AN ABSTRACT OF THE THESIS OF

Corina D. Constantinescu for the degree of Master of Science in Mathematics presented on June 11, 2003.

Title: Ruin Theory under Uncertain Investments

Redacted for Privacy

Abstract approved: ______________________

Enrique Thomann

An insurance company, having an initial capital u, receives premiums continuously and pays claims of random sizes at random times. A classical result states that if the rate of premium, c, exceeds the average of the claims paid per unit time, $\lambda\mu$, then the ruin probability decays exponentially fast as $u \to \infty$. However, if the insurance company invests in a risky asset whose price follows a geometric Brownian motion with drift a and volatility $\sigma > 0$, it is known that the probability of ruin decays at best algebraically, under a specific model for claim size distribution. In this thesis, the result is shown to be valid for claim size distributions having moment generating functions defined in a neighborhood of the origin.

Ruin Theory under Uncertain Investments

by

Corina D. Constantinescu

A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Master of Science

Completed June 11, 2003
Commencement June 2004

Master of Science thesis of Corina D. Constantinescu presented on June 11, 2003

APPROVED:

Redacted for Privacy

Major Professor, representing Mathematics

Redacted for Privacy

Chair of the Department of Mathematics

Redacted for Privacy

Dean of the Graduate School

I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

Redacted for Privacy

Corina D. Constantinescu, Author

ACKNOWLEDGEMENT

I would like to thank my advisor, Professor Enrique Thomann, without whose careful attention and encouragement I could not possibly have succeeded. Special thanks also to Professors Edward Waymire and Mina Ossiander for their helpful comments at all times during the preparation of this research. I would like to acknowledge my friend Paul for his technical support and my friend Sam for the proofreading. Finally, I would like to thank the "Eastern Block", Diana, Andra, Angi, Cami, Kinga, and all my friends in the department for their continual support.

TABLE OF CONTENTS

RUIN THEORY UNDER UNCERTAIN INVESTMENTS

1. INTRODUCTION

This is a paper about money. The solvency of an insurance company is analyzed. Namely, the asymptotic behavior of the ruin probability is studied under different claim size distribution scenarios, when the capital of the company is invested into a risky asset whose price follows a geometric Brownian motion.

The idea of this research is the following: if an insurance company, with an initial capital u, simply cashes premiums and pays claims, then the probability of attaining ruin decays to zero exponentially fast, as $u \to \infty$, provided certain conditions on the distribution of the claim sizes are met[7]. However, if the company invests in a risky asset whose price follows a geometric Brownian motion, then the probability of ruin either decays algebraically fast as $u \to \infty$ or the ruin is certain for all $u > 0$ [9]. This result was shown by Frolova et. al [9] in the very special case of exponentially distributed claim sizes. A generalization of the latter result for the case of claim size distributions with moment generating functions defined on a neighborhood of the origin is presented in this paper.

The method of deriving the asymptotic behavior of the ruin probability, $\Psi(u)$, introduced here consists in applying asymptotic methods for differential equations to the infinitesimal generator equation, $A\Psi(u) = 0$ [12].

Section 2 contains concepts and preliminary results useful in the course of developing our method. The infinitesimal generator is defined and then derived for diffusions and jump processes. As examples, the infinitesimal generators are calculated for both the classical Cramer Lundberg model, $X_t = u + ct - \sum_{k=1}^{N(t)} \xi_k$, and for a model combining this jump process and a geometric Brownian motion. Also in this section, basic notions of regular variation in Karamata's sense are introduced and the Karamata Tauberian Theorem and the Monotone Density Theorem are stated. These results are crucial for the derivation of the asymptotic behavior.

Section 3 presents an application of the method to classical Cramer Lundberg model. The well-known result of the exponentially distributed claim sizes case is re-derived. The behavior of the probability of ruin is determined as the initial capital u tends to infinity. The probability of ruin presents an exponential decay, $\Psi(u) = \Psi(0)\exp(-\frac{c-\lambda\mu}{\mu c}u)$, with the exponent depending on the average claim size, μ, average time of occurrence, λ, and premium rate, c [7].

Section 4 contains a generalization of Frolova et al.'s result [9] regarding the behavior of the probability of ruin when the insurance company invests in an asset with the price modelled by a geometric Brownian motion. In this Cramer-Lundberg with investments type model, the risk process is given by $X_t = u + a\int_0^t X_s\,ds + \sigma\int_0^t X_s\,dW_s + ct - \sum_{k=0}^{N(t)} \xi_k$. In Frolova et al. [9], only the exponential claim size distribution is consider. In this case, the decay is at best algebraic and depends only on the drift and volatility of the price of the investment. In this section is presented an extension of this decay structure under very general conditions on the claim size distribution. Specifically, if the claim size distribution has a moment generating function defined in a neighborhood of the origin, then applying Karamata's Tauberian theorem, the same result is obtained.

2. PRELIMINARIES

In this section the notions of infinitesimal generator and regular variation are introduced. The infinitesimal generator is derived in some particular cases.

2.1. Concepts

Definition 2.1..1. *A probability space is a triple $(\Omega, \mathcal{F}, P)$, where Ω is a nonempty set of all possible "outcomes" of an experiment, $\mathcal{F}$ is a set of "events", and P: $\mathcal{F} \to [0,1]$ is a function that assigns probabilities to events. $\mathcal{F}$ is a σ-field, meaning a (nonempty) collection of subsets of Ω that satisfies*

(i) if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$, and

(ii) if $A_i \in \mathcal{F}$ is a countable sequence of sets then $\bigcup_i A_i \in \mathcal{F}$.

Since $\bigcap_i A_i = (\bigcup_i A_i^c)^c$, it follows that a σ-field is closed under countable intersections.

Definition 2.1..2. *$(\Omega, \mathcal{F})$ is called a* ***measurable space****, a space on which one can define a measure. A measure is a nonnegative, countably additive set function $\mu : \mathcal{F} \to \boldsymbol{R}$ with the following properties:*

(i) $\mu(A) \geq \mu(\emptyset) = 0$ for all $A \in \mathcal{F}$, and

(ii) if $A_i \in \mathcal{F}$ is a countable sequence of disjoint sets then $\mu(\bigcup_i A_i) = \Sigma_i \mu(A_i)$.

Definition 2.1..3. *If $\mu(\Omega) = 1$, then μ is called a* ***probability measure****.*

Definition 2.1..4. *Given a set Ω and a collection A of subsets of Ω then there is a smallest σ-field containing A, called* ***the σ-field generated by A*** *and denoted by $\sigma(A)$.*

Let $\mathbf{R}^n = \{(x_1, \ldots, x_n) : x_i \in \mathbf{R}\}$. R^n = the Borel subsets of $\mathbf{R}^n$ is defined to be the σ-field generated by the open subsets of $\mathbf{R}^n$.

Definition 2.1..5. *A real valued function X defined on Ω is said to be a* ***random variable*** *provided that events of the form $\{X \in I\} := \{\omega \in \Omega : X(\omega) \in I\}$ are in $\mathcal{F}$ (i.e. measurable), for all intervals I.*

X is a measurable function on Ω with respect to $\mathcal{F}$.

Definition 2.1..6. *Given an index set I, a* ***stochastic process*** *indexed by I is a collection of random variables $\{X_\lambda : \lambda \in I\}$ on a probability space $(\Omega, \mathcal{F}, P)$ taking values in a set S. The set S is called the state space of the process.*

Definition 2.1..7. *A stochastic process $X_0, X_1, \ldots, X_n, \ldots$ has the Markov property if, for each n and m, the conditional distribution of $X_{n+1}, \ldots, X_{n+m}$ given $X_0, X_1, \ldots, X_n$ is the same as its conditional distribution given X_n alone. A process having the Markov property is called a* ***Markov process*** *[1].*

If, in addition, the state space of the process is countable, then the Markov process is called a **Markov chain** [1].

Definition 2.1..8. *$p(t; x, dy)$ is a transition probability on $(S,\mathcal{S})$, for $t > 0$, $x \in S$ if:*

(i) $\forall t > 0$, $x \in S$, $p(t; x, dy)$ is a probability measure on $(S, \mathcal{S})$;

(ii) $\forall t > 0$, $B \in \mathcal{S}$ the function $x \mapsto p(t; x, B)$ is Borel measurable;

(iii)(Chapman-Kolmogorov Equation) $\forall t > 0$, $s > 0$, $x \in S$, $B \in \mathcal{S}$,

$$p(t + s; x, B) = \int p(s; y, B)p(t; x, dy). \tag{2.1}$$

Definition 2.1..9. *Define the transition operator T_t by*

$$T_t f(x) = \int f(y)p(t; x, dy) = \mathbf{E}[f(X_t) \mid X_0 = x] \tag{2.2}$$

for all f for which the integral is finite, e.g., for the space $B(S)$ of all bounded measurable functions, or on the space $C_b(S)$ of all bounded continuous functions on S.

