

# Using Three-Dimensional Spacetime Diagrams in Special Relativity

Tevian Dray\*

Department of Mathematics, Oregon State University, Corvallis, OR 97331

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We provide several examples of the use of geometric reasoning with three-dimensional spacetime diagrams, rather than algebraic manipulations using three-dimensional Lorentz transformations, to analyze problems in special relativity.

## I. INTRODUCTION

Spacetime diagrams are a standard tool for understanding the implications of special relativity. A spacetime diagram displays invariant physical content geometrically, making it straightforward to determine what is happening in each reference frame, not merely the one in which the diagram is drawn.

Not surprisingly, most such diagrams are two-dimensional, showing one dimension of space, and one of time. Such two-dimensional spacetime diagrams are sufficient to describe most elementary applications, such as length contraction or time dilation, and even some more advanced applications, such as many of the standard paradoxes. However, some physical situations are inherently three-dimensional. Such situations can of course be represented using two-dimensional diagrams, capturing the appropriate information as needed using different views. Each such view is however nothing more than a projection of a single, three-dimensional spacetime diagram, either “horizontally” into a two-dimensional spacetime, or “vertically” into a purely spatial diagram.

We argue here by example that the starting point when representing three-dimensional configurations in special relativity should be a three-dimensional spacetime diagram. The essential physics can often be determined directly from such diagrams, or they can be used to construct more detailed (and easier to draw) two-dimensional diagrams, ensuring that they are correctly and consistently drawn.

Three-dimensional spacetime diagrams are not new, although we are not aware of any systematic treatments in the physics literature. An early insightful example of such diagrams is their use in the Annenberg video<sup>1</sup>. However, such diagrams have been discussed from the point of view of computer graphics; for a recent survey, see<sup>2</sup> and references cited there.

We adopt throughout a geometric view of special relativity in terms of “hyperbola geometry”, using hyperbolic trigonometry rather than algebraic manipulations involving  $\gamma^{-1} = \sqrt{1 - v^2/c^2}$ , although we avoid setting  $c = 1$ . This point of view, discussed in the first edition<sup>3</sup> of Taylor and Wheeler but removed from the second edition<sup>4</sup>, is extensively discussed (in two dimensions) in our recent book<sup>7</sup>, a (prepublication) version of which is available online<sup>8</sup>.

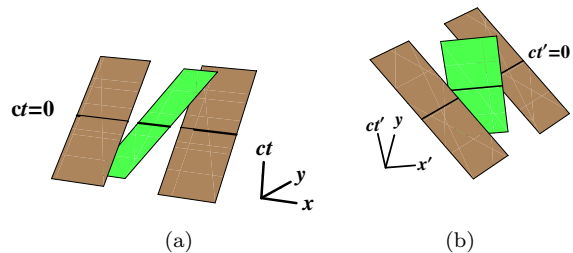


FIG. 1. The meter stick, (a) as observed in the laboratory frame, and (b) according to the stick.<sup>11</sup>

## II. THE RISING MANHOLE

*A meter stick lies along the  $x$ -axis of the laboratory frame and approaches the origin with constant velocity. A very thin metal plate parallel to the  $xz$ -plane in the laboratory frame moves upward in the  $y$ -direction with constant velocity. The plate has a circular hole with a diameter of one meter centered on the  $y$ -axis. The center of the meter stick arrives at the laboratory origin at the same time in the laboratory frame as the rising plate arrives at the plane  $y = 0$ . Does the meter stick pass through the hole?*

This problem appears in Taylor and Wheeler<sup>3</sup> as “The meter-stick paradox”, but is originally due to Shaw<sup>9</sup>, simplifying a problem previously posed by Rindler<sup>10</sup>. Taylor and Wheeler<sup>4</sup> later dubbed this problem “The rising manhole” for obvious reasons.

From the point of view of the metal plate (representing the manhole), the meter stick (representing the manhole cover) is very short, and should therefore easily fit through the hole. However, from the point of view of the stick, the hole in the plate is very short, so the plate should bump into the stick. Which of these scenarios is correct?

