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Title: A Free Boundary Value Problem Modeling Streambed Erosion

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Our purpose is twofold, to derive a 2-dimensional model of streambed erosion and to develop a solution procedure to solve the equations of our model. The flow domain, which varies in time, is bounded above by a free surface and is bounded below by an erodable streambed. An initial flow and streambed configuration are given and we would like to determine the streambed evolution.

Our basic approach to modeling this process is to combine the interaction of a viscous flow with boundary constraints which represent an erodable streambed.

Flow over a bed of sand causes the bed to undergo a distinct evolution. At slow flows, the bed remains flat. With increasing flow velocity, the bed passes through a set of stages: smooth, ripples, dunes, flat with chaotic flow, waves and finally antidunes. One of our main purposes is to obtain a model which has the potential of addressing this evolutionary process.

We obtain a model which exhibits the general nature of

streambed erosion. Specifically, we obtain classic dune migration. One indirect but significant result is that the vorticity of the flow near the streambed is shown to be a fundamental factor in the erosion process. Another indirect result is the wealth of open questions which have arisen as a result of this work.

A Free Boundary Value Problem Modeling Streambed Erosion

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A List of the Symbols Used

In general, variables subscripted with s refer to the sediment while those subscripted with f refer to the fluid. The subscript c is used for a critical value. Numbers as subscripts are for empirical constants. No subscript indicates a generic variable.

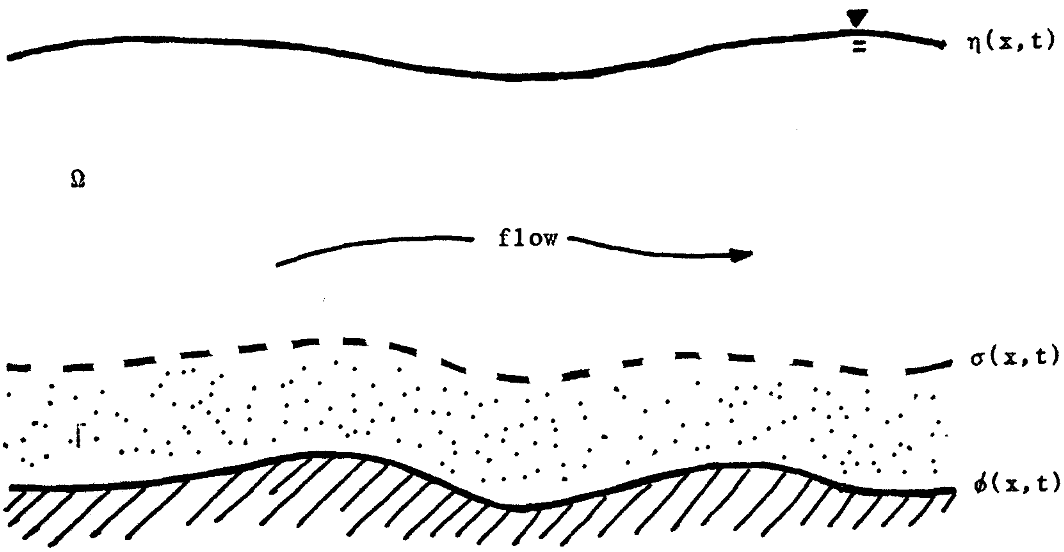
a	constant; inflow at $x=a$
A	matrix block in $\mathbf{A}$
$\mathbf{A}$	large matrix
b	constant; outflow at $x=b$
B	matrix block in $\mathbf{A}$
c	wave speed; $c=c(y^*-\phi)$
c_1	constant, empirical
c_2	constant, empirical
c_p	bed packing ratio
c_s	sediment concentration
C	matrix block in $\mathbf{A}$
d	function; $d=-\{\phi'+yh'\}/h$
da	element of area
dir	local flow direction
dr	$dr=[1+(\phi')^2]^{-1}$
d_s	sediment diameter
ds	element of arclength
D	flow domain; $D=D(t)=\Omega+\Gamma$
$\mathbf{D}$	rate of deformation tensor
e	efficiency factor
e_c	critical efficiency factor
e^i	1-d basis element
e^{ij}	2-d basis element
E	kinetic energy
$\dot{E}$	energy dissipation rate of fluid due to friction
$\dot{E}_a$	net rate energy available for erosion work rate
$\dot{E}_c$	critical energy rate level
$\dot{E}_f$	rate energy convertable to erosion work rate
f	general function
$\mathbf{f}$	vector valued function
Fr	Froude number; $Fr=U^2/Lg$
f_s	force on sediment particle
g	magnitude of gravity
$\mathbf{g}$	gravity vector
h	flow depth; $h=\eta-\phi$
κ	v. Karmen constant; $=.417$
k_i	constants

k_s	conversion factor; $k_s=c_p c_s[1+(\phi')^2]^{-1/2}$
K	arbitrary function; $K=K(x,\phi(x))$
L	characteristic length
n	unit normal vector
p	pressure, variable
p_i	nonfundamental parameters
q_f	volumetric fluid discharge
q_s	sediment discharge
r	resistance term
Re	Reynolds number; $Re=UL/\nu$
S	sediment particle
s	curve parameter
t	time, variable
T	stress tensor
u	horizontal component of **u**
u	fluid velocity, vector
u_s	sediment velocity along ϕ
u*	shear velocity; $u^*=\sqrt{\tau_0/\rho_f}$
U	mean fluid velocity
U_0	mean flow
U_c	critical flow
U_s	mean sediment velocity
U_σ	fluid velocity along σ
v	vertical component of **u**
V	arbitrary volume
V_c	critical velocity for sediment movement
V_s	sediment velocity neglecting resistence
w	test function
W	set of test functions
W_s	weight of particle
x	coordinate in Ω
$\tilde{x}$	coordinate in R
y	coordinate in Ω
$\tilde{y}$	coordinate in R
z	normal distance from ϕ
z_0	stationary flow depth
z	$\mathbf{z}=(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\delta},\boldsymbol{\varepsilon})$
α	coeficient for u; $u=\alpha_{ij}e^{ij}$
$\boldsymbol{\alpha}$	$\boldsymbol{\alpha}=(\alpha_{11}\ldots\alpha_{ij}\ldots\alpha_{IJ})$
β	coeficient for v; $v=\beta_{ij}e^{ij}$
$\boldsymbol{\beta}$	$\boldsymbol{\beta}=(\beta_{11}\ldots\beta_{ij}\ldots\beta_{IJ})$
θ	stream gradient, degrees
$\dot{\theta}$	particle rotation rate
θ_c	angle of repose
∂V	surface of V
δ	coeficient for η; $\eta=\delta_i e^i$
$\boldsymbol{\delta}$	$\boldsymbol{\delta}=(\delta_1\ldots\delta_i\ldots\delta_I)$
Δ	del operator; $\Delta=\nabla^2$

Δx	grid width
Δy	grid height
ε	coeficient for ϕ; $\phi=\varepsilon_i e^i$
$\boldsymbol{\varepsilon}$	$\boldsymbol{\varepsilon}=(\varepsilon_1 \ldots . \varepsilon_i \ldots \varepsilon_I)$
∇	gradient operator
γ	direction; down
Γ	streambed flow domain; $\Gamma=\Gamma(t)$
Λ	map; $\Lambda:\Omega\text{->}R$
η	upper surface; $\eta=\eta(x,t)$
ϕ	streambed; $\phi=\phi(x,t)$
ϕ_0	initial bed form
π	3.14159...
ρ	density; $\rho=\rho_f+\rho_s$
ρ_f	fluid density
ρ_s	sediment density
σ	common boundary of Ω and Γ
τ	shear stress
τ_0	boundary shear stress
τ_c	critical shear
μ	dynamic viscosity
ν	kinematic viscosity; $\nu=\mu/\rho_f$
ω	vorticity, scalar
ξ	coeficient for h; $h=\xi_i e^i$
λ	coefficient for w; $w=\lambda_{ij}e^{ij}$
$\boldsymbol{\lambda}$	$\boldsymbol{\lambda}=(\lambda_{11} \ldots . \lambda_{ij} \ldots \lambda_{IJ})$
λ_s	dune wave length
Ω	mainstream flow domain; $\Omega=\Omega(t)$

A FREE BOUNDARY VALUE PROBLEM MODELING STREAMBED EROSION

INTRODUCTION



The Erosion Problem Schematic

Fig 1.0

The Problem

Our purpose is twofold, to derive a model of streambed erosion and to develop a solution procedure to solve the equations of our model. The 2-dimensional flow domain, $D(t)$, which varies in time, consists of two distinct regions, $D = \Omega + \Gamma$. The mainstream flow region, $\Omega=\Omega(t)$, is bounded above by an air/fluid free surface, $\eta=\eta(x,t)$, while the eroding boundary region, $\Gamma=\Gamma(t)$, is bounded below by an erodable streambed, $\phi=\phi(x,t)$. Ω and Γ share the common boundary, σ, which separates the mainstream flow from the transport region. (see Fig 1.0). At time t_0, the fluid velocity in D_0 is given and we would like to determine the flow and the region $D(t)$ for $t > t_0$. We refer to the above problem as 'the problem' or as 'the erosion problem'.

The General Approach

Our basic approach in modeling streambed erosion is to combine the interaction of a viscous flow with boundary constraints which represent an erodable streambed. The viscous flow is characterized by the Navier-Stokes equation. The boundary condition is obtained by balancing sediment mass, momentum, and energy.

In order to examine the erosion process mathematically, we must first have concise terminology which will be common throughout our discussion. Streambed erosion, as a result of fluid flow, falls under the general category of fluid dynamics. Thus, we begin with the background and terminology of fluid dynamics as it relates to the problem.

Though the problem is set in two dimensions, we use three dimensions for all derivations. All ambiguity is removed by considering the problem to be constant in the third dimension.

The theory of viscous fluids is presented in chapter II. We wish to describe flows in an open channel. We explain why we choose the Navier-Stokes equation to describe our flows. In general, we require some initial conditions as well as some boundary conditions.

How and why we choose such conditions is included in our discussion.

The second part of chapter II highlights sedimentation theory. Empirical results are outlined and we describe various phenomena associated with streambed erosion. The major and minor streambed evolutionary changes are defined. A brief history of sedimentation theory is included.

Flow over a bed of sand causes the bed to undergo a distinct evolution. At slow flows, the bed remains flat. With increasing flow velocity, the bed passes through a set of stages: ripples, then dunes, then flat again with chaotic flow, then waves and finally antidunes. Past models, which have not considered a general flow, cannot address these evolutionary changes. One of our main purposes is to have a model potentially capable of addressing this evolutionary process.

We make a distinction between sediment particle transport and streambed shape evolution. The various stages of transport, which we discuss in chapter II, are rolling, saltation and suspension/ dispersion. Due to the mathematical difficulties associated with turbulence we shall concentrate our study on erosion resulting from rolling and saltation.

The formation of ripples is usually attributed to bed irregularities or fluctuating turbulence. It is our hope that, eventually, our model will show this evolution to be a bifurcation phenomenon. We note model requirements which will achieve this aim.

In chapter III we derive our model. First, we give the precise version of the Navier-Stokes system to be used to describe our flows. The second half of our modeling process involves depicting the flow/streambed interaction. We derive the streambed boundary condition. The result is a 2-dimensional, free boundary value problem with two free boundaries.

Generally, past models have been heavily empirical with relations formed from a large variety of nondimensional parameters. These models usually attempt to match a specific erosion process. Our model is derived strictly from balance laws. Our empirical relations are left undetermined and constants are used. We obtain a model which exhibits the general nature of streambed erosion. Our emphasis is towards the interaction between the flow and the streambed. Our goal is to have a model which can be studied to better understand this interaction.

In the process of obtaining the boundary condition, we assume that the streambed is homogeneous and consists of small, noncohesive, relatively spherical particles. We allow the transport rate to vary but the concentration of sediment is assumed to remain constant. We show that under these assumptions, the model remains broad enough that we obtain some nice results. Specifically, we obtain classic dune movement.

The complicated nature of the general model makes it prudent to consider first some special cases of flows. In chapter IV we consider almost uniform, steady flow. We also consider a potential flow in Ω combined with a logarithmic velocity profile in Γ. These flows yield the classic dune migration behavior.

In chapter V we consider a general flow. This case is solved using time dependent finite elements. We transform our free domain to a fixed domain. This fixed boundary value problem is converted into a system of ordinary differential equations which are then solved numerically. The solution to this system contains the boundary evolution as well as the flow evolution. No inverse transform is required to obtain the boundary evolution directly. This 'solution' is then presented graphically.

The last chapter presents some ideas on upgrading the model. We also contrast our model with previous theory. One main point is that we have the vorticity of the flow near the bottom as a fundamental factor in the erosion process. Another main point is that this model will have the potential to address some open questions which most other models cannot.

We conclude our work with a discussion of some of these open questions which have arisen as a result of this work. For example: How can the numerical scheme be improved? How can the boundary evolution be described as a bifurcation phenomenon?

BACKGROUND OF FLUID DYNAMICS AND SEDIMENTATION

The process of erosion has been around since before man and was studied by primitive societies. The Anasazi experimented with erosion control on Mesa Verde. Fluid Dynamics, the scientific study of flows, has been around for quite some time too. Although Euler is the father of modern fluid dynamics, it was Archemedies who postulated the law of buoyancy which could be considered a first step in the study of sedimentation. We use the theory of viscous fluids to examine the forces which can induce erosion. The Navier-Stokes equation is the standard description of a general viscous fluid.

A fairly complete discussion of fluid dynamics may be found in [18], [22], [24] and [33]. We highlight those aspects pertinent to the erosion problem.

Fluid Dynamics

The Navier-Stokes Equation

The Navier-Stokes equation is coupled with a conservation of mass equation which is often refered to as the continuity equation.

The continuity equation states that the net flux of mass into a volume must be the same as the rate of change of mass in that volume.

We derive all our relations in a 3-dimensional setting. To avoid any ambiguities, our problem, which is set in 2-dimensions, may be thought of as uniform in the third dimension.

Let V be an arbitrary, but fixed, volume, in the fluid domain D. Let ∂V denote the surface of V. Let points in the domain be denoted by $\mathbf{x}=(x_1,x_2,x_3)$. Generally, we use only $\mathbf{x}=(x,y)$. Let $\mathbf{u}=(u(\mathbf{x},t),v(\mathbf{x},t))$ be the velocity field and let $\rho(\mathbf{x},t)$ be the density of the fluid. Equating the rate of change of mass to the mass flux across the surface of V we obtain:

$$d/dt \int_V \rho dv = \int_V \{\partial\rho/\partial t\} dv = -\int_{\partial V} \{\rho\mathbf{u}\cdot\mathbf{n}\} da = -\int_V \{\mathrm{div}\rho\mathbf{u}\} dv \qquad (2.0)$$

The first term is the total change of mass in V. The second term is the result of pulling the differentiation inside the integral. The third term represents the flux across the surface of V. The last term is obtained by applying the Gauss divergence theorem.

Since V is arbitrary, we obtain the mass balance equation in D:

$$\rho_t + \mathrm{div}\ \rho\mathbf{u} = 0 \qquad (2.1)$$

For the type of flows we will be considering, density is relatively constant. Such incompressible or divergence free flows are described by a velocity field which satisfies

$$\mathrm{div}\ \mathbf{u} = 0. \qquad (2.2)$$

The Navier-Stokes equation, describe viscous flows. They are derived from the Balance of Momentum Law: the rate of change of momentum of an arbitrary volume must be balanced by the stress over the surface of the volume plus the body forces acting on the volume. [33].

Let V be an arbitrary, but fixed, volume, moving with the fluid, in the fluid domain D. V is also refered to as a material volume. Let $\mathbf{u}=(u(\mathbf{x},t),v(\mathbf{x},t))$ be the velocity field, let $p=p(\mathbf{x},t)$ be the pressure field and let $\rho(\mathbf{x},t)$ be the density of the fluid. The surface stresses are given by $\mathbf{Tn}$ (Cauchy's stress hypothesis) where $\mathbf{n}$ is the outward normal to V and $\mathbf{T}$ is the stress tensor. [33]. Let $\mathbf{B}$ be the body forces per unit mass on V. Equating the time rate of change of momentum to the forces on V we obtain:

$$\int_V \{D\rho\mathbf{u}/Dt\} dv = \int_{\partial V} \mathbf{Tn} da + \int_V \rho\mathbf{B} dv = \int_V \{\text{div } \mathbf{T} + \rho\mathbf{B}\} dv \qquad (2.3)$$

where

$$D\mathbf{u}/Dt = \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} \qquad (2.4)$$

is called the material derivative.

The first term is the momentum change of V. The two terms on the right of the first equal sign are the surface forces and body forces respectively. The last term is obtained by applying the divergence theorem to the surface integral.

We assume that, for an incompressible fluid, the stress tensor $\mathbf{T}$ depends linearly on the pressure and on the rate of strain of the fluid $\mathbf{D}$:

$$\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D}, \qquad \mathbf{D} = (\nabla\mathbf{u} + \nabla\mathbf{u}^T)/2 \qquad (2.5)$$

The first term is the normal stress and the second is the shear stress. We assume that viscosity, μ, is constant, thereby neglecting dependence on temperature, sediment concentration, and boundary turbulence. [33]. The only body force for our problem is gravity. We let $\mathbf{g}$ denote the gravity vector. We will use a standard rectilinear coordinate system with the upwards direction positive.

Since V is arbitrary, we obtain in D:

$$D\rho\mathbf{u}/Dt = \text{div } \mathbf{T} - \rho\mathbf{g}. \tag{2.6}$$

Taking ρ to be constant produces the classical Navier-Stokes equation:

$$\rho(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}) = -\nabla p + \mu\Delta\mathbf{u} - \rho\mathbf{g} \tag{2.7}$$

With suitable boundary conditions, the system (2.1), (2.6) will describe moderate erosion inducing flows. If the fluid/sediment mixture density does not vary greatly from the pure fluid density, then we can describe the flow by the much simpler incompressible Navier-Stokes equations (2.2), (2.7). Hereafter, this complete system will be refered to as the N-S system or the Navier-Stokes equations with the density clarified by the context.

