This paper is a classification of 181 problems, each of which involves the evaluation or application of determinants. The problems are those which have appeared in the "Problems and Solutions" sections of the American Mathematical Monthly, Deutsche Mathematiker-Vereinigung Jahresbericht, Mathematics Magazine, and School Science and Mathematics. The problems are classified, with suitable modifications, much in the style of A Treatise on the Theory of Determinants, by W. H. Metzler. For easy reference, the last chapter contains a description of each problem. The classification is preceded by a brief introduction to the history of the development of determinant theory.
A CLASSIFICATION OF DETERMINANTS
WHOSE EVALUATIONS HAVE APPEARED
IN PERIODICAL LITERATURE

by

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A CLASSIFICATION OF DETERMINANTS
WHOSE EVALUATIONS HAVE APPEARED
IN PERIODICAL LITERATURE

I. INTRODUCTION

Statement of Purpose

The purpose of this thesis, as stated in the title, is to classify determinants whose solutions have appeared in periodical literature. To this end all volumes of four periodicals were used. These periodicals are the American Mathematical Monthly [AMM], Deutsche Mathematiker Vereinigung Jahresbericht [DMVJ], Mathematics Magazine [MM], and School Science and Mathematics [SSM].

Almost without exception, this classification is restricted to the "Problems and Solutions" sections. There are several reasons for this restriction. Generally, the problems and their solutions are not indexed adequately for easy access to a particular type of problem. I hope to accomplish this goal for problems which deal primarily with determinants. Articles are generally ignored because, first of all, they are indexed for easy reference, and secondly, because they generally deal with classes of determinants rather than specific problems. Applications are generally omitted unless some insight into the evaluation of a particular determinant is included.

All references are given in the following form: (1) name of
periodical (see abbreviations above), (2) volume number, (3) the pages on which the solution appears, and (4) the problem number. In case no solution is given, the reference is given for the magazine in which the problem was first proposed. An attempt has been made to include all substantially different solutions to a given problem. In case of a solution which has been omitted, a comment to that effect is given following the solutions which have been included.

Basic definitions are given in the remainder of this chapter. They are generally those given in Muir and Metzler, *A Treatise on the Theory of Determinants*, with suitable modifications for those whose usage has changed since this book was first published. Chapter II is a brief historical background in which no claim for completeness is made. Chapter III is a classification of problems in much the same style as that of Muir. Finally, an individual description of each problem is included in Chapter IV, insofar as the problem has some characteristic to distinguish it from others. The names of the persons whose solutions appear in this thesis are also included.

**Preliminary Definitions**

**Definition**

If we have \( n^2 \) quantities arranged in a square of \( n \) rows
and \( n \) columns, then the sum of all the terms that can be formed by taking the product of \( n \) quantities, one from each row and one from each column, the sign preceding any term being determined by writing in succession the number of the rows from which the quantities composing it have come, and in a separate series the numbers of the columns, and taking \(+\) or \(-\) according as the total number of inversions of order in these two series is even or odd, is called the determinant of these quantities.

**Definition**

When the elements of the first row or column of a determinant are all functions of one variable, the elements of the second row or column are like functions of a second variable, and so on, the determinant is called an alternant.

**Definition**

If, in an \( n \)th order determinant the element in the \( r \)th row and \( s \)th column is equal to the element in the \( s \)th row and \( r \)th column, for every choice of \( r \) and \( s \) (\( s = 1, 2, \ldots, n \)) we say the determinant is symmetric.

**Definition**

A determinant such that each line perpendicular to the main
diagonal has all its elements alike is called **orthosymmetric** (per-
symmetric).

**Definition**

A determinant such that any row is obtained from the preceding row by passing the last (first) element over all others to the first (last) position is called a **circulant**.

**Definition**

A **continuant** is a determinant all of whose elements are zero except those in the main diagonal and in the two adjacent diagonals parallel to and on either side of the main diagonal.

**Definition**

A **recurrent**, in this thesis, is one of a set of determinants which may be evaluated by means of a \( k \)th order linear recurrence relation. Such a determinant generally has one of the following forms:

\[
\begin{array}{c|c}
\begin{array}{ccccccc}
 x & x & . & . & . & . & . \\
 x & x & x & . & . & . & . \\
 x & x & x & x & . & . & . \\
 x & x & x & x & x & . & . \\
\end{array}
 & \begin{array}{ccccccc}
 x & x & x & x & x & & . \\
 x & . & . & . & . & . & . \\
 x & . & . & . & . & . & . \\
 x & . & . & . & . & . & . \\
\end{array}
\end{array}
\]
with all other elements being zero.

Definition

If we consider \( n \) differentiable functions, all of the same \( n \) variables, the determinant which in every case has the element in its \( r \)th row and \( s \)th column equal to the differential coefficient of the \( r \)th function with respect to the \( s \)th variable is called the Jacobian of the set of functions with respect to the given variables.

Definition

The Jacobian of the first differential coefficients of a function of \( n \) variables which is twice continuously differentiable is called the Hessian of the function. In symbols

\[
H(u) = \begin{vmatrix}
\frac{\partial^2 u}{\partial x_1 \partial x_2}, & \frac{\partial^2 u}{\partial x_1 \partial x_3}, & \ldots, & \frac{\partial^2 u}{\partial x_1 \partial x_n} \\
\frac{\partial^2 u}{\partial x_2 \partial x_1}, & \frac{\partial^2 u}{\partial x_2 \partial x_3}, & \ldots, & \frac{\partial^2 u}{\partial x_2 \partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 u}{\partial x_n \partial x_1}, & \frac{\partial^2 u}{\partial x_n \partial x_2}, & \ldots, & \frac{\partial^2 u}{\partial x_n \partial x_n}
\end{vmatrix}
\]

Since \( \frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial^2 u}{\partial x_j \partial x_i} \), the Hessian is symmetric.

