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Title STIFFNESS MATRIX ANALYSIS OF STRUCTURES BY ELECTRONIC DIGITAL COMPUTER

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This paper presents a matrix method for analyzing civil engineering structures. Emphasis has been given to the use of stiffness influence coefficients.

Part 1 gives a brief account of the profound influence the high speed electronic computers have upon engineering calculation. Stated also are the general assumptions involved in the theory of the stiffness matrix analysis of structures.

The matrix algebra fundamentals applied to structural analysis are included in part 2.

Part 3 presents the theory of the stiffness matrix analysis preceeded by a brief discussion of the matrix force and deformation methods which serve as a background.

Part 4 includes the solution of a simple rigid frame bent worked through by means of a desk calculator. A step-by-step procedure is given in order to gain a working knowledge of the

matrix theory which will serve as a basis on which to formulate the use of high speed computers.

The analysis of a four-paneled Vierendeel truss is included in part 5. It is first analyzed by the Stiffness Matrix method and then by Moment Distribution. The stresses of both methods are tabulated for the sake of clarity and convenience in checking the agreement of the results.

Appendix A includes the IBM 1620 FORTRAN programming involved in the solution of the Stiffness Matrix equations formulated in this paper. Included also are the input and output data of the computer involved in solving these equations.

Although the stiffness influence coefficients for an individual beam element are readily obtained from structural handbooks, a derivation is provided for in Appendix B using the strain energy principles.

STIFFNESS MATRIX ANALYSIS OF STRUCTURES BY ELECTRONIC DIGITAL COMPUTER

by

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Part 1

INTRODUCTION

Many numerical methods have been developed in recent years in an attempt to eliminate the labor involved in the analysis of complex structures. The advent of electronic computers in particular has presented new horizons to the structural engineer. A new era has thus been marked with the faster than thought electronic giants. Although these computers have been successfully used in the aircraft industry, their use to civil engineering structures has been somewhat less familiar. The high cost of such computers has been the main reason for their non-availability to consulting firms specializing in Civil Engineering structures. In the last few years, however, this situation has changed considerably with the development of the Computer Program Exchange (7) which makes available to almost anyone warranting the use of a particular program.

Relaxation methods have been used successfully in the past to calculate complex elastic systems. The first application of the method to structural and other physical systems was made in 1935 by Southwell (31) inspired by the techniques of Hardy Cross (10) for rigid jointed frameworks. The basic computational process of the relaxation method when applied to frameworks can most easily be described in relation to the solution of equations which may also be solved by the more standard structural methods. Today, with the advent of electronic computers the iteration techniques of solution of simultaneous equations is becoming less desirable. The electronic digital computer of today is not employed merely for the solution of equations but also for the complete process of structural analysis.

Matrix methods of structural analysis are today the greatest tool in the numerical analysis of engineering structures. For clarification it may be emphasized here that speaking of matrix methods of analysis it is not meant merely the solution of a set of equations for the unknown coefficients by applying matrix algebra. What is meant and what is in fact essential here is the formulation and consistent development of structural analysis in matrix language, starting with the compilation of the initial data to the final stress distribution. This approach allows systematization and simplification of the calculations which otherwise would be impossible. Once the initial matrices are formulated and assembled, the operations that follow involve elementary matrix algebra. Besides, the matrix formulation is the ideal language for the electronic computer.

Several authors have shown the matrix formulation of structural theory and its application to structural problems. The stiffness matrix type of structural analysis was discussed in a paper by

G. Kron (19) in 1944. Papers by Langefors (21) and others followed. Argyris (2, 3) has given a thorough treatment of energy methods in matrix form. Lang and Bisplinghoff (20) have presented Levy's (23) application of the strain energy theory to a complex aircraft structure. Notable contributions were also made by Benscoter (5), Denke (11), Lansing and Wehle (22), Clough (8, 9), Turner (32), Martin (24), Klein (18), and many others.

1.1 General Assumptions

This thesis will be concerned only with structures composed of straight uniform members joined at their ends. The problems to be analyzed hereinafter will be limited only to those in which external loads consist of forces and couples applied at the joints. The justification of this assumption is based on the principle that any loading of a member at points between its ends may be replaced by equivalent fixed end forces and moments at the joints without causing any change of stresses in the rest of the structure (28, p. 236). The actual stresses in the loaded member may be determined by simple superposition after having evaluated the displacements and stresses due to the equivalent loading. All stresses, joint rotations and displacements are assumed to be linear functions of the applied loads. This implies that all displacements and rotations are small as compared with the dimensions of the framework. It is further implied that the joint loads and joint displacements and rotations are interrelated by a set of linear simultaneous algebraic equations.

1.2 Purpose of Thesis

The purpose of this thesis is to present a matrix method for analyzing civil engineering structures. It is of particular interest where automatic digital computing equipment is available to solve large numbers of simultaneous equations in matrix array. Emphasis has been given to the use of stiffness influence coefficients. This method will yield structural data of sufficient accuracy to be adequate for analyses of complex structures. Basic conditions of continuity and equilibrium requirements can be satisfied by use of a tabular procedure for writing the stiffness matrix of the entire structure. The method is illustrated by its application to a simple rigid frame bent and to a Vierendeel truss.

Part 2

MATRIX ALGEBRA APPLIED TO STRUCTURAL ANALYSIS

2. 1. Definitions and Terminology

This section is intended to point out some of the high-lights which will illustrate the advantages of the use of matrices. The matrix algebra approach presents the most convenient method of systematizing structural calculations for computer solution. Conveniently, all manufacturers of electronic digital computers have the basic operations of matrix algebra, such as matrix inversion, multiplication, addition and subtraction, already programmed (4, p. 73). These machine programs of matrix operations can be incorporated into the solution, thus reducing the amount of programming required for a solution.

The matric terminology used in this thesis is limited to those portions of matrix algebra which have been used to clarify their use in the subsequent discussion.

A <u>matrix</u> is defined as a rectangular array of elements arranged in rows and columns. It should be noted at the beginning that a matrix has no numerical value. It is simply a convenient way of representing arrays of numbers.

The *order* of a matrix refers to its size. A matrix containing m rows and n columns is said to be of order (m x n).

A *row* matrix is one containing a single row. Consider the equation

$$2x + 3y - 4z = 0$$

The coefficients of the unknowns; x, y, and z may be written as an array of numbers appearing in a row.

$$\begin{bmatrix} 2 & 3 & -4 \end{bmatrix}$$

This is defined as a row matrix. Its order is (1 x n), the square brackets denoting the fact that this is a matrix.

A *column* matrix is one containing a single column. The unknowns x, y, and z of the above equation may be written as an array of numbers appearing in a column.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This is defined as a column matrix and may also be written in the form

$$\left\{ x \quad y \quad z \right\}$$

where $\{\ \}$ denotes the fact that this is a column matrix appearing as a row for the sake of convenience or for conserving space.

A *square* matrix is one that has the same number of rows as columns.

A unit matrix is one which has unit elements on the diagonal and zeros all other elements. The unit matrix is denoted by I and serves the same function in matrix algebra as unity does in ordinary algebra.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

This is a unit matrix of order (3 x 3)

A symmetrical matrix is a square matrix which is symmetrical about its diagonal.

$$\begin{bmatrix} 2 & 3 & -4 \\ 3 & 5 & 6 \\ -4 & 6 & 8 \end{bmatrix}$$

This is a symmetrical matrix of order (3 x 3)

2. 2. Addition and Subtraction of Matrices

If the elements of one matrix are added or subtracted from the corresponding elements of another, the resulting elements form a third matrix, which is the sum or the difference of the first two. An example of matrix addition would be:

$$\begin{bmatrix} 6 & 8 \\ 4 & -3 \end{bmatrix} + \begin{bmatrix} 2 & -4 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 7 & 4 \end{bmatrix}$$

2.3 Matrix Multiplication

Consider the matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$[A]\ [B] = [C]$$

$$[C] = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

It may be noted that each element of the product matrix "C" is the inner product of its corresponding row and column. This process is known as matrix multiplication.

Consider the three simultaneous equations

$$\begin{aligned} 3x_1 + 2x_2 - 4x_3 &= C_1 \\ 2x_1 - 3x_2 - 5x_3 &= C_2 \\ x_1 \qquad\quad + 2x_3 &= C_3 \end{aligned}$$

Putting these equations in matrix form

$$[K]\{X\} = \{C\}$$

where

$$[K] = \begin{bmatrix} 3 & 2 & -4 \\ 2 & -3 & -5 \\ 1 & 0 & +2 \end{bmatrix} \quad \{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \quad \{C\} = \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix}$$

a square matrix of coefficients, a column matrix of unknowns, and column matrix of numbers. Multiplying each row of the square matrix by the elements of the column matrix of unknowns will restore the equations to their original algebraic form.

In order to obtain the solution for the unknowns x_1, x_2, and x_3 both sides of the matrix equation would be multiplied by the inverse of the square matrix of coefficients.

$$[K]^{-1} [K] \{X\} = [K]^{-1} \{C\}$$

or

$$\{X\} = [K]^{-1} \{C\}$$

There are several methods for obtaining the inverse of a matrix. One method is described briefly in the following pages as will be used later to illustrate the solution of a structural problem by the matrix method of analysis.

2.4. <u>Matrix Inversion</u> (15, p. 29-34)

Consider a square matrix of order n (let n=3 for simplicity)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{2.4.1}$$

Place a unit matrix to the right of the original matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{bmatrix} \tag{2.4.2}$$

Next we perform algebraic processes on the rows and columns of the composite matrix in such a manner as to reduce the original matrix to a unit matrix. When this has been accomplished, the original unit matrix portion has been converted to a reciprocal matrix, thus completing the matrix inversion as follows:

1) Divide each element of the first row by its leading element (a_{11}), thus obtaining:

$$\begin{bmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \frac{1}{a_{11}} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{bmatrix} \tag{2.4.3}$$

2) Subtract a_{21} times the first row from the respective elements of the succeeding rows $(i = 2, 3)$:

$$\begin{bmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \frac{1}{a_{11}} & 0 & 0 \\ 0 & a_{22} - a_{21}\left(\frac{a_{12}}{a_{11}}\right) & a_{23} - a_{21}\left(\frac{a_{13}}{a_{11}}\right) & 0 - a_{21}\left(\frac{1}{a_{11}}\right) & 1 & 0 \\ 0 & a_{32} - a_{31}\left(\frac{a_{12}}{a_{11}}\right) & a_{33} - a_{31}\left(\frac{a_{13}}{a_{11}}\right) & 0 - a_{31}\left(\frac{1}{a_{11}}\right) & 0 & 1 \end{bmatrix} \tag{2.4.4}$$

The reduced matrix of (2. 4. 4) may be written in the form

$$\begin{bmatrix} a'_{22} & a'_{23} & b'_{24} & 1 & 0 \\ a'_{32} & a'_{33} & b'_{34} & 0 & 1 \end{bmatrix} \tag{2.4.5}$$

Applying to (2. 4. 5) the same transformations as in steps 1 and 2, we obtain

$$\begin{bmatrix} 1 & \frac{a'_{23}}{a'_{22}} & \frac{b'_{24}}{a'_{22}} & \frac{1}{a_{22}} & 0 \\ 0 & a'_{33} - a'_{32}\left(\frac{a'_{23}}{a'_{22}}\right) & b'_{34} - a_{32}\left(\frac{a'_{23}}{a'_{22}}\right) & 0 - a'_{32}\left(\frac{1}{a'_{22}}\right) & 1 \end{bmatrix} \tag{2.4.6}$$

The reduced matrix of (2. 4. 6) may be written

$$\begin{bmatrix} a''_{33} & b''_{34} & b''_{35} & 1 \end{bmatrix} \tag{2.4.7}$$

Step 1 need by applied to (2. 4. 7) to obtain

$$\begin{bmatrix} 1 & \frac{b''_{34}}{a''_{33}} & \frac{b''_{35}}{a''_{33}} & \frac{1}{a''_{33}} \end{bmatrix} \tag{2.4.8}$$

The transformations outlined leave the original matrix in a triangular form:

$$\begin{bmatrix} 1 & \dfrac{a_{12}}{a_{11}} & \dfrac{a_{13}}{a_{11}} & \dfrac{1}{a_{11}} & 0 & 0 \\ 0 & 1 & \dfrac{a'_{23}}{a'_{22}} & \dfrac{b'_{24}}{a'_{22}} & \dfrac{1}{a'_{22}} & 0 \\ 0 & 0 & 1 & \dfrac{b''_{34}}{a''_{33}} & \dfrac{b''_{35}}{a''_{33}} & \dfrac{1}{a''_{33}} \end{bmatrix} \qquad (2.4.9)$$

which may be written:

$$\begin{bmatrix} 1 & \bar{a}_{12} & \bar{a}_{13} & \bar{b}_{14} & 0 & 0 \\ 0 & 1 & \bar{a}_{23} & \bar{b}_{24} & \bar{b}_{25} & 0 \\ 0 & 0 & 1 & \bar{b}_{34} & \bar{b}_{35} & \bar{b}_{36} \end{bmatrix} \qquad (2.4.10)$$

3) In order to reduce (2.4.10) to the unit matrix the following transformations are performed on rows 1, 2 of (2.4.10) Multiply an element of the fourth column by the elements of the last row and subtract the products from the respective row thus yielding (2.4.11) (i.e. $\bar{a}_{ij} - \bar{a}_{i3}\,\bar{a}_{3j}$; $i = 1, 2$; $j = 1, 2, \ldots 6$). This transformation leaves columns 1, 2 unchanged:

$$\begin{bmatrix} 1 & \bar{a}_{12} & 0 & \bar{b}_{14}-\bar{a}_{13}\bar{b}_{34} & \bar{b}_{15}-\bar{a}_{13}\bar{b}_{35} & \bar{b}_{16}-\bar{a}_{13}\bar{b}_{36} \\ 0 & 1 & 0 & \bar{b}_{24}-\bar{a}_{23}\bar{b}_{34} & \bar{b}_{25}-a_{23}b_{35} & \bar{b}_{26}-\bar{a}_{23}\bar{b}_{36} \end{bmatrix} \qquad (2.4.11)$$

Rewriting (2. 4. 11) with $\overline{\overline{b}}_{14}$ = element of fourth column, and $\overline{a}_{ij} = \overline{\overline{a}}_{ij}$, we have

$$\begin{bmatrix} 1 & \overline{\overline{a}}_{12} & 0 & \overline{\overline{b}}_{14} & \overline{\overline{b}}_{15} & \overline{\overline{b}}_{16} \\ 0 & 1 & 0 & \overline{\overline{b}}_{24} & \overline{\overline{b}}_{25} & \overline{\overline{b}}_{26} \end{bmatrix} \quad (2.4.12)$$

To obtain the next matrix a transformation similar to step 3 is applied to (2. 4. 12)

$$\begin{bmatrix} 1 & 0 & 0 & \overline{\overline{b}}_{14} - \overline{\overline{a}}_{12}\overline{\overline{b}}_{24} & \overline{\overline{b}}_{15} - \overline{\overline{a}}_{12}\overline{\overline{b}}_{25} & \overline{\overline{b}}_{16} - \overline{\overline{a}}_{12}\overline{\overline{b}}_{26} \end{bmatrix} \quad (2.4.13)$$

Rewriting (2. 4. 13) as before we obtain

$$\begin{bmatrix} 1 & 0 & 0 & \overline{\overline{\overline{b}}}_{14} & \overline{\overline{\overline{b}}}_{15} & \overline{\overline{\overline{b}}}_{16} \end{bmatrix} \quad (2.4.14)$$

Rewriting the entire matrix as before, the inverse results:

$$\begin{bmatrix} 1 & 0 & 0 & \overline{\overline{\overline{b}}}_{14} & \overline{\overline{\overline{b}}}_{15} & \overline{\overline{\overline{b}}}_{16} \\ 0 & 1 & 0 & \overline{\overline{b}}_{24} & \overline{\overline{b}}_{25} & \overline{\overline{b}}_{26} \\ 0 & 0 & 1 & \overline{b}_{34} & \overline{b}_{35} & \overline{b}_{36} \end{bmatrix}$$

Part 3

THE STIFFNESS MATRIX ANALYSIS OF STRUCTURES

3.1 Matrix Force and Deformation Methods

The matrix method of analysis of structures is generally formulated around three basic conditions which must be satisfied (32, p. 807):

(1) The applied forces must be in equilibrium with the internal forces

(2) The deformations of the members must be compatible (i.e. consistent with each other and with the boundary conditions)

(3) The forces and deflections in each member must be related in accordance with the stress-strain relationship assumed for the material.

The force-deformation relations are divided into two methods: (a) the force method in which forces or stress resultants (or generalized forces) are taken as unknowns and (b) the deformation method in which deflections or slopes (or generalized displacements) are taken as unknowns (3, p. 5).

When using the force method a flexibility matrix is first formulated by applying a unit force to each node (joint), one node at a time, and computing the deformations due to the unit force. The resulting matrix is then multiplied by a column matrix of known forces to produce the desired displacements in a column matrix. Using matrix notation

$$\{u\} = [f]\{P\}$$

where

$\{u\}$ is the displacement matrix of the structure units

$[f]$ is the flexibility matrix

$\{P\}$ is the matrix of forces and moments on the units of the structure

When using the deformation method a stiffness matrix is formulated by displacing a node of the structure, one node at a time, while restraining all other nodes, and computing the resulting moments and shears. In matrix form

$$\{P\} = [K]\{u\}$$

where $[K]$ is the stiffness matrix

Elastic Spring

Perhaps the simplest example that can be used to illustrate the method is the elastic spring. According to Hooke's Law the

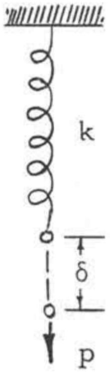


Figure 3.1.1. Elastic Spring.

relationship between force and displacement is

$$P = k\delta \qquad (3.1.1)$$

where P is the axial load, δ is the spring deflection, and k is the force required to produce a unit deflection. Since k is a measure of the stiffness of the spring, it can be considered to be a stiffness influence coefficient. Equation (3.1.1) can also be written

$$\delta = k^{-1}P = fP \qquad (3.1.2)$$

where f is the deflection due to a unit force. As such, f is a measure of the flexibility of the spring; therefore it can be considered to be a flexibility influence coefficient. It is useful to interpret k^{-1} as the deflection due to a unit load.

