A RECURSIVE DEFINITION OF ORDINAL ARITHMETIC

by

PATRICIA MOIRA PRENTER

A THESIS

submitted to

OREGON STATE UNIVERSITY

in partial fulfillment of
the requirement for the
degree of

MASTER OF ARTS

June 1962

APPROVED

[redacted]

Professor of Mathematics

in Charge of Major

[redacted]

Chairman of Department of Mathematics

[redacted]

Chairman of School Graduate Committee

[redacted]

Dean of Graduate School

Date thesis is presented April 17, 1962

Typed by Barbara Ottinger

TABLE OF CONTENTS

A RECURSIVE DEFINITION OF ORDINAL ARITHMETIC

CHAPTER I

INTRODUCTION

There are many instances of recursive definition in mathematics. Indeed, all definitions are in a certain sense recursive since a definition depends recursively on meaning being attached to the words or objects used in the definition. A most striking and simple example of recursive definition is encountered in the formal approach to the development of natural number arithmetic. On this subject, at least two approaches are possible--one via Peano's axioms and the other via set theoretic foundations. No matter which formal byway one selects, axiomatics are entailed. The object of this paper will be to give a recursive definition of ordinal arithmetic. The axiomatic foundations will be those of Von Neumann-Fraenkel set theory as presented in Halmos' text <u>NAIVE SET THEORY</u>--that is with the exception of one axiom, that of replacement. The axiom of replacement has been rewritten to avoid the logician's technicalities entailed in Suppes' formulation and the cumbersome sequence functions of type-a encountered in Halmos' formulation.

The ordinal numbers and their arithmetic have been a topic of interest to many well-known mathematicians such as Cantor, Sierpinski, Fraenkel, Von Neumann and Hausdorff. Cantor's method of dealing with these numbers is very beautiful in its simplicity and was more than likely a major source of interest in the subject. However, with the advent of Russell's paradox and other paradoxes lying in the then current theory of aggregates, a need was seen for an axiomatic foundation for the theory of sets. Although Cantor's development thus became axiomatically unstable, it remains one of the most intuitive approaches to the theory of numbers. For this reason, an outline of his development of ordinal arithmetic is included in Chapter V.

Once the axiomatic foundations of set theory were established, they lent themselves beautifully to at least part of their original purpose--the definition of number. Halmos obtains the natural numbers through the axioms of Von Neumann and Fraenkel. A brief outline of this method of obtaining the natural numbers is the topic of Chapter III. He extends the natural numbers, which are themselves ordinal numbers via a unary operation and the axiom of substitution. He then defines ordinal numbers and proves some basic theorems concerning them. However, rather than define ordinal arithmetic recursively as he did natural number arithmetic, he defines it by Cantor's method with the axiomatic flaws dusted away.

This paper will take the other approach and develop ordinal arithmetic recursively. The axioms of Von Neumann-Fraenkel set theory through that of infinity are assumed. It is also assumed that the reader is well acquainted with them and with their uses in the theory of numbers. Those axioms and theorems which are prerequisite to the developments of the paper are listed in Chapter II. Chapter IV lays the foundations necessary for this paper's development of ordinal arithmetic (that is, the foundations necessary beyond the material mentioned in Chapter II). The rest is self-explanatory.

CHAPTER II

BACKGROUND

This chapter will be no more than a listing of the axioms of the Von Neumann-Fraenkel system up to but not including the axiom of replacement, and of those theorems which will be used in this paper. The axioms are stated but no attempt is made to discuss them; we assume that the reader is already familiar with them and with their implications. Also included is a list of those abbreviations and definitions which are perhaps out of the ordinary from the classic foundations. Other new definitions and abbreviations will be introduced as they become relevant.

Axioms:

1. Extension: Two sets are equal iff they have the same elements.
2. Specification: To every set A and to every condition $S(x)$ there corresponds a set B whose elements are exactly those elements x of A for which $S(x)$ holds.
3. Existence: There exists a set.
4. Pairing: For any two sets there exists a set to which they both belong.

5. Union: For every set S of sets there exists a set that contains all the elements that belong to at least one set of S.
6. Powers: For each set S there exists a set of sets that contains among its elements all the subsets of the given set S.
7. Infinity: There exists a set containing 0 and containing the successor of each of its elements.
8. Choice: If S is a set of disjoint, non-empty sets S_α, then $\exists$ a set R which has as its elements exactly one element x_α of each S_α.

Theorems:

Principle of Mathematical Induction:

If $S \subset \omega$ and if $0 \varepsilon S$ and if $n \varepsilon S \Rightarrow J(n) \varepsilon S$ then $S = \omega$. $J(n)$ is the successor of n.

The Recursion Theorem:

If a is an element of a set X and if f is a function from X into X, then there exists a function u from ω into X such that

$$u(0) = a$$

$$u(Jn) = f[u(n)] \text{ for all } n \varepsilon \omega.$$

An application of the recursion theorem is called definition by induction.

Principle of Transfinite Induction:

If X is a well ordered set and if $S \subset X$ and if $s(x) \subset S \Rightarrow x \in S$, then $S = X$.

Transfinite Recursion Theorem:

If W is a well ordered set and if f is a sequence function of type W in a set X, then there exists a unique function u from W into X such that

$$u(a) = f(u^a)$$

for each $a \in \overline{W}$. An application of the transfinite recursion theorem is called definition by transfinite induction.

Theorem A:

Given two well ordered sets $\overline{X}$ and $\overline{Y}$, either $\overline{X}$ is order isomorphic to $\overline{Y}$ or one of these two sets is order isomorphic to a strict initial segment of the other.

Abbreviations:

$\forall$ = for every

$s(x) = \{y \in A \mid y < x$ in the ordering of $A\}$

$\overline{s(x)} = \{y \in A \mid y \leq x$ in the ordering of $A\}$

ω = the minimal successor set = the natural numbers

$(A \tilde{=} B)$ = the set A is similar to the set B

$J(n)$ = the (immediate) successor of n

Definitions:

Well Ordered:

A set $\overline{W}$ is well ordered iff every non-empty subset of $\overline{W}$ has a least element.

Successor:

Let n be a set. $J(n) = n \cup \{n\}$ is called the successor (immediate) of n.

A sequence of type a in X:

is a function from $s(a)$, where $a \in W$ and W is well ordered, into a set X. If u is a function from W to X then the restriction of u to $s(a)$, written $u|s(a)$ or u^a is an example of a sequence function of type a in X.

A sequence function of type W in X:

where W is well ordered and X is any set is a function f whose domain is all sequences of type-a in X for all $a \in W$ and whose range is included in X.

CHAPTER III

A PARTIAL RÉSUMÉ OF NATURAL NUMBER ARITHMETIC

All of the axioms of set theory are essentially tools for set manufacture. Of principal concern to us is the axiom of infinity for it gives us the natural numbers and their arithmetic. A brief résumé of the construction is instructive. Axiom three states that there exists a set; call the set B. Use specification on B to obtain the empty set 0; that is:

$$\phi = 0 = \{x \varepsilon B \mid x \neq x\}.$$

By the axiom of pairing $\{0\}$ exists; that is $\{0,0\} = \{0\}$. This singleton has a special name; it is called the number one. Similarly, for any set A, the axiom of pairing tells us that $\{A\}$ exists and from a second application of pairing we discover that the set $\{A,\{A\}\}$ exists; whence by the axiom of unions, the set $A \cup \{A\}$ exists. This set we give a special name, $J(A)$.