Definition

Consider \( n \) functions of the same variable \( x \) which are \( n \) times continuously differentiable. The determinant which has, in every case, for the element of its \( r \)th row and \( s \)th column
the \((r-1)^{\text{st}}\) differential of the \(s^{\text{th}}\) function is called the Wronskian of the functions with respect to \(x\). Thus the Wronskian of \(y_1', y_2', y_3\) is

\[
\begin{vmatrix}
    y_1 & y_2 & y_3 \\
    \frac{dy_1}{dx} & \frac{dy_2}{dx} & \frac{dy_3}{dx} \\
    \frac{d^2y_1}{dx^2} & \frac{d^2y_2}{dx^2} & \frac{d^2y_3}{dx^2}
\end{vmatrix}.
\]

**Definition**

A determinant \(H\) is said to be **Hermitian** if its elements are complex and if \(H\) equals the transpose of the conjugate of \(H\).

In symbols, \(H = \overline{H}'\), where \(\overline{H}\) is the conjugate of \(H\), \(H'\) is the transpose of \(H[\overline{(h_{ij})} = (h_{ji})]\) and \(H'\) is the transpose of \(H[\overline{(h_{ij})} = (h_{ji})]\).
II. HISTORICAL BACKGROUND

The discovery of determinants is generally ascribed to G.W. Leibnitz (1693); however, this belief is by no means universally held. There is evidence that the Chinese prior to this time had "developed the idea of subtracting columns and rows as in the simplification of a determinant" (Smith, 1958, p. 475) in their methods of representing the coefficients of the unknowns of linear equations "by means of rods on a calculating board" (Smith, 1958, p. 475).

The great seventeenth century Japanese mathematician, Seki Kōwa (1683), used the idea of determinants in eliminating a quantity from two linear equations in his work. Kōwa did not apply the idea generally to sets of simultaneous linear equations.

A difficulty in establishing the first use of determinants arises in the dual meaning of the word "determinant." Most contemporary mathematicians have based their definitions of determinant on the existence of a square array, or matrix. This approach more or less implies that the square matrix is an essential part of a determinant (Miller, 1930). If we accept this definition completely, then Cayley must be regarded as the founder of determinant theory, for he was the first (1841) to have used the idea of a square array between vertical bars.

If we accept a more general notion of determinants, in which
the word determinant also refers to a certain polynomial form which arises in the solution of systems of linear equations, then the origins of this theory can be ascribed to Leibnitz and Kôwa. From this standpoint, it is interesting to note that Leibnitz "considered these forms solely with reference to simultaneous equations" (Smith, 1958, p. 476). He also used number pairs for coefficients in much the same way as we use double subscripts today. It is, in fact, partly on this basis that he is given credit for the discovery of determinants (Miller, 1930).

Vandermonde (1771) may be considered the formal founder of determinant theory (Smith, 1958), as he improved the notation, studied certain independent functions subsequently associated with his name, and gave the first systematic account of what little was known (Bell, 1940). It should be noted in passing that Cramer had earlier employed in rudimentary form the idea of solving linear equations with the aid of determinant forms.

Almost simultaneously with Vandermonde, Laplace (1772) and Lagrange (1773) made a number of important contributions to the development of determinant theory. Laplace is credited with the rule for expansion of a determinant in terms of its minors. Lagrange used determinants for other purposes than solving equations, and developed identities which were "later recognized as very special cases of the characteristic property of reciprocal
determinants" (Bell, 1940, p. 400).

In contrast to earlier work which was primarily French, the Germans began to make useful contributions to the development of the theory after 1800. Notable among these were Wronski (1811) and Gauss (1801). Gauss was the first to use the word "determinant," but not in the present sense. He applied determinants to number theory (Smith, 1958) and was the first to associate a matrix with a corresponding polynomial (Miller, 1930). This early work of the German mathematicians, however, was hampered by a "thick notation" and cumbersome style.

A long step forward was taken by Binet (1812) "in the rule for multiplication which, under suitable hypotheses, suffices to define determinants" (Bell, 1940, p. 400).

Of the earlier contributors, Cauchy (1812) was easily the most dominant force. He used the word "determinant" for the first time in its present sense (Smith, 1958). He summarized the theory, simplified the notation, improved upon the proof of Binet's multiplication theorem, laid the foundations for the study of alternants and compound determinants, and generally gave respectability to the study and use of determinant theory.

The German contributions to the now rapidly expanding theory continued to increase until the monumental advances of 1841. Wronski (1812) first suggested the idea of a determinant of infinite
order, and Schweins (1825) was the first to formally state this concept. The theory of infinite order determinants was subsequently developed in detail by Fursteneau (1860). Wronski also discussed several special forms, one of which bears his name, but his work attracted little attention because of its "singularly uncouth disguise" (Bell, 1940, p. 400). Hesse (1843) studied forms involving two differential quotients of a variable.

It is interesting to observe that two of the three mathematicians most influential in the development of determinant theory, Jacobi and Cayley, made major contributions to its development in the same year (1841). Jacobi was responsible for the final acceptance of the word "determinant" with its present meaning (Smith, 1958). He fruitfully studied alternants and functional forms which bear his name. In 1841, for the first time, the theory of determinants was made readily accessible through his extended memoirs in the widely read Crelle's Journal. Of the ten forms of determinants which had already made their appearance, Jacobi studied six extensively.