The dual relations described briefly above are easily seen from the following relationships:

Force Method	Deformation Method
Force (stress)	Displacement (strain)
Displacement (strain)	Force (stress)
Flexibility $= \dfrac{\text{Displacement}}{\text{Force}}$	Stiffness $= \dfrac{\text{Force}}{\text{Displacement}}$

The duality of these two relationships has been treated in detail by Argyris (2, p. 176).

In this paper we will be concerned with the stiffness matrix method of analysis.

3.2 Stiffness Matrix Method

The Stiffness Matrix Method is based essentially on the assumption that a complex structure can be analyzed as an assemblage of elements whose elastic behavior is known (25, p. 537). It is further assumed that the loads acting on the structure are represented by discrete concentrated loads acting only at the nodal points of the structure. These loads are referred to as generalized forces although they may actually be forces, moments, or both. Consequently, when the elastic behavior of a structural element is known the generalized forces acting on it may be written as linear functions of the generalized nodal displacements. Thus:

$$\begin{aligned} p_1 &= k_{11}u_1 + k_{12}u_2 + k_{13}u_3 \cdots\cdots\cdots + k_{1n}u_n \\ p_2 &= k_{21}u_1 + k_{22}u_2 + k_{23}u_3 \cdots\cdots\cdots + k_{2n}u_n \\ &\cdot \\ &\cdot \\ p_n &= k_{n1}u_1 + k_{n2}u_2 + k_{n3}u_3 \cdots\cdots\cdots + k_{nn}u_n \end{aligned} \qquad (3.2.1)$$

where u's are the generalized displacements which may be either displacements or rotations acting in the same directions as the corresponding generalized forces. The k's are the stiffness coefficients; that is, k_{ij} is the force p_i produced by a unit displacement $u_j = 1$. Once the stiffness coefficients of the individual elements of the structure are determined, the stiffness coefficients of the entire structure may be obtained by merely adding together at each node the stiffness coefficients of the adjacent elements. These may be written for simplicity in matrix form

$$\{P\} = [K]\{u\} \tag{3.2.2}$$

where $\{P\}$ is a column matrix of the generalized forces, $[K]$ is a symmetrical square matrix of the stiffness coefficients and $\{u\}$ is a column matrix of the generalized displacements.

By inverting the stiffness matrix one obtains an influence matrix which gives the nodal displacements as a function of the external forces or loads acting on the structure

$$\{u\} = [K]^{-1}\{P\} \tag{3.2.3}$$

It is believed that the conceptual basis for the stiffness method can be easily understood by developing this technique for a typical beam structure by utilizing the strain energy method (see Appendix B) for determining the stiffness influence coefficients. The nodal displacements and the forces acting on the element of the beam are as shown in Figure 3. 2. 1.

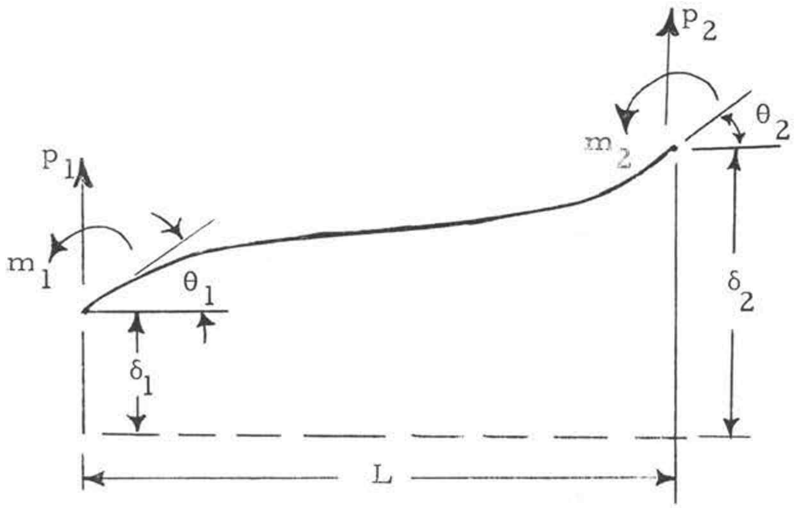


Figure 3. 2. 1. Assumed Elastic Curve

Substituting appropriate values in place of the u's and p's in Equation 3. 2. 1, we obtain

$$
\begin{aligned}
p_1 &= k_{11}\delta_1 + k_{12}\theta_1 L + k_{13}\delta_2 + k_{14}\theta_2 L \\
\frac{m_1}{L} &= k_{21}\delta_1 + k_{22}\theta_1 L + k_{23}\delta_2 + k_{24}\theta_2 L \\
p_2 &= k_{31}\delta_1 + k_{32}\theta_1 L + k_{33}\delta_2 + k_{34}\theta_2 L \\
\frac{m_2}{L} &= k_{41}\delta_1 + k_{42}\theta_1 L + k_{43}\delta_2 + k_{44}\theta_2 L
\end{aligned}
\qquad (3.2.4)
$$

Writing these equations in matrix form, we obtain

$$\{P\} = [k]\{u\} \qquad (3.2.5)$$

In order to make the generalized forces and displacements dimensionally consistent, the moments and rotations have been divided and multiplied respectively by L, the length of the beam segment. Thus, the stiffness coefficients become dimensionally consistent as well.

Having made certain simplifying assumptions in regards to the elastic curve and the permissible strain patterns it has been shown (see Appendix B3) that a stiffness matrix for the beam segment of equations (3. 2. 4) and (3. 2. 5) is as follows:

$$\left[K^{o}_{ij}\right] = \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}$$

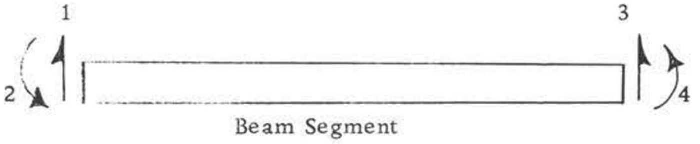
where K^{o} represents non-dimensional parts of K.

Figure 3.2.2 shows the Individual Element Stiffness Matrix of a beam segment, whereas Figure 3.2.3 summarizes the deflection conditions for a restrained beam segment. Shown in Figure 3.2.3 are also the values of the bending moments and shearing forces at the fixed-end beams, when a unit rotation is applied at each end or a unit translation at one end with respect to the other.

3. 2. 1. Study of a Cantilever Beam

From Equation (3. 2. 4) and in conjunction with Figure 3. 2. 3

Individual Element Stiffness Matrix (1, p. 5)

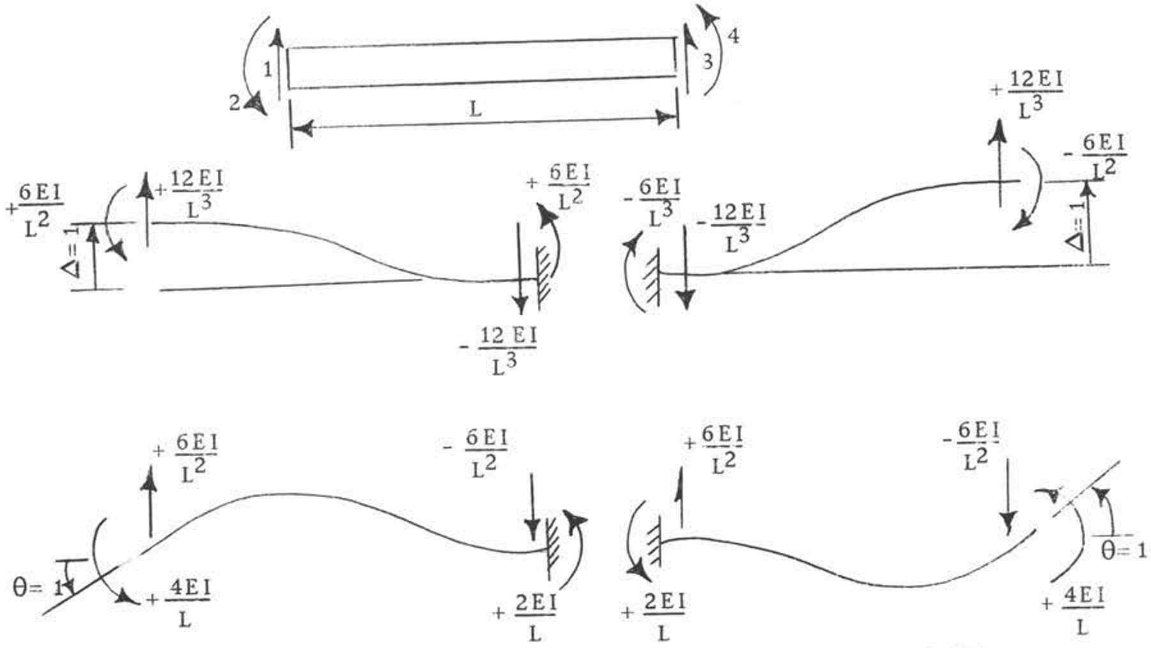


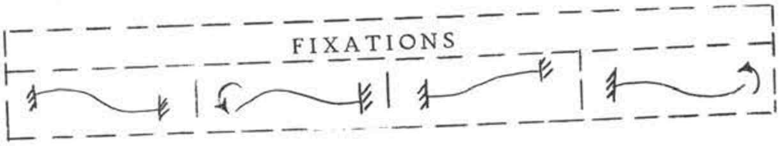
Symmetry

		FIXATIONS			
		Translation	Rotation	Translation	Rotation
		$\Delta = 1$	$\theta = 1$	$\Delta = 1$	$\theta = 1$
RESULTANT FORCES		1	2	3	4
Shear	1	k_{11}	k_{12}	k_{13}	k_{14}
Moment	2	k_{21}	k_{22}	k_{23}	k_{24}
Shear	3	k_{31}	k_{32}	k_{33}	k_{34}
Moment	4	k_{41}	k_{42}	k_{43}	k_{44}

Figure 3.2.2. Individual Element Stiffness Matrix.

Deflection Conditions for Restrained Beam Segment with no Axial Load





RESULTANT FORCES		1	2	3	4
Shear	1	$+\frac{12EI}{L^3}$	$+\frac{6EI}{L^2}$	$-\frac{12EI}{L^3}$	$+\frac{6EI}{L^2}$
Moment	2	$+\frac{6EI}{L^2}$	$+\frac{4EI}{L}$	$-\frac{6EI}{L^2}$	$+\frac{2EI}{L}$
Shear	3	$-\frac{12EI}{L^3}$	$-\frac{6EI}{L^2}$	$+\frac{12EI}{L^3}$	$-\frac{6EI}{L^2}$
Moment	4	$+\frac{6EI}{L^2}$	$+\frac{2EI}{L}$	$-\frac{6EI}{L^2}$	$+\frac{4EI}{L}$

Figure 3. 2. 3. Basic Beam Stiffness Matrix.

the forces and deflections for the beam shown in Figure 3.2.1 are related by the following equation:

$$\begin{Bmatrix} p_1 \\ m_1 \\ p_2 \\ m_2 \end{Bmatrix} = \begin{array}{c} \begin{array}{cccc} \delta_1 & \theta_1 & \delta_2 & \theta_2 \end{array} \\ \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \end{array} \begin{Bmatrix} \delta_1 \\ \theta_1 \\ \delta_2 \\ \theta_2 \end{Bmatrix} \qquad (3.2.7)$$

Suppose a cantilever beam is to be studied; this can be obtained by requiring joint 2 be fixed. Hence $\theta_2 = \delta_2 = 0$, and we may eliminate third and fourth columns as well as third and fourth rows of $[K]$ of Equation (3.2.7). The following rule may be stated:

> Strike out the row and the column in the stiffness matrix corresponding to any rotation (or displacement) which is completely restrained by a support. (28, p. 234).

For our problem at hand, after having complied with the above rule, the stiffness matrix is contracted to

$$\left\{ \begin{array}{c} p_1 \\ m_1 \\ \hline p_2 \\ m_2 \end{array} \right\} = \left[\begin{array}{c|c} k_{11} & k_{12} \\ \hline k_{21} & k_{22} \end{array} \right] \left\{ \begin{array}{c} \delta_1 \\ \theta_1 \\ \hline 0 \\ 0 \end{array} \right\} \qquad (3.2.8)$$

where p_2 and m_2 have become reactions, p_1 and m_1 become applied loads, and δ_1 and θ_1 are unknown deflections.

Expanding Equation (3.2.8) leads to the two equations:

$$\begin{Bmatrix} p_1 \\ m_1 \end{Bmatrix} = \begin{bmatrix} k_{11} \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \theta_1 \end{Bmatrix} \qquad (3.2.9)$$

$$\begin{Bmatrix} p_2 \\ m_2 \end{Bmatrix} = \begin{bmatrix} k_{21} \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \theta_1 \end{Bmatrix} \qquad (3.2.10)$$

Equation (3.2.9) relates all possible applied loads to all possible displacements for the cantilever under study.

3.2.2. Study of Plane Rectangular Framed Structures

The stiffness method of analysis is especially suited to the study of rectangular structures since, rotations of the joints and possible translations normal to the axes of the beams being locked, the structure becomes a system of fixed-end beams, the behavior of which is well-known. The number of fixations is consequently equal to the number of joints that can rotate plus the number of possible translations (6, p. 197).

In a framework of beams such as a simple portal frame or a Vierendeel truss, the rotations and translations of the ends of the respective beam elements will describe the entire deformation behavior of the structure, provided elementary beam theory is considered for each beam and the given external loads consist of a set of concentrated loads applied at the nodal points. In this manner, the entire stress pattern may be determined. If on the other hand, the given external loads are distributed along the beams, the end point slopes and deflections of the beam elements are not sufficient in determining precisely the entire stress and deformation pattern of the structure. Under these circumstances the distributed loading may be replaced with a statically equivalent set of concentrated generalized forces applied to the node points (28, p. 236). Thus, the stress condition of the framework may be adequately approximated from the end loads and the end moments due to these concentrated loads.

3.2.3. Procedure

A systematic procedure for the analysis of structures using the stiffness matrix method is outlined below:

(1) Assign consecutive numbers 1, 2, 3 ... to each unknown displacement (linear or angular).

(2) Compute the stiffness matrix for the individual structural elements

(3) Merge matrices into a stiffness matrix for the composite

structure by adding corresponding stiffness coefficients for adjacent sections

(4) Invert the stiffness matrix to obtain the flexibility matrix for the structure

(5) Calculate the deflections by determining the matrix product of the inverted matrix and the applied load vector

$$\{\delta\} = [K]^{-1} [P]$$

(6) Calculate the forces or moments in the individual elements using the stiffness matrices obtained in step 2. This would be given as the product of the deflections obtained in step 5 and the matrices of step 2.

The foregoing procedure is satisfactory as long as the deformations due to axial load effects are negligible.

Part 4

RIGID FRAME BENT

An illustrative example worked through by means of a desk calculator will be presented in this section. Thus a working knowledge of matrix theory will be gained as a necessary prerequisite or basis on which to formulate the use of high speed computer.

In Figure 4. 1 a simple rigid frame bent is shown. A lateral load is applied at B. The dimensions and moment of inertia for each member are as indicated in structure. The effect of axial forces will be neglected.

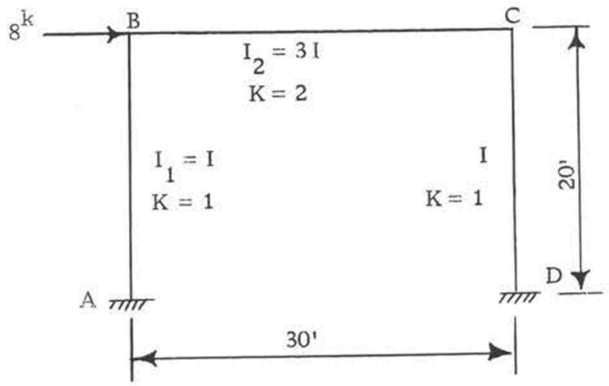


Figure 4. 1. Rigid Frame Bent

Step 1. - Coordinate identification numbers are assigned in compliance with outlined procedure as follows:

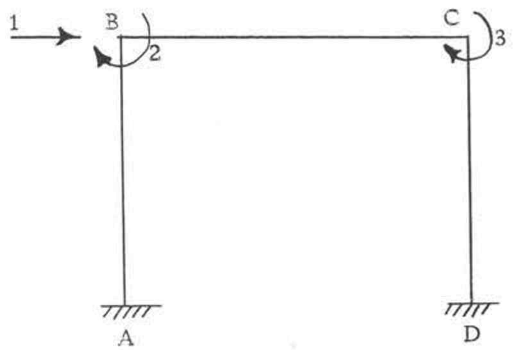


Figure 4. 2. Assigned Coordinate Numbers

There are therefore three loads assumed to be acting on the structure. All of these loads are known. Note that we do not assume deflections at points that cannot deflect. Thus, a constrained end can have neither an angular deflection nor a linear one.