<u>Definition 1</u>:

For any set A, $J(A) = A \cup \{A\}$, $J(A)$ is called the immediate successor of A. Any set obtained from A by repeated applications of J is called a successor set of A.

Now applying J to 0 and its successors we obtain:

$\phi = 0$

$1 = \{0\}$

$2 = \{0,1\}$

$3 = \{0,1,2\}$

etc.

Note that for each n so manufactured, $n = s(n)$. This property is of particular importance to us. We can continue manufacturing these "numbers" ad infinitum with no more than the axioms of pairing and union; but the critical question is, does there exist a set to which they all belong? By continued pairing and unions we can manufacture a set $C = \{0,1,2,\ldots,n\}$ and so on step by step; but nothing in our axioms before that of infinity tells us that this process repeated ad infinitum yields a set whose members are all the possible successors of 0 and 0 itself. The axiom of infinity is created to provide such a set. The minimal successor set of 0 is symbolically denoted ω and its members are called natural numbers. Thus ω is the set of all natural numbers. With this definition of the natural numbers, the principle of mathematical induction is a theorem rather than an axiom as it is in Peano's postulates. To define the elementary arithmetic operations on the natural numbers the recursion theorem is applied. For example:

<u>Definition of natural number addition</u>:

Let ω be the natural numbers

Let $X = \omega$ in the recursion theorem

Let $f = J$ in the recursion theorem

Then, for each $m \in \omega$ $\exists$ a unique $s_m \ni$

$$s_m(0) = m$$

$$s_m(Jn) = J[s_m(n)]$$

where $s_m\colon \omega \xrightarrow{\text{into}} \omega$

Here s_m corresponds to u in the recursion theorem. Define $s_m(n) = m + n$.

This is an instance of recursive definition. We will want to use such a recursive definition to define ordinal arithmetic. Ordinal numbers being special well ordered sets, it would seem that the transfinite recursion theorem suffices to define ordinal arithmetic; but such is not the case for several reasons. The first reason is that the transfinite recursion theorem maps a well ordered set into a set X where no structure whatsoever is imposed on X. However, the ordinal numbers are closed under addition; thus, if addition is to be regarded as some sort of recursive function, defined in a well ordered set, the range of this function must itself be well ordered. The exact difficulties encountered will be explained later. Suffice it to say that, in the case of natural number arithmetic operations, closure is readily obtained by specifying that the range set of the recursive function u be ω itself. No such gimmick is possible in the case of ordinal numbers for the set of all ordinal numbers does not exist. The man responsible for this skeleton in the mathematical closet was Burali-Forti; the proof that such a set does not exist is called the Burali-Forti paradox. Being void of a fitting and proper set of all ordinals to recurse from and to, we will have to resort to a more clandestine form of attack than that directly offered by the transfinite recursion theorem.

CHAPTER IV

ORDINAL NUMBERS AND SOME BASIC THEOREMS CONCERNING THEM

Recall that a well ordered set is a simply ordered set in which every non-empty subset has a smallest element. The definition of an ordinal number is based on a particular type of well ordered set. First recall definition one.

<u>Definition 1</u>

$J(A)$ is a unary operation on a set A such that

$$J(A) = A \cup \{A\}.$$

$J(A)$ is called the <u>immediate successor</u> of A. Any set obtained from A by repeated applications of J is called a successor set of A.

<u>Definition 2</u>

An <u>ordinal number</u> W is a set well ordered by point set inclusion such that $s(a) = a$ for all $a \varepsilon W$ and such that $W = s(W)$. It follows that if $a \varepsilon W$, a is an ordinal number.

<u>Definition 3</u>

If $y = J(x)$, x is the <u>immediate predecessor</u> of y.

Definition 4

An ordinal number γ is a limit ordinal if $\gamma \neq Jy$ for any ordinal number y.

On the basis of these definitions it should be obvious that the natural numbers are themselves ordinal numbers. The set of all natural numbers ω is also an ordinal number which is a limit ordinal since $\omega \neq Jx$ for any natural number x.

Two well ordered sets are similar if they are order isomorphic. It is a basic fact about well ordered sets (theorem A) that if X and Y are well ordered, then either X and Y are similar or one of them is similar to an initial segment of the other. Several important theorems about ordinal numbers are founded on this theorem.

Theorem 1

Given any two ordinal numbers a and b either $a \subsetneq b$ or $b \subsetneq a$ or $b = a$.

Proof: By theorem A either $a \cong b$ or $b \cong s(x)$ for some $x \in a$ or $a \cong s(y)$ for some $y \in b$ under an order isomorphism f.

<u>Case One</u>. $a \cong b$

Let $G \subset a$; $G = \{x \varepsilon a | f(x) = x \varepsilon b\}$.

Suppose $s(x) \subset G$.

Since $a \cong b$, $\exists$ an element $y \varepsilon b \ni x$ is mapped into y by f. That is, $f(x) = y$,

but $x = s(x)$ and $s[f(x)] = f|s(x)$,

$$\begin{aligned} \therefore s(y) &= s[f(x)] \\ &= f|s(x) \\ &= s(x) = x \\ \text{so } y &= f(x) = x \end{aligned}$$

$\therefore x \varepsilon G$ and $G = a$

$\therefore f\colon a \xrightarrow[1-1]{\text{onto}} b$ and $a = b$.

<u>Case Two</u>.

If $a \cong s(x)$ for some $x \varepsilon b$ then, by case one, under our order isomorphism, $f(a) = a = s(x)$ for some $x \varepsilon b$.

$\therefore a \varepsilon b \Rightarrow a \subset b$.

That $a \neq b$ follows from our initial condition

$a \not\cong b \Rightarrow a \underset{\neq}{\subset} b$.

Case Three.

$b \cong s(y)$ for some $y \varepsilon a$.

Proof: parallel to case two.

Implicit in the definition of an ordinal number W is the fact that if $a \varepsilon W$ then $a \subset W$ as well as the fact that if $a \varepsilon W$ then $a \neq W$. The first fact follows immediately from definition since $a = s(a) \Rightarrow a \subset W$. The second fact is not quite so immediate. If $a \varepsilon W$ then $a \subsetneq W$; for if it were true that $a = W$, then $a = s(a) = W \varepsilon W = s(W)$ which is impossible since $W \subsetneq W$ is impossible. Theorem one is no more than the trichotomy law for ordinal numbers.

Definition 5

An ordinal number a is less than an ordinal number b (written $a < b$)

$$\Leftrightarrow a \subsetneq b .$$

Hence theorem one states that for any two ordinal numbers a and b, either $a < b$, or $b < a$, or $a = b$.