Nearly simultaneously with the appearance of Jacobi's memoirs, the English mathematician Cayley introduced the present determinant notation. This most fortunate innovation led rapidly to a wide diversity of special forms, of which "a few [were] useful, the majority merely curious" (Bell, 1940, p. 400).

Cayley's work, together with that of Sylvester, brought the
English to the forefront as contributors to the development of the
theory. In the two decades following 1841, these two men launched
the most important phase of recent development of determinant
theory (Smith, 1958). Cayley studied symmetric, skew, and ortho-
gonal determinants, while Sylvester developed his "unbral notation"
and introduced continuants (1853). Sylvester was also concerned with
the maximum values of determinants. He studied compound deter-
minants, which were later studied more extensively by Kronecker.

By 1860, the foundations of determinant theory had been well
developed, and numerous special forms received much attention.
There was a tremendous interest in the study of alternants, com-
pound determinants, and determinants whose elements are combina-
torial numbers. Orthogonants received much attention as the associ-
ation between determinants and transformations from one set of
axes to another was firmly established. Following 1860 the amount
of genuinely original material declined, and many new textbooks made
their way into publication.

Toward the close of the nineteenth century, English remained
the dominant language of new publications in the field of determinant
theory. This was partly due to the influence of American mathe-
maticians. Alternants, circulants, and multilineants continued to
receive increased attention. There was continued interest in the
study of Wronskians as a test for linear independence, but little
contribution to the knowledge of determinant theory resulted. Study of other forms either declined or remained at about the same level as earlier.

Several later developments are noteworthy. In 1893 the approximation theorem of Hadamard appeared. This theorem states "that the square of a determinant is never greater than the norm-product of the lines" (Cajori, 1922, p. 340). Also, a "class of determinants which have the same importance in linear integral equations as do ordinary determinants for linear equations with \( n \) unknowns was worked out by E. Fredholm... and again by D. Hilbert who reaches them as limiting expressions of ordinary determinants" (Cajori, 1922, p. 341). Finally, in the second decade of the twentieth century, G. A. Miller used the idea of the rank of a determinant to present the theory of the solution of a system of linear equations in a particularly elegant form (Cajori, 1922).

At this point, we conclude our brief historical account, and turn our attention to the classification of determinants.
III. DETERMINANTS AND SOLUTIONS

General Properties

Problem 1. [AMM, Vol. 3, p. 116-117, Prob. 57 (Alg)]

Find the quotient of

\[
\begin{vmatrix}
(s-a_1)^2 & a_1^2 & a_1^2 & \ldots & a_1^2 \\
a_1^2 & (s-a_2)^2 & a_2^2 & \ldots & a_2^2 \\
a_2^2 & a_2^2 & (s-a_3)^2 & \ldots & a_3^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_n^2 & a_n^2 & a_n^2 & \ldots & (s-a_n)^2 \\
\end{vmatrix}
\]

\[
= \begin{vmatrix}
(s-a_1) & a_1 & a_1 & \ldots & a_1 \\
a_1 & s-a_2 & a_2 & \ldots & a_2 \\
a_2 & a_2 & s-a_3 & \ldots & a_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_n & a_n & a_n & \ldots & s-a_n \\
\end{vmatrix}
\]

Solution: Let \( Q \) be the quotient, and as we can interchange row for column without altering the value, we get

\[
Q = \begin{vmatrix}
(s-a_1)^2 & a_1^2 & a_1^2 & \ldots & a_1^2 \\
a_1^2 & (s-a_2)^2 & a_2^2 & \ldots & a_2^2 \\
a_2^2 & a_2^2 & (s-a_3)^2 & \ldots & a_3^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_n^2 & a_n^2 & a_n^2 & \ldots & (s-a_n)^2 \\
\end{vmatrix}
\]

\[
= \begin{vmatrix}
(s-a_1) & a_1 & a_1 & \ldots & a_1 \\
a_1 & s-a_2 & a_2 & \ldots & a_2 \\
a_2 & a_2 & s-a_3 & \ldots & a_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_n & a_n & a_n & \ldots & s-a_n \\
\end{vmatrix}
\]

All the elements in the \( i^{th} \) column of the numerator being \( a_i^2 \), of the denominator \( a_i \), except in the \( i^{th} \) row which is \( (s-a_i)^2 \) for the numerator, and \( (s-a_i) \) for the denominator. Hence, we have
Multiply the first column of the numerator by $a_i^2$, of the
denominator by $a_i$ and subtract from the $i^{th}$ column; do this for
each column and the value is unaltered.

\[
Q = \begin{vmatrix}
1 & -a_1^2 & -a_2^2 & -a_3^2 & \cdots \\
1 & 0 & s(s-2a_2) & 0 & \cdots \\
1 & 0 & 0 & s(s-2a_3) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
1 & 0 & 0 & 0 & \cdots \\
\end{vmatrix}
\]

\[
1 -a_1^2 -a_2^2 -a_3^2 \cdots \\
1 \cdot 0 \cdot s(s-2a_2) \cdot 0 \cdot \cdots \\
1 \cdot 0 \cdot 0 \cdot s(s-2a_3) \cdot \cdots \\
\vdots \cdot \vdots \cdot \vdots \cdot \vdots \cdot \ddots \\
1 \cdot 0 \cdot 0 \cdot 0 \cdot \cdots \\
\]

Let $u = (s-2a_1)(s-2a_2)(s-2a_3) \cdots (s-2a_n)$ and

\[
\sum_{i=1}^{n} \frac{a_i^2}{s-2a_i} = \frac{a_1^2}{s-2a_1} + \frac{a_2^2}{s-2a_2} + \frac{a_3^2}{s-2a_3} \cdots .
\]

