendicular to each other, and the matrix puts all the vectors on one of these lines, but he was not sure. He was asked to check his conjecture. He decided to use the system of equations representation of a matrix and created two lines instead of graphing the lines from eigenvectors:

I: *How did you conjecture that idea?*
C: *I guess we could use this:*

$$\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1X+\sqrt{3}y \\ \sqrt{3}X-1y \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

$$y = \frac{-1}{\sqrt{3}}X + C_1$$
$$y = \frac{\sqrt{3}}{1}X + C_2$$

Figure 4.4 Crosby's Example of a System of Equations-2

and get it to some constant, which we can say is 0 because we want it to go through the origin. But we could change the intercept and it would still be parallel. So then we get an equation, y equals [inaudible], mx plus b, so minus 1 over root 3 plus c sub one and then this one would be root 3 over 1 x plus c sub 2. So I guess in that case, a slope of -1

> *over root 3, and, that is this way, and 1.7. OK, this one [calculating the slope of the line created by one of the eigenvectors] would be -1.7, 1. So those aren't the same at all. So maybe I was just completely wrong. I seem to remember that, maybe I'm confusing it with scrinching matrices.*
>
> [I.2, lines 170-177]

It seemed that he thought the lines formed by the eigenvectors would coincide with the lines obtained from the system of equations. Once he noticed that they did not, he mentioned that he could be confusing the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ with a "scrinching" matrix (a matrix with determinant zero was called a "scrinching" matrix in LAW). He started to check his calculations and eigenvectors he had found and did not talk about the matrix for a while. Then, he was again asked to investigate his ideas proposed earlier. He said he could multiply six different vectors with the matrix to check his idea. After he was done with the multiplication, he drew the original and the resultant vectors on a Cartesian plane. He then recognized that the matrix reflected and scaled vectors. He was convinced with his answer that the matrix reflected and scaled; however, he was still bothered that the lines from the system of the equation did not match the lines he had formed from eigenvectors towards the end of the interview.

Crosby no longer talked about matrices representing systems of equations by the third interview. Then, he stated that a given matrix was rotating vectors, but he did not explicitly state that he viewed matrices as linear transformations.

On the post quiz and final exam, Crosby used terminology indicating that he was aware of the linear transformation representation of matrices. On both exams he stated a matrix “operated on the vector” or “acted on the vector”.

Overall, Crosby viewed matrices as both representing systems of equations and as linear transformations. It seemed, however, that he was more comfortable, at least initially, with the system of equations representation.

Actor-Oriented Transfer Episode Related to Matrix Representation: The second and third interviews data were analyzed by implementing the Actor-Oriented Transfer (AOT) framework. The episodes which have evidence of AOT were then categorized according to the goals. The following episode describes the AOT of Crosby that seemed to be related to the first goal. The episode is described briefly; then, the evidence of AOT is discussed.

Episode 1. This episode was observed during the second interview in which Crosby investigated what the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ represented and how vectors were changed when the matrix acted on these vectors. The investigation continued through the whole interview with some interruptions to find eigenvalues and eigenvectors of the matrix. The ideas that were only related to matrix representation are shared here.

Crosby’s second interview took place two weeks after LAW while he was enrolled in the first winter term physics course, Spin and Quantum Measurements. The second interview started with his answering the question on one interesting thing that he had learned during LAW which he did not know before.

C: *Definitely the activity we did with – I think there were six, maybe eight matrices. We just tested them on a couple of set of vectors to see what they did and then we related back to their determinants. That was interesting.*
I: *What was interesting about that?*
C: *I have never seen it before, so I didn't know – I thought determinants were only to show if a matrix is singular. I didn't realize you could find out more about it with a determinant.*
I: *OK. What can you tell me more about it?*
C: *A determinant of 1, it rotates, I think. If it is -1, it reflects, maybe. Then if it is some other value, it scrinches, what she [the professor] called it. That's all I know. There was something about 0 but I don't remember.*
I: *Anything else?*
C: *No.*
[I.2, lines 60-72]

After his explanation, the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ was given to him, and he was asked what it did to vectors. He said it changed the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} 1x+\sqrt{3}y \\ \sqrt{3}x-1y \end{bmatrix}$. Notably, he did not implement the one interesting thing he had learned during LAW.

After finding eigenvalues and eigenvectors of the matrix, he proposed that eigenvectors formed lines perpendicular to each other, and the matrix put the vectors on one of these lines. When he was asked to check his conjecture, he went back to using the system of equations representation of a matrix and created two equations of two lines. He found the slopes of these lines and compared them with the slopes of the lines formed by eigenvectors. It seemed that he thought the lines formed by the eigenvectors would coincide with the lines obtained from the system of equations. Then, he proposed that the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ could be a "scrinching" matrix. He was asked to check his proposal, and then he decided to multiply an assortment of vectors with the matrix and drew the original and the resultant vectors on a Cartesian plane. Two of these vectors were eigenvectors of the matrix that he had found earlier. As he mentioned at the beginning of the second interview, they had done a similar activity in class. When he was done with his calculations and drawing, he stated that the matrix was not "putting the vectors on the line" as he had proposed. He said the matrix seemed to reflect and scale vectors. Then he found the axis of reflection, and he stated that he was convinced of his answer. However, he said he was bothered that the lines from the system of equations did not match with the lines formed from eigenvectors. Towards the end of the interview when he was asked what else he had learned during LAW, he started to recall more ideas from LAW related to this particular idea.

> ***C:*** *Let's see, we went over bra and kets, daggers, conjugates, transpose. She specifically said not to think of it as system of equations, which is exactly what I did there. So maybe that is why I am getting confused. That is all I remember.*
> ***I:*** *Not to think what in terms of system of equations?*
> ***C:*** *Matrices.*
> ***I:*** *What are you going to think of them in terms of then?*
> ***C:*** *Maybe multipliers. I guess she probably just meant that we won't use them as system of equations, but if it is multipliers or augmenters or something.*
> [I.2, lines 392-402]

He seemed to remember a discussion that happened during the second day of LAW. The discussion was about representing a system of equations with a matrix and what it meant for this matrix to have a determinant of zero. Professor Clayton then explicitly stated that matrices were also used as operators, where they act on vectors. She also said matrices represent linear transformation (LAW, day 2).

AOT Analysis of Episode 1: At the beginning of the second interview, Crosby indicated that the activity in which students found a relationship between the determinant of a matrix and what it did as a linear transformation was very interesting. This particular activity occurred on the second day of LAW. Crosby's group was asked to investigate the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. As a group they decided that the matrix rotated the vectors by $\pi/2$ counterclockwise; its determinant was 1, and all the vectors got changed by the matrix since they would all be rotated.

Even though Crosby thought this activity was interesting, he did not initially choose to use any of these ideas for the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$. Instead he multiplied the matrix with the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ to decide what the matrix did. If this piece of Episode 1 data were analyzed by using a traditional transfer research framework, one could conclude that he was not transferring the ideas from the activity to the interview question. On the other hand, the actor-oriented transfer framework suggests looking for evidence of personal connections that a student makes between his previous experience and the questions asked rather than focusing on one pre-determined piece of information being transferred or not. So, the analysis of Episode 1 explores the question of whether Crosby's reasoning in this episode has roots in his prior experiences.

Crosby's first interview revealed some of his other previous experiences with matrix representation which he may have been relating to during Episode 1. In the first interview, Crosby was asked a similar question. He was asked what the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ represented, and again he multiplied the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ with the matrix and stated that geometrically the matrix transforms the vectors by making all new x values into –y values and new y values into x values; hence, one gets the transformed vector $\begin{bmatrix} -y \\ x \end{bmatrix}$. He did not like this answer, and it seemed that he was looking for some other geometrical representation. “I feel like I should be able to do this one from having been in the matrix and power series methods course because I know this was on our final, one just like this”(Interview 1).

As seen in Episode 1, Crosby revisited the same idea. When he was asked what the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ did, he used the same process to answer the question and concluded that the generic vector will be transformed to $\begin{bmatrix} 1x + \sqrt{3}y \\ \sqrt{3}x - 1y \end{bmatrix}$. This could constitute evidence of actor-oriented transfer because Crosby seemed to bring in his previous experience with matrix representation, which was also observed at the first interview.

When he conjectured the matrix mapped all the vectors to the same line, he was asked to check his idea. Then, it was observed that he still did not use the ideas from the activity done on the second day of LAW, but instead he used the matrix representation of the system of equations. He wrote two equations from the matrix and found the slope of the lines. A similar approach was observed at the first interview, when he described what matrices represented. It seemed that in

Episode 1 he was reaching out to his particular experience and applying the similar ideas to the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ by creating lines.

However, his attempt to use the system of equations did not help with his conjecture. He, then, seemed to begin to recall ideas from the activity done during LAW. He first said he had some confusion related to the new idea he had learned, the “scrinching” matrix. He stated that the matrix scrinches when its determinant is different than 1,-1 and 0. So, according to what he remembered, he concluded that the given matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ was a “scrinching" matrix. He decided to check his idea by multiplying some vectors with the matrix.

Even though he did not state explicitly that the idea of trying different vectors came from the activity they had done in class, the way that he progressed through his solution was exactly the same way they did the activity in class. For example, they were asked to use different colored pens when they were drawing the original vectors and the transformed vectors, and he did the exact same thing during the interview. He used color green for the original vector and numbered them, and then used black for the transformed vectors and then numbered them respectively.

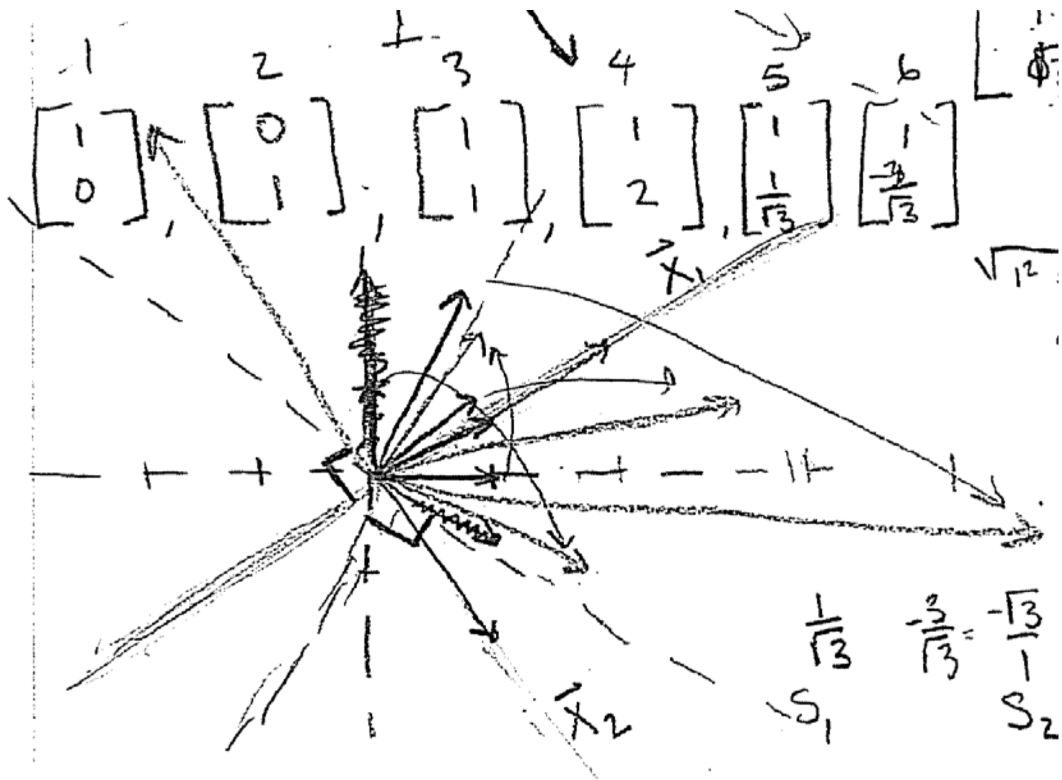


Figure 4.5 Crosby's vectors and graphs

This part of Episode 1 could constitute evidence of actor-oriented transfer. He was reaching back to his experience in the LAW week to answer the interview question. He, however, did not want to use this particular experience until the end of the interview. He, instead, wanted to use the idea of systems of equations. There could be a few reasons for this. He may have felt more comfortable using his pre-existing knowledge used many times before; or he may have not fully recalled the ideas presented with the activity during LAW, so he did not feel confident to apply it in an interview setting.

It may also be possible that the given matrix did not look similar initially to the ones in the class activity. The matrices used in the activity had integer entries and most of which were 0, 1 or -1. The given matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ did not look similar to any of the matrices from the activity. Having different entries for the matrix in the interview question gave it a different surface feature, but still the question was asking what the matrix did to vectors. As a result, the structural feature of the interview question was the same as the ones in the activity. It may be possible that the different surface feature caused Crosby to assume that these problems, the one at the interview and the ones from the class activity, were different. This could be a possible reason for his failure not to employ the ideas from the activity until the end of the interview.

In this episode it was also observed that he kept trying to remember what the professor had said in class to resolve the conflict he was having. He finally stated that the professor had told the students to think of matrices as multipliers, not as systems of equations. He seemed to remember a class discussion where the professor told them that the matrices also represented linear transformations, and they would refer to matrices as an operator in LAW and in subsequent physics courses.

Crosby seemed to implement the same ideas during the first and second interviews. It seemed that he was reorganizing some of his previous experiences with matrix representation to answer the questions in both interviews. He also referred to his experience during LAW while he worked on the second interview question. For all these reasons, this episode constitutes evidence of actor-oriented transfer.

Goal 2: Finding Eigenvalues and Eigenvectors

Another goal of LAW was that students should become fluent at finding eigenvalues and eigenvectors of a matrix or an operator. In this section the repeating ideas that describe Crosby's fluency in finding eigenvalues and eigenvectors are discussed. An episode exploring actor-oriented transfer is also described after the repeating ideas.

<u>Ideas on Finding Eigenvalues and Eigenvectors:</u> There were different ideas observed in Crosby's data relating to finding eigenvalues and eigenvectors; however, none of these ideas seemed to be repeated.

Initially, Crosby claimed that he "kind of" recalled how to find eigenvalues and eigenvectors using the determinant. However, he did not know how to find the eigenvectors once the eigenvalues were found, and he did not know what eigenvalues represented at the first interview. Crosby was enrolled in the linear algebra course during his first interview, and the topic of eigenvalues and eigenvectors had not been discussed at the time of the interview.

Crosby reported later that the linear algebra class worked for a week on the concept of eigenvalues and eigenvectors at the end of the fall term. According to him, they did not spend much time on how to find eigenvalues and eigenvectors in class, but maybe they did an example once for an assignment. He said their main focus was on "its use and proving it" [I.2].

When Crosby took the pre-quiz on the first day of LAW, his ideas on eigenvalue and eigenvectors seemed to be more developed than in the first interview; however, he still could not find eigenvectors of a two-by-two matrix.

Students were asked to find eigenvalues and eigenvectors of a two-by-two matrix $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$ on the pre-quiz. He was able to find the eigenvalues, and he attempted to find the eigenvectors using the eigenvalue equation. However, he only attempted to find one of the eigenvectors, and he could not find it correctly. It seemed that he was mixing up the two methods of finding eigenvectors. The eigenvalue equation $Ax=\lambda x$ and the equation of $(A-\lambda I)x = 0$ both could be used to find eigenvectors; however, it seemed that Crosby was using some parts from each equation (see Figure 4.6).

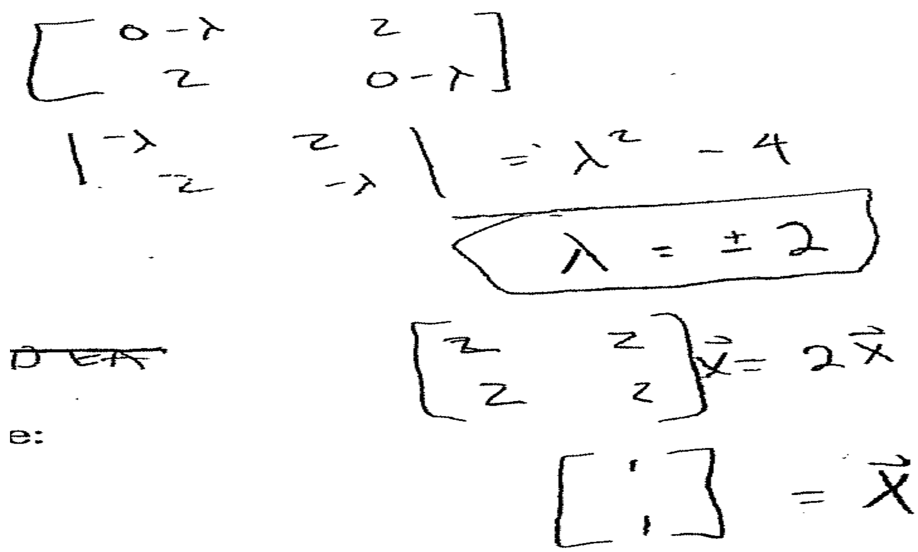


Figure 4.6 Crosby's Pre-Quiz

On the last day of LAW, students were given a post-quiz and asked to find eigenvalues and eigenvectors of a two-by-two complex matrix $\begin{bmatrix} 2i & 3 \\ 0 & -7 \end{bmatrix}$. Crosby was able to find eigenvalues correctly; however, he had made a sign error on one

of the eigenvectors. He had written $\left|v_2\right\rangle = \begin{bmatrix} 1 \\ -\frac{7}{3}+\frac{2}{3}i \end{bmatrix}$ instead of $\left|v_2\right\rangle = \begin{bmatrix} 1 \\ -\frac{7}{3}-\frac{2}{3}i \end{bmatrix}$.
It was observed that he used the eigenvalue equation to find the eigenvectors.

At the second interview, he again made a sign error on one of the eigenvectors when he was asked to find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$, although he later found his error. He used the eigenvalue equation to find the eigenvectors, and he seemed very comfortable using this equation when finding eigenvectors.

At the third interview, the researcher did not ask Crosby to find the eigenvalues and eigenvectors of a matrix, but he was asked to describe how one could find eigenvalues and eigenvectors. He was able to describe how they would be found using the characteristic equation and the eigenvalue equation; however, he did not state the names of the equations and said, “I immediately realized that I had forgotten everything I had learned, already – the second time around because we have all taken math 341 and physics which had applied the eigenvectors and eigenvalues” [I.3, lines 169-173].

Overall Crosby seemed to know how to find eigenvalues and eigenvectors, and at the second interview he seemed to be fluent. Since he was not asked to find eigenvalues and eigenvectors at the third interview but only asked to describe how to find them, no conclusion could be made on his fluency of finding them. Also, there were no actor-oriented transfer episodes related to this particular goal.

Goal 3: The Eigenvalue Equation and Its Interpretations

One of the goals of LAW was to introduce the eigenvalue equation, $A\vec{v} = \lambda\vec{v}$, and its algebraic and geometric interpretations to students. Students were expected to learn both interpretations of the equation and to apply them when appropriate.

In this section, Crosby's apparent understanding of eigenvalues and eigenvectors and his reasoning while he worked with them before and after LAW are described through ideas repeated at each interview. Then, an episode exploring actor-oriented transfer is discussed.

<u>Ideas on the Eigenvalue Equation and Its Interpretations:</u> One idea was repeated by Crosby as he described eigenvalues and eigenvectors. This repeated idea was directly related to the algebraic interpretation of the eigenvalue equation, so the researcher decided to refer to this idea as the algebraic interpretation.

At the beginning of the first interview, Crosby stated that he did not "remember much about" eigenvalues and eigenvectors, but he did remember "something about some multiple of some sort." At the end of the interview, he stated the same idea, but he said he did not know what that meant.

As previously mentioned, Crosby reported that in his linear algebra course the concept of eigenvalues, and eigenvectors were discussed at the end of the term and it was included on the final exam. They did not focus much on how to find them but mostly proved theorems related to eigenvalues and eigenvectors.

When Crosby took the pre-quiz on the first day of LAW, his description of eigenvalues and eigenvectors seemed to be more developed than in the first interview. When he was asked to define an eigenvalue, he wrote, “It’s a value that you multiply a vector by which does the same thing as if you had multiplied that specific vector by a specific matrix; Ax = λx” (Pre-quiz). He also stated that the eigenvector is the vector stated in this description. Crosby seemed to have started developing the algebraic interpretation of the eigenvalue equation prior to LAW.

On the last day of LAW, the students were given a post-quiz and again asked to describe eigenvalues and eigenvectors. Crosby’s description of eigenvalues and eigenvectors did not change much from the one he had given on the pre-quiz. He wrote, “It [eigenvalue] is a value that has the same effect when multiplied on a specific vector as when a certain matrix operates on that specific vector. $A|v\rangle = \lambda|v\rangle$.” and “It [eigenvector] is a vector that when operated on by a certain matrix yields the same result as if operated on by its eigenvalue” (Post-quiz).

He also had some ideas on how eigenvectors are used in physics. He said, “We use them in performing and investigating reflections and rotations and scrinches. We will apply them to quantum spin mathematics” (Post-quiz). He considered the linear transformations investigated during LAW as physics applications of eigenvectors rather than linear algebra topics.

In both quizzes, Crosby described eigenvalues and eigenvectors using his understanding of the eigenvalue equation which seemed to have evolved around the algebraic interpretation of the eigenvalue equation. The idea of using the equation and its interpretation was probably developed in the linear algebra course. He kept referring to the equation and its interpretation when he was asked to explain eigenvalues and eigenvectors in the second and third interviews and on the final exam. He also used the equation when he was reasoning through questions related to eigenvalues and eigenvectors at the second and third interviews.

At the second interview, he was asked to think back to the ideas he had learned in LAW and describe eigenvalues and eigenvectors. He once again provided descriptions for eigenvalues and eigenvectors that were similar to what he had written on the post quiz. The researcher asked him to find eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$. After finding the eigenvalues and eigenvectors, he was asked what they represented.

> ***C:*** *This is the vector that was used in this equation [pointing at the eigenvalue equation] with this eigenvalue, yields the same result as if it was used with the matrix. This is the vector that was used with the second eigenvalue, negative two.*
>
> [I.2, line 128-130]

He was also asked what the eigenvectors represented in relation to the given matrix. Crosby referred to the eigenvalue equation and its algebraic interpretation.

> ***C:*** *I'm not sure. Well, so statement [pointing to $Ax=\lambda x$] is saying that our original matrix when times by this, this specific – or this one – this specific vector, let's say, x1 is the same as our eigenvalue, two times by the same vector. So when this matrix is multiplied by this vector, it has to be the same thing as this matrix multiplied, or the scalar multiplied by the vector.*
>
> [I.2, line 199-205]

Even though he kept referring to the eigenvalue equation and its algebraic interpretation, there seemed to be a gap in his understanding of the equation and its algebraic interpretation at the second interview. For example, he found the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ and stated that the matrix reflected and scaled vectors. He was then asked what happened to the eigenvectors. He worked through the matrix multiplication to get the eigenvalue times the eigenvector. He graphed the resultant, the scaled eigenvector. He did not seem to realize that he could use the eigenvalue equation and its algebraic interpretation that he kept repeating earlier; but instead he did the matrix multiplication to get the answer of λ times the eigenvectors. He then graphed the eigenvectors and the scaled eigenvectors. He was also asked about the scaling factor. To figure this out, he decided to pick one of the eigenvectors and again did the matrix multiplication instead of using the eigenvalue equation or its algebraic interpretation.

C: *No, there has got to be a certain number. You can't just scale randomly. I suppose it is in the matrix, though. I don't know what it is. So it goes from, so from the length of two on number six [he picked the eigenvector to look at]two [mumblings], two, four, about, I think. I think it is four, about sixteen – oh, no, this is about twelve point twenty five, that is with rounding. So about four, so it doubles it in length. That looks right on the graph, too, about doubled. I bet I could find the exact value, but. .*

I: *You don't want to?*

C: *Yeah.*

[I.2, line 292-303]

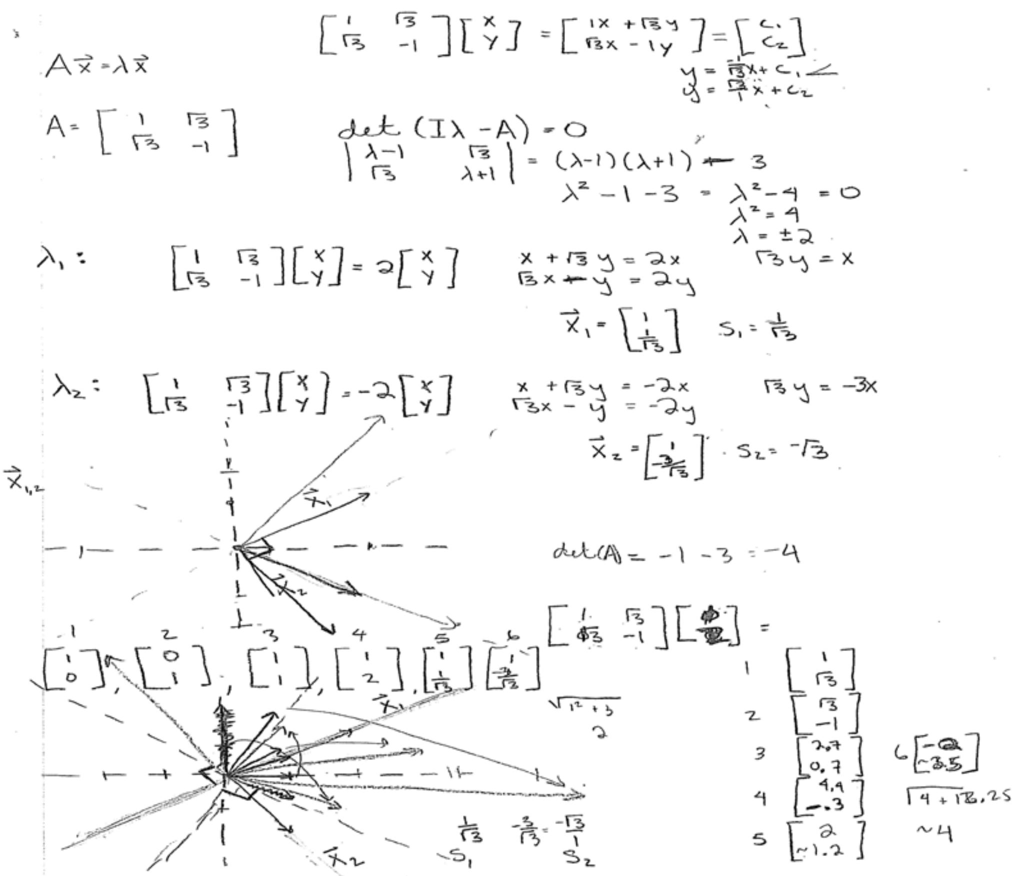


Figure 4.7 Crosby's solution in second interview

The eigenvalue equation was not used, even though it was on the same page. In both cases, he did not use the eigenvalue equation which could suggest that his understanding of the equation and its algebraic interpretation had some gaps.

The algebraic interpretation of the eigenvalue equation was observed again on Crosby's final exam. He wrote, "An eigenvalue is a particular number (scalar) which can replace a matrix when operating on a particular eigenvector. In quantum, eigenvalues represent possible measurements." He continued, "An eigenvector is a particular vector which when a certain matrix acts upon it, is the same as a scalar (its corresponding eigenvector) acting upon it. In quantum, eigenvectors represent the state of a particle" (Final Exam, Question 1).

He referred to the eigenvalue equation during the third interview, which was conducted during the Spring Term, approximately three months after the second interview and a month after the final exam. Once again, he referred to the algebraic interpretation of the equation.

> ***C:*** *OK, what I know about eigenvectors – a matrix can have an eigenvector, which is a very specific vector that when the matrix is multiplied by it, it is the same as if the matrix were just a scalar.* ***So basically the matrix acts along that direction of the eigenvector.*** *I know that the dimensions of the matrix, [inaudible] however dimensions they are equal to the number of eigenvectors there are and eigenvalues. That is it. That is all I know.*
>
> [I.3 lines 149-154]

In this episode it seemed that he could be aware of the geometric interpretation of the eigenvalue equation. However, his wording showed some confusion (**bold part**). The wording of the sentences prior to the bolded one suggests that he looks at only one hand side of the equation and states that what it does could be replaced by the other hand side of the *Eigenvalue Equation,* $A\vec{x} = \lambda\vec{x}$. He seemed to try the same method to include some geometrical interpretation. He could have noticed that on the right hand side of the equation the eigenvector's direction did not change since it was multiplied by a scalar. He wanted to show that this idea related to the left hand side of the equation, as well. However, his wording over generalized what the left hand side of the equation showed. With his sentence he was claiming that when a matrix acted on any vector, the result would be along the direction of the eigenvector. However, this is true only when the matrix acts on its eigenvectors.

There were only a few other instances during the last two interviews that could be indications of his being aware of the geometric interpretation of the eigenvalue equation. For example, in the second interview after finding eigenvectors of the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$, he was asked if any scalar multiple of these two eigenvectors were again eigenvectors of the matrix. He said they were again eigenvectors and explained further.

> ***C:*** *I think it is because the eigenvector is a line on the graph, so what you are doing with this matrix is putting, let's see, you are putting things on this line, or you are using the line to make something, so the scalar value doesn't matter.*
>
> [I.2, lines 150-152]

Even though he stated that eigenvectors were all on the same line, his further explanation seemed to imply that he assumed the matrix was mapping the vectors to a line, and the eigenvectors were also on this line. His explanation did not indicate if scalar multiples of an eigenvector and eigenvectors were always on the same line or if they happened to be on the same line because of this particular transformation.

Another example of his using the geometric interpretation of the eigenvalue equation was observed at the third interview. Crosby was given two eigenvalues with their corresponding eigenvectors of a two-by-two matrix, and then he was asked to tell everything he could about the matrix. He first drew the eigenvectors (Figure 4.8), and then he conjectured that the matrix rotated the axes. The researcher asked him to explain his conjecture.