Boundary Conditions

We wish to describe flows in an open channel, that is, one with an unconstrained upper surface. We call this free surface η, and postulate that the stress is continuous across η, the upper surface of D. If the atmospheric pressure is normalized to zero, the viscous effects of the air are neglected, and the surface tension of the fluid is considered negligible, we obtain on η

$$\mathbf{Tn}=0. \tag{2.8}$$

No constraint is placed directly on u on η. We obtain $\eta(x,t)$ from:

$$\partial\eta/\partial t = v(x,\eta,t). \tag{2.9}$$

On a fixed surface, the standard no slip boundary condition is imposed. We call the 'solid' streambed boundary $\boldsymbol{\partial}$. For fixed $\boldsymbol{\partial}$, we obtain on $\boldsymbol{\partial}$:

$$\mathbf{u}=\mathbf{0} \quad \text{or} \quad u=0=v \tag{2.10}$$

For a solid but moving object (such as a ship) the no slip condition would still apply resulting in $\mathbf{u}$ equal to the object velocity.

Initial Conditions

In general, for open channel flow, we require an initial velocity field on an initial domain. The initial domain, D_0, is required to satisfy (2.9). The pressure field can be calculated from the velocity field.

A flow that is independent of a spatial direction is called uniform flow and flow that is independent of time is called steady flow. The N-S equations are greatly simplified when the flow is uniform or steady. For steady flows, no initial condition is required, and the absence of an explicit expression for $\partial p/\partial t$ is no longer troublesome.

The kinematics of steady flow are often similar to average flow especially for slow open channel flows. In the case of erosion, however, even though the flow appears steady and uniform, the streambed is in motion.

We can consider the special case where the bulk of the flow is almost steady and uniform but we are forced to treat a general flow as unsteady and nonuniform. There is always at least a region of unsteady, nonuniform flow near the streambed. This time dependence requires that we have some initial conditions which we choose to be

$$\mathbf{u}(x,y,t)=\mathbf{u}_0(x,y,0) \quad \text{on } D_0=\{(x,y) \mid \eta(x,0) < y < \delta(x,0)\} \qquad (2.11)$$

Dimensional Analysis

Dimensional analysis is a useful tool in fluid dynamics. Processes which are not well understood, for which the physics are too complicated, or for which a scaled model is required are often reformulated by using the Buckingham Pi Theorem. [17]. The independent dimensional variables, which are assumed to affect the flow, are mixed and matched to yield a characteristic invariant of the correct dimensional units. The remaining variables are incorporated in non-dimensional invariants in a new functional relation. The combination of the first invariant (with dimension) and the function of the remaining invariants (without dimension) will yield the fundamental form for the unknown if all the necessary dependencies were initially included.

By way of example, suppose that the magnitude of the force, F, trying to roll a particle is a result of the fluid moving around the particle with velocity, $\mathbf{u}$, and viscosity, μ. Taking $\mathbf{u}$ as constant about a disk shaped particle of diameter d_s and applying Stokes' Theorem we have

$$F=k\ \omega\mu(d_s)^2 \qquad (\omega=\text{curl } \mathbf{u}) \qquad (2.12)$$

where k is the appropriate constant.

Now, suppose instead that we assume a form for F, such as $F=F(\omega,\mu,d_s,x_1,x_2,\ldots)$. Force has units mass length/time^2. So, $[F]=ML/T^2$, where [.] denotes dimension, M denotes dimensions of mass, L denotes dimensions of length, and T denotes dimensions of time.

We choose three variables as the set of fundamental parameters. Each succeeding variable will be represented in terms of the fundamental set. Choosing ω, μ, and d_s as the fundamental set yields: $ML/T^2=[\omega]^a[\mu]^b[d_s]^c$.

There is always a unique quantity formed if the parameters are dimensionally independent. In this case, $ML/T^2=[\omega][\mu][d_s]^2$. Thus $F=\omega\mu(d_s)^2 f(II_{x_1},II_{x_2},\ldots)$. Where II_{x_i} is an invariant or non-dimensional quantity using ω, μ, d_s, and x_i. The functional relation f, which is non-dimensional, is determined empirically. Reducing the functional relation to a simple constant, k, yields $F=k\,\omega\mu(d_s)^2$ as before. To get a final working relation, k would be determined empirically.

The Reynolds Number and Other Parameters

For Navier-Stokes flows involving sedimentation, the Reynolds number, $Re=UL\rho/\mu$, and the Froude number, $Fr=U^2/Lg$, are taken to be the non-dimensional parameters which best characterize the flow. U is a representative velocity, L is a representative length, and g is the magnitude of **g**. The Reynolds number essentially measures the ratio of the inertial to the viscous forces while the Froude number measures the ratio of the inertial to the gravity forces.

Although Re and Fr are the only nondimensional parameters appearing in the Navier-Stokes system, another nondimensional quantity is often encountered in sedimentation theory.

Let ν be the kinematic viscosity μ/ρ. Let τ_0 be the shear stress on the boundary. Then, the 'local' Reynolds number is given by $Re^*=u^*L/\nu$, where u^* is called the shear velocity and L is a local length. $u^*=\sqrt{\tau_0/\rho_f}$. Re^* is assumed to be a good characteristic parameter for several unknown functions in sedimentation theory.

One usually sets $L=d_s$ and $\tau_0=du/dy$ where the flow has been simplified to $u=u(y)$. The bottom line is: how well do unknown functions match empirical data? In the previous section, the functional relation f quite often becomes $f=f(Re,Re^*,Fr,\ldots)$.

In nondimensional form the incompressible Navier-Stokes system becomes:

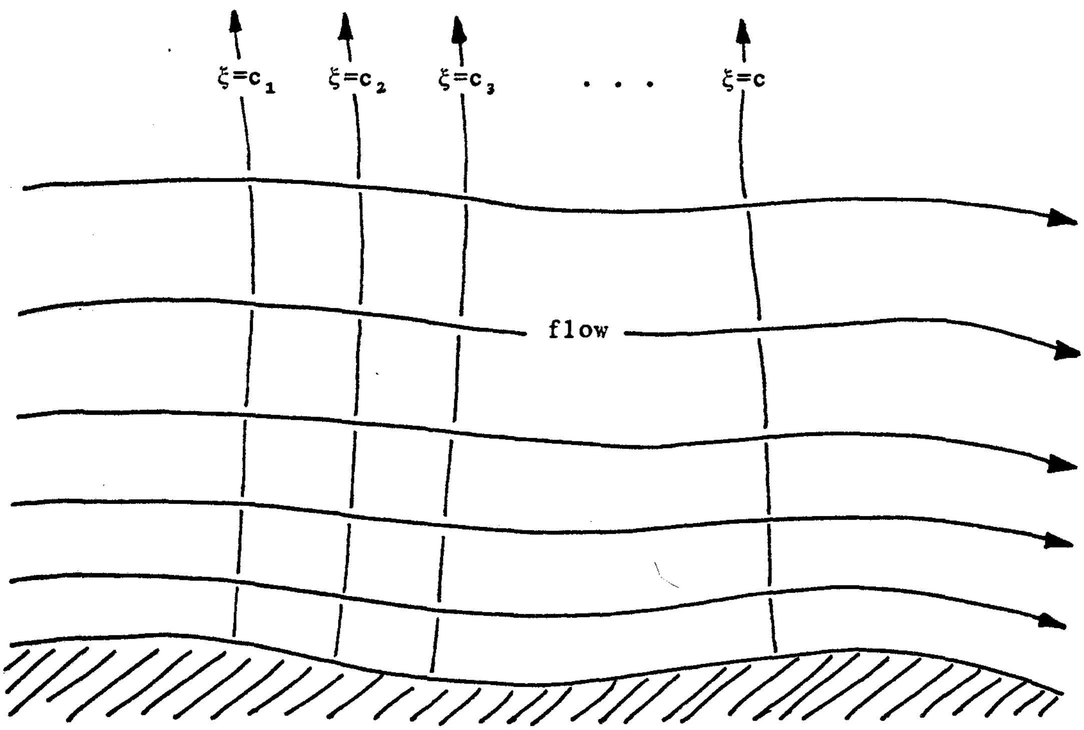
$$D\mathbf{u}/Dt = -\nabla p + 2\mathbf{D}/Re - \boldsymbol{\gamma}/Fr \qquad (2.13a)$$

$$\text{div } \mathbf{u} = 0 \qquad (2.13b)$$

where $\boldsymbol{\gamma}$ is the direction of the gravity vector.

Vorticity and Potential Flow

The vorticity of a flow is given by ω, the curl of the velocity field (recall that in 2-d flow ω is a scalar). The vorticity is a measure of the rotational strength or mixing strength (circulation) of the flow. Irrotational flows, flows for which $\omega=0$, have some special properties. An irrotational flow is circulation free. The integral of **u** around any closed curve is zero by Stokes' Thoerem. Thus, **u** must be the gradient of a function ξ or equivalently **u** is a potential. $\xi=c$ are the equipotential curves for **u**. The stream lines, curves along which the fluid moves, are orthogonal to ξ. (see Fig 2.0) ξ may be found from $\Delta\xi=0$, a direct consequence of $\omega=0=\text{div}\mathbf{u}$. [22]. The pressure can be recovered using Bernoulli's equation. [22].



Potential Flow

Fig 2.0

Even for rotational flows, it may be advantageous to express the Navier-Stokes system in a single equation. This is done by applying the curl operator to (2.13). [33]. We obtain the vorticity transport equation

$$D\omega/Dt = \Delta\omega/Re \quad (2.14a)$$

$$\omega=0 \text{ on } \eta \quad (2.14b)$$

$$u=0=v \text{ on } \phi \quad (2.14c)$$

$\mathbf{u}$ is easily recovered from ω. [33]. The lower boundary condition, however, is in terms of $\mathbf{u}$ rather than ω. [33].

Although mainstream flow may be approximated by potential flow, the flow in the boundary region cannot. Near the boundary, we must either use the full Navier-Stokes equation, some simplified version, or an empirical relation.

The Law of the Wall and Boundary Layers

Our sediment transport process (that is non-constant transport) will be restricted to the region 'near' the streambed. This region is often considered the boundary layer. There is a branch of fluid dynamics devoted strictly to boundary layer theory. [31]. We refer to this as the classical boundary layer theory. Our terminology, although similar to that of classical boundary layer theory, has a slightly different interpretation. We reserve the application of classical boundary layer theory to the erosion problem for future works.

One standard empirical approach to boundary layer flow is given by the logarithmic velocity profile or the 'law of the wall'. It was developed by v. Karmen and Prandtl. [32]. The normal velocity gradient, du/dz, is taken as a function of ρ, τ, and z, the density, stress, and normal distance respectivly. By dimensional analysis, the only combination of these parameters which can occur is

$$du/dz=\sqrt{(\tau/\rho)}/kz \tag{2.15}$$

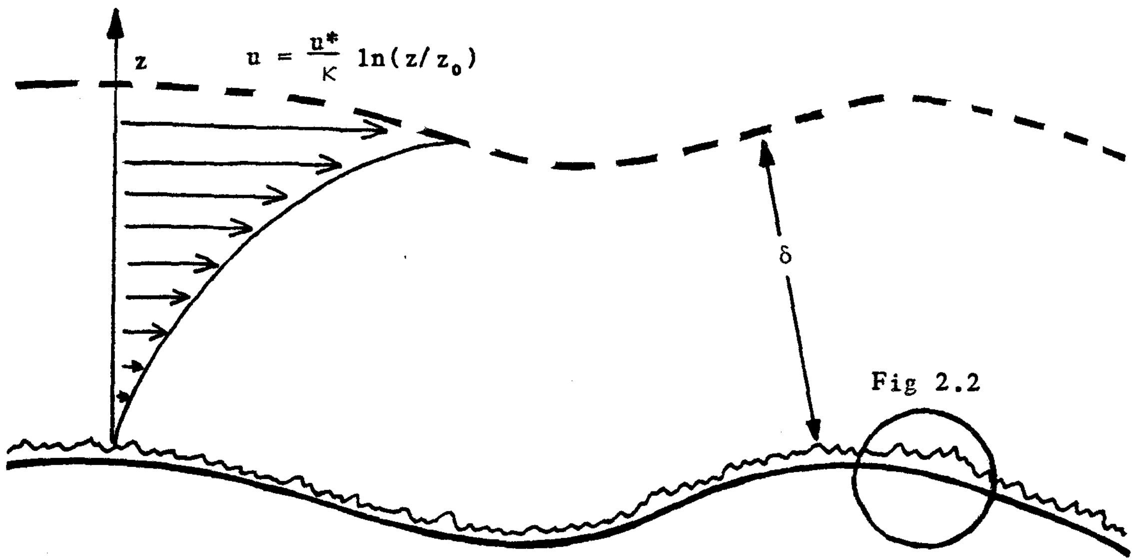
where k is an empirical constant. This constant was empirically determined by v. Karmen. [32]. Upon integration we obtain the Law of the Wall

$$u = \frac{u^*}{\kappa} \ln(z/z_0) \tag{2.16}$$

where u* is the shear velocity mentioned earlier, κ is the von Karmen constant 0.417, z is the normal distance and z_0 is the thickness of stationary fluid. (see Fig 2.1)

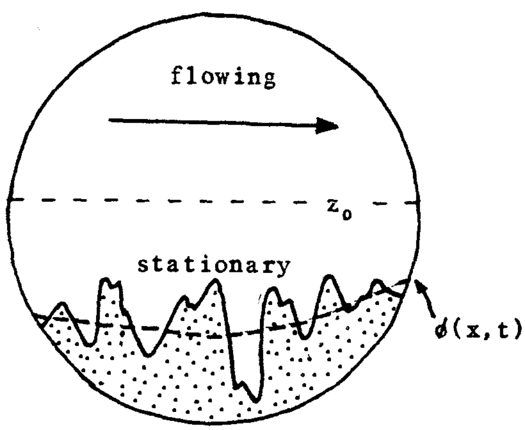
For a boundary composed of sediment, there is always a region of stationary fluid. [28]. (see Fig 2.2) $z_0=\mu/9\rho u^*$ is generally used for smooth boundaries and $z_0=d_s/30$ for sandy boundaries where d_s is the diameter for which 65% of the sand is finer. [28].

The obvious advantage of (2.16) is its independence of the boundary shape. Note, however, that only the dominant term for u is given while assuming laminar flow conditions.



The Law of the Wall

Fig 2.1



The Streambed Close up

Fig 2.2

Energy Dissipation

A discussion of energy dissipation is given by Serrin, [33]. A more complete discussion is given by Lamb, [21]. We give the highlights here.

The kinetic energy, in an arbitrary volume, V, of the fluid, is given by

$$E = \int_V \{\rho \mathbf{u}^2/2\} dv \qquad (2.17)$$

where $\mathbf{u}^2 = \mathbf{u}\cdot\mathbf{u}$. The fluid volume has potential energy as well. Such energy is not dissipated but rather converted to kinetic energy. That is, all the energy dissipated directly into the fluid and surroundings, due to frictional effects, must come from the kinetic energy. We have in V

$$\dot{E} = dE/dt = \frac{d}{dt}\{ \frac{\rho}{2} \int_V u^2 dv\} = \frac{\rho}{2} \int_V \partial u^2/\partial t dv = \rho \int_V \mathbf{u}\cdot\mathbf{u}_t dv. \qquad (2.18)$$

Substituting for $\mathbf{u}_t$ from N-S we obtain

$$\dot{E} = -\int_V \{div[\rho\mathbf{u}(\mathbf{u}^2/2 + p/\rho) - 2\mu\mathbf{u}\cdot\mathbf{D}] - 2\mu\mathbf{D}:\nabla\mathbf{u}\} dv. \qquad (2.19)$$

The term in parentheses is the energy flux due to fluid transport and is the same as for an ideal fluid. [33]. The term $2\mu\mathbf{u}\cdot\mathbf{D}$ is the velocity times the stress tensor and is the energy flux due to the frictional forces. The last term represents the actual dissipation which is lost to heat.

The energy is transfered from large eddies to smaller ones and it is there that the energy is finally dissipated to heat energy or available to the sediment.

Applying the divergence theorem to (2.19) and using $\mathbf{u}=0$ at infinity or $\mathbf{u}\cdot\mathbf{n}=0$ at finite surfaces we obtain in V

$$\dot{E} = -2\mu\int_V \mathbf{D}:\nabla\mathbf{u}\,dv = -2\mu\int_V \mathbf{D}^2\,dv \qquad (2.20)$$

where $\mathbf{D}^2=\mathbf{D}:\mathbf{D}$. (2.20), which is a standard form for the expression for energy dissipation, may be rewritten as the Bobyleff-Forsythe formula

$$\dot{E} = -\mu\int_V \omega^2\,dv - 2\mu\int_{\partial V} \left(\frac{D\mathbf{u}}{Dt}\cdot\mathbf{n}\right) da. \qquad (2.21)$$

In the case of irrotational flow, we have in V

$$\dot{E} = -2\mu\int_{\partial V} \{\mathbf{u}\cdot\mathbf{D}\mathbf{n}\}\ da = -2\mu\int_V \{\nabla\mathbf{u}\}^2\ dv. \qquad (2.22)$$

Some consequences of (2.20), (2.21) and (2.22):

1) In the regions of irrotational flow there is energy dissipation if and only if the flow is nonuniform.

2) The greatest dissipation occurs where the fluid has the largest gradients.

3) If the entire flow is irrotational, then there is no energy dissipation and hence no erosion could take place.

4) A flow which minimizes energy loss must tend towards potential flow.

5) Frictional dissipation of the fluid does not allow for energy to be transfered across a solid no slip boundary.

6) If the flow across ∂V is steady and uniform then the dissipation depends solely on the vorticity in V.

Concluding Remarks- Fluid Dynamics

Equations (2.4) and (2.7) with boundary conditions (2.9) and

(2.10) determine open channel flow. For steady, uniform flow they may be solved quite easily. [10]. Allowing for erosion generally eliminates the steady flow or uniform flow cases. Moderate suspension of sediment does not greatly affect the kinematics of a flow. That is, such flows are quite similar to the case for a pure fluid.

For the unsteady Navier-Stokes system, existence and uniqueness depend upon the boundary configuration and the Reynolds number. Large Reynolds number and complex boundaries restrict the period of existence and uniqueness. [18]. Typical erosion flows are extremely turbulent. We will restrict ourselves to very moderate erosion flows.

For a moderate but general erosion flow, we expect the Navier-Stokes system to be a suitable description of the flow. This system must be combined with a suitable boundary constraint which incorporates erosion.

Sedimentation

A fairly complete discussion of the current theory of sediment transport and related processes may be found in [1], [3], [8], [9], [28] and [39]. We highlight those aspects which are pertinent to the erosion problem.