Theorem 2

If a and b are ordinal numbers, then $a \cup b \cup \{a\} \cup \{b\}$ is an ordinal number to which both a and b belong.

Proof: By theorem 1, either $a = b$ or $a < b$ or $b < a$.

Case One.

If $a = b$ then $a \cup b = a = b$ is an ordinal number. Thus $a \cup b \cup \{a\} \cup \{b\} = a \cup \{a\}$ $= b \cup \{b\} = J(a) = J(b)$.

For every $x \varepsilon J(a)$, either $x = a = s(a)$ or $x \varepsilon a$, in which case $x = s(x)$ and clearly $J(a) = s[J(a)]$.

$\therefore J(a)$ is an ordinal number to which $a = b$ belongs.

Case Two.

$a < b$

$a < b \Rightarrow a \subsetneq b$

$\therefore a \cup b = b$

$J(a) = b \Rightarrow a \cup \{a\} \cup b = b \cup b = b$

$J(a) \neq b$ and $a < b \Rightarrow J(a) < b$ since $J(a)$ is an ordinal number.

(If $J(a) > b$, then $b = a$ or $b \subsetneq a$).

Again $a \cup \{a\} \cup b = b$ and $J(b)$ is an ordinal number to which b and a both belong.

Case Three.

$b < a$.

Proof: parallel to that of case 2.

Corollary 1:

If a is an ordinal number, $J(a)$ is an ordinal number to which a belongs.

Theorem 3

Let $\{\alpha_i\}$ be a set of ordinal numbers indexed by a well ordered set I. Then $\bigcup_{i \varepsilon I} \alpha_i$ is a unique ordinal number.

Proof: By the trichotomy of ordinal numbers the α_i form a simple chain under point set inclusion. The least element of each α_i is 0. Let $\bigcup_{i \varepsilon I} \alpha_i = A$. Let $a \varepsilon \bigcup_{i \varepsilon I} \alpha_i$; for some $j \varepsilon I$, $a \varepsilon \alpha_j \Rightarrow a = s(a)$. It remains to prove that A is well ordered. Let $B \subset A$. $b \varepsilon \alpha_j \Rightarrow b \varepsilon \alpha_m$ for all $m \varepsilon j$ and $m \varepsilon I$. Order B by point set inclusion (again the trichotomy of the ordinals) B must have a least element.

Proof: B is a simple chain. Let b_k be any member of B where $b_k \varepsilon \alpha_q$. If $b_k = 0$, b_k is the least element of B. If b_k is not zero, then $s(b_k) \subset \alpha_q$ <u>in the ordering of B</u>.

<u>Case One</u>.

If $s(b_k) = 0$, b_k is the least element of B.

<u>Case Two</u>.

If $s(b_k) \neq 0$, being a subset of the well ordered set α_q, $s(b_k)$ has a least element (<u>again in the ordering of B</u>) b_0 which is a least element of B.

Hence A is well ordered and is an ordinal number.

Q.E.D.

Having examined some basic properties of ordinal numbers, the next task will be to define an arithmetic for them. We have seen that the natural numbers and ω itself are ordinal numbers; but they are certainly only a small segment of the ordinal population. What other ordinal numbers are there? Corollary one states that if a is an ordinal number then $J(a)$ is also an ordinal. Similarly $J(\omega)$ is an ordinal which is given the name $\omega+1$ for reasons which will become apparent later. Given any ordinal, we can keep taking successors to obtain more and more ordinal numbers. How far can this process be continued; and, even more critical, can all these ordinals manufactured by the unary operation J be gathered together into one set? If they could, then by theorem 3, the union of this set would itself be an ordinal number. However, it happens that none of the tools for set manufacture which we have at hand to this point, including the axiom of infinity, is sufficient to guarantee that all the successors obtained from an initial ordinal number a by application of J ad infinitum belong together in a single set. To accomplish this desired end we need a new axiom, the axiom of replacement. Thus, just as the axiom of infinity was needed to manufacture ω, so the axiom of replacement is needed to make an appreciable extension of the ordinals beyond ω.

Once having the axiom of replacement, the somewhat curious process of defining ordinal arithmetic recursively can begin. As was mentioned earlier, natural number arithmetic was readily obtained by definition by induction from ω to ω. Here the range of the recursive function was well ordered. However, definition by transfinite induction of ordinal arithmetic is unfeasible since the transfinite recursion theorem as developed by Halmos imposed no structure whatsoever on the range of the recursive function. In order to define ordinal arithmetic recursively, we need a structured set for our recursive function so as to close the ordinal arithmetic operations.

The technique of recursive definition will be to start with any domain set upon which our function (binary operation) is to be defined, manufacture a well ordered range by the axiom of replacement and define f accordingly so that arithmetic operations will be closed. The whole process is rather like refusing to gamble unless you are sure that you are going to win. We want the sum of two ordinals to be a particular ordinal number; so we define the range of f first and then define f itself. In this manner, we extend the ordinals and define ordinal arithmetic simultaneously.

Once having done this, we could turn back and twist our f about to fit the transfinite recursion theorem and thus conjecture that our definition of ordinal arithmetic was an instance of definition by transfinite recursion. But this is totally unnecessary; so transfinite recursion will be abandoned and recursive definition will suffice.

It might be mentioned that there is a perhaps simpler way of obtaining ordinal arithmetic than that proposed. This other method amounts to ordering strings of disjoint well ordered sets appropriately. In any case, replacement is involved. The latter method being more intuitive, it will be examined briefly in a generalized form in the next chapter to gain insight into the recursive approach.

CHAPTER V

ORDER-TYPES, ORDINAL NUMBERS AND THEIR ARITHMETIC

Mainly to have at our command an operational understanding of order-types, we shall use Cantor's intuitively beautiful but axiomatically unstable approach to the order-type of a chain and the arithmetic of such order-types. The step to ordinal numbers is then simple; for ordinal numbers are merely the order-types of particular well ordered sets.

According to Cantor, given a set A which is a chain, the order-type $\overline{A}$ of A is defined as:

$$\{B \mid B \text{ is a set order-isomorphic to } A\}.$$

By B is order-isomorphic to A, we mean that there exists a function f which maps A onto B, one to one; and that for $x, y \varepsilon A$, where A is ordered by R_1 and B is ordered by R_2,

$$xR_1y \iff f(x)R_2f(y).$$

Consequently, $\overline{A} = \overline{B}$ if and only if $A \varepsilon \overline{B}$ (and $B \varepsilon \overline{A}$).

Now suppose you have two disjoint simply ordered sets, A and B, ordered by R and S respectively, where $\overline{A} = \alpha$ and $\overline{B} = \beta$. Then, $\alpha + \beta$ is the order type of $A \cup B$ ordered by $R \cup S \cup (A \times B)$. In general, the order-type of the $\sum_{i \in I} \alpha_i$ of order-types α_i of pairwise disjoint sets A_i ordered by R_i is:

$$\sum_{i \in I} \alpha_i = \text{order-type of } \bigcup_{i \in I} A_i \text{ ordered by}$$

$$[\bigcup_{i \in I} R_i] \cup [\bigtimes_{i \in I} A_i].$$

This, more or less, says that the order-type $\overline{A} + \overline{B}$ is the order-type of the set B glued on to the tail of the set A.