> ***I:*** *So there is this matrix M and then you are telling me M rotates the y and x axes?*
>
> ***C:*** *Yes.*
>
> ***I:*** *To where? How many degrees does it rotate?*
>
> ***C:*** *It would be like forty five degrees clockwise. I think so.*
>
> ***I:*** *How did you decide that it is doing that rotation?*
>
> ***C:*** *I suppose it just looks like it might. I am not really sure.*

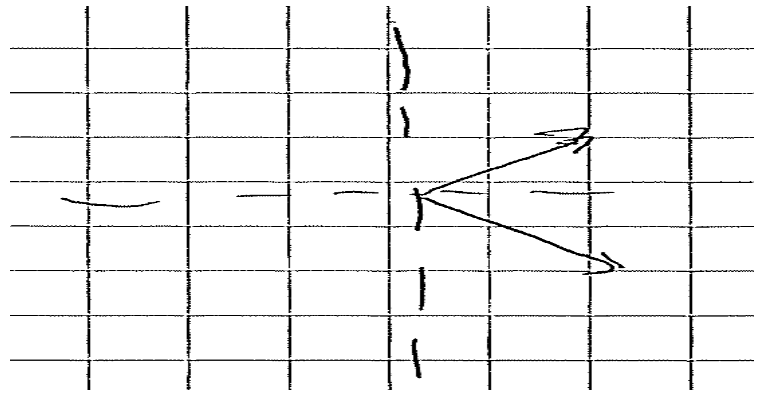


Figure 4.8 Crosby's graph in third interview

I: *Looks like? How do you decide it looks like that? What was the reason behind that observation?*

C: *I think I've seen the same, similar matrix before, perhaps, in spins. Well, what I think the matrix is doing is reassigning these [eigenvectors], because these are in Cartesian, or something similar. So it is reassigning those directions to be forty five degrees clockwise from where they were before.*

[I.3, 88-100]

He proposed that the unknown matrix was reassigning the standard basis vectors (1,0) and (0, 1) of R^2 to the given eigenvectors (1,1) and (1,-1) by rotating the standard basis vectors 45º degrees clockwise. Another question was posed to understand what he meant by the reassignment:

C: *I guess the only way I know to answer that is with the equation, which is [wrote $M|\varphi_x\rangle = \lambda|\varphi_x\rangle$], which $M|\varphi_x\rangle$ is just the same as multiplying it[the vector] by a scalar value, so it would either just lengthen it or shorten it, change the length and possibly switch the direction if there is a negative, switch the – not change the direction, but switch it from positive to negative.*

[I.3, lines 104-124]

This was the first time that he had mentioned the geometric interpretation of the eigenvalue equation. Even though he used the geometric interpretation to answer the question, his answer conflicted with his conjecture of M being a rotation. With the interviewer's clarifying question, Crosby decided that his conjecture about matrix M might be incorrect, but he did not provide any further explanation on what the matrix did.

Crosby's attempt to use some geometric interpretation during the third interview could be an indication of his being aware of this interpretation of the equation. It seemed that he preferred to use the algebraic interpretation of the equation when he described eigenvalues and eigenvectors.

Actor-Oriented Transfer Episodes Related to the Eigenvalue Equation and its interpretations: Data from the second and third interviews were analyzed by implementing the AOT framework. The episodes constituting evidence of AOT were then categorized according to these goals, and the following episode describes the AOT of Crosby that seemed to be related to the third goal. The episode is described briefly; then, evidence of AOT is discussed.

Episode 2. This episode occurred during the second interview. Crosby was asked to describe eigenvalues and eigenvectors and he said, "An Eigenvalue is a number that when acting on a certain vector does the same thing as if you used a matrix, a specific matrix. You start with a matrix, and you add a vector that you use the matrix on. An eigenvalue does the same thing. It is just easier." Then he continued explaining, "So with a specific eigenvector, you use the scalar eigenvalue, and it replaces what this matrix does" and wrote " $A\vec{x} = \lambda\vec{x}$."

He later found eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$.

While he was trying to find eigenvalues and eigenvectors, he stated that they did not spend much time on finding eigenvalues and eigenvectors during the linear algebra course, but they had talked about what they were used for and had proven some theorems related to eigenvalues and eigenvectors. He was prompted to explain the uses of eigenvalues and eigenvectors.

> ***C:*** *To replace this matrix to make it easier or cheaper for computers.*
> ***I:*** *What do you mean by that?*
> ***C:*** *I guess it is really expensive or it takes a lot of memory to use matrices in computers, big ones, so if we can find an eigenvalue to replace it, it is easier and cheaper.*
> ***I:*** *Eigenvalues to replace it?*
> ***C:*** *Yes.*
> ***I:*** *Is that from your instructor or is that from another class or a book?*
> ***C:*** *That is from the instructor and the book told me that.*
> ***I:*** *Linear algebra course you mean?*
> ***C:*** *Yes. I guess Professor Clayton said it, too. She said something about if you have a giant matrix – no, she said if you have a giant diagonal matrix and you were trying to find the eigenvalues, you could just use diagonals which is much easier. [Continued with the calculation of eigenvectors]*
>
> [I.2, lines 109-123]

After finding eigenvalues and eigenvectors of $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$, he decided that the matrix reflected and scaled the vectors. He later indicated that one of the eigenvectors was the axis of reflection by looking at the graph he drew earlier. Even though he was convinced of his answer, he was bothered that the lines from the system of equations did not coincide with the lines from the eigenvectors.

> ***C:*** *No. I still feel like these slopes [from eigenvectors] should somehow correspond to these[slopes of lines from the system], and I might be completely wrong. It seems right.*
> ***I:*** *Why do you think that?*
> ***C:*** *I guess I seem to remember something that Professor Clayton said on the board about them corresponding somehow, but I might have just made it up in my head. That was the first time I ever heard of scrinching matrices, so something I remember from there seems to make me think that they should. I guess maybe that wasn't a good way of looking at it, because this is definitely what I remember*

> *seeing on the board, is that it reflects across this eigenvector. So maybe that is what I'm thinking of. I think so.*
>
> [I.2, lines 319-327]

It seemed that he tried to resolve this conflict between his two different representations of matrices. At the end of the second interview, Crosby was asked if LAW had changed his ideas about eigenvalues and eigenvectors, and he stated that it had added onto what he had already learned in the linear algebra course in terms of the definitions.

AOT Analysis of Episode 2. In this episode Crosby was referred to the eigenvalue equation $A\vec{x} = \lambda\vec{x}$ and its algebraic interpretation as he tried to answer questions related to eigenvalues and eigenvectors at the second interview. His description of eigenvalues and eigenvectors was formed from the algebraic interpretation of the equation.

To investigate his previous experience with eigenvalues and eigenvectors, the first interview data, pre- and post-quiz data and Crosby's LAW data were analyzed. It was noticed that Crosby stated similar descriptions of eigenvalues and eigenvectors on the pre and post quizzes. He also stated that he learned the use of eigenvalues and eigenvectors when he took the linear algebra course and during LAW.

It was also noticed that when he was trying to resolve his representational conflict, he explicitly stated that he recalled that during LAW they worked on a reflection matrix and one of the eigenvectors was the axis of reflection. The reason for his conflict seemed to be that he was again trying to recall a transformation that they had done during LAW. This particular transformation mapped the vectors to a line, and when represented with a matrix the determinant of the matrix was zero. As a class they had decided to call this type matrix a “scrinching” matrix. He seemed to be referring to this experience and trying to find similarities between that idea and the second interview question.

As Crosby tried to answer the second interview question, he implemented his experience with the eigenvalue equation and its algebraic interpretation. He also tried to reorganize his experience to build similarities between the interview question and the “scrinching” matrices. For these reasons, this episode constitutes evidence of actor-oriented transfer. Even though Crosby did not use the eigenvalue equation consistently, it seemed that it became part of his understanding, and probably it was still developing at the time of the second interview.

Goal 4: Basis Vectors

One of the ideas that Professor Clayton expected students to understand after completing the winter term physics courses was that any vector could be linearly expanded with eigenvectors. In other words, she wanted them to realize that eigenvectors could be used as the basis vectors of the vector space.

In this section, Crosby’s emerging understanding of eigenvectors as basis vectors is described through repeating ideas and an investigation of whether Crosby used his experience from LAW and winter term physics courses at the second and third interviews is discussed.

<u>Ideas on Basis Vectors</u>: Crosby did not mention the idea of eigenvectors as basis vectors until the third interview. Also, no idea relating to eigenvectors as basis vectors was found in any other data source. For this reason, no repeating ideas

were formed, but Crosby did mention the superposition idea and talked about eigenvectors representing basis vectors at the third interview when he was asked.

At the beginning of the third interview Crosby was asked if the sum of two eigenvectors of a matrix A was again an eigenvector of A. He recalled a similar question that was asked in one of the winter term physics courses, but he said he could not remember the answer. Later he said the sum was just a superposition of two states. It was observed that he changed the wording of the question; the researcher used the word "eigenvectors," but he replied with "states." He explained that the sum of two eigenstates was not an eigenstate because there were two eigenstates for the matrix, and they were orthogonal. The combination of two orthogonal eigenstates would not be orthogonal to both of these eigenstates, so he claimed that the third state was not an eigenstate.

Crosby was asked to explain what he meant by two eigenstates being orthogonal. While he was explaining his thinking, ideas that are slightly related to the basis vectors appeared. He gave an example of two eigenvectors being on x and y axes so that the angle between them was 90º degrees. When he was asked to assume that one of the eigenvectors was not on either of those axes, he was confused. However, he then started to talk about how eigenvectors define new directions:

> ***I:*** *What if an eigenvector is not on the x axis? Then what would happen?*
>
> ***C:*** *I suppose that would be a super-position, a summation of two eigenvectors. I don't know,* ***because I think when you are talking about a matrix, you cannot necessarily assign it to Cartesian coordinates, especially when you are talking about eigenvector. From what I understand is that the eigenvectors themselves are the axis y and z direction.***

So by finding the eigenvectors of a matrix, you are defining those directions.

I: *A new xy?*

C: *Right, so you can't define them by an x, y, and z, unless you are saying your matrix isn't Cartesian coordinates.*

[I.3, lines 61-70]

When Crosby said "***because I think when you are talking about a matrix, you cannot necessarily assign it to Cartesian coordinates"*** he was probably referring to a matrix representation of a linear transformation with non-standard basis for R^3. He could have meant to say one could choose a different set of basis vectors and represent a linear transformation matrix with these new basis vectors. His following statement "***From what I understand is that the eigenvectors themselves are the axis y and z direction***" suggests that he was thinking that eigenvectors could be used as new basis vectors for R^3. However, there is no further evidence to conclude that he made a connection between eigenvectors and basis vectors.

He did not mention or refer to the basis vectors idea again in the third interview. However, he was asked to talk about what kind of ideas tie together when he thought about eigenvalues and eigenvectors. He said, "I think now I realize – I'm not so sure that anyone knows how to describe eigen stuff well in math, and relate it to physics. But when I hear someone say eigen anything, it just makes me think that there is going to be a set of possible solutions, and then of those sets we can have combinations of them. I don't necessarily think of the math at all but the idea of having a set of solutions and a possible – or a combination of those sets that can work. It ties a lot of physics together" [I.3]. It could be possible that seeing the eigenvectors in three different physics settings during three winter term physics courses, he was beginning to see the parallels between them.

The researcher later explicitly asked him if he was referring that the eigenvectors were similar to basis vectors.

> **I:** *I heard somebody's explanation saying that eigenvectors are like basis vectors. So I was curious if you were referring to basis vectors?*
>
> **C:** *Yes, yes that is my opinion. Because when I think of eigenvectors I think of like making your own space with eigenvectors being the basis vectors and then the super-positions are where whatever you are talking about actually lies in that spaces. So for the spin one half system, you have a two dimensional space because there are two options, and then whatever super-position of the spins you get ends up being somewhere within that space, which is the superposition of the basis. Just like a point in Cartesian, a super-position of the basis is x and y, or x, y, z. But I don't think I learned that in linear algebra 1 [math course]. Maybe a little.*
>
> [I.3, lines 319-336]

Since Crosby used examples from one of the winter term physics courses he took after LAW, he was asked if he knew any other examples from the other two physics courses. He said that "[…] the different allowed frequencies for the system would be like the basis or the eigenvectors" in the second course, Waves. He could not think of an example from the third course, Central Forces for a while. However, he had some examples later, towards the end of the interview.

> **I:** *Do you feel comfortable talking about Hamiltonians, what are they?*
>
> **C:** *I know Hamiltonian is the energy of the system. That is about it. We definitely learned like two different ways of writing it. Both work in an eigen statement, the way of writing it as a matrix, and then the way of writing it as an operator, derivative, differential. I don't know which is applied when, but I know that it works. I guess that would be more applied math, is when you – because when we*

learned this, we learned it as a matrix with solutions that are column vectors or row vectors – I can't remember which one it was. When we actually used it with the Hamiltonian, we used differential operators rather than matrices. Then our solutions were states that usually were $e^{\theta i}$ something.. So both are math, but the second one is more applied, more realistic, maybe.

I: *How does the basis [vector] idea work in the realistic case?*

C: *I suppose the eigenvectors that you get out of the differential operator are still basis, but maybe not so much, not so easy to think of them as in the geometric sense.*

[I.3, lines 429-441]

It seemed that Crosby started to develop an understanding of eigenvectors as basis vectors and the vector could be expanded linearly with eigenvectors.

Actor-Oriented Transfer Episodes Related to Using Eigenvectors as Basis Vectors:

The second and third interviews data were analyzed by implementing the AOT framework. The following episode describes the AOT of Crosby that seemed to be related to the fourth goal. The episode is described briefly; then, evidence of AOT is discussed.

Episode 3. This episode occurred at the beginning of the third interview. Crosby was asked if the sum of two eigenvectors would be an eigenvector of the same matrix. Crosby had a smile as soon as he heard the question, and he said that he recalled a similar question from one of the physics courses he took after LAW.

C: *I can't remember what class I was in. We had a very difficult question. I don't remember who was teaching it. Someone asked is the sum of two eigenstates an eigenstate itself. I don't remember the answer.*

I: *You just remember the question.*

C: *I don't think it is. It is not, it is just a superposition of two states. I don't remember. But so you are asking specifically if I have two eigenvalues. . .*

[I.3, lines 9-14]

Crosby needed some clarification on the question.

I: *What happens is a general question; a piece of it is if it is an eigenvector again.*
C: *I guess the piece I do know how to answer is I don't believe it is an eigenvector. It is just a super-position of two, but I don't know what you mean by what happens. Like mathematically?*
I: *Sure, mathematically is fine. This is a general question. What can you tell me about this sum vector?*
C: *So the physics that we have learned, if it is a super-position of the two, it has a probability of being in one or the other. But mathematically I have no idea what that relates to.*
I: *You said it was not an eigenstate, is that what you said?*
C: *Yes.*
I: *How do you know or why do you have that feeling that it is not an eigenstate?*
C: *I think my answer was that it was an eigenstate and I was wrong, if I remember it. I don't remember – I guess it can't be an eigenstate, because an eigenstate would be either one or the other, since there are two eigenstates for the matrix, so it can't be both, because they are orthogonal. So the two of them combined can't be an eigenstate, or you have three states. That wouldn't make any sense. Is that right?*
I: *I don't know.*
C: *I don't know. I guess I would think about how they are orthogonal, so you add them and it just wouldn't make sense for them to be an eigenstate or an eigenvector.*
[I.3, lines 17-39]

Crosby was then asked to give an example illustrating his ideas. He assumed $|\varphi_1\rangle$ and $|\varphi_2\rangle$ were two eigenstates of a matrix A. He said the superposition would just be the combination of these two eigenstates, and it would have some coefficients, c_i and wrote $|\psi\rangle = c_1|\varphi_1\rangle + c_2|\varphi_2\rangle$. After writing the equation, he did not say anything further. The researcher also asked him how he knew two eigenvectors were orthogonal to each other. He was not sure why they were orthogonal to each other and thought it could be a property of eigenvectors.

<u>AOT Analysis of Episode 3</u>. In this episode it was observed that Crosby was recalling an example that was presented in one of the winter term physics

courses. Even though he did not explicitly state which course and example, it was noted that in the second course of the winter term students spent some time on a similar question. Students were asked to figure out if the function

$$\phi(x) = \sqrt{\frac{1}{5}}\phi_1 + \sqrt{\frac{3}{5}}\phi_2 + \sqrt{\frac{1}{5}}\phi_3$$

was an eigenstate (eigenvector) of the Hamiltonian operator, when each ϕ_i for i=1,2,3 was an eigenstate (eigenvector) of the Hamiltonian operator. Students discussed the questions in small groups, and later there was a brief whole-class discussion. It may be possible that Crosby found the interview question similar to this particular activity and used his experience from the course to answer the question at the interview. Crosby could not remember all the details of this example; he only recalled his answer that it was an eigenstate, and his answer was wrong. The way that Crosby thought these two situations could be similar constitutes evidence of actor-oriented transfer. However, this does not imply that Crosby completed the interview question correctly. He was able to recognize that he had done a similar example and reconstructed his experience from the physics courses to form an answer at the interview; however, he could not provide a full reason for his answer.

He decided that the sum cannot be an eigenvector. He said if you add two eigenvectors of a matrix, and if the sum is an eigenvector, then the matrix would end up having three eigenvectors, which he said was not possible. He did not mention the dimension of the matrix, but it may be possible that he assumed the matrix was a two-by-two.

It was noticed that Crosby did not include the case of having eigenvalues with multiplicities bigger than one, in other words, he did not ask if eigenvalues e_1 and e_2 were the same eigenvalues. The analysis of this case of would conclude that the sum of the eigenvectors would be an eigenvector again. However, it was observed that in LAW students did spend much time on multiplicities of eigenvalues, and this idea did not come up again in the other physics courses. This could be one of the reasons why Crosby did not think about the multiplicity of eigenvalues.

It was also observed that Crosby thought all eigenvectors were perpendicular to each other. Since students mainly worked with Hermitian operators during the winter term physics courses after LAW, they were familiar with eigenvectors that were always perpendicular to each other. It seemed that Crosby generalized this idea and thought the eigenvectors of all operators were perpendicular. He referred to this experience when he was reasoning through the question in this episode. Crosby implemented his experience of eigenvectors being orthogonal to each other in the interview question without checking to see if the operator was Hermitian, and this again could constitutes evidence for actor-oriented transfer.

Summary

Crosby viewed matrices as representing systems of equations. He was also aware that matrices represented linear transformations. However, it seemed that he was not aware of these representations being different from each other, and he wanted to use both of them at the same time. Crosby seemed to adapt the terminology of linear transformations when he talked about matrices after LAW. For example, he stated that matrices acted or operated on the vectors, and he was consistent and accurate with his use.

Crosby recalled how to find eigenvalues but not the eigenvectors at the first interview. A similar trend was observed on the pre-quiz even after being introduced to eigenvalues and eigenvectors at the end of the fall term in the linear algebra course. He stated that in the linear algebra course, they did not spend time on finding eigenvalues and eigenvectors. It may be possible that for this reason, he was not fluent in finding eigenvectors on the pre-quiz. He, however was able to find eigenvalues and eigenvectors correctly on the post-quiz. Similarly, he seemed to be fluent during the second interview. However, there were no instances to observe fluency at the third interview.

Crosby implemented the algebraic interpretation of the eigenvalue equation throughout the study. He initially described eigenvalues and eigenvectors as some multiple of some sort (Interview 1) and then his description of an eigenvalue on the pre- quiz was developed into: "It's a value that you multiply a vector by which does the same thing as if you had multiplied that specific vector by a specific matrix; $Ax = \lambda x$." It seemed that he was introduced to the algebraic interpretation of the eigenvalue equation prior to LAW. He seemed to develop this idea further and tried to use it as he answered questions at the second interview. However, the algebraic interpretation of the eigenvalue equation idea did not seem to develop fully. For example, when he was given eigenvalues and eigenvectors of a two-by-two matrix at the third interview, he could not use the algebraic interpretation of the equation or just the equation to find the matrix.

Crosby also showed few signs of being aware of the geometric interpretation of the eigenvalue equation; however, it was unclear if it was part of his emerging understanding of eigenvalues and eigenvectors. For example, after spending a long time on calculations and also prompts from the interviewer during the second interview, he mentioned seeing something similar to the geometric interpretation in the class. Even though he mentioned something similar to the geometric interpretation of eigenvectors at the third interview, he did not use the idea in any of his work.

A similar trend was seen with the idea that eigenvectors could be used as basis vectors. Crosby did not mention anything related to this until the third interview. He was explicitly asked about the relationship between eigenvectors and basis vectors. He gave examples to represent his thinking on the relationship. He talked about the superposition idea at the third interview, but again the idea of eigenvectors as basis vectors was not brought up. It was unclear if Crosby viewed eigenvectors as basis vectors, but he was aware of the idea.

Overall, it seemed that Crosby's apparent understanding of eigenvalues and eigenvectors were being developed around the algebraic interpretation of the eigenvalue equation. Eigenvectors as basis vectors idea was also becoming part of his emerging understanding of eigenvalues and eigenvectors.

Deniz

Deniz was a junior-year physics student. He completed all of the mathematics course requirements for his degree before his junior year. Instead of taking a matrix and power series methods course, he took the third course in the calculus sequence. Deniz stated that in the third calculus sequence course they focused on sequences and series, Taylor's formula and power series; however, the linear algebra topics were not covered in this course. He finished the vector calculus sequence (two courses), a differential equations course, and an introduction to linear algebra course. He hoped to take the second linear algebra course sometime in the future. During the data collection period of this study, Deniz took the three modules of physics courses each term but no mathematics courses.

During his junior year (spring term, 2008), he started to work on two different research projects; one of them was with a professor in the Physics Department and the other one was with a chemistry professor. He was hoping to continue working on one of them during summer term as a Research Experience for Undergraduate (REU) students.

Deniz seemed to be very involved in small-group activities during LAW. He contributed to the discussions, asked questions, and answered other group members' questions. However, similar to Crosby, he was very quiet during whole-class discussions. He did not volunteer to present the group work on the board.

Another interesting observation about Deniz was that he took notes all the time, even during small group activities (Course Observation Notes, LAW).

During the interviews, he asked many times if he could look at his notes or a book. For example, at the beginning of the first interview he asked what the research was about, and as soon as he heard that it was on linear algebra topics he said without the book he would not remember much. Similar comments were made during all three interviews, and he also wrote on the pre quiz that if he had the linear algebra book, he would know how to do matrix multiplication in 5-10 seconds. It seemed that he was very dependent on his notes and books.

At the first interview Deniz was asked to describe eigenvalues and eigenvectors, and he stated that he used to crunch them out because there was a formula or algorithm. When he was asked to talk about matrix multiplication, he again mentioned that matrix multiplication was “another one of those (referring back to finding eigenvalues and eigenvectors) where there’s an algorithm for it, and as long as you memorized it, you could get away with it without really totally understanding the chapter” [I. 1, lines 111-113]. However, he could not recall the algorithm. It seemed that he was aware of some algorithms or ideas related to matrix algebra, but he could not remember them completely.

A similar pattern was observed when he was asked to talk about vectors; however, this time he was able to remember the procedures, but he could not figure out why the procedures worked. For example, he was asked to find the magnitude of a vector and calculated correctly; however, he did not know the reason behind the procedures. He said he “kind of remembered” how to do it but did not know why it worked, and he did not “have any good words for it” [I.1, lines 283-288].

Next the researcher asked him to talk about multiplication of two vectors. Once again he said he remembered some pictures from a book and was not sure what those pictures represented. When he talked about cross multiplication of two vectors, he said that there was a mnemonic device and he was “all about mnemonic devices.” At the end of the first interview he was asked if he would like to add anything else to what was talked about, and he stated that all of these made him “want to go back and look at the book really badly” [I.1, line 459].

It seemed that Deniz had a hard time recalling what he had learned about matrix algebra in earlier math courses which he had taken at least a year before the first interview. Since Deniz had taken linear algebra a year earlier, his experience was somewhat different than the other participants of this study. Most of the other students had taken linear algebra one or two terms prior to the winter term of their junior year. This was one of the reasons Deniz was included in the case studies.

The following section describes the repeating ideas observed in Deniz’s data as they fit under the four chosen goals of LAW and the winter term physics courses. Then, Deniz’s attempts to use his experience from LAW and courses he took after LAW at the second and third interviews are also analyzed and described by implementing the Actor-Oriented Transfer Framework as they fit under the four chosen goals of LAW and winter term physics courses

Goal 1: Matrix representation

As reported previously, Professor Clayton expected students to view matrices as linear transformations during the winter term courses, and for this reason she usually used an activity called linear transformations on the second day of LAW. In this section Deniz's view of matrices before and after LAW as they were repeated during the interviews are described as well as Deniz's use or non-use of his experience from LAW and winter term physics courses at the second and third interviews are discussed.

<u>Ideas on Matrix Representation</u>: Only one idea related to matrix representation was repeated in Deniz's data and it was the idea that matrices represented lines.

Even though Deniz talked about different transformations when he was working with matrices at the second and third interview, he never stated that those matrices represented the transformations. Also, he did not seem to recognize the phrase "an operator acting on a vector" at the second interview. For example, the researcher asked him what would happen if the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ was operated on the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. He asked back if it was "like multiplied with it" [I.2, line 146].

Deniz seemed to view matrices as representing lines at the first interview, and he also mentioned that they could represent vectors. It seemed that these two ideas were the same thing for him. For example, at the first interview he was asked what he could tell about matrices and what they represented; he said, "Oh. [pause] Lines, for one, because, linear algebra and all that" [I.1, line 83]. Then, he wrote a three-by-three matrix as an example.

D: *Each one of this [pointing at each column] is a something that means something.*
I: *Okay*
D: *Each one of this is a line. Right, each, uh, so this is [writing on whiteboard] columns and rows. So, columns and rows, and each column is a vector all by itself.*
[I.1, lines 85-93]

To clarify the "lines" and "column vectors" ideas, the researcher asked him to talk more about the "lines" idea. He was given the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$ and asked what the lines were in this matrix.

D: *Right, and that's where I confused myself, too. Right, but I know it should be something like either one, or you know one direction or the other. I want it to be the columns really bad for some reason. [He started to draw] But then it would be like x and y, and so then we'd have a little origin here and*

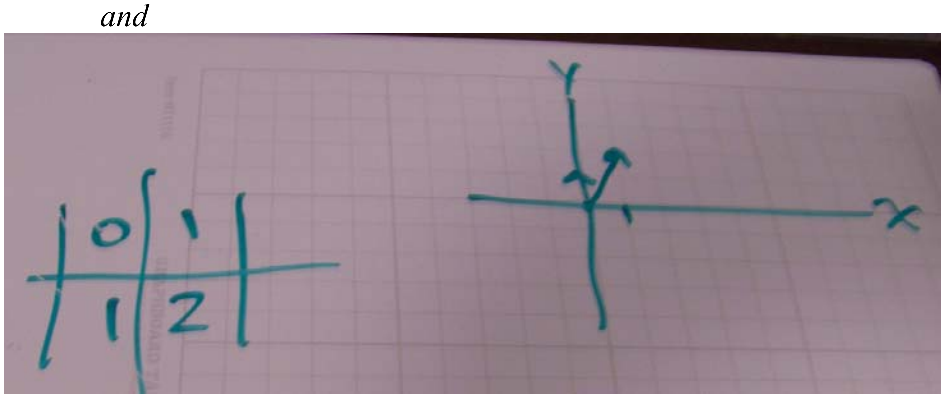


Figure 4.9 Deniz's matrix example

D: *(continued), so then this one would be the vector, the points, and x is zero and y equals one. Alright, so it'd be like that. And then this one would be for one, two.[pause] And then there's something…[inaudible]. Yeah, I don't remember anything about that graphically.*
I: *Yeah, I am just trying to understand what you mean by the line. So this is what you were trying to say?*
D: *Yeah something like that, something that happens in here, yeah, conceptually I'm really struggling with it.*
[I.1, lines 137-157]

It seemed that he meant each column vector represented the direction of a line. It was not clear if he was recalling the word “lines” from the system of equations representation of a matrix, but it would seem that it was possible.

At the second interview, the same idea was observed when he was asked what the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ did to vectors. He first stated that he was not sure, and he did not know. In the beginning of the second interview Deniz stated that how determinants could tell the behaviors of “those” such as rotation was one of the interesting things that he learned during LAW. He was asked to implement that idea.

The researcher asked him to implement the determinant idea to see what the matrix did. He found the determinant of the matrix and said that it was negative, so “it reflects, so it does not rotate.” However, he continued to say that he really did not know what that meant. He was asked to check his conjecture on $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$, where the negative value of the determinant indicated reflection. It seemed that he did not know what to do in this situation, and he again referred to column vectors.

> ***D:*** *I think this where I am not really sure what’s going on, because I don’t really know how to see that. I don’t understand why, I vaguely do, but just in a hand-wavy sort of way, but other than that, [I] don’t really understand it. [pause] I think you could, I think the idea is that these are two vectors [pointing at the column vectors of the matrix], so I’m just guessing now. [pause]*
>
> [I.2, lines 140-144]

Then, the researcher suggested that he operate on the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with the matrix

$\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$. He multiplied the vector with the matrix and noticed that the length of the vector was changed. He said this implied that "it" [possibly the matrix] was not just a rotation but could be a rotation and a stretch. He then decided that "it" could be a reflection. He did not ever say explicitly that he viewed matrices as linear transformations.

<u>Actor-Oriented Transfer Episode Related to Matrix Representation:</u> Data from the second and third interview were analyzed by implementing the Actor-Oriented Transfer framework. The episodes which have evidence of AOT were then categorized according to these four chosen goals of the courses. The following episode describes the AOT of Deniz that seemed to be related to the first goal. The episode is described briefly; then, the evidence of AOT is discussed.

*<u>Episode 1.</u>*This episode was observed at the second interview. Deniz was given the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ and asked what it did to vectors. As described in the previous section, he decided that it could be a reflection. He was asked if he was certain that it was a reflection matrix. He said he was not sure, but he based this on his idea that it was not a rotation because the transformed vector was stretched.

D: *[pause] We could have, I don't remember the conditions exactly where you can have a stretch and a rotation, a rotation is the same as two reflections, reflect the line x and then you reflect the line y, is identical to a rotation.*

I: *Okay.*

D: *So, I suppose that's probably possible here, because we're [wrote the identity matrix], that's the identity, I don't* *remember now. I know we were doing it on the first couple of days, in class, if you look along this [pointing at the non-diagonal entries of the identity matrix] and it's one and one [wrote the matrix* $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$*], than it is just a rotation, if it's negative one, one, yeah I don't remember what the combinations were, but one of them would just be a rotation, one of them would just be, if it was negative, negative, or if it was just negative one, then it would be a flip along the x and then if was negative, negative it would be a flip along the x and a flip along the y which is the same as a rotation.*

I: *Uh-um. What would be the other cases?*

D: *There's a just stretching, and a combination of all the other things and then there was scrinching, something like that.*

I: *What was that?*

D: *That one I'm fuzzy on, I don't remember that very well. That's where compressing and stretching, everything got smashed down onto one line, all of the vectors, no matter what you did to it, would all end up on the line, on the line similar to axis of reflection and then they'd all be smashed or stretched, but they all ended up mashed on that line as a projection, so you lose some information when that happens.*

[I.2, lines 219-238]

Then he was asked to give an example of the latter idea about which he had talked. He proposed that the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ could be an example. He was asked to check his proposition, and he said that he needed to find eigenvectors of it and also needed "some random vectors, or operator and multiply it, and it should end up the same thing, no matter what it did" [Interview 2, lines 253-255]. Even though he suggested these ideas, he later decided that he did not know how to test his idea.

<u>AOT Analysis of Episode 1</u>: Deniz seemed to be recalling some ideas related to the geometric meaning of determinant of matrices. He explicitly stated during the first couple of days of LAW that the students had worked on these

ideas. He could not recall the details of the activity, but the ideas he proposed seemed to be from the activity called linear transformations which was done on the second day of LAW. One of the goals of this activity was to introduce the matrices as linear transformations. During this activity students and Professor Clayton decided to call the matrices with determinant zero "scrinching" matrices, and it seemed that Deniz was recalling this part of the activity. He also recalled the part where the students were asked to operate on an assortment of vectors with a given matrix to figure out what the matrix did to these vectors. Even though Deniz could not execute the ideas that he had recalled, it seemed that he was trying to make connections between the interview question and his experience in LAW. For this reason, this episode could constitute evidence of actor-oriented transfer.