We draw a three-dimensional spacetime diagram, as shown in Figure 1a in the laboratory frame. The  $z$ -direction is suppressed; only one spatial dimension of the plate is shown. The plate is therefore represented in two pieces, with a hole in the middle; world sheets are shown for each of the two line segments in the plate along the  $x$ -axis that connect the hole to the edge of the plate. This

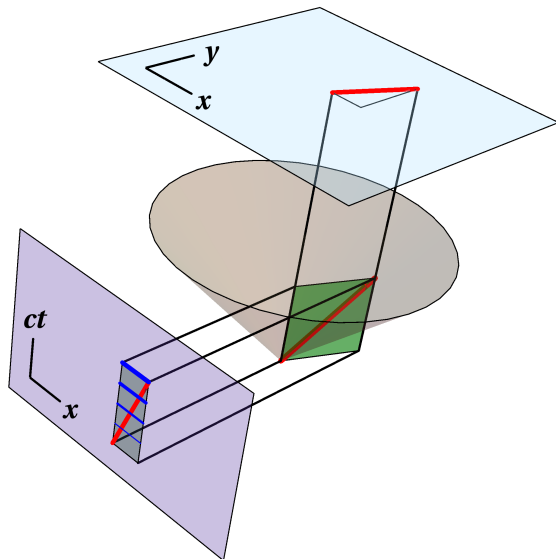


FIG. 2. A lightray traveling along the mast of a moving boat, observed in the reference frame of the boat. The worldsheet of the mast is represented by the vertical rectangle, and the heavy line along the lightcone represents the lightray. Horizontal lines represent the mast at several different instants of time.

two-part worldsheet is tilted backward, indicating motion upward (in space, that is, in the  $y$ -direction). The stick is moving toward the right, and clearly does fit through the hole, as shown by the heavy lines indicating the positions of the stick and the plate at  $t = 0$ . So what is observed in the stick's frame? A plane of constant time *according to the stick* is not horizontal in the laboratory frame; since this plane is tilted, so is the plate! Redraw the spacetime diagram in the reference frame of the stick, as shown in Figure 1b. The worldsheet of the stick is now vertical, and the worldsheet of the plate is tilted both to the left and back. Considering the heavy lines indicating the positions of the stick and the plate at  $t' = 0$ , yields the traditional “snapshot” of the situation in the stick's frame; the stick is indeed longer than the hole, but passes through it at an angle.

### III. THE MOVING SPOTLIGHT

*A sailboat is manufactured so that the mast leans at an angle  $\theta$  with respect to the deck. A spotlight is mounted on the boat so that its beam makes an angle  $\theta$  with the deck. If this boat is then set in motion at speed  $v$ , what angle  $\theta'$  does an observer on the dock say the beam makes with the deck? What angle  $\theta''$  does this observer say the mast makes with the deck?*

This problem appears in Griffiths<sup>12</sup>; an alternate ver-

sion<sup>7</sup> replaces the boat with a spaceship, and the mast with an antenna.

A three-dimensional spacetime diagram in the reference frame of the boat is shown in Figure 2. The heavy line represents the lightray, and lies along the lightcone as shown. The worldsheet of the mast is also shown; it is vertical, since the boat is at rest. Since the spotlight is pointed along the mast, the lightray must lie within this worldsheet, as shown. As in Griffiths<sup>12</sup>, the mast is tilted toward the back of the boat, so that  $\theta$  is measured in the direction opposite to the motion of the boat.

Projecting this diagram vertically yields the spatial diagram shown at the top of Figure 2, with the lightray superimposed on the mast — which both clearly make the same angle with the deck. The triangle shown in this projection represents the displacements  $\Delta x$  and  $\Delta y$  traveled by the lightray in moving from one end of the mast to the other.