The theory of sedimentation has largely focused on two areas: the total sediment load in the stream or the microscopic characteristics of the sediment particles.

Generally, sedimentation models are heavily empirical. Expressions for transport parameters are formed from a large variety of nondimensional parameters with Re, Re*, Fr, and energy slope being the most common. The energy slope is simply the average stream gradient which is a measure of the stream's potential to perform work. Most models attempt to deal with a particular erosion phenomenon. We give a brief overview of sedimentation phenomena and some standard models.

In giving some empirical results of sediment transport, we make a distinction between sediment particle transport and boundary shape evolution.

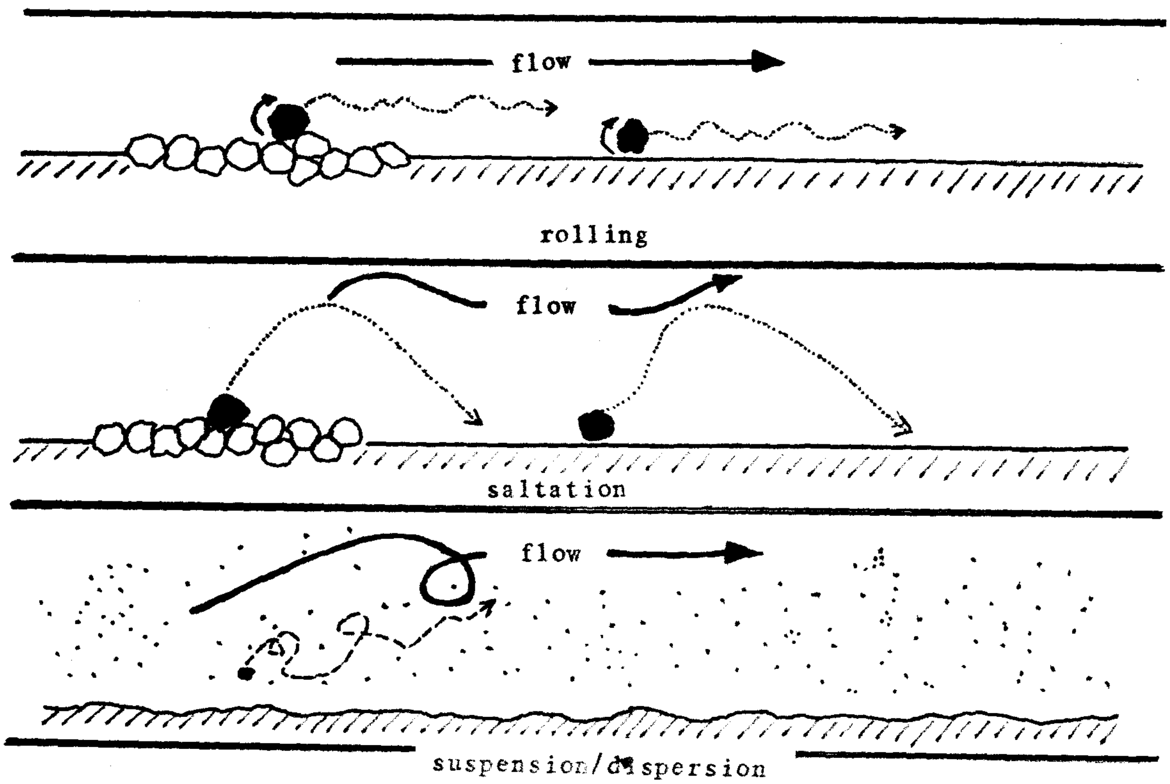
Empirical Results- Sediment Transport

There are three broad classifications of erosion processes with distinct characteristics.

(1) rolling: the particles 'roll' along the streambed,

(2) saltation: the extended bouncing of particles along the streambed,

(3) suspension/dispersion: large quantities of the sediment are carried into the mainstream flow.



Classification of Sediment Transport Mechanisms

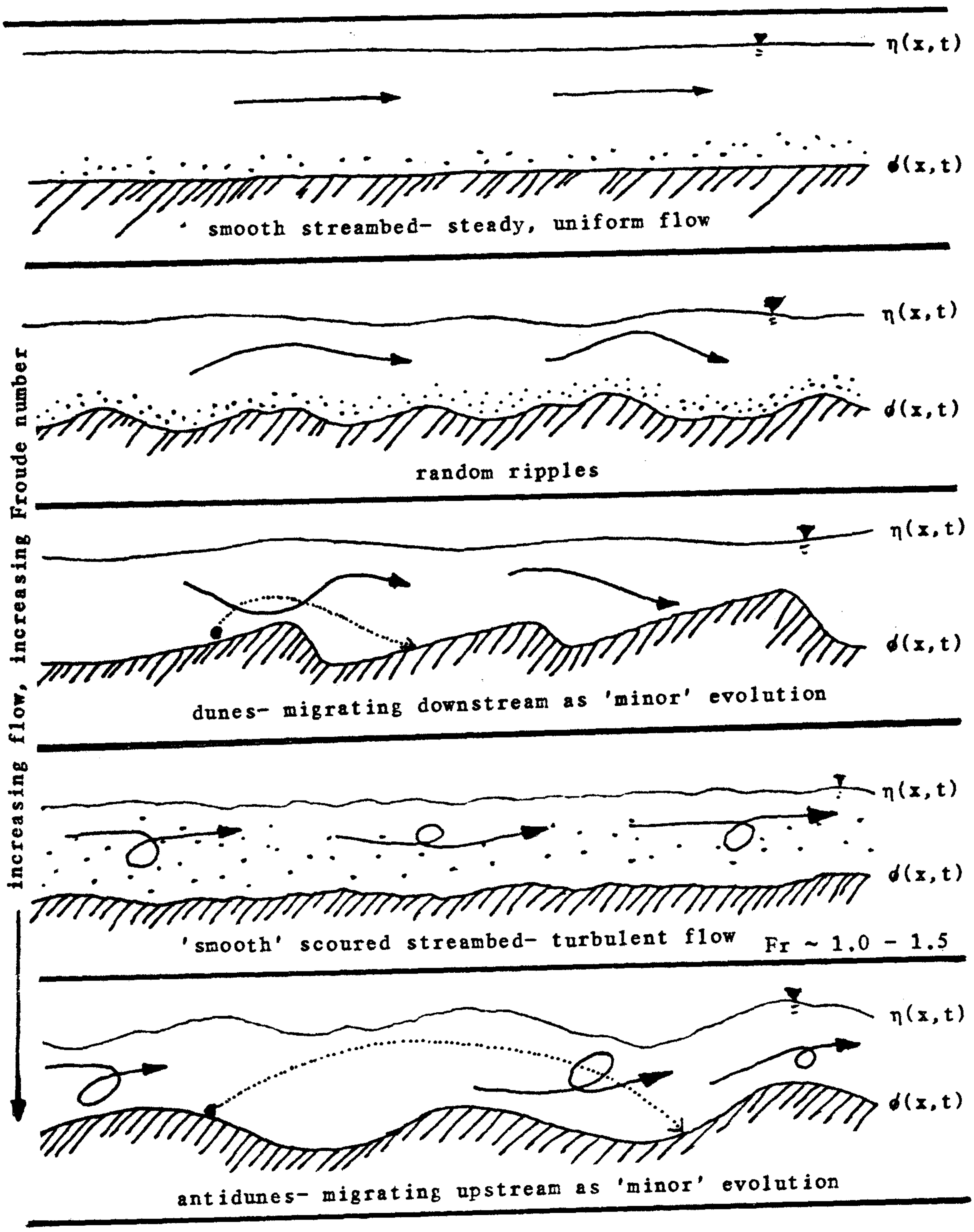
Fig 2.4

Rolling and saltation occur in all eroding flows with the heavier particles (relatively speaking) primarily transported in this manner. (see Fig 2.3) Suspension/dispersion is primarily associated with turbulent flows. Due to the mathematical difficulties associated with turbulence we shall concentrate our study on erosion resulting from rolling and saltation.

Empirical Results- Boundary Evolution

The flow over a bed of sand will cause the bed to undergo a distinct evolution. We distinguish between 'major' evolutionary changes and 'minor' evolutionary changes. At slow flows, the bed remains flat. With increasing flow velocity, the bed passes through a set of stages: ripples, then dunes, then flat again with chaotic flow, then waves and finally antidunes. These are the 'major' changes. Ripples are distinguished from dunes in that dunes have a distinctive form and they migrate downstream. Antidunes migrate upstream. This migration and the classic forms which arise are 'minor' changes. (see Fig 2.4)

The formation of ripples is usually attributed to bed irregularities or fluctuating turbulence. [28]. In the ripple and dune stages, rolling and moderate saltation are the primary transport mechanisms. In the transition stage, saltation is increased and suspension/dispersion becomes nontrivial. In the antidune stage, saltation and dispersion are extensive.



'Major' and 'Minor' Streambed Evolution Stages

Fig 2.4

Few models have made an attempt to account for all these changes. Once dunes form, however, their migration is easy to explain. Antidunes eventually form when the pressure gradient on the lee side of the dune becomes strong enough to cause extensive particle lift. These stages appear to be strongly correlated with the Froude number. The boundary shape evolves as the Froude number increases with antidunes forming when $Fr \sim 1.0$-1.5.

We shall concentrate our study on this evolution phenomenon. Our model will attempt to account for these evolutionary stages as well as giving an explaination for the initial dune formation.

A Brief History of Sedimentation Theory

A fairly complete history of recent sedimentation transport models is given by Allen, [1], and Vanoni, [39]. Bed load transport models have generally been developed from one of five main themes:

(1) average fluid velocity,

(2) streambed shear stress,

(3) probabilistic inferences,

(4) bedform celerity relationships,

(5) energetics.

Probably the first formula involving an erosion process is due to Brahms, 1753, and involves the average stream velocity. Brahms found the empirical relation, $V_c = kW_s^{1/6}$, between a critcal velocity and the weight of an object it can move. This relation may also be derived from some basic assumptions. [4]. It is used today primarily in engineering design.

Gilbert, 1914; Donat, 1929; Straub, 1939; Meyer-Peter and Muller, 1948; et. al. developed formulae of the form $q_s=q_s(U^n)$, $q_s=kU^2(U^2-U_c)$, etc. Other similar models assume that sediment discharge properties may be characterized by the mainstream Reynolds number, local Reynolds number, and Froude number (i.e. $q_s=q_s(Re, Re^*, Fr)$). These are empirical formulae which depend only on characteristic values and not local values. The sediment discharge rate correlates well with these parameters but models arising from these expressions do not deal with distinct boundary movement but only the average effect of the stream on the sediment load.

It was the idea of a critical force which led DuBoys, in 1879, to derive a general equation $q_s=k\tau_0(\tau_0-\tau_c)$ for the sediment discharge, q_s, in terms of a critical tractive force, τ_c, where τ_0 is the bed shear and k is an empirical constant. A number of refinements to his model exist, most of which concentrate on improving the expression for τ_c. [38].

Most of the theory of sediment transport from a probabilistic approach is attributed to H. A. Einstein, 1942, 1952 and Kalinske, 1947. Einstein wanted to account for the initiation of ripples on a flat streambed without resorting to the irregularity arguement. He assumed normally distributed turbulence fluctuations to account for the initial uneven erosion.

The bedform celerity models such as those of Kennedy, 1960; Crickmore, 1970; Korchokha, 1972; Willis and Kennedy, 1977; at. el.

are strictly empirical formulae relating dune characteristics (height, length, etc) to the dune celerity. Kennedy's work is perhaps the most extensive and one of his models is given here. [20]. $\lambda_s = (2\pi/g)U^2$ gives the dune wavelength, λ_s, in terms of a mean velocity, U. g is the magnitude of gravity.

Models based on the energy of the flow are largely due to Bagnold, 1966. These models are derived from theoretical considerations and are generally of the form $q_s=k(\tau_0U_0 - \tau_cU_c)$ or $q_s=k\tau_0U$. Gilbert, 1914; Meyer-Peter, 1948; Straub, 1939; et. al. have developed models based on the energy slope (stream gradient).

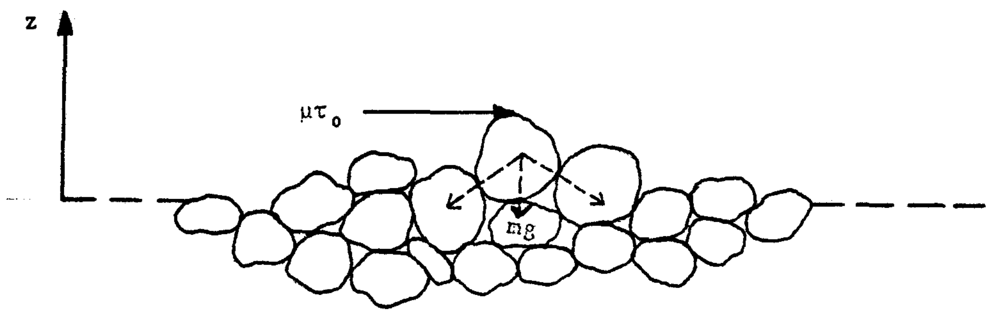
In summary, we note that, dimensional analysis is often employed to arrive at the fundamental dependence of the unknowns. Of the present models, vorticity does not appear to have been considered as a fundamental parameter. The mathematical difficulties associated with turbulence makes it difficult to deal with the true flow field of typical erosion flows. The flow field is often buried in a constant such as the average stream velocity allowing for no local variation. The probabilistic models are the only ones which seem to deal with the boundary evolution stage from theoretical considerations.

Forces on the Sediment Particle

The forces on the sediment are usually expressed in either global terms or very specific terms. In global terms, the frictional force on a smooth flat surface is given by

$$\mu\tau_0=\mu du/dz \tag{2.23}$$

where z is normal to the boundary. The specific forces on individual particles are also considered. They try to pivot and lift the particle out of its position as shown in Fig 2.5. The moment is written in terms of $\mu\tau_0$, the particle weight, and the pivot angle. In actuallity, the fluid forces should apply to the entire surface. We will be using a slightly different force diagram in our model derivation.



Classical Force Diagram for a Sediment Particle

Fig 2.5

Concluding Remarks- Sedimentation

The theory of sediment transport, though related, is distinct from the theory of boundary evolution. Most of the work has been concentrated in sediment transport theory. The models, though developed initially from physical considerations, become very empirical in nature. It appears that only the probabilistic models have addressed the question of initial dune formation arising from a smooth streambed. There does not appear to be any accepted theory of the actual process of dune formation. The generally held belief of coagulation around some irregularity in the surface does not stand up to close scrutiny. Increased sediment homogeniety still yield ripples and dunes. A model which can address the major evolutionary changes must combine a general flow with some reasonable streambed erosion constraint.

THE EROSION MODEL

Our basic approach in modeling streambed erosion is to combine the interaction of a flow with some boundary constraints which represent an erodable boundary. The flow is characterized by a general viscous flow (i.e. the Navier-Stokes equation). The boundary condition is obtained by balancing sediment mass, momentum, and energy.

The erosion model consists of open channel flow together with its boundary condition representing erosion. We give a brief overview of the problem along with some terminology. The basic assumptions pertaining to the model are noted along with the degree of their validity. The erosion constraint as a boundary condition is derived. Some special cases of flows are considered in chapter IV and a general flow is considered in chapter V.

We use the term erosion to encompass both the deposition and the scouring, or lifting up, of sediment. The entire fluid domain, D, is divided into two regions: the mainstream region, Ω, and the region in which sediment is transported, Γ. We assume negligible sediment transport in Ω. The sediment in Ω behaves as if it is fully dissolved and does not itself affect the sediment load in Γ. Γ contains a mixture of both fluid and sediment. We will refer to this mixture as a fluid. The upper surface of Ω, the air/fluid interface, is free, and is denoted by η. The lower surface of Γ, the streambed, is also free, and is denoted by ϕ. 'Surface' will refer to η while the 'bed' will refer to the streambed, ϕ. The Ω/Γ interface is denoted by σ. Refer to Fig 1.0 for a schematic.

The Flow Field

We assume that the erosion process is a local phenomenon, the result of the interaction between the fluid and sediment along the streambed. Erosion depends on the local flow field which we take to be governed by the Navier-Stokes Equations for open channel flow.

Difficulties arise concerning the conservation of mass. The sediment particles are discrete rigid masses but they cannot easily be modeled as dense points of the continuum. Fortunately, the kinematics of the flow do not depend strongly upon small amounts of sediment in solution. [28]. The sediment transport is assumed concentrated near the boundary. This allows us to govern our flow in Ω by the incompressible Navier-Stokes equation for constant density.

Let $\mathbf{x}=(x_1,x_2,x_3)$ with the upwards direction positive. Throughout our discussion we assume the flow to be constant in the x_3 direction. Thus, we equate $\mathbf{x}$ with (x,y) where any ambiguity will be clarified.

We let $\mathbf{u}=(u(\mathbf{x},t),v(\mathbf{x},t))$ be the velocity field and $p=p(\mathbf{x},t)$ be the pressure field in D. We normalize the pressure so that atmospheric pressure is zero. We let $\rho=\rho(\mathbf{x},t)$ denote the density of the fluid (mixture) and the density of the bed sediment is denoted by ρ_s. We assume ρ is constant in Ω and ρ_s is constant in the streambed. The flow in Ω is then governed by:

$$\rho D\mathbf{u}/Dt = \text{div } \mathbf{T} + \rho\mathbf{g} \quad \text{in } \Omega(t) \tag{3.0a}$$

$$\text{div } \mathbf{u} = 0 \quad \text{in } \Omega(t) \tag{3.0b}$$

$$p=0 \quad \text{on } \eta(x,t) \tag{3.0c}$$

$$\mathbf{Tn=0} \quad \text{on } \eta(x,t) \tag{3.0d}$$

$$\eta_t = v|_{y=\eta} \tag{3.0e}$$

$$u_0,\ v_0,\ D_0 \text{ given at } t_0 \tag{3.0f}$$

where $D/Dt:=\partial/\partial t + (\mathbf{u}\cdot\nabla)$ is the material or spatial derivative, and

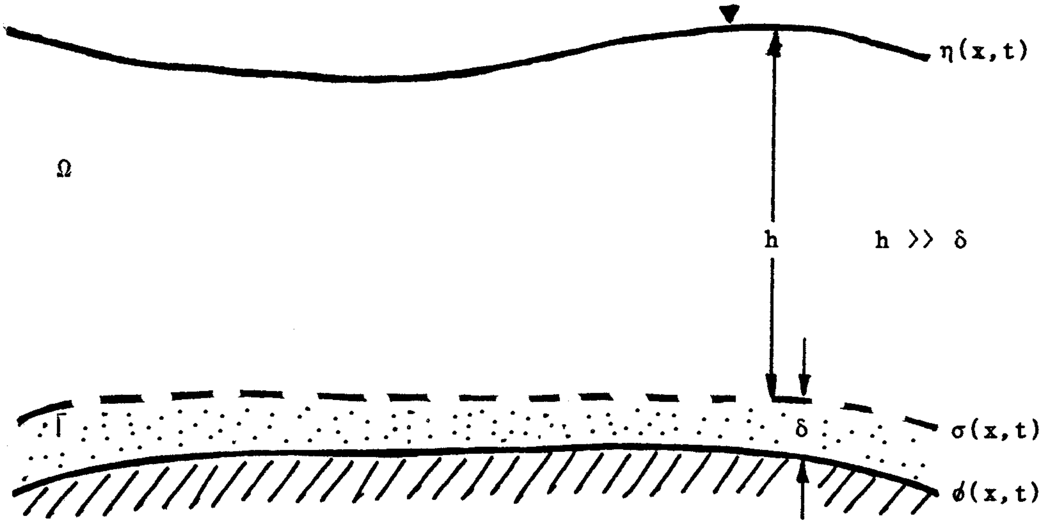
$$\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D}, \qquad \mathbf{D} = (\nabla\mathbf{u} + \nabla\mathbf{u}^T)/2 \tag{3.0h}$$

defines the stress-strain relationship of the fluid.