Goal 2: Finding Eigenvalues and Eigenvectors

Another goal of LAW was that students are expected to be fluent in finding eigenvalues and eigenvectors of a matrix or an operator. In this section the repeating ideas that describe Deniz's fluency in finding eigenvalues and eigenvectors are discussed. An episode exploring actor-oriented transfer is also described after the repeating ideas.

Ideas on Finding Eigenvalues and Eigenvectors: In Deniz's data, different ideas arose from each interview. These ideas were not necessarily repeated in each interview, but they all seemed to be related to the idea of finding eigenvalues and eigenvectors by following an algorithm.

At the first interview, Deniz was asked to describe eigenvalues and eigenvectors. He said he only "vaguely understood them the first time." He used to "crunch it out" because he knew a "formula thing, algorithm" for finding them, but he did not recall the algorithm anymore.

On the pre-quiz Deniz could not find eigenvalues and eigenvectors of a two-by-two matrix; however, on the post-quiz he was able to find eigenvalues of a two-by-two matrix with complex number entries accurately. It was noted that when he tried to find eigenvectors from the eigenvalue equation, he did not write the equation correctly. Thus, he could not find eigenvectors correctly.

At the second interview Deniz found eigenvalues and eigenvectors of $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ with no errors even though he had some hesitations in his abilities. Once he found the eigenvalues, he was asked to talk about what he was doing while he was working. As he explained his work, it seemed that Deniz was following the ideas on how to find eigenvectors that were presented by Professor Clayton on the fourth day of LAW. (See Goal 2 of this section for detailed discussion on the method.)

I: *So what are you doing right now?. What is your goal?*
D: *Um, I'm looking to, I'm supposed to be finding the eigenvectors, right?*
I: *Uh-um..*
D: *So I'm solving the puzzle for v here [writes the equation Av=λv], for that guy, where this [pointing at* $\begin{bmatrix} x \\ y \end{bmatrix}$*] is v.*
I: *And what's lambda?*
D: *And lambda in this case right here is one of these two choices[pointing at the two eigenvalues] apparently.*
I: *Okay.*
D: *I'm just picking one. [pause and did calculations] So then one of the eigenvectors would be, not normalized*
I: *Okay.*

D: *And then I could go through and do the same thing for the negative and that would be [started to calculate the second eigenvector]*

[I.2, lines 82-94]

At the second interview, he was again able to find eigenvalues and eigenvectors of another matrix correctly. It seemed that he was fluent in finding eigenvalues and eigenvectors during the second interview.

At the third interview, he was not directly asked to find eigenvalues and eigenvectors of a matrix. However, the idea of how to find them came up a couple of times during the third interview. At the beginning of the third interview, he was asked to comment on the sum of two eigenvectors. He immediately thought about the steps of finding eigenvalues and eigenvectors of a matrix in general.

D: *The sum of those? Square matrix, two eigenvectors, which presumably come from the eigenvalues. Then the sum of the two eigenvectors?[pause] I am imagining when you are talking the eigenvalues, finding the eigenvalues and assigning eigenvectors to them, I want to say that it is 1, intuitively, but [mumbling]. Then, I haven't thought about this for awhile. A is the square matrix, two by two or something. Then det of that [writing det (λI-A)=0] equals zero, that is how I find the eigenvalues. The eigenvalue, that's [inaudible]. Something like that, the eigenvalues lambda one and lambda two. Something like that [wrote A=λv]. So two by two, you just do the [mumbling] two things, not necessarily the same thing. [Wrote* $\lambda_1\begin{bmatrix} x \\ y \end{bmatrix}$ and $\lambda_2\begin{bmatrix} x \\ y \end{bmatrix}$*] Something like that.*

[I.3, lines 10-17]

It was noticed that when he wrote the equation A=λv he had forgotten to write the eigenvector v on the left-hand side of the equation. However, later in the same interview when he was working on a different problem he was able to write the eigenvalue equation correctly. At the third interview, he mentioned that he had "gotten messed up in the past," when he was finding eigenvectors from eigenvalues, and he thought that there was a "jump from eigenvalues to eigenvectors."

The researcher asked him how he would describe the concepts of winter term physics courses to a student if he were the teacher. He was told that he could assume that the students knew about matrix algebra. He stated that it was all about definitions and that the students needed to know what the words meant, and after that "it is almost entirely mechanics."

Overall it seemed that Deniz was fluent in finding eigenvalues and eigenvector at the time of the second interview, and he still seemed to recall the steps for finding them at the third interview. He knew that the characteristic equation would be used to find eigenvalues, and the eigenvalue equation would be used to find eigenvectors. There were no actor-oriented transfer episodes related to this particular goal.

Goal 3: The Eigenvalue Equation and Its Interpretations

One of the goals of LAW was to introduce the eigenvalue equation, $A\vec{v} = \lambda\vec{v}$ and its algebraic and geometric interpretations to students. The students were expected to learn both interpretations of the equation and apply them when appropriate.

In this section, Deniz's apparent understanding of eigenvalues and eigenvectors and his reasoning while he worked with them before and after LAW are described through ideas repeated at each interview. Then, episodes exploring actor-oriented transfer are discussed.

Ideas on the Eigenvalue Equation and Its Interpretations. No ideas seemed to be repeated through Deniz's data, but two different ideas related to the interpretations of the eigenvalue equations were observed separately at the second and third interview. Deniz did not mention the eigenvalue equation at the first interview.

Deniz seemed to be trying to use a geometric interpretation of eigenvectors when he answered the questions during the second interview. However, his first attempt to use a geometric interpretation of eigenvectors first seen in the data was on the post quiz. He described an eigenvector as "a vector along the line that when a reflection occurs the vector does not change direction" [Post-quiz]. He seemed that he was trying to describe an eigenvector through a reflection transformation. He stated that the direction of the eigenvector did not change; however, it was not clear if he knew this idea was true for all the other linear transformations. It was noted that his group had worked on a reflection matrix during the Linear Transformation activity on the second day of LAW.

At the second interview (approximately a week after the post-quiz), Deniz seemed to be trying to express eigenvectors geometrically, but he could not recall how to do it.

> ***I:*** *Eigenvalues and eigenvectors, what are they?*
>
> ***D:*** *Yes, there are the simplest vectors that are independent, linearly independent. They can be built, they…let's see…they point in some specific direction but I can't geometrically figure that out right now.*
>
> [I.2, lines 49-52]

After his explanation he was given the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ and asked to find its eigenvalues and eigenvectors first. He found eigenvalues and eigenvectors correctly, and then he was asked what the matrix did to the vectors. He proposed that the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ was a reflection, but he was not sure if his proposal was correct and how to check it. He was asked what would happen to eigenvectors under the reflection transformation. Deniz said that if it were "a reflection, then eigenvectors would all point along the axis of reflection" [I.2, lines 183-184]. He later continued explaining that eigenvectors were perpendicular to each other in this situation because the vectors along the line which was perpendicular to the axis of reflection would still point "along that line"; however, it was not clear which line he meant when he said "along that line." The researcher asked him to find the axis of reflection after his explanation.

D: *The axis of reflection? I know that the information is here [pointing to the pages].*
I: *Here, you mean on this page [pointing to the page he was working on], or all these pages?*
D: *Kind of all of the above, it's present. [pause] That's probably along this line. [pointing at* $\frac{-x}{\sqrt{3}} = y$ *which was formed when calculating the second eigenvector]*
I: *Why do you think so?*
D: *Because this [eigenvector 2] is a scalar multiple of this [line].Which would presumably, it reflects across that, is where we ended up.*
I: *And the other [eigenvector] one is….where will it be?*

D: *The other one should be perpendicular to it.*
[I.2, lines 195-212]

It seemed that he was trying to implement the geometric interpretation of eigenvectors in this question. To investigate what he knew about eigenvectors if the transformation was different than reflection, he was asked to give an example of a "scrinching" matrix. He did not seem to recall that the determinant of such matrix was zero; however, he stated that "everything got smashed onto one line, all of the vector no matter what you did to it." He said he was "fuzzy" about the "scrinching" matrix and tried to come up with an example of it. He proposed that $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ could be an example of such a matrix. He was asked how he could check if this was what he wanted.

> ***D:*** *I'm sure you could take the eigenvectors of it and then just pick some random vector, or operator, and multiply it and it should end up the same thing, no matter what it did. The same direction, rather, not the magnitude, because the magnitude's just a scalar multiple in that direction.*
>
> [I.2, lines 249-256]

His explanation was not clear. He seemed to talk about eigenvectors that were pointing in the same direction of the line to where all the vectors were mapped, but it may be possible that he was just talking about all the vectors, not just eigenvectors. As a result, , he was asked to multiply random vectors and eigenvectors with the matrix to see what happened to them. He multiplied the eigenvectors with the matrix. He got the eigenvalue multiple of the eigenvector back, then he seemed puzzled.

> ***D:*** *[pause] So this one, so, I suppose that the line of whatever is along y=x, so it ends up being like something like that. I suppose I could test it, so that would be….[pause] So I guess it would be, I don't know, we'll say that or something. [pause] There. [pause] Something is funny about that.. Maybe. I don't know the matrix multiplication, but…*

[pause] Yeah, I'm not really sure how to test that, I don't like that very much.

I: *Okay. So you don't know what that matrix does? Or…*

D: *Yeah, I can't really think of a way of testing it.*

I: *But you found the eigenvectors and eigenvalues?*

D: *Yeah, and they end up pointing like that.*

I: *Okay.*

D: *Where it goes from there, don't understand.*

[I.2, lines 260-273]

Deniz did not realize that the matrix he had picked was a reflection matrix; not a "scrinching" one, but, instead, he thought his matrix multiplication was incorrect. It seemed that he was not sure how the geometric interpretation of eigenvectors could be used in a situation like this one.

On the final exam and at the third interview Deniz seemed to use the algebraic interpretation of eigenvectors. He wrote: "A vector that; when operated on returns itself times a constant. In the eigenvalue equation ($E|\psi\rangle = \lambda|\psi\rangle$), it is ψ." A similar idea was observed at the third interview when he was asked how he could check if a vector was a superposition not an eigenstate. He stated that he did not remember at first but that he knew there was a way, and then he said, "When something or another comes back… when you saw it come back again, it is part of the equation when it returns itself; then, it is an eigenvalue or an eigenstate." It seemed that he was restating the algebraic interpretation of the eigenvalue equation. He later implemented this idea when he was solving another problem. He was given the eigenvalues and eigenvector of an unknown matrix M and asked to tell everything he could about this matrix. He wrote the eigenvalue equation.

D: *Because of the – I don't know what it is called – the eigenvalue, one of the eigen equations. Whoa, stop. Just* *like that [writing* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$*] , so those must be the same[pointing at the vectors* $\begin{bmatrix} x \\ y \end{bmatrix}$].

[I.3, lines 77-79]

Deniz did not mention the geometric interpretation of eigenvectors and the eigenvalue equation at the third interview. He referred to the algebraic interpretation of the eigenvalue equation at the third interview.

Overall, he seemed to be aware of the eigenvalue equation and its interpretations after LAW. He tried to implement the geometric interpretation of eigenvectors at the second interview and the algebraic interpretation of the equation at the third interview.

It was also noted that he did not say much about eigenvalues at the interviews. At first Deniz said it was explained to him that the "eigenvalue was like the magnitude of the eigenvector," but he was not really sure what it meant by that [I.1, lines 74-75]. Even after being introduced to eigenvalues and eigenvectors in LAW, the same idea appeared on the post quiz. He wrote "a magnitude of the eigenvector" when he was asked what an eigenvalue was [Post Quiz]. In the second or third interview, he did not bring up the magnitude idea. However, on the final exam (two weeks before the third interview), he again wrote the same answer, "A magnitude of the eigenvector, given by the eigenvalue relation E $|\psi\rangle$=λ$|\psi\rangle$ where λ is the eigenvalue of E" [Final exam, 1a]. Since it did not happen during the interviews, it was unclear what he had meant by "magnitude."

<u>Actor-Oriented Transfer Episodes Related to the Eigenvalue Equation and Its Interpretations:</u> Data from the second and third interviews were analyzed by implementing the AOT framework. The episodes constituting evidence of AOT were then categorized according to these goals, and the following episode describes the AOT of Deniz that seemed to be related to the third goal. The episode is described briefly; then, evidence of AOT is discussed.

*<u>Episode 2</u>.*This episode occurred at the second interview while Deniz was working on the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$. He was asked to find eigenvalues and eigenvectors of the matrix and also explain what the matrix did to vectors. Deniz found the eigenvalues and eigenvectors correctly. He conjectured that the matrix was reflecting the vectors, so he was asked what would happen to eigenvectors under this transformation. He said the eigenvectors would point along the axis of reflection.

D: *The eigenvectors, if this is a reflection, and let's just assume that it is, then the eigenvectors would all point along the axis of reflection*

I: *Okay. Both of them?*

D: *And then, one of them would be perpendicular to it because the other one is just negative one times the other, its just a scalar multiple, which would make it point in the other direction, but still along that line. And that's the most that I remember out of classes last week, but that might only be for rotation. And, I'm not totally clear what's going on there.*

[I.2, lines 185-193]

Then, he continued to talk about the axis of reflection. The researcher asked him to find the reflection line, and he said that the second eigenvector he had found was the reflection axis.

AOT Analysis of Episode 2. It was observed that Deniz was recalling some ideas from activities done during LAW. When Deniz was asked what would happen to the eigenvectors under the proposed transformation, he said that they would point along the axis of reflection. It was not clear if both eigenvectors were on the axis of reflection. In his further explanation, he tried to clarify it, but it seemed that he had some conflicting ideas. He said that the eigenvectors were perpendicular to each other but still both of them being along the same line: "And then, one of them would be perpendicular to it because the other one is just negative one times the other; it's just a scalar multiple, which would make it point in the other direction but still along that line."

It seemed like he was trying to remember some ideas from class because he finished his explanation by stating that this was the most he could remember from class. The researcher's in-classroom observation notes indicated that Deniz and his group were working on a reflection matrix, and it may have been possible that Deniz was trying to reconstruct ideas by relating the interview question to that particular group activity.

This was not the only time Deniz tried to explain eigenvectors using the reflection example from class. For example, when he was asked what an eigenvector was on the post quiz, he wrote: "A vector along the line that when a reflection occurs the vector does not change direction" (Post-Quiz)

In this episode Deniz explicitly stated that he was trying to recall ideas from LAW. His reconstruction of ideas at the interview was not spontaneous;

rather, they seemed to be rooted in his earlier experience during LAW. For this reason, this episode constitutes evidence of actor-oriented transfer.

Goal 4: Basis Vectors

One of the ideas that Professor Clayton expected students to understand after completing the winter term physics courses was that any vector could be linearly expanded with eigenvectors; in other words, she wanted them to realize that eigenvectors could be used as the basis vectors of the vector space. In this section, Deniz's emerging understanding of eigenvectors as basis vectors is described through repeating ideas and the result of an investigation of the AOT is discussed.

Ideas on Basis Vectors*:* Deniz repeated the idea that "eigenvectors are basis vectors" in all three interviews.

In the first interview, Deniz stated that eigenvectors were "kind of like the basis vector", but it seemed that he was not sure what that meant. To clarify his basis vector idea he was asked to give an example, but he said he could not provide one. Later in the second interview he was asked to comment on this particular episode from the first interview. It seemed that during the first interview Deniz could not fully recall how basis vectors were related to eigenvectors.

> ***I:*** *[plays back tape of previous interview] So, that's the part that I was going to ask you, what did you mean when you said it had something to do with basis vectors?*
>
> ***D:*** *Oh, that was just trying to pull from memory, using words that I remembered were involved, so, there was a base somewhere and that's about all I could remember. Yeah, it was kind of just repeating everything that was running back here[pointing to his head], it was just a vague memory.*
>
> [I.2, lines 316-324]

Deniz was able to talk more about eigenvectors as basis vectors at the second interview. He was asked to describe eigenvalues and eigenvectors, and he immediately said that the eigenvectors were basis vectors. It was noted that he did not say that they were "like" basis vectors, but declared that they "were" basis vectors. He continued his explanation by adding that eigenvectors were the "simplest vectors that are independent, linearly independent." He also included the geometric interpretation of eigenvectors in his description.

He was then asked to find the eigenvalues and eigenvectors of the matrix $A=\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ and talk about what they represented. After finding them, he stated that they were the basis of the matrix and that the matrix could be built by the eigenvectors. However, he was not certain about this idea and stated, "There I start to get fuzzy" [I.2, lines 97-102]. It seemed that he was recalling some ideas related to diagonalization of a matrix. However, there was no further evidence if he was thinking about diagonalization.

He also knew that basis vectors are linearly independent and span the vector space, and he brought up both of these ideas when he talked about eigenvectors as basis vectors. For example, he was asked about the relationship between the eigenvectors, $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ which he had found earlier at the second interview. He said, "They should be linearly independent. I can't make this one out of that one, and I can't make that one out of this one" [I.2, lines 112-113].

Similar ideas were observed at the third interview. He was asked if the sum of two eigenvectors was again an eigenvector. He started the question by first stating that eigenvectors needed to be orthogonal and linearly independent. The researcher asked him to explain further what he meant by linearly independent and orthogonal. He claimed that if two vectors were linearly independent that they would be orthogonal. He did not provide any further explanation on this idea and started to talk about the sum of two eigenvectors. He stated that the sum would not necessarily be an eigenvector. He said he was thinking about the "spanning" idea and gave an example from one of the winter term physics courses. He said "A wave could be a sum of eigenvectors, but it is not necessarily an eigenvector… but [the wave] itself is not necessarily an eigenvector, but it can be represented by a sum of them, a superposition" [Interview 3, lines 32-34].

Even though Deniz's understanding of eigenvectors seemed to evolve around basis vectors, it was not clear if he knew the distinction between basis vectors and eigenvectors. It seemed that he thought eigenvectors were another way of saying basis vectors.

> ***D:*** *From what I gather, more or less they are just a root, another word in order not to say basis again. They are what makes – they are, I guess, main points and for what I am not really particular sure. I guess that is kind of the point – for whatever your subject is. Is there a root of A [matrix]? Where it tends to, I suppose. You could build a basis. Eigenvectors in a three space, I would guess are x, y, and z.*
>
> ***I:*** *OK, x hat, y hat and z hat. So correct me if I misunderstood you. You said these are eigenvectors of R three?*
>
> ***D:*** *I am just trying to put together what I understand so far.*
>
> ***I:*** *Building onto that idea that you have, can I take like this vector, another vector here and another vector here [drawing three arbitrary vectors]. Let's pick these vectors. Can they be eigenvectors, too? Or can they be basis vectors?*
>
> ***D:*** *If we are to rotate it, yes.*
>
> ***I:*** *Rotate it?*

D: *Our origin, more of less. They could be just as long as they are all orthogonal to each other. As long as we can build anything out of them. We can add them all together and make anything anywhere.*

[I.3, lines 121-149]

It was unclear if Deniz was aware of the relationship between a matrix and its eigenvectors as basis vectors. He seemed to assume that eigenvectors of any matrix could form a basis of the vector space on which the matrix was defined. For example, let's take the matrix M= $\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, which is defined on R^3 (i.e., M takes vectors from R^3 to R^3). The eigenvalues of $\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ are 4 and 3 with multiplicities of 1 and 2, respectively, and the corresponding eigenvectors are $\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\}$ and $\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\}$. They only form a basis for a subspace of R^3 but not all of R^3. It seemed that Deniz did not consider this situation. Nevertheless, it seemed that Deniz's understanding of eigenvalues and eigenvectors was evolving around the idea that eigenvectors are basis vectors.

<u>Actor-Oriented Transfer Episodes Related to Basis Vectors:</u> The second and third interview data were analyzed by implementing the AOT framework. The following episode describes the AOT of Deniz that seemed to be related to the fourth goal. The episode is described briefly; then, evidence of AOT is discussed.

<u>Episode 3</u>. This episode was observed at the beginning of the third interview. Deniz was asked to comment on the sum of two eigenvectors if it would

be an eigenvector. He first talked about how eigenvalues of a matrix could be found using the characteristic equation. Then he mentioned that eigenvectors were orthogonal to each other and also linearly independent. Then he provided an example from one of the winter term physics courses he took after LAW.

> ***D:*** *Yeah.[pause] I'm still a bit fuzzy about even what eigenvectors are. So then if that is the case, I have two vectors that are linearly independent then, they would be orthogonal. If I were to add them – adding vectors – let's see. Just for kicks [mumbling], like that. This would be the sum of the vectors in that case, and is it an eigenvector? No, just the sum of eigenvectors, not necessarily an eigenvector. I am thinking about this whole spanning. It could be a sum of them. You could be given a wave that could be a sum of eigenvectors, that it is not necessarily an eigenvector, but it can be represented by a sum of them, super-position.*
>
> [Interview 3, lines 27-34]

He continued explaining the wave functions idea. He said that in one of the courses they looked at the wave function as a solution, and then they checked if the given wave function was an eigenstate. If they got 1, it meant the wave function goes to unity, which meant the wave function was an eigenstate. He was not explicit about how they check the wave function. He continued saying, "Sometimes it [wave function] wouldn't, unless you broke this one thing into two pieces. OK, I recognize one of the pieces, so it could be a sum of eigenvectors. Then, when you add them together, that is the super-position thing, and they can add together and create a new thing. That new thing is not necessarily an eigenvector or an eigenstate."

<u>*AOT Analysis of Episode 3.*</u> It was observed that Deniz first reminded himself about eigenvalues by stating how they were found through a characteristic

equation, and later, he focused on eigenvectors and thought of them as being perpendicular to each other. The same idea surfaced in the second interview, and it seemed like it became a part of his emerging understanding of eigenvectors. He seemed to refer to this idea when he was thinking about eigenvectors.

He also mentioned that he was thinking about the "spanning" idea as he worked on this third interview question. Even though he did not explicitly say that he was thinking of "eigenvectors as basis vectors", he had been proposing the basis vectors idea in all three interviews, and it seemed that his "spanning" idea was part of it. He seemed to be reorganizing his 'basis vectors" idea to provide an answer.

Deniz later gave an example of a wave function by stating that it could be the sum of two eigenvectors. This example did not seem to be a spontaneous construction and that it was unconnected to his previous experience. During the Waves course, which was offered during the fifth and seventh week of the winter term, a very similar example was discussed. Students were asked if the function

$$\phi(x) = \sqrt{\frac{1}{5}}\phi_1 + \sqrt{\frac{3}{5}}\phi_2 + \sqrt{\frac{1}{5}}\phi_3$$

was a eigenstate of the Hamiltonian, when each ϕ_i for i=1,2,3 was an eigenstate of the Hamiltonian. The students discussed the questions in small groups, and later there was a brief whole-class discussion. Deniz could be relating the third interview question to his experience in this class that day, and this could constitute evidence for actor-oriented transfer.

Summary

Deniz mentioned that matrices represented “lines” at the beginning of the first interview. He later referred to the columns of a matrix as column vectors and seemed to think that these vectors were forming the directions of the “lines” which were represented by the matrix. Even after being introduced to the idea of a matrix representing a linear transformation, he referred back to the lines idea at the second interview.

Deniz mentioned that the determinant idea that students discussed on the second day of LAW was one of the most interesting things he had learned that he did not know before. He, however, did not try to implement this idea when he was asked what the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ did to vectors. After being reminded, he decided to use it, but he was not sure if he recalled it correctly. Even though he mentioned that the determinant informed him on the behaviors of rotation and others, he did not use an explicit language indicating that he viewed matrices as linear transformations. It was unclear if Deniz viewed matrices as linear transformations.

Deniz mentioned at the first interview that he used to find eigenvalues and eigenvectors through an algorithm. He could not recall the algorithm at the first interview. It was also noted that he did not recall it on the pre-quiz at the beginning of LAW. After being introduced to the way of finding eigenvalues and eigenvectors in LAW, he was able to find eigenvalues correctly on the post-quiz of LAW. He, however, did not find the eigenvectors correctly. It seemed that he did not write the eigenvalue equation correctly on the post-quiz. At the second and third interviews, he was able to find eigenvalues and eigenvectors correctly. He also mentioned that he believed there was a conceptual jump between finding eigenvalues and eigenvectors, and he made mistakes previously when finding eigenvectors. He did not make any mistakes during the last two interviews.

Deniz seemed to be more familiar with the eigenvalue equation at the second and third interview. It seemed that his understanding of the eigenvalue equation was more developed at the third interview. Deniz was probably exposed to the equation prior to LAW week; however, he did not mention it during the first interview or on the pre-quiz. In the second and third interview, Deniz used the equation many times; it seemed that he was attempting to connect the interview question to his previous experience with the equation. From the researcher's perspective, Deniz was transferring his experience from the courses to the interviews.

A geometric interpretation of eigenvectors was observed in Deniz's second interview data. Deniz did not seem to know a geometric interpretation for eigenvalues and eigenvectors prior to the LAW week, but it may be possible that he could not recall it at the time of the first interview. He, however, was aware of the geometric interpretation of eigenvectors at the second interview. On the post-quiz and during the second interview, he had attempted to answer problems using the geometric interpretation of eigenvectors.

He recalled the algebraic interpretation of the eigenvalue equation at the third interview. He did not mention the geometric interpretation of the eigenvalue equation at the third interview. One reason for not observing the geometric interpretation may be that the third interview questions could be solved without this interpretation. It seemed that Deniz's understanding of the eigenvalue equation and its interpretation was not comprehensive, but it seemed to be developing.

Deniz described eigenvectors as basis vectors during all three interviews. At the first interview he stated that eigenvectors were like basis vectors, but he was not sure what it really meant. It seemed that his experience in LAW and the courses after LAW helped him to develop this idea further. Even though it was not clear if he knew the distinction between eigenvectors and basis vectors, he kept referring to eigenvectors as basis vector when he got stuck with other descriptions of eigenvectors. Overall, Deniz preferred to think of eigenvectors as basis vectors, and his understanding of eigenvectors as basis vectors seemed to develop through his experiences.

Gus

Gus was a junior engineering physics major and expected to graduate at the end of spring term 2009. He had taken a calculus sequence (three courses), a vector calculus course, and a differential equations course during his first two years in college. These were the only mathematics courses he took prior to the study. He said that he had planned to take either the second vector calculus course or a linear algebra course during his junior year because he thought these courses might help him in graduate school, which he was considering. He took the second course of a vector calculus sequence during the spring term of his junior year. He took all three physics courses the during fall term of 2007. Table 4.3 shows the courses in which he was enrolled during this study.

Table 4.3 Courses Gus took during the study

Course Name	**Term Taken**
3 Modules of Physics Courses	Fall Term 2007
3 Modules of Physics Courses	Winter Term 2008
Mechanical Engineering Course-Dynamics	Winter Term 2008
Engineering course-Numerical analysis and DE	Winter Term 2008
3 Modules of Physics Courses	Spring Term 2008
Engineering Economy	Spring Term 2008
Vector Calculus 2	Spring 2008

At the first interview, Gus mentioned that the second Vector Calculus course was required for the fall term physics courses, but without taking it he had “made it through them.” Later, in the same interview when he was talking about the dot and cross product of vectors, he tried to explain these concepts using a vector differential operator. After struggling for some time, he said these were the topics covered in the second Vector Calculus course and in then physics course he had taken at the beginning of fall term. He said he still did not “seem to know” them.

The examples Gus used to explain a concept during the interviews and in class were all contextualized. For example, when he was asked what a matrix represented, he gave an example from his engineering course with beam forces and wrote:

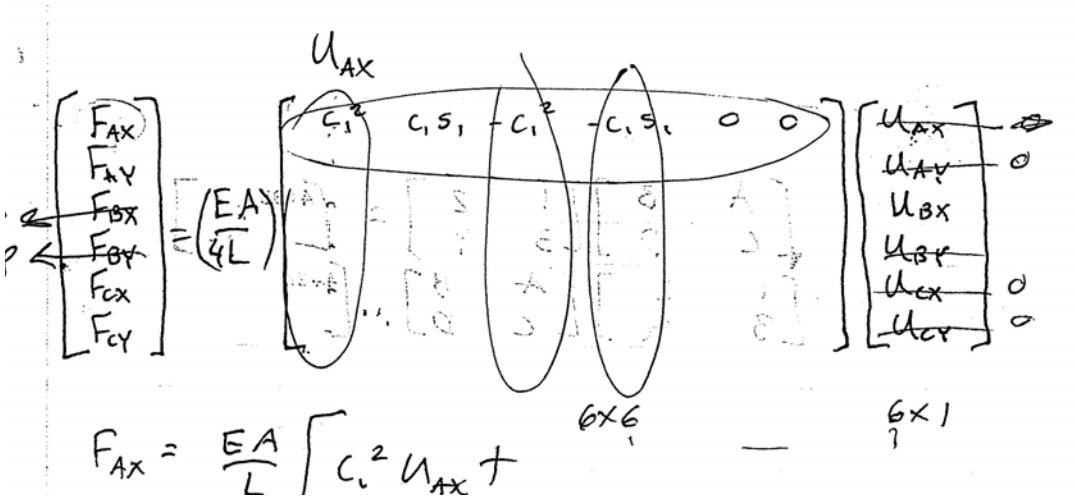


Figure 4.10 Gus' matrix example

During the same interview he tried to explain dot product by taking the divergence of the electric field to find the electrical potential.

Gus was an active student during LAW. He participated in whole-class discussions and in small-group activities. For example, on the second day of LAW the students were trying to come up with a hypothesis which related the determinants of matrices to their actions on vectors. After discussing three examples, he pointed out that the matrix with determinant value 1 rotated the vectors and a determinant of negative 1 reflected them. Then, the students worked on this hypothesis and added more ideas to it (Classroom Observation Notes of LAW).

Gus was also active in the small-group activities. He presented the results of both small activities and answered Professor Clayton's questions during his presentation. During the second small-group activity, he had some questions related to finding eigenvectors. He wanted to know the reasons for finding the free variable and what they meant, but his group members did not know the answer. He did not ask that question to Professor Clayton. Gus seemed to be the only student who had questions on free variables.

The following section describes the repeating ideas observed in Gus's data as they are categorized according to the four chosen goals of LAW and the winter term physics courses. Then, Gus's attempts to use his experience from LAW and the courses he had taken after LAW at the second and third interviews are also analyzed and described by implementing the *Actor-Oriented Transfer Framework* as they fit under the four chosen goals of LAW and winter-term physics courses. Gus was one of the few students who did not take a linear algebra course prior to the winter term of 2008.

Goal 1: Matrix representation

One of the goals of LAW was that students were expected to view matrices as linear transformations. In this section, Gus' view of matrices before and after LAW as they repeated during the interviews are described. Also the exploration of whether Gus used his experience from LAW and winter term physics courses at the second and third interviews is reported.

<u>*Ideas on Matrix Representation*</u>*:* Gus repeated two phrases, "transforms" and "operate on," as he talked about matrices throughout his second and third interviews. These words were not used in the first interview.