Expand by minors of the first row and factor. Then
\[ Q = \frac{s^{n-1} u \left\{ s + \sum_{i=1}^{n} \frac{a_i}{s-2a_i} \right\}}{u \left\{ 1 + \sum_{i=1}^{n} \frac{a_i}{s-2a_i} \right\}} \]

\[ = s^{n-1} \frac{s + \sum_{i=1}^{n} \frac{a_i}{s-2a_i}}{1 + \sum_{i=1}^{n} \frac{a_i}{s-2a_i}} \]

**Problem 2.** [AMM, Vol. 27, p. 81, Prob. 2810]

In the expansion of the following determinant, find the number of terms, and the number of terms having the coefficients +1, -1, +2, -2, +3, -3, +4, -4, +5, -5, +6, -6, +10, respectively:

\[
\begin{vmatrix}
  a & b & c & d & e & 0 & 0 & 0 \\
  0 & a & b & c & d & e & 0 & 0 \\
  0 & 0 & a & b & c & d & e & 0 \\
  0 & 0 & 0 & a & b & c & d & e \\
  A & B & C & D & E & 0 & 0 & 0 \\
  0 & A & B & C & D & E & 0 & 0 \\
  0 & 0 & A & B & C & D & E & 0 \\
  0 & 0 & 0 & A & B & C & D & E \\
\end{vmatrix}
\]

**Solution:** No solution to this problem has been published.
Problem 3. [AMM, Vol. 33, p. 278-279, Prob. 3184]

Reduce to a product the determinant

\[
\begin{vmatrix}
... & 1 & n & ... & n^p_i & ... \\
... & a_i & (n+1)a_i & ... & (n+1) a_i & ... \\
... & a_i^2 & (n+2)a_i^2 & ... & (n+2) a_i^2 & ... \\
... & a_i^{r-1} & (n+r-1)a_i^{r-1} & ... & (n+r-1) a_i^{r-1} & ...
\end{vmatrix}
\]

where the \( i \)th group of \( p_i+1 \) consecutive columns are indicated, where there are \( k \) such groups such that \( p_1 + p_2 + ... + p_k + k = r \), and where \( n \) is any number, which may, in fact, have a value in any group of columns different from that in any other group of columns without affecting the result.

Solution: No solution to this problem has been published.

Problem 4. [AMM Vol. 33, p. 526-527, Prob. 3148]

\[ \Delta = |x_1 y_2 z_3| \neq 0 \] and \( X_i \) the cofactor \( x_i \), \( Y_i \) the cofactor of \( y_i \), etc.

---

Prove that if 
\[
\begin{vmatrix}
1/x_1 & 1/x_2 & 1/x_3 \\
1/y_1 & 1/y_2 & 1/y_3 \\
1/z_1 & 1/z_2 & 1/z_3 \\
\end{vmatrix} = 0
\]
and conversely.

Solution: Denote the second determinant by \( \Delta' \) and the third by \( D' \). Then on expanding these determinants, we find

\[
\Delta' = \lambda_1 \left[ x_2 x_3 y_1 z_1 x_1 + x_3 x_1 y_2 z_2 x_2 + x_1 x_2 y_3 z_3 x_3 \right],
\]

(1)

\[
D' = \lambda_2 \left[ X_2 X_3 Y_2 Z_1 x_1 + X_3 X_1 Y_2 Z_2 x_2 + X_1 X_2 Y_3 Z_3 x_3 \right]
\]

where \( \lambda_1 \) and \( \lambda_2 \) are two finite factors each different from zero--these facts being implied by the forms in which \( \Delta' \) and \( D' \) are given. Now since

\[
\sum_{i=1}^{3} z_i x_i = 0
\]

and

\[
\sum_{i=1}^{3} Z_i x_i = 0,
\]

by a well-known elementary property of determinants, we can write (1) in the form

\[
\Delta' = -\lambda_1 (x_2 z_2 X_2 Z_3 - x_2 z_3 X_3 Z_2),
\]

(2)

\[
D' = -\lambda_2 (x_3 z_2 X_2 Z_3 - x_2 z_3 X_3 Z_2).
\]
Since $\Delta \neq 0$ by hypothesis, and $\lambda_1$ and $\lambda_2$ do not vanish in this problem, we see that $\Delta' = 0$ if, and only if, $D' = 0$.

**Problem 5.** [AMM, Vol. 35, p. 495-496, Prob. 2938]

Show that the determinant

\[
\begin{vmatrix}
    a_1 & a_2 & \cdots & a_n & -b_1 & 0 & \cdots & 0 \\
    0 & a_1 & \cdots & a_{n-1} & -b_2 & -b_1 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & a_1 & -b_n & -b_{n-1} & \cdots & -b_1 \\
    b_1 & b_2 & \cdots & b_n & a_1 & 0 & \cdots & 0 \\
    0 & b_1 & \cdots & b_{n-1} & a_2 & a_1 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & b_1 & a_n & a_{n-1} & \cdots & a_1
\end{vmatrix}
\]

has the property that every $n$-rowed determinant of the $n$ left-hand columns is equal to its cofactor.