Chapman-Kolmogorov equation (2.1) becomes the semigroup property:

$$\begin{aligned} T_{t+s}f(x) &= \int (T_s f)(y)p(t;x,dy) \\ &= \int f(z)p(t;y,dz)p(t;x,dy) \\ &= T_t T_s f(x). \end{aligned}$$

Definition 2.1..10. *The infinitesimal generator of* $\{T_t, t > 0\}$*, or of the Markov process* X_t*, is the linear operator* $\mathbf{A}$ *defined by:*

$$\mathbf{A}g(x) = \lim_{h \to 0} \frac{T_h g(x) - g(x)}{h}$$

for all real-valued, bounded, Borel measurable functions g *defined on* S*,* $g \in \mathbf{B}(S)$ *such that the right side converges to some function uniformly in* x*. The class of all such functions* g *comprises the domain* $\mathcal{D}_A$ *of* $\mathbf{A}$*.*

2.2. Cramer-Lundberg model

Consider an insurance company ABC, that disposes of an initial capital u, receives premiums continuously at a given fixed rate c and pays claims ξ_k of random sizes occurring at random times. The company stays solvent as long as the initial capital together with the incoming premiums exceeds the claims to be paid.

Definition 2.2..1. *Cramer-Lundberg model is a risk process* X *defined as [7]:*

$$X(t) = u + ct - S(t), \quad t \geq 0$$

where

$$S(t) = \xi_1 + \xi_2 + \ldots + \xi_{N(t)},$$

$N(t)$ *represents the number of claims in the interval* $[0, t]$*,*

$$N(t) = \sup_{n \geq 1}\{t_n \leq t\}, \quad t \geq 0$$

and $(\xi_k)_{k\in N}$ *are the claim sizes, positive, independent, identically distributed random variables with finite mean,* μ, *and finite variance. Claims occur at random instants of time* $0 < t_1 < t_2 < \dots$ *a.s. and the inter-arrival times*

$$Y_1 = t_1, \quad Y_k = t_k - t_{k-1}, \quad k = 2, 3, \dots$$

are independent, exponentially distributed random variables with finite mean λ. *The sequences* (ξ_k) *and* (Y_k) *are independent of each other.*

Given that the inter-arrival times are exponentially distributed, $N(t)$ is a homogeneous Poisson process with intensity $\lambda t > 0$, i.e.

$$\mathbf{P}(N(t) = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}, \quad k = 0, 1, 2, \dots$$

2.3. Infinitesimal generator for the classical Lundberg model

Considering the classical Cramer-Lundberg model:

$$X(t) = u + ct - \sum_{k=1}^{N(t)} \xi_k,$$

where $N(t)$ is a Poisson process with parameter λt, then $X(t)$ is a Markov process since it has stationary and independent increments. The state space S is $\mathbf{R}$. Consider that the transition probability distribution $\mathbf{P}(h; u, dy)$ has a density $p(h; u, y)$. Then for Borel subsets B of S, one define the transition probability $\mathbf{P}(h; u, B)$ of $X(t)$ as

$$\begin{aligned}\mathbf{P}(h, u, B) &= \int_B p(h; u, y)dy \\ &= \sum_{k=0}^{\infty} \frac{(\lambda h)^k}{k!} e^{-\lambda h} \mathbf{1}(u + ch - \sum_{j=1}^{k} \xi_j \in B).\end{aligned}$$

Proposition 2.3..1. *In the case of a Markov process* $X(t)$, *with*

$$X(t) = u + ct - \sum_{k=1}^{N(t)} \xi_k,$$

the infinitesimal generator of the process is given by

$$\mathbf{A}g(u) = cg'(u) + \lambda \int_0^\infty (g(u-y) - g(u))dF(y)$$

where g bounded, real-valued, differentiable function and F is the distribution of the claim sizes ξ.

Proof. Consider the semigroup associated to $X(t)$ given by

$$\begin{aligned} T_h g(u) &= \sum_{k=0}^{\infty} g(u + ch - S(h))\mathbf{P}(h, u, B) \\ &= \sum_{k=0}^{\infty} \frac{(\lambda h)^k}{k!} e^{-\lambda h} E(g(u + ch - S(h))|N(h) = k) \end{aligned}$$

for $x \in \mathbf{R}$ and $g \in M_b(\mathbf{R})$. Expanding the sum after k

$$T_h g(u) = e^{-\lambda h} g(u + ch) + \lambda h e^{-\lambda h} E(g(u + ch - \xi_1)|N(h) = 1) + o(h), \tag{2.3}$$

where $\lim_{h\to\infty} \frac{o(h)}{h} = 0$, then

$$\frac{T_h g(u) - g(u)}{h} = \frac{e^{-\lambda h} g(u + ch) - g(u)}{h} + \lambda e^{-\lambda h} E(g(u + ch - \xi_1)|N(h) = 1) + \frac{o(h)}{h} \tag{2.4}$$

has the limit

$$\lim_{h\to 0} \frac{T_h g(u) - g(u)}{h} = cg'(u) - \lambda g(u) + \lambda \int_0^\infty g(u - y)dF(y), \tag{2.5}$$

that can also be written as

$$\lim_{h\to 0} \frac{T_h g(u) - g(u)}{h} = cg'(u) + \lambda \int_0^\infty (g(u - y) - g(u))dF(y), \tag{2.6}$$

where F represent the distribution of the claim sizes ξ, $\int_0^\infty dF(y) = 1$. □

2.4. Diffusion process

Diffusion processes are an appropriate description of the stochastic properties of stock prices or interest rates. The time evolution of a diffusion process can be modelled by a stochastic differential equation of the form

$$dX(t) = a(t, X(t))dt + \sigma(t, X(t))dW_t, \tag{2.7}$$

where W_t represents the usual Brownian Motion [13].

Definition 2.4..1. *A Markov process $X(t), t \geq 0$ on the state space $S = \{(a,b), -\infty \leq a < b \leq \infty\}$ is said to be a diffusion with drift $a(t,x)$ and diffusion coefficient $\sigma^2(t,x) > 0$, if it has continuous sample paths, and the following relationships hold for all $\epsilon > 0$:*

$$E((X_{s+t} - X_s)1_{[|X_{t+s}-X_s|\leq\epsilon]}|X_s = x) = ta(t,x) + o(t)$$

$$E((X_{s+t} - X_s)^2 1_{[|X_{t+s}-X_s|\leq\epsilon]}|X_s = x) = t\sigma^2(t,x) + o(t)$$

$$P((|X_{s+t} - X_s| > \epsilon)|X_s = x) = o(t)$$

as $t \to 0^+$, where $a(t,x)$ and $\sigma^2(t,x) > 0$ are continuous differentiable with bounded derivatives on S. Also, σ'' exists and is continuous, and $\sigma^2 > 0$ for all x [1].

The stochastic differential equation

$$dX(t) = a(t, X(t))dt + \sigma(t, X(t))dW_t, \tag{2.8}$$

is equivalent to the equation

$$X(t) = X(0) + \int_0^t a(s, X(s))ds + \int_0^t \sigma^2(s, X(s))dW_s, t \geq 0. \tag{2.9}$$

If the solution is unique, then the process $X(t)$ is called a diffusion process with infinitesimal drift function $a(t,x)$ and infinitesimal variance $\sigma^2(t,x)$ at (t,x), provided that $\sigma^2(t,x) > 0$ for all $t \geq 0$ and $x \in S$, where $S \subset \mathbf{R}$ is the state space of $X(t)$ [13].