Step 2. - Compute the stiffness matrices for the individual structural elements:

MEMBER AB

$$
\begin{array}{c|cc}
i \backslash j & 1 & 2 \\
\hline
\begin{array}{c} 1 \\ \\ 2 \end{array} &
\multicolumn{2}{c}{\left[\begin{array}{cc} +\dfrac{12\,E\,I_1}{L^3} & +\dfrac{6\,E\,I_1}{L^2} \\[2ex] +\dfrac{6\,E\,I_1}{L^2} & +\dfrac{4\,E\,I_1}{L} \end{array}\right]}
\end{array}
\qquad (4.1)
$$

$$
\begin{array}{c}
V_A \\ \\ M_A
\end{array}
\quad
\left[\begin{array}{cc} -\dfrac{12\,E\,I_1}{L^3} & -\dfrac{6\,E\,I_1}{L^2} \\[2ex] +\dfrac{6\,E\,I_1}{L^2} & +\dfrac{2\,E\,I_1}{L} \end{array}\right]
\qquad (4.2)
$$

Evaluate coefficients shown above to obtain numerical values

$$\frac{12\,E\,I_1}{L^3} = \frac{12\,E\,I_1}{(240)^3} = 0.868 \times 10^{-6}\ E\,I_1$$

$$\frac{6\,E\,I_1}{L^2} = \frac{6\,E\,I_1}{(240)^2} = 1.0417 \times 10^{-4}\ E\,I_1$$

$$\frac{4\,E\,I_1}{L} = \frac{4\,E\,I_1}{240} = 1.6667 \times 10^{-2}\ E\,I_1$$

$$\frac{2\,E\,I_1}{L} = \frac{2\,E\,I_1}{240} = 0.8334 \times 10^{-2}\ E\,I_1$$

Hence, Equations (1) and (2) become

$$\begin{array}{c|c} i \backslash j & \begin{array}{cc} \quad 1 \quad\quad\quad & \quad\quad\quad 2 \quad \end{array} \\ \hline \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} +0.8680 \times 10^{-6} & +1.0417 \times 10^{-4} \\ +1.0417 \times 10^{-4} & +1.6667 \times 10^{-2} \end{bmatrix} E I_1 \\ \begin{array}{c} V_A \\ M_A \end{array} & \begin{bmatrix} -0.8680 \times 10^{-6} & -1.0417 \times 10^{-4} \\ +1.0417 \times 10^{-4} & +0.8334 \times 10^{-2} \end{bmatrix} E I_1 \end{array}$$

(4.1a)

(4.2a)

Note that V_A and M_A represent the shear and Moment at the constrained end A respectively

<u>MEMBER BC</u>

All terms are caused by rotational displacements only $\theta = 1$. No linear displacement ($\Delta = 0$).

$$\begin{array}{c|cc} i \backslash j & 2 & 3 \\ \hline \begin{array}{c} 2 \\ \\ 3 \end{array} & \multicolumn{2}{c}{\begin{bmatrix} +\dfrac{4 E I_2}{L_2} & +\dfrac{2 E I_2}{L_2} \\ \dfrac{2 E I_2}{L_2} & \dfrac{4 E I_2}{L_2} \end{bmatrix}} \end{array} \qquad (4.3)$$

Evaluate coefficients shown above to obtain numerical values

$$\frac{4 E I_2}{L_2} = \frac{4 E (3 I_1)}{360} = \frac{12 E I_1}{360} = 3.3334 \times 10^{-2} E I_1$$

$$\frac{2 E I_2}{L_2} = \frac{2 E (3 I_1)}{360} = \frac{6 E I_1}{360} = 1.6667 \times 10^{-2} E I_1$$

Hence, Equation (4.3) becomes

$$\begin{array}{c|cc} i \backslash j & 2 & 3 \\ \hline 2 & +3.334 \times 10^{-2} & +1.6667 \times 10^{-2} \\ 3 & +1.6667 \times 10^{-2} & +3.3334 \times 10^{-2} \end{array} \; EI_1 \qquad (4.3\ a)$$

MEMBER CD

The stiffness influence coefficients as computed for Member AB. Hence,

$$\begin{array}{c|cc} i \backslash j & 1 & 3 \\ \hline 1 & +0.8680 \times 10^{-6} & +1.0417 \times 10^{-4} \\ 3 & +1.0417 \times 10^{-4} & +1.6667 \times 10^{-2} \end{array} \; EI_1 \qquad (4.4)$$

$$\begin{array}{c|cc} V_D & -0.8680 \times 10^{-6} & -1.0417 \times 10^{-4} \\ M_D & +1.0417 \times 10^{-4} & +0.8334 \times 10^{-2} \end{array} \; EI_1 \qquad (4.5)$$

The Shear and Moment at the constrained end D are represented by V_D and M_D respectively.

Step 3. - Merge Stiffness Matrices

Three unknown displacements are indicated in Figure 4-2; specifically, Δ_1, θ_2, and θ_3. Hence, the stiffness matrices obtained in step 2 must merge in order to formulate three simultaneous equations, as there are only three unknowns. The combined matrix must include the effects which result from the redundant deflections in question. Hence,

$$k_{11} = \left[+0.8680 \times 10^{-6} + 0.8680 \times 10^{-6}\right] E\,I_1 = +1.7360 \times 10^{-6} E\,I_1$$

$$k_{12} = +1.0417 \times 10^{-4}\ E\,I_1$$

$$K_{13} = +1.0417 \times 10^{-4}\ E\,I_1$$

$$k_{21} = +1.0417 \times 10^{-4}\ E\,I_1$$

$$k_{22} = \left[+1.6667 \times 10^{-2} + 3.3333 \times 10^{-2}\right] E\,I_1 = +5.0000 \times 10^{-2} E\,I_1$$

$$k_{23} = +1.6667 \times 10^{-2}\ E\,I_1$$

$$k_{31} = +1.0417 \times 10^{-4}\ E\,I_1$$

$$k_{32} = +0.6667 \times 10^{-2}\ E\,I_1$$

$$k_{33} = \left[+3.3337 \times 10^{-2} + 1.667 \times 10^{-2}\right] E\,I_1 = +5.0000 \times 10^{-2} E\,I$$

which may be written compactly as follows:

$$\begin{array}{c|c} i \backslash j & \begin{array}{ccc} \quad 1 \quad & \quad\quad 2 \quad\quad & \quad 3 \quad \end{array} \\ \hline \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} +1.7360 \times 10^{-6} & +1.0417 \times 10^{-4} & +1.0417 \times 10^{-4} \\ +1.0417 \times 10^{-4} & +5.0000 \times 10^{-2} & +1.6667 \times 10^{-2} \\ +1.0417 \times 10^{-4} & +1.6667 \times 10^{-2} & +5.0000 \times 10^{-2} \end{bmatrix} EI_1 \end{array} \qquad (4.6)$$

<u>Step 4.</u> - Invert Stiffness Matrix

The procedure outlined in Part 2.4 has been used for the inversion of the composite matrix (4.6) of the structure. The results representing the step-by-step procedure are shown in Table 4.1 for the sake of clarity and for conserving space.

Table 4.1. Matrix Inversion

$+1.7360\times10^{-6}$	$+1.0417\times10^{-4}$	$+1.0417\times10^{-4}$	1	0	0
$+1.0417\times10^{-4}$	$+5.0000\times10^{-2}$	$+1.6667\times10^{-2}$	0	1	0
$+1.0417\times10^{-4}$	$+1.6667\times10^{-2}$	$+5.0000\times10^{-2}$	0	0	1
1	+60.0058	+60.0058	+576040.	0	0
0	+ 0.043749	+ 0.010416	-60.0061	1	0
0	+ 0.010416	+ 0.043749	-60.0061	0	1
	1	+ 0.2381	-1371.5990	+ 22.8572	0
	0	+ 0.041269	- 45.7195	- 0.2381	1
		1	+1107.8410	- 5.7695	+ 24.2313
1	+60.0058	+60.0058	+576040.	0	0
0	1	+ 0.2381	- 1371.5990	+ 22.8572	0
0	0	1	- 1107.8410	- 5.7695	+ 24.2313
1	+60.0058	0	+642516.8855	+ 346.2035	-1454.0185
0	1	0	- 1107.8410	+ 24.2313	- 5.7695
1	0	0	+708993.7710	-1107.8410	-1107.8410
1	0	0	+708993.7710	-1107.8410	-1107.8410
0	1	0	- 1107.8410	+ 24.2313	- 5.7695
0	0	1	- 1107.8410	- 5.7695	+ 24.2313

The inverse of the matrix may now be written in its complete form

$$[K]^{-1} = \frac{1}{EI}\begin{bmatrix} 708{,}993.771 & -1107.841 & -1107.841 \\ -1{,}107.841 & +\ 24.2313 & -\ \ 5.7695 \\ -1{,}107.841 & -\ \ 5.7695 & +\ 24.2313 \end{bmatrix} \tag{4.7}$$

Step 5. - Compute the deflections

Equation 3.2.3 may now be written in expanded forms as below

$$\begin{Bmatrix} \Delta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \frac{1}{EI}\begin{bmatrix} 708{,}993.771 & -1107.841 & -1107.841 \\ -1{,}107.841 & +\ 24.2313 & -\ \ 5.7695 \\ -1{,}107.841 & -\ \ 5.7 & +\ 24.2313 \end{bmatrix}\begin{Bmatrix} -8{,}000 \\ 0 \\ 0 \end{Bmatrix} \tag{4.8}$$

From Equation (4.8) the deflections are determined as follows

$$\Delta_1 = \frac{708,993.771(-8,000)}{EI} = \frac{-5,671,950,168}{EI} \text{ inches}$$

$$\theta_2 = \frac{-1107.841\ (-8000)}{EI} = \frac{8,862,728}{EI} \text{ radians}$$

$$\theta_3 = \frac{-1107.841\ (-8000)}{EI} = \frac{8,862,728}{EI} \text{ radians}$$

Step 6. - Calculate member forces and moments

MEMBER AB

From Equations (4.1a) and (4.2a) the forces and moments are found to be

$$V_A = -0.8680 \times 10^{-6} EI \left(\frac{-5,671,950,168}{EI}\right) - 1.0417 \times 10^{-4} EI \left(\frac{8,862,728}{EI}\right)$$

$$= 4000 \text{ lb.}$$

$$M_A = \frac{1}{12}\left[+1.0417 \times 10^{-4} EI \left(\frac{-5,671,950,168}{EI}\right) + 0.8334 \times 10^{-2} EI \left(\frac{8,862,728}{EI}\right)\right]$$

$$= -43,082 \text{ ft. lb.} \quad = 0.516984 \times 10^{6} \text{ in. lb.}$$

$$V_B = +0.8680 \times 10^{-6} EI \left(\frac{-5,671,950,168}{EI}\right) + 1.0417 \times 10^{-4} EI \left(\frac{8,862,728}{EI}\right)$$

$$= -4000 \text{ lb.}$$

$$M_B = \frac{1}{12}\left[+1.0417 \times 10^{-4} EI \left(\frac{-5,671,950,168}{EI}\right) + 1.6667 \times 10^{-2} EI \left(\frac{8,862,728}{EI}\right)\right]$$

$$= -36,928 \text{ ft. lb.} \quad = -0.443136 \times 10^{6} \text{ in. lb.}$$

MEMBER BC

From Equation (4. 3a) we obtain

$$M_B = \frac{1}{12}\left[+3.3334 \times 10^{-2} EI \left(\frac{8,862,728}{EI}\right) + 1.6667 \times 10^{-2} EI \left(\frac{8,862,728}{EI}\right)\right]$$

$$= 36,928 \text{ ft. lb.} \quad = 0.443136 \times 10^{6} \text{ in. lb.}$$

$$M_C = \frac{1}{12}\left[+1.6667 \times 10^{-2} EI \left(\frac{8,862,728}{EI}\right) + 3.3334 \times 10^{-2} EI \left(\frac{8,862,728}{EI}\right)\right]$$

$$= 36,928 \text{ ft. lb.} \quad = 0.443136 \times 10^{6} \text{ in. lb.}$$

MEMBER CD

From Equations (4. 4) and (4. 5) we obtain

$$V_C = +0.8680 \times 10^{-6} EI \left(\frac{-5,571,950,168}{EI}\right) + 1.0417 \times 10^{-4} EI \left(\frac{8,862,728}{EI}\right)$$

$$= -4000 \text{ lb.}$$

$$M_C = \frac{1}{12}\left[+1.0417\times 10^{-4} EI\left(\frac{-5,671,950,168}{EI}\right) + 1.6667\times 10^{-2} EI\left(\frac{8,862,728}{EI}\right)\right]$$

$$= -36928 \text{ ft. lb.} = -0.443136 \times 10^{6} \text{ in. lb.}$$

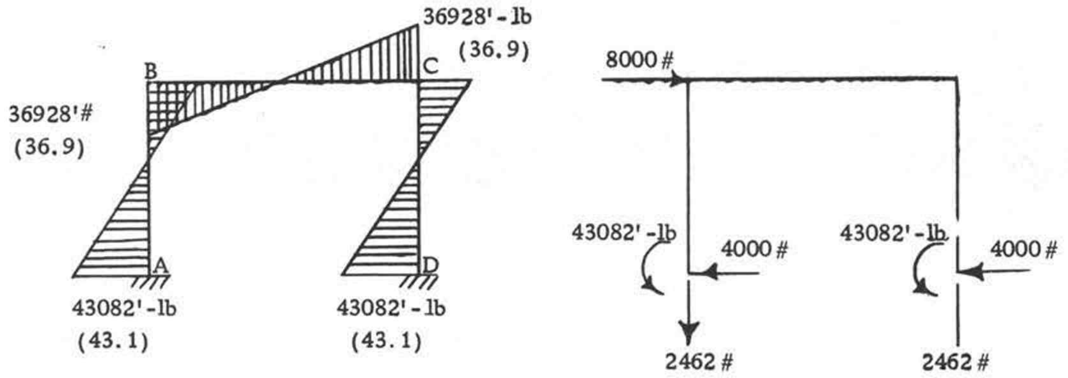
$$V_D = -0.8680\times 10^{-6}\, EI\ \frac{-5,671,950,168}{EI} - 1.0417\times 10^{-4} EI\ \frac{8,862,728}{EI}$$

$$= 4000 \text{ lb.}$$

$$M_D = \frac{1}{12}\left[+1.0417\times 10^{-4} EI\left(\frac{-5,671,950,168}{EI}\right) + 0.8334\times 10^{-2} EI\left(\frac{8,862,728}{EI}\right)\right]$$

$$= -43,082 \text{ ft. lb.} = -0.516984 \times 10^{6} \text{ in. lb.}$$

The final moments and reactions are shown in Figure 4.3, (a) and (b) respectively. The moments in parentheses are expressed in ft.kips.



(a) (b)

Figure 4.3. Final Moments and Reactions

Note: The results of this problem agree with those obtained by conventional methods. For comparison, the numbers in parentheses are those obtained by Moment Distribution (27, p. 295, Example 2).

Part 5

ANALYSIS OF A VIERENDEEL TRUSS

5.1. By Stiffness Matrix Method

In this section the case of a Vierendeel truss will be considered for analysis by the stiffness matrix method. The computed answers in this analysis will include the rotation and translation of each end of the chords and verticals as well as the bending moments and shears in these members.

Figure 5.1.1. shows the center line dimensions of the truss, the assumed loading and moments of inertia. Also, assume $E = 30 \times 10^6$ psi.

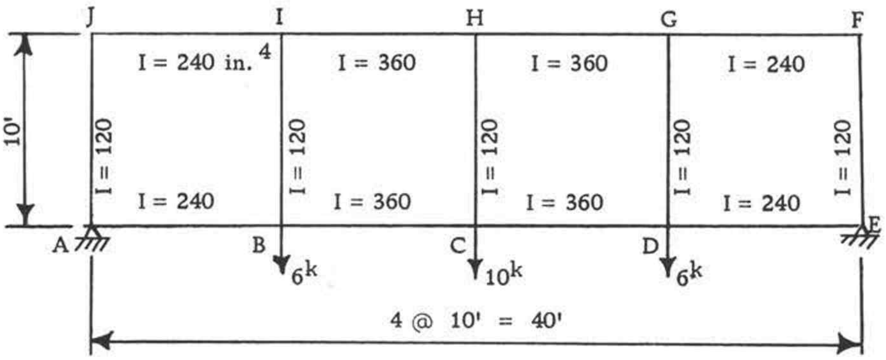


Figure 5.1.1. Symmetrically Loaded Vierendeel Truss

The procedure outlined preceeding the analysis of the simple frame bent will also be used for the analysis of the Vierendeel truss. Hence,

Step 1. - Assign stiffness matrix coordinates as shown in Figure 5. 1. 2.