Similarly, Cantor defined the product of two order-types, $\overline{A}$ and $\overline{B}$, as the order-type of the set obtained by gluing the set A successively on to its own tail as many times as the cardinality of B. That is:

$$\overline{A} \cdot \overline{B} = \sum_{b \in B} \overline{A}_b \text{ where } A_b = A.$$

To define exponentiation, let

$$\overline{A}^{\sigma} = \prod_{i \in \sigma} \overline{A}_i, \text{ where } \overline{A}_i = \overline{A} \text{ for all } i \in \sigma.$$

<u>Examples</u>:

1) Let $A = \{1,2,3\}$ and let $B = \{a,b\}$.

Then: $\overline{A} + \overline{B} = \{\overline{1,2,3,a,b}\}$

$\overline{A} \cdot \overline{B} = \{\overline{1_a,2_a,3_a,1_b,2_b,3_b}\}$

$\overline{A}^{\overline{B}} = \overline{A} \cdot \overline{A} = \{1_1,2_1,3_1,1_2,2_2,3_2\}$

2) If $\omega = \{1,2,3,\text{-----}\}$ and $\overline{W} = \aleph_0$

and if $A = \{1\}$ and $\overline{A} = 1$,

then $\aleph_0 + 1 = \overline{\{1,2,3,\text{------},1\}}$

whereas $1 + \aleph_0 = \overline{\{1',1,2,3,\text{-------}\}}$

$= \overline{\{1,2,3,\text{----}\}} = \aleph_0$,

and hence $\aleph_0 + 1 \neq 1 + \aleph_0$.

3) Similarly if $A = \{a,b\}$ and $\overline{A} = 2$,

then $\aleph_0 \cdot 2 = \overline{\{1,2,3,\text{----};\ a_1,a_2,a_3,\text{----}\}} \neq \aleph_0$

whereas $2 \cdot \aleph_0 = \overline{\{a_1,b_1,a_2,b_2,\text{-----}\}} = \aleph_0$

since

$$f(n) = \left\{ \begin{array}{ll} a_{\frac{n+1}{2}} & \text{if } n \text{ odd} \\ b_{\frac{n}{2}} & \text{if } n \text{ even} \end{array} \right\} \text{is an order-isomorphism.}$$

Although this method gives us a simple algebra of order-types, we still cannot successfully extend the ordinal numbers beyond ω without the axiom of replacement and a unary successor operation.

The axiomatic difficulty with Cantor's definition of order-type is the usual one. That is, we have no axiom that guarantees that the set $\overline{A}$ exists. One sneaks out of this difficulty by first defining a specific set B to have a fixed order-type $\overline{B}$ and then by specifying that any set which is order-isomorphic to B has order-type $\bar{B}$. In our case, we will be concerned only with the order-types of well ordered sets. Our fixed well ordered sets will be the ordinal numbers and we will obtain the order-type of a particular well ordered set by finding the ordinal number to which it is order isomorphic.

CHAPTER VI

THE AXIOM OF REPLACEMENT

In Chapter IV a method for extending the ordinal numbers beyond ω was discussed. Thus $J\omega$, $J[J\omega]$, etc. are all ordinal numbers. The critical problem was, given an ordinal number b, can all the successors of b be gathered together in one set. We know by the axiom of infinity that the set of all successors of 0 exists--it is ω. We next give ourselves an axiom that at first sight might not seem independent at all but which, in fact, is independent from the axiom of infinity (no proof will be given). This is the axiom of replacement.

Definition 6

Axiom of Replacement: Given any set A such that with each $\alpha \varepsilon A$ there is associated a set B_α, then $\exists$ a set B whose members are all the B_α. The association between each α and B_α will be called an indexing of the B_α's by A--written $B_\alpha = f(\alpha)$. The set B will be called a replacement set of the set A.

Note that with this axiom, we are permitted to manufacture functions to our heart's delight. If the set A is finite, the axiom is unnecessary, but if A is not finite, the axiom is mandatory. For example:

1) Suppose A is the ordinal number 3.

   $3 = \{0,1,2\}$

   Suppose A,B,C are any three sets.

   We can index A,B, and C by A in the following manner:

   $f(0) = A$

   $f(1) = B$

   $f(2) = C$

   By combining effects of the axioms of pairing and unions we can manage to get A,B, and C together in a set $H = \{A,B,C\}$ which is the replacement set of A. Now that H is a set, f becomes a function. Replacement was not needed to obtain the function f from the indexing f.

2) Now suppose $A = \omega$ and that we have a "collection" of sets B_α indexed by ω. For example:

   $f(0) = B_0$

   $f(1) = B_1$ where each B_i is a set.

   $f(2) = B_2$

   etc.

Then by no gymnastics with the axioms of union and pairing are we guaranteed a set containing all the B_α's. By the axiom of replacement, we are guaranteed such a set · B. Now that B is a set, our indexing f becomes a function.

With this new tool we extend the ordinals and obtain ordinal arithmetic by recursive definition.

CHAPTER VII

ORDINAL NUMBER ARITHMETIC

<u>Theorem 4</u>

Given any ordinal number $\overline{W}$, for each $a \varepsilon \overline{W}$, $\exists$ a unique function h_a mapping $\overline{W}$ onto a replacement set B :

$$h_a(0) = a$$

$$h_a(Jx) = J[h_a(x)]$$

$h_a(\gamma) = \bigcup_{\beta \varepsilon \gamma} h_a(\beta)$ when γ is a limit ordinal.

Proof:

<u>First Induction</u>.

$$\text{Let } G = \left\{ b\varepsilon\overline{W} \;\middle|\; \begin{array}{l} h_b \text{ exists and for each } x\varepsilon\overline{W}, \\ h_b(x) \text{ is a uniquely defined} \\ \text{ordinal number.} \end{array} \right\}$$

Let $s(a) \subset G$.

We must prove $a \varepsilon G$; to accomplish this end we use the axiom of replacement and a double induction.

With each $x \varepsilon \overline{W}$ associate a set by an indexing f of $\overline{W}$ as follows:

$f(0) = a$ where $a \varepsilon \overline{W}$ and hence is a set which is a unique ordinal number

$f(Jx) = J(f(x))$

$f(\gamma) = \bigcup_{\beta \varepsilon \gamma} f(\beta)$ where γ is a limit ordinal.

To prove $f(y)$ exists and is a set for each $y \varepsilon \overline{W}$, use a second induction.

<u>Second Induction</u>.

Let $G' \subset \overline{W}$; $G' = \left\{ c \ \varepsilon \ \overline{W} \,\middle|\, f(c) \text{ is a unique ordinal number.} \right\}$

Let $s(x) \subset G'$; then $x \varepsilon G'$

Proof:

<u>Case One</u>.

If $x = Jy$ then $y \varepsilon s(x)$ and so $f(y)$ is a uniquely defined ordinal number.