At the first interview, Gus described a matrix as representing a system of equations. He said that it was a tool that made the solving of the equations easier. He said it was hard for him to think about matrices without a problem, so he was asked to come up with an example. He then used an example from one of his engineering courses to explain what he meant by a system of equations (see Figure 4.10). He mentioned that the matrix representation condensed the original equations for the force vector and also made it easier to enter in the computer. He also pointed out that even though he said this was how he thought about matrices, it was how the computer thought about them. The matrix in Figure 4.10 can be thought of as an operator acting on the displacement vectors to find the resultant force at a given point. To investigate if Gus was aware of this idea, he was asked to explain his example. In his explanation, it became clear that he used the matrix to represent the given system, and he did not seem to be aware of the idea of matrices acting on a vector. Gus did not mention the idea of a matrix representing a system of equations in the second and the third interviews.

At the beginning of the second interview, as he was explaining eigenvectors, Gus started to use the word "transform" for the matrix. He said that when the determinant of a matrix was zero that it transformed the vectors onto a line, and the vectors that were already on this particular line would not change.

He was consistent with his use of "transform" throughout the second interview.

For example, he wrote the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ to represent a rotation of 90º, and to check his work he said that he would transform $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$ using the matrix. Later, in the second interview, he was asked to give an example of the transformation that he had talked about at the beginning of the interview, the one that transformed vectors on to a line. Then, he proposed the matrix $\begin{pmatrix} 1 & 5 \\ 2 & 10 \end{pmatrix}$, and he multiplied an assortment of vectors with this matrix. As he was multiplying the vectors, he used the word "transforming." He later found the eigenvalues of the matrix $\begin{pmatrix} 1 & 5 \\ 2 & 10 \end{pmatrix}$. As he was explaining the unknowns of the eigenvalue equation for this particular problem, he said, "Well we know λ now and we know A. So, v_1 which so this is like if you transform v_1 using A, it is the same as transforming v_1. And that would find the…that would if you knew λ and if you knew A, it would give you your eigenvectors" [I.2, lines 385-387].

He also summarized the interdependence of a matrix, its determinant and the vectors upon which it acts (what matrix does to vectors according to its determinant). He said if a matrix had a positive determinant, it rotated the vectors, and if the matrix had a negative determinant it reflected the vectors.

In the third interview he consistently used the word "operate on" as he talked about matrices. For example, he was given the eigenvalues and eigenvectors of an unknown matrix M and asked what he could say about this matrix.

G: *OK. These matrices are hard to work with now. That was back with what we did with spins. That was the first half of the course that we used those, [inaudible] functions. They are way back there in the memory. So what can I tell you? I am trying to tell you about the operator, so.*

I: *I heard some people like to call it a matrix. Some people like to call it an operator. Whatever you wish.*

G: *An operator seems to work better. So, let's see. [Writing] An operator, as far as I know, has to act on something, so it has to act on these two, but maybe, yeah, so it has to act on these two and return this eigenvalue. So if it is [mumbling]. So it somehow looks like it is giving me that thing,*

> *somehow. I am not sure how but it seems to be [inaudible] out here, too. And how are we giving me that value*
> [I.3, lines 162-172]

He was later asked once more if he considered matrices as operators. He said, "Yes, so I think, like the poly-spin matrices as operators. Those are operators." He continued to explain that these matrices acting on vectors gave back vectors. After some calculations, he said that the matrix M reflected vectors over the $y = x$ line, and he checked this claim by applying it to another vector.

Overall, Gus seemed to treat matrices as linear transformations that transform vectors at the second interview after LAW, and his matrix representation idea then formed into an "operator" that acts on vectors at the third interview.

<u>Actor-Oriented Transfer Episode Related to Matrix Representation</u>*:* The second and third interview data were analyzed by implementing the actor-oriented transfer framework. The following episode describes the AOT of Gus that seemed to be related to the first goal. The episode is described briefly; then, the evidence of AOT is discussed.

*<u>Episode 1.</u>*This episode was observed during the second interview. Gus started to talk about matrices with determinants of zero when he was explaining eigenvectors. He was then asked to give an example of a matrix with determinant zero. He looked at the eigenvalue equation that he had written previously and said that A was the transformation and that he wanted the determinant of A, ad-bc, equal to zero. He later assigned trigonometric functions to each entry of the matrix A.

$$
\begin{aligned}
a &= \cos\theta \\
b &= \sin\theta \\
c &= \cos\theta \\
d &= \sin\theta
\end{aligned}
$$

He was asked why he chose those values for the matrix.

> ***G:*** *Well we did it, I was trying to remember we had a homework problem that dealt with this but. It was a little different it wasn't exactly this, but it made me think of this. I think it was like if you have, I think it was like, um it was ... (pause) It doesn't choose, you have to choose the a,b,c,d so there are a equal [inaudible] It ends up giving you the um a cosine squared of theta plus sin squared of theta. So it's positive, which is the rotation matrix.*
>
> [I.2, lines 78-82]

He checked his work by multiplying the vector $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$ with the matrix and got the transformed vector $\begin{pmatrix} -5 \\ 1 \end{pmatrix}$. He was asked how he knew that $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$ rotated 90 degrees would give the vector $\begin{pmatrix} -5 \\ 1 \end{pmatrix}$. He said these vectors were perpendicular to each other, and he checked it by taking the dot products of these vectors.

Then, he was reminded of the question on which he was working, and that was to find a matrix representation of the transformation that would take the vectors to a line. He wrote the matrix $\begin{pmatrix} 1 & 5 \\ 2 & 10 \end{pmatrix}$ and said, "Two vectors with the same slope. So, they are linearly independent, right? Is that a good way to put that, I think?" He was asked to check his ideas. He first calculated the determinant of the matrix and said he was trying to think about formulas. He said that he was trying to remember what they did in class, and "for some reason it is not coming back to me very quickly." He glanced at the eigenvalue equation and read the equation aloud. He was asked if he wanted to use that equation, and he said that he would use that but "To check that [what the matrix does] don't I have to come up with vectors and multiply them to see. I guess that is what I was going to do, but that wasn't the right formula to use. So, let's use one like $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and one with like." He then wrote three more vectors and multiplied them with the matrix.

After finding the transformed vectors, he graphed all the vectors, the initial and the transformed ones. He later decided to try the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and said this was an eigenvector. Then, he decided to transform the vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ too. He claimed that $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ would transform to the zero vector.

> ***G:*** *Because it starts out why I do that, I am so bad at that. 2,-1. Its slope is the, it's orthogonal to the line that everything gets put onto. So we are going to see what happens to it, if it goes to zero or if it goes to the other side. I already know it is going to zero. I am going to do it anyways sometimes you get surprised in physics. (pause) hum, maybe it didn't go to zero. Did I do that right? (pause) I did it wrong, didn't I? No I did. I would have thought that all the vectors brought onto this line and magnitude changes. Since this doesn't have any piece on this line that it would go to zero but apparently that is not true.*
>
> [I.2, lines 202-209]

He noticed that the vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ was transformed to negative three times the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. He was confused with this result and stated that he thought about the matrices with determinant zero (these matrices were called "scrinch" in LAW) as projections.

> ***G:*** *So, when we use the determinant, the term scrinch in class how I thought about it is you take this vector and dot it onto your scrinch, or scrinch line or whatever. And then it was the size changed some, but it seemed to be kind of proportional to two things: 1. How close your original vector was to the scrinch line, so how much was dotted on there and how the magnitude of the original line. So if you were using all unit vectors it would just be based on how close they were to the line, because they all have the same magnitude. But these ones I guess they don't have, there not well they are as far from the line as you can get so I guess it is based only on their magnitude. So maybe how close you are to the line doesn't matter, that is what I thought about when I thought of scrinch, because you scrinch them onto the line. That I what I think about when I think of that. So it seems like it rotates vectors orthogonal to scrinch line by pi over two, or negative pi over two, maybe plus or minus, I don't know. And it possibly reflects vectors on scrinch line and then it dots other vectors onto the scrinch line, I guess. Dots other vectors onto the scrinch line. You would need more examples to prove these. That is just yeah.*

[I.2, lines 328-341]

Then, he was asked to find eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 1 & 5 \\ 2 & 10 \end{pmatrix}$.

<u>*AOT Analysis of Episode 1.*</u>Gus was referred to the ideas that were introduced in the activity, Linear Transformations, on the second day of LAW. He brought up the determinant idea when he was explaining eigenvalues and eigenvector, and he used this idea throughout the second interview. He initially mentioned the matrices with determinants zero, and when he was asked to give an example, he tried to recall his previous experience with these ideas. He assigned trigonometric functions to the entries of the matrix A, which was done in class on the third day of LAW and was a part of the first assignment. He explicitly stated

that the interview situation made him think about an assignment in which they had looked at the matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. It seemed that Gus was trying to connect to his previous experiences, and as he thought about them, he was trying to build similarities between them and the interview situation.

His transfer, however, was not restricted to the connection he had made to the assignment. As he worked on the question further, he was referring to ideas used in the Linear Transformation Activity. For example, he seemed to recall the discussion in class on linearly dependency of column vectors in a matrix and the determinant being zero. This idea was discussed on the second day of LAW during the Linear Transformation activity, when a group had presented their results. The matrix under consideration was $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ and Professor Clayton pointed out that determinant zero was also related to the idea of vectors being linearly dependent. It seemed that Gus was trying to refer to this conversation, but he used the term independent instead. Since the actor-oriented transfer framework attempts to build an understanding about the student's uses of their previous experiences as they work on the problems at the interview setting, this part of Episode 1 constitutes evidence of actor-oriented transfer.

Gus picked four vectors which were very similar to the ones from the Linear Transformation activity and transformed them. He then drew the initial and transformed vectors. All of these ideas arose during the Linear Transformation activity on the second day of LAW. As he realized that the vectors were mapped to the line y=2x, he decided to pick the vector $\begin{pmatrix}1\\2\end{pmatrix}$ and a vector $\begin{pmatrix}2\\-1\end{pmatrix}$ that was perpendicular to it. He knew that $\begin{pmatrix}1\\2\end{pmatrix}$ would be mapped to a scalar multiple of itself and he assumed that $\begin{pmatrix}2\\-1\end{pmatrix}$ would be mapped to the zero vector. As he worked on the calculations, he realized that it was mapped to negative three times the vector $\begin{pmatrix}1\\2\end{pmatrix}$, in other words, to a vector on the same line. Then, he referred back to the ideas that were related to the matrices with determinant zero. As a class, they had decided to call these types of matrices "scrinch" matrices because Professor Clayton pointed out that not all matrices with determinant zero were projections. She said that when the projection matrix was squared it equaled itself, and the students checked this idea with the matrix $\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}$ on the second day of LAW. It seemed that Gus was not recalling these discussions. His reasoning indicated he had assumed that the matrix $\begin{pmatrix}1 & 5\\2 & 10\end{pmatrix}$ would do an orthogonal projection of vectors onto the y=2x line instead of assigning them to the line y=2x.

As he explained what he thought about the "scrinch" matrices during class and the situation at the interview, he seemed to be reconstructing his experience with "scrinch" matrices.

Overall, during Episode 1 it was observed that Gus found the interview question similar to the assignment he had completed during LAW. He also seemed to recall a discussion on the linear dependency from the second day of LAW; in other words, he referred to his previous experience. As he referred to the ideas related to "scrinch" matrices, he seemed to reorganize his experience as he answered the interview question. For all of these reasons, Episode 1 constitutes evidence for actor-oriented transfer.

Goal 2: Finding Eigenvalues and Eigenvectors

Another goal of LAW was that the students were expected to be fluent in finding eigenvalues and eigenvectors of a matrix or an operator. It was noticed that there were no repeating ideas that were related to finding eigenvalues and eigenvectors in Gus's data.

At the first interview Gus did not recall anything related to finding eigenvalues and eigenvectors. On the pre-quiz, students were asked to find eigenvalues and eigenvectors of the matrix, $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ and Gus wrote the matrix $\begin{pmatrix} 0-\lambda & 2 \\ 2 & 0-\lambda \end{pmatrix}$ and then the equation $\lambda^2 - 4 = 0$, which was followed by $\lambda = \pm 2$. It seemed that Gus was aware of how to find eigenvalues; however, he did not attempt to find the eigenvectors on the pre-quiz.

On the post-quiz, Gus tried to find the eigenvalues; however, he had made an error when taking the determinant of the matrix $\begin{pmatrix} \lambda - 2i & -3 \\ 0 & \lambda + 7 \end{pmatrix}$ and wrote $(\lambda - 2i)(\lambda + 7) + 3 = 0$. He wrote the eigenvalue equation with the matrix of this particular question on the quiz. It seemed that Gus was planning to use the equation to find eigenvectors.

At the second interview, Gus wrote the characteristic equation and said this was how eigenvalues could be found, and then he wrote the eigenvalue equation $A\vec{v} = \lambda\vec{v}$. He wrote these equations while he was trying to come up with a matrix that had determinant zero. Once, when he proposed the matrix $\begin{pmatrix} 1 & 5 \\ 2 & 10 \end{pmatrix}$, he was asked about the eigenvectors of $\begin{pmatrix} 1 & 5 \\ 2 & 10 \end{pmatrix}$. He tried to find the eigenvector of this particular matrix by graphing vectors and their images under this transformation. He claimed that $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ was an eigenvector of the matrix. He was asked if there were any other eigenvectors or how he could check to see if the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ was an eigenvector. He insisted on looking at the graph and did not use the eigenvalue equation to check his answer. Even though the researcher suggested "using an equation", Gus decided to use the graph.

He was first asked if he knew any of the eigenvalues of this matrix and then asked to find it. He said he needed to use the equation $\det(\lambda I - A) = 0$ instead of the eigenvalue equation, and when asked why, he replied, "I am sure there is a way to do it from here, but I don't know it." [I.2, lines 365] He found eigenvalues 0 and 11; however, he was surprised with the value 0.

G: *Um, oops. I see that it all drops out which means the determinant is, ok hold on. So, oops, that zero that is what I just told myself. Well that will just be 11 so wow 11/2 – 11/2, which will be huh that is weird. So you just get zero.*

I: *Why?*

G: *Well because it is self plus or minus itself. I just think that is weird, I am not sure why but I guess. Well um so your eigen values are 0 and 11 and (inaudible) so I think. So*

> *$Av=\lambda v$, that is kind of weird. Um, I don't know if I am doing this right, but.*
>
> ***I**: What is bothering you?*
>
> ***G:** Um, well in class we didn't, we used this one $Av=\lambda v$ where we didn't know v or λ, and I don't know how to use the formula apparently because I am confused on what you are trying to get out of this, I guess.*
>
> [I.2, lines 372-382]

He was then asked what were the unknowns of the equation $Av=\lambda v$, and he stated that he knew λ and A. It seemed that he was puzzled by getting a zero value for λ. Later he pointed out that that eigenvalue did not tell him anything.

A similar situation was observed later in the second interview when he was working with a reflection matrix. He again graphed the vectors and their images under the transformation and tried to find eigenvectors on the graph. However, he could not find them on the graph this time. He stated that he could find a potential area where they could be according to his graph; however, he did not try to find them using the eigenvalue equation. He could not complete this question because there was not enough time. At the third interview, he did not talk much about finding eigenvalues and eigenvectors. During the first question of the third interview, he wrote the characteristic equation and the eigenvalue equation and stated these equations would be used to find eigenvalues and eigenvectors, respectively.

Overall, it seemed that Gus was aware of how to find eigenvalues using the equation, $\det(\lambda I - A) = 0$ but he did not choose to follow the algorithm to find eigenvectors during the second interview. He rather used graphs to find eigenvectors. It may be possible that he did not feel comfortable using the equation to find eigenvectors; however, no conclusion could be made with this data.

In Gus's data, there were no episodes with evidence of actor-oriented transfer; however, the researcher observed an episode in which Gus seemed to transfer a suggestion made by Professor Clayton during the fourth day of LAW. In this episode, Gus was working on the matrix $\begin{pmatrix} 1 & 5 \\ 2 & 10 \end{pmatrix}$ in the second interview, and he was asked what the eigenvectors of this transformation were. He said the only eigenvector was $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ because its direction was unchanged. He stated that the "slope" of each vector he tried was all changed. He was asked how he could be sure that there were no other eigenvectors.

> ***G:*** *Oh well, this would be so looking at the graph I would know because knowing everything got scrinched to, the slope of that line. The original, the original vectors would have to, their slope would have to, would have been the same as the slope of this line[y=2x]. Or, well even if they are negative it could be the same thing I guess. I guess it would just have to be based on their slope, because the size doesn't matter just the slope. So I would just look at, just look at the rise over run and see if they are the same as the original, I guess.*
>
> [I.2, lines 350-356]

He did not do any further calculations to check his work and stated that he was convinced graphically. Gus seemed to use the geometric interpretation of the eigenvalue equation to reason through his answer for eigenvectors. He graphed vectors and figured the line $y=2x$ to which vectors were mapped, and then he found a vector which had the same slope as this line. Since the vector he found $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ did not change its slope, he claimed this was an eigenvector. It seemed that he decided no change in slope implied no direction change of the vector.

Even though in this episode Gus did not explicitly state that he was referring to any activities or ideas from LAW, it seemed that he was using some ideas from day four of LAW. On the fourth day of LAW, Professor Clayton stopped Gus right before he presented his group's results on the eigenvalues and eigenvectors activity. He was about to present their calculations for finding eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Professor Clayton asked him what their matrix did and which were the unchanged vectors. He answered these questions with the help of the class, and later Professor Clayton suggested that before doing any calculations, they should think about the possible answers; in other words, they should figure out what the matrix did and the vectors that were unchanged. She later stated that this way they would make fewer algebra mistakes. Since this conversation took place during Gus's presentation, it seemed possible that he recalled her suggestion. He found what the matrix $\begin{pmatrix} 1 & 5 \\ 2 & 10 \end{pmatrix}$ did and geometrically found an eigenvector. For this reason Episode 2 could be considered as evidence of actor-oriented transfer.

Goal 3: The Eigenvalue Equation and its interpretations

One of the goals of LAW was to introduce the eigenvalue equation, $A\vec{v} = \lambda\vec{v}$ and its algebraic and geometric interpretations to students. Students were expected to learn both interpretations of the equation and apply them when appropriate. In this section, Gus's descriptions of eigenvalues and eigenvectors and his reasoning with these descriptions before and after LAW are described through the repeating ideas. Then, an episode exploring actor-oriented transfer is discussed.

Ideas on the Eigenvalue Equation and Its interpretations. One repeating idea was observed in Gus's data. He repeatedly referred to the geometric interpretation of eigenvectors as he described eigenvectors during both the second and third interviews. He also used the algebraic and geometric interpretation of the eigenvalue equation; however, these ideas were not observed separately in the second and third interview. Gus attempted to interpret eigenvalues geometrically and algebraically during the second interview.

In the first interview, Gus recalled that he had seen the concept of eigenvalues and eigenvectors in an electrical engineering course when they were working on matrices. He could not recall what they were, but he said that he knew the variable λ was used to represent one of them.

At the second and third interview, Gus described eigenvectors as the vector that is unchanged by the transformation (operator).

> ***G:*** *[...] When you transform vectors the one that is either not change or the exact opposite is the eigenvector. So, like when you transform, let's see here, if your determinant is zero then it transforms, it takes either all the y's or all the x, or it dots it onto a certain line. I guess it could be any line. The one that is already aligned along that line doesn't lose any information or get changed, that one right there is an eigenvector.*
>
> [I.2, lines 9-14]

He continued his explanation by stating that he did not actually know what the last part meant, but he knew that the one that did not change direction was an eigenvector. He kept talking about the example he provided in his initial description at the second interview. He stated that the matrix with determinant zero did not rotate or flip a vector, but the matrix put it on a certain line. As he explained these ideas, he started to graph the vectors on a Cartesian plane.

He explained that the vectors were sent to the dotted line, and the one that is already on that line would be an eigenvector because it would not change direction since it was already on the line to where the vectors were sent. When he talked about eigenvectors he referred to his graph and gave examples from it. Later he was asked to suggest a matrix that could be used as an example of the situation to which he was referring, and he suggested the matrix $\begin{pmatrix} 1 & 5 \\ 2 & 10 \end{pmatrix}$. He transformed some vectors and graphed the initial and transformed vectors. Once he was done graphing, he decided to transform the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. He stated correctly that this vector was an eigenvector because its slope did not change, and it was on the line to which all the vectors were sent.

He was later asked about other types of transformations. He suggested the matrix $\begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix}$ which has a negative determinant (-2) and claimed that it was a reflection matrix. He was asked to check his claim and to find eigenvalues and eigenvectors. He again decided to transform some vectors and graph the initial and transform vectors. While he was working on that idea, he was asked if he had any guess where eigenvectors would be. He claimed that the line with the slope -2 (the determinant of the matrix) would be the reflection axis and an eigenvector would lie on this line. He continued stating that the line perpendicular to the y=1/2x would also have an eigenvector.

> ***G:*** *Because it will do this type of thing, it will reflect it over. So the line with the eigenvector goes along the one with the slope. So if you have the line along here the one that is orthogonal to that line since this is a reflection should be reflected right over that line and then would be negative. The slope would be the negative reciprocal of the original matrix.*
>
> [I.2, lines 564-568]

It was not clear why he thought the determinant of the matrix would be related to the slope of the reflection line, and he did not have enough time to finish this problem. However, it was noticed that he was aware of the idea that the reflection line and the line perpendicular to it would have eigenvectors.

The geometric interpretation of eigenvectors was again observed at the third interview. For example, he was give the eigenvalues and eigenvectors of an unknown matrix M and asked what he could say about this matrix. After trying some matrices, he used the eigenvalue equation to find the matrix M=$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. He explained that the vectors that lie along the direction of eigenvectors of the matrix M would get scaled when transformed by the matrix, and their direction would not be flipped. He then stated that the matrix M was reflecting vectors over the $y = x$ line and demonstrated that a vector which did not have the same direction with eigenvectors would be reflected over the $y = x$ line.

It was also observed that during the third interview, Gus was referring to the algebraic interpretation of the eigenvalue equation. For example, when he was asked what eigenvectors were, he stated that if an operator acted on a vector, it returned an eigenvalue times this particular vector, and he wrote the eigenvalue equation $\hat{H}[A] = E[A]$. He later changed [A] to $\vec{v}_1$ and rewrote the equation as $\hat{H}\vec{v}_1 = E\vec{v}_1$. Later, he used the eigenvalue equation and its algebraic interpretation to find the matrix M. As he was trying different matrices in the equation, he stated that he "wanted to get the same thing back." He was asked why he had wanted the same thing back.

> **G:** *I am struck on this whole thing that if you act an operator – I don't know if this is true, but when you act an operator on something – when you act an operator on a state that is an eigenvector, eigenfunction, whatever, of that operator, you get back the eigenvalue times that eigenvector. These are eigenvalues?*
> **I:** *Yes.*
> **G:** *Yeah, so that is an eigenvector. So when you operate on an eigenvector, then you will get that same eigenvector back times the constant of some sort. So I am trying to use something like familiar to think about this example.*
> [I.3, lines 195-202]

In the second interview, Gus seemed to attempt to make sense of eigenvalues using the eigenvalue equation. He found an eigenvector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ of the matrix $\begin{pmatrix} 1 & 5 \\ 2 & 10 \end{pmatrix}$ by acting the matrix on the vector. Later he was asked about the eigenvalues of $\begin{pmatrix} 1 & 5 \\ 2 & 10 \end{pmatrix}$. He correctly calculated eigenvalues using the characteristic equation. He said there were two eigenvalues 0 and 11, but he was confused by a zero for an eigenvalue.

> **G:** *So this one is kind of confusing because this one [λ=0] doesn't really tell me anything, because that just multiply by zero you would get zero. Well, yeah well at least anything I can think of right now, or anything that matters in this problem. So that, see that is what I was getting. So if my*

> *value, if you multiply a vector by zero you would just get, it just doesn't do anything for me. Yeah so I don't know how that helps, I am not sure. But the eleven, I can deal with the eleven and, where is this one[searching through his previous calculations]. The one that will lie, oh yeah this makes sense now. The one that will lie along the line, this vector that was unchanged was scaled up by eleven, sounds about right.*
>
> *So what I possibly thought earlier was right, that eleven there would scale up factor is an eigenvalue.*
>
> [Interview 2, lines 400-408]

He decided later that zero was not "a real eigenvalue." [I.2, line 418].When he earlier transformed the vector $\begin{pmatrix}1\\2\end{pmatrix}$, he found 11 times this vector. He stated that $\begin{pmatrix}1\\2\end{pmatrix}$ was an eigenvector; he initially did not say anything about the scalar 11. After his calculation of eigenvalue, he recognized it in his calculations and said the scale that he had found earlier was an eigenvalue. Gus later mentioned that the eigenvalue zero still did not make any sense while he was reasoning through the eigenvalue equation.

> ***G:*** *And, but the thing is that this, so when I took a few, when I made up this matrix and took just a few different vectors I pretty quickly found out that the eigen vector wasn't going to be one two. And so then I could use that one a, v_1 use the v_1 that I know and do that multiplication, get my eigenvalue. But, that still doesn't give me my zero, so maybe that one is not a real eigenvalue. Maybe that just doesn't work. I just don't see how it would really make any sense.*
>
> [I.2, lines 414-419]

He kept talking more about eigenvalues, and he said that he did not know "what the significance of how much" an eigenvector scaled up and why the scaling value would be important.

Overall, it was observed that Gus repeatedly used the geometric interpretation of eigenvectors in both the second and third interviews. It was noticed that he was trying to use an algebraic interpretation of the eigenvalue equation in the third interview. This may be because students were working on functions and differential operators during the last two courses of the winter term physics courses, so an algebraic interpretation might have made more sense to them.

Actor-Oriented Transfer Episodes Related to the Eigenvalue Equation and its interpretations: The second and third interviews data were analyzed by implementing actor-oriented transfer framework. The episodes constituting evidence of AOT were then categorized according to these goals and the following episode describes the AOT of Gus that seemed to be related to the third goal. The episode is described briefly; then, evidence for AOT is discussed.

Episode 2. This episode was observed during the second interview. Gus suggested the matrix $\begin{pmatrix} 1 & 5 \\ 2 & 10 \end{pmatrix}$ and was asked to figure out what it did to vectors and to find its eigenvalues and eigenvectors. He wrote four vectors, transformed them, and then drew the graph of the initial and transformed vectors. He later added a fifth vector, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ which he seemed to obtain by looking at the *y=2x* line. He transformed this vector and noticed that it was scaled 11 times. When he was asked which vector was an eigenvector, he said $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ was an eigenvector. He mentioned that it was on the line to which all vectors were sent, and the slope of the eigenvector did not change. Later in the interview he decided to add the vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and claimed that since it was perpendicular to $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, it would be an eigenvector. As he was transforming the vector, $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ he said that he knew it would be mapped to the vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and he did not need to do all of the calculations. However, he then said that he would do it because "sometimes you get surprised in physics." He then found that the vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ was not mapped to the zero vector but rather mapped onto the line $y=2x$. He was surprised with the result.

> **G:** *OK, I know in class there was a lot of weird stuff in class when the determinant is zero, you know it puts them on these lines. But I remember being, so maybe that is why. But I still don't like that.*
>
> [I.2, lines 227-229]

He continued stating that he thought the projection of the vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ onto the line would be zero. Then he decided to rewrite the vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ using the basis vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and operate on the components to convince himself with the result.

<u>AOT Analysis of Episode 2</u>. It was observed that Gus initially worked on four vectors, and as he was graphing these vectors, he decided to add a fifth vector. He chose that vector to be $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, and while he transformed it, he noticed that it got

11 times bigger. It seemed that as he was graphing the transformed vectors; he realized that all the vectors were sent to the line $y = 2x$ (and he put a dotted red line for y=2x), and then he chose his fifth vector to be on this line. At the beginning of the second interview, as he was explaining one interesting thing that he learned form LAW, Gus gave a similar example and stated that the eigenvectors of "scrunching" matrices would be on the line on which all of the vectors were mapped. It seemed that by choosing the fifth vector on this line, he was implementing his idea.

He later decided to pick the vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ as a possible eigenvector because it was perpendicular to the line $y = 2x$. Once he realized that his claim was not correct, he seemed to be confused and immediately thought about his class experience. It seemed that Gus was reconstructing the geometric interpretation of eigenvectors of the "scrunching" matrix experience from LAW during the interview. Since Gus' way of solving this problem had roots in his previous experience in LAW, this episode constitutes evidence for actor-oriented transfer.

Goal 4: Using Eigenvectors as Basis Vectors

One of Professor Clayton's goals was that students should understand that any vector could be linearly expanded with eigenvectors; in other words, she wanted students to realize that eigenvectors could be used as the basis vectors of the vector space.

It was noted that Gus did not mention that eigenvectors could be used as basis vectors during the first two interviews. At the third interview, ideas related to expanding any states with eigenstates were observed. Since he had only mentioned these ideas during the third interview, these ideas were not considered to be repeating ideas. However, this does not imply that Gus was not aware of the idea of expansion of any state with eigenvectors.

At the beginning of the third interview, Gus was asked if the sum of two eigenvectors was again an eigenvector of the same matrix. It was noted that Gus was using vocabulary from the winter term physics courses as he worked on the problem. For example, instead of the matrix he used the word Hamiltonian operator or Hamiltonian and instead of eigenvectors he said eigenfunctions or eigenstates.

He initially stated that the sum of the eigenvectors was not an eigenvector of the Hamiltonian. He was asked to explain his thinking further.

> ***G:*** *Because, when you [he pointed at the equation* $\hat{H}\vec{v}_1 = E\vec{v}_1$ *he wrote earlier], say this eigenvalue is* E_1*, if you have the same thing with a* v_2 *here, you would get an* E_2*. When you act the Hamiltonian,* v_1 *plus* v_2*, you would get a different – like if you called this* v_3*, you would get an* E_3*. I guess depending on the E's, I guess there is a chance that they would add up to be the same thing, because they are just numbers. You could add* v_1 *plus* v_2 *equals* v_3*, then it would be, but it doesn't seem like that would always be true.*
>
> [I.3, lines 72-73]

Gus seemed to be taking the eigenvalues into consideration as he thought about the sum of the eigenvectors. However, he did not refer back to the eigenvalues again when he worked more on the problem. Also, he did not ask if the given eigenvectors were from different eigenvalues.

Gus decided in the third interview to give an example to show when the sum would not be an eigenvector, so he chose the differential operator $\frac{d}{dx}$. He stated that e^{5x} and e^{2x} were eigenfunctions with eigenvalues 5 and 2, respectively. He then operated on the sum $e^{5x}+ e^{2x}$ $(\frac{d}{dx}(e^{5x}+e^{2x}))$ and stated that he would get seven and said the sum was an eigenvector. It was not clear how he got seven, and he did not seem satisfied with his answer either. He said that he thought he would get something else and did not understand why he got this answer. The researcher suggested that maybe he could check his example by explicitly stating what the operator and eigenvectors were; however, this exercise did not help him any further. He could not think of any examples from his physics courses, so the interview was continued with another question. However, similar ideas came up later during the interview. For example, he was asked to describe eigenvalues and eigenvectors to an imaginary student who was planning to take the winter-term physics courses.