(a) is the Navier-Stokes equation with $\mathbf{T}$ being the stress tensor and $\mathbf{g}$ being the gravity vector. (b) is the Continuity equation for constant density. (c) represents normalized atmospheric pressure. (d) represents continuous stress accross η. (e) is an auxiliary condition resulting from (d). (f) gives the initial conditions.

The system (3.0) is not complete but requires an additional boundary condition. This condition cannot be given on σ, the Ω/Γ boundary, since the flow moves accross that interface arbitrarily. It is the sediment, not the fluid, which is trapped in Γ.

We would like to extend (3.0) to include Γ. To accomplish this, we assume that the depth of Γ is 'small'. Let δ denote the thickness of Γ and let h denote the flow depth. We assume δ is 'small' relative to h. (see Fig 3.0) It is important to note that δ is not being used in the sense of classical boundary layer theory.



The Flow Regions Ω and Γ

Fig 3.0

We interpret the region Γ as follows:

The region Γ is infintesimal with respect to the region Ω. This is a mathematical simplification which may be physically interpreted as the sediment load moving along the bed as an impulse load. Thus, the sediment load is no longer a function of the vertical displacement but only a function of position along ϕ.

The mainstream flow kinematics and boundary layer kinematics inherent in the Navier-Stokes equation are taken to be the primary influence on the transport process. The classical boundary layer theory uses simplifications to arrive at the same effects. The trade off at this point is between matching two sets of equations; one set for the main stream and one set for the boundary layer versus one set of equations and an infintesimal boundary layer.

There is a natural boundary condition on ϕ, namely, the no slip condition which we adopt. Since δ is 'small', the flow in Γ can be considered as a flow only along the streambed. We no longer distinguish between D and Ω. Physically, the flow in D is totally dominated by the flow in Ω. The system (3.0) becomes:

$$\rho D\mathbf{u}/Dt = \text{div}\,\mathbf{T} + \rho g \quad \text{in } D(t) \tag{3.1a}$$

$$\text{div}\,\mathbf{u} = 0 \quad \text{in } D(t) \tag{3.1b}$$

$$p=0 \quad \text{on } \eta(x,t) \tag{3.1c}$$

$$\mathbf{Tn}=\mathbf{0} \quad \text{on } \eta(x,t) \tag{3.1d}$$

$$\eta_t = v|_{y=\eta} \tag{3.1e}$$

$$u=0=v \quad \text{on } \phi(x,t) \tag{3.1f}$$

$$u_0,\ v_0,\ D_0 \text{ given at } t_0 \tag{3.1g}$$

The flow domain Ω has been extended to D and the no slip boundary constraint, (f) has been added. For a given ϕ, the flow is now completely determined by (3.1). This no slip condition does not preclude erosion but rather states that there is a level at which the bed is fixed and, at that level, the fluid adheres to the bed. Realistically, the bed has fluid moving through it and Γ is an agitated mixture of the fluid and sediment moving along very close to the bottom. We have simplified the picture by assuming that the sediment moves along so close to the streambed that $D \to \Omega$. The 'boundary' is made up of a movable solid.

Streambed Erosion as a Boundary Condition

We now have the flow given by (3.1) and require a boundary condition, which reflects the process of erosion, to complete the model. We obtain this condition from the conservation of mass law for the sediment.

Let V be an arbitrary volume in Γ. Let c_s be the concentration of sediment (mass/volume) in V. The total sediment mass in V is given by

$$\int_V c_s \, dv \tag{3.3}$$

and the rate of mass gain is given by

$$\frac{d}{dt} \int_V c_s \, dv. \tag{3.4}$$

For fine sediment, c_s is a continuous function. So,

$$\frac{d}{dt} \int_V c_s \, dv = \int_V \partial c_s/\partial t \, dv. \tag{3.5}$$

Let $\mathbf{u}_s$ denote the velocity of sediment particles. Then, the mass flux into V is given by

$$-\int_{\partial V} c_s \mathbf{u}_s \cdot \mathbf{n} \, da \tag{3.6}$$

where ∂V denotes the boundary of V and $\mathbf{n}$ is the outward unit normal. By the Divergence Theorem we have

$$-\int_{\partial V} c_s \mathbf{u}_s \cdot \mathbf{n} \, da = -\int_V \operatorname{div} c_s \mathbf{u}_s \, dv. \tag{3.7}$$

Since mass is conserved, we have in Γ

$$\int_V \partial c_s/\partial t \, dv = -\int_V \operatorname{div} c_s \mathbf{u}_s \, dv. \tag{3.8}$$

We let $\delta \rightarrow 0$, the width of Γ shrinks, and let ∂V coincide with $\mathscr{B}$, the bed. We assume that the sediment is concentrated in a thin layer along ϕ and is uniform in that layer. (3.8) becomes

$$\delta\int_{\partial\phi} \partial c_s/\partial t \, dv = -\delta\int_{\partial\phi} \{\partial(c_s u_s)/\partial\xi + k_s \phi_t\} \, dv \qquad (3.9a)$$

$$k_s = c_p c_s [1 + (\phi')^2]^{-1/2} \qquad (3.9b)$$

where $\partial\phi$ denotes the portion of ∂V which coincides with ϕ. ξ is the unit tangent vector to ϕ and u_s is the sediment velocity parallel to ϕ. $\partial/\partial\xi$ refers to the change along ϕ. k_s relates the bed density (100% concentration) to c_s, for a given change in ϕ, by c_p, the bed packing ratio.

Since V was arbitrary, $\partial\phi$ is arbitrary and we have

$$\partial c_s/\partial t = -\partial(c_s u_s)/\partial\xi - k_s \phi_t. \qquad (3.10)$$

The term on the left represents the total increase in sediment mass due to locally unsteady flow. The first term on the right represents the tangential flux of sediment due to locally nonuniform (tangential) flow. The last term represents the normal flux of sediment which is a source/sink term due to erosion.

We would like explicit forms for c_s, and u_s. We restrict ourselves to quasiequilibrium erosion where changes in the sediment concentration are 'small'. A significant sediment flux accross ϕ is always accounted for by a relative increase in the transport rate. In this way, changes in the sediment concentration remain small. In this case, c_s is constant and (3.10) becomes

$$-c_p \phi_t = du_s/dx \qquad (3.11)$$

where $d/dx = \{\partial/\partial x + \partial/\partial y \, dy/dx\}|_{y=\phi(x)}$.

We cannot assume u_s to be constant for if we were to make such an assumption, no erosion would occur since the right side of (3.11) vanishes. There are a variety of possibilities for determining u_s

including:

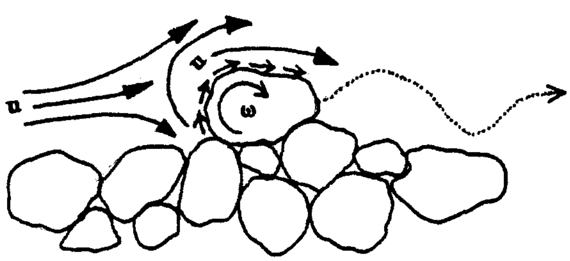
(1) Assume that the sediment rolls with the fluid.

(2) Assume the rate of work in transporting the sediment depends on the energy dissipated by the fluid.

(3) Use dimensional analysis on $u_s(p_1, p_2, \ldots)$ where p_i are parameters influencing sediment transport.

We consider both (1) and (2) and make some remarks concerning (3).

Method (1): <u>Obtaining an expression for u_s by treating the fluid as a collection of rolling, spinning fluid volumes</u>



Sediment Particle Transport Mechanism

Fig 3.1

The sediment is made up of small homogeneous particles with $\rho \sim \rho_s$. They are spun, rolled or agitated in much the same manner as a similar volume of fluid. If we neglect resistance, a ball immersed in a flow would quickly assume the same local angular velocity as the fluid. We derive u_s for the case where the sediment rolls and spins with the fluid. That is, we treat the fluid as a conglomeration of rolling, spinning volumes. The sediment is assumed to roll and spin in a similar manner. (see Fig 3.1)

Consider a particle S of mean diameter d_s rolling and spinning in the fluid. Let U_s be the mean surface velocity of S. Let $\dot{\theta}$ be the mean rotational velocity of S. Let $\mathbf{u}$ be the fluid velocity, the fluid being adhered to the particle. Then,

$$U_s = (1/\pi d_s) \int_{\partial S} \mathbf{u} \cdot d\mathbf{s} \qquad (3.12)$$

where ∂S denotes the boundary of S. Applying Stokes' Theorem we have

$$U_s = (1/\pi d_s) \int_S \text{curl } \mathbf{u} \cdot d\mathbf{a} \qquad (3.13)$$

But $u_s = \pi d_s \dot{\theta}$ and $\dot{\theta} = U_s/\pi d_s$. An application of the Mean Value Theorem yields

$$U_s \sim \omega d_s/4\pi \qquad (3.14)$$

where ω is the magnitude of the vorticity at $(x, \phi(x))$.

We obtained (3.14) by neglecting resistance. We include the presence of resistance in as simple a way as possible by assuming U_c to be a critical threshold which U_s must exceed. We obtain the expression for u_s

$$u_s = \text{dir}[U_s - U_c]^+ \qquad (3.15)$$

where $\text{dir} = -\omega/|\omega|$. dir denotes the direction of u_s. $[.]^+$ denotes the positive part of $[.]$.

In general, $U_c = U_c(d_s, \phi', \gamma_s, \ldots)$ gives the resistance to movement for the sediment. That is, the resistance is a function of such parameters as the particle size, the particle shape, the particle weight, the bed slope, etc. For simplicity, we take

$$U_c = k(\phi' + \theta_c) \qquad (3.16)$$

where θ_c is the angle of repose for the sediment and k is an empirical constant. This form includes only bed slope effects. A more general relation could be obtained by applying dimensional analysis.

Recall D'Alembert's Paradox which says that there is no force on a particle in a perfect fluid. As the sediment becomes more distinct from the fluid, u_s must depend on the viscosity as well as particle size and weight. Recall that U_s was derived by neglecting all resistance. The term U_s should incorporate an empirical constant which takes these properties into account. The first generalization of (3.15) is

$$u_s = \mathrm{dir}[c_1\omega - c_2(\phi' + \theta_c)]^+. \qquad (3.17)$$

Method (2): Obtaining an expression for u_s by equating the energy dissipated by the fluid with the work rate of sediment being transported

The energy dissipated by the fluid must be either converted to heat or transfered to the sediment.

Energy dissipation is carried out in the following manner: Energy is transfered from the larger eddies, as they decompose, to smaller ones with little loss. The small eddies then transfer their kinetic energy, usually to heat but also to sediment transport.

Within the large eddies, the flow is almost irrotational while vorticity dominates in the smaller eddies. [21]. This explanation is consistent with our interpretation of (2.21). From (2.21), we deduced that energy dissipation was minimized by potential flows.

Let $\dot{E}_f$ be the rate energy is dissipated by the fluid. Then, the rate energy becomes available to the sediment, for transport, is given by

$$\dot{E}_a = \{\text{efficiency}\} \times \{\dot{E}_f - \text{unavailable and lost energy}\}. \quad (3.18)$$

Clearly, some energy is unavailable or lost due to fluid properties or to sediment/boundary characteristics. For example: Energy is 'unavailable' due to the heating of fluid which is independent of the erosion process. Energy is 'lost' due to the friction on a cohesive boundary. Obviously, only a portion of this remaining energy is available to contribute to the rate of work performed by the sediment due to the 'efficiency' of the energy transfer process.

Let $\dot{E}_c$ be that critical, minimum amount of energy initially lost or unavailable and let e be the efficiency of energy transfer. $\dot{E}_c$ accounts for the energy which must be dissipated regardless whether or not erosion takes place. (3.18) becomes:

$$\dot{E}_a = e[\dot{E}_f - \dot{E}_c]^+ \quad (3.19)$$

The total energy dissipated by the fluid is given by (2.21) or (2.22) and may be alternatively written as

$$\dot{E} = -\mu\int_V \{\omega^2 - 4\det\nabla\mathbf{u}\}\, dv \quad (3.20)$$

where μ is the viscosity of the fluid.

The term $\det \nabla \mathbf{u}$ measures the dissipation due to deforming V and we assume this energy to be 'unavailable', leaving:

$$\dot{E}_a = e[\mu\omega^2 - \dot{E}_c]^+ \tag{3.21}$$

With regards to the sediment, we assume that potential energy changes are negligible relative to the kinetic energy changes. The sediment is lifted only an infintesimal distance. Losses due to heating are absorbed in $\dot{E}_c$ and e. Thus, all the work is contained in transporting the sediment and is given by

$$u_s f_s \tag{3.22}$$

where f_s is the force on the sediment particles which causes them to roll and spin.

There are a variety of ways to arrive at an expression for f_s. We give two. First, we assume $f_s=f_s(\omega,d_s,\mu,p_1,p_2,\ldots)$. ω,d_s, and μ are obvious choices as fundamental parameters. By dimensional analysis, the only quantity of force using ω,d_s, and μ is given by $\mu\omega$. Thus,

$$f_s=\mu\omega f \tag{3.23}$$

where f represents $f(II_1,\ldots)$ II_i being the nondimensional quantities formed from μ, ω, d_s, and p_i. The simplest form for f_s is then

$$f_s=k\mu\omega \tag{3.24}$$

We obtain the same expression for f_s by assuming the force to be a result of rolling a particle where the surface force is given by $\mu f(\mathbf{u})$. Applying Stokes' Theorem to $f(u) \sim k_1 u$ gives the force per particle as

$$k_1\mu\omega. \tag{3.25}$$

The number of particles is constant so we again obtain (3.24).

With f_s now given, we relate the work rate to the available energy to obtain

$$u_s = e[\mu\omega^2 - \dot{E}_c]^+/f_s \tag{3.26}$$

which becomes

$$u_s = dir[c_1\omega - r]^+ \tag{3.27}$$

where we interpret those terms grouped into r as a result of resistance factors.

We have obtained a similar expression for u_s by method (1).

Method (3): <u>Obtaining an expression for u_s from strictly empirical considerations</u>

Another standard method for determining u_s would be to use dimensional analysis. The only question would be the choice of fundamental parameters. Consider that the vorticity plays a role in the fluid energy dissipation, is a measure of the local turbulence, is a measure of the local agitation at the bottom, and is related to the rolling and spinning of the fluid and the sediment. Clearly, ω would be a good choice for a fundamental parameter of streambed erosion. However, ω does not appear to have been previously used as a fundamental parameter. One obvious explanation for this is the implied uniform flow field inherent in most models. Countless expressions for u_s could be obtained using ω as a fundamental parameter. Additionally, improving the physical reality of the model would be best achieved by replacing empirical constants by empirical relations using such parameters. We leave such additional versions for later studies.