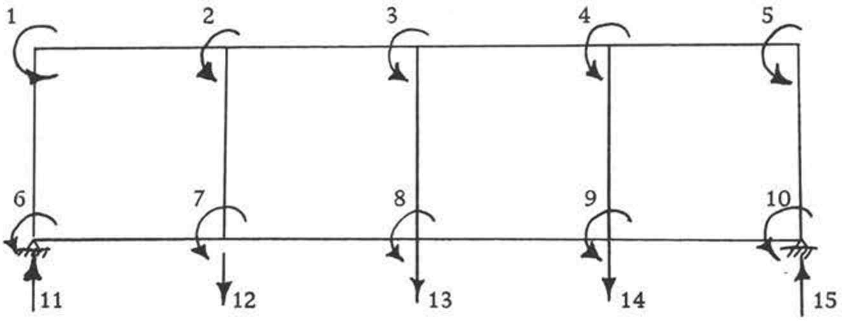


Figure 5. 1. 2. Assigned Coordinate Numbers

Step 2. - Stiffness Matrix for Individual Members

For convenience, the stiffness influence coefficients will be determined prior to investigating its individual members

For all vertical Members

$$k_{11} = \frac{4\,EI}{L} = \frac{4\,(30 \times 10^{6})\,120}{120} = +\,1.2 \times 10^{8}$$

$$(k_{11} = k_{22} = k_{33} = k_{44} = k_{55})$$

$$k_{16} = \frac{2\,EI}{L} = \frac{2\,(30 \times 10^6)\,120}{120} = +\,0.6 \times 10^8$$

$$(k_{16} = k_{27} = k_{38} = k_{49} = k_{5.10})$$

$$k_{61} = \frac{2\,EI}{L} = \frac{2\,(30 \times 10^6)\,120}{120} = +\,0.6 \times 10^8$$

$$(k_{61} = k_{72} = k_{83} = k_{94} = k_{10.5})$$

$$k_{66} = \frac{4\,EI}{L} = \frac{4\,(30 \times 10^6)\,120}{120} = +\,1.2 \times 10^8$$

$$(k_{66} = k_{77} = k_{88} = k_{99} = k_{10.10})$$

<u>For Members AB, DE, JI, and GF</u>

$$k_{11.11} = +\frac{12\,EI}{L^3} = +\frac{12\,(30 \times 10^6)\,240}{(120)^3} = +\,5.0 \times 10^4$$

$$k_{11.6} = +\frac{6\,EI}{L^2} = \frac{6\,(30 \times 10^6)\,240}{(120)^2} = +\,3.0 \times 10^6$$

$$k_{11.12} = \frac{12\,EI}{L^3} = -\frac{12\,(30 \times 10^6)\,240}{(120)^3} = -\,5.0 \times 10^4$$

$$k_{11.7} = +\frac{6\,EI}{L^2} = +\frac{6\,(30 \times 10^6)\,240}{(120)^2} = +\,3\ 0 \times 10^6$$

$$k_{6.11} = +\frac{6\,EI}{L^2} = +\frac{6\,(30 \times 10^6)\,240}{(120)^2} = +\,3.0 \times 10^6$$

$$k_{6.6} = + \frac{4\,EI}{L} = + \frac{4\ (30 \times 10^6)\ 240}{120} = + 2.4 \times 10^8$$

$$k_{6.12} = - \frac{6\,EI}{L^2} = - \frac{6\ (30 \times 10^6)\ 240}{(120)^2} = - 3.0 \times 10^6$$

$$k_{6.7} = + \frac{2\,EI}{L} = \frac{2\ (30 \times 10^6)\ 240}{120} = + 1.2 \times 10^8$$

$$k_{12.11} = - \frac{12\,EI}{L^3} = \frac{12\ (30 \times 10^6)\ 240}{(120)^3} = - 5.0 \times 10^4$$

$$K_{12.6} = - \frac{6\,EI}{L^2} = - \frac{6\ (30 \times 10^6)\ 240}{(120)^2} = - 3.0 \times 10^6$$

$$k_{12.12} = + \frac{12\,EI}{L^3} = + \frac{12\ (30 \times 10^6)\ 240}{(120)^3} = + 5.0 \times 10^4$$

$$k_{12.7} = - \frac{6\,EI}{L^2} = - \frac{6\ (30 \times 10^6)\ 240}{(120)^2} = - 3.0 \times 10^6$$

$$k_{7.11} = + \frac{6\,EI}{L^2} = + \frac{6\ (30 \times 10^6)\ 240}{(120)^2} = + 3.0 \times 10^6$$

$$k_{7.6} = + \frac{2\,EI}{L} = + \frac{2\ (30 \times 10^6)\ 240}{120} = + 1.2 \times 10^8$$

$$k_{7.7} = + \frac{4\,EI}{L} = + \frac{4\ (30 \times 10^6)\ 240}{120} = + 2.4 \times 10^8$$

For Members BC, CD, IH, and HC

$$k_{12.12} = + \frac{12\,EI}{L^3} = + \frac{12\ (30 \times 10^6)\ 360}{(120)^3} = + 7.5 \times 10^4$$

$$k_{12.7} = + \frac{6\,EI}{L^2} = + \frac{6\ (30 \times 10^6)\ 360}{(120)^2} = + 4.5 \times 10^6$$

$$k_{12.13} = - \frac{12\,EI}{L^3} = - \frac{12\ (30 \times 10^6)\ 360}{(120)^3} = - 7.5 \times 10^4$$

$$k_{12.8} = + \frac{6\,EI}{L^2} = + \frac{6\ (30 \times 10^6)\ 360}{(120)^2} = + 4.5 \times 10^6$$

$$k_{7.12} = + \frac{6\,EI}{L^2} = + \frac{6\ (30 \times 10^6)\ 360}{(120)^2} = + 4.5 \times 10^6$$

$$k_{7.7} = + \frac{4\,EI}{L} = + \frac{4\ (30 \times 10^6)\ 360}{120} = + 3.6 \times 10^8$$

$$k_{7.13} = - \frac{6\,EI}{L^2} = - \frac{6\ (30 \times 10^6)\ 360}{(120)^2} = - 4.5 \times 10^6$$

$$k_{7.8} = + \frac{2\,EI}{L} = + \frac{2\ (30 \times 10^6)\ 360}{120} = + 1.8 \times 10^8$$

$$k_{13.12} = - \frac{12\,EI}{L^3} = - \frac{12\ (30 \times 10^6)\ 360}{(120)^3} = - 7.5 \times 10^4$$

$$k_{13.7} = -\frac{6\,E\,I}{L^2} = -\frac{6\ (30 \times 10^6)\ 360}{(120)^2} = -4.5 \times 10^6$$

$$k_{13.13} = +\frac{12\,E\,I}{L^3} = +\frac{12\ (30 \times 10^6)\ 360}{(120)^3} = +7.5 \times 10^4$$

$$k_{13.8} = -\frac{6\,E\,I}{L^2} = -\frac{6\ (30 \times 10^6)\ 360}{(120)^2} = -4.5 \times 10^6$$

$$k_{8.12} = +\frac{6\,E\,I}{L^2} = +\frac{6\ (30 \times 10^6)\ 360}{(120)^2} = +4.5 \times 10^6$$

$$k_{8.7} = +\frac{2\,E\,I}{L} = +\frac{2\ (30 \times 10^6)\ 360}{120} = +1.8 \times 10^8$$

$$k_{8.13} = -\frac{6\,E\,I}{L^2} = -\frac{6\ (30 \times 10^6)\ 360}{(120)^2} = -4.5 \times 10^6$$

$$k_{8.8} = +\frac{4\,E\,I}{L} = +\frac{4\ (30 \times 10^6)\ 360}{120} = +3.6 \times 10^8$$

5.1.1. Stiffness Influence Coefficients for Individual Elements

MEMBER AJ

All terms caused by rotational displacements only, $\theta = 1$.

Linear displacement $\Delta = 0$.

$$\left[k\right]_{AJ} = \begin{array}{c} \\ 1 \\ 6 \end{array} \begin{array}{c} \begin{array}{cc} \quad 1 \quad & \quad 6 \quad \end{array} \\ \begin{bmatrix} +1.2 \times 10^8 & +0.6 \times 10^8 \\ +0.6 \times 10^8 & +1.2 \times 10^8 \end{bmatrix} \end{array}$$

MEMBER AB

All terms due to unit displacements ($\theta = 1$, $\Delta = 1$)

$$\begin{array}{cc}
 & \begin{matrix} 11 & \quad 6 & \quad 12 & \quad 7 \end{matrix} \\
\begin{matrix} 11 \\ 6 \\ 12 \\ 7 \end{matrix} &
\begin{bmatrix}
+5.0\times10^{-4} & +3.9\times10^{6} & -5.9\times10^{4} & +3.0\times10^{6} \\
+3.0\times10^{6} & +2.4\times10^{8} & -3.0\times10^{6} & +1.2\times10^{2} \\
-5.0\times10^{4} & -3.0\times10^{6} & +5.0\times10^{4} & -3.0\times10^{6} \\
+3.0\times10^{6} & +1.2\times10^{8} & -3.0\times10^{6} & +2.4\times10^{8}
\end{bmatrix}
\end{array}$$

Since the end at coordinate 11 is constrained, cross-out first column and first row in compliance with the rule stated in part 3. 2. 1.

MEMBER JI

All terms due to unit displacements ($\theta = 1$, $\Delta = 1$)

$$\begin{array}{cc}
 & \begin{matrix} 11 & \quad 1 & \quad 12 & \quad 2 \end{matrix} \\
\begin{matrix} 11 \\ 1 \\ 12 \\ 2 \end{matrix} &
\begin{bmatrix}
+5.0\times10^{4} & +3.0\times10^{6} & -5.0\times10^{4} & +3.0\times10^{6} \\
+3.0\times10^{6} & +2.4\times10^{8} & -3.0\times10^{6} & +1.2\times10^{8} \\
-5.0\times10^{4} & -3.0\times10^{6} & +5.0\times10^{4} & -3.0\times10^{6} \\
+3.0\times10^{6} & +1.2\times10^{8} & -3.0\times10^{6} & +2.4\times10^{8}
\end{bmatrix}
\end{array}$$

Cross-out first column and first row as in Member AB, above.

MEMBER BI

All terms due to rotational displacements only, $\theta = 1$.

Linear displacement $\Delta = 0$

$$[k]_{BI} = \begin{array}{c|cc} & 2 & 7 \\ \hline 2 & +1.2\times10^{8} & +0.6\times10^{8} \\ 7 & +0.6\times10^{8} & +1.2\times10^{8} \end{array}$$

MEMBER BC

All terms due to unit displacements ($\theta = 1$, $\Delta = 1$)

$$\begin{array}{c|cccc} & 12 & 7 & 13 & 8 \\ \hline 12 & +7.5\times10^{4} & +4.5\times10^{6} & -7.5\times10^{4} & +4.5\times10^{6} \\ 7 & +4.5\times10^{6} & +3.6\times10^{8} & -4.5\times10^{6} & +1.8\times10^{8} \\ 13 & -7.5\times10^{4} & -4.5\times10^{6} & +7.5\times10^{4} & -4.5\times10^{6} \\ 8 & +4.5\times10^{6} & +1.8\times10^{8} & -4.5\times10^{6} & +3.6\times10^{8} \end{array}$$

MEMBER IH

All terms due to unit displacements ($\theta = 1$, $\Delta = 1$)

$$\begin{array}{c|cccc} & 12 & 2 & 13 & 3 \\ \hline 12 & +7.5\times10^{4} & +4.5\times10^{6} & -7.5\times10^{4} & +4.5\times10^{6} \\ 2 & +4.5\times10^{6} & +3.6\times10^{8} & -4.5\times10^{6} & +1.8\times10^{8} \\ 13 & -7.5\times10^{4} & -4.5\times10^{6} & +7.5\times10^{4} & -4.5\times10^{6} \\ 3 & +4.5\times10^{6} & +1.8\times10^{8} & -4.5\times10^{6} & +3.6\times10^{8} \end{array}$$

MEMBER CH

All terms caused by rotational displacement only, $\theta = 1$.

Linear displacement $\Delta = 0$.

$$[k]_{CH} = \begin{array}{c} \\ 3 \\ 8 \end{array} \begin{array}{c} \begin{array}{cc} 3 & 8 \end{array} \\ \begin{bmatrix} +1.2 \times 10^{8} & +0.6 \times 10^{8} \\ +0.6 \times 10^{8} & +1.2 \times 10^{8} \end{bmatrix} \end{array}$$

MEMBER CD

All terms due to unit displacements $(\theta = 1,\ \Delta = 1)$

$$\begin{array}{c} \\ 13 \\ 8 \\ 14 \\ 9 \end{array} \begin{array}{c} \begin{array}{cccc} 13 & 8 & 14 & 9 \end{array} \\ \begin{bmatrix} +7.5 \times 10^{4} & +4.5 \times 10^{6} & -7.5 \times 10^{4} & +4.5 \times 10^{6} \\ +4.5 \times 10^{6} & +3.6 \times 10^{8} & -4.5 \times 10^{6} & +1.8 \times 10^{8} \\ -7.5 \times 10^{4} & +4.5 \times 10^{6} & +7.5 \times 10^{4} & -4.5 \times 10^{6} \\ +4.5 \times 10^{6} & +1.8 \times 10^{8} & -4.5 \times 10^{6} & +3.6 \times 10^{8} \end{bmatrix} \end{array}$$

MEMBER HG

All terms due to unit displacements $(\theta = 1,\ \Delta = 1)$

$$\begin{array}{c} \\ 13 \\ 3 \\ 14 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 13 & 3 & 14 & 4 \end{array} \\ \begin{bmatrix} +7.5 \times 10^{4} & +4.5 \times 10^{6} & -7.5 \times 10^{4} & +4.5 \times 10^{6} \\ +4.5 \times 10^{6} & +3.6 \times 10^{8} & -4.5 \times 10^{6} & +1.8 \times 10^{8} \\ -7.5 \times 10^{4} & -4.5 \times 10^{6} & +7.5 \times 10^{4} & -4.5 \times 10^{6} \\ +4.5 \times 10^{6} & +1.8 \times 10^{8} & -4.5 \times 10^{6} & +3.6 \times 10^{8} \end{bmatrix} \end{array}$$

MEMBER DG

All terms caused by rotational displacement only.

$$[k]_{DG} = \begin{array}{cc} & \begin{array}{cc} \quad 4 \quad & \quad 9 \quad \end{array} \\ \begin{array}{c} 4 \\ 9 \end{array} & \begin{bmatrix} +1.2\times10^{8} & +0.6\times10^{8} \\ +0.6\times10^{8} & +1.2\times10^{8} \end{bmatrix} \end{array}$$

MEMBER DE

All terms due to unit displacements ($\theta = 1$, $\Delta = 1$)

$$\begin{array}{cc} & \begin{array}{cccc} \quad 14 \quad & \quad 9 \quad & \quad 15 \quad & \quad 10 \quad \end{array} \\ \begin{array}{c} 14 \\ 9 \\ 15 \\ 10 \end{array} & \begin{bmatrix} +5.0\times10^{4} & +3.0\times10^{6} & -5.0\times10^{4} & +3.0\times10^{6} \\ +3.0\times10^{6} & -2.4\times10^{8} & -3.0\times10^{6} & +1.2\times10^{8} \\ -5.0\times10^{4} & -3.0\times10^{6} & +5.0\times10^{4} & -3.0\times10^{6} \\ +3.0\times10^{6} & +1.2\times10^{8} & -3.0\times10^{6} & +2.4\times10^{8} \end{bmatrix} \end{array}$$

Since the end at coordinate 15 is constrained, cross-out third column and third row in accordance with the rule stated in part 3.2.1.

MEMBER GF

All terms due to unit displacements ($\theta = 1$, $\Delta = 1$)

$$\begin{array}{cc} & \begin{array}{cccc} \quad 14 \quad & \quad 4 \quad & \quad 15 \quad & \quad 5 \quad \end{array} \\ \begin{array}{c} 14 \\ 4 \\ 5 \\ 15 \end{array} & \begin{bmatrix} +5.0\times10^{4} & +3.0\times10^{6} & -5.0\times10^{4} & +3.0\times10^{6} \\ +3.0\times10^{6} & +2.4\times10^{8} & -3.0\times10^{6} & +1.2\times10^{8} \\ -5.0\times10^{4} & -3.0\times10^{6} & +5.0\times10^{4} & -3.0\times10^{6} \\ +3.0\times10^{6} & +1.2\times10^{8} & -3.0\times10^{6} & +2.4\times10^{8} \end{bmatrix} \end{array}$$

Cross-out third column and third row as in Member DE, above.

MEMBER EF

All terms due to rotational displacement only.

$$[k]_{EF} = \begin{array}{c|cc} & 5 & 10 \\ \hline 5 & +1.2\times10^{8} & +0.6\times10^{8} \\ 10 & +0.6\times10^{8} & +1.2\times10^{8} \end{array}$$

5.1.2. Deflections and Rotations

Once the stiffness matrix Equation for the composite structure has been established, Table 5.1.1, the next step would be

$$[K] \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 12 \\ 13 \\ 14 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -6,000 \\ -10,000 \\ -6,000 \end{bmatrix}$$

or

$$[K]\{\Delta\} = \{P\}$$

to determine the joint deflections and rotations of the truss members by the use of the IBM 1620 computer.