> ***G:*** *Well, eigen – let's see – the first thing that comes to mind when I think of eigenvectors are stable states. So if something – what is a good example – like a hydrogen atom, if something is in orbit it will stay in that orbit. So eigenvectors and eigenfunctions are stable states. I have a better example, like a pendulum. Two pendulums that are connected by a spring. So it is just two pendulums [inaudible] and they are connected by just one spring. They have symmetric and anti-symmetric modes. So if you pull them apart the same distance and let them go, they will stay in that motion forever. Or if you pull them the same distance to the side, they will stay in that motion forever. Those are what I refer to as stable.* ***But if you pull one a little farther than the other, like all the way back and do the same motion over a certain period of time, so that would be sort of described as the sum of the different eigenfunctions****. If you have many different eigenfunctions, you could have those in different – you could scale each eigenfunction and add them all up and describe the evolution in some way.* ***So it is sort of like your basis*** *– I*

> *don't know if that is a good way to put it. It is the – I don't know how to describe it actually.*

[I.3, lines 461-475]

It seemed that in his description he was referring to the superposition of eigenvectors. He also seemed to refer to eigenvectors as being basis vectors. He later continued to talk more on superposition of eigenvectors.

G: *[…] When I think of an eigenvector I think of a stable state. So it starts then. Unless it is acting on something else, it is going to stay there.* ***If it is in a super position or an addition of even eigenstates, then that is not necessarily an eigenstate.*** *It is a super position of one, so your motion of something, whatever you are trying to describe, won't be uniform, necessarily, because it is changing in different ways.*

If it is in just one eigenstate, then it will be uniform or periodic or act like a simple [inaudible] or something like that.

[I.3, lines 497-503]

Even though he stated that if a state was a superposition of eigenstates it was not necessarily an eigenstate; he did not realized what he was referring to in his quote was similar to the first question of the third interview. It seemed that Gus did not find these situations to be similar and connect these ideas. It may also be possible that he forgot about the first interview question by the time he was talking about these ideas.

Gus was asked when he learned about the pendulum idea and he said a similar example was discussed in one of his physics courses during spring term, 2009.

Overall, Gus seemed to be aware of the superposition of eigenvectors and could describe it with a physical example. However, it was not clear if he knew that eigenvectors could be used as basis vectors.

Actor-Oriented Transfer Episodes Related to Using Eigenvectors as Basis Vectors*:* The third interview data were analyzed by implementing the actor-oriented transfer framework. The following episode describes the AOT of Gus that seemed to be related to the fourth goal. The episode is described briefly; then, evidence of AOT is discussed.

Episode 3. This episode was observed at the third interview. Gus was asked if the sum of eigenvectors would be an eigenvector of the same matrix. After working on the problem, Gus decided that according to his example, the sum was an eigenvector (see the previous section on his solution). However, he did not seem to be convinced with his answer.

> ***I:*** *What happened? What is bothering you?*
> ***G:*** *I was thinking that when I did this, when I did these two right here, they would not end up to be that, but I guess when you choose the e to ax, this comes down in front, so if you choose – in this situation it is always going to add up but it seems like it would always work, because I don't think it always worked when we did it in the physics course last term but I can't think of any examples that we used in there to show that it didn't work. We used [mumbling]. I can't see why I'm thinking, because it seems pretty obvious that it does work, but I can't remember an example. I am just thinking that it didn't work. I am trying to remember why it didn't work, but it seems to going pretty well here, of course.*
> [I.3, lines 95-104]

He later was asked to go over the problem by stating the operator and its eigenvectors.

AOT Analysis of Episode 3. In this episode, Gus was not convinced of his answer, and he thought it was not correct. He did not check his calculations, but he still thought his final conclusion was incorrect. He did not know exactly why he thought the answer was incorrect. He tried to connect his thinking to his experience in the physics courses. He recalled that the examples done in the physics courses during winter term had a different result; the sum was not an eigenvector. He could not remember an example from any of the courses, but he was sure that the sum was not an eigenvector.

In this episode even though Gus could not give an example using his previous experience, he recalled the results from examples and this was enough for him to be suspicious about his result at the interview. It seemed that Gus referred to his experience from the winter-term course. Moreover, later in the same interview Gus gave an example from a physics course in which he was enrolled at the time of the third interview. Even though he did not use an example from a course from winter term, he was still referring back to his experience from another course. It seemed that Gus was aware of the superposition idea, and he tried to explain it using examples from his previous experience. For this reason, this episode constitutes evidence of actor-oriented transfer.

Summary

Gus initially said that a matrix represented a system of equations and this representation made it easier to enter the equations into the computer and also to solve the system of equations. However, this idea was not observed again at the second and third interviews. He used the word “transform” as he explained what a matrix did to vectors at the second interview. At the third interview, it was observed that he used the word “operates on” a vector as he explained what a matrix did to vectors. He said that he considered matrices as operators. However, he also mentioned that it had been a while since he used matrices as operators since they had focused more on Hamiltonian operators during the last two physics courses in the winter term.

Gus used the idea that eigenvectors do not change direction to find eigenvectors of a matrix instead of calculating them through the eigenvalue equation. In both examples he graphed the vectors and their images: then, using the graph he tried to figure out where eigenvectors would be. Even though he wrote the eigenvalue equation when he transformed vectors, he did not seem to use it to find eigenvectors. Notably, he did not recognize the eigenvalue when he found the transformed vectors. For example, once he realized that a vector was an eigenvector and it got eleven times bigger, he did not recognize that eleven was an eigenvalue. He calculated eigenvalues using the characteristic equation during the second interview.

Gus seemed to use the geometric interpretation of eigenvectors often as he worked on problems during second and third interviews. He stated the algebraic interpretation of the eigenvalue equation at the third interview, but this interpretation did not arise during the second interview.

Gus seemed to be aware of the superposition idea, that any vector could be expanded linearly in terms of eigenvectors. He, however, used the vocabulary from his physics courses stating that any state could be expanded in terms of eigenfunctions. It was not clear that he knew that eigenvectors could form basis for the vector space. Since he did not take any linear algebra course, it is possible that he may not have been familiar with the concept of basis vectors.

Conclusion

The four case studies in this chapter represent four different junior level physics undergraduate students who took an intensive linear algebra review week and a series of three 3-week intensive physics courses in a 10-week period, namely, the linear algebra review week (LAW), spin and quantum measurements, waves and central forces. Each student's transfer was analyzed and described using qualitative research methods and the researcher does not claim that these students represent a wider population of physics students. These four participants, however, were students of the physics courses during the winter term of 2008 and represent the potentially diverse nature of the course.

There were similarities and differences in the students' emerging understandings and the experiences they transferred to the interviews. A cross-case analysis was conducted on similarities and differences, and the results are discussed in the next chapter.

CHAPTER FIVE

CROSS-CASE ANALYSIS

The purpose of this study is to investigate seven third-year college physics students' transfer of learning of the concept of eigenvalues and eigenvectors from an intensive linear algebra review week and a series of three 3-week intensive physics courses in a 10-week period (namely, the linear algebra review week (LAW), spin and quantum measurements, waves and central forces) to interviews in which the students participated during and after these courses. Transfer of learning for each student is explored by implementing the Actor-Oriented Transfer (AOT) framework.

To better understand what each student transferred to the interviews from the physics courses, first each student's emerging understanding of the concept of eigenvalues and eigenvectors was analyzed according to the course goals that were related to the concept. Then each student's implementation of his/her experiences was described with episodes that provided evidence of AOT. Results in the form of case studies of four participants, Milo, Gus, Deniz and Crosby, comprised Chapter Four.

This chapter presents a cross-case analysis of the episodes of actor-oriented transfer using data from all seven participants. The results are organized according to the four goals in the physics courses and are summarized to address the research questions of this study.

All seven students were included in the cross-case analysis: the four case study students and three third year students, Joey, Ozzy and Tom. Tom was a philosophy major with a minor in physics. He attempted to take all the prerequisite mathematics courses for his minor even though this was not required. He took the matrix and power series methods course prior to his junior year. Joey also took the same course prior to her junior year. Ozzy was enrolled in the matrix and power series methods course at the time of his first interview for this study.

In the cross-case analysis, the seven participants' episodes containing evidence of AOT were analyzed according to the four goals of the linear algebra week (LAW) and the three physics courses in the nine weeks following LAW. The episodes which did not contain evidence of AOT were also analyzed to provide a description of transfer from the "researcher's point of view."[8] The results are described next in two sections under each goal.

GOAL ONE

Students were expected to view matrices as linear transformations. Toward this goal, Professor C introduced a linear transformation activity on the second day of LAW. There were four main ideas addressed in this activity and three of them were related to Goal One. The fourth idea was related to the third goal and is discussed under that goal later in this chapter. These four ideas were

1. The terminology applied to matrices as linear transformations: "A matrix operates or acts on vectors";

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[8] The actor-oriented transfer framework suggests that the researchers take the subject's point of view while analyzing the data for transfer.

2. Operating on an assortment of vectors with matrices to determine what the transformations were;

3. Relating the determinant of a matrix to its action as a transformation, and

4. Finding the vectors that are unchanged by the transformations.

Students formed small groups and each group was assigned a matrix from the activity. Tom, Ozzy and Milo were in the same group during LAW and on the second day of LAW they worked on the matrix $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$. Gus and Crosby were in the same group and worked on the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Joey and Deniz were in different groups and in their groups they worked on the matrices $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, respectively. Once students operated on an assortment of vectors, they drew the initial and the transformed vectors on a Cartesian plane to figure out what kind of transformation the matrix represented. Then they found the determinant of the matrix and the vectors that were unchanged by the transformation. Each group presented their results and as they were presenting Professor C helped students relate the determinants to the transformations. As a class they decided that matrices with positive determinants rotated and scaled the vectors and matrices with negative determinants reflected and scaled the vectors. The matrices with determinant zero mapped all the vectors to a certain line and scaled the vectors; they named this type of matrix as a “scrunching” matrix.

The episodes which constitute evidence of AOT from all seven students were analyzed and the ones related to the first goal are described next.

AOT Episodes for Goal One

In the first interview, Joey viewed matrices as representing linear transformations. Four students-Deniz, Gus, Ozzy, and Tom- did not seem to view matrices as linear transformations. Three of these students-Gus, Ozzy and Tom- said that a matrix represented a system of equations. Deniz, the fourth student said that matrices represented lines and the columns vectors of the matrix were the direction of these lines. Two other students-Crosby and Milo- first mentioned that matrices represented systems of equations but later during the same interview they stated that linear transformations could be represented with matrices.

After being introduced to linear transformations during LAW, five of the participants seemed to view matrices as linear transformations. Deniz, however, still seemed to view matrices as representing lines. The researcher asked him to "operate on a vector with a given matrix" and he was puzzled by this wording. He asked if it meant to multiply the matrix with the vector. Also Ozzy, even though he stated that matrices represented transformations during the second interview, was unclear in his explanation of transformation. For him it seemed that rotation was not a transformation.

***O**:* *So that is just, I guess that could be a transformation, too, but that looks sort of like a rotation, too.*
I: *So, the word transformation?*
O: *Well, a transformation is just kind of the same thing.*
I: *Same thing with?*
O: *Well, I mean, it's not a very descriptive word, because it is just changing it basically.*
I: *Okay.*

O: *That's why it doesn't seem right to say that this is a transformation.*
I: *Uh-um.*
O: *Maybe it was a flip, that doesn't sound right either, we called it something else.*
[Ozzy, I.2, lines 160-171]

Ozzy also wrote that eigenvalues and eigenvectors were "characteristic(s) of a system of equations/matrix" on the post quiz. It was unclear if he viewed matrices as representing both systems of equations and linear transformation after LAW.

Crosby, one of the five who said matrices were linear transformations at the second interview, but also referred to them as system of equations and he seemed to want to make a connection between the two representations.

The second interviews with the participants were conducted a week after LAW. At the beginning of the interview all students were asked to talk about one of the new and most interesting things they had learned during LAW. Crosby, Deniz and Tom mentioned that relating the determinant value of a matrix to what kind transformation the matrix represented was the most interesting thing. The other students mentioned other aspects of the week.

Milo, Deniz, Crosby, Tom, and Joey were given the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ to work with during the second interview. They were asked to discuss what this matrix did to vectors and to find its eigenvalues and eigenvectors. Gus and Ozzy worked with different matrices ($\begin{pmatrix} 1 & 5 \\ 2 & 10 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, respectively) because they had chosen these matrices as examples. For instance, Gus stated that the eigenvectors of a matrix with determinant zero would lie on the line to which all the vectors were mapped and gave the matrix $\begin{pmatrix} 1 & 5 \\ 2 & 10 \end{pmatrix}$ as an example. The researcher continued asking the questions from the interview protocol but focused on his matrix. Similarly, Ozzy mentioned the rotation matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and worked on this matrix during the second interview.

Six students-Gus, Milo, Deniz, Crosby, Ozzy and Tom- all referred to the linear transformation activity when they were asked "what does this matrix do?" It seemed that the ideas from the activity became part of their experiences with matrices. For example, when Ozzy was trying to determine what the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ did to vectors, he said he should know this from class. He decided to look at the determinant of the matrix and found it to be positive. Then he concluded that the matrix was a rotation. He decided to operate on the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with the matrix and drew the initial and the transformed vectors. He seemed confused and tried to make sense of the situation.

> ***O:*** *Huh. Yeah, it rotated it which makes sense. I am curious why, I mean, if you think of these as x's and y's [pointing at the columns of the matrix], then I would think it would've changed it this way, made the y negative, but then, yeah, but the rotation part sounds right from class. I don't know, this way, I guess, and this is, should, yeah, that wouldn't quite keep it the same, would it?*
>
> [Ozzy, I.2, lines 136-140]

He later continued to talk about his experience in class. He said the matrix he worked on during the small group activity put the vectors on some kind of a line and it had a determinant of zero. He also mentioned that this type of matrix was called "scrinch or something like that."

Ozzy explicitly referred to his experience during the Linear Transformation activity when he was trying to solve the interview question. Episodes similar to this one occurred during the second interviews with Milo, Gus, and Tom. Deniz also referred to the determinant idea that was introduced during the Linear Transformation activity. He stated that he found this idea to be one of the most interesting things he learned during LAW. He said the determinant of the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ was negative and so it reflected vectors. He was unsure if he recalled it correctly so he was asked to check this idea; however he was not sure how to check this idea. The researcher suggested maybe he could operate on some vectors with the matrix. He was confused with the wording and asked if he needed to multiply the vector by the matrix. Even though Deniz recalled the determinant idea, it seemed that he was not sure how it was formed in class and he seemed unfamiliar with the terminology.

Crosby also found the determinant idea to be one of the most interesting things he learned during LAW. He, however did not implement this idea when he was asked about the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$. He instead multiplied the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ with the matrix and concluded that the matrix transformed the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} 1x + \sqrt{3}y \\ \sqrt{3}x - 1y \end{bmatrix}$. (In the interviews with math professors one of the mathematics professors used a similar method and gave the same answer. She did not mention that this matrix was a reflection matrix either.) Crosby later decided to look at the determinant of the matrix when he was trying to determine how eigenvectors were related to the matrix. After looking at the determinant, he thought the matrix mapped the vectors to a certain line. He could not recall which determinant resulted in what kind of transformation. The researcher suggested that he check his idea and then he decided to operate on an assortment of vectors with this matrix. After his calculations he concluded that the matrix reflected and scaled the vectors. While he was working on this problem, Crosby referred to the ideas from LAW. It was interesting that Crosby implemented all the other ideas he knew first and then tried the ideas from LAW. It seemed that during this interview Crosby preferred using his previous experiences in mathematics courses to the experiences from LAW.

Joey did not explicitly refer to the linear transformation activity during the second interview. She used the determinant idea as she tried to figure out what the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ represented as a transformation. She said the determinant was negative so the matrix reflected the vectors but then she said she was not sure. She looked at the basis vectors to see what the matrix did to them. She then concluded that the matrix was not rotating because the lengths of the transformed vectors were changed. She also stated that the vectors did not seem to be reflected either. She said she was not sure what the matrix did, but probably rotated the vectors. She did not know how to check her idea further. Although Joey did not refer to the second day of LAW, when she was asked to give examples of eigenvalues and eigenvectors, she referred to the reflection transformation. The researcher had observed that she had worked with a reflection matrix during the small group activity on the second day of LAW and she was referring to the same example during the interview. She kept referring to the reflection transformation over the *y=x* line.

Four students-Gus, Deniz, Crosby and Ozzy- mentioned the linear transformation which mapped all the vectors onto a line. It seemed that they were interested in this type of transformation. For example, Gus created a matrix with determinant zero to explain what eigenvalues and eigenvectors represented in this situation. Crosby seemed to be interested in this particular transformation also. He initially though the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ was one of them, a matrix that mapped the vectors to some line.

Overall the data from interview two with Deniz, Crosby, Ozzy, Gus, Milo and Tom indicated that these students were explicitly referring to ideas from the linear transformation activity. It seemed that these students found the second interview question similar to the ones done in this activity and reconstructed their experiences with the activity to address the interview question. In each of the observed episodes, the participants referred to the activity explicitly. For all these reasons, these participants' data provide evidence for actor-oriented transfer. In other words, according to the actor-oriented transfer framework, these students seemed to transfer their experiences related to matrix representation to the interview setting. It is possible that Joey could have been listed with the other six students but she did not mention the activity explicitly or seem to implement many of the ideas from the activity.

Researcher's Perspective

The researcher also observed during the second and third interviews that five students-Ozzy, Gus, Milo, Joey and Tom- were using the terminology: "operating (acting) on a vector with matrices". Crosby and Deniz did not use this terminology during both interviews. The researcher observed that Professor C also used this terminology during LAW and further she pointed out to students that she was specifically using this terminology because she wanted the students to view matrices as linear transformations. Even though these participants of this study did not mention explicitly why they were using this terminology, these students used it during both interviews when they talked about matrices. Since students did not explicitly state that their use of terminology was from LAW or three physics courses and it was only the researcher's observations, this did not become evidence for actor-oriented transfer. Students could be assumed to transfer the language used in class to the interview settings from the researcher's point of view. However, this does not constitute actor-oriented transfer, because students were not explicitly asked to talk about the terminology during the interviews. It may also be possible that students were only imitating Professor C's way of talking about matrices without adding any meaning to the words.

GOAL TWO

Professor C expected students to be fluent in finding eigenvalues and eigenvectors. On the fourth day of LAW she demonstrated how to find eigenvalues and eigenvectors of a matrix. She started with the eigenvalue equation $A|v\rangle = \lambda|v\rangle$ and explained how the equation *det (A-λI)=0* was obtained to find the eigenvalues of *A*. Then she demonstrated how to find eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 1 & 2 \\ 9 & 4 \end{pmatrix}$ step by step. She used the eigenvalue equation $A|v\rangle = \lambda|v\rangle$ while she was finding eigenvectors. After she finished her example, students were assigned to groups and each group was given a matrix from the Eigenvalues and Eigenvectors activity. Students were asked to find eigenvalues and eigenvectors of their matrix and once they were done they presented their results. During each presentation Professor C asked each group what kind of transformation their matrix represented and what the unchanged vectors were under the transformation prior to discussing the calculations. When Gus was about to present, Professor C stated that the process of figuring out the transformation prior to the calculations would help students to make fewer algebraic errors.

Students were given a quiz on finding eigenvalues and eigenvectors of a two by two complex matrix on the fifth day of LAW. The episodes from all seven students, which had evidence of AOT and seemed to be related to the second goal, were analyzed and the results are described next.

AOT Episode for Goal Two

At the first interview only Ozzy was able to find eigenvalues and eigenvectors of a matrix. He was enrolled in the matrix and power series methods course during the fall term and his first interview was conducted after he was introduced to the concept. Milo and Crosby were enrolled in the linear algebra course however the concept of eigenvalues and eigenvectors was introduced at the end of the fall term and their first interviews were scheduled prior to this. Crosby and Joey were able to find eigenvalues but they did not know how to find eigenvectors. Except for Deniz, all students mentioned that the variable λ was used to represent the eigenvalue; however none of the students recalled what the eigenvalue was used for or what it meant.

Students were given a pre quiz on the first day of LAW before they were introduced to the linear algebra topics. One of the questions was on finding eigenvalues and eigenvectors of a two by two matrix $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$. Milo and Ozzy were the only students on the pre quiz who were able to find eigenvalues and eigenvectors correctly. They both used the equation $(A\text{-}\lambda I)v=0$ to find eigenvectors of the matrix. Gus, Joey and Crosby were able to find eigenvalues accurately; however Crosby used an inaccurate equation when he tried to find the eigenvectors. He seemed to use a mixture of the equations $(A\text{-}\lambda I)v=0$ and $A\vec{v} = \lambda\vec{v}$. Tom and Deniz both wrote that they did not remember how to find eigenvalues and eigenvectors.

Participants of the study stayed in the same groups after the first day. Ozzy, Tom and Milo were in the same group and found eigenvalues and eigenvectors of the matrix $A_5 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Crosby and Gus worked on the matrix $A_{2=}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in the same group. Joey and her group found eigenvalues and eigenvectors of the matrix $A_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Deniz and his group worked on the

matrix $A_6 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. During small group activities, Gus had some questions related to assigning free variables when finding eigenvectors. He asked his group members and they could not help him. He did not seem to be familiar with this idea and he had not taken linear algebra prior to the winter term physics courses.

A post quiz was given on the fifth day of LAW and students were asked to find eigenvalues and eigenvectors of a two by two complex matrix $\begin{bmatrix} 2i & 3 \\ 0 & -7 \end{bmatrix}$. Milo was able to find eigenvalues and eigenvectors correctly. Crosby made a sign error in one of the eigenvectors, instead of $\begin{bmatrix} 1 \\ -\frac{7}{3} - \frac{2i}{3} \end{bmatrix}$ he wrote $\begin{bmatrix} 1 \\ -\frac{7}{3} + \frac{2i}{3} \end{bmatrix}$. Deniz and Ozzy found eigenvalues accurately. Ozzy was able to find one of the eigenvectors correctly but he made an algebraic error while finding the second eigenvector. He used the eigenvalue equation $A|v\rangle = \lambda|v\rangle$ instead of $(A-\lambda I)v=0$ when finding eigenvectors. Deniz did not write the correct eigenvalue equation when finding eigenvectors. Both Joey and Gus did not calculate the determinant of $A-\lambda I$ accurately; however both of them wrote the eigenvalue equation indicating that if they had found the eigenvalues, they would have used that equation to find eigenvectors. Similarly, Tom did not find the eigenvalues accurately, however he also wrote the eigenvalue equation to find eigenvectors.

Five students-Milo, Tom, Ozzy, Crosby and Deniz- were able to find eigenvalues and eigenvectors accurately at the second interview. Gus also found eigenvalues and eigenvectors but he used a different method when finding eigenvectors-the geometric interpretation of the eigenvalue equation. For example, for the matrix $\begin{bmatrix} 1 & 5 \\ 2 & 10 \end{bmatrix}$ he first determined the line to which the vectors were mapped and then stated that the vectors on this line would be eigenvectors because they did not change direction. He tried to apply the same method of finding eigenvectors of a different matrix; however he was not successful. He did not mention using the eigenvalue equation to find eigenvectors. Joey was the only student who did not successfully find the eigenvectors of a matrix; however she was able to find eigenvalues accurately.

During the third interview one student-Milo- successfully found eigenvalues and eigenvectors. Three students-Tom, Joey and Ozzy- were not able to remember the correct equations to find eigenvalues and eigenvectors. The remaining three students-Deniz, Crosby and Gus- wrote the correct equations to find eigenvalues and eigenvectors and explained the equations; however, they were not asked to find eigenvalues and eigenvectors during the third interview. Four students-Milo, Deniz, Crosby and Gus- were able to find eigenvalues and eigenvectors one term after they were introduced to it in LAW. Milo seemed to be fluent in finding eigenvalues and eigenvectors; however no such conclusion could be made for Deniz, Crosby and Gus.

All the episodes with evidence of AOT were categorized according to the physics courses' goals and it was noticed that there were not many episodes indicating evidence for AOT related to the second goal. Even though it was observed that students fluency of finding eigenvalues and eigenvectors differed over a time, students did not explicitly mentioned using their previous experience as they were finding eigenvalues and eigenvectors. There were only two similar AOT episodes in the data for Ozzy, Crosby and Milo.

Episodes. Ozzy indicated at the second interview that one of the most interesting things he learned during LAW was to find eigenvectors using the eigenvalue equation $A\vec{v} = \lambda\vec{v}$ instead of the equation $(A-\lambda I)\vec{v} = \vec{0}$. He said he learned to find eigenvectors using the equation $(A-\lambda I)\vec{v} = \vec{0}$ in the matrix and power series methods course; but Professor C used the eigenvalue equation $A\vec{v} = \lambda\vec{v}$ to find eigenvectors. According to him these two ways of finding eigenvectors were different, so it was interesting to him to see this second way. He later stated that the way he learned in LAW was “very similar” to the one from the math course “but it was not quite the same”. Ozzy used the equation $(A-\lambda I)\vec{v} = \vec{0}$ on the pre quiz and used the eigenvalue equation $A\vec{v} = \lambda\vec{v}$ on the post quiz to find eigenvectors.

Similar changes were observed in Milo’s and Crosby’s post quizzes. It was unclear which equation Crosby tried to use when finding eigenvectors on the pre quiz, but he wrote the eigenvalue equation on the post quiz. Milo used the equation $(A-\lambda I)\vec{v} = \vec{0}$ on the pre quiz and then switched to the eigenvalue equation on the post quiz. Milo kept using the eigenvalue equation every time he calculated eigenvectors. He was asked if he used the eigenvalue equation all the time and he said he should write it every time “because I always end up staring at the ceiling and realized I should write that down and not have to think about it so hard".

It seemed that Ozzy, Milo, and Crosby were all using their experience with finding eigenvectors from LAW at the second and/or third interviews. For this reason the data from these three students provide evidence of AOT. In other words, these students transferred their learning of finding eigenvalues and eigenvectors from LAW to the interviews. However, this conclusion does not imply that these students were fluent in finding eigenvalues and eigenvectors. Notably, only Milo was able to find eigenvalues and eigenvectors accurately throughout the study. Crosby and Ozzy both were implementing a new way of solving for eigenvectors and they were fluent during the second interview. Ozzy could not recall the eigenvalue equation correctly in the third interview and Crosby only described how one would find eigenvalues and eigenvectors.

Researcher's Perspective:

The researcher observed that even though Deniz and Tom did not explicitly refer to their experience in LAW while finding eigenvalues and eigenvector, they were able to find them during the second interview. Both of these students could not recall how to find eigenvalues and eigenvectors in their first interviews. When traditionally analyzed, the researcher could conclude that these students were transferring the way of finding eigenvalues and eigenvectors to the second interviews. However, Tom could not recall the correct equations to find eigenvalues and eigenvectors at the third interview. He stated that using an equation with determinant one could find the eigenvalues and he said it was similar to *det (Ax-I)=0* and then to find eigenvectors one needed to plug the eigenvalues to another equation to find eigenvectors and he thought the equation to find eigenvectors might be something like $A|v\rangle = c|v\rangle$. Even though he wrote the eigenvalue equation accurately, he was not sure where to plug the results from eigenvalues. Joey also stated similar ideas, she said after finding eigenvalues one needed to plug the results back to some equation, and proposed $A\lambda=\lambda v$. She stated that this equation did not look correct because the left hand side was still a matrix and the right hand side was a vector, however she could not find a way to fix the equation. This observation suggests that Tom and Joey transferred the memory of the actions of finding eigenvalues and eigenvectors, not the concrete equations of finding eigenvalues and eigenvectors.

Deniz was able to find eigenvalues and eigenvectors accurately on the second interview and he recalled the correct equations to find eigenvalues and eigenvectors and explained how these equations could be used to find eigenvalues and eigenvectors. Even though he did not work on a specific example to find eigenvalues and eigenvectors, Deniz seemed to be fluent in finding eigenvalues and eigenvectors.

The researcher also observed that Gus was implementing the geometric interpretation of eigenvectors to find eigenvectors at the second interview. Gus mentioned that he thought the geometric interpretation of eigenvectors was one of the most interesting things he learned during LAW. To find eigenvectors of the matrix $\begin{bmatrix} 1 & 5 \\ 2 & 10 \end{bmatrix}$ he operated on an assortment of vectors with the matrix and determined first that the vectors were all mapped to the line $y = 2x$. He claimed that the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ was an eigenvector because it was already on the line (because its slope was the same as the line's) and also the vector did not change its direction. Professor C had suggested that students should determine what kind of transformations the matrix represented and find the unchanged vectors before doing any calculations. It seemed that Gus was following this suggestion as he found the eigenvectors. It was unclear if Gus could find eigenvectors fluently using the eigenvalue equation at the second interview. Even though Gus did not use the eigenvalue equation to find eigenvectors during the second interview, he mentioned it in his third interview.

GOAL THREE

Professor C expected students to understand the concept of eigenvalues and eigenvectors, the eigenvalue equation $A\vec{v} = \lambda\vec{v}$ and its algebraic and geometric interpretations and to use these interpretations when solving problems. The concept of eigenvalues and eigenvectors were first introduced in the linear transformations activity on the second day of LAW. Students first were asked to decide what kind of transformation the matrix represented and they were asked to find the determinant of the matrix. Students also were asked to find the vectors that were unchanged by transformations without being told that these vectors were eigenvectors. In other words, students were introduced to the geometric interpretation of the eigenvectors in the linear transformation activity on the second day of LAW. During the wrap up discussion of the activity on the second day, Professor C asked students to comment on what they observed about the vectors that were unchanged by the transformations. One of the students asked if the unchanged vectors had anything to do with “eigen things”. This was the first time the word “eigen” was used in class and Professor C asked students to write an equation on their whiteboards that had something to do with eigenvalues and eigenvectors. She then picked three students’ whiteboards and explained what was written on each whiteboards. The first whiteboard she showed had the eigenvalue equation $Av_1=\lambda v_1$. She stated that the equation told them “[pointing to the left hand side of the equation] if you multiplied the transformation with the vector, which means that if you put the vector in the black box and transformed it, [pointing to the right hand side of the equation] you would get a scalar times the same vector again.” She also stated that this meant for a particular transformation there might be vectors “when you transform them you get back the same thing except it might be multiplied by a scalar” She also mentioned that when she asked about the vectors that were unchanged during the activity, she wanted students to find eigenvectors of the transformation. She briefly reviewed the eigenvectors of each matrix from the linear transformation activity. She then asked if the scalar multiples of the eigenvectors were again eigenvectors and students agreed that they would be eigenvectors of the same operator. She then pointed to the eigenvalue and stated that eigenvectors get scaled by the eigenvalue. She explicitly stated by pointing to the eigenvalue equation that students needed to think geometrically about the equation and geometrically it told them the direction of the eigenvectors were unchanged but eigenvectors were multiplied by a scalar which made them stretch, shrink or, in the case of a negative scalar it change to the opposite direction. Students should start thinking of “north and south” or “west and east” as being the same direction. She also said that the equation det (λI-A)=0 was used for finding eigenvalues.

At the beginning of the third day Professor Clayton asked students what they had learned so far in LAW and one of the idea that was mentioned by students was eigenvectors and how their directions were not changed by a transformation they belonged to, but they might be scaled by eigenvalues.