We choose (3.17) as our working version for u_s for the remainder of this work. Our final version of our model becomes

$$\rho D\mathbf{u}/Dt = \text{div}\ \mathbf{T} + \rho\mathbf{g} \quad \text{in } D(t) \tag{3.28a}$$

$$\text{div}\ \mathbf{u} = 0 \quad \text{in } D(t) \tag{3.28b}$$

$$p=0 \quad \text{on } \eta(x,t) \tag{3.28c}$$

$$\mathbf{Tn=0} \quad \text{on } \eta(x,t) \tag{3.28d}$$

$$\eta_t = v|_{y=\eta} \tag{3.28e}$$

$$u=0=v \quad \text{on } \phi(x,t) \tag{3.28f}$$

$$-c_p\phi_t = du_s/dx \tag{3.28h}$$

$$u_s = \text{dir}[c_1\omega - c_2(\phi' + \theta_c)]^+ \tag{3.28g}$$

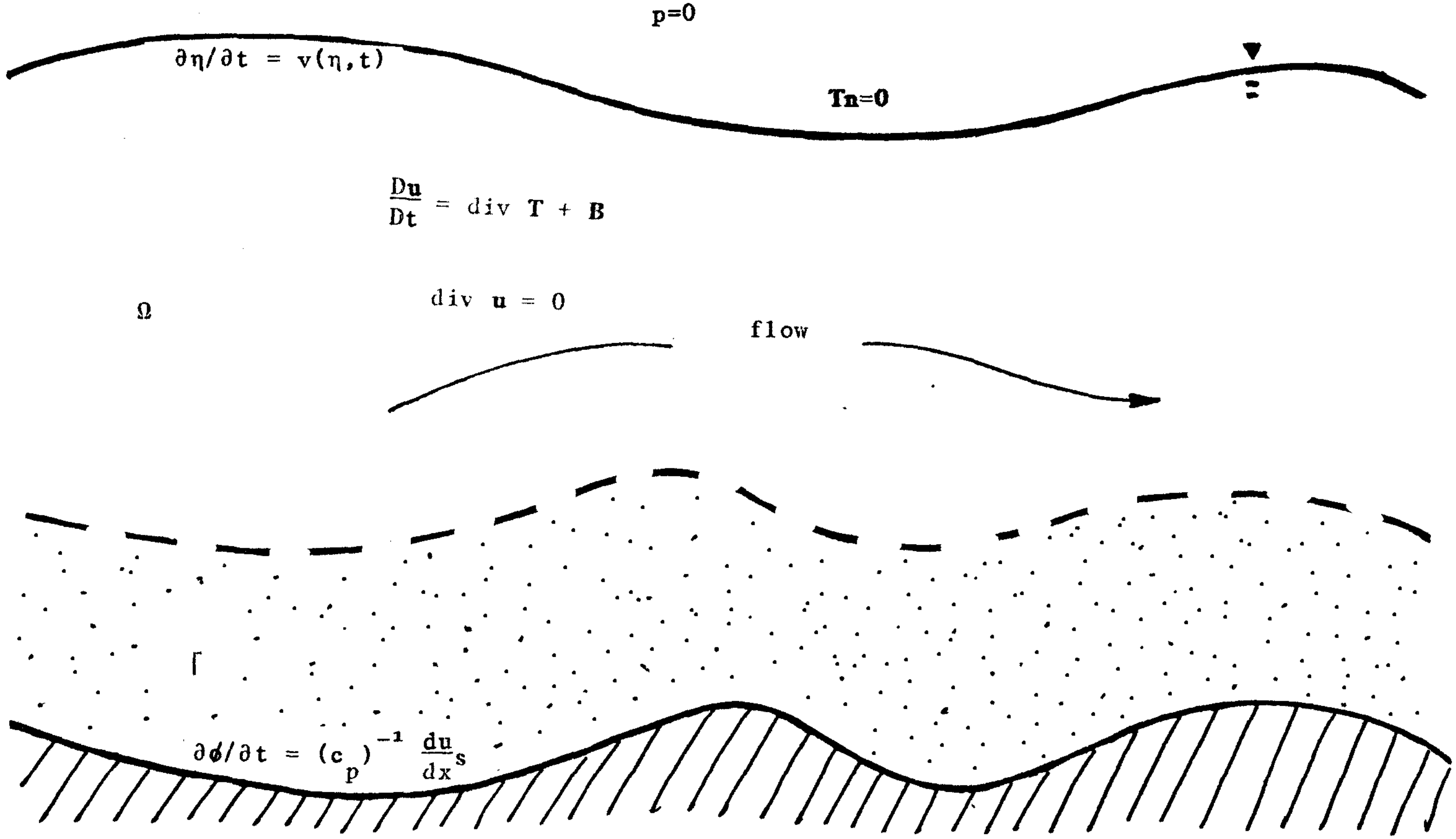
$$u_0,\ v_0,\ D_0 \text{ given at } t_0 \tag{3.28h}$$

where $d/dx=\{\partial/\partial x + \partial/\partial y\ dy/dx\}|_{y=\phi(x,t)}$.

Concluding Remarks- The Model

The complete model, consisting of a flow field and a streambed boundary condition, is given by (3.28). Given below are the highlights of the model and its derivation.

1. The erosion process consists of a flow interacting with an erodable boundary.



The Erosion Model- An Overview

Fig 3.2

2. The erosion process is obtained by balancing sediment mass, energy and momentum.
3. The erosion producing flow is governed by the Navier-Stokes equations for open channel flow.
4. The sediment transport occurs in a 'thin' layer along the stream bed.
5. Our model is for a relatively fine, round sediment. Large discreet particles would invalidate the mass balance derivation as well as the assumptions used to obtain u_s.
6. The sediment transport must be in a state of quasiequilibrium. Increased sediment flux, across the boundary, results in an increased transport rate rather than an increased sediment concentration.
7. The vorticity of the flow near the streambed is a fundamental factor for the erosion process in our model.
8. The model will accept a variable flow field. Such a flow field is necessary for variable erosion.
9. An overview of the complete model is shown in Fig 3.2.

SPECIAL FLOWS

We consider three flow cases:

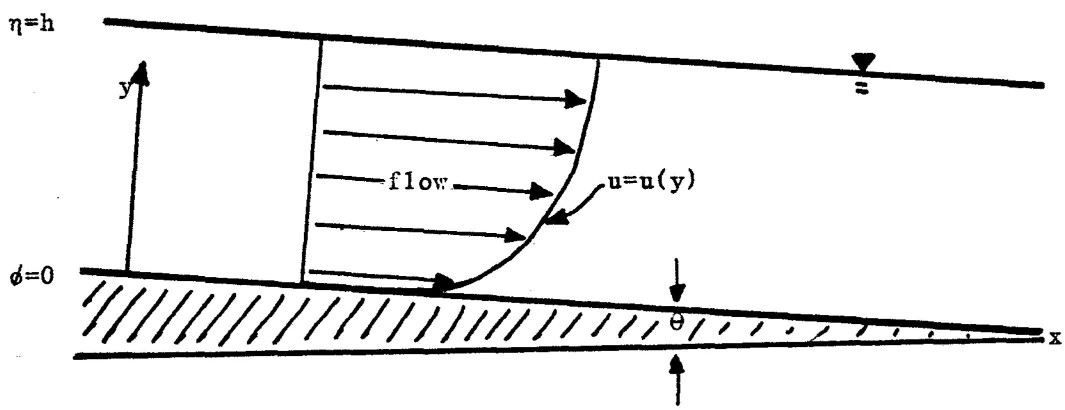
(1) almost uniform, steady flow,

(2) potential flow in Ω with 'law of the wall' in Γ,

(3) a general flow.

Due to the complexity of a general flow, (3) is dealt with in chapter V.

Almost Steady, Uniform Flow



Steady, Uniform Flow

Fig 4.0

If we replace the erosion condition, (3.27g), in our model, (3.27), with a fixed streambed condition, we have standard open channel flow. If we assume a steady, uniform flow over a smooth bottom, η would be flat also. (see Fig 4.0) In this case, $u=u(y)$ and $p=p(y)$ and our model, (3.27), becomes:

$$0 = \mu u_{yy} + \rho g \sin\theta \qquad (4.0a)$$

$$0 = -p_y - \rho g \cos\theta \qquad (4.0b)$$

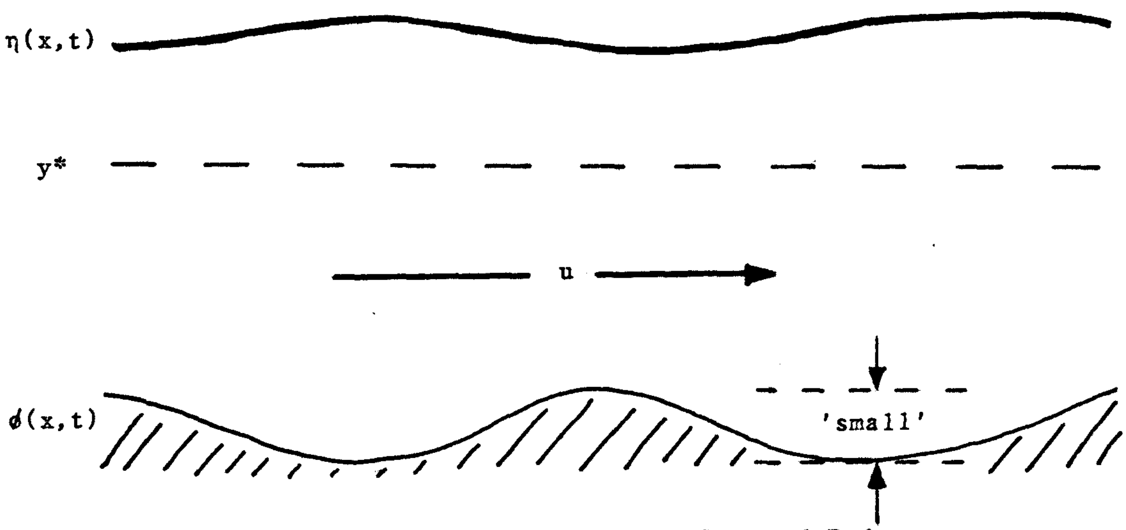
$$u=0 \text{ on } y=0 \qquad (4.0c)$$

$$\mathbf{Tn}=\mathbf{0} \text{ on } y=h \qquad (4.0d)$$

We solve this exactly and give the solution below.

$$u = \frac{\rho g}{2\mu}(\sin\theta)y(2h-y) \qquad (4.1a)$$

$$p = \rho g(\cos\theta)(h-y) \qquad (4.1b)$$



Almost Steady, Uniform Flow- A Rippled Bed

Fig 4.1

In actuality, such flows are unstable and will at least experience boundary turbulence. Suppose, however, that we use this flow as an approximation to the flow in the case where the streambed is relatively smooth and flat; that is, where ϕ' is 'small'. We are, in essence, assuming this flow to be approximated by the stable flow given by (4.1).

We consider almost steady, uniform flow over a rippled boundary. (see Fig 4.1) Of necessity, the bed gradients must be small. If the flow is too slow, then u_s will be zero and no erosion will take place at all. If the flow is too strong, our approximation will be unrealistic. In between these extreems, u_s is non-zero and approximated by

$$u_s \sim k_0 + k_1 du/dy. \qquad (4.2)$$

Assume now that the flow is deep enough that there exists y* such that for y < y* the fluid discharge is approximately constant. Then,

$$q_f \sim (1/2)(y^* - \phi)^2 du/dy \qquad (4.3)$$

where q_f is the volumetric fluid discharge in the region $[\phi, y^*]$. This gives

$$du/dy \sim \text{constant}/(y^* - \phi)^2. \qquad (4.4)$$

Substituting (4.4) into (4.2) we obtain

$$du_s/dx \sim k/(y^* - \phi)^3 \partial\phi/\partial x \qquad (4.5)$$

where k is the appropriate constant.

Our erosion constraint (3.17) becomes

$$\phi_t + c\phi_x = 0 \qquad (4.6a)$$

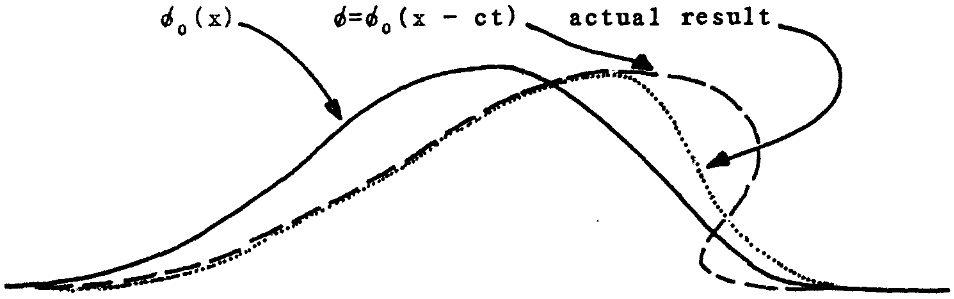
$$c = k/(y^* - \phi)^3 \qquad (4.6b)$$

which is a quasi-linear hyperbolic wave equation.

This equation may be solved by the method of characteristics. [41]. We obtain a solution of the form

$$\phi = \phi_0(x - ct) \qquad (4.7)$$

where $\phi_0(x)$ is the initial boundary. The initial dune moves downstream with speed c. (4.6a) indicates that the top of the dune moves faster than the bottom. In actuality, the lee side sluffs off at the angle of repose of the sediment. (see Fig 4.2).



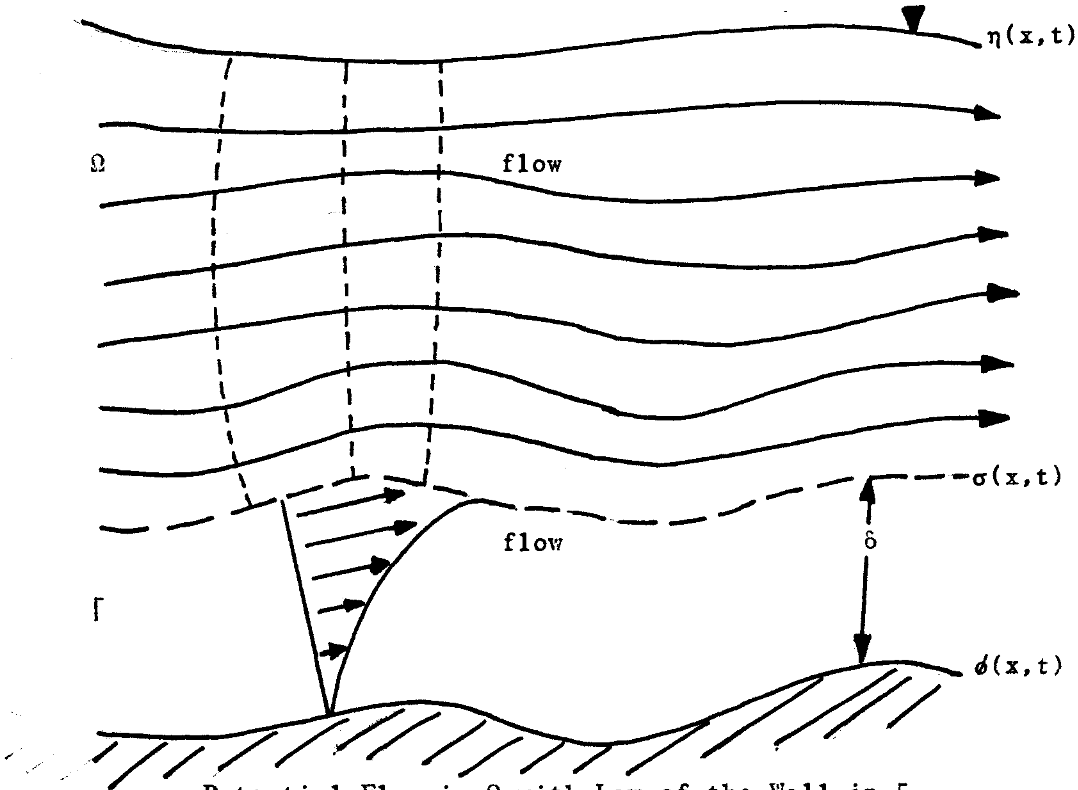
Dune Migration- Almost Steady, Uniform Flow

Fig 4.2

Thus, with almost steady, uniform flow, ripples migrate downstream and assume the classical dune shape. In this scenario, if the bottom is initally flat, then no ripples would form.

<u>Potential Flow in Ω with Law of the Wall in Γ</u>

We consider the case where the flow is given by a logarithmic velocity profile in a region near the boundary and given by potential flow outside that region. We use the 'Law of the Wall' for the logarithmic velocity profile. The region near the boundary is denoted by Γ and the potential flow region is denoted by Ω. Γ is a constant width δ. (see Fig 4.3).



Potential Flow in Ω with Law of the Wall in Γ

Fig 4.3

$$\Gamma = \{(x,y) \mid \phi \leq y \leq \sigma\} \quad (4.8)$$

$$\Omega = \{(x,y) \mid \sigma \leq y \leq \eta\} \quad (4.9)$$

$$\sigma = \{(x',y') \mid x' = x - \delta\phi'/dr, \quad y' = \phi(x) + \delta/dr\} \quad (4.10)$$

$$dr = [1 + (\phi')^2]^{-1/2} \quad (4.11)$$

The flow in Ω is determined in one of three ways:

Method (1): Approximate uniform flow $\mathbf{u}=\mathbf{u}(x)$,

Method (2): Conformal mapping,

Method (3): Numerically solving $\Delta\xi=0$, where $\mathbf{u}=\text{grad } \xi$.

If η is assumed apriori and ϕ' is 'small' (hence $\sigma'$is 'small') then we can approximate the flow on σ by

$$U_\sigma = q_f/h, \qquad h = \eta(x) - \sigma(x), \tag{4.12}$$

where q_f is the volumetric fluid discharge and h is the depth of Ω.

In the general case, the conformal maps must be obtained numerically. However, if we assume some particular geometric form for the bed, in particular, one for which the conformal map is known, or one which is well approximated by a standard conformal map, then we can 'conviently' use conformal mapping. In the first case, as the bed evolves, the map changes. In the second case, we continuously approximate the bed by the same standard simple geometric form.

With the direct numerical approach, the numerical solution depends on whether η is assumed apriori or taken as the continuation of a streamline. In the first case, we solve directly

$$\Delta\xi=0 \quad \text{in } \Omega \tag{4.13a}$$

subject to the boundary conditions:

$$\partial\xi/\partial n=u_{normal}=0 \quad \text{at } \eta \tag{4.13b}$$

$$\partial\xi/\partial n=u_{normal}=0 \quad \text{at } \sigma \tag{4.13c}$$

In the second case, we solve the same problem but iterate on η until Bernoulli's condition,

$$u^2/2 + p/\rho + gy = \text{constant} \quad \text{on } \eta, \tag{4.14}$$

is satisfied.

Method (3) is the most versatile while method (1) is the simplest. For deep flows relative to ripple height, η is almost flat and both method (1) and (3) are easy to implement. In all cases, some inflow and outflow constraints must be assumed. Here we use uniform flow, method (1).

The flow in Γ and has a local logarithmic form. The velocity, parallel to ϕ, is given by

$$U = K \log[(z + z_0)/z_0] \qquad (4.15)$$

where U is the flow magnitude, z is the normal distance above the bed, z_0 is that distance where the fluid begins movement, and $K = K(x,\phi(x))$ is necessary to match the velocity in Γ with the velocity in Ω. z_0 is usually assumed to be of the order $d_s/30$ for sand where 65% of the sediment is finer than d_s. However, we take $z=z_0$ to corresponds to ϕ.

Let $U = U(z;x,\phi(x))$ denote the velocity a distance z from the point $(x,\phi(x))$. Let U_σ denote the velocity on σ determined by the potential flow in Ω. Continuity of the flow requires

$$U_\sigma = U(\delta;x,\phi(x)) \qquad (4.16)$$

and $K(x,\phi(x))$ is determined from

$$K(x,\phi(x)) = U_\sigma/\log[(\delta + z_0)/z_0] \qquad (4.17)$$

We obtain u and v in Γ by

$$u = U/dr, \qquad v = U\phi'/dr \qquad (4.18)$$

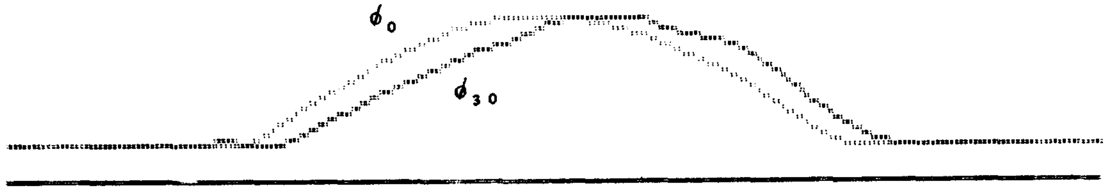
where ds is given by (4.11). u_s is then easily obtained and hence ϕ_t.