$\therefore J[f(y)]$ is a uniquely defined ordinal number by corollary 1.

But $J[f(y)] = f[Jy]$

$\therefore f(x)$ is a uniquely defined number.

$\therefore x \varepsilon G'$.

Case Two.

If x is a limit ordinal, then for every $y \varepsilon x$, $f(y)$ is a uniquely defined ordinal ($y \varepsilon x \Rightarrow y \varepsilon s(x)$). But then, by theorem 3, $\bigcup_{y \varepsilon x} f(y)$ is a unique ordinal number since x is a well ordered set. But by definition

$f(x) = \bigcup_{y \varepsilon x} f(y)$

$\therefore$ $f(x)$ is a unique ordinal number.

$\therefore$ $x \varepsilon G'$.

This completes the second induction.

But then, $\forall y \varepsilon \overline{W}$, $f(y)$ is a set which is a unique ordinal number.

$\therefore$ The axiom of replacement applies and the replacement set for $\overline{W}$ exists.

$\therefore$ f is a function.

Define $h_a = f$.

Then h_a exists and for each $x \varepsilon \overline{W}$, $h_a(x)$ is a uniquely defined ordinal number.

$\therefore$ $a \varepsilon G \Rightarrow G = \overline{W}$.

This completes the first induction and proves the theorem.

Definition 7

Definition of Ordinal Addition: $a + b = h_a(b)$

Lemma 1:

Given any two ordinal numbers α and β, $\alpha + \beta$ is an ordinal number.

Proof: This follows immediately from the inductive proof of theorem 4.

Lemma 2:

If β, γ, α are ordinals such that $\beta < \gamma$ then $\alpha + \beta < \alpha + \gamma$ and conversely:

let $G = \left\{ x \varepsilon \overline{W} - \overline{s(\beta)} \;\middle|\; \begin{array}{l} \alpha + \beta < \alpha + x \text{ where} \\ \beta < x. \end{array} \right\}$

Let $s(y) \subset G$.

Consider $\alpha + \beta$ and $\alpha + y$ where $\beta < y$.

Case One.

$y = Jx$

Then $\alpha + y = \alpha + Jx = h_\alpha[Jx] = J[h_\alpha(x)]$

$= (\alpha + x) \cup \{\alpha + x\}$.

$\beta < x \Rightarrow \alpha + \beta < \alpha + x$

So $\alpha + \beta \;\varepsilon\; \alpha + x \subset (\alpha + x) \cup \{\alpha + x\}$ and clearly

$\alpha + \beta \;\varepsilon\; J(\alpha + x) = \alpha + y$

$\therefore \alpha + \beta < \alpha + y$.

Case Two.

y is a limit ordinal.

$h_\alpha(y) = \bigcup_{\gamma \varepsilon y} h_\alpha(\gamma)$

but since $\beta < y \Rightarrow \beta \varepsilon y$

$\alpha + \beta \ \varepsilon \bigcup_{\gamma \varepsilon y} h_\alpha(\gamma)$

$\therefore \alpha + \beta \ \varepsilon \ \alpha + y \Rightarrow \alpha + \beta < \alpha + y$

hence $y \varepsilon G \Rightarrow G = \overline{W} - \overline{s(\beta)}$.

Proof of Sufficiency: If $\alpha + \beta < a + \gamma$ then $\beta < \gamma$

$\alpha + \beta < \alpha + \gamma \Rightarrow h_\alpha(\beta) \subsetneq h_\alpha(\gamma)$

Suppose $\beta \not< \gamma$.

Then by theorem 1, $\beta = \gamma$ or $\gamma < \beta$

Case One.

$\beta = \gamma$

Since ordinal addition is unique, it follows that

$h_\alpha(\beta) = h_\alpha(\gamma)$ contradicting hypotheses.

$\therefore \beta \neq \gamma$.

Case Two.

$\gamma < \beta$

By the necessity proof of the first part of this lemma it follows that

$h_\alpha(\gamma) < h_\alpha(\beta)$

but $\alpha + \gamma < \alpha + \beta$ and $\alpha + \beta < \alpha + \gamma$

is again a contradiction to the theorem!

$\therefore \gamma < \beta$ by theorem 1. Q.E.D.

<u>Lemma 3</u>:

Given α, β, γ are ordinal numbers and $\alpha < \beta$ it does not follow that $\alpha + \gamma < \beta + \gamma$.

Proof: Let $\alpha = 2,\ \beta = 3,\ \gamma = \omega$

$2 + \omega = \omega$

$3 + \omega = \omega$

but $\omega \not< \omega$.

<u>Lemma 4</u>:

Given any ordinal number α

$\alpha + 0 = 0 + \alpha$

Proof: $\alpha + 0 = h_{\alpha}(0) = \alpha$

$0 + \alpha = h_0(\alpha) = ?$

Let $G = \{x \varepsilon \overline{W} \mid x + 0 = 0 + x\}$

Let $s(x) \subset G$.

<u>Case One</u>:

$y = Jx \Rightarrow x \varepsilon G$

$h_y(0) = y$

$h_0(y) = h_0[Jx] = J[h_0(x)] = J[0+x] = J[x+0]$

$= J(x) = h_{Jx}(0) = h_y(0).$

Case Two:

y is a limit ordinal $\Rightarrow x \varepsilon G$, $\forall x \varepsilon y$

$$h_0(y) = \bigcup_{\gamma \varepsilon y} h_0(\gamma) = \bigcup_{\gamma \varepsilon y} h_\gamma(0)$$

$$= \bigcup_{\gamma \varepsilon y} \gamma = y = h_y(0)$$

$\therefore y \varepsilon G \Rightarrow G = \overline{W}$. Q.E.D.

Lemma 5:

Ordinal addition does not commute.

Proof: Let $\alpha = \omega$, $\beta = 1$.

$$\alpha + \beta = \omega + 1 = h_\omega(1) = J[h_\omega(0)] = J(\omega)$$

$$\beta + \alpha = 1 + \omega = h_1(\omega) = \bigcup_{\gamma \varepsilon \omega} h_1(\gamma) = \omega$$

and $\omega \neq J\omega$.

Lemma 6:

β is a limit ordinal iff $\alpha + \beta$ is a limit ordinal for any ordinal α.

Proof of Necessity: Let $G = \{x \varepsilon \overline{W} \mid x + \beta$ is a limit ordinal$\}$

Let $s(y) \subset G$.

Consider $y + \beta$.

Suppose $y + \beta$ is not a limit ordinal.

Then $y + \beta = J(\gamma)$ for some ordinal $\gamma \Rightarrow \gamma \varepsilon J(\gamma)$

but $y + \beta = h_y(\beta) = \bigcup_{x \varepsilon \beta} h_y(x)$

$\therefore \gamma \in \bigcup_{x \in \beta} h_y(x)$

$\therefore \exists \delta \in \beta \ni \gamma = y + \delta$

$\therefore J(\gamma) = J(y + \delta) = J[h_y(\delta)] = h_y[J\delta]$

but $J\delta \in \beta$ so that $J(\gamma) \in h_y(\beta)$

$\therefore (y + \beta) \in (y + \beta) \Rightarrow (y + \beta) \in s(y + \beta)$

which is nonsense.