**Solution:** Let $m$ be the minor determinant obtained by taking the $n$-left-hand columns in their original order and associating them with rows numbered $i_1, i_2, \ldots, i_n$, in order of naming of the given determinant $D$. The cofactor $M$ of $m$ in $D$ may be written as a $2n$-rowed determinant whose $n$ right-hand columns are the same as those of $D$, while the elements in its $n$ left-hand columns are all zeros except those forming the leading diagonal of $m$; that is, those in row $i_1$, column $1$; row $i_2$, column $2$; etc.; and
these elements are each unity. Representing \( M \) in this way, multiply it by the determinant (equal to \( a_1^n \)) whose left half is identical with that of \( D \), and whose right half has all of its elements zero except those in the leading diagonal of the lower half of it, which are each unity. On carrying out the multiplication in such a way that the element in row \( p \) and column \( q \) of the product is the result of combining column \( p \) of the determinant representing \( M \) with column \( q \) of the determinant representing \( a_1^n \), we get a determinant whose upper left quarter is precisely \( m \), whose lower right quarter is the same as the upper left of \( D \), while the lower left quarter consists of zeros. Hence \( a_1^n m = a_1^n M \). Therefore when \( a_1 \) is not zero, \( M = m \).

An additional comment is made concerning implications of this solution when a number of the \( a' \)'s and \( b' \)'s vanish.


Let \( D_n \) denote a determinant of order \( n \) whose elements are all zeros and ones, and which has two ones and \( n-2 \) zeros in every row and column. Show that:

(a) For every \( n \), \( D_n = \pm 2^m \), where \( m \) and \( n \) are both even or both odd, or \( D_n = 0 \).

(b) If \( D_5 = 0 \), then two rows are identical, and conversely.

(c) If two rows of \( D_n \) are alike, then two columns are alike,
and conversely.

(d) \(3m \leq n\)

(e) Show that property (b) does not hold except for \(D_2, D_3, D_5\).

Solution: By rearrangement of the order of rows and columns, \(D_n\) reduces to

\[
D_n = \begin{pmatrix}
A_1 & 0 & 0 & \ldots & 0 \\
0 & A_2 & 0 & \ldots & 0 \\
0 & 0 & A_3 & \ldots & 0 \\
& \ldots & \ldots & \ldots & \ldots \\
& & & \ldots & A_m
\end{pmatrix}
\]

where the \(A_k\) are square arrays of order \(n_k\) \((n_k > 1)\) with elements \(a_{ii} = a_{n_k} = 1\), \(a_{i+1,i} = a_{i,i+1} = 1\) \((i = 1, \ldots, n_{k-1})\), and with all other elements being zero.

Any \(A_k\) is at once seen to simplify to \(B_k\) in which \(b_{ii} = (-1)^{i+1}\) for \(i = 1, 2, \ldots, n_{k-1}\), and \(b_{n_k}\) \(n_k\) equals 0 or 2 according as \(n_k\) is even or odd, \(b_{i,i+1} = 1\) for \(i = 1, 2, \ldots, n_{k-1}\), and all other elements are zero. Thus \(B_k\) expands at once and we have:

\[
A_k = B_k = 0 \text{ for } n_k \equiv 0, \text{ mod } 2
\]

\[
A_k = B_k = 2 \text{ for } n_k \equiv 1, \text{ mod } 4
\]

\[
A_k = B_k = -2 \text{ for } n_k \equiv 3, \text{ mod } 4.
\]

Further, if \(n\) is odd, \(D_n \neq 0\) only when \(m\) and every \(n_k\) are
odd; and if \( n \) is even, \( D \neq 0 \) only when \( m \) is even and every \( n_k \) is odd. Hence:

(a) For \( n \) odd, \( D_n = \pm 2^m \) where \( m \) is odd, or \( D_n = 0 \); and for \( n \) even \( D_n = \pm 2^m \) where \( m \) is even, or \( D_n = 0 \).

(b) After the first reduction above, \( D_5 \) will have only one \( A_k \) (of order 5) or will have two \( A_k \), one of order three and the other of order two. In the first case \( D_5 \neq 0 \), and in the second \( D_5 = 0 \), and by definition of an \( A_k \) of order two we find that two rows, and two columns, of \( D_5 \) are identical. The converse is obvious.

(c) If two rows of \( D_n \) are identical the two ones must fall in corresponding columns, hence the two columns are identical, and conversely.

(d) Since to be different from zero, \( A_k \) must have order \( n_k > 3 \), then obviously

\[
n = \sum_{k=1}^{m} n_k > 3m .
\]

(e) By (b), \( D_5 \) necessitates two identical rows, and for \( D_2 \) it is trivial,

\[
D_3 = \pm \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \pm \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \mp 2 .
\]
Hence $D_3$ cannot be zero, and part (e) does not apply.

Any other $D_n = 0$ ($n = 4, n > 5$) may be due to the appearance of an $A_k$ of order $4p$, whose vanishing does not necessitate two identical rows in $D_n$. The converse of (b) is always true.


Prove that

$$
\begin{vmatrix}
  e_{11} & 0 & 0 & \ldots & 0 & A_1 \\
  e_{21} & e_{22} & 0 & \ldots & 0 & A_2 \\
  e_{31} & e_{32} & e_{33} & \ldots & 0 & A_3 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  e_{n1} & e_{n2} & e_{n3} & \ldots & e_{n,n-1} & A_n
\end{vmatrix}
= A_n - \sum A_{n/p_i} + \sum A_{n/p_ip_j} \ldots \sum \frac{1}{(-1)^{s}} A_{n/p_ip_j \ldots p_s}
$$

where (1) $e_{ij} = 1$ if $j$ is a divisor of $i$ and $e_{ij} = 0$ if $j$ is not a divisor of $i$; (2) $p_1, p_2, \ldots$ are distinct prime factors of $n$.