Some solutions (graphically represented) are shown in Fig 4.4.

Concluding Remarks- Special Case Flows

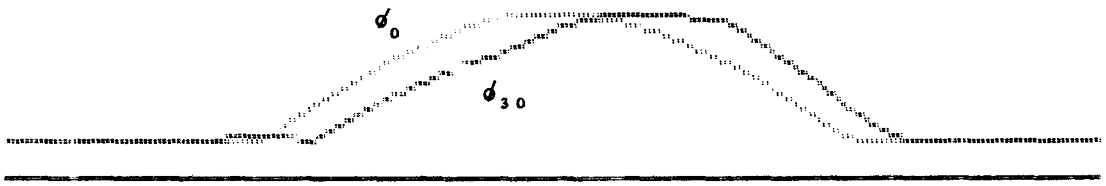
We have shown from these two cases that our model does indeed embody some of the major facets of dune migration. We have refered to these facets as 'minor' evolutionary processes. In contrast, 'major' evolutionary changes are evolution from one 'major' characterization to another. (e.g. smooth bed -> ripples -> dunes) (refer to Fig 2.4, page 23)

In the above cases we have replaced the general flow with a simpler flow. Under these simplifications, our model is similar to most others in that no local flow variation is allowed. It is clear that when the local flow cannot vary, neither can the local erosion rate. Thus, although we get classical dune shape and migration, unless we attach a general flow to our erosion constraint our model will not have the potential to address 'major' evolutionary changes in the streambed.



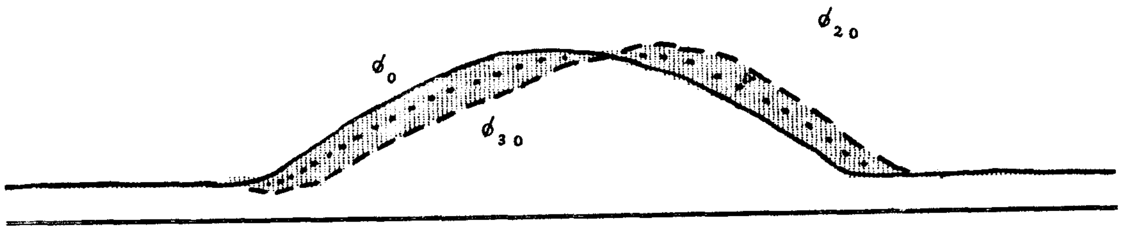
Dune Migration- Potential Flow with Law of the Wall

Fig 4.4a



Dune Migration- Potential Flow with Law of the Wall

Fig 4.4b



Dune Migration- Potential Flow with Law of the Wall

Fig 4.4c

A GENERAL FLOW

The Flow Domain

The general model for erosion was given in chapter III, (3.28). In considering the special case flows of chapter IV, we tacitly assumed uniform flow at infinity. A similar adjustment is necessary here also since we cannot deal numerically with a domain of infinite length. For the remainder of this discussion, we take the domain, D=D(t), to be the finite region

$$D(t)=\{(x,y) \mid a \leq x \leq b, \quad \phi(x,t) \leq y \leq \eta(x,t)\}. \tag{5.0}$$

For simplicity, we assume uniform flow outside [a,b], that is, outside the respective inflow and outflow boundaries. We are requiring uniform flow at the entrance and exit boundaries. With respect to the inflow, this artificial constraint is not very limiting. With respect to the outflow, however, this additional constraint is quite limiting. We are excluding those flows which do not tend towards a uniform flow near the exit boundary. For some flows (in particular, those not tending towards a uniform flow) numerical 'reflections' off this artificial exit boundary may exist. We choose only flows where this reflection appears to be negligible.

The erosion model on a finite interval, denoted by (*), is given by:

$$\rho D\mathbf{u}/Dt = \operatorname{div} \mathbf{T} + \rho \mathbf{g} \quad \text{in } D(t) \tag{5.1a}$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } D(t) \tag{5.1b}$$

$$p=0 \quad \text{on } \eta(x,t) \tag{5.1c}$$

$$\mathbf{Tn}=0 \quad \text{on } \eta(x,t) \tag{5.1d}$$

(*)

$$\eta_t = v|_{y=\eta} \tag{5.1e}$$

$$u=0=v \quad \text{on } \phi(x,t) \tag{5.1f}$$

$$-c_p \phi_t = du_s/dx \tag{5.1g}$$

$$u_s = \text{dir}[c_1\omega - c_2(\phi' + \theta_c)]^+ \tag{5.1h}$$

$$\partial u/\partial x=0=\partial v/\partial x \quad \text{at } x=a \text{ and } x=b \tag{5.1i}$$

$$u_0,\ v_0,\ D_0 \text{ given at } t_0 \tag{5.1j}$$

where $du_s/dx = \{\partial u_s/\partial x + \phi'\partial u_s/\partial y\}|_{y=\phi(x,t)}$.

Solution Procedure- Reducing to a System of Initial Value Problems

The system (*) depicts nonuniform, unsteady flow. Ideally we would like to have a time dependent equation for each of the variables: u, v, p, ϕ, and η, and only time dependent equations. The only exceptions are p, the pressure variable and (b), the equation for incompressibility. We can use equation (b) to eliminate the pressure term from (a). The 'improved' system will have exclusively: time dependent equations, one for each remaining variable.

Reducing (*) to a system of initial value problems is a result of choosing finite elements as our solution method. There are two main reasons for choosing finite elements over another method:

(1) We have seen from almost steady, uniform flow that our erosion boundary condition becomes a nonlinear hyperbolic wave equation. 'Shocks' develop in such equations and the

natural approach for this phenomenon is to reformulate the problem in its weak setting. [23], [41]

(2) The free surface condition is easily incorporated into our method and it simplifies the final equations.

Having decided upon finite elements, we apply a natural transformation, to a fixed domain, which simplifies certain aspects of our problem. The resulting ordinary differential equation system requires some initial conditions. Numerical considerations force us to choose starting values consistent with the eventual flow.

We give a brief outline of our method for solving (*):

Step (1): 'Remove' the pressure term and the continuity equation, (b), by applying the divergence operator to the Navier-Stokes equation, (a).

Step (2) Generate the weak form of this system.

Step (3) Map the system from the free domain, D, to a fixed domain R.

Step (4) Apply time dependent finite elements to the problem in the R domain.

Step (5) Determine 'suitable' initial values.

The result of steps (1)-(5) is a system of initial value problems set in the fixed domain, R.

Step (1): <u>Modifying (*) to 'Remove' p and equation (5.1b)</u>

Taking the divergence of (5.1 a), we obtain in D(t):

$$\mathrm{div}\{\rho[\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}]\} = \mathrm{div}\{-\nabla p + \mu\Delta\mathbf{u} - \rho\mathbf{g}\}. \quad (5.2)$$

Holding ρ constant, applying (5.1 b) and simplifying, we have in D(t):

$$\rho\ \mathrm{div}\{(\mathbf{u}\cdot\nabla)\mathbf{u}\} = -\Delta p. \quad (5.3)$$

Thus, given $\mathbf{u}$ at time t, we have p implicitly given.

Equations (a) and (b) in (*) are replaced by

$$\rho\{\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}\} = -\nabla p + \mu\Delta\mathbf{u} - \rho\mathbf{g} \quad \text{in } D(t). \quad (5.4)$$

where p is now the implicitly known function given by the Poisson equation:

$$\Delta p = 2\rho\ \det\nabla\mathbf{u} \quad \text{in } D(t) \quad (5.5a)$$

$$p = 0 \quad \text{on } \eta(x,t) \quad (5.5b)$$

$$p=\rho g(\eta-\phi)\cos\theta \quad \text{on } \phi(x,t) \quad (5.5c)$$

$$\partial p/\partial x=0 \quad \text{at } x=a \quad (5.5d)$$

$$\partial p/\partial x=0 \quad \text{at } x=b \quad (5.5e)$$

The method of solution for (5.5) is independent of the method for the other portions of our problem. Problems such as (5.5) are easily solved. [6], [29] Consequently, we proceed as if p is a known function, albeit implicitly.

The system (*) has been altered to this new version of our problem which we denot by (#).

$$\rho\{\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}\} = -\nabla p + \mu\Delta\mathbf{u} - \rho\mathbf{g} \quad \text{in } D(t) \tag{5.6a}$$

$$\mathbf{Tn}=0 \quad \text{on } \eta(x,t) \tag{5.6b}$$

$$\eta_t = v|_{y=\eta} \tag{5.6c}$$

$$u=0=v \quad \text{on } \phi(x,t) \tag{5.6d}$$

(#)

$$-c_p\phi_t = du_s/dx \tag{5.6e}$$

$$u_s = \text{dir}[c_1\omega - c_2(\phi' + \theta_c)]^+ \tag{5.6f}$$

$$\partial u/\partial x=0=\partial v/\partial x \quad \text{at } x=a \text{ and } x=b \tag{5.6g}$$

$$u_0,\ v_0,\ D_0 \text{ given at } t_0 \tag{5.6h}$$

where $du_s/dx = \{\partial u_s/\partial x + \phi'\partial u_s/\partial y\}|_{y=\phi(x,t)}$ and p is given by (5.5).

Step (2): <u>The Weak Form of (#)</u>

We let $\langle f,g\rangle_D$ be the inner product given by $\iint_D fg\,da$. We also use the equivalent notation $\langle f,g\rangle$ when no ambiguity will result. We let W be our class of test functions defined on D. The exact restrictions on W will be given later. The weak form of the problem (*) is given as:

For $t > t_0$, find u and v such that for all $w \in W$

$$\langle \mathbf{u}_t, w\rangle = \langle -(\mathbf{u}\cdot\nabla)\mathbf{u}, w\rangle + \langle \rho^{-1}\text{div } \mathbf{T}, w\rangle + \langle \mathbf{g}, w\rangle \tag{5.7a}$$

$$\langle \eta_t, w\rangle = \langle v|_{y=\eta}, w\rangle \tag{5.7b}$$

$$\langle \phi_t, w\rangle = \langle -c_p^{-1} du_s/dx, w\rangle \tag{5.7c}$$

subject to:

$$\mathbf{Tn}=0 \quad \text{on } \eta(x,t) \tag{5.8a}$$

$$u=0=v \quad \text{on } \phi(x,t) \tag{5.8b}$$

$$u_s = \text{dir}[c_1\omega - c_2(\phi' + \theta_c)]^+ \tag{5.8c}$$

$$\partial u/\partial x=0=\partial v/\partial x \quad \text{at } x=a \text{ and } x=b \tag{5.8d}$$

$$u_0,\ v_0,\ D_0 \text{ given at } t_0 \tag{5.8e}$$

and where p is given by (5.5).

We now incorporate the constraints (5.8) into (5.7). Recall Green's theorem:

$$\int_D (\mathrm{div}\ \mathbf{T}) w\ da = -\int_D \nabla w\ \mathbf{T}\ da + \int_{\partial D} (\mathbf{Tn}) w\ ds \tag{5.9}$$

We apply Green's theorem, (5.9), to (5.7a) and obtain

$$\langle \mathbf{u}_t, w\rangle = \langle -(\mathbf{u}\cdot\nabla)\mathbf{u}, w\rangle + \langle \mathbf{g}, w\rangle - \langle \rho^{-1}\mathbf{T}, \nabla w\rangle + \langle \rho\mathbf{Tn}, w\rangle_{\partial D} \tag{5.10}$$

where the last two products are defined by the right hand side of (5.9).

The test functions, w, are usually chosen to vanish on the boundary. For problems with non-zero boundary conditions, a simple substitution is made to get zero boundary data. Such a proceedure eliminates products of the form $\langle .,.\rangle_{\partial D}$. In our case, the standard procedure does not quite work. The boundaries are free making the usual (constant for all time) substitution awkward. We achieve a similar end from a slightly different approach.

We consider our boundary term

$$\langle \rho\mathbf{Tn}, w\rangle_{\partial D} = \langle \rho\mathbf{Tn}, w\rangle_{\partial D(\eta)} + \langle \rho\mathbf{Tn}, w\rangle_{\partial D(\phi)} + \langle \rho\mathbf{Tn}, w\rangle_{\partial D(a)} + \langle \rho\mathbf{Tn}, w\rangle_{\partial D(b)} \tag{5.11}$$

where the parentheses indicate the portion of boundary to which we are restricting ourselves.

The constraint (5.8a) assures $\mathbf{Tn}=0$ on η so that

$$\langle \rho\mathbf{Tn}, w\rangle_{\partial D(\eta)} = 0. \tag{5.12a}$$

$\mathbf{Tn}$ does not vanish on ϕ but $\mathbf{u}$ does. Since we need not solve (5.5) in the same manner as (5.8), we need not choose a basis for w which admits p. Our test functions must only admit u and v.

Therefore, we can require that all functions in W vanish on ϕ. Thus, w=0 on $\phi(x,t)$ yields

$$\langle \rho \mathbf{T}\mathbf{n}, w \rangle_{\partial D(\phi)} = 0. \qquad (5.12b)$$

The uniformity of the flow at x=a and x=b allows us to determine the actual flow there. (In fact, we will choose uniform planar flow which we determine analytically.) We need not generate a finite element equation for these edges since they are known. We need only solve (*) in the horizontal interior of D. By choosing w to vanish at x=a and x=b we are in essence throwing out the discrepancy generated by the boundary reflection mentioned above. There is no reflection when the flow becomes uniform as it reaches the outflow boundary x=b. We choose w=0 at x=a and x=b and obtain

$$\langle \rho \mathbf{T}\mathbf{n}, w \rangle_{\partial D(a)} = 0 = \langle \rho \mathbf{T}\mathbf{n}, w \rangle_{\partial D(b)}. \qquad (5.12c)$$

The result from (5.12a), (5.12b) and (5.12c) is

$$\langle \rho \mathbf{T}\mathbf{n}, w \rangle_{\partial D} = 0. \qquad (5.13)$$

Our problem is now:

Let W be given by

$$W = \{ w(x,y) \mid w \in C^1(D),\ w(x,\phi)=0,\ w(a,y)=0,\ w(b,y)=0 \}. \qquad (5.14)$$

Given u_0, v_0, D_0, find u and v such that for $t > t_0$, and for all $w \in W$,

$$\langle \mathbf{u}_t, w \rangle = \langle -(\mathbf{u}\cdot\nabla)\mathbf{u}, w \rangle + \langle \mathbf{g}, w \rangle - \langle \rho^{-1}\mathbf{T}, \nabla w \rangle \qquad (5.15a)$$

$$\langle \eta_t, w \rangle = \langle v|_{y=\eta}, w \rangle \qquad (5.15b)$$

$$\langle \phi_t, w \rangle = \langle -c_p^{-1} du_s/dx, w \rangle \qquad (5.15c)$$

where p is given implicitly by (5.5).

It is this problem, (5.15), which is to be mapped to the fixed domain R.

Step (3): <u>Mapping D(t) to R</u>

The problem (5.15) has two free boundaries making it awkward to fix a basis for W. Mapping D to a fixed domain will allow us to fix a basis for W. The drawbacks are, dealing with the inverse map and additional nonlinearities. We can eliminate the need to invert our map but we must suffer the consequences of additional nonlinearities.

Let R be the rectangle given by

$$R=\{(\tilde{x},\tilde{y}) \mid a < \tilde{x} < b, \quad \phi < \tilde{y} < \eta\} \tag{5.16}$$

and let $\Lambda : D \to R$ be the map given by

$$\tilde{x} = x \tag{5.17a}$$

$$\tilde{y} = (y - \phi)/(\eta - \phi). \tag{5.17b}$$

The Jacobian of Λ is given by

$$J[\Lambda] = J = \eta - \phi = h \tag{5.18}$$

which never vanishes. We are primarily interested in the sediment boundary evolution, that is $\phi(x,t)$. Since $\tilde{x}=x$ and ϕ is independent of y, $\Lambda^{-1}(\tilde{\phi}(\tilde{x},t) = \phi(x,t)$. We need not perform a complicated calculation to obtain $\phi(x,t)$ from R. If u is desired, however, then such a calculation must be performed.

Our inner products must also be mapped to R. $\langle f,g\rangle_D \to \langle \tilde{f},\tilde{g}\rangle_R$ where

$$\langle \tilde{f},\tilde{g}\rangle_R = \iint_R \widetilde{fg}\ J[\Lambda]\ da. \tag{5.19}$$

We now suppress the R and the tilda notation and give the present version of our problem.

$$\langle u_t J, w\rangle = \{2\langle v_y, w_x\rangle - 2\langle v_y, w_y d\rangle - \langle h^{-1} u_y, w_y\rangle - \langle v_x, w_y\rangle$$
$$+ \langle v_y d, w_y\rangle\}/Re + \langle uv_y, w\rangle - \langle vu_y, w\rangle + \langle ph, w_x\rangle$$
$$- \langle pdh, w_y\rangle + \langle(\sin\Theta)h, w\rangle/Fr \qquad (5.20a)$$

$$\langle v_t J, w\rangle = \{-\langle u_y, w_x\rangle + \langle u_y, w_y d\rangle - \langle v_y dh, w_x\rangle + \langle v_x dh, w_y\rangle$$
$$+ \langle v_y dh, w_x\rangle - \langle v_y d^2 h, w_y\rangle - 2\langle h^{-1} v_y, w_y\}/Re$$
$$- \langle uhv_x, w\rangle + \langle uv_y dh, w\rangle - \langle vv_y, w\rangle + \langle p, w_y\rangle$$
$$- \langle(\cos\Theta)h, w\rangle/Fr \qquad (5.20b)$$

$$\langle \eta_t, w\rangle = \langle v|_{y=\eta}, w\rangle = \langle v(x,\eta), w\rangle \qquad (5.20c)$$

$$\langle \phi_t, w\rangle = -c_p^{-1}\{\langle u_y d, w\rangle + \langle \phi' h^{-1} u_y, w\rangle\}|_{y=\phi} \qquad (5.20d)$$

where

$$h = h(x,t) = \eta(x,t) - \phi(x,t) \qquad (5.20e)$$

$$d = d(x,y,t) = -\{\phi' + yh'\}/h \qquad (5.20f)$$

and p is known implicitly.