Lemma 2.4..1 (Ito's lemma). *Let X_t be a process given by*

$$dX(t) = a(t, X(t))dt + \sigma(t, X(t))dW_t. \tag{2.10}$$

Let $g(t,x) \in C^2([0,\infty) \times \mathbf{R})$, i.e. g is twice continuously differentiable in $[0,\infty) \times \mathbf{R}$. Then $Y_t = g(t, X_t)$ is also a diffusion process and [11]

$$dY_t = (a(X_t)f'(X_t) + \frac{\sigma^2(X_t)}{2}f''(X_t))dt + \sigma(X_t)f'(X_t)dW_t. \tag{2.11}$$

Definition 2.4..2. *A Brownian motion with drift a and diffusion coefficient σ^2 is a stochastic process $X_t : t \geq 0$ having continuous sample paths and independent Gaussian increments. The increments $X_{t+s} - X_t$ have mean sa and variance $s\sigma^2$.*

Definition 2.4..3. *A Brownian motion with drift zero and diffusion coefficient of 1 is called standard Brownian motion.*

Definition 2.4..4. *Let $X_t = X_0 + t\mu + \sigma W_t$, $t \geq 0$ where W_t is a standard Brownian motion starting at zero and independent of X_0. Then the process*

$$Z_t = Z_0 e^{ta + \sigma W_t}$$

with $Z_0 = e^{X_0}$ is the geometric Brownian motion.

2.5. Infinitesimal generator for a diffusion process

Proposition 2.5..1. *Let $\{X_t\}$ be a diffusion process on $S = (a, b)$:*

$$dX(t) = a(t, X(t))dt + \sigma(t, X(t))dW_t, \tag{2.12}$$

Then, for all twice continuously differentiable g, vanishing outside a closed bounded subinterval of S, and belonging to $\mathcal{D}_A$, the infinitesimal generator is given by

$$\mathbf{A}g(x) = a(t, x)g'(x) + \frac{1}{2}\sigma^2(t, x)g''(x). \tag{2.13}$$

Remark 2.5..1. *The drift of $Y_t = g(X_t)$ is $a(X_t)g'(X_t) + \frac{\sigma^2(X_t)}{2}g''(X_t)$, so $\mathbf{A}g(X_t)$ can be interpreted as the drift term of the function g of the diffusion process (X_t). In the special case of $a(t, X(t)) = aX(t)$ and $\sigma(t, X(t)) = \sigma X(t)$ the infinitesimal generator is given by*

$$\mathbf{A}g(x) = axg'(x) + \frac{\sigma^2}{2}x^2 g''(x), \tag{2.14}$$

Proof. In order to prove this result, let $x \in S$ fix and $\delta > 0$. One can find an ϵ s.t. $\mid g''(x) - g''(y) \mid < \delta$ for all y with $\mid x - y \mid < \epsilon$. Writing

$$T_t g(x) = \mathbf{E}(g(X_t)\mathbf{1}_{|X_t - x| < \epsilon}|X_0 = x)) + \mathbf{E}(g(X_t)\mathbf{1}_{|X_t - x| \geq \epsilon}|X_0 = x)) \tag{2.15}$$

Taking a Taylor expansion on $g(X_t)$ around x and get

$$\mathbf{E}(g(X_t)\mathbf{1}_{|X_t-x|<\epsilon}|X_0=x)) =$$

$$= \mathbf{E}(g(x)+(X_t-x)g'(x)+\frac{(X_t-x)^2}{2}g''(x)+\frac{(X_t-x)^2}{2}(g''(\delta_t)-g''(x))\mathbf{1}_{|X_t-x|<\epsilon}|X_0=x)) =$$

$$= g(x)+ta(t,x)g'(x)+\frac{\sigma^2(t,x)}{2}g''(x)+o(t)$$

where δ_t is some number between X_t and x. The remainder term is less then $\frac{\delta\sigma^2(x)t}{2}+o(t)$. Getting

$$\limsup_{t\to 0} \mid \frac{T_t g(x)-g(x)}{t} - (a(t,x)g'(x)+\frac{\sigma^2(t,x)}{2}g''(x)) \mid <$$

$$\frac{\delta\sigma^2(t,x)}{2} + \limsup_{t\to 0}(\frac{M}{t}+\mathbf{E}(\mathbf{1}_{|X_t-x|\geq\epsilon}|X_o=x))$$

and since $\mathbf{E}(\mathbf{1}_{|X_t-x|\geq\epsilon}|X_o=x)) = \mathbf{P}(|X_t-x|\geq\epsilon|X_0=x)) = o(t)$ then

$$\limsup_{t\to 0} M\frac{\mathbf{P}(|X_t-x|\geq\epsilon|X_0=x))}{t} = \limsup_{t\to 0}\frac{o(t)}{t} = 0.$$

Given that δ is arbitrary, it follows that the limit is zero, implying

$$\lim_{t\to 0}\frac{T_t g(x)-g(x)}{t} = a(t,x)g'(x)+\frac{1}{2}\sigma^2(t,x)g''(x).$$

□

2.6. Infinitesimal generator for our model

Considering the process X_t, given by the equation,

$$X_t = u + a\int_0^t X_s\,ds + \sigma\int_0^t X_s\,dW_s + ct - \sum_{k=0}^{N(t)}\xi_k, \tag{2.16}$$

i.e. a sum of a diffusion and a process with jumps. One can find its infinitesimal generator as the sum of the corresponding infinitesimal generators of these two processes. Here ξ_k represents the size of the k-th claim, with a probability distribution F on $(0,\infty)$, c represents the fixed rate of premium, u is the initial capital, a is the drift and σ^2 is the volatility of a geometric Brownian motion. Using Ito's formula [11], one find the infinitesimal generator for X given by the sum of (2.14) and (2.6), as in [12]:

$$Ag(u) = aug'(u) + \frac{\sigma^2}{2}u^2 g''(u) + cg'(u) + \lambda \int_0^\infty (g(u-y) - g(u))\, dF(y). \tag{2.17}$$

A useful result for the analysis of the ruin probability behavior is Paulsen's theorem 2.1 from the paper [12]:

Theorem 2.6..1 (Paulsen). *If $\Psi(u)$ is a bounded and twice continuous differentiable function defined for $u \geq 0$ that solves $\mathbf{A}\Psi(u) = 0$ on $u > 0$ together with the boundary conditions:*

$$\Psi(u) = 1, \quad for\ u < 0$$

$$\lim_{u \to \infty} \Psi(u) = 0$$

then the solution is

$$\Psi(u) = \boldsymbol{P}(T_u < \infty).$$

Proof. This is a sketch of the proof. Let $T^b = \inf(t : X_t \geq b)$, $b > u$ and let Ψ_n be twice a function continuous and differentiable such that $\Psi_n = \Psi$ on $(-\infty, -\frac{1}{n}] \cup [0, \infty)$. The idea of the proof is the following: by Ito's lemma

$$\mathbf{E}(\Psi_n(X_{t \wedge T_u \wedge T^b})) = \Psi_n(u),$$

and then by bounded convergence theorem

$$\mathbf{E}(\Psi(X_{t \wedge T_u})) = \Psi(u),$$

as $b \to \infty$ and $n \to \infty$. But,

$$\begin{aligned}
\mathbf{E}(\Psi(X_{t \wedge T_u})) &= \mathbf{E}(\Psi(X_{t \wedge T_u})\mathbf{1}_{T_u \leq t}) + \mathbf{E}(\Psi(X_{t \wedge T_u})\mathbf{1}_{T_u > t}) \\
&= \mathbf{E}(\Psi(X_{T_u})\mathbf{1}_{T_u \leq t}) + \mathbf{E}(\Psi(X_t)\mathbf{1}_{T_u > t}) \\
&= \mathbf{P}(T_u \leq t) + \mathbf{E}(\Psi(X_t)\mathbf{1}_{T_u > t}).
\end{aligned}$$

Finally as $t \to \infty$, $X_t \to \infty$, and then

$$\Psi(u) = \mathbf{E}(\Psi(X_{t\wedge T_u})) = \mathbf{P}(T_u \leq \infty) + \mathbf{E}(\Psi(X_t)\mathbf{1}_{T_u > \infty}) = \mathbf{P}(T_u \leq \infty).$$

□

It follows from the theorem that the boundary conditions together with the boundedness assumptions are sufficient to determine Ψ uniquely, provided the solutions exist.

2.7. Regular variation

Definition 2.7..1. *Let l be a positive measurable function, defined in some neighborhood $[M, \infty)$ of infinity, and satisfying*

$$l(\lambda x)/l(x) \to 1, \quad \text{as} \quad x \to \infty, \quad \forall \lambda > 0,$$

then l is said to be slowly varying in Karamata's sense [3].