Solution: In the given determinant subtract from the $n$th row each of the rows $n/p_i$, add each of the rows $n/p_ip_j$, and so on, and denote the resulting determinant by $D$. Thus $D$ is equal to the original determinant. Let $e_{nj}$ ($j = 1, 2, \ldots, n$) denote the elements of the $n$th row of $D$. Then if we let $e_{ij} = 0$ ($i < j$), we may write
\[ a_{nk} = e_{nk} - \sum \frac{e_{n}}{p_{i}^{k}}, k + \sum \frac{e_{n}}{p_{i}p_{j}}^{k}, \ldots + (-1)^{s} \frac{e_{n}}{p_{i}p_{j} \cdots p_{s}}, k, \]

\((k = 1, 2, \ldots n-1)\)

and

\[ a_{nn} = A_{n} - \sum A_{n}/p_{i} + \sum A_{n}/p_{i}p_{j} - \ldots + (-1)^{s} A_{n}/p_{i}p_{j} \cdots p_{s}. \]

If \( k \) is not a divisor of \( n \), clearly \( a_{nk} = 0 \). Suppose that \( k \) divides \( n \). Let \( n = p_{1}^{t_{1}} p_{2}^{t_{2}} \cdots p_{s}^{t_{s}}, k = p_{1}^{r_{1}} p_{2}^{r_{2}} \cdots p_{s}^{r_{s}}, \)

and suppose that \( r_{i} < t_{i} \) for just \( m \) values of \( i \). Then

\[ a_{nk} = 1 - m + \binom{m}{2} - \ldots + (-1)^{m} = (1 - 1)^{m} = 0, (k = 1, 2, \ldots, n-1). \]

Hence \( a_{nn} \) is the only nonvanishing element in the last row of \( D \), and since \( e_{ii} = 1 \), we have \( D = a_{nn}. \)


A determinant of order \( n + 1 \) has for elements of its first and second rows, respectively, the successive powers of \( a \) and \( x \) with exponents from 0 to \( n \) inclusive. The third row is the derivative of the second row, and the elements of any row after that are given by \( a_{i+1,j} = \frac{1}{i-1} \frac{d a_{ij}}{d x}, \quad i = 2, 3, \ldots, n. \) Prove that the determinant has value \((x-a)^{n}\).

**Solution:** This problem is an easy corollary of the following
general theorem. Let

\[
F(x) = \begin{vmatrix}
    f_0(x) & f_1(x) & \ldots & f_n(x) \\
    f_0(a) & f_1(a) & \ldots & f_n(a) \\
    f'_0(a) & f'_1(a) & \ldots & f'_n(a) \\
    f''_0(a) & f''_1(a) & \ldots & f''_n(a) \\
    \vdots & \vdots & \ddots & \vdots \\
    f_{(n-1)}(a) & f_{(n-1)}(a) & \ldots & f_{(n-1)}(a) \\
\end{vmatrix},
\]

where \( f_i(x), i = 0, 1, \ldots, n, \) are \( n+1 \) distinct polynomials in \( x \) and \( f_i^k(x) \) is the \( k \)th derivative of \( f_i(x) \). If \( m \), the highest degree of any one of these polynomials, is not less than \( n \), then \( F(x) \) contains \( (x-a)^n \) as a factor.

**Proof:** It is evident that \( F(x) \) is a polynomial degree \( m \) in \( x \) whose \( k \)th derivative is

\[
F^k(x) = \begin{vmatrix}
    f_0^{(k)}(x) & f_1^{(k)}(x) & \ldots & f_n^{(k)}(x) \\
    f_0^{(k)}(a) & f_1^{(k)}(a) & \ldots & f_n^{(k)}(a) \\
    f'_0^{(k)}(a) & f'_1^{(k)}(a) & \ldots & f'_n^{(k)}(a) \\
    f''_0^{(k)}(a) & f''_1^{(k)}(a) & \ldots & f''_n^{(k)}(a) \\
    \vdots & \vdots & \ddots & \vdots \\
    f_{(n-1)}^{(k)}(a) & f_{(n-1)}^{(k)}(a) & \ldots & f_{(n-1)}^{(k)}(a) \\
\end{vmatrix}.
\]
Now both $F(x)$ and $F^{(k)}(x)$ vanish when $x=a$, provided $k < n$. It therefore follows from a well-known theorem regarding multiple roots that $F(x)$ contains $(x-a)^n$ as a factor.

Cor. 1: If $m = n$, $F(x) = c(x-a)^n$, where $c$ is a constant which may be determined by comparing coefficients of like powers of $x$ of the two members of the equation.

Cor. 2: If $f_i(x) = x^i$, then $c = 1!2!3! \ldots (n-1)!(-1)^n$.

Cor. 3: If $f_i(x) = x^i$ and the coefficients of the rows after the second are those specified in the problem, then $c = (-1)^n$.

Cor. 4: If in Cor. 3, we interchange $x$ and $a$, we have the determinant, as originally stated, equal to $(-1)^n(a-x)^n = (x-a)^n$.


Do there exist determinants of order $n > 2$, such that all $n!$ terms of the expansion are positive?