Step (4): <u>Applying Time Dependent Finite Elements</u>

The general idea behind finite elements is the Ritz-Galerkin method. [7], [29] Our variables should be $C^1(D)$. The finite version of our variables should become 'arbitrarily close' to the true variables as the number of basis elements increase. Orthogonality is of course desirable as is compact support. Finally, the basis chosen must suffice for both the test functions and the variables. We choose a special finite basis for W. This is the basis of piecewise linear functions.

The 'problem' is written in terms of the finite basis elements. The test functions now contain only a finite number of unknowns as do the variables. In our case, the coefficients will also be time dependent. The hope is that, as the number of basis elements increase, the finite version of the variables converges to the true weak solution. Under certain conditions, such convergence can be proven. [7]

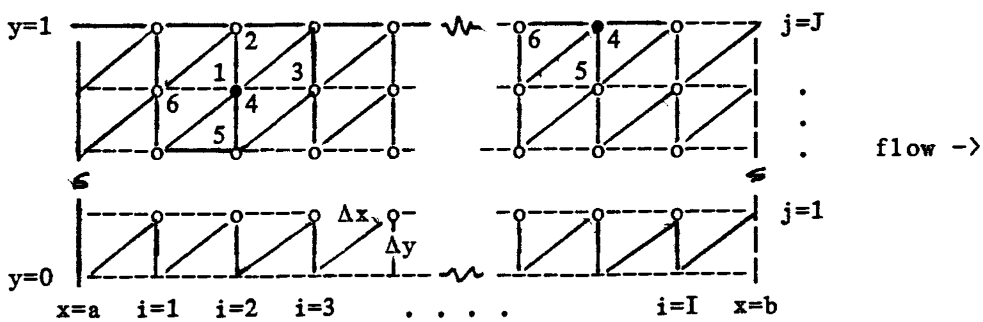
We proceed as follows:

(4a) choose a 'suitable' finite basis for W

(4b) write the entire problem in terms of the finite basis

(4c) evaluate the inner products and form a system of ordinary differential equations

Step (4a): The Grid on R



The Grid on R

Fig 5.0

We let e^i denote the following basis for i=1,...I

$$e^i = \begin{cases} 1-i + x/\Delta x, & x \varepsilon \ [(i-1)\Delta x, i\Delta x] \\ 1+i - x/\Delta x, & x \varepsilon \ [i\Delta x, (i+1)\Delta x] \\ 0 & \text{otherwise} \end{cases} \qquad (5.21)$$

where Δx is $(b-a)/(I+1)$.

We let e^{ij} denote the following basis for i=1,...I; j=1,...J

$$e^{ij}(x,y) = \begin{cases} 1+j-i + x/\Delta x - y/\Delta y, & x \varepsilon \text{ region } 1 \\ 1+j - y/\Delta y, & x \varepsilon \text{ region } 2 \\ 1+i - x/\Delta x, & x \varepsilon \text{ region } 3 \\ 1+i-j - x/\Delta x + y/\Delta y, & x \varepsilon \text{ region } 4 \\ 1-j + y/\Delta y, & x \varepsilon \text{ region } 5 \\ 1-i + x/\Delta x, & x \varepsilon \text{ region } 6 \\ 0 & \text{otherwise} \end{cases} \qquad (5.22)$$

where Δy is $1/J$.

There are numerous properties of this basis. A typical such example being

$$e^{ij}\ e^{mn} = 0 \text{ if } m > i+1 \text{ or respectively } n > j+1$$

Such properties will be used without explicitly explaining each step. The following representation for our variables is used throughout where we employ the standard summation notation.

$$u(x,y,t)=\alpha_{ij}(t)\ e^{ij} \qquad (5.23a)$$

$$v(x,y,t)=\beta_{ij}(t)\ e^{ij} \qquad (5.23b)$$

$$\eta(x,t)=\delta_i(t)\ e^i \qquad (5.23c)$$

$$\phi(x,t)=\varepsilon_i(t)\ e^i \qquad (5.23d)$$

$$(\eta-\phi)=h=\xi_i(t)\ e^i \qquad (5.23e)$$

$$w(x,y)=\lambda_{ij}\ e^{ij} \qquad (5.23f)$$

$$i=1,...I;\ j=1,...J$$

All our functions are piecewise linear. Piecewise differentiation may be easily carried out. The following notation will be common during the remaining discussion.

$$\partial u/\partial x = \alpha_{ij}\ \partial e^{ij}/\partial x = \alpha_{ij}\ e_x^{\ ij} \tag{5.24}$$

$$\partial u/\partial t = \partial \alpha_{ij}/\partial t\ e^{ij} = \alpha_{ij}\ e^{ij} \tag{5.25}$$

Greek letters as coefficients may be thought of as a vector. We shall use the notation $\boldsymbol{\alpha} = (\alpha_{11}, \alpha_{12}, \ldots, \alpha_{ij}, \ldots \alpha_{IJ})$.

Step (4b): The Discrete Version of Our Problem

Given $\boldsymbol{\alpha}_0$, $\boldsymbol{\beta}_0$, $\boldsymbol{\delta}_0$, $\boldsymbol{\varepsilon}_0$, find $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\delta}$, $\boldsymbol{\varepsilon}$ such that for all $\boldsymbol{\lambda}$

$$\lambda\langle IP_1\rangle\dot{\mathbf{z}} = \lambda\langle IP_2\rangle \tag{5.26}$$

where $\mathbf{z}=(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\delta},\boldsymbol{\varepsilon})$. The full form of the inner products $\langle IP_1\rangle$ and $\langle IP_2\rangle$, in their discreet version, are given in appendix 1.

Step (4c): Evaluation of the inner products to form the system of ordinary differential equations

Our evaluation is analogous to a much simpler case which we explain first.

Given the following sort of 1-dimensional inner product, which must hold for all a_i. We could simplify in this way:

$$\langle a_i e^i, b_j e^j\rangle = a_i\{b_{i-1}\langle e^i, e^{i-1}\rangle + b_i\langle e^i, e^i\rangle + b_{i+1}\langle e^i, e^{i+1}\rangle\}$$
$$= a_i\{.\} \tag{5.27}$$

The terms in the brackets, {...}, may all be evaluated in terms of **b** and this example reduces to the form

$$\mathbf{aEb} = \mathbf{a}\{.\} \quad \text{no summation for } a_i \tag{5.28}$$

where **E** represents the matrix formed from evaluating the bracketed terms in (5.26).

Since the a_i (test functions) appear on both sides of the final equation, we end up with a linear system to solve.

$$\mathbf{Eb} = \{.\} \tag{5.29}$$

If **b** was time dependent, we would replace the constant coefficients in (5.23) with time dependent coefficients. The end result would be the system of ordinary differential equations in place of (5.29).

$$\mathbf{E\dot{b}} = \{.\} \tag{5.30}$$

Our problem is analogous to this example except that we have four variables instead of one and we are set in 2-d domain. If we momentarily denote $\mathbf{z} = (\alpha,\beta,\delta,\varepsilon)$ then we will ultimately obtain the form:

$$\mathbf{A\dot{z}} = \{.\} \tag{5.31}$$

where **A** is the counterpart to **E** above.

We skip all the intermediate steps and give only the final form of the problem:

$$\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & C \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\delta} \\ \dot{\varepsilon} \end{pmatrix} = \begin{pmatrix} f_1(\alpha,\beta,\delta,\varepsilon) \\ f_2(\alpha,\beta,\delta,\varepsilon) \\ f_3(\beta) \\ f_4(\alpha,\beta,\delta,\varepsilon) \end{pmatrix} \tag{5.32}$$

or

$$\mathbf{A\dot{z}} = \mathbf{f}$$

where A, B, C, and f_i are given in appendix 1.

Our problem has now been reduced to solving the ordinary differential equation system (5.32). We need only add suitable initial conditions to complete the problem.

Step (5): <u>The Initial Condition for the Initial Value Problems</u>

Presumably, we could attach any initial conditions to our system (5.32). Although most fluid flow is unstable it may (numerically) eventually smooth out. In practice, it is impractical to use initial conditions which are too incompatible with the flow. With this in mind, we take for our initial conditions the solution to a related problem: the steady flow generated when we momentarily fix the sediment boundary.

We fix the bottom with some initial shape and solve the resulting steady flow problem. This solution becomes our initial condition at the time we turn the sediment loose. We refer to these initial conditions simply as $\mathbf{u}_0$, p_0 and D_0.

<u>Solution Procedure- Solving the Initial Value Problems Numerically</u>

Initial vlaue problems such as (5.32) may be attacked with various methods. [5], [7], [29], [35]. The method of choice should deal well with nonlinear equations. We choose Adams-Bashforth since it is an implicit/explicit multi-step method. These methods are more stable than the convenient, explicit, single step methods such as Runga-Kutta.

We give the outline for solving (5.32) numerically.

(1) Generate the initial conditions,

(2) Set up initial steps of multi-step start,

(3) Use Adams-Bashforth predictor/corrector,

(4) Find the new pressure distribution,

(5) Go to (3).

We give the highlights of the procedure.

Step (1): <u>Generating the Initial Conditions</u>

Generating the initial conditions is a two step process. We first read in all the initial data which predetermined. This allows us to set the parameters for solving the steady flow problem which is the second initialization step. We use a predetermined domain and solve the fixed boundary problem by finite elements. That is, we reduce to a linear equation system. Our output is the initial flow, the initial domain being already known.

Step (2): <u>Set Up Initial Steps of Multi-step Start</u>

The multi-step method requires the solution for two time periods. These are generally obtained by using a single step, explicit method. In our case, we want to avoid the single step, explicit methods. Our method benefits us here twice. The steady flow (fixed bed) is supposed to gradually go into an eroding flow. Thus, by using the steady flow at both $t=0$ and $t=t_1$, we satisfy the multi-step requirements as well as making a smooth transition.

Step (3): Using Adams-Bashforth

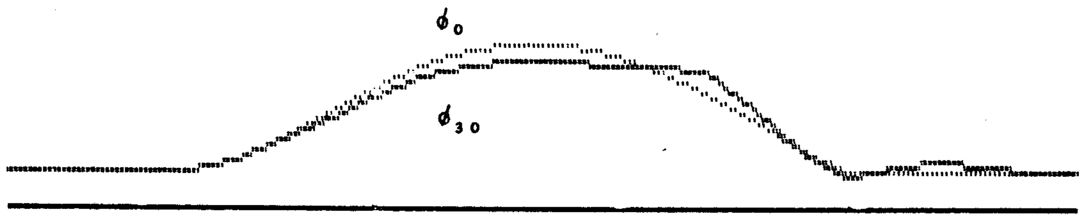
Applying Adam-Bashforth is easy at this stage since we have the initial values at t^n and t^{n-1}. The slopes are given by simple function evaluation. We simply run these values through our loop. The only nonstandard calculation is the implicit calculation of the pressure after each pass through the loop.

Step (4): Find the New Pressure Distribution

The pressure distribution is necessary in evaluating the slopes. This is obtained by solving the Poisson equation for the pressure in terms of u^n, η^n, and ϕ^n. We used finite elements here. Thus, we must solve the linear equation system.

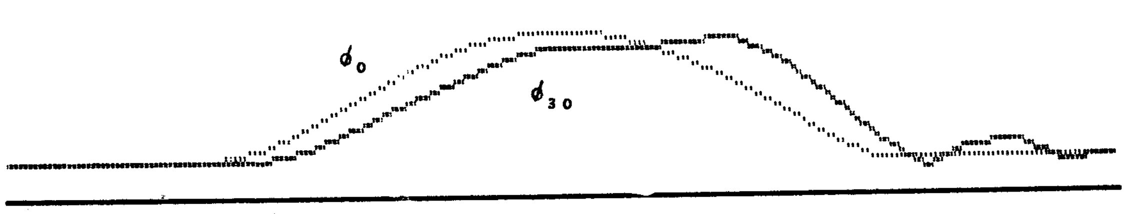
Results- Graphical

The results are presented graphically in Fig 5.1a and Fig 5.1b.



Dune Migration- A General Flow

Fig 5.1a



Dune Migration- A General Flow

Fig 5.1b

Concluding Remarks- The General Solution

We have developed a 'reasonable' procedure for solving our model for a general flow. Given an initial dune, similar to the special case flows' initial dune, we get a similar migration.

An equilibrium flow was also used for the initial flow. However, in that case, the flow either becomes unstable or does not undergo any appreciable erosion. The difficulty there lies with the large variation in time scales between the evolution rate and the flow rate. If we increase the flow and the erodability of the bed to promote evolution' and even up' the rates, then we get instability at an increased rate. Certainly, the first improvement is a finer grid for faster flows.

Our solution method for the general flow has some limitations which we note here.

(1) The flow rate must not be too excessive.

(2) The initial configuration must be fairly smooth.

(3) The outflow must tend towards uniform flow.

SUMMARY, OPEN QUESTIONS AND CONCLUDING REMARKS

Summary

In chapter III we derived a model for streambed erosion. Our basic approach was to combine the interaction of a viscous flow with boundary constraints which represented an erodable streambed. The viscous flow was characterized by the Navier-Stokes equation and the boundary condition was obtained by balancing sediment mass, momentum, and energy. The highlights of the model are:

1. The erosion process consists of a flow subject to a boundary condition which represent an erodable streambed.
2. The general erosion producing flow is governed by the Navier-Stokes equations for open channel flow.
3. The boundary condition is obtained by balancing sediment mass, energy and momentum.
4. The sediment transport occurs in a 'thin' layer along the streambed. This is not the classical boundary layer.
5. Our model is for relatively fine, noncohesive, homogeneous, spherical sediment particles. Large discrete particles would invalidate the mass balance derivation as well as the assumptions used to obtain u_s, the sediment transport velocity.

6. The sediment transport must be in a state of quasi-equilibrium. Increased sediment flux, across the boundary, results in an increased transport rate rather than an increased sediment concentration.
7. The vorticity of the flow near the bottom is a fundamental factor for the erosion process.

The erosion model we obtained was a 2-dimensional free boundary value problem with two free boundaries. We examined some special flows as well as a general flow. The special flows were discussed in chapter IV. The first case was: Almost steady, uniform flow. The second case was: A logarithmic velocity profile flow in conjunction with potential flow.

Results from the first special case flow show the inherent shock nature of the model equations. Also, from this case we, found the flow to be too restrictive to allow for initial dune formation. However, the classical dune migration pattern for an existing dune is obtained.

Results from the second special case flow are similar to the first case. The 'shock' forms as the angle of repose is attained and the solution begins to breakdown. As in the previous case, we cannot account for initial dune formation but the classic dune migration patterns are obtained.

The main difference between case one and case two is that we are using an empirical flow in case two. The importance of this case is that it would allow us to upgrade our empirical constants to empirical relations.

In chapter V we considered a general flow. We achieved dune migration but our method has some limitations which produces an unstable flow while attempting to achieve 'major' evolutionary changes. We attributed this to: one, the time scale variation between flow rate and bed evolution rate and two, the relative coarseness of the grid for high flows or flows with an artificially high erodability.

Some Remarks- The Model and its Solutions

One of the major distinctions of this model aside from the general flow field is the appearance of the vorticity, near the streambed, as a fundamental parameter affecting erosion.

The largest particle a stream can move is called the competence of the stream. Recall Brahms' formula, $V_c = kW_s^{1/6}$, which relates the midstream velocity to the weight of the particle. We show that a similar expression is obtained using ω as a fundamental parameter. Let f_s be the force trying to roll a particle of diameter d_s. $\mu\omega$ is of the same order as τ_0 and τ_0 is known to be proportional to the square of the midstream velocity, V_c. So,

$$(V_c)^2 \sim \tau_0 \sim \mu\omega \sim f_s \sim d_s \sim W_s^{1/3} \qquad (4.17)$$

Flow over a bed of sand causes the bed to undergo a distinct evolution. At slow flows, the bed remains flat. With increasing flow velocity, the bed passes through a set of stages: ripples, then dunes, then flat again with chaotic flow, then waves and finally antidunes. One of our main purposes was to have a model capable of addressing these 'major' evolutionary changes.

Erosion producing flows are generally turbulent, or at the very least unsteady and nonuniform. For the unsteady Navier-Stokes system, existence and uniqueness depend upon the boundary configuration and the Reynolds number. Large Reynolds number, that is, turbulent flows, and complex boundaries restrict the period of existence and uniqueness. It is just such flows which occur during the 'major' evolutionary changes.

We have imposed restrictions on the class of flows with the goal of attaining a better mathematical understanding of the erosion process over the goal of attaining a model which matches data more precisely. This restricted class of flows still yields classic erosion patterns.

The results of our model and its solutions are summarized here.

1. Our special case flows have a flow field similar to many models.
2. Our model can accept a general flow.
3. We obtain dune migration in both the special case flows and the general flow.
4. A fundamental parameter in our model is the vorticity near the streambed.

<u>Improvements- The Model</u>

The main improvements which we could make to the model are:

1. Replace empirical constants with empirical relations

in the derivation of the boundary condition

2. Consider the case c_s nonconstant (this could be done by assuming a simple exchange law)

3. Consider more involved expressions for the resistence terms

4. Take the pressure lift force into account

5. Allow for nonconstant density in Ω

Improvements- The Solution

The main improvements to the solution procedure as it now stands are:

1. Use a finer, variable mesh

2. Use a more sophisticated method in the elimination of the pressure variable and the continuity equation

Open Questions

The open questions which arise from this problem are of two types: Those which deal with the model and its implications and those which deal with the solution procedures. Some of the more interesting open questions are given below.

1. Can the model display the other 'minor' evolutionary processes, namely antidune migration?

2. Can the model display 'major' evolutionary stages?

3. Can the model be used to show that the evolutionary

stages are, in fact, bifurcation phenomenon?

4. Can we apply the classical boundary layer theory and retain enough flow realism and still get the bifurcation effects? Note that in BLT one often assumes a boundary of constant curvature.

5. What is the best solution approach? Is there another reasonable solution method? This problem is unique so that there is no set solution method.

6. How does one deal with the outflow boundary condition when the flow does not become uniform?

7. How can the difference in natural time scales for the streambed versus the flow region be dealt with more profitably?

Concluding Remarks

We have derived a new model for streambed erosion. The model exhibits classical dune migration behavior. The model accepts a general flow field. The pressence of the general flow is necessary for certain aspects of nonuniform erosion. Thus, this model is potentially capable of addressing erosion as a bifurcation phenonenon.