Definition 2.7..2. *If $U : \mathbf{R} \to \mathbf{R}$ has locally bounded variation, is right continuous, and vanishes on $(-\infty, 0)$, we define its Laplace-Stieltjes transform*

$$\hat{U}(s) := \int_{-\infty}^{\infty} e^{-sx} dU(x) = \int_{0}^{\infty} e^{-sx} dU(x)$$

where the integral converges absolutely for $s > \sigma$, where σ may be $+\infty$.

Remark. Define

$$U(u) = \begin{cases} 0 & \text{if} \quad u < 0 \\ \int_0^u \Psi(x)dx & \text{if} \quad u \geq 0. \end{cases}$$

Then the Laplace Stieltjes transform of $U(u)$, $\hat{U}(s)$ is equal to the Laplace transform of $\Psi(u)$, $\mathcal{L}(\Psi(u))(s)$, i.e.

$$\hat{U}(s) = \int_0^{\infty} e^{-su} dU(u) = \int_0^{\infty} e^{-su} \Psi(u) du = \mathcal{L}(\Psi(u))(s). \tag{2.18}$$

Theorem 2.7..1 (Karamata Tauberian Theorem[3]). *Let U be a non-decreasing right-continuous function on $\boldsymbol{R}$ with $U(x) = 0$ for all $x < 0$. If l varies slowly and $c \geq 0, \rho \geq 0$ the following are equivalent:*

$$U(x) \sim cx^{\rho}l(x)/\Gamma(1+\rho), \quad (x \to \infty),$$

$$\hat{U}(s) \sim cs^{-\rho}l(1/s), \quad (s \to 0_{+}).$$

In this theorem $\hat{U}$ denotes the Laplace-Stieltjes transform.

Definition 2.7..3. *A function f is ultimately monotone if there exists y such that for any $x > y$, $f(x)$ is monotone.*

Theorem 2.7..2 (Monotone Density Theorem [3]). *Let $U(x) = \int_0^x u(y)dy$. If*

$$U(x) \sim cx^{\rho}l(x), \quad x \to \infty,$$

where $c \in \mathbf{R}$, $l \in \mathbf{R}_0$, and if u is ultimately monotone, then

$$u(x) \sim c\rho x^{\rho-1}l(x), \quad x \to \infty.$$

3. RUIN PROBABILITY IN THE CLASSICAL C-L CASE

In this section, the method is demonstrated for the classical Cramer-Lundberg model. The ruin probability is derived in the special case of exponentially distributed claim sizes.

3.1. The model

As presented before, a Cramer- Lundberg model is described by a risk process[7]:

$$X(t) = u + ct - S(t), \quad t \geq 0$$

where

$$S(t) = \xi_1 + \xi_2 + \ldots + \xi_{N(t)},$$

represents the total claim amount process, and $N(t)$ represents the number of claims in the interval $[0, t]$,

$$N(t) = \sup_{n \geq 1}\{t_n \leq t\}, \quad t \geq 0.$$

Here $(\xi_k)_{k \in N}$ are the claim sizes, positive, independent, identically distributed random variables with finite mean, μ, and finite variance. As a consequence of the independence of claim sizes and inter-arrival times, $S(t)$, defined as [4]

$$S(0) = 0, \quad as \quad N(0) = 0$$

and

$$S(t) = \sum_{i=1}^{N(t)} \xi_i, \quad for \quad N(t) > 0$$

has the mean $\mathbf{E}(\sum_{j=1}^{N(t)} \xi_j) = (\lambda t)(\mu)$.

Definition 3.1..1. *The difference between the premium rate and the average cost of the claims represents the "safety loading", or the "risk premium rate". The net profit condition refers to a positive safety loading.*

Over a period of time $[0, t]$ the premium income is ct and the average cost of the claim is $\lambda\mu t$. The net profit condition says that the expected value of incoming premiums is greater than the expected value of the paid claims, i.e.

$$E(ct) > E(\sum_{j=1}^{N(t)} \xi_j)$$

$$ct > \lambda\mu t$$

$$c/\lambda\mu - 1 > 0.$$

Denoting $\rho = \frac{c}{\lambda\mu} - 1$ the net profit condition reduces to $\rho > 0$.

Definition 3.1..2. *The ruin probability in finite time is defined as*

$$\Psi(u, T) = P(X(t) < 0, \quad \text{for some} \quad t \leq T), \quad \text{where} \quad 0 < T < \infty, \quad u \geq 0 \tag{3.1}$$

and, in infinite time:

$$\Psi(u) = \Psi(u, \infty).$$

Definition 3.1..3. *Let $\bar{F}(x) = 1 - F(x)$ denotes the tail of the distribution. Then the integrated tail of the distribution is*

$$F_I(x) = \frac{1}{\mu} \int_0^x \bar{F}(y) dy.$$

Theorem 3.1..1 (Cramer-Lundberg Theorem). *Consider the Cramer-Lundberg model*

$$X_t = u + ct - \sum_{k=1}^{N(t)} \xi_k,$$

including the net profit condition $\rho > 0$. Assume that there exists a $\nu > 0$ such that

$$\int_0^\infty e^{\nu x} dF_I(x) = \frac{c}{\lambda\mu} = 1 + \rho$$

(i)Then for all $u \geq 0$

$$\Psi(u) \leq e^{-\nu u}.$$

(ii)If the size of the claim is exponentially distributed, $F(x) = 1 - e^{-x/\mu}$, the ruin probability has the form:

$$\psi(u) = \frac{1}{1+\rho} exp(-\frac{\rho}{\mu(1+\rho)} u), u \geq 0$$

Proof. i) The new method is introduced. Consider the ruin probability as a solution of the infinitesimal generator equation $\mathbf{A}\Psi(u) = 0$ [12]. Using the result (2.6) regarding jump processes, the infinitesimal generator can be derived as follows:

$$\mathbf{A}\Psi(u) = \lim_{h \to 0} \frac{T_h\Psi(u) - \Psi(u)}{h} = c\Psi'(u) + \lambda \int_0^\infty (\Psi(u-y) - \Psi(u))dF(y).$$

Accordingly, the equation is:

$$c\Psi'(u) + \lambda \int_0^\infty (\Psi(u-y) - \Psi(u))dF(y) = 0.$$

3.2. The method

The convolution type integral suggests the use of the Laplace transform, since the convolution transforms into a product. The solution of the Laplace transform of this equation is the Laplace transform of the desired ruin probability. In order to analyze the solutions of the equation

$$\mathbf{A}\Psi(u) = c\Psi'(u) + \lambda \int_0^\infty (\Psi(u-y) - \Psi(u))dF(y) = 0 \tag{3.2}$$

with the boundary conditions:

$$\lim_{u \to \infty} \Psi(u) = 0$$
$$\Psi(0) = \frac{\lambda\mu}{c}$$

the solutions of the Laplace transform of this equation are derived and analyzed. Let $\mathcal{L}\Psi(u)(s) = \hat{U}(s)$. Using the following properties of the Laplace transform

$$\mathcal{L}(\Psi'(u))(s) = s\hat{U}(s) - \Psi(0)$$

$$\mathcal{L}\int_0^u \Psi(u-y)dF(y)(s) = \hat{U}(s)\mathcal{F}(s)$$

and

$$\mathcal{L}(1-F(u))(s) = \frac{1}{s} - \frac{\mathcal{F}(s)}{s},$$

then the equation (3.2) is equivalent to:

$$s\hat{U}(s) - \Psi(0) = \frac{\lambda}{c}\hat{U}(s) - \frac{\lambda}{c}\hat{U}(s)\mathcal{F}(s) - \frac{\lambda}{c}(\frac{1}{s} - \frac{\mathcal{F}(s)}{s}).$$

Rearrangements of this equation, progressively lead to the following forms:

$$cs^2\hat{U}(s) - sc\Psi(0) = \lambda s\hat{U}(s) - \lambda s\hat{U}(s)\mathcal{F}(s) - \lambda + \lambda\mathcal{F}(s)$$

$$\hat{U}(s)(cs^2 - \lambda s + \lambda s\mathcal{F}(s)) = sc\Psi(0) - \lambda + \lambda\mathcal{F}(s),$$

where $\mathcal{F}(s) = \mathcal{L}(f(u))(s)$ will have the following power series expansion:

$$\mathcal{F}(s) = \int_0^\infty e^{-sx}f(x)dx = \mathbf{E}(e^{-sY}) = \mathbf{E}(\sum_{k=0}^\infty \frac{(-sY)^k}{k!}) = \sum_{k=0}^\infty \frac{(-s)^k}{k!}\mathbf{E}(Y^k).$$

Here Y is a random variable having the distribution F. Consequently the equation becomes:

$$\hat{U}(s)(cs^2 - \lambda s + \lambda s(1 - \mu s + s^2\mathcal{F}_2(s))) = sc\Psi(0) - \lambda + \lambda(1 - \mu s + s^2\mathcal{F}_2(s)) \quad (3.3)$$

where $\mathcal{F}_2(0)$ is non-zero. Then $\hat{U}(s)$ will have the form:

$$\hat{U}(s) = \frac{c\Psi(0) - \lambda\mu + s\lambda\mathcal{F}_2(s)}{s(c + \lambda\mu + \lambda s\mathcal{F}_2(s))} = \frac{\lambda\mathcal{F}_2(s)}{c + \lambda\mu + \lambda s\mathcal{F}_2(s)} \quad (3.4)$$

since $\Psi(0) = \frac{\lambda\mu}{c}$.