Solution: No. Any $2 \times 2$ minor would necessarily have three terms of like sign, viz., either three $+$'s and a $-$ or three $-$'s and a $+$. But it is already impossible to arrange a $3 \times 2$ matrix whose three $2 \times 2$ minors all have this structure.
Problem 10. [AMM, Vol. 52, p. 523-524, Prob. 4125]

Prove that

\[
\begin{vmatrix}
\sin \theta_1 & -i\theta_1 & 0 & 0 & \ldots & 0 & 0 \\
\sin \theta_2 & e^{i\theta_2} & -i\theta_2 & 0 & \ldots & 0 & 0 \\
\sin \theta_3 & 0 & e^{i\theta_3} & -i\theta_3 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\sin \theta_n & 0 & 0 & 0 & \ldots & e^{i\theta_n} & 0 \\
\end{vmatrix}
= \sin(\theta_1 + \theta_2 + \ldots + \theta_n).
\]

Solution: Put each \( \sin \theta = (e^{i\theta} - e^{-i\theta})/2i \) and remove the factor \( 1/2i \) outside the determinant. Subtract from each element of the first column the sum of all the other elements in its row. The determinant then becomes

\[
\begin{vmatrix}
\frac{i\theta_1}{2i} & -i\theta_1 & 0 & 0 & \ldots & 0 & 0 \\
\frac{i\theta_2}{2i} & e^{-i\theta_1} & -i\theta_2 & 0 & \ldots & 0 & 0 \\
\frac{i\theta_3}{2i} & 0 & e^{-i\theta_2} & -i\theta_3 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{i\theta_n}{2i} & 0 & 0 & 0 & \ldots & e^{-i\theta_{n-1}} & -i\theta_n \\
\frac{-i\theta_n}{2i} & e^{i\theta_n} & 0 & 0 & \ldots & 0 & e^{-i\theta_n} \\
\end{vmatrix}
\]

Expand by elements of the first column. The minor of \( e^{i\theta_1} \) is \( i(\theta_2 + \theta_3 + \ldots + \theta_n) \), since all elements below the main diagonal of this minor are zero. The minor of \( e^{-i\theta_n} \) is
\((-1)^{n-1} e^{i(\theta_1 + \theta_2 + \ldots + \theta_n)}\), since all the elements above its main diagonal are zero. Thus the determinant equals

\[
\frac{1}{2i} \left[ e^{i(\theta_1 + \theta_2 + \ldots + \theta_n)} - e^{-i(\theta_1 + \theta_2 + \ldots + \theta_n)} \right] = \sin(\theta_1 + \theta_2 + \ldots + \theta_n).
\]

A discussion of possible alternate solutions is also given.


Show that a pandiagonal magic square of order 4, when regarded as a determinant, has value zero.

**Solution:** Following Kraitchik's *Mathematical Recreations* (p. 186), we designate the general square as follows:

\[
\begin{array}{cccc}
A & B & C & D \\
E & F & G & H \\
I & J & K & L \\
M & N & O & P \\
\end{array}
\]

Let \(S\) be the common sum of each of the rows, columns, and diagonals, including the broken diagonals such as \(A, N, K, H\). These 16 sums yield 12 independent relations making it possible to eliminate all but four of the unknowns. Thus elimination yields the following square:
\[
\begin{array}{cccc}
A & B & C & S-A-B-C \\
E & S-A-B-E & A-C+E & B+C-E \\
S/2-C & A+B+C-S/2 & S/2-A & S/2-B \\
S/2-A+C-E & S/2-B-C+E & S/2-E & A+B+E-S/2 \\
\end{array}
\]

Now, subtracting the first column from the third, the second column from the fourth, and factoring out C-A and S-A-2B-C, leaves columns two and four equal. Hence, the determinant of the square is zero.


Do all fifth order pandiagonal magic squares have zero determinants?

Solution: Represent the general square determinant by

\[
\begin{array}{cccccc}
A & B & C & D & E \\
F & G & H & I & J \\
K & L & M & N & O \\
P & Q & R & S & T \\
U & V & W & X & Y \\
\end{array}
\]

and let Z be the common sum of the rows, columns, and diagonals including broken diagonals such as B, F, O, S, W. These 20 sums
yield 17 independent relations making it possible to eliminate all
but 8 of the elements. We have

\[
\begin{array}{cccc}
A & B & C & D \\
F & G & H & I \\
\end{array}
\begin{array}{cccc}
-A+H+I & -B-F-G-H+Z & -C-G-H-I+Z & -D+F+G \\
-A-B-C-D & -F-G-H-I+Z & A+B+C+D+G & +H-Z \\
-C-D-H-I+Z & A+B+C+F+G & B+C+D+G+H & -A-B-F \\
+H-Z & +I-Z & A-B-I & -B-C-G-H \\
C+D-F & -A-B-C-G+Z & -C-D-H-B+Z & A+B-I \\
& & B+C+F+G+H & +I-Z \\
\end{array}
\]

as the general pandiagonal square. Although the general form may
be of some interest, its determinant is not identically zero, as is
shown by

\[
\begin{array}{cccc|c}
10 & 18 & 1 & 14 & 22 \\
4 & 12 & 25 & 8 & 16 \\
23 & 6 & 19 & 2 & 15 \\
17 & 5 & 13 & 21 & 9 \\
11 & 24 & 7 & 20 & 3 & \hline
\end{array}
\]

\[
\begin{array}{c}
= -4,680,000.
\end{array}
\]

Hence, not all pandiagonal magic squares of order five have
zero determinants.

Given the 18 numbers \( a_i, b_i, c_i, \ i = 1, 2, 3, 4, 5, 6 \) such that

\[
\begin{vmatrix}
(bc)_{13} & (ca)_{13} & (ab)_{13} \\
(bc)_{25} & (ca)_{25} & (ab)_{25} \\
(bc)_{46} & (ca)_{46} & (ab)_{46}
\end{vmatrix}
= \begin{vmatrix}
(bc)_{24} & (ca)_{24} & (ab)_{24} \\
(bc)_{16} & (ca)_{16} & (ab)_{16} \\
(bc)_{35} & (ca)_{35} & (ab)_{35}
\end{vmatrix} = 0,
\]

where \( (mn)_{ij} = \begin{vmatrix} m_i & m_j \\ n_i & n_j \end{vmatrix} \), show that

\[
\begin{vmatrix}
(bc)_{12} & (ca)_{12} & (ab)_{12} \\
(bc)_{34} & (ca)_{34} & (ab)_{34} \\
(bc)_{56} & (ca)_{56} & (ab)_{56}
\end{vmatrix} = 0.
\]