On the fourth day students were introduced to methods of finding eigenvalues and eigenvectors. Professor C started with the eigenvalue equation and explained how the equation, det (A-λI)=0 was obtained to solve for eigenvalues. After she demonstrated how to find eigenvalues and eigenvectors on an example, students started to work on a small group activity. Each group was given a different matrix for which to find eigenvalues and eigenvectors and once all the groups were done they presented their results. At the beginning of each presentation Professor Clayton asked students to first identify what the matrix represented as a transformation and explained the vectors that are unchanged. Especially during Gus’s presentation Professor C said student should practice finding what the matrix represented and the vectors that were unchanged before doing any calculations because it would help them to avoid any algebra mistakes. The goal of this particular activity and presentations was to help students to relate the algebraic and geometric interpretations of eigenvectors.

The following episodes constitute evidence of AOT of the concept of eigenvalues and eigenvectors, the eigenvalue equation and its algebraic and geometric interpretations are described next.

AOT Episodes for Goal Three

Participants were asked to describe eigenvalues and eigenvector in all three interviews, on the pre and post quizzes and on the final exam. Both the first interview and pre quiz were conducted prior to the introduction to eigenvalues and eigenvectors in LAW. The post quiz was conducted on the last day of LAW and the second interviews started a week after LAW. The final exam was given at the end of the third physics course and the third interviews began four weeks after the end of the final physics course.

In six participants' data there were episodes that constituted evidence of actor-oriented transfer of the concept of eigenvalues and eigenvectors, the eigenvalue equation and its algebraic or geometric interpretations. It seemed that these six participants attempted to reconstruct their experiences from LAW or the three physics courses to address the questions at the interviews. The seventh student Ozzy did not explicitly refer to the interpretations of the eigenvalue equation during the interviews and no evidence of actor-oriented transfer was found in his data.

The following episodes from different participants give the flavor of each participant's actor-oriented transfer of the concept of eigenvalues and eigenvectors and the interpretations of the eigenvalue equation from different interviews.

<u>*Episode 1*</u>.Joey provided a geometric interpretation of eigenvectors in her second interview stating that eigenvectors were the vectors that did not change direction. She used the example of a reflection transformation over the $y=x$ line. She said when vectors were reflected over the line $y=x$ the vectors along this line would not change direction and these vectors were eigenvectors. She was asked if there were any other eigenvectors which were not on the $y=x$ line.

> ***J:*** *Well, apparently there is the one perpendicular to that line. I trust them but I don't really believe them[instructors of the course], because its direction changed, it is opposite from itself, which, okay, fine. I mean, I understand where the whole opposite from itself is technically the same direction, but it is not. They told us to trust them, and I am. So that's okay.*
>
> [Joey, I.2, lines 63-67]

She was aware that the vectors along the $y = -x$ were also eigenvectors of the operator and stated explicitly she was told these were eigenvectors. She thought that the directions of these vectors were changed by the operator, since these vectors would be mapped to the vectors pointing the opposite direction. She seemed to be puzzled with the "opposite direction" idea from class discussions. She was making connection to her experience in class while she answered the interview question. The idea of "opposite direction" was not a spontaneous idea which was created at the time of the interview. She knew this idea from LAW and was told by "them" that "the opposite direction" was the same direction. This particular idea was part of her experience in LAW. The researcher also observed that during the small group activity on the second day of LAW, Joey was working with the matrix representing the reflection over the $y=x$ line and she used the same transformation as an example during the second interview. This example seemed to have originated from her experience in LAW. For these reasons this episode constitutes evidence of actor-oriented transfer of the geometric interpretation of eigenvectors.

Joey also mentioned that the concept of eigenvalues and eigenvectors were "easiest to think" in "vector form" but after the first physics course of the winter term they started to use different "forms". Once she started to talk about different forms, she used the algebraic interpretation of the eigenvalue equation. She stated in the second physics course that eigenvectors were functions and operators were generally differentials, "so you have to get back to the same form, with only a scalar change. It is not a change in direction. It is a change in form. You have to make sure that they don't change form". Since Joey connected her experience with eigenvectors to forms in the second physics courses explicitly and she implemented the algebraic interpretation of eigenvalue equation to explain, this episode also constitutes evidence of actor-oriented transfer.

Episode 2. Deniz also referred to the geometric interpretation of eigenvectors using the reflection transformation during the second interview. Deniz was asked to find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ and found them accurately. He conjectured that the matrix was reflecting vectors because the determinant of the matrix was negative; however he was not sure. He was asked what would happen to eigenvectors when the transformation applied to the eigenvectors. He said the eigenvectors would point along the axis of reflection.

D: *The eigenvectors, if this is a reflection, and let's just assume that it is, then the eigenvectors would all point along the axis of reflection*

I: *Okay. Both of them?*

D: *And then, one of them would be perpendicular to it because the other one is just negative one times the other, its just a scalar multiple, which would make it point in the other direction, but still along that line. And that's the most that I remember out of classes last week, but that might only be for rotation. And, I'm not totally clear what's going on there.*

[Deniz, I.2, lines 185-193]

Then he continued to talk about the axis of reflection. The researcher asked him to find the axis of reflection and he stated accurately that the second eigenvector he found was the reflection axis. Deniz seemed to be recalling ideas from the activities done in LAW. Deniz was not sure if the ideas he stated were for a reflection or a rotation transformation but he explicitly stated the ideas were from class. On the second day of LAW, Deniz and his group had worked on a reflection matrix and it may be possible that Deniz was trying to reconstruct ideas he learned during the activity in order to answer the interview question. This was not the only time Deniz tried to explain eigenvectors using the reflection transformation from class. For example, when he was asked what an eigenvector was on the post quiz, he wrote "A vector along the line that when a reflection occurs the vector does not change direction."

In this episode Deniz explicitly stated that he was trying to recall the geometric interpretation of eigenvectors using the reflection transformation. His reconstruction of ideas on eigenvectors at the interview was not spontaneous but seemed to be rooted in his earlier experience with reflection transformation during LAW. For this reason, this episode constitutes evidence of actor-oriented transfer.

Episode 3. Deniz tried to use the eigenvalue equation when he was working on one of the third interview questions. In this third interview problem Deniz was given two eigenvalues and two eigenvectors that were associated with the eigenvalues of an unknown operator M and he was asked to tell everything he could about M with the given information. He said eigenvalues and eigenvectors probably came from a matrix like this one: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then he mentioned that he knew "an easy way" to do this problem and he had seen one of his classmates from the physics course did a similar problem and wrote $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. He stated that $\begin{pmatrix} x \\ y \end{pmatrix}$ will be $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ because of the eigenvalue equation and wrote $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$. He continued working on the problem using the eigenvalue equation. Deniz did not mention the algebraic or geometric interpretation of the eigenvalue equation but he knew he could use the equation to solve the problem. He wrote $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and claimed that the matrix was the identity matrix. He later said the other eigenvector would not satisfy the eigenvalue equation if M was an identity matrix and it would be "weird" to have two different matrices; so M could not be the identity matrix. He also wrote $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -1\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ but he had a sign error. He could not find what M was and gave up on the problem. However, it was noticed that Deniz found the third interview question similar to a previous question he worked on with a classmate during the winter term.

The researcher noticed that students did a similar problem during one of the spin and quantum measurements course and in the second assignment of this course. It may be possible that Deniz recalled this assignment. He seemed to use his experience of working on a similar problem during the third interview and he explicitly stated that he had seen a classmate working on a similar problem. Since Deniz was trying to implement his previous experience of using the eigenvalue equation at the third interview, this episode seems to provide evidence of actor-oriented transfer of the eigenvalue equation and its appropriate use.

Episode 4. In the first interview, Tom stated that he could not recall what eigenvalues and eigenvectors meant but he continued explaining that eigenvectors were linearly independent solutions of something. He also said during the first interview that in his math course they mostly computed eigenvalues and eigenvectors. At the second interview, however, he mentioned that the eigenvectors were vectors that were unchanged by transformation and maybe their magnitudes were changed but the direction remained the same. He also stated the algebraic interpretation of the eigenvalue equation during the second and third interviews.

In the second interview Tom was asked what the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ did to vectors; he claimed that the matrix reflected vectors because its determinant was negative. To convince the researcher he decided to operate on the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ with this matrix.

T: *One point seventy three. So how much of this can I do? So now seven point nineteen and twenty four, so it stretched. You can't tell – it made y a lot smaller and x a lot bigger. Well, yeah, so it did stretch it. It stretched the magnitude and maybe it rotated it or maybe it reflected it about the eigenvector of this matrix. I remember hearing talk of that in class.*

I: *You were talking about eigenvectors?*

T: *Yeah. I remember we had a discussion about, like when we had these weird transformations that we didn't know how to interpret, if we did multiple transformations and it looked like there was some point about which everything was reflecting, it was suggested that maybe they were reflecting about the eigenvector. But I don't remember a conclusion, a hypothesis.*

[Tom, I.2, lines 176-185]

He started to express the ideas presented during the linear transformations activity on the second day of LAW. The researcher suggested maybe he could check these proposed ideas and he said he should find the eigenvectors and if one of them was between the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and its image, then the matrix might possibly be reflecting vectors over the eigenvector. He found the eigenvectors and stated that the matrix was reflecting over the second eigenvector. He was also asked what would happen to these eigenvectors when the matrix was operated on them. He said eigenvectors should stretch but their direction would not change. Notably, he was referring to the geometric interpretation of the eigenvalue equation.

Tom talked about ideas from the linear transformation activity and tried to implement them to solve the second interview problem. As he worked on the problem, it was noticed that he was implementing the geometric interpretation of eigenvectors. Even though he could not recall the conclusions from class, he was able to implement the ideas he learned to answer the interview question. He also seemed to make a connection between the interview question and the class activity. For these reasons this episode was considered to provide evidence for actor-oriented transfer of the geometric interpretation of the eigenvectors.

Episode 5. Gus was the only participant who did not take a linear algebra courses prior to LAW. He stated he did not know what eigenvalues and eigenvectors were but he heard about them in one of his engineering courses. In the first interview he said that the variable λ was the eigenvalue but he did not know what eigenvalue meant. He described eigenvalues algebraically and eigenvectors geometrically on the post quiz. He used the geometric interpretation of the eigenvectors during both second and third interviews.

In the third interview Gus was given two eigenvalues (1 and -1) and two eigenvectors ($\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$) which associated with the given eigenvalues of an unknown operator M and he was asked to describe the operator M as much as he could. Gus wrote the eigenvalue equation first and then plugged in the given eigenvalues and eigenvectors into the equation as seen in Figure 5.1.

$$\boxed{\begin{aligned}\hat{M}\vec{\lambda}_1 &= \lambda_1\vec{\lambda}_1\\ \hat{M}\vec{\lambda}_2 &= \lambda_2\vec{\lambda}_2\end{aligned}}$$

$$\hat{M}\begin{pmatrix}1\\1\end{pmatrix} = 1\begin{pmatrix}1\\1\end{pmatrix}$$

$$\hat{M}\begin{pmatrix}1\\-1\end{pmatrix} = -1\begin{pmatrix}1\\-1\end{pmatrix}$$

Figure 5.1 Gus' Third Interview Question

Then he started to try different matrices with entries 0 and 1 as M. He said since the eigenvalues and eigenvectors all had one, he did not think the matrix M would have an entry greater than one. He guessed and checked a couple of different matrices. Then he thought M was one of the poly-spin matrices that the class had worked with in the spin and quantum measurements course.

Gus tried to recall the poly-spin matrices and then decided to rewrite M as a generic two by two matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and to solve for the entries using the eigenvalue equation. Then he found the matrix M and said that it represented a reflection over the $y=x$ line. Gus recalled his experience in spin and quantum measurements course and seemed to reconstruct ideas from this course to solve the third interview problem. He also mentioned that the eigenvectors of the matrix made sense, because the first eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ was on the reflection axis and if the second eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ was reflected over the line $y=x$, it would become -1 multiple of itself, in other words the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ was not changing its direction. Notably, he implemented the geometric interpretation of the eigenvectors. Since Gus referred to his experience from one of the physics courses, this episode constitutes evidence of actor oriented transfer of the eigenvalue equation.

Episode 6. Milo could not recall eigenvalues and eigenvectors at the beginning of the first interview even though he wrote the eigenvalue equation $Ax=\lambda x$. He stated that λ was the eigenvalue but he could not recall what an eigenvector was. Initially he did not seem to know that the variable *x* represented a vector in his equation. However, he was able to decide what each variable represented and construct the algebraic interpretation of the eigenvalue equation at the first interview. Milo referred to the geometric interpretation of the eigenvectors during the second and third interviews. Both algebraic and geometric interpretations of the eigenvalue equation were observed during the third interview with Milo.

In the second interview Milo was given the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ and asked what it did to vectors. Milo stated that this matrix reminded him the things that were done in class with determinants. He found the determinant of the matrix and said it was a reflection matrix and that they had talked about this idea in LAW. He was not sure if he recalled correctly if the matrix was a reflection so the interviewer suggested that he check his conjecture. He multiplied the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ with the matrix and he could not decide if it was reflected or rotated. He operated on the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ too and decided that the matrix was reflecting because the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ did not seem to be rotated by the same degree. Then he stated if the matrix was a reflection there should be a reflection axis lying between the initial vectors and the transformed ones. He proposed a line between the vectors, and when he was asked to find the reflection line he stated that he needed to find the eigenvectors of the matrix and provided the geometric interpretation of eigenvectors as his reason. When he found the first eigenvector, he was asked if he had any intuition about where the second eigenvector could be, he stated it would be orthogonal to the first one. Then as Joey had, he mentioned the “opposite direction” idea from LAW.

> ***M:*** *I think it will be orthogonal to this vector [pointing to the eigenvector he found]. Because if it is a reflection then a vector along the line of reflection will be an eigenvector and any line orthogonal to the line of reflection will be an eigenvector because its direction isn’t changed, just the way its pointing. But it is said in physics course; well the north and south are facing the same direction.*
>
> [Milo, I.2, lines 122-126]

Unlike Joey, Milo seemed to be comfortable with this idea. Milo referred to his experience related to the “opposite direction” of eigenvectors from LAW as he explained his thinking about the relationship between two eigenvectors. Since Milo recalled his experiences in LAW as he was working on this problem and referred to the “opposite direction” idea from LAW, the episode provides evidence of actor-oriented transfer of geometric interpretation of eigenvectors.

<u>*Episode 7.*</u> Crosby was one the only participants who implemented the algebraic interpretation of the eigenvalue equation throughout the study except for the first interview. At the first interview he could not recall what eigenvalues and eigenvectors were and stated that they were “something about some multiple of some sort.” On the pre and post quizzes and the final exam and during the second and third interviews he referred to the algebraic interpretation of the equation. Crosby was asked if he knew the use of the algebraic interpretation of the eigenvalue equation and he stated that if the dimension of the matrix was big, it would be easier to replace the matrix by a scalar for computers. He then stated that

he learned this example in his linear algebra course and also Professor C had mentioned a similar idea. It seemed that the examples provided from his math course and LAW were very accessible to Crosby and he referred to them during the interview to justify the algebraic interpretation of the equation. Deniz seemed to be attached to the algebraic interpretation and he used it throughout the study to describe eigenvalues and eigenvectors. Since Crosby referred to his experience of algebraic interpretation of the eigenvalue equation from his previous mathematics course and combined it with his experience with it in LAW, this episode provided evidence of actor-oriented transfer of the algebraic interpretation of the equation. Even though during the second interview Crosby's emerging understanding of the algebraic interpretation was incomplete, it still provided evidence for AOT.

Episode 8. Ozzy was the only student who did not describe explicitly what eigenvalues and eigenvectors were in the interviews. There was only one episode during the second interview in which he referred to eigenvectors and attempted to describe them using the geometric interpretation, but it was unclear if he was aware of the geometric interpretation. When he was asked to describe eigenvalues and eigenvectors, he said he did not know what they were and started to explain the linear transformation activity done on the second day of LAW. He said in his group they worked on the reflection matrix which reflected the vectors over the line $y=x$.

> **O:** *[...]Like we ended up with line y=x for the matrix that we had, and the eigenvector is going to be one, a scalar multiple of those and then one that is perpendicular and he kind of said that in math class, but we never really, we did get to some projection of stuff, but we never really revisited the eigenvalues and vectors, it was more of a statement and*

> *then we solved for a bunch of values and vectors, so that was kind of nice to do that.*
>
> [Ozzy, I.2, lines 75-81]

He described the linear transformation activity but what he meant was unclear when he started to talk about the eigenvectors. For example he said eigenvectors were scalar multiple of "those" and it was unclear what "those" represented. The researcher did not interrupt him for clarification, but later during the same interview when he was working on a rotation matrix she asked what he could say about the eigenvalues and eigenvectors. He immediately said he could find them and found the eigenvalues and eigenvectors. After his calculations he was asked what eigenvalues and eigenvectors represented with respect to the given matrix and he said he did not know. Even though he referred to the linear transformation activity to recall eigenvalues and eigenvectors, it seemed that he was mostly recalled the things that were done in the activity rather than the concepts discussed. He did not use the ideas related to the concept of eigenvalues and eigenvectors from the linear transformation activity. There was not enough evidence to conclude that he was transferring from the actor-oriented perspective.

Overall six students-Milo, Gus, Tom, Deniz, Crosby and Joey- developed an algebraic or geometric interpretation of eigenvalues and eigenvectors after being introduced to the concept of eigenvalues and eigenvectors, the eigenvalue equation and its interpretations in LAW and the three physics courses. Joey and Milo explicitly mentioned the idea that a vector's direction was assumed to be unchanged if it pointed in the opposite direction after being transformed by the matrix. Milo seemed to be comfortable with this idea, however Joey said only that she “trusted them [the instructors]”.

Joey, Tom and Deniz recalled a class discussion on reflection in which it was stated that the vectors that were on the reflection axis would be eigenvectors and the ones perpendicular to the reflection axis would also be eigenvectors. The implementation of this idea and the geometric interpretation of eigenvectors were observed in their reasoning.

Gus and Deniz referred to their experience in the spin and quantum measurements courses as they worked on the same third interview question. Deniz stated that he had watched a classmate as they worked on a similar question and he tried to implement his experience from that situation. Gus also solved the problem using the eigenvalue equation and as he was working on it he recalled working on a similar matrix during the spin and quantum measurements course. Only Milo mentioned that the matrix M looked “an awful lot like one of those” poly-spin matrices however he did not seem to use his observation to work on this problem. He implemented the eigenvalue equation to find the matrix M.

Crosby was the only student who implemented the algebraic interpretation of the eigenvalue equation throughout the second and third interviews, on the pre and post quizzes and the final exam. Crosby also mentioned the geometric interpretation but he did not implement the idea in any of the questions.

Researcher’s perspective:

The researcher also observed that some students implemented the algebraic and geometric interpretation of the equation but did not explicitly refer to LAW or any other experiences; and some of these students seemed to have misunderstandings related to implementing the algebraic and geometric interpretations. For example, Joey mentioned the geometric interpretation of eigenvectors again during the third interview. She said eigenvectors were vectors of an operator that did not change direction when the operator acted on them, but their magnitudes could change. She tried to implement this idea when she worked on the matrix M with the eigenvalues 1 and -1 and the eigenvectors vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. She suggested that the matrix M was a reflection matrix over the x-axis and when she was asked to justify her suggestion, she said eigenvectors did not change direction when reflected over the x-axis. For example when the eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ reflected over x-axis it would be transformed to the other eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The way Joey tried to implement the geometric interpretation of the eigenvector in this question was inaccurate; however she did use the geometric interpretation to solve this particular problem. She mentioned eigenvalues infrequently during the second interview, only stating that eigenvalues were the values that scaled the vector. During the third interview she did not describe eigenvalues, however she said "you can't just know something is an eigenvalue. You have to know an eigenvalue of **what**." Her emphasis of the "what" seemed to indicate that she recalled some ideas related to eigenvalues from LAW or the three physics courses, but she did not mention anything further on this topic. From the researcher's perspective, Joey was still aware of the geometric interpretation of the eigenvectors during the third interview and some ideas related to eigenvalues; however she seemed to need more time to develop these ideas.

The researcher observed that Milo and Tom seemed to connect the geometric interpretation of eigenvectors to what matrices represent. For example, both of these students proposed that the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ was a reflection matrix and when they were asked where the reflection axis was they both stated that they needed to find the eigenvectors.

The researcher also observed that more participants were referring to the algebraic interpretation of the eigenvalue equation during the third interview than during the second interview. One of the reasons for this could have been that students were introduced to operators other than matrices during the last two winter term physics courses and the geometric interpretation did not make as much sense when working with functions as it had with vectors. So, instead of thinking about unchanged direction of a vector, their instructor suggested that students think about the unchanged form of a function.

GOAL FOUR

Students were expected to understand that eigenvectors could be chosen as basis vectors of a vector space and any vectors from this space could be expanded linearly with the eigenvectors. Professor C expected that students would develop this understanding after completing the three physics courses. In all three physics courses students were introduced to this idea through examples. For example, during the Waves course students were asked if the function

$$\phi(x) = \sqrt{\frac{1}{5}}\phi_1 + \sqrt{\frac{3}{5}}\phi_2 + \sqrt{\frac{1}{5}}\phi_3$$

was an eigenstate (eigenvector) of the Hamiltonian operator, when each ϕ_i for i=1,2,3 was an eigenstate (eigenvector) of the Hamiltonian operator. Students discussed the questions in small groups and later there was a brief whole class discussion of why $\phi(x)$ was not an eigenstate.

The episodes which constitute evidence of AOT from all seven students were analyzed and the ones related to the fourth goal are described next.

AOT Episodes for Goal Four

During the first two interviews, participants were not asked questions on the relationship between basis vectors and eigenvectors unless they specifically mentioned it themselves. In the third interviews all participants were asked to comment on the sum of two eigenvectors, v_1 and v_2, in particular if the sum, v_1+v_2, was again an eigenvector of the same operator. Students were not told if the eigenvectors v_1 and v_2 were associated with the same eigenvalue and students were expected to argue all possible cases. However, none of the participants explored the different possible cases and all of them mentioned the "superposition" idea they had learned in their physics courses.

<u>*Episode 1.*</u>Joey and Deniz were the only two participants who mentioned the basis vectors throughout the study. At the first interview they both mentioned that eigenvectors were the basis vectors or like basis vectors. Joey provided a more

detailed description than Deniz. Joey said "Eigenvectors are the basis vectors, e_1 and e_2, basically corresponding to the x and y axis. You can switch them to be different axes, so you can pick different basis for your math problems. You use them to find transformations. In order to do transformations, you find what happens, what the transformation does to each eigenvector and that tells you what it does to the space."[Joey, I.1] Further, in the interview she was asked more questions on eigenvalues and eigenvectors and she seemed to be confusing eigenvectors with the standard basis vectors of R^2. For example, the researcher asked her to find eigenvectors and eigenvalues of the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and she pointed to the first column of the matrix and said it represented the x- eigenvector and the second column represented the y-eigenvector. It was unclear what she meant by x-eigenvector and y-eigenvector, however it seemed that she was confusing the standard basis vectors with eigenvectors.

In the second interview Joey again mentioned the basis vectors idea when she was asked how eigenvalues and eigenvectors were used in physics. She did not know how they were used in physics but she said "The bases are important, but those are not eigenvectors, the basis vector, which are different. They are obviously the same sometimes, but is important to know what your basis, and you have to work in different basis so you can change and all that. I know those are important. Eigenvectors, I don't." Then she was asked to watch the clip in which she described eigenvalues and eigenvectors in the first interview. After watching the clip, she said she was confusing eigenvectors with the basis vectors.

In the third interview, when she was asked about the sum of two eigenvectors, she said the sum should not be an eigenvector and tried to explain her answer using the geometric interpretation of eigenvectors. She was adding two eigenvectors which were on the same line.

> **J:** *Well, the sum shouldn't be an eigenvector. I don't think so because the eigenvectors are things that do not change direction, only magnitude. But if they are two different eigenvectors then they are two different directions. If you sum them, well, no. [Writing] Because the sum of this would be that and they only change the magnitude but the direction will be the same. So yes, it will be an eigenvector.*
>
> [Joey, I.3, lines 13-18]

It was noticed that on the final exam, prior to the third interview Joey wrote any state could be written as a combination of eigenstates. Notably, she did not refer to the idea from the final exam during the third interview. Later at the end of the third interview she was asked if she had anything she would like to add about eigenvalues and eigenvectors. As she was explaining more ideas she stated that "all the other vectors are the sum of eigenvectors". Then she was asked what she thought about the question from the beginning of the interview. She noticed that there was a contradiction and the researcher probed her more on this idea.

> **J:** *But I know that in the class we do sums of eigenvectors.*
>
> **I:** *Did anyone tell you anything about that sum being an eigenvector again, or did you just find the sum of eigenvectors?*
>
> **J:** *I think we had to prove it or something, because you sum up all these different things and they have the same form, so you want to end up with the same form. And if you do, then it is, but if you don't, then, I don't know. We probably proved that it was and then just were never told, by the way, this is what you just proved. Either that or I just don't remember what they told us.*

[Joey, I.3, lines 280-287]

Joey could not recall exactly what was done in class but she decided that they must had proven the sum was an eigenvector which did not conflicted with her result for the first question in the interview. Notably, she did not try to come up with an example or alternative justification but rather she assumed her answer to the first question was correct and backed it up by "altering" the memory of her class experience. She did not explore if her reasoning was correct in the first interview question.

Later Joey was asked what she thought about the idea that "eigenvectors were basis vectors" at the end of the third interview.

> ***J:*** *No, because I had that confused when I first did the whole matrix thing. Eigenvectors are sometimes basis vectors, because in your normal Cartesian xy, your basis vectors are (1, 0), and (0, 1). And you can make anything in here [pointing to Cartesian plane] with them and that is the same sort of thing that eigenvectors are. But eigenvectors are properties of a matrix. Basis vectors are properties of a coordinate system, so they are not necessarily the same. They coincide a lot of times, but they are not the same.*

[Joey, I.3, lines 379-384]

She was asked to give an example demonstrating the difference between eigenvectors and basis vectors. She tried to use the second question of the third interview and stated that the vectors $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$ were eigenvectors of the matrix M and they were not basis vectors of the Cartesian plane. She said she heard that bases could be changed but she was not sure about that. Even though she did not provide a proper example demonstrating the difference, she seemed to be aware that there was a difference.

It was observed that Joey was referring back to her experience in class especially when she talked about the sum of eigenvectors. Even though she seemed to alter her recollection of what was done in class to fit her answer at the interview, her attempt of trying to connect her result to the class experience was considered to be evidence of actor-oriented transfer.

Episode 2. As did Joey, Deniz also stated that eigenvectors were “like basis vectors. However he was not able to provide any further explanation. During the second interview Deniz described eigenvectors as basis vectors and he did not say that they were “like” basis vectors, but declared that they “were” basis vectors.

> ***D:*** *Um, they’re basis vectors. They’re the..the hard part about eigenvectors is the certain arbitrariness that is built into them, something about that just drives me nuts and if since they’re the basis vectors, they can be any scalar multiple of this set of vectors, so when you first deriving them you can already see that there’s an infinite, or that there’s infinite solutions, that its not just zero, so then if there’s infinite solutions then which one’s do you pick, and that’s the certain arbitrariness, if you pick the simplest one’s possible then let the arbitrariness be alpha.*
> [Deniz, I.2, lines 38-44]

He continued his explanation by stating that eigenvectors were linearly independent. At the third interview similar ideas were observed. He was asked if the sum of two eigenvectors was again an eigenvector. He started the question by first stating that eigenvectors needed to be orthogonal and linearly independent. He stated that the sum would not necessarily be an eigenvector and he was thinking about the “spanning” idea. He gave an example from one of the three physics courses. He said “a wave could be a sum of eigenvectors but itself is not necessarily an eigenvector, but it can be represented by a sum of them, superposition" He seemed to recall the activity from the second physics course. As previously mentioned students had been given similar examples in all three physics courses and Deniz seemed to prefer an example from the waves course. Deniz could be relating the third interview question to his experience in this class and this could constitute evidence for actor-oriented transfer because Deniz was addressing the interview questions with an example from his previous experience.

Episode 3. Tom said that eigenvectors were linearly independent during the first interview, however he was not sure what linearly independent meant for eigenvectors. He stated that eigenvectors were solutions to equations and he was not sure what the equations were. He said that when he studies eigenvalues and eigenvectors in his math course, they learned how to find them but not what they were. During the second interview "eigenvectors are linearly independent" was not mentioned.

On the final Tom wrote that "In QM eigenvectors form the set of orthonormal bases that correspond to states of a given system [and] eigenvalues are the possible results of measurements made on those systems." [Final Exam] The basis vectors idea was also mentioned at the end of the third interview however he did not implemented it when he was asked if the sum of two eigenvectors was an eigenvector at the beginning of the third interview. He rather tried to recall a statement made in one of the physics courses.

> ***T:*** *I recall a statement, the sum of eigenvectors is never an eigenvector, but – and I don't remember the rest of the sentence. I remember for all of our purposes that we never run into a case where the sum of the two eigenvectors was*

> *an eigenvector itself. I think – I never really thought about it beyond that, because it wasn't really necessary to. I was thinking about other stuff. I mean, you could – the only think I would know how to do is in a specific case, take specific eigenvectors, superpose them an then insert them into the eigenvalue equation and see if they worked, or something like that.*
>
> [Tom, I.3,lines 7-13]

As Tom mentioned the case of eigenvalues with multiplicities never occurred in their physics courses and Professor C mentioned that it was a topic covered in senior level physics courses. Tom later tried to recall come examples from two of the physics courses that showed the sum was not an eigenstate. He said "a quantum wave function can be represented as – it can be dissolved into a sum of components of eigenvectors, so you have some constant times – I don't remember what we used. Any wave can be represented as the sum of the orthogonal components or the eigenvectors of state, so to speak"[I.3] and he also tried to give an example from the spin and quantum measurements course but he could not recall it completely. Tom seemed to find the third interview question similar to his experience in the three physics courses and tried to reconstruct his previous experience to answer the interview question. For this reason this episode constitutes evidence of actor-oriented transfer. Even though Tom did not mentioned basis vectors when working on this problem, he stated that any state could be written as a sum of eigenvectors.

The relationship between basis vectors and eigenvectors arose later in the third interview. Tom was asked to give a mini lecture to a pretend student who would be taking the courses he had just completed. He asked if the student knew about vector spaces and the researcher said he could assume that.

T: *OK, then you can – a vector space has basis vectors, which, I mean, a three-dimensional vector space will have three basis vectors. Basis vectors are vectors out of which any vector in that vector space can be constructed. So they are mutually orthogonal, linearly independent, and the eigenvectors of a matrix give you the bases for that matrix. I don't know if that terminology makes sense. I think it is all kind of abstract. It is most useful to think about vectors – so once you have the eigenvalues and the eigenvectors of whatever you are working with, you can express anything in this space, whatever you are working in. Somehow I think in quantum mechanics there is something that starts with an h – Hilbert spaces, but I'm not sure because that word has never been mentioned in class.*

[Tom, I.3, lines 216-225]

He was asked what he knew about Hilbert spaces and he said he read about it in his philosophy courses and it was an abstract vector space. Then he completed his explanation stating that any vector in the vector space could be expressed as the sum of eigenvectors.