Remark 3.2..1. *As $s \to 0$ the asymptotic behavior of $\hat{U}(s)$ is:*

$$\hat{U}(s) \sim \frac{k\lambda}{(c + \lambda\mu) + k\lambda s},$$

where k is a real constant.

It will be the subject of future research to show that under certain conditions (to be determined) regarding the distribution F of the claim sizes, the Laplace inverse of $\hat{U}(s)$, namely $\Psi(u)$, has an exponential decay as $u \to \infty$, i.e.

$$\Psi(u) \sim Ke^{-\frac{c+\lambda\mu}{k\lambda}}.$$

So $\Psi(u) \leq M$, for large n. □

3.3. Exponentially distributed claim sizes case

Proof. ii) Assume the distribution of the claim sizes is exponential with parameter μ. Then $\mathcal{F}(s) = \mathcal{L}(f(u))(s) = \frac{1}{s\mu+1}$. The Laplace transform of the equation (2.6) is:

$$\hat{U}(s)(cs^2 - \lambda s + \lambda s\frac{1}{s\mu+1}) = sc\Psi(0) - \lambda + \frac{1}{s\mu+1} \tag{3.5}$$

$$\hat{U}(s)[cs^3\mu + cs^2 - \lambda s^2\mu - \lambda s + \lambda s] = s^2\mu c\Psi(0) + sc\Psi(0) - \lambda\mu s - \lambda + \lambda \tag{3.6}$$

Since $\Psi(0) = \frac{\lambda\mu}{c}$, after simplification the solution of the Laplace equation looks like:

$$\hat{U}(s) = \frac{\Psi(0)}{s + \frac{c-\lambda\mu}{\mu c}}. \tag{3.7}$$

Consequently, $\Psi(u)$, the inverse of the Laplace transform is:

$$\Psi(u) = \Psi(0)\exp(-\frac{c-\lambda\mu}{\mu c}u) = \frac{1}{\rho+1}\exp(-\frac{\rho}{\mu(\rho+1)}u), \tag{3.8}$$

i.e. the well-known result of the exponentially distributed claim size distribution in the case of net profit condition. □

4. RUIN PROBABILITY UNDER UNCERTAIN INVESTMENTS

This section contains the analysis of the changes in the asymptotic behavior of the ruin probability if the insurance company invests in a risky asset whose price follows a geometric Brownian motion, with drift a and volatility σ. In contrast to the classical Cramer-Lundberg case with no investments, Section 3, where the ruin probability decays exponentially as $u \to \infty$, in Frolova et al.'s paper [9] it is shown that the ruin probability either decays algebraically or equals one, depending on the parameters a and σ of the asset only. In their paper the result is established only for exponentially distributed claim sizes. Their method of proof relays on the fact that the derivative of an exponential is an exponential. In this section, a generalization of the result for distributions of the claim sizes having moment generating functions defined on a neighborhood of the origin is presented.

4.1. The model

Let us consider a process X having the form

$$X_t = u + a\int_0^t X_s\, ds + \sigma \int_0^t X_s\, dW_s + ct - \sum_{k=0}^{N(t)} \xi_k, \tag{4.1}$$

where ξ_k represents the size of the k-th claim, with a probability distribution F on $(0, \infty)$, c is the fixed rate of premium and u is the initial capital. X_t describes the evolution of the capital of an insurance company which is continuously invested in an asset, with the price following a geometric Brownian motion. Relative price increments are $adt + \sigma dW_t$.

4.2. The method

The behavior of the probability of ruin is analyzed, using the following method. Recall that the ruin probability in infinite time is given by

$$\Psi(u) = \Psi(u,\infty) = \mathbf{P}(X(t) < 0, \quad \text{for some} \quad t < \infty) = \mathbf{P}(T_u < \infty).$$

Assume $\Psi(u)$ is a monotone decreasing function. In order to study the asymptotic behavior of the ruin probability as u approaches infinity, one analyzes the asymptotic behavior of the solution of the infinitesimal generator equation $A(\Psi(u)) = 0$, by means of the Laplace transform of this equation (Step 1). Define

$$U(u) = \begin{cases} 0 & \text{if} \quad u < 0 \\ \int_0^u \Psi(x)dx & \text{if} \quad u \geq 0. \end{cases}$$

and its Laplace Stieltjes transform, $\hat{U}(s)$. This is equal to the Laplace transform of $\Psi(u)$, $\mathcal{L}(\Psi(u))(s)$, i.e.

$$\hat{U}(s) = \int_0^\infty e^{-su} dU(u) = \int_0^\infty e^{-su}\Psi(u)du = \mathcal{L}(\Psi(u))(s).$$

To derive the asymptotic behavior of $\mathcal{L}(\Psi(u))(s) = \hat{U}(s)$, as $s \to \infty$, one needs to consider the non-homogeneous equation $\mathcal{L}(A(\Psi(u)))(s) = g(s)$ (Step 2). Since $U(u)$ is a non-decreasing, right-continuous function in $\mathbf{R}$ with $U(u) = 0$ for all $u < 0$, the Karamata Tauberian Theorem applies. Accordingly, the asymptotic behavior of $\hat{U}(s)$ as $s \to 0_+$, determines the asymptotic behavior of $U(u)$ as $u \to \infty$ (Step 3). The last step uses the Monotone Density Theorem (Step 4), relating the asymptotic behavior of $U(u)$ and $\Psi(u)$ as $u \to \infty$.

4.3. The result

Theorem 4.3..1. *Assume the moment generating function of the claim size distribution, F, is define on a neighborhood of the origin. Assume that $\sigma > 0$.*

If the ruin probability decays at infinity then $\rho = 2a/\sigma^2 > 1$.

If $1 < \rho < 2$ *then for some* $K > 0$,

$$\Psi(u) = Ku^{1-\rho}(1 + o(1)), \quad as \quad u \to \infty.$$

Remark 4.3..1. *It is conjectured that if* $\rho \le 1$, *then* $\Psi(u) = 1$ *for all* u.

Proof. In order to prove that $\Psi(u) = Ku^{1-\rho}(1 + o(1))$, i.e. that $\Psi(u)$ behaves asymptotically at ∞ as $u^{1-\rho}$, the function

$$U(u) = \begin{cases} 0 & \text{if} \quad u < 0 \\ \int_0^u \Psi(x)dx & \text{if} \quad u \ge 0. \end{cases}$$

is introduced and it is proved that the asymptotic behavior of the Laplace Stieltjes of U, $\hat{U}$, is given by $cs^{\rho-2}l(1/s)$ as $s \to 0$. The proof follows the method described above. Consider the process $X_t = u + a\int_0^t X_s\, ds + \sigma\int_0^t X_s\, dW_s + ct - \sum_{k=0}^{N(t)} \xi$ as a sum of a diffusion and a process with jumps. Note that the infinitesimal generator for $\Psi(u)$ has the form:

$$A\Psi(u) = \frac{\sigma^2}{2}u^2\Psi''(u) + (au + c)\Psi'(u) + \lambda\int_0^\infty (\Psi(u - y) - \Psi(u))\, dF(y). \tag{4.2}$$