$\therefore y + \beta$ is a limit ordinal

$\therefore y \in G \Rightarrow G = \overline{W}$. Q.E.D.

Proof of Sufficiency: Given $\alpha + \beta$ is a limit ordinal.

Suppose β is not a limit ordinal.

Then $\beta = Jy \Rightarrow h_\alpha[Jy] = J[h_\alpha(y)]$.

But $J[h_\alpha(y)]$ is not a limit ordinal. Contradiction.

$\therefore \beta$ is a limit ordinal. Q.E.D.

<u>Lemma 7</u>:

Given any three ordinal numbers α, β, γ

$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

Proof: Let $G = \{x \in \overline{W} \mid (\alpha + \beta) + x = \alpha + (\beta + x)\}$

Let $s(y) \subset G$

Consider $(\alpha + \beta) + y$.

Case One.

If $y = Jx$, then

$$(\alpha + \beta) + y = h_{\alpha+\beta}(y) = h_{\alpha+\beta}(Jx) = J[h_{\alpha+\beta}(x)]$$
$$= J[h_\alpha(\beta + x)] = h_\alpha[J(\beta + x)]$$
$$= h_\alpha[J(h_\beta(x)] = h_\alpha[h_\beta(Jx)]$$
$$= h_\alpha[h_\beta(y)] = h_\alpha(\beta + y)$$
$$= \alpha + (\beta + y).$$

Case Two.

If y is a limit ordinal, then $\beta + y$ is a limit ordinal.

$$\therefore \alpha + (\beta + y) = \bigcup_{(\beta+\delta)\varepsilon(\beta+y)} h_\alpha(\beta + \delta)$$
$$= \{\eta \mid (\exists\delta)((\beta+\delta)\varepsilon(\beta+y) \text{ and } \eta = a + (\beta+\delta)\}$$

But $\beta + \delta \;\varepsilon\; \beta + y \Rightarrow \delta\varepsilon y$ by lemma 2.

$$\therefore \alpha + (\beta+y) = \{\eta \mid (\exists\delta)(\delta\varepsilon y \text{ and } \eta = \alpha + (\beta+\delta))\}$$
$$= \bigcup_{\delta\varepsilon y} \alpha + (\beta + \delta) = \bigcup_{\delta\varepsilon y} (\alpha + \beta) + \delta$$
$$= (\alpha + \beta) + \gamma$$

$\therefore y\varepsilon G \Rightarrow G = W.$ Q.E.D.

Lemma 8:

For any ordinal number A and $\forall x\varepsilon A$ and $B_x = h_x(A)$, $A \cup B_x$ is an ordinal number.

Proof: $\forall y \;\varepsilon A \cup B_x$, $y = s(y)$ by lemma 1 and $A \cup B_x$ being well ordered follows from lemma 2.

It is readily seen that theorem four gives us a systematic way of extending the ordinal numbers by manufacturing entire replacement sets of them. For example, the replacement set B for $\overline{W} = \omega + 1$ where $B = h_\omega(\overline{W})$ is

$$B = \{\omega, \omega+1, \omega+2, \ldots\}$$

and by lemma 8

$$\overline{W} \cup B = \{0,1,2, \ldots, \omega,\omega+1, \ldots\} = \omega + \omega = \omega \cdot 2$$

is again an ordinal number. We can extend the ordinals even more quickly by defining ordinal multiplication. We again want to define the operation recursively. The most natural way to do this would seem to be to define a function p_m on a set which is an ordinal number containing m so that:

$$p_m(Jx) = h_m[p_m(x)] .$$

A less natural way of defining p_m would be to let

$$p_m(Jx) = h_{p_m(x)}(m).$$

It turns out, however, that the latter method is the correct one since (for reasons which will become clear shortly)

$$p_3(\omega + 1) \neq h_3[p_3(\omega)] = h_3(\omega) = \omega$$

whereas

$$p_3(\omega + 1) = h_{p_3(\omega)}(3) = h_\omega(3) = \omega + 3.$$

Theorem 5. Second Existence Theorem

Given an ordinal number $\overline{W}$, $\forall m \varepsilon \overline{W}$ $\exists$ a unique function p_m: $\overline{W} \to B$ (B a replacement set) $\ni$

$$p_m(0) = 0$$

$$p_m(Jx) = h_{p_m(x)}(m)$$

$$p_m(\beta) = \bigcup_{\gamma \varepsilon \beta} p_m(\gamma) \quad \text{when } \beta \text{ is a limit ordinal.}$$

First Induction.

Let $s(m) \subset G$ where

$$G = \left\{ \begin{array}{l} b\varepsilon\overline{W} \mid p_b \text{ exists and } \forall x\varepsilon\overline{W} \;\; p_b(x) \text{ is a} \\ \text{unique ordinal number} \end{array} \right\}$$

Does p_m exist?

We want to manufacture a replacement set for $\overline{W}$ via an indexing which we shall call P_m.

Define $P_m(0) = 0$. 0 is certainly a set and is a unique ordinal number.

Second Induction.

Let $G' = \{c \;\varepsilon\; \overline{W} \mid P_m(c)$ is a unique ordinal number$\}$.

Let $s(y) \subset G'$; does $y\varepsilon G'$?

Case One.

y is a limit ordinal.

Define $P_m(y)$ recursively as

$$P_m(y) = \bigcup_{\beta \varepsilon y} P_m(\beta)$$

This, by the axiom of replacement, is a set since each $P_m(\beta)$ is a unique ordinal number and the $P_m(\beta)$ are indexed by the set y. That is, the replacement set $y = \{P_m(\beta) | \beta \varepsilon y\}$ exists and hence, axiomatically by the axiom of unions, $\bigcup_{\beta \varepsilon y} P_m(\beta)$ exists as a set. Furthermore, by theorem 3, $P_m(y)$ is a unique ordinal number.

Case Two.

$y = Jb \Rightarrow b \varepsilon s(y) \Rightarrow P_m(b)$ is an ordinal number.

Define $P_m(y)$ recursively as follows:

$$P_m(y) = P_m(Jb) = h_{P_m(b)}(m) = P_m(b) + m$$

In order that we be permitted to do this, we need to know that $\exists$ an ordinal number to which both m and $P_m(b)$ belong in order that addition be defined. By theorem 2 $P_m(b) \cup \{P_m(b)\} \cup m \cup \{m\}$ is such an ordinal number.

Hence, $h_{P_m(b)}(m)$ is defined and by theorem 4 is a unique ordinal number.

Thus, $y \varepsilon G' \Rightarrow G' = \overline{W}$; which completes the second induction. But now, P_m is defined on all of $\overline{W}$ and for each $a \varepsilon \overline{W}$, $P_m(a)$ is an ordinal number and, hence, a set. Thus, by the axiom of replacement, $\exists$ a replacement set B for $\overline{W}$ obtained by the replacement function P_m indexed over $\overline{W}$.