The $[K]$ matrix of stiffness coefficients are punched onto control data cards. The computer is then instructed to invert the

Table 5.1.1. Combined Stiffness Matrix for Entire Structure

i \ j	1	2	3	4	5	6	7	8	9	10	12	13	14
1	$+3.6x10^{8}$	$+1.2x10^{8}$	0	0	0	$+0.6x10^{8}$	0	0	0	0	$-3.0x10^{6}$	0	0
2	$+1.2x10^{8}$	$+7.2x10^{8}$	$+1.8x10^{8}$	0	0	0	$+0.6x10^{8}$	0	0	0	$+1.5x10^{6}$	$-4.5x10^{6}$	0
3	0	$+1.8x10^{8}$	$+8.4x10^{8}$	$+1.8x10^{8}$	0	0	0	$+0.6x10^{8}$	0	0	$+4.5x10^{6}$	0	$-4.5x10^{6}$
4	0	0	$+1.8x10^{8}$	$+7.2x10^{8}$	$+1.2x10^{8}$	0	0	0	$+0.6x10^{8}$	0	0	$+4.5x10^{6}$	$-1.5x10^{6}$
5	0	0	0	$+1.2x10^{8}$	$+3.6x10^{8}$	0	0	0	0	$+0.6x10^{8}$	0	0	$+3.0x10^{6}$
6	$+0.6x10^{8}$	0	0	0	0	$+3.6x10^{8}$	$+1.2x10^{8}$	0	0	0	$-3.0x10^{6}$	0	0
7	0	$+0.6x10^{8}$	0	0	0	$+1.2x10^{8}$	$+7.2x10^{8}$	$+1.8x10^{8}$	0	0	$+1.5x10^{6}$	$-4.5x10^{6}$	0
8	0	0	$+0.6x10^{8}$	0	0	0	$+1.8x10^{8}$	$+8.4x10^{8}$	$+1.8x10^{8}$	0	$+4.5x10^{6}$	0	$-4.5x10^{6}$
9	0	0	0	$+0.6x10^{8}$	0	0	0	$+1.8x10^{8}$	$+7.2x10^{8}$	$+1.2x10^{8}$	0	$+4.5x10^{6}$	$-1.5x10^{6}$
10	0	0	0	0	$+0.6x10^{8}$	0	0	0	$+1.2x10^{8}$	$+3.6x10^{8}$	0	0	$+3.0x10^{6}$
12	$-3.0x10^{6}$	$+1.5x10^{6}$	$+4.5x10^{6}$	0	0	$-3.0x10^{6}$	$+1.5x10^{6}$	$+4.5x10^{6}$	0	0	$+25.0x10^{4}$	$-15.0x10^{4}$	0
13	0	$-4.5x10^{6}$	0	$+4.5x10^{6}$	0	0	$-4.5x10^{6}$	0	$+4.5x10^{6}$	0	$-15.0x10^{4}$	$+30.0x10^{4}$	$-15.0x10^{4}$
14	0	0	$-4.5x10^{6}$	$-1.5x10^{6}$	$+3.0x10^{6}$	0	0	$-4.5x10^{6}$	$-1.5x10^{6}$	$+3.0x10^{6}$	0	$-15.0x10^{4}$	$+25.0x10^{4}$

Table 5.1.2. Flexibility Matrix

Row 1	Row 2	Row 3	Row 4	Row 5
0.37587090×10^{-8}	-0.28632452×10^{-9}	-0.17729004×10^{-9}	-0.35635246×10^{-9}	-0.22121566×10^{-9}
-0.28632458×10^{-9}	0.20488247×10^{-8}	-0.33218021×10^{-9}	-0.35546965×10^{-9}	-0.35635265×10^{-9}
-0.17729002×10^{-9}	-0.33218023×10^{-9}	0.17588598×10^{-8}	-0.33218047×10^{-9}	-0.17729027×10^{-9}
-0.35635246×10^{-9}	-0.35546966×10^{-9}	-0.33218045×10^{-9}	0.20488250×10^{-8}	-0.28632437×10^{-9}
-0.22121567×10^{-9}	-0.35635266×10^{-9}	-0.17729027×10^{-9}	-0.28632440×10^{-9}	0.37587093×10^{-8}
0.14340643×10^{-9}	0.41859955×10^{-9}	-0.35180998×10^{-9}	-0.30502301×10^{-9}	-0.24174740×10^{-9}
0.41859959×10^{-9}	0.28651492×10^{-9}	0.10411984×10^{-9}	-0.48379323×10^{-9}	-0.30502324×10^{-9}
-0.35181004×10^{-9}	0.10411987×10^{-9}	0.27543972×10^{-9}	0.10411967×10^{-9}	-0.35181028×10^{-9}
-0.30502305×10^{-9}	-0.48379321×10^{-9}	0.10411965×10^{-9}	0.28651486×10^{-9}	0.41859972×10^{-9}
-0.24174744×10^{-9}	-0.30502324×10^{-9}	-0.35181025×10^{-9}	0.41859977×10^{-9}	0.14340688×10^{-9}
0.10912694×10^{-6}	0.55966058×10^{-7}	-0.41598208×10^{-7}	-0.63081542×10^{-7}	-0.45634934×10^{-7}
0.89285493×10^{-7}	0.10714286×10^{-6}	0.31000000×10^{-13}	-0.10714296×10^{-6}	-0.89285740×10^{-7}
0.45634927×10^{-7}	0.63081570×10^{-7}	0.41598253×10^{-7}	-0.55966080×10^{-7}	-0.10912700×10^{-6}

Row 6	Row 7	Row 8	Row 9	Row 10
0.14340643×10^{-9}	0.41859961×10^{-9}	-0.35181004×10^{-9}	-0.30502305×10^{-9}	-0.24174744×10^{-9}
0.41859952×10^{-9}	0.28651491×10^{-9}	0.10411990×10^{-9}	-0.48379322×10^{-9}	-0.30502326×10^{-9}
-0.35180998×10^{-9}	0.10411987×10^{-9}	0.27543972×10^{-9}	0.10411964×10^{-9}	-0.35181028×10^{-9}
-0.30502302×10^{-9}	-0.48379324×10^{-9}	0.10411966×10^{-9}	0.28651488×10^{-9}	0.41859977×10^{-9}
-0.24174741×10^{-9}	-0.30502327×10^{-9}	-0.35181026×10^{-9}	0.41859974×10^{-9}	0.14340691×10^{-9}
0.37587085×10^{-8}	-0.28632427×10^{-9}	-0.17729004×10^{-9}	-0.35635243×10^{-9}	-0.22121566×10^{-9}
-0.28632428×10^{-9}	0.20488249×10^{-8}	-0.33218021×10^{-9}	-0.35546972×10^{-9}	-0.35635267×10^{-9}
-0.17729006×10^{-9}	-0.33218021×10^{-9}	0.17588600×10^{-8}	-0.33218042×10^{-9}	-0.17729025×10^{-9}
-0.35635242×10^{-9}	-0.35546972×10^{-9}	-0.32218041×10^{-9}	0.20488250×10^{-8}	-0.28632427×10^{-9}
-0.22121564×10^{-9}	-0.35635267×10^{-9}	-0.17729023×10^{-9}	-0.28632426×10^{-9}	0.37587093×10^{-8}
0.10912692×10^{-6}	0.55966068×10^{-7}	-0.41598209×10^{-7}	-0.63081544×10^{-7}	-0.45634937×10^{-7}
0.89285683×10^{-7}	0.10714287×10^{-6}	0.29000000×10^{-13}	-0.10714286×10^{-6}	-0.89285750×10^{-7}
0.45634921×10^{-7}	0.63081575×10^{-7}	0.41598251×10^{-7}	-0.55966082×10^{-7}	-0.10912700×10^{-6}

Table 5.1.2. Flexibility Matrix (Continued)

Row 11	Row 12	Row 13
$0.10912694 \times 10^{-6}$	$0.89285695 \times 10^{-7}$	$0.45634926 \times 10^{-7}$
$0.55966052 \times 10^{-7}$	$0.10714285 \times 10^{-6}$	$0.63081568 \times 10^{-7}$
$-0.41598206 \times 10^{-7}$	$0.35000000 \times 10^{-13}$	$0.41598255 \times 10^{-7}$
$-0.63081540 \times 10^{-7}$	$-0.10714286 \times 10^{-6}$	$-0.55966079 \times 10^{-7}$
$-0.45634935 \times 10^{-7}$	$-0.89285750 \times 10^{-7}$	$-0.10912700 \times 10^{-6}$
$0.10912692 \times 10^{-6}$	$0.89285689 \times 10^{-7}$	$0.45634922 \times 10^{-7}$
$0.55966064 \times 10^{-7}$	$0.10714287 \times 10^{-6}$	$0.63081570 \times 10^{-7}$
$-0.41598211 \times 10^{-7}$	$0.30000000 \times 10^{-13}$	$0.41598252 \times 10^{-7}$
$-0.63081541 \times 10^{-7}$	$-0.10714286 \times 10^{-6}$	$-0.55966079 \times 10^{-7}$
$-0.45634934 \times 10^{-7}$	$-0.89285750 \times 10^{-7}$	$-0.10912699 \times 10^{-6}$
$0.17516412 \times 10^{-4}$	$0.16785712 \times 10^{-4}$	$0.89121525 \times 10^{-5}$
$0.16785711 \times 10^{-4}$	$0.26547621 \times 10^{-4}$	$0.16785718 \times 10^{-4}$
$0.89121525 \times 10^{-5}$	$0.16785719 \times 10^{-4}$	$0.17516423 \times 10^{-4}$

$[K]$ matrix and multiply the result of this inversion with the $\{P\}$ matrix. The deflections and rotations are thus determined as the product of the flexibility matrix and the applied load vector

$$[K]\{\Delta\} = \{P\}$$
$$\{\Delta\} = [K]^{-1}\{P\}$$

When the structure is subjected to a different loading condition the stiffness matrix remains unchanged because it is a function only of the geometric and elastic properties of the structure. The different loading condition only changes the values of fixed-end forces and moments. Consequently, only the values of $\{P\}$ matrix need be changed in the input data if an analysis of the structure is desired under any number of loading conditions. Hence, the inversion process of the stiffness matrix of the structure would be by-passed because the stiffness matrix would remain unchanged.

Table 5.1.3. Deflections

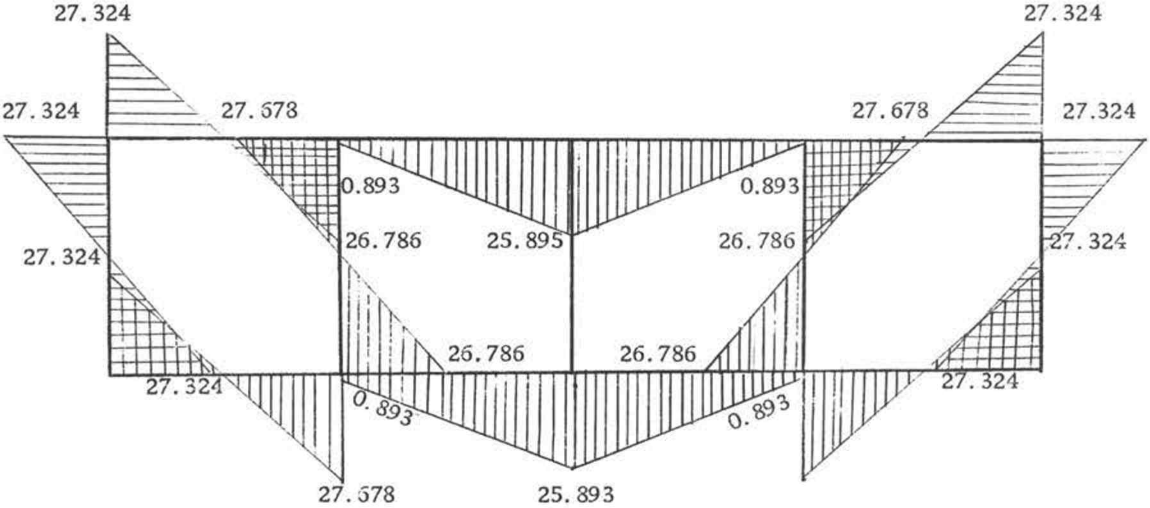
Coordinate	Deflection	Coordinate	Deflection
1	$-0.18214281 \times 10^{-2}$ rad	8	$-0.57750000 \times 10^{-9}$ rad
2	$-0.17857143 \times 10^{-2}$ rad	9	$0.17857142 \times 10^{-9}$ rad
3	$-0.57750000 \times 10^{-9}$ rad	10	$0.18214291 \times 10^{-9}$ rad
4	$0.17857142 \times 10^{-2}$ rad	12	-0.32642849 inches
5	$0.18214290 \times 10^{-2}$ rad	13	-0.46690479 inches
6	$-0.18214279 \times 10^{-2}$ rad	14	-0.32642863 inches
7	$-0.17857145 \times 10^{-2}$ rad		

5.1.3. End Moments and Shears

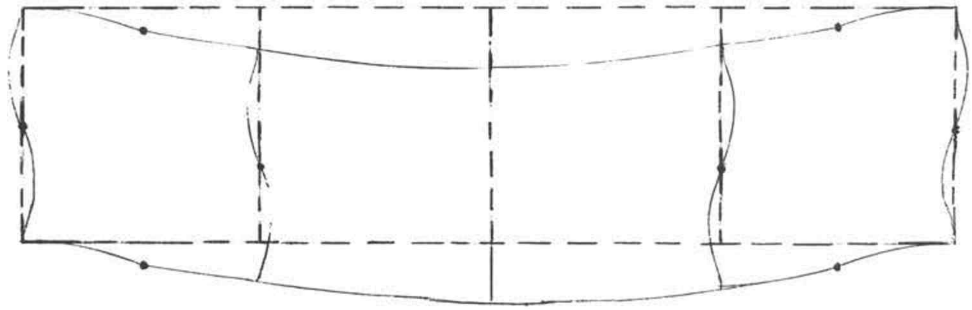
Having determined the values of rotations and deflections as outlined previously, the end shears and moments are determined by multiplying the original stiffness matrices of the individual members with the appropriate deflections. Appendix A shows the input and output data of the computer used for these operations. For clarity, the computed end shears and moments have been conveniently converted to kips and ft. kips respectively and tabulated in Table 5.1.4.

Table 5.1.4. Computed End Shears and Moments

Member	Moment (l. e.) ft. - kp.	Shear kp.	Moment (r. e.) ft. - kp.
AJ	- 27.324	---	- 27.324
AB	+ 27.324	± 5.500	+ 27.678
IJ	+ 27.678	± 5.500	+ 27.324
BI	- 26.786	---	- 26.786
BC	- 0.893	± 2.500	+ 25.893
HI	- 0.893	± 2.500	+ 25.893
CH	- 0.008	---	- 0.008
CD	- 25.893	± 2.500	+ 0.893
GH	- 25.893	± 2.500	+ 0.893
DG	+ 26.786	---	+ 26.786
DE	- 27.324	± 5.500	- 27.678
FG	- 27.324	± 5.500	- 27.678
EF	+ 27.324	---	+ 27.324



(a)



(b)

Figure 5.1.3. (a) Moment diagram, and (b) Deflected structure

5.2. By Moment Distribution

The Vierendeel truss that has been analyzed in part 5.1 by the stiffness method will now be analyzed by the moment distribution method.

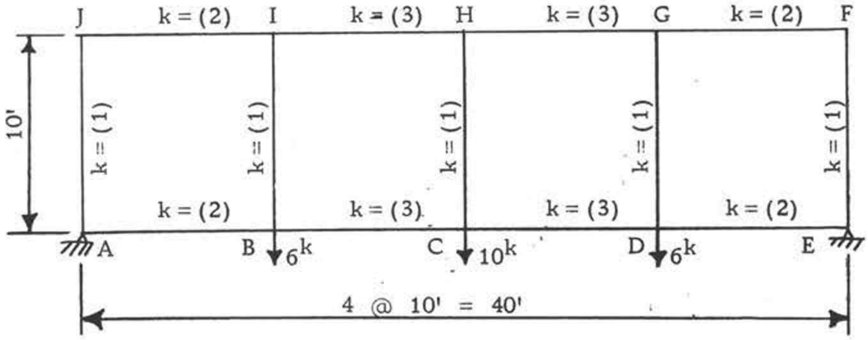


Figure 5.2.1. Symmetrically loaded Vierendeel Truss

DISTRIBUTION FACTORS, $r = \frac{k}{\Sigma k}$

Joints A, E, F, and J

$$r_{AJ} = r_{JA} = r_{EF} = r_{FE} = \frac{1}{3}, \quad r_{AB} = r_{ED} = r_{FG} = r_{JI} = \frac{2}{3}$$

Joints B, D, G, and I

$$r_{BA} = r_{DE} = r_{GF} = r_{IJ} = \frac{1}{3}, \quad r_{BI} = r_{IB} = r_{DG} = r_{JD} = \frac{1}{6}$$

$$r_{BC} = r_{DC} = r_{IH} = r_{GH} = \frac{1}{2}$$

Joints C and H

$$r_{CB} = r_{CD} = r_{HI} = r_{HG} = \frac{3}{7}, \quad r_{CH} = r_{HC} = \frac{1}{7}$$

CASE I - Let Joint B deflect $\Delta B_1 E = 10$

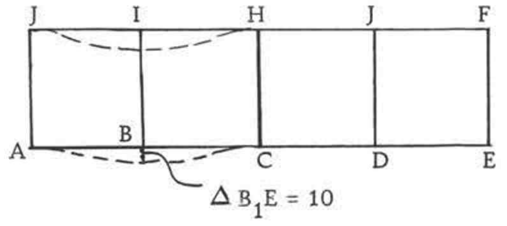


Figure 5. 2. 2. Allowing Joint B to Deflect

Fixed End Moments

$$M^F_{AB} = M^F_{BA} = M^F_{JI} = M^F_{IJ} = \frac{6\,E\,\Delta K}{L} = \frac{6\,(10)(2)}{10} = \overset{+}{12}\text{ k}$$

$$M^F_{BC} = M^F_{CB} = M^F_{IH} = M^F_{HI} = \frac{6\,E\,\Delta K}{L} = \frac{6\,(10)(3)}{10} = \overset{-}{18}\text{ k}$$

CASE II - Let Joint C deflect $\Delta C_1 E = 10$

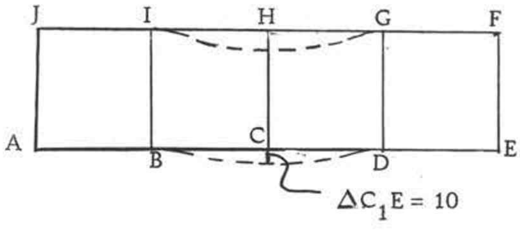


Figure 5. 2. 3. Allowing Joint C to Deflect

Fixed End Moments

$$M^F_{BC} = M^F_{CB} = M^F_{HI} = M^F_{IH} = \frac{6E\,\Delta K}{L} = \frac{6\ (10)(3)}{10} = 18^k$$

$$M^F_{CD} = M^F_{DC} = M^F_{HG} = M^F_{GH} = \frac{6E\,\Delta K}{L} = \frac{6\ (10)\ 3}{10} = 18^k$$

CASE III - Let Joint D Deflect $\Delta D_1 E = 10$

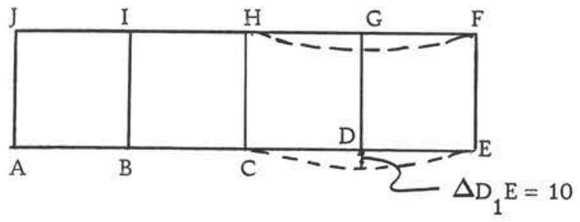


Figure 5. 2. 4. Allowing Joint D to Deflect

Since structure is symmetrically loaded, the deflections at joines B and D are equal. Hence, Moment Distribution of Case I may be applied in Case III provided proper consideration is given to sign changes of deflected structure.