Taking into account that $\Psi(u - y) = 1$ for any $u < y$ and that F is a distribution function, i.e. $\int_0^\infty dF(y) = 1$, one obtains:

$$A\Psi(u) = \frac{\sigma^2}{2}u^2\Psi''(u) + (au + c)\Psi'(u) + \lambda\int_0^\infty \Psi(u - y)\, dF(y) - \lambda\Psi(u).$$

$$A\Psi(u) = \frac{\sigma^2}{2}u^2\Psi''(u) + (au + c)\Psi'(u) - \lambda\Psi(u) + \lambda\int_0^u \Psi(u - y)dF(y) + \lambda\int_u^\infty dF(y)$$

Then, $A\Psi(u) = 0$ has the form

$$\frac{\sigma^2}{2}u^2\Psi''(u) + (au + c)\Psi'(u) - \lambda\Psi(u) + \lambda\int_0^u \Psi(u - y)\, dF(y) + \lambda(1 - F(u)) = 0 \tag{4.3}$$

Step 1, Laplace transform. Assume the distribution of the claims size has mean μ. The Laplace transform of the infinitesimal generator equation (4.3) is consequently:

$$\frac{\sigma^2}{2}\frac{d^2}{ds^2}[s^2\hat{U}(s)] + a(-\frac{d}{ds}[s\hat{U}(s)]) + cs\hat{U}(s) - \lambda\hat{U}(s) + \lambda\hat{U}(s)\mathcal{F}(s) + \frac{\lambda}{s}(1 - \mathcal{F}(s)) = c\Psi(0)$$

where $\hat{U}(s) = \mathcal{L}(\Psi(u)(s))$ denotes the Laplace transform of the $\Psi(u)$ and $\mathcal{F}(s) = \mathcal{L}(f(u))(s)$. The terms of the equation were obtained using the following elementary properties of Laplace transform:

$$\mathcal{L}(\Psi'(u)(s)) = s\hat{U}(s) - \Psi(0)$$

$$\mathcal{L}(u\Psi'(u)(s)) = -\frac{d}{ds}[s\hat{U}(s)]$$

$$\mathcal{L}(u^2\Phi''(u)(s)) = \frac{d^2}{ds}[s^2\hat{U}(s)]$$

$$\mathcal{L}(\int_0^u \Psi(u-y)f(y)\,dy) = \hat{U}(s)\mathcal{F}(s)$$

$$\mathcal{L}(1 - F_U(u)) = \frac{1}{s} - \frac{1}{s}\mathcal{L}(f(u))(s).$$

Since the distribution F of the claim sizes has the moment generating function defined on a neighborhood of the origin, then the Laplace transform of the density has the form:

$$\mathcal{L}(f(u)(s) = \mathbf{E}(e^{-sX}) = \sum_{k=0}^{\infty} \frac{(-s)^k}{k!}\mathbf{E}(X^k),$$

where X is a random variable having the density f. Performing calculations, the Laplace transform equation becomes:

$$\frac{\sigma^2}{2}\frac{d^2}{ds^2}[s^2\hat{U}(s)] + a(-\frac{d}{ds}[s\hat{U}(s)]) + (cs - \lambda + \lambda\mathcal{F}(s))\hat{U}(s) + \frac{\lambda}{s}(1 - \mathcal{F}(s)) = c\Psi(0)$$

Since

$$\frac{d}{ds}[s\hat{U}(s)] = \hat{U}(s) + s\hat{U}'(s)$$

$$\frac{d}{ds}[s^2\hat{U}(s)] = 2s\hat{U}(s) + s^2\hat{U}'(s)$$

$$\frac{d^2}{ds^2}[s^2\hat{U}(s)] = 2\hat{U}(s) + 4s\hat{U}'(s) + s^2\hat{U}''(s)$$

then the equation can be rewritten as:

$$\frac{\sigma^2 s^2}{2}\hat{U}''(s) + (2s\sigma^2 - as)\hat{U}'(s) + (cs - \lambda + \lambda\mathcal{F}(s) + \sigma^2 - a)\hat{U}(s) = c\Psi(0) - \frac{\lambda}{s}(1 - \mathcal{F}(s)) \quad (4.4)$$

Step 2, Asymptotic behavior of $\hat{U}$ At this step the asymptotic behavior of the solutions of the equation (4.4) is analyzed. Starting with the solutions for the homogeneous equation, $\hat{y}_1(s) = s^{-1}\gamma_1(s)$ and $\hat{y}_2(s) = s^{-2+\frac{2a}{\sigma^2}}\gamma_2(s)$ it is shown that the non-homogeneous equation has an additional solution $\hat{y}_3(s) = \gamma_3(s)$, where γ_1, γ_2, γ_3 are holomorphic in the disk $\mid s \mid < \frac{1}{\mu}$ with $\gamma_1(0) = \gamma_2(0) = \gamma_3(0) = 1$. Dividing (4.4) by $\frac{\sigma^2}{2}$ and denoting $\hat{U}(s) = y$ the following form is obtained:

$$s^2y'' + \frac{2(2\sigma^2 - a)}{\sigma^2}sy' + \frac{2(cs - \lambda + \sigma^2 - a + \lambda\mathcal{F}(s))}{\sigma^2}y = \frac{2c\Psi(0)}{\sigma^2} - \frac{2\lambda}{\sigma^2 s}(1 - \mathcal{F}(s)).$$

This is equivalent to:

$$s^2y'' + p(s)sy' + q(s)y = g(s), \tag{4.5}$$

where

$$p(s) = p_0 = \frac{2(2\sigma^2 - a)}{\sigma^2}$$

$$q(s) = q_0 + q_1(s) = \frac{2(\sigma^2 - a)}{\sigma^2} + q_1(s)$$

$$g(s) = g_0 + g_1(s) = \frac{2(c\Psi(0) - \lambda\mu)}{\sigma^2} + g_1(s)$$

$$\mathcal{F}(s) = 1 + \mu s + \mathcal{F}_2(s)$$

with q_1, g_1, $\mathcal{F}_2$ holomorphic in the disc $\mid s \mid < 1/\mu$ and $q_1(0) = g_1(0) = \frac{\mathcal{F}_2(s)}{s} \mid_{s=0} = 0$. Having $s = 0$ as a regular singular point of the homogeneous equation

$$s^2y'' + p(s)sy' + q(s)y = 0 \tag{4.6}$$

the solution has the form:

$$\hat{y}(s) = s^\rho \sum_{k=0}^{\infty} c_k s^k$$

i.e.

$$\hat{y}(s) = \sum_{k=0}^{\infty} c_k s^{\rho+k} \tag{4.7}$$

with $c_0 = 1$ [8]. Substituting this into (4.6), a recurrence system of equations is obtained:

$$c_0 f(\rho) = 0$$

$$c_1 f(\rho + 1) + c_0 f_1(\rho) = 0$$

$$\ldots$$

$$c_k f(\rho + k) + c_{k-1} f_1(\rho + k - 1) + \cdots + c_0 f_k(\rho) = 0$$

where $f(\rho) = \rho(\rho - 1) + p_0\rho + q_0$ and $f_k(\rho) = \rho p_k + q_k$. Since p and q are holomorphic functions and

$$p(s) = \frac{2(\sigma^2 - a)}{\sigma^2} = \sum_{k=0}^{\infty} p_k s^k$$

$$q(s) = \frac{2}{\sigma^2}(\sigma^2 - a + cs - \lambda + \lambda(1 + \mu s + \hat{F}_2(s)) = \sum_{k=0}^{\infty} q_k s^k$$

then the specific form of p_0 and q_0 are $p_0 = \frac{2(2\sigma^2 - a)}{\sigma^2}$ and respectively $q_0 = \frac{2}{\sigma^2}(\sigma^2 - a - \lambda + \lambda) = \frac{2}{\sigma^2}(\sigma^2 - a)$. Given that $c_0 = 1$ and $f(\rho) = \rho(\rho - 1) + p_0\rho + q_0$ [8], the equation $c_0 f(\rho) = 0$ becomes

$$\rho(\rho - 1) + p_0\rho + q_0 = 0$$

and can be solved for ρ, considering p_0 and q_0 calculated above:

$$\rho(\rho - 1) + \frac{2(2\sigma^2 - a)\rho}{\sigma^2} + \frac{2(\sigma^2 - a)}{\sigma^2} = 0$$

$$\rho^2 + (\frac{4\sigma^2 - 2a}{\sigma^2} - 1)\rho + \frac{2\sigma^2 - 2a}{\sigma^2} = 0$$

i.e.

$$\rho^2 + \frac{3\sigma^2 - 2a}{\sigma^2}\rho + \frac{2\sigma^2 - 2a}{\sigma^2} = 0.$$