Projecting the diagram horizontally, as shown at the front of Figure 2, yields a traditional, two-dimensional spacetime description, in which we have indicated the (projection of the) mast at several times by horizontal lines. Since the worldsheet is not parallel to the  $x$ -axis, the lightray does *not* travel at the speed of light *in the direction the boat is moving*! (Of course not, since the spotlight was not pointed directly forward.) How fast is it moving? Using the projected lightray as the hypotenuse of a right triangle, we have

$$\frac{u}{c} = \frac{\Delta x}{c\Delta t} = \tanh \alpha \quad (1)$$

But  $\Delta x$  is the same as before, and, since the (unprojected) lightray really does travel at the speed of light, we also have

$$(c\Delta t)^2 = (\Delta x)^2 + (\Delta y)^2 = \frac{(\Delta x)^2}{\cos^2 \theta} \quad (2)$$

so that

$$\tanh \alpha = \cos \theta \quad (3)$$

What is the situation in the dock's frame? As shown in Figure 3, the primary difference is that the worldsheet of the mast is no longer vertical, since it is moving. But the rest of the argument is the same, with  $\Delta x'$ ,  $\Delta y'$ , and  $\Delta t'$  again denoting the displacements of the lightray (which do not correspond to the sides of any triangles in Figure 3). The apparent speed of the light ray parallel to the boat is given by the Einstein addition law, which takes the form

$$\frac{u'}{c} = \frac{\Delta x'}{c\Delta t'} = \tanh(\beta - \alpha) \quad (4)$$

where  $\tanh \beta = v/c$ , so that

$$\cos \theta' = \tanh(\alpha - \beta) = \frac{\tanh \alpha - \tanh \beta}{1 - \tanh \alpha \tanh \beta} = \frac{\cos \theta - \tanh \beta}{1 - \cos \theta \tanh \beta} \quad (5)$$

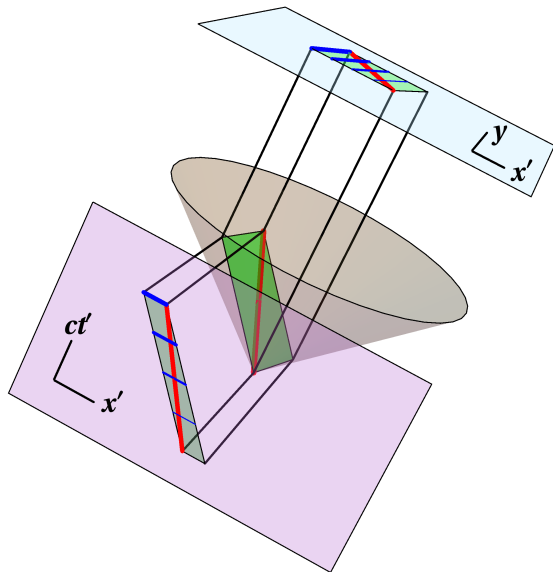


FIG. 3. A lightray traveling along the mast of a moving boat, according to an external observer. The worldsheet of the mast is represented by the tilted rectangle, and the heavy line along the lightcone represents the lightray. Horizontal lines represent the mast at several different instants of time.

or in more familiar language

$$\cos \theta' = \frac{c \cos \theta - v}{c - v \cos \theta} \quad (6)$$

Determining the angle of the mast is much easier. If the mast is at an angle  $\theta$  to the horizontal in the rest frame of the boat, as shown in Figure 4a, then

$$\tan \theta = \frac{\Delta y}{\Delta x} \quad (7)$$

In the dock frame,  $y$  is unchanged, but  $x$  is length-contracted, so

$$\tan \theta'' = \frac{\Delta y'}{\Delta x'} = \frac{\Delta y}{\Delta x / \cosh \beta} = \tan \theta \cosh \beta \quad (8)$$

Did we really need to refer to the upper, spatial projection in Figure 3 to help visualize this argument? Probably not.

But wait a minute. How can the lightray move along the mast if the angles are different? Now the spatial projection at the top of Figure 3, reproduced in Figure 4b, comes in handy, as it shows a “movie” of the lightray propagating along the (moving) mast despite the fact that the angles between the deck and the lightray and mast are clearly different. The same question could be raised in the Newtonian version of the problem, in which a ball is thrown along the mast. The resolution of this apparent paradox is the same in both contexts: Position and velocity transform differently between reference frames, and therefore so do “position angles” and “velocity angles”.