Episode 3. Ozzy mentioned ideas related to spanning of vectors when he was asked what eigenvectors represented during the first interview. It was unclear if he knew what the ideas he mentioned meant and if they were related to basis vectors.

O: *I don't know, we talked about, this isn't really your question but, uh, we talked about that if we have two eigenvectors that we solved, kind of like these, that we came out with v1 and v2, if we have two of them, we have a plane then.And then, they can be anything in that plane, multiplied by either one of them or times, or the addition of them? No, not the multiplication of them, the addition. [...]Well that, if you have two eigenvectors, then you know the solutions for the entire plane, versus if you had just one, then you know you can just multiply that, so it could be anywhere along that vector, but you can't, in three space, you can't go off of that anyway because you don't have a way to define a plane, so by having the two vectors, you now have a plane.*

[Ozzy, I.1, lines 245-280]

Ozzy mentioned that eigenvectors formed a plane again during the second interview. He was asked if any scalar multiple of an eigenvector was again an eigenvector. He said it would be because the scalar multiple of eigenvector would be on the same line with the eigenvector. He was asked how being on the same line implies being an eigenvector and he was not sure. He then started to talk about eigenvectors defining a plane idea that he had learned in a math course he took.

> ***O:*** *Yeah, just from math class, um, you know, we were saying that there is the trivial solution and then if it's the plane, you're going to have an infinite number of eigenvectors on that, because I remember solving for, well, I don't remember exactly, but I remember that if we had a plane and we solved for two eigenvectors, then it would be perpendicular because it would ask for us to transform them so that you'd have perpendicular eigenvectors on that plane, so that they represent the plane, which makes sense, because they were both eigenvectors, even after you transformed them, so I assume that if it is on the same line, that it is going to still be considered an eigenvector.*
>
> [Ozzy, I.2, lines 250-260]

Ozzy seemed to be implying that eigenvectors could be used as basis vectors to span a plane. He did not mention this idea again but he was asked from which course he knew this information and he said from the matrix and power series methods course that he took in the previous term. Even though it was unclear if he knew the connection between eigenvectors and basis vectors, he seemed to be referring to his experience from his math course rather than LAW or the physics courses.

<u>Episode 5.</u> Similar to Tom, Ozzy referred to his experience in class when he was asked if the sum of two eigenvectors was an eigenvector in

the third interview. He said that the instructor had said you could have a superposition of eigenvectors that was not necessarily an eigenvector. He could not remember the conditions. He was asked to give an example of a superposition of eigenvectors and he tried to give an example from the spin and quantum measurements course. Later when he was asked to give a mini lecture to a pretend student who planned to take these physics courses, he said he would introduce eigenvectors as the new coordinate system and he claimed that the new coordinate system would be obtained through the eigenvalue equation. He could not recall correctly what the eigenvalue equation was. He seemed to assume that when the operators acted on the eigenvector, the transformed vector would result in a new coordinate system. It seemed he was mixing the transforming vectors with eigenvectors idea. He later stated that he heard someone saying that eigenvectors would create a new system but it was unclear if he knew what that "new system" meant.

Even though Ozzy did not answer the interview questions correctly or completely, he referred back to the spin and quantum measurements course when he needed examples. His experience in the spin and quantum measurements course seemed to be more accessible to him, however the ideas related to eigenvectors being basis vectors were not developed fully. Since with the actor-oriented transfer framework students' attempts to solve problems were analyzed, not the completeness or correctness of the answers, Ozzy's interview three data seemed to provide evidence for actor-oriented transfer of ideas related to basis vectors.

Gus, Crosby, and Milo did not mention the relationship between eigenvectors and basis vectors during first two interviews. Gus and Milo both stated that all states could be expressed in terms of eigenstates on the final exam prior to the third interview. Milo also included that eigenvectors were "the basis" but did not specify the basis of what. In the third interview, Milo again mentioned that any state could be written as a sum of two eigenvectors. He also mentioned eigenvectors were similar to basis vectors.

Episode 6. Gus did not mention that eigenvectors could be used as basis vectors until the third interview and also referred to superposition of two eigenstates on the final exam. Unlike other participants, Gus tried to create his own example for the interview question which was related to the sum of two eigenvectors. He initially stated that the sum of two eigenvectors was not an eigenvector of the Hamiltonian. When he was asked why, he wrote the eigenvalue equation with two separate eigenvectors and eigenvalues. He said that the sum could possibly be an eigenvector sometimes because it could be possible to find a value for e_3 in the equation $e_3\ v_3 = e_1\ v_1 + e_2\ v_2$. It was unclear why he was adding the eigenvalues, so he was asked to justify his thinking and then he decided to look at an example with an operator d/dx. He said e^{5x} and e^{2x} would be eigenvectors of this operator and he added two eigenvectors inaccurately to get e^{7x} and he added the eigenvalues 5 and 2 and stated that the sum was also an eigenvector. He however seemed to be bothered by his conclusion. He immediately thought about his experience in class and said his answer did not seem right because he recalled it differently from class but he could not think of another way to figure out the problem. Gus referred to his experience from class when he got an unexpected answer. Even though he could not come up with example to disprove his initial answer, he stated more than once that his answer did not seem to be correct. It seemed that Gus referred to his experience from winter term courses. Moreover, later in the same interview Gus gave an example from a physics course he was enrolled in at the time of the third interview. Even though he did not use an example from a course from the winter term, he was still referring back to his experience during winter term. It seemed that Gus was aware of the superposition idea and he tried to explain it using examples from his previous experience. For this reason, this episode from Gus's third interview provides evidence of actor-oriented transfer.

Episode 7. Crosby also did not mention the relationship between eigenvectors and basis vectors until the third interview. When he was asked if the sum of two eigenvectors would be an eigenvector of the same matrix, he smiled as soon as he heard the question. He said that he recalled being asked a similar question in one of the physics courses he took after LAW and could not remember the answer. He stated that in his physics course it was mentioned that if a state was a superposition of two eigenstates then it had a probability of being in one or the other one. When he was asked to justify further, he stated that eigenstates were orthogonal to each other and the sum could not be orthogonal to both and there would be three eigenstates. It seemed that Crosby was thinking that an operator only had finite number of eigenvectors. Although he was not complete in his answers to the interview questions, he indicated that the interview question was similar to a question from class and tried to implement the ideas from class.

Crosby seemed to think all eigenvectors were perpendicular to each other. Since students worked with Hermitian operators during the winter term physics courses after LAW, they were familiar with eigenvectors that were always perpendicular to each other. It seemed that Crosby generalized this idea and thought the eigenvectors of all operators were perpendicular. He referred to this experience when he was reasoning through the question in this episode. Crosby implemented his experience of eigenvectors being orthogonal to each other in the interview question without checking to see if the operator was Hermitian. For all these reasons Crosby's third interview data provide evidence of actor-oriented transfer.

Overall all students except for Milo recalled their experiences with eigenvectors expanding any vectors in the vector space and referred to their experience in LAW or physics courses to give examples or to justify their answers. Milo also knew that eigenvectors could expand any state however he did not explicitly mentioned his experience in any of the courses.

Researcher's perspective

Milo did not mention the relationship between eigenvectors and basis vectors during first two interviews. In the third interview, Milo was asked to comment on the sum of two eigenvectors. He immediately said the sum was not an eigenvector and continued explaining that eigenvectors were linearly independent of each other and so it was not possible to have an eigenvector which was linearly dependent of the other two. His reason seemed to imply that he was aware of the idea that eigenvectors could be basis vectors. To investigate his reasoning further he was asked to give an example. He said he could not think of an example and when the researcher insisted on an example, he used the word "spin" and restated his answer as an example from spin and quantum measurements course.

> ***M:*** *OK, we talk about spin and if the two eigenvectors are eigenstates, then if you add them together, then all you have is a super-position of eigenstates. You don't have a whole new eigenstate.*
> [Milo, I.3, lines 17-19]

It was unclear if he was recalling any of his experience from spin and quantum measurements course. Later in the same interview, he stated that the fact that any state of a system could be expressed as the super-position of eigenstates was a very important idea but this time he provided a mathematical example.

> ***M:*** *Well, the eigenvectors – this probably isn't always true – but the eigenvectors make up a spanning set for this base you are working in. It is analogous to – we just did Cartesian coordinate system. If you have i, j, k – any point in space can be represented as xi+yj + z k. So any point is a super-position of the three eigenvectors, in a sense. That is really important. I am not doing it justice. It is really important.*
> [Milo, I.3, lines 257-262]

He also repeated that the superposition idea was a very important concept however he still could not give examples from his physics courses. He seemed to know that eigenvectors could be used as basis vectors however this does not imply that his apparent understanding of eigenvectors as basis vectors was fully developed. The third interview data also suggested that from the researcher's perspective Milo was transferring ideas relating to basis vectors to the interview setting. However he presented these ideas as facts rather than his understanding of eigenvectors as basis vectors.

As Milo had done, Joey, Deniz and Crosby tried to describe how eigenvectors could be basis vectors by providing examples from the vector spaces R^2 or R^3. It seemed that they were relating their experience in LAW and physics courses with basis vectors to these vector spaces. The fifth day of LAW, Professor C mentioned these vector spaces and explained that any vector could be written as a linear combination of the basis vectors. It was unclear if these students were referring to this particular experience or another earlier experience from a math course. However, from the researcher's perspective students were transferring the basis vector ideas.

Overall all students seemed to know that eigenvectors could be used as basis vectors and that any vector of the given vector space could be expanded by the eigenvectors.

Summary

The purpose of this study was to investigate junior-level physics students' transfer of learning of the concept of eigenvalues and eigenvectors from the winter term physics courses to the interviews in which they participated during and after these courses.

In particular, this study addresses the following research questions:

1. What characterizes upper-level physics students' emerging understanding of the concepts of eigenvalues and eigenvectors before, during and after studying the concepts in an intensive linear algebra review week and implementing them during a series of three 3-week intensive physics courses?
2. What do students transfer about the concepts of eigenvalues and eigenvectors from this series of courses to an interview setting? More precisely, what kind of experiences and views related to matrices, methods of finding eigenvalues and eigenvectors, the interpretation and use of the eigenvalue equation, and the relationship between basis vectors and eigenvectors do upper-level physics students transfer from their coursework to the interview setting?
3. In what ways do the experiences students choose to transfer relate to their emerging understanding of the concept of eigenvalues and eigenvectors?

Each student’s emerging understanding of the concept of eigenvalues and eigenvectors was analyzed according to his/her apparent understanding of the ideas from the course goals that are directly related to eigenvalues and eigenvectors. Data were analyzed by using the inductive analysis (Patton, 2002) and also by implementing a modified version of the Qualitative Hypothesis-Generating analysis (Auerbach & Sliverstein, 2003) and repeating ideas were created to describe the students emerging understanding of a given goal.

Transfer of learning of each student was explored by implementing the Actor-Oriented Transfer (AOT) framework. The AOT framework suggests that the researcher focuses on the participants' point of view and the participants' experiences. Episodes were also presented for which there was evidence of transfer from the researcher's perspective. In the next section these results are summarized to answer the research questions.

Question 1

The first research question of this study was the following.

> What characterizes upper-level physics students' emerging understanding of the concepts of eigenvalues and eigenvectors before, during and after studying the concepts in an intensive linear algebra review week and implementing them during a series of three 3-week intensive physics courses?

The concept of eigenvalues and eigenvectors were introduced through the linear transformation activity, so the students' views of matrices seemed to be important for their emerging understanding of the concept. There were three ideas that seemed to describe the participants' different views of matrices. These ideas were that matrices represent systems of equations; that matrices represent linear transformations and that matrices represent lines. Joey, Milo, Tom, and Gus seemed to view matrices as representing linear transformations after being introduced to the transformations in LAW. Crosby, on the other hand, seemed to view matrices as representing both systems of equations and linear transformations before and after being introduced to the linear transformation idea. Ozzy also mentioned that matrices represented systems of equations during both interviews, before and after LAW. In the second interview he seemed to adopt the linear transformation representation; however he did not repeat this idea after the second interview. Deniz was the only student who viewed matrices as representing lines during the first two interviews but it seemed that he was not sure what "representing lines" really meant.

Students were expected to find eigenvalues and eigenvectors fluently in LAW and in their physics courses. When students were introduced to finding eigenvalues and eigenvectors the connection between the geometric interpretation of eigenvectors and eigenvalues and algebraic computation was discussed. For this reason, the ideas related to finding eigenvalues and eigenvectors were also investigated. Three ideas appeared in the interviews that were related to finding eigenvalues. The ideas were the three equations used to find eigenvalues and eigenvectors: the eigenvalue equation, $Av=\lambda v$, $(A-\lambda I)v = 0$ and the characteristic equation $\det(A-\lambda I)=0$. Milo was the only student who used the eigenvalue equations and the characteristic equation to find eigenvalues and eigenvectors throughout the study and he was the only student who seemed to be fluent in finding eigenvalues and eigenvectors. Ozzy was the only student who used the equation $(A-\lambda I)v=0$ to find eigenvectors during the first two interviews. At the third interview he seemed to mix the equation $(A-\lambda I)v=0$ with the eigenvalue equation. He also stated during the second interview that the way in which the eigenvectors were found in LAW was "interesting" to him because he learned to use the equation $(A-\lambda I)v=0$ to find eigenvectors in his math course. Joey used the characteristic equation to find eigenvalues throughout the study however she could not recall how to find eigenvectors during the study. Similarly, Gus did not use the eigenvalue equation to find eigenvectors during the second interview and he mentioned the equation but not to find eigenvectors during the third interview. Tom was able to find eigenvalues and eigenvectors during the second interview; however he did not recall the characteristic equation correctly to find eigenvalues in the third interview.

Students were introduced to the eigenvalue equation and its algebraic and geometric interpretations on the second day of LAW and students were expected, when appropriate, to use these interpretations throughout the subsequent physics courses. The eigenvalue equation and its interpretations are related to students' emerging understandings of eigenvalues and eigenvectors and the ideas describing their apparent understanding were explored. Three repeating ideas appeared in the data: the geometric interpretation of eigenvectors, the algebraic interpretation of eigenvalue and the algebraic interpretation of the eigenvalue equation. The geometric interpretation of the eigenvalue equation was also observed but less frequently. Milo, Tom, Joey and Gus all used the geometric interpretation of eigenvectors. Milo also talked about the algebraic interpretation of eigenvalues. Milo and Tom both mentioned the algebraic and geometric interpretations of eigenvalue equation in the third interview. Gus mentioned the algebraic interpretation of the eigenvalue equation in the third interview. Crosby repeatedly used the algebraic interpretation of the eigenvalue equation during the last two interviews. Deniz tried to implement the geometric interpretation of eigenvectors during the second interview and the algebraic interpretation of the eigenvalue equation at the third interview. Ozzy did not use either the algebraic or geometric interpretation of the eigenvalue equation.

After completing the winter term courses students were expected to understand that eigenvectors could form a set of basis vectors for the vector space and any vector from this space could be expanded linearly with eigenvectors. For this reason the researcher search for statements related to this goal. Only Joey and Deniz mentioned that eigenvectors were basis vectors. However, all students stated that a vector (or a state) could be written as a sum of eigenvectors (eigenstates) during the third interview.

Question 2:

The second research question of this study was the following.

> What do students transfer about the concepts of eigenvalues and eigenvectors from this series of courses to an interview setting? More precisely, what kind of experiences and views related to matrices, methods of finding eigenvalues and eigenvectors, the interpretation and use of the eigenvalue equation, and the relationship between basis vectors and eigenvectors do upper-level

physics students transfer from their coursework to the interview setting?

All participants except for Joey seemed to transfer their experiences from the linear transformation activity on the second day of LAW to the second interview. In this interview students mentioned the activity when finding the determinant of a given matrix, when transforming vectors and when determining what a matrix represented as a transformation.

The researcher also observed during the second and third interviews that Ozzy, Gus, Milo, Joey and Tom were using the terminology of “operating (acting) on a vector with matrices”. Even though participants of this study did not mention explicitly why they were using this terminology, these students used it during both interviews when they used matrices. (Since students did not explicitly refer to LAW or the winter term courses and the use of this terminology did not become evidence for actor-oriented transfer.)

During the interviews Ozzy, Crosby and Milo seemed to transfer their experiences on finding eigenvectors from LAW. All three students changed their methods of finding eigenvectors between the first and the second interview. During the second interview Ozzy commented that finding eigenvectors using the eigenvalue equation was one of the most interesting things he learned in LAW. During the first interview he found eigenvectors using the equation $(A-\lambda I)v= 0$. However, this change seemed to confuse him later in the third interview because he could not recall the correct eigenvalue equation and wrote several different equations one of which resembled $(A-\lambda I)v= 0$.

The researcher observed that Gus implemented the geometric interpretation of eigenvectors to find eigenvectors during the second interview and he mentioned that the geometric interpretation of eigenvectors was one of the most interesting things he learned during LAW. Even though he did not mention explicitly why he was implementing the geometric interpretation of eigenvectors to find eigenvectors instead of the equation, he seemed to transfer one of the suggestions of Professor Clayton.

Tom and Joey could not recall the equations for finding eigenvalues and/or eigenvectors. The researcher, however, observed that Tom and Joey seemed to transfer the memory of the actions of finding eigenvalues and eigenvectors when they described how eigenvalues and eigenvectors were found once the equations were given.

Milo, Gus, Tom, Deniz, Crosby and Joey implemented their experience with the algebraic or geometric interpretation of eigenvalues and eigenvectors after being introduced to the concept of eigenvalues and eigenvectors, the eigenvalue equation and its interpretations in LAW and the subsequent physics courses. Joey and Milo explicitly mentioned the idea that a vector's direction was assumed to be unchanged if it pointed in the opposite direction after being transformed by the matrix. Milo seemed to be comfortable with this idea, however Joey said she "trust them [the instructors]".

Joey, Tom and Deniz recalled a class discussion on reflection in which it was stated that the vectors that were on the reflection axis would be eigenvectors and the ones perpendicular to the reflection axis would also be eigenvectors. The implementation of this idea and the geometric interpretation of eigenvectors were also observed in their reasoning.

Gus and Deniz referred to their experience in the spin and quantum measurements courses as they worked on the same third interview question. Deniz stated that he had watched a classmate as they worked on a similar question and he tried to recreate his experience from that situation. Gus also solved this problem using the eigenvalue equation and as he was working on it he recalled working on a similar matrix during the spin and quantum measurements course.

The researcher observed that some students implemented (sometimes incorrectly) the algebraic and geometric interpretations of the equation but did not explicitly refer to LAW or any physics course. Joey seemed to be aware of the geometric interpretation of the eigenvectors during the third interview and some ideas related to eigenvalues; however she seemed to need more time to develop these ideas.

The researcher also observed that more participants were referring to the algebraic interpretation of the eigenvalue equation during the third interview. One of the reasons for this could have been that students were introduced to operators other than matrices during the final two physics courses and the geometric interpretation did not make much sense when working with functions as the vectors. In these classes, instead of thinking about the unchanged direction of a vector, students were told to think about the unchanged form of a function.

All students except for Milo recalled their experiences with the superposition idea from their physics courses and seemed to transfer their experiences in LAW or the physics courses to the third interview when giving examples of justifying their answers.

Students seemed to find the second interview question “what does this matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ do?” similar to the linear transformation activity and except for Joey all students explicitly referred to their experience with this activity.

At the beginning of the third interview all students were asked if the sum of two eigenvectors was an eigenvector of the same operator. Students seemed to find this question similar to questions they had worked on in their physics courses. In all three physics courses students had worked examples where they needed to determine if a given state was an eigenstate of the operator. Six participants explicitly indicated that they had done a similar question in one their physics courses.

Deniz and Gus also stated that they had worked a similar problem to the second question of the third interview in which they were given the eigenvalues and eigenvectors of an operator and asked to comment on the operator. Deniz stated that he had worked on a similar problem with a classmate and Gus stated that he had seen something similar to this matrix in his spin and quantum measurements course. However, Joey and Ozzy indicated that they were not familiar with this type of question because they always found the eigenvalues and eigenvectors of a matrix and did not find the matrix when the eigenvalues and eigenvectors were given.

Overall, the actor-oriented transfer analysis of the participants' second and third interviews produced evidence that suggests the participants reconstructed their experiences from

- the linear transformations activity,
- the eigenvalues and eigenvectors activity,
- the superposition examples from some of the physics courses and
- the exercises done on constructing a matrix when eigenvalues and eigenvectors were given in the spin and quantum measurements course.

Question 3

The third research question of this study was the following.

> In what ways do the experiences students choose to transfer relate to their emerging understanding of the concept of eigenvalues and eigenvectors?

Students' emerging understanding of eigenvalues and eigenvectors described under each goal of the winter term courses seemed to describe their overall observable understanding of eigenvalues and eigenvectors. The experience student's choose to transfer, on the other hand, seemed to inform the researcher on the student's learning process of the goal. In other words, a student may not have any repeated ideas describing the student's emerging understanding of a goal, but it seems possible for the same student to transfer an experience related to that goal. For example, Deniz repeatedly mentioned that eigenvectors were basis vectors throughout the study. He also seemed to reconstruct his experience in the waves course to address an interview question related to the superposition idea. On the other hand, Deniz did not have any repeating ideas describing his emerging understanding of the interpretations of the eigenvalue equation but he referred to his experience with a reflection transformation from LAW during the second interview. Deniz repeatedly mentioned that matrices represented lines. Even though he seemed to know that matrices represented linear transformations, he did not repeat this idea during the interviews. It seems that Deniz's overall understanding of eigenvalues and eigenvectors evolved around the idea that eigenvectors could be used as basis vectors. He seemed to be developing his understanding of matrices as linear transformations and the interpretations of the eigenvalue equations.

When Milo's data was analyzed, there were no repeating ideas related to basis vectors and he also did not transfer any of his experiences relating to this idea. He seemed to view matrices as linear transformations and repeatedly mentioned and used the geometric interpretation of eigenvectors. The researcher also observed that these two repeating ideas seemed to be connected for Milo. He seemed to connect the geometric interpretation of eigenvectors to his view of matrix representation. For example, he proposed that the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ was a reflection matrix during the second interview and when he was asked about the reflection he stated that he needed to find the eigenvectors to find the reflection axis. It seems that Milo was starting to develop the basis vectors idea for eigenvectors and his understanding of eigenvalues and eigenvectors evolved more around the algebraic and geometric interpretations of the eigenvalues and eigenvectors.

Overall, each student's emerging understanding discussed under each goal seemed to help paint a picture of the student's overall observable understanding of eigenvalues and eigenvectors. The actor-oriented transfer analysis seemed to describe the student's learning process.

CHAPTER SIX

DISCUSSION

This chapter summarizes the results of the data analysis and provides implications and limitations of this study together with suggestions for future research studies in the area of transfer of learning of linear algebra topics. The chapter starts with a summary and synthesis of the results of this study.

Summary and Synthesis of the Results

Participants of this study were all junior level physics undergraduate students who took an intensive linear algebra review week and a series of three 3-week intensive physics courses in a 10-week period, namely, the linear algebra review week (LAW), spin and quantum measurements, waves, and central forces. Participants were invited to participate in interviews prior to, during, and after the 10 weeks of coursework. In each interview students were asked to describe the concept of eigenvalues and eigenvectors and to answer questions related to the concept of eigenvalues and eigenvectors. Three participants were enrolled in a linear algebra course during the fall term of this study. The other three students had taken a linear algebra course prior to the study and only one participant had not taken a linear algebra course.

Five students (Milo, Crosby, Tom, Gus, and Ozzy) initially viewed matrices as representing systems of equations and two students among these five also stated that linear transformations could be represented as matrices prior to

LAW. Joey seemed to view matrices as linear transformation throughout the study and Deniz viewed matrices as representing lines.

After being introduced to matrix representation in LAW, six students seemed to view matrices as linear transformations and Deniz still seemed to view them as representing lines. Crosby seemed to have a conflict between two representations and he seemed to be searching for a connection between the two representations. In the second interview it seemed that most students were reconstructing their experience using one of the activities which occurred during LAW. For this reason, according to the actor-oriented transfer framework, six of the students were assumed to be transferring what a matrix represented from LAW to the interviews.

Only one student knew how to find eigenvalues and eigenvectors prior to LAW and the rest of the students either were not successful in finding either eigenvalues or eigenvectors or both. All seven participants were able to find eigenvalues of a given matrix during the second interview after LAW and six of these students were also able to find eigenvectors. Joey could not recall how to find eigenvectors during the second interview, a week after being introduced to it in LAW. Gus was able to find eigenvectors in the second interview, however he implemented the geometric interpretation of eigenvectors and it was unclear if he knew how to find them algebraically. In the third interview he was able to describe how to find eigenvalues and eigenvectors, but he was not asked to find them. The fluency of finding eigenvalues and eigenvectors of each participant varied throughout the interviews. Only Milo seemed to be fluent in finding eigenvalues and eigenvectors after being introduced to them in LAW. However, the focus of the study was not to evaluate the fluency of finding eigenvalues and eigenvectors. It seemed that Ozzy, Milo and Crosby were all using the eigenvalue equation to find eigenvectors after being introduced to it in LAW and prior to LAW they were all using different equations to find eigenvectors. Ozzy explicitly stated that using the eigenvalue equation was one of the most interesting things he learned in LAW that he did not know before. He stated that in his linear algebra course they used another equation (A-λI) v=0 to find eigenvectors. It seemed that Ozzy did not see the connection between these two equations and assumed that they were two different equations. On the third interview Ozzy seemed to confuse these two equations and could not recall the eigenvalue equation for finding eigenvectors. He proposed the equation $A\lambda = \vec{v}$ to find eigenvectors. It seemed since he did not create a connection between two equations, the eigenvalue equation $A\vec{v} = \lambda\vec{v}$ and the equation $(A - \lambda I)\vec{v} = 0$ and later he could not recall either one of them.

Five students seemed to transfer their experiences using the eigenvalue equation to find eigenvectors to the interviews. Notably, Milo, Ozzy and Crosby changed their way of finding eigenvectors to the method introduced in LAW. Gus implemented the geometric interpretation of eigenvectors to find them during the second interview and he explained how to find eigenvalues and eigenvector using the characteristic equation and the eigenvalue equation during the third interview. It was unclear if he knew how to find eigenvectors during the second interview. However, even he did not recall how to find eigenvectors by using the eigenvalue equation, it was clear that he knew the geometric interpretation of the eigenvectors to find them. It seemed that he was reconstructing his experience with the geometric interpretation of eigenvectors for the interview question. For all these reasons, there seemed to be evidence of actor-oriented transfer related to finding eigenvalues and eigenvectors.

Students were introduced to the eigenvalue equation and its algebraic and geometric interpretations during LAW and expected to use these when appropriate. Five of the participants (Milo, Gus, Tom, Joey and Deniz) referred to their experience in class and seemed to reorganize the geometric interpretation ideas from class experience to address the interview questions. Crosby on the other hand referred to the algebraic interpretation of the eigenvalue equation which he seemed to be developing since the linear algebra course he had taken during the fall term. Milo took the same course and both students referred to the algebraic interpretation on the pre-quiz. However, Milo seemed to use both interpretations during the second and third interview and seemed to switch between the interpretations during the third interview.

Students provided evidence of actor-oriented transfer of the algebraic and/or geometric interpretations of the eigenvalue equation by reconstructing their experience in LAW. Notably, students were referring to their experiences during the linear transformations activity but they also seemed to refer to other ideas discussed on the third and fifth days of LAW.

Only two students (Joey and Deniz) suggested a relationship between eigenvectors and basis vectors during the first interview. These students were again the only students who explicitly referred to the relationship during the second interview. Tom and Ozzy talked about eigenvectors being linearly independent and spanning a plane or a space, however their ideas did not seem to be developed fully during the first interview. Students were not asked questions about the relationship of eigenvectors and basis vectors during the second interview but Joey and Deniz still mentioned the idea. During the third interview however all students were asked if the sum of two eigenvectors was an eigenvector of the same operator. Students were not given any information on the eigenvalues and expected to consider the cases when the eigenvectors associated with the same eigenvalue and with different ones. None of the students thought about the case in which given eigenvectors were from the same eigenvalue unless they were explicitly asked. When they were explicitly asked, they seemed not know what to do and the researcher suggested maybe they could find an example from their physics courses. Two of the students stated that they had seen something similar to this idea in an earlier physics course but not in any of the winter term physics courses. Later Professor Clayton commented that the eigenvalue with algebraic multiplicities would be discussed during the senior year physics courses.

All participants expect for Milo recalled their experience with the "superposition" idea from the winter term physics courses. Some students gave examples from one of the physics courses and some students only recalled the fact that the sum was not an eigenvector and discussed a way to check that idea. It seemed that these students were connecting to their experiences from the spins and quantum measurements course to answer the interview question. There seemed to be evidence of actor-oriented transfer of the superposition idea. However this does not imply that students knew either that eigenvectors could form a set of basis vectors or knew the distinction between eigenvectors and basis vectors.

The results of this study suggest the importance of exploring the issue of transfer by implementing the actor-oriented transfer framework. If the data from all participants were analyzed using the traditional transfer research paradigms, then the answer to the research question “Do students transfer the concept of eigenvalues and eigenvectors and other related ideas from winter term physics courses to the interviews?” would indicate that only one participant (Milo) seemed to transfer. However, the analysis with the actor- oriented transfer framework provides an in-depth exploration of students’ experiences which they seemed to connect to during interviews. The actor-oriented transfer framework seemed to allow for an investigation of the learning process whereas the traditional transfer paradigm looks only on the end product of learning.

Implications

The results of this study have both research and pedagogical implications for the field of mathematics education. The section contains a discussion on the attributes of the actor-oriented transfer framework in education research and a discussion on pedagogical implication concludes the section.

The Actor-Oriented Transfer Framework in Mathematics Education

Educational researchers (Bransford & McCarrell, 1974; Anderson, 1976; Gick & Holyoak, 1983; Lave, 1988; Detterman, 1993; Greeno, Smith & Moore, 1993; Bransford & Schwartz, 1999; Lobato, 1996; Beach, 1999; Mestre, 2005) have attempted to address questions related to transfer of learning for more than

100 years. Some researchers even suggested avoiding transfer of learning as a research construct (Carreher, & Schliermann, 2002) by stating that transfer is no different than learning and the influence of transfer tasks during transfer studies are neglected. The result of this study implies that investigating transfer of learning is a complicated task however it also addresses the aforementioned shortcomings. First, the results imply the importance of the alignment between the underlying learning theory of the implemented framework and the researcher's view of learning. As previously mentioned, there have been different transfer paradigms to investigate the issue of transfer. The underlying learning theories of these paradigms have varied from cognitive to social constructivism thus the definition of transfer and the methodologies to investigate transfer have also varied accordingly. The alignment between the underlying learning theory and the methodologies to investigate transfer is very important and for this reason in this study the actor-oriented transfer framework was chosen to analyze data. The researcher described her view of learning as a process of personal construction of concepts in two different planes, psychological and social and the actor-oriented transfer framework suggests that transfer happens in both planes. The results obtained by implementing this framework in this study provide insights on students' learning processes as students' reconstruct their experiences during interviews. In other words, the results of the analysis imply that learning and transfer are interrelated and the actor-oriented transfer paradigm helped the researcher to investigate the learning process in detail.