The discriminant of this equation is:

$$d = (\frac{3\sigma^2 - 2a}{\sigma^2})^2 - 4(\frac{2\sigma^2 - 2a}{\sigma^2}) =$$

$$= \frac{9\sigma^4 - 12\sigma^2 a + 4a^2 - 8\sigma^4 + 8\sigma^2 a}{\sigma^4} =$$

$$= \frac{\sigma^4 - 4\sigma^2 a + 4a^2}{\sigma^4} = \frac{(\sigma^2 - 2a)^2}{\sigma^4} > 0$$

for any σ and for any a, $\sigma \neq 0$. If $2\sigma^2 \neq a$, the equation has two distinct solutions:

$$\rho_{1,2} = \frac{1}{2}(-\frac{3\sigma^2 - 2a}{\sigma^2} \pm \frac{\sigma^2 - 2a}{\sigma^2})$$

namely:

$$\rho_1 = -1$$

and

$$\rho_2 = -2 + \frac{2a}{\sigma^2}.$$

The derivation of the solutions of the non-homogeneous equation uses the variation of parameters method. Consider the equation

$$s^2 y'' + \frac{2(2\sigma^2 - a)}{\sigma^2} s y' + \frac{2}{\sigma^2}(\sigma^2 - a + cs + \mu s + \mathcal{F}_2(s))y = g(s),$$

$$g(s) = \frac{2\Psi(0)c}{\sigma^2} + \frac{\lambda}{s}(\mu s + \mathcal{F}_2(s)).$$

The solutions of the homogeneous equation are

$$\hat{y}_1(s) = y_1(s)\gamma_1(s)$$

$$\hat{y}_2(s) = y_2(s)\gamma_2(s),$$

where

$$y_1(s) = s^{-1}$$

$$y_2(s) = s^{-2+\frac{2a}{\sigma^2}}$$

and $\gamma_1(0) = \gamma_2(0) = 1$. A particular solution of the non-homogeneous equation has the form

$$\hat{y}_p(s) = A_1(s)\hat{y}_1(s) + A_2(s)\hat{y}_2(s)$$

where A_1 and A_2 satisfy:

$$\begin{cases} A_1'(s)\hat{y}_1(s) + A_2'(s)\hat{y}_2(s) & = & 0 \\ A_1'(s)\hat{y}_1'(s) + A_2'(s)\hat{y}_2'(s) & = & \frac{g(s)}{s^2} \end{cases}$$

and $g(s) = \frac{2(\Phi(0)c - \lambda\mu)}{\sigma^2} - \frac{2\lambda\mathcal{F}_2(s)}{s}$. Solving the system of equations in A_1' and A_2' one gets

$$A_1'(s) = -\frac{g(s)}{s^2}\frac{y_2(s)\gamma_2(s)}{y_1(s)\gamma_1(s)[y_2(s)\gamma_2(s)]' - y_2(s)\gamma_2(s)[y_1(s)\gamma_1(s)]'}$$

$$A_2'(s) = \frac{g(s)}{s^2}\frac{y_1(s)\gamma_1(s)}{y_1(s)\gamma_1(s)[y_2(s)\gamma_2(s)]' - y_2(s)\gamma_2(s)[y_1(s)\gamma_1(s)]'}$$

where $\gamma_1(s) \neq 0 \neq \gamma_2(s)$. Reorganizing the expressions:

$$A_1' = \frac{g(s)}{s^2} \frac{1}{y_2(s)\gamma_2(s)(\frac{y_1(s)\gamma_1(s)}{y_2(s)\gamma_2(s)})'}$$

$$A_2' = \frac{g(s)}{s^2} \frac{1}{y_1(s)\gamma_1(s)(\frac{y_2(s)\gamma_2(s)}{y_1(s)\gamma_1(s)})'}$$

$$(\frac{y_1(s)\gamma_1(s)}{y_2(s)\gamma_2(s)})' = s^{-\frac{2a}{\sigma^2}}[(1-\frac{2a}{\sigma^2})(\frac{\gamma_1}{\gamma_2}) + s(\frac{\gamma_1}{\gamma_2})']$$

$$(\frac{y_2(s)\gamma_2(s)}{y_1(s)\gamma_1(s)})' = s^{-2+\frac{2a}{\sigma^2}}[(-1+\frac{2a}{\sigma^2})(\frac{\gamma_2}{\gamma_1}) + s(\frac{\gamma_2}{\gamma_1})']$$

Accordingly,

$$A_1'(s) = \frac{g(s)}{s^2} \frac{1}{s^{-2+\frac{2a}{\sigma^2}}\gamma_2(s)} \frac{1}{s^{-\frac{2a}{\sigma^2}}(1-\frac{2a}{\sigma^2})(\frac{\gamma_1}{\gamma_2}) + s(\frac{\gamma_1}{\gamma_2})'}$$

$$= \frac{g(s)}{(1-\frac{2a}{\sigma^2})\gamma_1(s) + s(\frac{\gamma_1(s)}{\gamma_2(s)})'} = \hat{\Gamma}_1(s),$$

with

$$\hat{\Gamma}_1(0) = \frac{g(0)}{1-\frac{2a}{\sigma^2}} = \frac{2(c\Psi(0)-\lambda\mu)}{\sigma^2(1-\frac{2a}{\sigma^2})} = \frac{2(c\Psi(0)-\lambda\mu)}{(\sigma^2-2a)}$$

and

$$A_2'(s) = \frac{g(s)}{s^2} \frac{1}{s^{-1}\gamma_1(s)} \frac{1}{s^{-2+\frac{2a}{\sigma^2}}(-1+\frac{2a}{\sigma^2})(\frac{\gamma_2}{\gamma_1}) + s(\frac{\gamma_2}{\gamma_1})'} =$$

$$= g(s)s^{1-\frac{2a}{\sigma^2}} \frac{1}{(-1+\frac{2a}{\sigma^2})\gamma_2(s) + s\gamma_1(s)(\frac{\gamma_2}{\gamma_1})'} = s^{1-\frac{2a}{\sigma^2}}\hat{\Gamma}_2(s)$$

with

$$\hat{\Gamma}_2(0) = \frac{g(0)}{-1+\frac{2a}{\sigma^2}} = \frac{2(c\Psi(0)-\lambda\mu)}{(2a-\sigma^2)}.$$

Integrating

$$A_1(s) = s\Gamma_1(s)$$

$$A_2(s) = s^{2-\frac{2a}{\sigma^2}}\Gamma_2(s)$$

where Γ_1 and Γ_2 are holomorphic in the disk $|s| < \frac{1}{\mu}$ functions, with $\Gamma_1(0) = \frac{c\Psi(0)}{2(\sigma^2-2a)}$ and $\Gamma_2(0) = \frac{c\Psi(0)}{2(2a-\sigma^2)(2-\frac{2a}{\sigma^2})}$.

Remark 4.3..2. *In order to integrate $s^{1-\frac{2a}{\sigma^2}}$, the exponent, $1-\frac{2a}{\sigma^2}$, should be greater than -1, i.e. $\frac{2a}{\sigma^2} < 2$.*