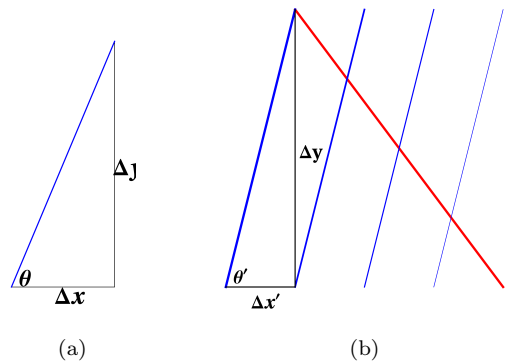


FIG. 4. The horizontal projections of the spacetime diagrams in Figures 2 and 3. In the boat’s frame (a), the mast is at rest, but in the dock’s frame (b) the boat moves to the left; the mast is shown at four discrete instants of time. The beam of light moves along the mast in both cases, as shown by the diagonal line in (b) (and not shown separately in (a)).

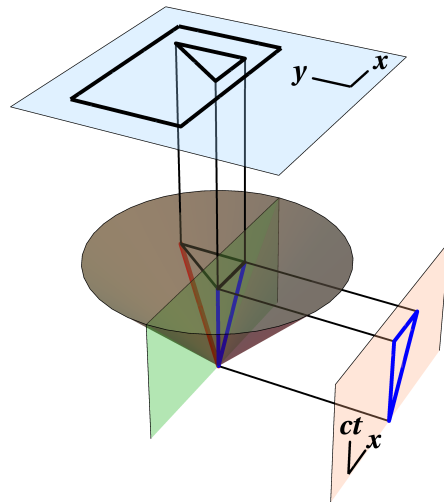


FIG. 5. A bouncing lightray on a moving train. One “tick” of the lightray is represented by the diagonal line along the lightcone. The vertical projection (top) yields the usual analysis of time dilation in terms of a *Euclidean* triangle; the corresponding *hyperbolic* triangle is obtained from the horizontal projection (right).

#### IV. THE LORENTZIAN INNER PRODUCT

Perhaps the most famous thought experiment ever is Einstein’s argument that the constancy of the speed of light leads to time dilation: A beam of light bouncing vertically in a (horizontally!) moving train travels different distances, and therefore takes different amounts of time, in the moving reference frame than in the rest frame. Thus, the “ticks” of a clock measure different time intervals in the two reference frames.

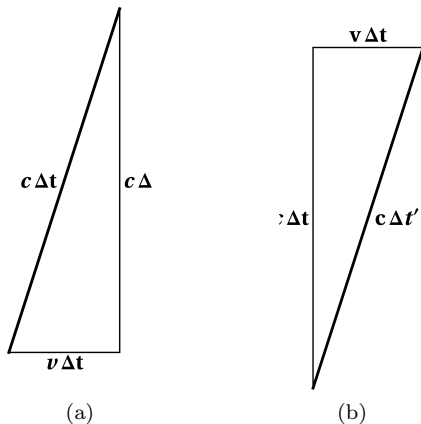


FIG. 6. The (a) ordinary and (b) spacetime Pythagorean theorem for a bouncing light beam on a moving train.

This scenario is usually drawn as a purely spatial “movie”, but is inherently three-dimensional. Figure 5,<sup>13</sup> shows the worldlines of a lightray (along the lightcone), an observer at rest (vertical), and the lightsource on the floor of the moving train (tilted to the right).

The projection into a horizontal plane  $t = \text{constant}$ , as shown at the top of Figure 5, is precisely the standard spatial “movie”, shown in Figure 6a. This triangle shows the distance  $v\Delta t$  traveled by the train (base), the distance  $c\Delta t$  traveled by the beam of light according to the observer on the platform (hypotenuse), and the height of the train (side), that is, the distance  $c\Delta t'$  traveled by the beam of light according to the observer on the train. Projecting instead into the vertical plane  $y = 0$ , as shown on the right of Figure 5, one obtains the spacetime diagram shown in Figure 6b.