Table 5.2.1. Moment Distribution - Case I

J	
JA	JI
1/3	2/3
0	+12.00
-4.00	- 8.00
-2.00	+ 1.00
+0.33	+ 0.67
+0.17	- 0.06
-0.04	- 0.07
-0.02	+ 0.05
-0.01	- 0.02
-5.57	+ 5.57

I		
IJ	IB	IH
1/3	1/6	1/2
+12.00	0	-18.00
+ 2.00	+1.00	+ 3.00
- 4.00	+0.50	+ 3.86
- 0.12	-0.06	- 0.18
+ 0.34	-0.03	- 0.60
+ 0.10	+0.05	+ 0.14
- 0.04	+0.02	+ 0.27
- 0.08	-0.04	- 0.13
+10.20	+1.44	-11.64

H		
HI	HC	HG
3/7	1/7	3/7
-18.00	0	0
+ 7.71	+2.58	+7.71
+ 1.50	+1.29	0
- 1.20	-0.40	-1.19
- 0.09	-0.20	-0.96
+ 0.54	+0.18	+0.53
+ 0.07	+0.09	+0.23
- 0.17	-0.06	-0.16
- 9.64	+3.48	+6.16

G		
GH	GD	GF
1/2	1/6	1/3
0	0	0
0	0	0
+3.86	0	0
-1.93	-0.64	-1.29
-0.60	-0.32	0
+0.46	+0.15	+0.31
+0.26	+0.08	+0.22
-0.28	-0.09	-0.19
+1.77	-0.82	-0.95

F	
FG	FE
2/3	1/3
0	0
0	0
0	0
0	0
-0.65	0
+0.43	+0.22
+0.16	+0.11
-0.18	-0.09
-0.24	+0.24

A	
AJ	AB
1/3	2/3
0	+12.00
-4.00	- 8.00
-2.00	+ 1.00
+0.33	+ 0.67
+0.17	- 0.06
-0.04	- 0.07
-0.02	+ 0.05
-0.01	- 0.02
-5.57	+ 5.57

B		
BA	BI	BC
1/3	1/6	1/2
+12.00	0	-18.00
+ 2.00	+1.00	+ 3.00
- 4.00	+0.50	+ 3.86
- 0.12	-0.06	- 0.18
+ 0.34	-0.03	- 0.60
+ 0.10	+0.05	+ 0.14
- 0.04	+0.02	+ 0.27
- 0.08	-0.04	- 0.13
+10.20	+1.44	-11.64

C		
CB	CH	CD
3/7	1/7	3/7
-18.00	0	0
+ 7.71	+2.58	+7.71
+ 1.50	+1.29	0
- 1.20	-0.40	-1.19
- 0.09	-0.20	-0.96
+ 0.54	+0.18	+0.53
+ 0.07	+0.09	+0.23
- 0.17	-0.06	-0.16
- 9.64	+3.48	+6.16

D		
DC	DG	DE
1/2	1/6	1/3
0	0	0
0	0	0
+3.86	0	0
-1.93	-0.64	-1.29
-0.60	-0.32	0
+0.46	+0.15	+0.31
+0.26	+0.08	+0.22
-0.28	-0.09	-0.19
+1.77	-0.82	-0.95

E	
ED	EF
2/3	1/3
0	0
0	0
0	0
0	0
-0.65	0
+0.43	+0.22
+0.16	+0.11
-0.18	-0.09
-0.24	+0.24

Table 5.2.2. Moment Distribution - Case II

J	
JA	JI
1/3	2/3
0	0
0	0
0	-3.00
+1.00	+2.00
+0.50	+0.25
-0.25	-0.50
-0.13	-0.19
+0.11	+0.21
+1.23	-1.23

I		
IJ	IB	IH
1/3	1/6	1/2
0	0	+18.00
-6.00	-3.00	- 9.00
0	-1.50	0
+0.50	+0.25	+ 0.75
+1.00	+0.13	0
-0.37	-0.19	- 0.57
-0.25	-0.10	0
+0.12	+0.06	+ 0.17
-5.00	-4.35	+ 9.35

H		
HI	HC	HG
3/7	1/7	3/7
+18.00	0	-18.00
0	0	0
- 4.50	0	+ 4.50
0	0	0
+ 0.38	0	- 0.38
0	0	0
- 0.28	0	+ 0.28
0	0	0
+13.60	0	-13.60

G		
GH	GD	GF
1/2	1/6	1/3
-18.00	0	0
+ 9.00	+3.00	+6.00
0	+1.50	0
- 0.75	-0.25	-0.50
0	-0.13	-1.00
+ 0.57	+0.19	+0.37
0	+0.10	+0.25
- 0.17	-0.06	-0.12
- 9.35	+4.35	+5.00

F	
FG	FE
2/3	1/3
0	0
0	0
+3.00	0
-2.00	-1.00
-0.25	-0.56
+0.50	+0.25
+0.19	+0.13
-0.21	-0.11
+1.23	-1.23

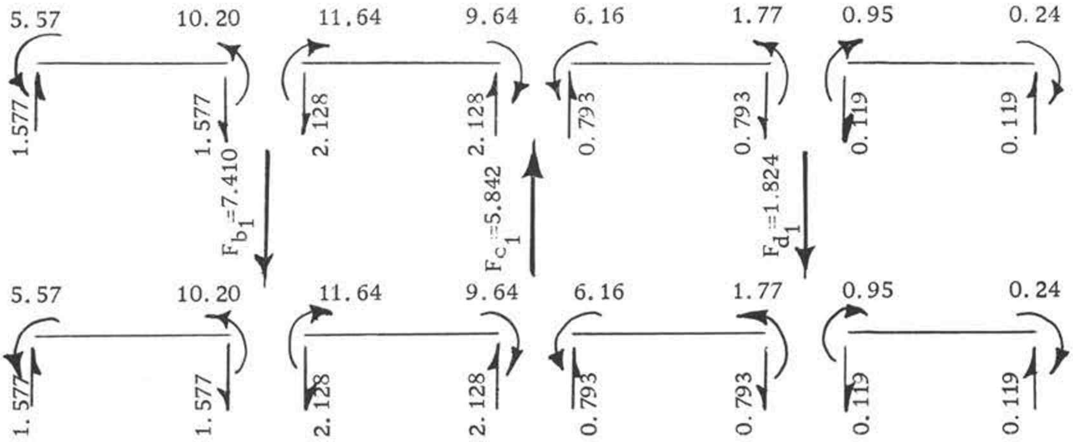
A	
AJ	AB
1/3	2/3
0	0
0	0
0	-3.00
+1.00	+2.00
+0.50	+0.25
-0.25	-0.50
-0.13	-0.19
+0.11	+0.21
+1.23	-1.23

B		
BA	BI	BC
1/3	1/6	1/2
0	0	+18.00
-6.00	-3.00	- 9.00
0	-1.50	0
+0.50	+0.25	+ 0.75
+1.00	+0.13	0
-0.37	-0.19	- 0.57
-0.25	-0.10	0
+0.12	+0.06	+ 0.17
-5.00	-4.35	+ 9.35

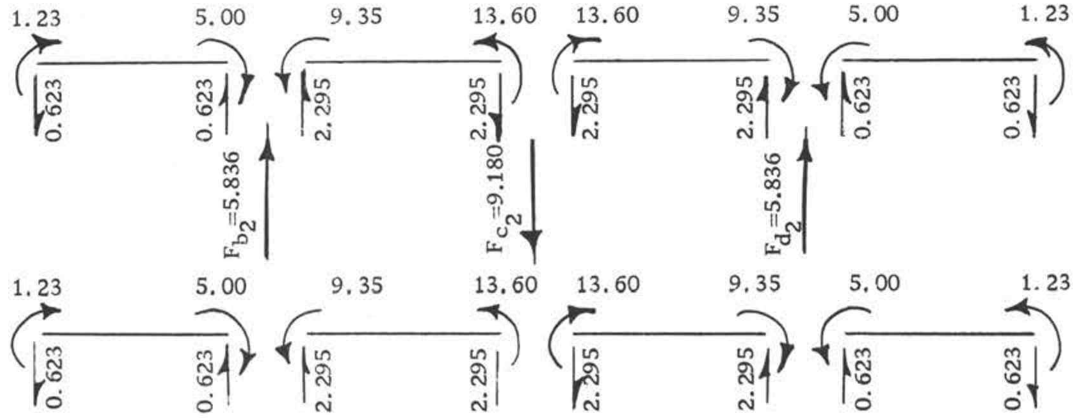
C		
CB	CH	CD
3/7	1/7	3/7
+18.00	0	-18.00
0	0	0
- 4.50	0	+ 4.50
0	0	0
+ 0.38	0	- 0.38
0	0	0
- 0.28	0	+ 0.28
0	0	0
+13.60	0	-13.60

D		
DC	DG	DE
1/2	1/6	1/3
-18.00	0	0
+ 9.00	+3.00	+6.00
0	+1.50	0
- 0.75	-0.25	-0.50
0	-0.13	-1.00
+ 0.57	+0.19	+0.37
0	+0.10	+0.25
- 0.17	-0.06	-0.12
- 9.35	+4.35	+5.00

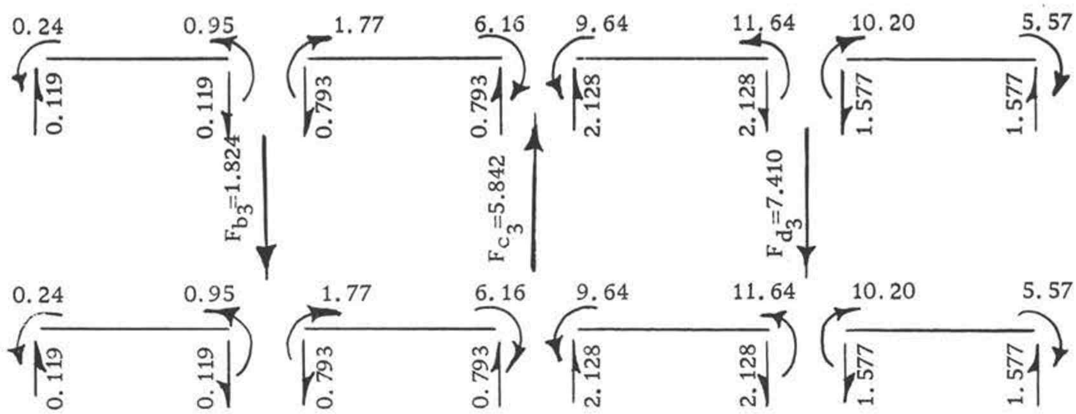
E	
ED	EF
2/3	1/3
0	0
0	0
+3.00	0
-2.00	-1.00
-0.25	-0.50
+0.50	+0.25
+0.19	+0.13
-0.21	-0.11
+1.23	-1.23



(a) Case I



(b) Case II



(c) Case III

Figure 5. 2. 5 Girder Shears

<u>Solve For Correction Factors</u>

The following equations may be formed

$$\Sigma F_b \qquad 7.410\, k_1 - 5.836\, k_2 + 1.824\, k_3 = 6 \qquad (1)$$

$$\Sigma F_c \qquad -5.842\, k_1 + 9.180\, k_2 - 5.842\, k_3 = 10 \qquad (2)$$

$$\Sigma F_d \qquad 1.824\, k_1 - 5.836\, k_2 + 7.410\, k_3 = 6 \qquad (3)$$

$$k_1 - 0.788\, k_2 + 0.246\, k_3 = 0.810 \qquad (1a)$$

$$-k_1 + 1.571\, k_2 - k_3 = 1.712 \qquad (2a)$$

$$+ 0.783\, k_3 - 0.754\, k_3 = 2.522 \qquad (4)$$

$$-k_1 + 1.571\, k_2 - k_3 = 1.712 \qquad (2a)$$

$$k_1 - 3.200\, k_2 + 4.063\, k_3 = 3.290 \qquad (3a)$$

$$- 1.629\, k_2 + 3.063\, k_3 = 5.002 \qquad (5)$$

$$k_2 - 0.963\, k_3 = 3.221 \qquad (4a)$$

$$-k_2 + 1.880\, k_3 = 3.071 \qquad (5a)$$

$$0.917\, k_3 = 6.292 \qquad (5a)$$

$$\boxed{k_3 = 6.842}$$

$$k_2 - 0.963\,(6.862) = 3.221 \qquad (4a)$$

$$k_2 - 6.576 = 3.221$$

$$\boxed{k_2 = 9.797}$$

$$k_1 - 0.788\,(9.829) + 0.246\,(6.862) = 0.810 \qquad (1a)$$

$$k_1 - 7.723 + 1.691 = 0.810$$

$$\boxed{k_1 = 6.842}$$

Table 5.2.3. Final Moments

Member	I	II	III	IV	V	VI	Final Moments
	$M_{case\ I}$	$M_{case\ II}$	$M_{case\ III}$	$M_I \times 6.842$	$M_{II} \times 9.797$	$M_{III} \times 6.842$	IV + V + VI
JA	- 5.57	+ 1.23	- 0.24	-38.110	+ 12.050	- 1.642	-27.702
JI	+ 5.57	- 1.23	+ 0.24	+38.110	- 12.050	+ 1.642	+27.702
IJ	+10.20	- 5.00	+ 0.95	+69.788	- 48.985	+ 6.500	+27.303
IB	+ 1.44	- 4.35	+ 0.82	+ 9.852	- 42.617	+ 5.610	-27.155
IH	-11.64	+ 9.35	- 1.77	-79.641	+ 91.602	-12.110	- 0.149
HI	- 9.64	+13.60	- 6.16	-65.957	+133.239	-42.147	+25.135
HC	+ 3.48	0	- 3.48	+23.810	0	-23.810	0
HG	+ 6.16	-13.60	+ 9.64	+42.147	-133.239	+65.957	-25.135
GH	+ 1.77	- 9.35	+11.64	+12.110	- 91.602	+79.641	+ 0.149
GD	- 0.82	+ 4.35	- 1.44	- 5.610	+ 42.617	- 9.852	+27.155
GF	- 0.95	+ 5.00	-10.20	- 6.500	+ 48.985	-69.788	-27.303
FG	- 0.24	+ 1.23	- 5.57	- 1.642	+ 12.050	-38.110	-27.702
FE	+ 0.24	- 1.23	+ 5.57	+ 1.642	- 12.050	+38.110	+27.702
AJ	- 5.57	+ 1.23	- 0.24	-38.110	+ 12.050	- 1.642	-27.702
AB	+ 5.57	- 1.23	+ 0.24	+38.110	- 12.050	+ 1.642	+27.702
BA	+10.20	- 5.00	+ 0.95	+69.788	- 48.985	+ 6.500	+27.303
BI	+ 1.44	- 4.35	+ 0.82	+ 9.852	- 42.617	+ 5.610	-27.155
BC	-11.64	+ 9.35	- 1.77	-79.641	+ 91.602	-12.110	- 0.149
CB	- 9.64	+13.60	- 6.16	-65.957	+133.239	-42.147	+25.135
CH	+ 3.48	0	- 3.48	+23.810	0	-23.810	0
CD	+ 6.16	-13.60	+ 9.69	+42.147	-133.239	+65.957	-25.135
DC	+ 1.77	- 9.35	+11.64	+12.110	- 91.602	+79.641	+ 0.149
DG	- 0.82	+ 4.35	- 1.44	- 5.610	+ 42.617	- 9.852	+27.155
DE	- 0.95	+ 5.00	-10.20	- 6.500	+ 48.985	-69.788	-27.303
ED	- 0.24	+ 1.23	- 5.57	- 1.642	+ 12.050	-38.110	-27.702
EF	+ 0.24	- 1.23	+ 5.57	+ 1.642	- 12.050	+38.110	+27.702

Comparison of Methods

To facilitate comparison of methods, Table 5.2.4 has been prepared showing the computed end shears and moments of the Vierendeel truss, (a) by the stiffness matrix method, and (b) by the moment distribution method.