The actor-oriented transfer framework also addresses the second shortcoming of traditional transfer research by searching for learning that might occur during transfer tasks (for example during interviews). Researchers can also investigate transfer between transfer tasks. This particular kind of transfer occurred during an episode in one of the interviews. Milo made a spontaneous "discovery" of the algebraic interpretation of the eigenvalue equation during the first interview and he implemented his "discovery" on the pre-quiz. (The details of this episode were described in Chapter Four.) This result implies that interview settings (transfer tasks) should be investigated thoroughly as suggested by the framework.

The actor-oriented transfer framework considers the learner's perspective to describe what students transfer. In this study as well as in the results from the learner's perspective, the results from the researcher's perspective are provided. The researcher's perspective helped in the investigation of each student's emerging understanding and also revealed possibly transfer ideas. For example, the researcher observed that four participants of this study used terminology from LAW or the other three physics courses. This result might imply that students were transferring the "language" used in their courses related to the concepts of eigenvalues and eigenvectors to the interviews. Since students did not explicitly state why they chose to use the terminology, these observations were not considered as evidence of actor-oriented transfer. Further investigation is needed to conclude if such observations could be considered as transfer, however the researcher's perspective reveals this possibility. In other words, as the actor-oriented transfer framework focuses on the participant's point of view, including the researcher's perspective seems to provide an overview of the ideas that seem to be transferred to a new setting. The results of this research imply that two approaches compliment each other and provide a comprehensive image of student's learning process at the time of the study. It needs to be emphasized that the researcher's perspective still explores the answers to the question "what do student transfer" rather than "do they transfer".

The Pedagogical Implication of Results

Several different issues which impact teaching and learning of linear algebra topics, especially the concepts of eigenvalues and eigenvectors seem to have emerged from this study. First, it seems that learning is a very *slow* process and educators should be aware of it as they design their courses. In this study the researcher observed that students were still struggling with the concepts of eigenvalues and eigenvectors even after working with the concepts more than ten weeks and being introduced to them prior to the ten weeks. This finding suggests that educators should have "realistic" expectations and assess students' learning accordingly. For example, one of the goals of the courses student took was that students were expected to understand the algebraic and geometric interpretations of eigenvalues and eigenvectors. The researcher observed that students were aware of the interpretations by explicitly stating what the interpretations were however they still needed some time to implement and restructure these interpretations accurately.

A second issue related to teaching and learning is that the actor-oriented transfer paradigm helps us to understand the learning process of students as students find similarities between a current situation and earlier experiences. As seen in the results of this study, most of the participants referred to one of the activities done in LAW and used their individual experiences from this activity during the interviews. The activity recalled by most of the participants was a small group activity at the end of which students had a whole group discussion. During the whole group discussion, with the help of the instructor, students developed a hypothesis on the relationship between determinants and the matrix representation of a linear transformation. Even though most students could not recall the exact hypothesis from the activity during the interviews, they were able to reconstruct the ideas from the activity and form the necessary results for the interview questions. The other experiences that students seemed to refer to were again small group activities done in the waves and central forces courses. In each of these courses students worked in groups to decide if the sum of eigenstates was an eigenstate. These results underline the social aspects of learning processes. The learning environment in these physics courses seemed to help students to connect to the activities and make them become a part of their experiences, however further investigation on social aspects of learning during these courses is required.

Other than the social component of these activities, the researcher observed that these activities also shared some structural similarities. Each activity seemed to start with connecting a new concept to a previously discussed idea. In some activities this particular structure is done explicitly and in some activities it is implicit. The researcher also observed that each activity emphasized multiple representations of the concept. It seemed possible that these similar structures of each “transferred” activity helped the participants reconstruct their experiences from these activities during the interviews. However, further activity task analysis is required to explore the connection between transfer and the structure of activities.

The results of this study also imply the importance of the connection made between pre-existing knowledge and new knowledge. As previously discussed, Crosby seemed to view matrices as representing systems of equations and then during LAW he was introduced to linear transformations. During the second interview a conflict between these two different representations arose and Crosby tried to find a way to connect these two different ideas. He also referred to an in-class discussion on these two different representations although he could not recall all the details. Ozzy seemed to have a conflict between his existing knowledge for finding eigenvectors and the method introduced in LAW. He transferred his experience from LAW during the second interview when he was finding eigenvectors and also mentioned that he found this “new” way very interesting. On the third interview however he did not recall either way of finding eigenvectors. These actor-oriented transfer episodes suggested that students pre-existing knowledge is very important and when new knowledge is introduced its assimilation into existing knowledge might take more time. The results suggest that the students could benefit from activities which allow students to make explicit connections between pre-existing knowledge and new ideas. Furthermore, educators might consider explicitly referring to previously discussed idea to help students to connect the pre-existing knowledge to the new ones.

As previously mentioned, one of the results of this study indicated that students' fluency in finding eigenvalues and eigenvectors changed over time. Notably, one student (Gus) did not find eigenvectors of a matrix using the eigenvalue equation but rather he implemented the geometric interpretation. It is known that the geometric interpretation may not be feasible for all situations, however as it is seen in Gus' situation, it provided a conceptual way of solving a seemingly procedural question. Overall the algebraic and geometric interpretations of the eigenvalue equation provide a conceptual way to understand eigenvalues and eigenvectors. The results of this study offer preliminary evidence on the importance of teaching conceptual ideas even when the procedural understanding is the primary goal.

Limitations of the Study

The purpose of this study was to investigate the junior-level physics students' transfer of learning of the concept of eigenvalues and eigenvectors from the winter term physics courses to the interviews in which they participated during and after taking these courses. Transfer of learning of each student was explored by implementing the Actor-Oriented Transfer (AOT) framework. Each student's transfer was analyzed and described using qualitative research methods and the researcher does not claim that these students represent a wider population of physics students. Rather, these seven participants were students who volunteered to be in this study. Data would have more actor-oriented transfer episodes if there were more students involved in the study however the data analysis would take longer under such circumstances. By studying only a small number of cases, the researcher was able to understand the experiences of each student and to provide a rich description of their emerging understanding of the concept of eigenvalues and eigenvectors and the actor-oriented transfer episodes. However, if the researcher was able to gather more data on the social learning aspects of a student's experiences, then the student's process of learning might be better understood. For example, it would be valuable to know how these participants interacted with each other outside of class when working on their assignments and studying for the final exams.

A second limitation of the study was the structure of the interviews. The researcher observed that students worked in groups during the winter term courses. It was also noticed that some students worked together on homework assignments. It seemed that active engagement with others was a big part of these students' learning experience. However during the interview students worked alone and did not have any interaction with anyone other than the researcher who did not engage in the tasks as a fellow student would have done. This lack of interaction during the interviews may have prohibited observing all that was possible to know about each participant's understanding of the concept of eigenvalues and eigenvectors. An additional group interview with the participants might provide more insights on these participants' process of learning and emerging understanding of the concept because such an interview setting would be closer to their familiar learning situation.

A third limitation of this study may have been the participants' perspective of mathematics, the fact that the interview questions were designed to have no context and the fact that the participants were all physics students. The questions were designed to have minimum context so that the surface features of a question would not be distracting and also so that students could use any context to interpret the question. However, for some students this created an issue of needing clarification on the questions: did the researcher want to know the mathematical ideas or physics ideas. Although the researcher said she was looking for all possible ideas, the issue of what it meant to the students to have a mathematical or a physical interpretations of a given question was not explored. The students' perspective of mathematics seemed to be related to the issue of what students choose to transfer to the interviews. However, the analysis of the students' perspectives of mathematics was out of the scope of this study.

Fourth, as with any study, the researcher brings certain biases and beliefs that can potentially impact the research results. The researcher's experiences with linear algebra and her personal beliefs about how these topics should be taught and understood may have impacted what was "noticed" and how that data were analyzed. In this study, the researcher made an effort to minimize this limitation. First, the researcher kept a personal journal of what she expected from students on each interview question during the creation of the interview questions, prior to the interview. She observed the classes during LAW prior to analyzing the data and she included her expectations from students prior to the interviews. If the researcher's expected ideas and the ones that were noticed during the data analysis overlapped, a second analysis was conducted. Second, the episodes constituting evidence of actor-oriented transfer were restricted to ones in which students explicitly referred to an experience from their previous courses. Even though other episodes seemed to provide some evidence, these were only included under the researcher's perspective.

Recommendations for Future Research

The results of this study provided four in-depth case analysis of four students' emergent understanding, evidence of their actor-oriented transfer and a cross-case analysis of seven participants' actor-oriented transfer. Data collected on the other three participants were not presented as case studies in this study and their case studies could be presented in the future.

Data were collected during the participant's third year of college physics courses. Students were interviewed before, during and after they were introduced to the concept of eigenvalues and eigenvectors during the period of the study. As previously mentioned, some of the students' ideas related to the concept of eigenvalues and eigenvectors seemed to be developing during the interview process. For example, it seemed that Deniz's apparent understanding of the relationship of eigenvectors and basis vectors was developed partially. Some ideas that are related to eigenvalues and eigenvectors were not discussed during the third year physics courses and were part of fourth year courses. For these reasons, further interviews on the concept of eigenvalues and eigenvectors with these seven students during their senior year would provide more insights on what experiences students transfer after being introduced to more ideas related to eigenvalues and eigenvectors and after spending more time working with eigenvalues and eigenvectors.

A second possibility for future research arises from one of the limitations of this study (see page 292). It would be interesting to investigate the distinction that physics students' make between "doing mathematics" and "doing physics".

A third question which builds from the results of this study is to explore the connection between the students' perspective on what they learned during a linear algebra course and what they transfer. The results of this study suggested that some students did not want to share some of their experiences from course (for example Milo) because they seemed not be confident in their understanding. This type of research would also require more in depth analysis on the connection between learning theories and actor-oriented transfer and also metacognition.

Another future research question arises from the chosen topic of this research, the concept of eigenvalues and eigenvectors. The focus of this study was to investigate the physics' students' transfer of learning of the concept as they were introduced and widely used the concept in their winter term physics courses. This setting was chosen because it was realized that students would be first introduced to the concept during a linear algebra week and then exposed to its application during the subsequent weeks of physics. It would be interesting to provide a similar linear algebra week to mathematics students and follow them as they take subsequent mathematics courses in which they would be introduced to the application of eigenvalues and eigenvectors. It would be interesting to explore in what ways mathematics students transfer their learning from the linear algebra week to their subsequent math courses and to other settings.

Conclusion

Research studies in the area of transfer of learning of mathematics topics from one setting to another one have been focused on the topics from lower-level undergraduate courses, such as trigonometric functions and algebra skills (Cui, 2006; Ozimek et.al., 2004). The purpose of this study was to explore the transfer of learning of topics from an upper-level mathematics courses, namely the concept of eigenvalues and eigenvectors, to an interview setting. The previous studies which were conducted to explore transfer of learning had been under the influence of traditional transfer paradigms. Recently new studies have included new approaches to the research construct of transfer. One of the new approaches of investigating students' transfer has been proposed by Lobato (1996), the actor-oriented transfer, and in this study this new approach was implemented.

The results of this study underlined one of the most helpful features of the actor-oriented transfer framework. This framework could inform the researchers about the learning process rather than to merely observe the end result of learning. The actor-oriented transfer framework seems to focus a lens on how students connect their previous experiences (for example the experiences during teaching) to new ones (for example the experiences in the interviews) as they find explicit or implicit similarities between the experiences. The results of this study also confirmed some of the findings from studies done in teaching and learning of linear algebra, for example students' difficulties with multiple representations (Kaput, 1992; Hillel, 2000; Stewart & Thomas, 2003). However, it also added that students could implement the algebraic and geometric interpretations of the eigenvalue equation and transfer their experiences with these interpretations when they think it would be appropriate.

Also a researcher's perspective was included in this study and it seemed that it complimented the actor-oriented transfer framework. While the actor-oriented transfer framework provides details of the learning process, the researcher's perspective seemed to provide the other possible ideas students might have been attempting to transfer. Moreover, each student's emerging understanding of discussed under each goal seemed to help recognizing the student's overall observable understanding of eigenvalues and eigenvectors.

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APPENDICES

APPENDIX A

PILOT STUDY QUESTIONS

Interview Questions:

{Show the signed informed consent form and explain the study}

1. What is an eigenvalue? What is an eigenvector? Examples?

2. Tell me about your experience in linear algebra (or physics course) class so far. How is it the same or different from other math courses you took before or taking right now?

3. How would you determine an angle between two vectors?

4. What is a matrix multiplication? How does it work, how does the procedure work? (For further prompting use this: Here are two matrices, could you please do the following multiplications A*B; B*A?
   A=$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ B=$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Also ask about content that they would do a matrix multiplication.)

5. Let's T be a linear transformation in R^2 that reflects vectors across x-axis. What are eigenvalues and the eigenvectors? How can we find them? What do they represent?
6. What the following matrices represent (choose one if no time)? Find its eigenvalues and eigenvectors?
   $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

7. Let λ_1 and λ_2 are eigenvalues of the same operator. What can you tell me about the sum of these eigenvalues?

APPENDIX B

INFORMED CONSENT DOCUMENT

Project Title: **Students' Understanding and Actor-Oriented Transfer Framework: Focusing on Eigenvalues and Eigenvectors in Physics Settings**
Principal Investigator: **Barbara Edwards, Mathematics Department**
Co-Investigator(s): **Gulden Karakok (Student Researcher) Department of Science and Mathematics Education**

WHAT IS THE PURPOSE OF THIS STUDY?

You are being invited to take part in a research study designed to investigate students' understanding of eigenvalues and eigenvectors and the evolution of their understanding during different learning contexts throughout the Fall Term 2007 and Winter Term, 2008. The results will help us to improve the curricula of math courses and physics courses. Faculty in each discipline could be informed by the finding of this study in ways that could allow them to make improvements in their respective curricula. The results will be used in student researcher's, Gulden Karakok, thesis and for further publications, presentation and to conduct further studies.

WHAT IS THE PURPOSE OF THIS FORM?

This consent form gives you the information you will need to help you decide whether to be in the study or not. Please read the form carefully. You may ask any questions about the research, the possible risks and benefits, your rights as a volunteer, and anything else that is not clear. When all of your questions have been answered, you can decide if you want to be in this study or not.

WHY AM I BEING INVITED TO TAKE PART IN THIS STUDY?

You are being invited to take part in this study because you are enrolled in Paradigms Program and you **are at least18 years old**.

WHAT WILL HAPPEN DURING THIS STUDY AND HOW LONG WILL IT TAKE?

If you decide to participate, you will be asked to fill out the survey that is handed to you today. The survey will take at most 10 minutes of your time. You are also invited to participate in an interview that will be conducted this term, Fall Term of 2007. Please see the table below for more details of each portion of this study.

If you wish to decide later, please see the student researcher's, Gulden Karakok, information at the end of this form. You will have the next couple of days to decide but please do not take more than a week. If you wish to participate, please make a copy of an informed consent form and sign it and either bring it to the student researcher's office or mail it to the student researcher. Then a survey will be emailed or mailed to you upon the arrival of the informed consent form.

If you are agreed to participate for an interview, it may take an hour or as much as 90 minutes of your time. It will be an individual interview. The interview time will be set up according to your schedule after the third week of this term. The interview will take place in the student researcher's (Gulden Karakok's) office in Weniger 332-H. The interview will be audio and video taped. In the interview you will be asked to solve linear algebra problems and explain your thinking aloud while you are solving the problems. Getting the right answers is not the focus of this study. We want to explore your way of solving problems in linear algebra. The focus of this study is <u>not to evaluate</u> you.

After the first interview, you might be contacted for second and third interviews during winter term 2008 and spring term 2008. Each interview might take an hour or as much as 90 minutes of your time. These students will be selected according to the answers they provide during the initial interview. The answers that represent different approaches to the questions will be selected to represent the different types of understanding of the topics.

There is a chance you may be contacted further to clarify some of aspects of the interviews. This contact is planned to be done via email during spring term of 2008.

If you agree to participate, your responses will be included in the student researcher's thesis and in articles about the course you are taking to be used as examples of student understanding of eigenvalues and eigenvectors.

What?	When?	Who?	How long?
Informed Consent form and Survey	Third week of Fall 2007	All the students in Paradigms in Physics fall term 2007 will be invited.	Survey- 10 minutes
Interview 1	Contact: Third week of Fall 2007 Conduct: Start fourth week and finish until all the volunteers are interviewed during fall term 2007.	All the volunteers from Paradigms in Physics course	Interview #1: 60 to 90 minutes
Interview 2	Contact: At the end of Fall term 2007 and during the winter break. Conduct: Second week of Winter 2008	Selected students (up to 9)	Interview #2: 60 to 90 minutes
Interview 3 ($25 will be given after the completion of the third interview.)	Contact: At the end of Winter Term, 2008. Conduct: First week of Spring 2008	Same students from interview 2.	Interview #3: 60 to 90 minutes

WHAT ARE THE RISKS OF THIS STUDY?

There are no foreseeable risks to participants who will be involved in this research. Other than the researchers of this study, no one will know that you are participating in this study. However, there is always a chance for the surveys or emails to get lost through the mailing or emailing transmission. This may cause a lost of confidentiality**.** Also, you may feel discomfort with taping (audio and/or video) and discomfort for being observed while working out problems.

WHAT ARE THE BENEFITS OF THIS STUDY?

There is no direct benefit. You may benefit indirectly from this study if you volunteer for an interview and find that solving problems help your learning process. We hope that, in the future, other people might benefit from this study because, the results of this study will help mathematics educators improve Linear Algebra curriculum for the prospective Linear Algebra students.

WILL I BE PAID FOR PARTICIPATING?

You will be paid for different portion of this study.

1. You will **<u>not</u>** be paid for filling out the survey only.
2. You will **<u>not</u>** be paid for the first interview.
3. If you are selected for the second and third interviews, you will be paid $25 in cash upon the completion of the third interview. There will be no partial payment after the second interview. The amount of $25 will be paid right after the third interview.

The student researcher, Gulden Karakok will be in charge of payments. There will be no non-monetary compensation.

WHO WILL SEE THE INFORMATION I GIVE?

The information you provide during this research study will be kept confidential to the extent permitted by law. To help protect your confidentiality, we will first collect one of the copies of this informed consent form and the survey regardless of you deciding to participate or not so that your participation will be anonymous to your friends in this classroom and to you professor in the classroom.

To help to protect your confidentiality further, we will assign random code numbers to the surveys. These code numbers will be used throughout the study instead of your real name. The file that has the codes with corresponding names will be kept in a secure location separate from the first and second pages of the surveys and audio/video tapes. If the participants submitted an electronic version of the survey then the survey will be printed and the above process will be applied to it. The email will be deleted afterwards. All the files will be held in a locked storage area.

Details of interview process: One aspect of this study involves making audio recording and video recording of you during the interview. We are making these recordings because they can convey your understandings of linear algebra concepts more vividly than the written descriptions. Student researcher Gulden Karakok will transcribe the audio recordings and these transcriptions will be kept for indefinitely but tapes will be destroyed two years after the interviews have been conducted. Student researcher Gulden Karakok will always have the access to the transcripts and tapes until they are destroyed. However, the student researcher (Gulden Karakok) and the principal investigator (Dr. Barbara Edwards) will meet to discuss the analysis of the data. It is possible that the principal investigator will get to see some or all of the transcripts and video tapes during these meetings. The tapes of the interviews will be archived (in a locked file cabinet in student researcher’s office) for two more years after the interviews to make sure all the aspects of students’ understanding are documented. All the tapes will be destroyed two years after they’ve been conducted. The transcripts will be kept in student researcher’s computer in a folder with a password which will be known by the student researcher only.

It is possible that we may want to include video clips from the interview in our presentations at conferences and your quotes in student researcher’s thesis and in other publications. Your name and information will not be identified in any place. If you do not wish the researchers to use data that include you image, utterances or writing, please let the researcher know by checking the appropriate part at the end of this document.

If the results of this project are published your identity will not be made public.

DO I HAVE A CHOICE TO BE IN THE STUDY?

If you decide to take part in the study, it should be because you really want to volunteer. You will not lose any benefits or rights you would normally have if you choose not to volunteer. You can stop at any time during the study and still keep the benefits and rights you had before volunteering. Your decision to participate or not to participate will not affect your grade in this course.

You will not be treated differently if you decide to stop taking part in the study. If you volunteer for the survey and the interview, you are free to skip any questions you prefer not to answer. If you choose to withdraw from this project before it ends, the researchers may keep information collected about you and this information may be included in study reports.

WHAT IF I HAVE QUESTIONS?

If you have any questions about this research project, please contact: **Barbara Edwards; 541-737-5179;** edwards@math.oregonstate.edu and/or **Gulden Karakok, 541-908-6604, office weniger 332-H; gkarakok@science.oregonstate.edu**

If you have questions about your rights as a participant, please contact the Oregon State University Institutional Review Board (IRB) Human Protections Administrator, at (541) 737-4933 or by email at IRB@oregonstate.edu.

There is a chance you may be contacted in the future to participate in an additional study related to this project, which will require the researchers to retain your contact information after this study has been completed. If you would prefer not to be contacted, please let the researchers know, at any time. **If you are contacted, you can choose whether or not to participate.**

Your signature indicates that this research study has been explained to you, that your questions have been answered, and that you are older than 18 years old and you agree to take part in this study. You will receive a copy of this form.

Name (printed): ____

(Signature of Participant) (Date)

Please fill out the best way to contact you:

(Email Address)

(Mailing Address)

Please Check All that Applies:

☐**Yes, I agree to participate in the survey only.**
☐**Yes, I agree to participate in both survey and all the interviews.**
☐**Not sure about the interviews, ask me later during the term.**
☐ **I do not want to participate in this study.**

Also, check the appropriate boxes in 1 and 2 below.

1.

	Yes, I agree that data including my image utterances, or writing may be included in publications and/or presentations.
	No, I do not agree that data including my image, utterances, or writing may be included in publications and/or presentation.

2.

	Yes, I am willing to be contacted for future studies.
	No, please do not contact me for future studies.

Student Researcher's contact information:
Gulden Karakok (Office- Weniger 332-H; gkarakok@science.oregonstate.edu)

Mailing Address:

Science and Math Education Department,
OSU, Weniger 239,
Corvallis, OR, 97331

Phone: 908 6604 or 737-1824 *Fax*: 737-1817

APPENDIX C

BACKGROUND SURVEY

Name:

Email address(es):

Mailing Address:

Which is the best way to contact you- email or mail?

1. Please tell us about your planned undergraduate degree:

Major: _________________________

Minor:__________________________

Circle the most suitable one for Fall Term 2007: Freshman Sophomore Junior Senior

The expected graduation term and year: ___________________________

2. Please circle all the courses you've taken from the list below (please include the term you've taken it, for example, write Winter 2006 next to the circled course):

Math 251 Calculus 1
Math 252 Calculus 2
Math 253 Sequences and Series
Math 254 Vector Calculus 1
Math 255 Vector Calculus 2
Math 256 Differential Equations
Math 306 Matrix and Power Series Methods
Math 311 Introduction to Real Analysis
Math 341 Linear algebra 1
Math 342 Linear algebra 2

3. Please list the math courses you've taken that are not on the list above- including this term.

4. Will you take all the physics courses in Winter Term, 2008?

6. Do you plan to take any other math courses next term? If so, please list them.

APPENDIX D

PRE and POST QUIZZES

Pre Quiz:

1. Do the following matrix multiplication:

$$\begin{pmatrix} 2 & 1 & 0 \\ -7 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

2. Find the determinant of the following matrix.

$$\begin{pmatrix} 3 & 2 & 0 \\ 4 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3. Find the eigenvalues and eigenvectors of the following matrix.

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

4. Briefly answer the questions on the back of this page:

What is an eigenvalue?

What is an eigenvector?

How do we use eigenvectors in physics?

Post Quiz:

1. Find the eigenvalues and eigenvectors of the following matrix. You do **not** need to normalize the eigenvectors.

$$\begin{bmatrix} 2i & 3 \\ 0 & -7 \end{bmatrix}$$

2. Briefly answer the questions on the back of this page:

What is an eigenvalue?

What is an eigenvector?

How do we use eigenvectors in physics?

APPENDIX E

LINEAR ALGEBRA WEEK ACTIVITIES

Linear Transformations

1) Using colored pencils, draw the initial vectors below on a single graph on the top half of the graph paper provided.

$$|\text{red}\rangle = 1|\hat{\imath}\rangle + 0|\hat{\jmath}\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\text{green}\rangle = 0|\hat{\imath}\rangle + 1|\hat{\jmath}\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad |\text{blue}\rangle = 1|\hat{\imath}\rangle + 1|\hat{\jmath}\rangle \doteq \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|\text{black}\rangle = 1|\hat{\imath}\rangle - 1|\hat{\jmath}\rangle \doteq \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad |\text{purple}\rangle = 1|\hat{\imath}\rangle + 3|\hat{\jmath}\rangle \doteq \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

2) Each group will be assigned one of the following matrices. Operate on the initial vectors with your group's matrix and graph the transformed vectors on a single graph on the bottom half of the graph paper provided.

$$A_1 \doteq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad A_2 \doteq \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad A_3 \doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad A_4 \doteq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad A_5 \doteq \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A_6 \doteq \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \quad A_7 \doteq \begin{pmatrix} 1 & 2 \\ 9 & 4 \end{pmatrix} \quad A_8 \doteq \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad A_9 \doteq \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad A_{10} \doteq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

3) Find the determinant of your matrix.

4) Make note of any differences between the initial and transformed vectors. Specifically, look for rotations, inversions, length changes, anything that is different. Are there any vectors which are left unchanged by your transformation? Your group should be prepared to report to the class about your transformation.

5) When your group is done, put a sketch of your transformed vectors on the chalkboard. State what your matrix does, give the determinant of your matrix, and mention any unchanged vectors.

by Jason Janesky & Corinne Manogue

Eigenvalues and Eigenvectors

1) Each group will be assigned one of the following matrices. Find the eigenvalues and (unnormalized) eigenvectors of your matrix. When you are finished, write your solutions on the board.

$$A_1 \doteq \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad A_2 \doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad A_3 \doteq \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad A_4 \doteq \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

$$A_5 \doteq \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad A_6 \doteq \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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APPENDIX F

INTERVIEW QUESTIONS

Interview 1

- Greeting: "Welcome and thanks for coming. My name is Gulden and I'm a PhD student."
- Show the signed the Informed Consent forms for the interviewee and explain the research and conditions from the consent form.
- Ask about the math courses they've taken and courses they are taking now and will take in the future. And then ask if they have any questions before we start.

Explain the interviewee the protocol: "Please try to say as much as you can while solving the problems. I might ask you questions just for clarification, it doesn't mean you are doing something wrong."

Give the first problem and ask them to read it out loud and again ask him/her to verbalize their problem solving process as much as possible, anything they think of about this problem.

Problems:

1. What is an eigenvalue?

(Depending on their answer, if they mention eigenvector then ask what that is too. When they are done and if they don't mention eigenvector, ask the following)

2. What is an eigenvectors?

Some students may not know any of these questions. If this is the case, refer to the optional interview 1 questions

3. Could you please give me examples of an eigenvalue and an eigenvector and explain your example?

(Ask why they choose this example, was there a particular reason, and what they represent. Also, if not mentioned ask if there is a geometric interpretation of the example)

4. Let's T be a linear transformation in R^2 that reflects vectors across x-axis. What are eigenvalues and the eigenvectors? How can we find them? What do they represent?

(Probe more: Some students might want to write the matrix representation of this linear transformation. How is this related to your explanation above? How do you know? Are you sure? How can we be sure? Does this problem look familiar to you? How you done a similar problem before? When and where?)

Optional Interview 1 Questions:

1. What is a vector? Could you please give some examples?
2. How can we multiply two vectors? Could you please give some examples?
3. What is a matrix? What does it represent? Could you give me some examples?
4. How does matrix multiplication work?
5. Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. What does this matrix represent? Do you know any geometric description of matrix?

(Please keep asking these questions when possible: Do any of these questions seem familiar to you? Have you done anything similar before? When and where?)

<u>*Interview 2*</u>
Use the same protocol from Interview 1, make sure that they see their signed copy of consent form and emphasize that when you ask a question it is for clarification purposes.

{Start with asking about their break and last term so that they will be warmed up. Then explain them the interview protocol- please talk as much as you can, no one other than me and/or my advisor will see any of the data that reveals your name, show them their IRB, you can use all you want and calculator is here, I might ask you more questions as you speak it doesn't mean that you are doing or saying something wrong}

1. Please think about last week's paradigm course, all in-class activities, lectures, pre-quiz, the other two quizzes and the both assignments. Could you please tell me what the most interesting thing was that you've learned from last week's class that you didn't know before? {prompt more, from which aspect of class-assignment, in class activity, discussion from friends. Give me examples.}

2. {If they don't say anything about eigenvalues} What is an eigenvalue? What is an eigenvector?

3. {For students who I need them to watch their first interview episode}Did anything from this course make you change what you've been thinking about eigenvalues and eigenvectors? { Give them some time then proceed} Well, let's see, this is what you've told me about eigenvalues and eigenvectors {watch the clip} anything you want to change or add?

4. Let $A = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$
    a. What does A do?
    b. What are the eigenvalues and eigenvectors of this matrix?
    c. What do they represent?
    d. In relation to what it represents what do eigenvalues and eigenvectors represent?
    e. Any other characteristics of this transformation that you know?

5. Does any nonzero multiple of an eigenvector is again an eigenvector? Why or why not?

Interview 3

{Start with asking about their break and last term so that they will be warmed up. Then explain them the interview protocol- please talk as much as you can, no one other than me and/or my advisor will see any of the data that reveals your name, show them their signed consent form and explain, you can use all you want and calculator is here, I might ask you more questions as you speak it doesn't mean that you are doing or saying something wrong. You will receive $25. I had your permission to get copies of your exams and homework assignments before graded, I want to double check that you know that and you are OK with that. I might contact you again to ask clarifying questions but other than that thank you for your help!}

0. Which classes are you taking this term? Which class did you take last term?

1. Let A be a square matrix (operator) and let e_1 and e_2 be the eigenvectors of this matrix, what can you tell me about the sum $e_1 + e_2$? In particular is it an eigenvector of this operator? Under which conditions it will be?

2. Let $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ be an eigenvector that associates with the eigenvalue of 1 and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ with the eigenvalue of -1 of an operator M. Tell me everything you can about this operator.

3. Well, you've taken all three physics courses. Tell me everything you know about eigenvalues and eigenvectors. Pretend that I'm a new student who will take these courses and tell me what I should know. Tell me what will I know after taking all these courses. Give me a mini-lecture.

4. Then: I'm working on this problem. Can you please help me?

   Give the general solution to the differential equation

$$i\frac{d}{d\varphi}f(\varphi)-af(\varphi)=0$$

   subject to the condition $f(\varphi)=f(\varphi+2\pi)$

   (Follow up questions: what does it mean to give a general solution? Does this remind you of anything you've done in physics courses?)