Remarkably, the edges of the two triangles formed by the horizontal and vertical projections have the same lengths. It is apparent from Figure 5 that they share one edge. The four remaining legs consist of two pairs, each of which forms a right triangle whose hypotenuse is the lightray. So long as one measures time and distance in the same units (thus setting  $c = 1$ ), each such triangle is isosceles; the legs have the same length. This argument justifies the labeling used in Figure 6b, and can in fact be used to *derive* the hyperbolic Pythagorean theorem, which says that

$$(c\Delta t')^2 = (c\Delta t)^2 - (v\Delta t)^2, \quad (9)$$

or equivalently, that

$$t' = \frac{t}{\cosh \beta}, \quad (10)$$

where  $\tanh \beta = v/c$  is the speed of the train. Moving clocks therefore run slow, by a factor of precisely  $\cosh \beta$ .

Figures 6a and 6b thus present the same content, but in quite different ways; two of the sides appear to have

been interchanged. In Figure 6a, the vertical *distance* traveled by the lightray was  $c\Delta t'$ , whereas in Figure 6b, the *time* taken by the beam to reach the top of the train is  $c\Delta t'$ .

## V. CONCLUSION

We have considered three scenarios in special relativity that lend themselves to an analysis using three-dimensional spacetime diagrams. In the first scenario, the qualitative aspects of the diagram immediately yielded insight into the counterintuitive effects of transforming velocities between reference frames, and projections of the three-dimensional diagram were used to recover more familiar, two-dimensional diagrams of the same scenario. In the second scenario, the two-dimensional diagrams played a more fundamental role, and the three-dimensional spacetime diagram was used primarily to ensure that the two-dimensional diagrams were consistent. Finally, in the third scenario, well-known two-dimensional diagrams were combined into a single three-dimensional diagram, providing new insight into a standard thought experiment about the Lorentzian inner product of special relativity. Each of these somewhat different uses of three-dimensional spacetime diagrams demonstrates the usefulness of such diagrams.

So when are three-dimensional spacetime diagrams useful? Diepstraten *et al.*<sup>5</sup> list some of the features of such diagrams, including the presence of motion in more than one spatial direction, as in the rising manhole example, and the ability to present spatial angles, as in the moving spotlight example. We would add to this list the ability to interpolate between traditional spatial “movies”, and (two-dimensional) spacetime diagrams, as in our last example. However, our goal here has not been to present general criteria for determining when to (or not to) use such diagrams, but rather to demonstrate by example that they are appropriate in some situations. A more systematic treatment would also address the differences between *seeing* (using light) and *observing* (using an army of observers), thus describing what things look like, not merely what they do.

## ACKNOWLEDGMENTS

The geometric approach to special relativity described here and in our previous work<sup>6–8</sup> grew out of class notes for a course on *Reference Frames*<sup>14</sup>, which in turn forms part of a major upper-division curriculum reform effort, the *Paradigms in Physics* project<sup>15,16</sup>, begun in the Department of Physics at Oregon State University in 1997, and supported in part by NSF grants DUE-965320, 0231194, 0618877, and 1023120.

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\* [tevian@math.oregonstate.edu](mailto:tevian@math.oregonstate.edu)

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- <sup>11</sup> Online versions of these and subsequent figures are available as rotatable Java applets at <http://physics.oregonstate.edu/coursewikis/GSR/book/updates/3d>.
- <sup>12</sup> David J. Griffiths, *Introduction to Electrodynamics*, 3rd edition (Prentice Hall, Upper Saddle River, 1999).
- <sup>13</sup> This figure appears opposite the title page of Ref. 7, and is used with permission; the following discussion is adapted from §6.4 of that reference.
- <sup>14</sup> Tevian Dray, Reference Frames course website, <http://physics.oregonstate.edu/portfolioswiki/courses:home:rhome>.
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