Table 5.2.4. Stresses

	STIFFNESS METHOD			MOMENT DISTRIBUTION METHOD		
	Moment (l.e)* ft. - kp	Shear kp.	Moment (r.e) ft. ~ kp	Moment (l.e.) ft. ~ kp	Shear kp.	Moment (r.e.) ft. ~ kp
AJ	- 27.324	---	- 27.324	- 27.702	---	- 27.702
AB	+ 27.324	± 5.500	+ 27.678	+ 27.702	± 5.500	+ 27.303
IJ	+ 27.678	± 5.500	+ 27.324	+ 27.702	± 5.500	+ 27.303
BI	- 26.786	---	- 26.786	- 27.155	---	27.155
BC	- 0.893	± 2.500	+ 25.893	- 0.149	± 2.499	+ 25.135
HI	- 0.893	± 2.500	+ 25.893	- 0.149	± 2.499	+ 25.135
CH	- 0.008	---	- 0.008	0	---	0
CD	- 25.893	± 2.500	+ 0.893	- 25.135	± 2.499	+ 0.149
GH	- 25.893	± 2.500	+ 0.893	- 25.135	± 2.499	+ 0.149
DG	+ 26.786	---	+ 26.786	+ 27.155	---	+ 27.155
DE	- 27.324	± 5.500	- 27.678	- 27.303	± 5.500	- 27.702
FG	- 27.324	± 5.500	- 27.678	- 27.303	± 5.500	- 27.702
EF	+ 27.324	---	+ 27.324	+ 27.702	---	+ 27.702

* l.e. and r.e. refer to "left end" and "right end" respectively.

CONCLUSIONS

It has been demonstrated how complex structures can be analyzed by taking advantage of the capabilities of electronic computers. Great care must, however, be given to the preparation of data and punching of cards because the computer cannot check whether the data is right or wrong. A wrong data may lead to correct solution of a wrong problem.

For plane structures the stiffness matrix of a beam is a 4 x 4 matrix defined by Figure 3.2.3. From this, the analysis of structures with side-sways, i.e. Vierendeel trusses and tall buildings, becomes straight forward, and requires no supplementary equation.

Once the basic elastic and geometric properties of the structure and the loading patterns have been specified in matrix form, the complete solution of stresses and deflections may be obtained from strictly numerical matrix operations.

The economy of the matrix method of analysis will depend entirely on the availability of a high speed digital computer, and then only when a repetitive solution is required.

Any set of loading conditions can be inserted as the last step in the computations. Here is where the biggest advantage of the stiffness method occurs; it enables a complex structure subjected to any desired pattern of loading to be analyzed with a minimum of effort.

Further, changes in design are accounted for by locally correcting the stiffness matrix. As a result, several design configurations can be investigated without undue effort.

In concluding this paper, it may be said that electronic computers have great future for solving the problems of structural analysis and design. The ability, and skill, and experience of the engineer is in every way as essential when an electronic computer is used as when it is not used; but that skill and ability and experience has to be applied in a new way, enabling him to undertake analyses that would have been prohibitive in the past.

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APPENDICES

APPENDIX A

A-1. IBM 1620 FORTRAN Programming

The word "FORTRAN" is a contraction of "Formula Translation". It is an algebraic language compiler which has become available on the IBM 1620, 704, 709, and 650 as well as other machines of various manufacturers. It is, in a sense, a language designed for the description of algebraic problems and their solutions. Or, it may be said to be a language that is a compromise between the language of the computer and the language of the engineer (17, p. 5).

A FORTRAN program as prepared by the programmer is just another way of saying "series of instructions and fixed data." The program must be thorough, and every conceivable combination of circumstances must be explicitly stated as to what the computer is to do with the input data. Some of these instructions may be algebraic formulas, whereas others may be English language statements. There may be informative statements, informing the compiler what meaning is intended by the programmer, or there may be imperative statement, ordering the computer to execute a particular statement next in line, or read cards, or print out desired information, etc. This sequence of statements defining the ultimate operations of the computer is called the " source program, " which is key-punched onto standard IBM cards.

In this thesis, FOR-TO-GO was used to compile the subsequent matrix multiplication and matrix inversion programs. FOR-TO-GO compiles the required object program directly into memory until the END card is processed or an error is found. Detection of an error terminates compilation, but the remainder of the program is still checked for errors. A message is typed out when the END statement is read to indicate whether or not the program is acceptable.

If the program is not acceptable, an error deck will have been produced. It will start with a copy of the control card found at the beginning of the program, and will contain two error cards for each error. One of these cards will give an error code and the statement number of the statement in error; the other card will be a copy of the statement in error.

If the program is accepted, the object program is executed immediately. When the program terminates because of a STOP or END statement or because of an error, a message is typed on the console typewriter.

A-2. Digital Programs

A-2.1. Matrix Multiplication

```
C  C   MATRIX MULTIPLICATION
       DIMENSION A(N, N ), B( N), R( N)
  101  READ 1, N
    1  FORMAT (13)
       READ 102, ((A(I, J), I=1, N), J=1, N)
  102  FORMAT (E15. 8)
       READ 102, (B(I), I=1, N)
       DO 104 I =1, N
       R (I) = 0.
       DO 103 J = 1, N
  103  R (I) = R(I) + A (I, J)* B(J)
  104  CONTINUE
       PUNCH 102, (R(I), I = 1, N)
       GO TO 101
       END
```

A-2.2. Matrix Inversion

```
C  C   MATRIX INVERSION
       DIMENSION A (N, N)
       READ 1, N
    1  FORMAT (13)
       READ 101, ((A(I, J), I=1, N), J=1, N)
  101  FORMAT (E12. 5)
       DO 105 K = 1, N
       COM=A(K,K)
       A (K,K) = 1
       DO 102 J = 1, N
  102  A (K, J) = A(K, J)/COM
       DO 105 I = 1, N
       IF (I-K)103, 105, 103
  103  COM=A(I,K)
       A(I,K) = . 0
       DO 104 J = 1, N
  104  A (I, J) = A(I, J) - COM* A(K, J)
  105  CONTINUE
       PUNCH 106, ((A (I, J), I=1, N), J=1, N)
```

```
106     FORMAT (E15. 8)
        STOP
        END
```

A-3. Rigid Frame Bent

Input Data for Matrix Inversion

+ 1. 73600E-06	+ 0. 104170E-04	+ 1. 04170E-04
+ 1. 04170E-04	+ 5. 000000E-02	+ 1. 66670E-02
+ 1. 04170E-04	+ 1. 666700E-02	+ 5. 00000E-02

INVERTED MATRIX

0. 70899E+06	- 0. 11078E+04	- 0. 11078E+04
- 0. 11078E+04	0. 24231E+02	- 0. 57692E+01
- 0. 11078E+04	- 0. 57692E+01	0. 24231E+02

LOADING DATA (INPUT)

- 8. 00000E+03
0. 00000E+00
0. 00000E+00

DEFLECTIONS
(= PRODUCT OF INVERTED MATRIX AND LOADING)

- 0. 56719E+10
0. 88624E+07
0. 88624E+07

Table A-3. 1. Computing End Shears and Moments

MATRIX MULTIPLICATION INPUT DATA	MATRIX MULTIPLICATION RESULTS (OUTPUT)	
2		
+0. 86800E-06		
+1. 04170E-04		
+1. 04170E-04		
+1. 66670E-02	0. 40000E+04	V_A
-0. 56719E+10	-0. 51698E+06	M_A
+0. 88624E+07	-0. 40000E+04	V_B
2	0.44313E+06	M_B
-0. 86800E-06	0. 44313E+06	M_B
+1. 04170E-04	0. 44313E+06	M_C
-1. 04170E-04	-0. 40000E+04	V_C
+0. 83340E-02	-0. 44313E+06	M_C
-0. 56719E+10	0. 40000E+04	V_D
+0. 88624E+07	-0. 51698E+06	M_D
2		
+3. 33340E-02		
+1. 66670E-02		
+1. 66670E-02	Shears - lb.	
+3. 33340E-02	Moments - in. lb.	
+0. 88624E+07		
+0. 88624E+07		
2		
+0. 86800E-06		
+1. 04170E-04		
+1. 04170E-04		
+1. 66670E-02		
-0. 56719E+10		
+0. 88624E+07		
2		
-0. 86800E-06		
+1. 4170E-04		
-1. 04170E-04		
+0. 83340E-02		
-0. 56719E+10		
+0. 88624E+07		

Table A-4.1. Input Data for 13 x 13 Matrix Inversion

(1)	(2)	(3)	(4)	(5)
+3.60000E+08	+0.00000E+00	+0.00000E+00	+0.00000E+00	+0.00000E+00
+1.20000E+08	+0.00000E+00	+0.60000E+08	+0.00000E+00	+0.00000E+00
+0.00000E+00	+1.80000E+08	+0.00000E+00	+0.00000E+00	-4.50000E+06
+0.00000E+00	+7.20000E+08	+0.00000E+00	0.00000E+00	-1.50000E+06
+0.00000E+00	+1.20000E+08	+0.00000E+00	+0.60000E+08	+3.00000E+06
+0.60000E+08	+0.00000E+00	+1.20000E+08	+0.00000E+00	+0.00000E+00
+0.00000E+00	+0.00000E+00	+7.20000E+08	+0.00000E+00	+0.00000E+00
+0.00000E+00	+0.00000E+00	+1.80000E+08	+0.00000E+00	-4.50000E+06
+0.00000E+00	+0.60000E+08	+0.00000E+00	+1.20000E+08	-1.50000E+06
+0.00000E+00	+0.00000E+00	+0.00000E+00	+3.60000E+08	+3.00000E+06
-3.00000E+06	+0.00000E+00	+1.50000E+06	+0.00000E+00	+0.00000E+00
+0.00000E+00	+4.50000E+06	-4.50000E+06	+0.00000E+00	-1.50000E+05
+0.00000E+00	-1.50000E+06	+0.00000E+00	+3.00000E+06	+2.50000E+05
+1.20000E+08	+0.00000E+00	+0.00000E+00	-3.00000E+06	Last
+7.20000E+08	+0.00000E+00	+0.00000E+00	+1.50000E+06	Card
+1.80000E+08	+0.00000E+00	+0.60000E+08	+4.50000E+06	
+0.00000E+00	+1.20000E+08	+0.00000E+00	+0.00000E+00	
+0.00000E+00	+3.60000E+08	+0.00000E+00	+0.00000E+00	
+0.00000E+00	+0.00000E+00	+0.00000E+00	-3.00000E+06	
+0.60000E+08	+0.00000E+00	+1.80000E+08	+1.50000E+06	
+0.00000E+00	+0.00000E+00	+8.40000E+08	+4.50000E+06	
+0.00000E+00	+0.00000E+00	+1.80000E+08	+0.00000E+00	
+0.00000E+00	+0.60000E+08	+0.00000E+00	+0.00000E+00	
+1.50000E+06	+0.00000E+00	+4.50000E+06	+2.50000E+05	
-4.50000E+06	+0.00000E+00	+0.00000E+00	+1.50000E+05	
+0.00000E+00	+3.00000E+06	-4.50000E+06	+0.00000E+00	
+0.00000E+00	+0.60000E+08	0.00000E+00	+0.00000E+00	
+1.80000E+08	+0.00000E+00	+0.00000E+00	-4.50000E+06	
+8.40000E+08	+0.00000E+00	+0.00000E+00	+0.00000E+00	
+1.80000E+08	+0.00000E+00	+0.60000E+08	+4.50000E+06	
+0.00000E+00	+0.00000E+00	+0.00000E+00	+0.00000E+00	
+0.00000E+00	+3.60000E+08	+0.00000E+00	+0.00000E+00	
0.00000E+00	+1.20000E+08	+0.00000E+00	-4.50000E+06	
+0.60000E+08	+0.00000E+00	+1.80000E+08	+0.00000E+00	
+0.00000E+00	+0.00000E+00	+7.20000E+08	+4.50000E+06	
+0.00000E+00	+0.00000E+00	+1.20000E+08	+0.00000E+00	
+4.50000E+06	-3.00000E+06	+0.00000E+00	-1.50000E+05	
+0.00000E+00	+0.00000E+00	+4.50000E+06	+3.00000E+05	
-4.50000E+06	+0.00000E+00	-1.50000E+06	-1.50000E+05	
(2)	(3)	(4)	(5)	

Table A-4. 2. Inverted Matrix (Output)

(1)	(2)	(3)	(4)	(5)
0. 37587090E-08	-0. 35635246E-09	0. 41859961E-09	-0. 24174744E-09	0. 45634926E-07
-0. 28632458E-09	-0. 35546965E-09	0. 28651491E-09	-0. 30502326E-09	0. 63081568E-07
-0. 17729002E-09	-0. 33218047E-09	0. 10411987E-09	-0. 35181028E-09	0. 41598255E-07
-0. 35635246E-09	0. 20488250E-08	-0. 48379324E-09	0. 41859977E-09	-0. 55966079E-07
-0. 22121567E=09	-0. 28632440E-09	-0. 30502327E-09	0. 14340691E-09	-0. 10912700E-06
0. 14340643E-09	-0. 30502301E-09	-0. 28632427E-09	-0. 22121566E-09	0. 45634922E-07
0. 41859959E-09	-0. 48379323E-09	0. 20488249E-08	-0. 35635267E-09	0. 63081570E-07
-0. 35181005E-09	0. 10411967E-09	-0. 33218021E-09	-0. 17729025E-09	0. 41598252E-07
-0. 30502305E-09	0. 28651486E-09	-0. 35546972E-09	-0. 28632427E-09	-0. 55966079E-07
-0. 24174744E-09	0. 41859977E-09	-0. 35635267E-09	0. 37587093E-08	-0. 10912699E-06
0. 10912594E-06	-0. 63081542E-07	0. 55966068E-07	-0. 45634937E-07	0. 89121525E-05
0. 89285693E-07	-0. 10714286E-06	0. 10714287E-06	-0. 89285750E-07	0. 16785718E-04
0. 45634927E-07	-0. 55966080E-07	0. 63081575E-07	-0. 10912700E-06	0. 17516423E-04
-0. 28632452E-09	-0. 22121566E-09	-0. 35181004E-09	0. 10912694E-06	Last Card
0. 20488247E-08	-0. 35635265E-09	0. 10411990E-09	0. 55966052E-07	
-0. 33218023E-09	-0. 17729027E-09	0. 27543972E-09	-0. 41598206E-07	
-0. 35546965E-09	-0. 28632437E-09	0. 10411966E-09	-0. 63081540E-07	
-0. 35635266E-09	0. 37587093E-08	-0. 35181026E-09	-0. 45634935E-07	
0. 41859955E-09	-0. 24174740E-09	-0. 17729004E-09	0. 10912692E-06	
0. 28651492E-09	-0. 30502324E-09	-0. 33218021E-09	0. 55966064E-07	
0. 10411987E-09	-0. 35181028E-09	0. 17588600E-08	-0. 41598211E-07	
-0. 48379321E-09	0. 41859972E-09	-0. 33218041E-09	-0. 63081541E-07	
-0. 30502324E-09	0. 14340688E-09	-0. 17729023E-09	-0. 45634934E-07	
0. 55966058E-07	-0. 45634934E-07	-0. 41598209E-07	0. 17516412E-04	
0. 10714286E-06	-0. 89285740E-07	0. 29000000E-13	0. 16785711E-04	
0. 63081570E-07	-0. 10912700E-06	0. 41598251E-07	0. 89121525E-05	
-0. 17729004E-09	0. 14340643E-09	-0. 30502305E-09	0. 89285695E-07	
-0. 33218021E-09	0. 41859952E-09	-0. 48379322E-09	0. 10714286E-06	
0. 17588598E-08	-0. 35180998E-09	0. 10411964E-09	0. 35000000E-13	
-0. 33218045E-09	-0. 30502302E-09	0. 28651488E-09	-0. 10714286E-06	
-0. 17729027E-09	-0. 24174741E-09	0. 41859974E-09	-0. 89285750E-07	
-0. 35180998E-09	0. 37587085E-08	-0. 35635243E-09	0. 89285689E-07	
0. 10411984E-09	-0. 28632428E-09	-0. 35546972E-09	0. 10714287E-06	
0. 27643972E-09	-0. 17729006E-09	-0. 33218042E-09	0. 30000000E-13	
0. 10411965E-09	-0. 35635242E-09	0. 20488250E-08	-0. 10714286E-06	
-0. 35181025E-09	-0. 22121564E-09	-0. 28632426E-09	-0. 89285750E-07	
-0. 41598208E-07	0. 10912692E-06	-0. 63081544E-07	0. 16785712E-04	
0. 31000000E-13	0, 89285683E-07	-0. 10714286E-06	0. 26547621E-04	
0. 41598253E-07	0. 45634921E-07	-0. 55966082E-07	0. 16785719E-04	
(2)	(3)	(4)	(5)	

Table A-4. 3. Computing Deflections for Vierendeel Truss

INPUT DATA (Matrix Multiplication)	OUTPUT (Computed Deflections)	
13 x 13 Inverted Matrix (Table A-4. 2)		
+	C C Matrix Multiplication	
. 00000000E+00	-0. 18214281E-02	(1)
. 00000000E+00	-0. 17857143E-02	(2)
. 00000000E+00	-0. 57750000E-09	(3)
. 00000000E+00	0. 17857142E-02	(4)
. 00000000E+00	0. 18214290E-02	(5)
. 00000000E+00	-0. 18214279E-02	(6)
. 00000000E+00	-0. 17857145E-02	(7)
. 00000000E+00	-0. 57750000E-09	(8)
. 00000000E+00	0. 17857142E-02	(9)
. 00000000E+00	0. 18214291E-02	(10)
-. 60000000E+04	-0. 32642849	(12)
-. 10000000E+05	-0. 46690479	(13)
-. 60000000E+04	-0. 32642863	(14)