The fundamental parameter in this model, which distinguishes it from most others, is the vorticity near the streambed. Such a term is basic to vortex sheeting which is possibly related to the initiation of the ripples. The model can be easily 'improved' by replacing empirical constants with relations.

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APPENDIX

APPENDIX

The Inner Products

$$A\dot{\alpha} = q_{11} - q_{12} + q_{13} - q_{14} - q_{15} - q_{16} + q_{17} + q_{p1}$$

$$A\dot{\beta} = -q_{21} + q_{22} - q_{23} + q_{24} + q_{25} - q_{26} + q_{27} + q_{28} - q_{29} - q_{210} - q_{211} + q_{p2}$$

$$B\dot{\delta} = q_{31} \qquad\qquad C\dot{\varepsilon} = q_{41} + q_{42}$$

$$a_{ij} = \Delta x \Delta y \begin{matrix} h_i/2 & h_i/12 & & & \\ h_i/12 & h_i/2 & h_i/12 & & \\ & & & & \\ & h_i/12 & h_i/2 & h_i/12 & \\ & & h_i/12 & h_i/2 & \end{matrix} \qquad i=1,\ldots I, \; j=1,\ldots J$$

$$A = \begin{matrix} a_{11} & a_{12} & & & \\ & & & & \\ & a_{ij-1} & a_{ij} & a_{ij+1} & \\ & & & & \\ & & & a_{IJ-1} & a_{IJ} \end{matrix}$$

$$
B_{ij} = \Delta x
\begin{bmatrix}
2/3 & 1/6 & & & & & & & \\
1/6 & 2/3 & 1/6 & & & & & & \\
 & 1/6 & 2/3 & 1/6 & & & & & \\
 & & & & 1/6 & 2/3 & 1/6 & & \\
 & & & & & & 2/3 & 1/6 & \\
 & & & & & & 1/6 & 2/3 & 1/6 \\
 & & & & & & & 1/6 & 2/3
\end{bmatrix}
$$

$$
C_{ij} = \Delta x
\begin{bmatrix}
\ldots 2/3 & \ldots 1/6 & & & \\
\ldots 1/6 & \ldots 2/3 & & & \\
 & \ldots 1/6 & & & \\
 & & \ldots 1/6 & & \\
 & & \ldots 2/3 & & \\
 & & \ldots 1/6 & & \\
 & & & \ldots 1/6 & \\
 & & & \ldots 2/3 & \ldots 1/6 \\
 & & & \ldots 1/6 & \ldots 2/3
\end{bmatrix}
$$

$$
q_{11} =
\begin{bmatrix}
\vdots \\
\Delta x \alpha_{ij} \{-\beta_{i-1j-1} + \beta_{i-1j} - 2\beta_{ij-1} + 2\beta_{ij+1} - \beta_{i+1j} + \beta_{i+1j+1}\}/6 \\
\vdots \\
\Delta x \alpha_{iJ} \{-\beta_{i-1J-1} + \beta_{i-1J} - 2\beta_{iJ-1} + 2\beta_{iJ}\}/6 \\
\vdots
\end{bmatrix}
$$

$i=1,\ldots I; \quad j=1,\ldots J$

$$q_{12} = \begin{cases} \vdots \\ \Delta x\beta_{ij} \{-\alpha_{i-1j-1} + \alpha_{i-1j} - 2\alpha_{ij-1} + 2\alpha_{ij+1} - \alpha_{i+1j} + \alpha_{i+1j+1}\}/6 \\ \vdots \\ \Delta x\beta_{iJ} \{-\alpha_{i-1J-1} + \alpha_{i-1J} - 2\alpha_{iJ-1} + 2\alpha_{iJ}\}/6 \\ \vdots \end{cases}$$

$i=1,\ldots I; \quad j=1,\ldots J$

$$q_{13} = \begin{cases} \vdots \\ \{-\beta_{i-1j-1} + \beta_{i-1j} + \beta_{ij-1} - \beta_{ij} + \beta_{ij+1} + \beta_{i+1j} - \beta_{i+1j+1}\}/2 \\ \vdots \\ \{-\beta_{i-1J-1} + \beta_{i-1J} + \beta_{iJ-1} - \beta_{iJ}\}/2 \\ \vdots \end{cases}$$

$i=1,\ldots I; \quad j=1,\ldots J$

$$q_{14} = \begin{cases} \vdots \\ \frac{\Delta x}{\Delta y}\, g_{ij} \{-\beta_{ij-1} + 2\beta_{ij} - \beta_{ij+1}\} \\ \vdots \\ \frac{\Delta x}{\Delta y}\, g_{iJ} \{-\beta_{iJ-1} + \beta_{iJ}\} \\ \vdots \end{cases}$$

$$\xi_i = \delta_i - \varepsilon_i$$

$$g_{ij} = \left[\frac{1}{2\Delta x\ \xi_i} \right] [\varepsilon_{i+1} - \varepsilon_{i-1} + j\Delta y(\xi_{i+1} - \xi_{i-1})]$$

$i=1,\ldots I; \quad j=1,\ldots J$

$$q_{15} = \begin{cases} \dfrac{\Delta x}{\Delta y} \{-\beta_{ij-1} + 2\beta_{ij} - \beta_{ij+1}\}/\xi_i \\ \dfrac{\Delta x}{\Delta y} \{-\beta_{iJ-1} + \beta_{iJ}\}/\xi_i \end{cases}$$

i=1,...I; j=1,...J

$$q_{16} = \begin{cases} \{-\beta_{i-1j-1} + \beta_{i-1j} + \beta_{ij-1} - \beta_{ij} + \beta_{ij+1} + \beta_{i+1j} - \beta_{i+1j+1}\}/2 \\ \{-\beta_{i-1J-1} + \beta_{iJ-1} - \beta_{iJ} + \beta_{i+1J}\}/2 \end{cases}$$

i=1,...I; j=1,...J

$$q_{17} = \begin{cases} \{\sin(\theta)\ \xi_i\ \Delta x \Delta y\}/Fr & i=1,...I;\ j=1,...J-1 \\ \{\sin(\theta)\ \xi_i\ \Delta x \Delta y\}/2Fr & i=1,...I;\ j=1,...J \end{cases}$$

$$f_{1j} = \{\varepsilon_2 - \varepsilon_1 + (j-1)\Delta y(\xi_2 - \xi_1)\}/\Delta x$$

$$f_{ij} = \{\varepsilon_{i+1} - \varepsilon_{i-1} + (j-1)\Delta y(\xi_{i+1} - \xi_{i-1})\}/2\Delta x$$

$$f_{Ij} = \{\varepsilon_I - \varepsilon_{I-1} + (j-1)\Delta y(\xi_I - \xi_{I-1})\}/\Delta x$$

$$a_{ij} = \Delta y\{\gamma_{i-1j-1}\xi_{i-1} + 2\gamma_{i-1j}\xi_{i-1} - \gamma_{ij-1}\xi_i + \gamma_{ij+1}\xi_i - 2\gamma_{i+1j}\xi_{i+1}$$

$$b_{ij} = \Delta x\{\gamma_{i-1j-1}f_{i-1j-1} - \gamma_{i-1j}f_{i-1j} + 2\gamma_{ij-1}f_{ij-1} - 2\gamma_{ij+1}f_{ij+1}$$

$$a_{iJ} = \Delta y\{\gamma_{i-1J-1}\xi_{i-1} + \gamma_{i-1J}\xi_{i-1} - \gamma_{iJ-1}\xi_i - \gamma_{i+1J}\xi_{i+1}\}/6$$

$$b_{iJ} = \Delta x\{\gamma_{i-1J-1}f_{i-1J-1} + 2\gamma_{iJ-1}f_{iJ-1} + 2\gamma_{iJ}f_{iJ} + \gamma_{i+1J}f_{i+1J}\}/6$$

$$c_{ij} = \{\gamma_{i-1j-1} - \gamma_{i-1j} + 2\gamma_{ij-1} - 2\gamma_{ij+1} + \gamma_{i+1j} - \gamma_{i+1j+1}\}/6$$

$$c_{iJ} = \{\gamma_{i-1J-1} + 2\gamma_{iJ-1} + 2\gamma_{iJ} + \gamma_{i+1J}\}/6$$

$$q_{p1} = \begin{array}{c} \vdots \\ a_{ij} + b_{ij} + c_{ij} \\ \vdots \\ a_{iJ} + b_{iJ} + c_{iJ} \\ \vdots \end{array}$$

$i=1,\ldots I;\ j=1,\ldots J$

$$q_{21} = \begin{array}{c} \vdots \\ \Delta y\alpha_{ij}\xi_i\ \{-\beta_{i-1j-1} - 2\beta_{i-1j} + \beta_{ij-1} - \beta_{ij+1} + 2\beta_{i+1j} + \beta_{i+1j+1}\}/6 \\ \vdots \\ \Delta x\alpha_{iJ}\xi_i\ \{-\beta_{i-1J-1} - \beta_{i-1J} + \beta_{iJ-1} + \beta_{i+1J}\}/6 \\ \vdots \end{array}$$

$i=1,\ldots I;\ \ j=1,\ldots J$

$$q_{22} = \begin{cases} \vdots \\ g\ \alpha_{ij}\ \{-\beta_{i-1j-1} + \beta_{i-1j} - 2\beta_{ij-1} + 2\beta_{ij+1} - \beta_{i+1j} + \beta_{i+1j+1}\}/6 \\ \vdots \\ g\ \alpha_{iJ}\ \{-\beta_{i-1J-1} + \beta_{i-1J} - 2\beta_{iJ-1} + 2\beta_{iJ}\}/6 \\ \vdots \end{cases}$$

$g_{ij} = \varepsilon_{i+1} - \varepsilon_{i-1} + (\xi_{i+1} - \xi_{i-1})(j-1)\Delta y/2$ $\qquad$ $i=1,\ldots I;\quad j=1,\ldots J$

$$q_{23} = \begin{cases} \vdots \\ \Delta x\beta_{ij}\ \{-\beta_{i-1j-1} + \beta_{i-1j} - 2\beta_{ij-1} + 2\beta_{ij+1} - \beta_{i+1j} + \beta_{i+1j+1}\}/6 \\ \vdots \\ \Delta x\beta_{iJ}\ \{-\beta_{i-1J-1} + \beta_{i-1J} - 2\beta_{iJ-1} + 2\beta_{iJ}\}/6 \\ \vdots \end{cases}$$

$i=1,\ldots I;\quad j=1,\ldots J$

$$q_{24} = \begin{cases} \vdots \\ \{-\alpha_{i-1j-1} + \alpha_{i-1j} + \alpha_{ij-1} - \alpha_{ij} + \alpha_{ij+1} + \alpha_{i+1j} - \alpha_{i+1j+1}\}/2 \\ \vdots \\ \{-\alpha_{i-1J-1} + \alpha_{i-1J} + \alpha_{iJ-1} - \alpha_{iJ}\}/2 \\ \vdots \end{cases}$$

$i=1,\ldots I;\quad j=1,\ldots J$

$$q_{25} = \begin{cases} \frac{\Delta x}{\Delta y}\, g_{ij}\, \{-\alpha_{ij-1} + 2\alpha_{ij} - \alpha_{ij+1}\} \\ \frac{\Delta x}{\Delta y}\, g_{iJ}\, \{-\alpha_{iJ-1} + \alpha_{iJ}\} \end{cases}$$

$$g_{ij} = \left[\frac{1}{2\Delta x\ \xi_i} \right] [\varepsilon_{i+1} - \varepsilon_{i-1} + j\Delta y(\xi_{i+1} - \xi_{i-1})]$$

$i=1,\dots I;\quad j=1,\dots J$

$$q_{26} = \begin{cases} \frac{\Delta y}{Re\Delta x}\, \{-\beta_{i-1j} + 2\beta_{ij} - \beta_{i+1j}\} \\ \frac{\Delta y}{2Re\Delta x}\, \{-\beta_{i-1j} + 2\beta_{ij} - \beta_{i+1j}\} \end{cases}$$

$i=1,\dots I;\quad j=1,\dots J$

$$q_{27} = \begin{cases} g_{ij}\, \{-\beta_{i-1j-1} + \beta_{i-1j} - \beta_{ij-1} + 2*\beta_{ij} + \beta_{ij+1} - \beta_{i+1j} + \beta_{i+1j+1}\} \\ g_{iJ}\, \{-\beta_{i-1j-1} + \beta_{ij-1} - \beta_{ij} + \beta_{i+1j}\} \end{cases}$$

$$g_{ij} = [\, \varepsilon_{i+1} - \varepsilon_{i-1} + (j-1)\Delta y(\xi_{i+1} - \xi_{i-1})\,]/(4\Delta x Re)$$

$i=1,\dots I;\quad j=1,\dots J$

$$q_{28} = \begin{matrix} \vdots \\ g_{ij}\ \{-\beta_{i-1j-1} + \beta_{i-1j} - \beta_{ij-1} + 2\beta_{ij} + \beta_{ij+1} - \beta_{i+1j} + \beta_{i+1j+1}\} \\ \vdots \\ g_{iJ}\ \{-\beta_{i-1j-1} + \beta_{ij-1} - \beta_{ij} + \beta_{i+1j}\} \\ \vdots \end{matrix}$$

$$g_{ij} = \{\ \varepsilon_{i+1} - \varepsilon_{i-1} + (j-1)dy(\xi_{i+1} - \xi_{i-1})\ \}/4\Delta x Re$$

$i=1,\ldots I;\quad j=1,\ldots J$

$$q_{29} = \begin{matrix} \vdots \\ h_{ij}\ \{-b_{ij-1} + 2{*}b_{ij} - b_{ij+1}\} \\ \vdots \\ h_{iJ}\ \{-b_{ij-1} + b_{ij}\} \\ \vdots \end{matrix}$$

$$g_{ij} = \varepsilon_{i+1} - \varepsilon_{i-1} + (j-1)\Delta y(\xi_{i+1} - \xi_{i-1}), \qquad h_{ij}=(g_{ij})^2/4\Delta x\Delta y Re\xi_i$$

$i=1,\ldots I;\quad j=1,\ldots J$

$$q_{210} = \begin{matrix} \vdots \\ \frac{\Delta x}{\Delta y}\ \{-\beta_{ij-1} + 2\beta_{ij} - \beta_{ij+1}\}/\xi_i \\ \vdots \\ \frac{\Delta x}{\Delta y}\ \{-\beta_{iJ-1} + \beta_{iJ}\}/\xi_i \\ \vdots \end{matrix}$$

$i=1,\ldots I;\quad j=1,\ldots J$

$$q_{211} = \begin{bmatrix} \vdots & \\ \{\cos(\theta)\ \xi_i\ \Delta x \Delta y\}/Fr & i=1,\ldots I;\quad j=1,\ldots J-1 \\ \vdots & \\ \{\cos(\theta)\ \xi_i\ \Delta x \Delta y\}/2Fr & i=1,\ldots I;\quad j=1,\ldots J \\ \vdots & \end{bmatrix}$$

$$q_{p2} = \begin{bmatrix} a_{11} \\ \vdots \\ a_{1j} \\ \vdots \\ a_{1J} \\ \vdots \\ a_{i1} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{iJ} \\ \vdots \\ a_{I1} \\ \vdots \\ a_{Ij} \\ \vdots \\ a_{IJ} \end{bmatrix}$$

$i=1,\ldots I;\ j=1,\ldots J$

$$a_{11} = -\gamma_{11}f_{11}/2 + \gamma_{1j}k1 + \gamma_{13}f_{13}/2 - \gamma_{21}k2f'_{21} +$$
$$\gamma_{31}f_{31}/2$$

$$a_{1J} = -\gamma_{1J-1}f_{1J}/2 + \gamma_{1J-1}k1 + \gamma_{1J}f_{1J}/2 - \gamma_{2J}k2f'_{2J} -$$
$$\gamma_{3J}f_{3J}/2$$

$$a_{1j} = -\gamma_{1j-1}f_{1j-1}/2 + \gamma_{1j}k1 + \gamma_{1j+1}f_{1j+1}/2$$

$$a_{i1} = -\gamma_{i-11}f_{i-11}/2 - \gamma_{i1}k2f'_{i1} + \gamma_{i+11}f_{i+11}/2 +$$
$$\gamma_{i+12}k1 - \gamma_{i+12}f_{i+12}/2$$

$$a_{ij} = \gamma_{i+1j-1}f_{i+1j-1}/2 + \gamma_{i+1j}k1 - \gamma_{i+1j+1}f_{i+1j+1}/2$$

$$a_{iJ} = \gamma_{i-1J}f_{i-1J}/2 - \gamma_{iJ}{-1}k2f'_{iJ} + \gamma_{i+1J-1}f_{i+1J-1}/2 +$$
$$\gamma_{i+1J-1}k1 - \gamma_{i+1J}f_{i+1J}/2$$

$$a_{I1} = -\gamma_{I-11}f_{I-11}/2 - \gamma_{I1}k2f'_{I1} + \gamma_{Ij-1}f_{Ij-1}/2$$

$$a_{Ij} = \gamma_{Ij-1}f_{Ij-1}/2 + \gamma_{Ij}k1 - \gamma_{Ij+1}f_{Ij+1}/2$$

$$a_{IJ} = \gamma_{I-1J}f_{I-1J}/2 - \gamma_{I-1J}k2f'_{I-1J} - \gamma_{IJ}f_{IJ}/2$$

$$f'_{1j} = 1.0 + \{\varepsilon 2 - \varepsilon 1 + (j-1)dy(\xi i{+}1 - \xi i) - (\varepsilon i{+}1 - \varepsilon i + (j-1)dy(\xi i{+}1 - \xi i))(\xi i{+}1 - \xi i))/4(\Delta x)^2\}$$

$$f_{,ij} = [1.0 + \{\varepsilon i{+}1 - 2\varepsilon i + \varepsilon i{-}1 + (j-1)dy(\xi i{+}1 - 2\xi i + \xi i{-}1) - (\varepsilon i{+}1 - \varepsilon i{-}1 + (j-1)dy(\xi i{+}1 - \xi i{-}1))(\xi i{+}1 - \xi i{-}1))/4(\Delta x)^2\}]/(\xi_i)^2$$

$$f'_{Ij} = 1.0 + \{-\varepsilon i + \varepsilon i{-}1 + (j-1)dy(-\xi i{+}\xi i{-}1) - (\varepsilon i - \varepsilon i{-}1 + (j-1)dy(\xi i - \xi i{-}1))(\xi i - \xi i{-}1))/4(\Delta x)^2\}$$