So,

$$y(s) = A_1(s)\hat{y}_1(s) + A_2(s)\hat{y}_2(s)$$

$$= s\Gamma_1(s)s^{-1}\gamma_1(s) + s^{2-\frac{2a}{\sigma^2}}\Gamma_2(s)s^{-2+\frac{2a}{\sigma^2}}\gamma_2(s)$$

$$= \Gamma_1(s)\gamma_1(s) + \Gamma_2(s)\gamma_2(s) = \gamma_3(s).$$

Asymptotic behavior of $\mathcal{L}\Psi(u)(s)$ In order to study the asymptotic behavior of $\mathcal{L}\Psi(u)(s)$, the first assumption is that $\rho_1 - \rho_2$ is not an integer. In this case the particular solutions of (4.5) will have the form:

$$\hat{y}_1 = s^{\rho_1}\gamma_1(s)$$

$$\hat{y}_2 = s^{\rho_2}\gamma_2(s)$$

$$\hat{y}_3 = \gamma_3(s)$$

where $\gamma_j(s)$ are holomorphic in the disk $|s| < \frac{1}{\mu}$ and $\gamma_j(0) = 1$, $j = 1, 2, 3$, therefore γ_1, γ_2, γ_3 are slowly varying functions. Using the computed value of ρ_1 and ρ_2, the solutions are:

$$\hat{y}_1 = c_1 s^{-1}\gamma_1(s)$$

$$\hat{y}_2 = c_2 s^{-2+\frac{2a}{\sigma^2}}\gamma_2(s)$$

$$\hat{y}_3 = c_3\gamma_3(s)$$

where $\gamma(s) = \sum_{-\infty}^{\infty} \gamma_k s^k$. The solution of the Laplace transform of the equation of the infinitesimal generator, i.e. the Laplace transform of the ruin probability, is a linear combination of these particular solutions. So, the solution of the Laplace transform of the (4.3) has the form:

$$\hat{y} = c_1\hat{y}_1 + c_2\hat{y}_2 + c_3\hat{y}_3$$

$$= c_1 s^{-1}\gamma_1(s) + c_2 s^{-2+\frac{2a}{\sigma^2}}\gamma_2(s) + c_3\gamma_3(s)$$

where γ_1, γ_2 and γ_3 are holomorphic functions. Determining asymptotic behavior of $U(u)$ as $u \to \infty$ requires knowledge of asymptotic behavior of $\hat{U}(s)$ as $s \to 0^+$, which is now considered. The asymptotic behavior of the solution as $s \to 0^+$ is dictated by the leading term of this linear combination. It can be distinguished two cases.

Case 1.

Suppose the leading term of the linear combination is s^{-1}. In this case the asymptotic behavior of $\mathcal{L}(\Psi(u))(s) = \hat{U}(s)$ is given by

$$\hat{U}(s) \sim cs^{-1}\gamma_1(s) \quad \text{as} \quad s \to 0.$$

Since $U(u)$, defined as the integral of the ruin probability, for positive values of u and zero otherwise, is non-decreasing, right continuous and equals to zero for negative values of u, $c \in \mathbf{R}$, $c \geq 0$, $\delta \geq 0$, Karamata Tauberian theorem can be applied (Step 3). Using the theorem, from the derived asymptotic behavior of $\hat{U}(s)$ as $s \to 0$ results that the asymptotic behavior of U is given by:

$$U(u) \sim cu\gamma(1/u)/\Gamma(2) \quad \text{as} \quad u \to \infty.$$

Applying the Monotone Density theorem (Step 4) for $U(u)$, implies that the asymptotic behavior of the ruin probability is given by

$$\Psi(u) \sim c\gamma(1/u)/\Gamma(2) \quad \text{as} \quad u \to \infty.$$

Hence, the asymptotic behavior of the ruin probability, decays to a constant as $u \to \infty$. Obviously, in this case, the function does not satisfy the boundary conditions from Paulsen's theorem, so it is not a solution that can be related to the ruin probability.

Case 2.

Considering the case of $s^{-2+\frac{2a}{\sigma^2}}$ as leading term, then the Laplace transform $\hat{U}(s) = \mathcal{L}(\Psi(u))(s)$ has an asymptotic behavior around zero given by:

$$\hat{U}(s) \sim cs^{-2+\frac{2a}{\sigma^2}}\gamma(s) \quad \text{as} \quad s \to 0.$$

The Karamata-Tauberian Theorem can be applied since all the conditions are fulfilled (Step 3). Given the asymptotic behavior of $\hat{U}(s)$ as $s \to 0$ is concluded that the asymptotic behavior of $U(u) = \int_0^u \Psi(x)dx$ is

$$U(u) \sim cu^{2-\frac{2a}{\sigma^2}}\gamma(1/u)/\Gamma(3-\frac{2a}{\sigma^2}), \quad \text{as} \quad u \to \infty.$$

In order to satisfy Paulsen's theorem condition $U(u)$ needs to decay as $u \to \infty$. If $2-\frac{2a}{\sigma^2} < 1$, then $u^{2-\frac{2a}{\sigma^2}}$ decays as $u \to \infty$, otherwise it grows. In other words, only for $\rho = \frac{2a}{\sigma^2} > 1$ $U(u)$ decays as $u \to \infty$. Again, since $\Psi(u)$ is monotone, $c \in \mathbf{R}$, $\rho \in \mathbf{R}$ and

$$U(u) \sim cu^{2-\frac{2a}{\sigma^2}}\gamma(1/u)/\Gamma(3-\frac{2a}{\sigma^2}),$$

as $u \to \infty$, Monotone Density Theorem applies (Step 4). Applying the theorem, it can be conclude that the derivative of $U(u)$, i.e. $\Psi(u)$, will have the asymptotic behavior

$$\Psi(u) \sim c(2-\frac{2a}{\sigma^2})u^{2-\frac{2a}{\sigma^2}-1}\gamma(1/u)/\Gamma(3-\frac{2a}{\sigma^2}), \quad \text{as} \quad u \to \infty$$

for $\frac{2a}{\sigma^2} > 1$. Combining this condition with the one in the Remark 2, it means that for $1 < \frac{2a}{\sigma^2} < 2$,

$$\Psi(u) \sim c(2-\frac{2a}{\sigma^2})u^{1-\frac{2a}{\sigma^2}}\gamma(1/u)/\Gamma(3-\frac{2a}{\sigma^2}), \quad \text{as} \quad u \to \infty$$

i.e.

$$\Psi(u) = Ku^{1-\frac{2a}{\sigma^2}}\gamma(1/u), \quad \text{as} \quad u \to \infty,$$

where $K = c(2-\frac{2a}{\sigma^2})/\Gamma(3-\frac{2a}{\sigma^2})$. Denoting $\frac{2a}{\sigma^2} = \rho$, implies that $\Psi(u)$ decreases at infinity as the power function $u^{1-\rho}$ for $1 < \rho < 2$. □

In the case $\rho = \frac{2a}{\sigma^2} < 1$ there is no decay of the ruin probability and it is conjectured that the probability of ruin is one, meaning that the ruin is certain.

5. CONCLUSION

The insurance companies should be very attentive to where they invest their capital. Investments in an asset with stochastic interest rate may be too risky for an insurance company. It can be mathematically justified that an investment in such an asset is not recommended. Disaster may arrive when the market value of the asset is low and it is not possible to be cover the losses just by selling these assets [9]. Investments in a stock with large volatility, namely when $\rho < 1$, lead to a ruin with probability one whatever is the initial capital. The decay of the ruin probability in case of investments with small volatility, $1 < \rho < 2$, depends only on the investment parameters a and σ. This suggests that the "insurance" part of the model does not influence the long term asymptotic behavior of the ruin probability.

BIBLIOGRAPHY

1. R.N. Bhattacharya and E. Waymire. *Stochastic Processes with Applications.* John-Wiley&Sons, 1990.

2. Patrick Billingsley. *Probability and Measure.* Wiley Series in Probability and Mathematical Statistics. JohnWiley&Sons, 1995.

3. N.H. Bingham, C.M. Goldie, and J.L. Teugels. *Regular variation.* Encyclopidia of Mathematics and its Applications. Cambridge University Press, 1987.

4. N.L.Bowers, H.U.Gerber, J.C.Hickman, D.A.Jones, and C.J.Nesbitt. *Actuarial Mathematics.* The Society of Actuaries, 1997.

5. Paul Embrechts. Actuarial versus financial pricing of insurance. *The Journal of Risk Finance*, Summer 2000.

6. Paul Embrechts, R. Frey, and H. Furrer. Stochastic processes in insurance and finance. *Handbook of Statistics*, 19, 2001.

7. P. Embrechts, C.Kluppelberg, and T. Mikosch. *Modelling Extremal Events for Insurance and Finance.* Springer-Verlag, Berlin, 1997.

8. M Fedoryuk. *Asymptotic Analysis.* Springer-Verlag, 1991.

9. A. Frolova, Y. Kabanov, and S. Pergamenshchikov. In the insurance business risky investments are dangerous. *Finance and stochastics*, (6):227–235, 2002.

10. Jan Grandell. *Aspects of risk theory.* Springer-Verlag, 199?

11. B. Oksendal. *Stochastic Differential Equations.* Universitext. Springer- Verlag, 1998.

12. J. Paulsen and H. Gjessing. Ruin theory with stochastic return on investments. *Advanced Applied Probability*, (29):965–985, 1997.

13. T.Rolski, H. Schmidli, V.Schmidt, and J.Teugels. *Stochastic Processes for Insurance and Finance.* Wiley Series in Probability and Statistics. JohnWiley&Sons, 2000.