Table A-4.4. Input data to compute end shears and moments

2	2	2	2	2
+1.20000E+08	+1.20000E+08	+1.20000E+08	+1.20000E+08	+1.20000E+08
+0.60000E+08	+0.60000E+08	+0.60000E+08	+0.60000E+08	+0.60000E+08
+0.60000E+08	+0.60000E+08	+0.60000E+08	+0.60000E+08	+0.60000E+08
+1.20000E+08	+1.20000E+08	+1.20000E+08	+1.20000E+08	+1.20000E+08
-0.18214281E-02	-0.17857142E-02	-0.57750000E-09	0.17857142E-02	0.18214290E-02
-0.18214279E-02	-0.17857145E-02	-0.57750000E-09	0.17857142E-02	0.18214291E-02
4	4	4	4	
+5.00000E+04	+7.50000E+04	+7.50000E+04	+5.00000E+04	
+3.00000E+06	+4.50000E+06	+4.50000E+06	+3.00000E+06	
-5.00000E+04	-7.50000E+04	-7.50000E+04	-5.00000E+04	
+3.00000E+06	+4.50000E+06	-4.50000E+06	+3.00000E+06	
+3.00000E+06	+4.50000E+06	+4.50000E+06	+3.00000E+06	
+2.40000E+08	+3.60000E+08	+3.60000E+08	+2.40000E+08	
-3.00000E+06	-4.50000E+06	-4.50000E+06	-3.00000E+06	
+1.20000E+08	+1.80000E+08	+1.80000E+08	+1.20000E+08	
-5.00000E+04	-7.50000E+04	-7.50000E+04	-5.00000E+04	
-3.00000E+06	-4.50000E+06	-4.50000E+06	-3.00000E+06	
+5.00000E+04	+7.50000E+04	+7.50000E+04	+5.00000E+04	
-3.00000E+06	-4.50000E+06	-4.50000E+06	-3.00000E+06	
+3.00000E+06	+4.50000E+06	+4.50000E+06	+3.00000E+06	
+1.20000E+08	+1.80000E+08	+1.80000E+08	+1.20000E+08	
-3.00000E+06	-4.50000E+06	-4.50000E+06	-3.00000E+06	
+2.40000E+08	+3.60000E+08	+3.60000E+08	+2.40000E+08	
0.00000000E+00	-0.32642849	-0.46690479	-0.32642863	
-0.18214279E-02	-0.17857145E=02	-0.57750000E-09	0.17857142E-02	
-0.32642849	-0.46690479	-0.32642863	0.00000000E+00	
-0.17857145E-02	=0.57750000E-09	0.17857142E-02	0.18214291E-02	
4	4	4	4	
+5.00000E+04	+7.50000E+04	+7.50000E+04	+5.00000E+04	
+3.00000E+06	+4.50000E+06	+4.50000E+06	+3.00000E+06	
-5.00000E+04	-7.50000E+04	-7.50000E+04	-5.00000E+04	
+3.00000E+06	+4.50000E+06	+4.50000E+06	+3.00000E+06	
+3.00000E+06	+4.50000E+06	+4.50000E+06	+3.00000E+06	
+2.40000E+08	+3.60000E+08	+3.60000E+08	+2.40000E+08	
-3.00000E+06	-4.50000E+06	-4.50000E+06	-3.00000E+06	
+1.20000E+08	+1.80000E+08	+1.80000E+08	+1.20000E+08	
-5.00000E+04	-7.50000E+04	-7.50000E+04	-5.00000E+04	
-3.00000E+06	-4.50000E+06	=4.50000E+06	-3.00000E+06	
+5.00000E+04	+7.50000E+04	+7.50000E+04	+5.00000E+04	
-3.00000E+06	-4.50000E+06	-4.50000E+06	-3.00000E+06	
+3.00000E+06	+4.50000E+06	+4.50000E+06	+3.00000E+06	
+1.20000E+08	+1.80000E+08	+1.80000E+08	+1.20000E+08	
-3.00000E+06	-4.50000E+06	-4.50000E+06	-3.00000E+06	
+2.40000E+08	+3.60000E+08	+3.60000E+08	+2.40000E+08	
0.00000000E+00	-0.32642849	-0.46690479	=0.32642863	
-0.18214281E-02	-0.17857143E-02	-0.57750000E-09	0.17857142E-02	
-0.32642849	-0.46690479	-0.32642863	0.00000000E+00	
-0.17857143E-02	-0.57750000E-09	0.17857142E-02	0.18214290E-02	

Table A-4. 5. Computed End Shears and Moments (Output of Table A-4. 4.

-0. 32785704E+06	M	AJ
-0. 32784704E+06	M	
0. 54999990E+04	V	AB
0. 32785703E+06	M	
-0. 54999990E+04	V	
0. 33214264E+06	M	
0. 54999990E+04	V	JI
0. 32785701E+06	M	
-0. 54999990E+04	V	
0. 33214267E+06	M	
-0. 32142859E+06	M	BI
-0. 32142860E+06	M	
0. 25000045E+04	V	BC
-0. 10713903E+05	M	
-0. 25000045E+04	V	
0. 31071460E+06	M	
0. 25000055E+04	V	IH
-0. 10713803E+05	M	
-0. 25000055E+04	V	
0. 31071470E+06	M	
-0. 10395000	M	CH
-0. 10395000	M	
-0. 25000010E+04	V	CD
-0. 31071444E+06	M	
0. 25000010E+04	V	
0. 10714210E+05	M	
-0. 25000010E+04	V	HG
-0. 31071444E+06	M	
0. 25000010E+04	V	
0. 10714210E+05	M	
0. 32142855E+06	M	DG
0. 32142855E+06	M	
-0. 55000030E+04	V	DE
-0. 33214299E+06	M	
0. 55000030E+04	V	
-0. 32785721E+06	M	
-0. 55000030E+04	V	GF
-0. 33214300E+06	M	
0. 55000030E+04	V	
-0. 32785723E+06	M	
0. 32785723E+06	M	EF
0. 32785723E+06	M	

APPENDIX B

B-1. Stiffness Coefficients as a Function of Strain Energy

Let F_1, F_2, F_3, and F_4 represent P_1, $\frac{M_1}{L}$, P_2, and $\frac{M_2}{L}$ respectively (Reference Figure B. 2. 1). Then the stiffness equation for the beam segment (13, p. 6) is

$$\left\{F_i\right\} = \left[K_{ij}\right] \left\{u_j\right\} \tag{B 1.1}$$

The element K_{ij} of the stiffness matrix is then a stiffness influence coefficient, or more explicitly the force F_i caused by a unit displacement, u_j. The general Equation for K_{ij} may be determined by applying a deflection u_j and then computing the internal moment due to u_j. Next in sequence, a virtual deflection ϵu_j is applied and the variation in the internal energy, δU, is computed as well as the external work, δW, done by F_i. Since the first variation of the total potential, $\delta (U - W)$, must vanish at equilibrium, it will suffice to compute $\delta U = \delta W$ which gives us

$$F_i = k_{ij} \; u_j \tag{B 1.2}$$

Internal moment due to u_j

$$M_j(x) = -EI \frac{d^2 y_i}{dx^2} = -EI \; u_j \frac{1}{L^2} \frac{d^2 \Phi_i}{d\psi^2} \tag{B. 1. 3}$$

Variation in internal energy due to u_i:

$$\delta U = -\int_0^L M_j(x)\left[\epsilon \frac{d^2 y_i}{dx^2}\right] dx$$

$$= -\int_0^1 EI\, u_j \frac{1}{L^2} \frac{d^2\Phi_i}{d\psi^2} (\epsilon u_j) \frac{1}{L^2} \frac{d^2\Phi_i}{d\psi^2} L \, d\psi$$

$$= \frac{EI}{L^3} u_j (\epsilon u_j) \int_0^1 \frac{d^2\Phi_i}{d\psi^2} \frac{d^2\Phi_j}{d\psi^2} d\psi \qquad \text{(B 1.4)}$$

Variation in external work:

$$\delta W = F_i (\epsilon u_i)$$

Then, since $\delta W = \delta U$

$$F_i (\epsilon u_i) = \frac{EI}{L^3} u_j (\epsilon u_i) \int_0^1 \frac{d^2\Phi_i}{d\psi^2} \frac{d^2\Phi_j}{d\psi^2} d\psi \qquad \text{(B 1.5)}$$

As the limit of $\epsilon \rightarrow 0$ Equation (B 1.5) becomes

$$F_i = \left[\frac{EI}{L^3} \int_0^1 \frac{d^2\Phi_i}{d\psi^2} \frac{d^2\Phi_j}{d\psi^2} d\psi\right] u_j \qquad \text{(B 1.6)}$$

The term in brackets in Equation (B 1.6) is the stiffness coefficient as dictated by Equation (B 1.2).

For simplicity, the stiffness coefficient may be written

$$k_{ij} = \frac{EI}{L^3} k^o_{ij} \qquad \text{(B 1.7)}$$

where

$$k^{o}_{ij} = \int_{o}^{1} \frac{d^2 \Phi_i}{d\psi^2} \frac{d^2 \Phi_j}{d\psi^2} \, d\psi \qquad \text{(B 1.8)}$$

B. 2 Particular Solutions of Elastic Curve Equation

Considering the beam shown in Figure B 2. 1, it is implied the usual assumptions of beam theory hold (small deflections, complete recovery in the elastic range after the load has been removed,

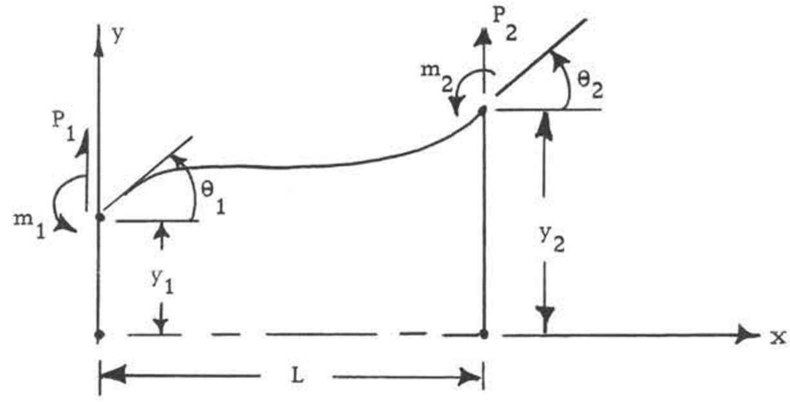

Figure B 2. 1. Deflected Beam Element.

a plane before bending remains a plane after bending, etc.). It is further assumed that the elastic curve of the beam element is given by a cubic equation

$$y(\psi) = a\psi^3 + b\psi^2 + c\psi + d \qquad \text{(B 2.1)}$$

in which ψ is a function of x; $\psi = \frac{x}{L}$

The constants a, b, c, and d are evaluated by the use of proper boundary conditions of the beam. Thus

(1) When the beam is given a unit deflection at the left or near the end of the beam as shown in Figure B 2. 2, the boundary conditions are:

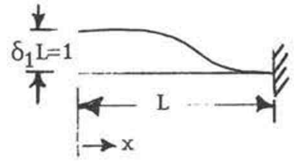

Figure B 2.2. Unit deflection at near end of beam

1) $y(0) = 1$ 3) $y(1) = 0$

2) $y'(0) = 0$ 4) $y'(1) = 0$

The first derivative of Equation B 2. 1 is

$$y'(\psi) = 3a\psi^2 + 2b\psi + c \tag{B 2. 2}$$

Appropriate applications of the conditions above to Equations (B 2. 1) and (B 2. 2) yields:

When $y(0) = 1$ then $d = 1$

$y'(0) = 0$ then $c = 1$

$$\left.\begin{array}{ll} y(1) = 0 & \text{then } a + b + 1 = 0 \\ y'(1) = 0 & \text{then } 3a + 2b = 0 \end{array}\right\} a = 2,\ b = -3.$$

Substituting these values into Equations (B 2. 1) we obtain

$$y_1 = 2\psi^3 - 3\psi^2 + 1 \tag{B 2. 3}$$

(2) When the beam is given a unit rotation at the near end as shown in Figure B 2. 3, the values of the constants a, b, c, and d are obtained by applying the new boundary conditions as outlined in (1) above. Thus,

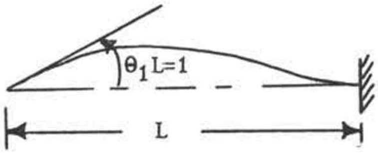

Figure B 2. 3. Unit Rotation at Near End of Beam

$$\text{When}\quad y\,(0) = 0 \quad \text{then} \quad d = 0$$

$$y'(0) = 1 \quad \text{then} \quad c = 1$$

$$\left.\begin{array}{lll} y\,(1) = 0 & \text{then} & a + b + 1 = 0 \\ & & \\ y'(1) = 0 & \text{then} & 3a + 2b + 1 = 0 \end{array}\right\} \quad a = 1,\ b = -2$$

Hence,

$$y_2 = \psi^3 - 2\psi^2 + \psi \qquad \text{(B 2. 4)}$$

(3) Unit deflection at the far end of beam. Same procedural sequence is followed as in (1) and (2) above.

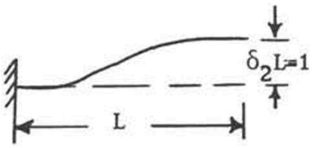

Figure B 2.4. Unit deflection at far end of beam

$$\text{When}\quad y\,(0) = 0 \quad \text{then} \quad d = 0$$

$$y'(0) = 0 \quad \text{then} \quad c = 0$$

$$\left.\begin{array}{lll} y\,(1) = 1 & \text{then} & a + b = 1 \\ y'(1) = 0 & \text{then} & 3a + 2b = 0 \end{array}\right\} \quad a = -2,\ b = 3$$

Hence,

$$y_3 = 3\psi^2 - 2\psi^3 \qquad \text{(B 2. 5)}$$

(4) Unit rotation at the far end of beam. Proceeding as

before we obtain

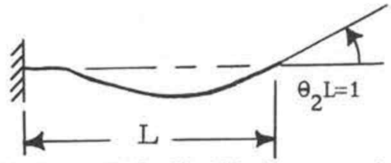

Figure B2.5. Unit rotation at far end of beam

When $y(0) = 0$ then $d = 0$

$y'(0) = 0$ then $c = 0$

$$\left.\begin{array}{ll} y(1) = 0 & \text{then } a + b = 0 \\ y'(1) = 1 & \text{then } 3a + 2b = 1 \end{array}\right\} \quad a = 1,\ b = -1$$

Hence,

$$y_4 = \psi^3 - \psi^2 \qquad \text{(B 2.6)}$$

B.3. Computing the Stiffness Influence Coefficients

The second derivatives of Equations (B 2.3), (B 2.4), (B 2.5), and (B 2.6) are respectively

$$y_1'' = 12\psi - 6 \qquad \text{(B 3.1)}$$

$$y_2'' = 6\psi - 4 \qquad \text{(B 3.2)}$$

$$y_3'' = 6 - 12\psi \qquad \text{(B 3.3)}$$

$$y'' = -2 + 6\psi \qquad \text{(B 3.4)}$$

Then the Stiffness Influence Coefficients are determined as shown below in compliance with Equation (B 1.8):

$$k_{11}^o = \int_o^1 y_1'' \; y_1'' \; d\psi = \int_o^1 (144\psi^2 - 144\psi + 36)\, d\psi = +12$$

$$k_{21}^o = \int_o^1 y_2'' \; y_1'' \; d\psi = \int_o^1 (72\psi^2 - 84\psi + 24)\, d\psi = +6$$

$$k_{22}^o = \int_o^1 y_2'' \; y_2'' \; d\psi = \int_o^1 (36\psi^2 - 48\psi + 16)\, d\psi = +4$$

$$k^o_{31} = \int_o^1 y_3'' \; y_1'' \; d\psi = \int_o^1 (-144\,\psi^2 + 144\,\psi - 36)\, d\psi = -12$$

$$k^o_{32} = \int_o^1 y_3'' \; y_2'' \; d\psi = \int_o^1 (\;72\,\psi^2 + \;84\,\psi - 24)\, d\psi = -\,6$$

$$k^o_{33} = \int_o^1 y_3'' \; y_3'' \; d\psi = \int_o^1 (\;36 - 144\psi + \;144\psi^2)\, d\psi = +12$$

$$k^o_{41} = \int_o^1 y_4'' \; y_1'' \; d\psi = \int_o^1 (\;72\,\psi^2 - 60\psi + 12)\; d\psi = +\,6$$

$$k^o_{42} = \int_o^1 y_4'' \; y_2'' \; d\psi = \int_o^1 (\;36\,\psi^2 - 36\psi + \;8)\; d\psi = +\,2$$

$$k^o_{43} = \int_o^1 y_4'' \; y_3'' \; d\psi = \int_o^1 (\,-72\,\psi^2 + 60\psi - 12)\; d\psi = -\,6$$

$$k^o_{44} = \int_o^1 y_4'' \; y_4'' \; d\psi = \int_o^1 (\;36\,\psi^2 + 24\psi + \;4)\; d\psi = +\,4$$

From Betti's Law or the generlized Maxwell's Law of Reciprocal Deflections (16, p. 376) the relation (B 3.5) was obtained

$$k_{ij} = k_{ji} \qquad \text{(B 3.5)}$$

Hence, the stiffness matrix for the beam segment may be written

$$\left[K^o_{ij}\right] = \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}$$

where $\left[K^o\right]$ represents non-dimensional parts of $\left[K\right]$.

APPENDIX C

NOTATION

Symbols are defined where they are first introduced in the paper, and the most important ones are also listed here for convenience.

E	modulus of elasticity
P_i	generalized force acting on structural element
I	moment of inertia
k_{ij}	stiffness coefficient (= force p_i produced by a unit displacement $u_j = 1$)
L	length of beam
m_3, m_2	moments at left and right ends of beam element
P_1, P_2	lateral forces at left and right ends of beam element
θ_1, θ_2	rotations at left and right ends of beam element
δ_1, δ_2	deflections at left and right ends of beam element
u_i	generalized displacement of structure or structural element at a node
f_{ij}	flexibility coefficient
$[K]$	stiffness matrix for structure
$[K^o]$	non-dimensional parts of K
$\{P\}$	generalized forces acting on structure