Since, for each $a \varepsilon \overline{W}$, $P_m(a)$ is uniquely defined, P_m is a function which maps $\overline{W}$ onto the replacement set B.

Let $p_m = P_m$.

Thus $m \varepsilon G \Rightarrow G = \overline{W}$.

This completes the second induction and thus proves the theorem.

Definition 8

Definition of Ordinal Multiplication:

For any ordinal number $\overline{W}$ and $\forall m, b \varepsilon \overline{W}$, define:

$$m \cdot b = p_m(b)$$

where $m \cdot b$ is called the product of m and b.

A few examples of ordinal multiplication follow.

Let $\overline{W} = \omega + 1$

$p_5(0) = 0$

$p_5(1) = h_{p_5(0)}(5) = h_0(5) = 5$

$p_5(2) = h_{p_5(1)}(5) = 5 + 5 = 10$

$p_5(\omega) = \bigcup_{\gamma\varepsilon\omega} p_5(\gamma) = \omega$

$p_\omega(1) = h_{p_\omega(0)}(\omega) = \omega$

$p_\omega(2) = h_{p_\omega(1)}(\omega) = \omega + \omega = \omega \cdot 2$

$p_\omega(7) = h_{p_\omega(6)}(\omega) = \omega \cdot 6 + \omega = \omega \cdot 7$

<u>Lemma 9</u>:

For any ordinal number $\overline{W}$ and for each $x\varepsilon\overline{W}$

$\bigcup_{y\varepsilon\overline{W}} p_x(y)$ is an ordinal number.

Proof: Theorem 3.

<u>Lemma 10</u>:

Ordinal multiplication is not commutative.

Proof: Let $\overline{W} = \omega + 1 \cdot \omega\varepsilon\overline{W}$ and $2\varepsilon\overline{W}$.

$p_\omega(2) = h_{p_\omega(1)}(\omega) = h_\omega(\omega) = \omega \cdot 2$

$p_2(\omega) = \bigcup_{\beta\varepsilon\omega} p_2(\beta) = \omega \neq \omega \cdot 2$

Lemma 11:

If $\alpha > 0$, then $\beta < \gamma \Leftrightarrow \alpha\beta < \alpha\gamma$.

Proof: Let $\beta \varepsilon \overline{W}$ an ordinal number.

Necessity: Let $G = \{x\varepsilon(\overline{W} - \overline{s(\beta)}) | \beta < x \Rightarrow \alpha\beta < \alpha x\}$

Let $s(x) \subset G$.

Case One.

$x = Jy \Rightarrow y\varepsilon s(x) \Rightarrow \alpha\beta < \alpha y$

Then $\alpha x = p_\alpha[Jy] = h_{p_\alpha(y)}(\alpha)$

$= p_\alpha(y) + \alpha$

$= \alpha y + \alpha$ where $\alpha > 0$

but then $\alpha y < \alpha x$ by lemma 2

(that is $0 < \alpha \Rightarrow \alpha y + 0 < \alpha y + \alpha$)

$\therefore \alpha\beta < \alpha y < \alpha x \Rightarrow \alpha\beta < \alpha x$

$\therefore x\varepsilon G \Rightarrow G = \overline{W}$

Case Two.

x is a limit ordinal.

Then $\alpha x = \bigcup_{\delta \varepsilon x} \alpha\delta$

but $\beta \varepsilon x$ since $\beta < x$

$\therefore \alpha\beta \varepsilon \alpha x \Rightarrow \alpha\beta < \alpha x$

$\therefore x\varepsilon G \Rightarrow G = \overline{W}$

Sufficiency: If $\alpha\beta < \alpha\gamma$ and $\beta \not< \gamma$ then $\beta = \gamma$ or $\gamma < \beta$.

$\beta = \gamma \Rightarrow \alpha\beta = \alpha y$ by Lemma 12.

$\gamma < \beta \Rightarrow \alpha\gamma < \alpha\beta$ by necessity portion of lemma 11.

$\therefore \beta < \gamma$ to avoid contradiction.

<u>Lemma 12</u>:

If $\alpha > 0$ and $\alpha\beta = \alpha\gamma$ then $\beta = \gamma$.

Proof: Suppose $\alpha\beta = \alpha\gamma$

but $\beta \neq \gamma$

then $\beta < \gamma$ or $\gamma < \beta$ by theorem 1.

$$\left.\begin{array}{l} \beta < \gamma \Rightarrow \alpha\beta < \alpha\gamma \\ \gamma < \beta \Rightarrow \alpha\gamma < \alpha\beta \end{array}\right\} \text{contradiction}$$

$\therefore \beta = \gamma$.

<u>Lemma 13</u>:

If $\alpha > 0$ and $\beta\alpha = \gamma\alpha$ it does not follow that $\beta = \gamma$.

Proof: Let $\beta = 1, \gamma = 2, \alpha = \omega$

$$\beta\alpha = p_1(\omega) = \bigcup_{x\varepsilon\omega} p_1(x) = \omega$$

$$\gamma\alpha = p_2(\omega) = \bigcup_{x\varepsilon\omega} p_2(\omega) = \omega$$

but $2 \neq 1$.

<u>Lemma 14</u>:

If $\alpha > 0$ and $\beta < \gamma$ it does not follow that $\beta\alpha < \gamma\alpha$.

Proof: Let $\beta = 1, \gamma = 2, \alpha = \omega$.

Lemma 15:

$\alpha \cdot \beta$ is a limit ordinal iff α is a limit ordinal or β is a limit ordinal.

Proof Necessity:

Part One.

Let β be a limit ordinal. Suppose $\alpha \cdot \beta$ is not a limit ordinal. Then $\alpha \cdot \beta = y$ which is not a limit ordinal.

$\therefore \exists\, m,n$ where $n = Jm \ni$

$\alpha \cdot m \leq y \leq \alpha \cdot n$ (i)

$\alpha \cdot \beta = \bigcup_{\gamma \varepsilon \beta} \alpha \cdot \gamma$

(i) $\Rightarrow \alpha \cdot m \;\varepsilon\; \alpha \cdot \beta \Rightarrow m \;\varepsilon\; \beta$ by lemma 11, but since β is a limit ordinal, $m \;\varepsilon\; \beta \Rightarrow n \;\varepsilon\; \beta$.

$\therefore \alpha \cdot n \;\varepsilon\; \alpha \cdot \beta$

so by the transitivity of ε, $\alpha\beta\varepsilon\; a \cdot \beta$ which is impossible.

$\therefore \alpha \cdot \beta$ is a limit ordinal.

Part Two.

Let α be a limit ordinal.

Case One.

$\beta = Jy$.

Then $\alpha \cdot \beta = h_{P(\alpha)(y)}(\alpha) = p_{\alpha}(y) + \alpha$

and $\alpha \cdot y + \alpha$ is a limit ordinal by lemma 6.

$\therefore\ \alpha \cdot \beta$ is a limit ordinal.

<u>Case Two</u>.

β is a limit ordinal.

Then $\alpha \cdot \beta$ is a limit ordinal by part one.

Sufficiency: Suppose α, β are not limit ordinals.

Then $\beta = Jy$

$$\alpha \cdot \beta = \alpha(Jy) = p_\alpha[Jy] = h_{p_\alpha(y)}(\alpha)$$

$$= \alpha \cdot y + \alpha$$ is not a limit ordinal by lemma 6.

<u>Lemma 16</u>:

Ordinal multiplication distributes over addition from the left, but <u>not</u> from the right. That is:

(i) $m(\alpha + \beta) = m\alpha + m\beta$

(ii) $(\alpha + \beta)m \neq \alpha m + \beta m$ (Proof omitted)

<u>Proof(i)</u>:

For $\alpha, \beta \varepsilon \overline{W}$ an ordinal number

Let $G = \{x \varepsilon \overline{W} \mid \alpha(\beta + x) = \alpha\beta + \alpha x\}$

Suppose $s(x) \subset G$

Then consider $\alpha(\beta + x)$.

Case One.

$x = Jy$

Then $\beta + Jy = h_\beta(Jy) = J[h_\beta(y)]$

$$\therefore \alpha(\beta + x) = \alpha(\beta + Jy) = \alpha \cdot J[\beta + y]$$
$$= p_\alpha[J(\beta + y)]$$
$$= h_{p_\alpha(\beta + y)}(\alpha)$$
$$= h_{\alpha\beta + \alpha y}(\alpha)$$
$$= h_{\alpha\beta}[h_{\alpha y}(a)]$$
$$= h_{\alpha\beta}[h_{p_\alpha(y)}(\alpha)]$$
$$= h_{\alpha\beta}[p_\alpha\{Jy\}]$$
$$= h_{\alpha\beta}[p_\alpha(x)]$$
$$= \alpha\beta + \alpha x \Rightarrow x \varepsilon G .$$

Case Two.

If x is a limit ordinal then by lemma 6, $\beta + x$ is a limit ordinal.

$$\alpha(\beta + x) = \bigcup_{\delta \varepsilon (\beta + x)} \alpha \cdot \delta$$
$$= \{c \mid c = \alpha\delta \text{ for } \delta \varepsilon \beta + x\}$$
$$= \{c \mid c = \alpha(\beta + \gamma) \text{ for } \gamma \varepsilon x\}$$
$$= \{c \mid c = \alpha\beta + \alpha\gamma \text{ for } \gamma \varepsilon x\} \text{ since } \gamma \varepsilon G$$

$= \{c \mid c = h_{\alpha\beta}(\alpha\gamma)$ for $\gamma \varepsilon x\}$

$= \{c \mid c = h_{\alpha\beta}(\phi)$ for $\phi \varepsilon \alpha x\}$ since αx is a limit ordinal

$= \bigcup_{\gamma \varepsilon x} h_{\alpha\beta}(\alpha\gamma)$

$= h_{\alpha\beta}(\alpha x)$

$= \alpha\beta + \alpha x$

$\therefore$ $x \varepsilon G \Rightarrow G = \overline{W}$. Q.E.D.

<u>Lemma 17</u>:

Given α, β, γ ordinals

$(\alpha\beta)\gamma = \alpha(\beta\gamma)$

Proof: Let $\alpha, \beta \varepsilon \overline{W}$ an ordinal number

Let $G = \{x \varepsilon \overline{W} \mid \alpha(\beta x) \equiv (\alpha\beta)\, x\}$

Suppose $s(x) \subset G$

<u>Case One</u>.

$x = Jy \Rightarrow y \varepsilon G$

$(\alpha\beta)x = (\alpha\beta)[Jy] = p_{\alpha\beta}[Jy]$

$= h_{p_{\alpha\beta}(y)}(\alpha\beta) = (\alpha\beta)y + \alpha\beta$

$= \alpha(\beta y) + \alpha\beta$ since $y \varepsilon G$

$= \alpha[\beta y + \beta]$ by lemma 15

$= \alpha[h_{\beta y}(\beta)] = \alpha[h_{p_\beta(y)}(\beta)]$

$= \alpha \cdot [\beta(Jy)] = \alpha(\beta x)$

$\Rightarrow x \varepsilon G$

<u>Case Two</u>.

x is a limit ordinal.

$\alpha(\beta x) = p_\alpha[p_\beta(x) = p_\alpha[\bigcup_{\gamma\varepsilon x} \beta\gamma]$

$= \bigcup_{\delta\varepsilon q} p_\alpha(\delta)$ since βx is a limit ordinal,

$\beta x = \{\beta\gamma | \gamma\varepsilon x\} = \{\delta\} = q$

$= \{m | m = \alpha\delta \text{ for } \delta\varepsilon q\}$

$= \{m | m = \alpha(\beta\gamma) \text{ for } \gamma\varepsilon x\}$ since $\delta = \beta\gamma$

$= \{m | m = (\alpha\beta)\gamma \text{ for } \gamma\varepsilon x\}$ since $\gamma\varepsilon G$

$= \bigcup_{\gamma\varepsilon x} p_{\alpha\beta}(\gamma) = (\alpha\beta)\, x \Rightarrow x\varepsilon G$

$\Rightarrow G = \overline{W}$. Q.E.D.

The object of this paper has thus, in part, been achieved. It is possible to extend the theory to obtain ordinal exponentiation, the theory of sequences of ordinals and so forth. For the details of such material the reader is referred to the works of Suppes and Sierpinski. It might be remarked that this paper has relied rather heavily upon the work of Suppes although it has avoided his axiom schema in order not to become entangled in the logic of quantifiers, free and bounded variables, etc.

If the functions defining addition and multiplication in any way vaguely resemble homomorphisms from sets with unary operations onto sets with unary operations, such is not a coincidence. This paper began as an extension of Leon Henkin's paper "On Mathematical Induction" to the domain of definition by transfinite recursion on well ordered sets. Henkin's paper emphasized the algebraic aspects of the recursive definition of natural numbers. He defined natural number addition for example as a homomorphism from a very special induction model (special in that it was a Peano model) with a unary operation into itself. Thus, when this paper was begun, the search was for homomorphisms.

BIBLIOGRAPHY

1. Cantor, George. The theory of transfinite numbers. La Salle, Ill., Open Court, 1941. 211 p.

2. Fraenkel, Abraham. Abstract set theory. Amsterdam, North Holland, 1953. 479 p.

3. Halmos, Paul. Naive set theory. New York, Van Nostrand, 1960. 104 p.

4. Hausdorff, Felix. Set theory. New York, Chelsea, 1957. 352 p.

5. Henkin, Leon. On mathematical induction. American Mathematical Monthly 67: 323-338. 1960.

6. Sierpinski, Waclau. Lecons sur les numbres transfinis. Paris, Gauthier-Villars, 1928. 240 p.

7. Suppes, Patrick. Axiomatic set theory. New York, Van Nostrand, 1960. 265 p.

8. Wilder, Raymond. Introduction to the foundations of mathematics. New York, Wiley, 1958. 305 p.