Solution: Let \((i, j, k)\) denote the determinant \( |a_{i,j}b_{j,k}| \) and \(|(ij), (k\ell), (mn)|\) the determinant \( |(bc)_{ij}(ca)_{k\ell}(ab)_{mn}| \). Then

\[
\begin{vmatrix}
0 & (413) & (513) \\
(125) & (425) & 0 \\
(146) & 0 & (546)
\end{vmatrix}
\]

In the same way we find that

(ii) \(|(24), (16), (35)|(145) = -(524)(416)(135)-(124)(516)(435),\)

and (iii) \(|(12), (34), (56)|(145) = (412)(534)(156) + (512)(134)(456) \).

Since by hypothesis both (i) and (ii) are zero,

and this implies that (iii) is zero. Thus, the required result is proved if \((145) \neq 0\). In a similar manner the result is true if \((236) \neq 0\).

Let \((145) = (236) = 0\). We may therefore assume that one of the vectors, say \((a_1, b_1, c_1)\), is a linear combination of the other two, \((a_4, b_4, c_4)\) and \((a_5, b_5, c_5)\). Accordingly, we write

\[
(a_1, b_1, c_1) = \rho (a_4, b_4, c_4) + \sigma (a_5, b_5, c_5)
\]

and \((a_2, b_2, c_2) = \lambda (a_3, b_3, c_3) + \mu (a_6, b_6, c_6)\).

Then \(|(13), (25), (46)| = |\rho (43) + \sigma (53), \lambda (35) + \mu (65), (46)|\)

\[(iv)\] 
\[= \rho \lambda |(43), (35), (46)| + \rho \mu |(43), (65), (46)|
+ \sigma \mu |(53), (65), (46)|,

\[(v)\] 
\[| (24), (1\theta), (35) | = \rho \lambda | (34), (46), (35) | + \lambda \sigma | (34), (56), (35) |
+ \mu \sigma | (64), (56), (35) |, \quad \text{and}

\[(vi)\] 
\[| (12), (34), (56) | = \rho \mu | (46), (34), (56) | + \lambda \sigma | (53), (34), (56) | .

If \((iv) = (v) = 0\), \((vi)\) is also zero.

Geometrically the result is well known. If \(a_i, b_i, c_i\) are the homogeneous coordinates of six points \(P_i\), the hypothesis states \(P_1 P_3, P_2 P_5, P_4 P_6\) are concurrent and that \(P_2 P_4, P_1 P_6, P_3 P_5\) are also concurrent. These two sets of lines determine two projective pencils and therefore \(P_1 P_2, P_3 P_4, P_5 P_6\) are concurrent.

Consider a determinant of order \( n \) whose elements are \( x \) in the main diagonal, and \( \pm 1 \) elsewhere. Find the smallest positive number \( a \), such that for \( x > a \), the determinant is positive for all choices of the \( \pm \) signs.

Solution: It is known that the determinant \( d(a_{ij}) \) is positive if

\[
a_{ii} > \sum_{i \neq j} |a_{ij}|, \quad \text{for } i = 1, 2, \ldots, n.
\]

It follows that for \( x > n-1 \) the determinant in question is certainly positive for all choices of signs. We can take \( a = n-1 \) because the determinant clearly vanishes if \( x = n-1 \) and all the signs are negative.

The result quoted is due to P. Furtwängler (Sitzunsber. Acad. Wiss., Wien, Abt. IIa, 145(1936), p. 527) and can be proved by induction.

Problem 15. [AMM, Vol. 56, p. 33-37, Prob. E813]

Let \( S \) be the sum of the integer elements of a magic square of order three, and let \( D \) be the value of the square considered as a determinant. Show that \( D/S \) is an integer.
Solution I:

\[
\begin{array}{ccc}
 a & b & c \\
 d & e & f \\
 g & h & i \\
\end{array}
\]

Let \(d + e + f\) have magic sum \(N = S/3\). Then

\[
N = (a + e + i) + (d + e + f) + (g + e + c) - (a + d + g) - (c + f + i) = 3e,
\]

and \(S = 9e\). Hence, adding rows and columns,

\[
\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc}
 a & b & c & a & b & c & a & b & 3e & a & b & e \\
 d & e & f & d & e & f & d & e & 3e & d & e & e \\
 g & h & i & 3e & 3e & 3e & 3e & 3e & 9e & 1 & 1 & 1 \\
\end{array}
\]

Solution II: Every magic square of the third order may be written in a form equivalent to

\[
\begin{array}{ccc}
 m+x & m-x-y & m+y \\
 m-x+y & m & m+x-y \\
 m-y & m+x+y & m-x \\
\end{array}
\]

(See Kraitchik, Mathematical Recreations, p. 148.) Considering the square as a determinant, expanding and simplifying, we find its value to be \(9m(y^2 - x^2)\); and, since \(S = 9m\), and all elements are integers, \(D/S\) is an integer.

If the square is transformed by rotation, the foregoing result remains unaffected; likewise, if it be transformed by reflection in a mirror, except that in the latter case \(x^2\) and \(y^2\) are interchanged.
Remarks by the Proposer: The property \( D/S = 1 \), an integer, may be extended to magic squares of higher orders with certain restrictions.

In the absence of a qualifying adjective, such as pandiagonal, semi-nasik, multiplicative, etc., there is a certain looseness in the use of the term "magic square." Sometimes the term is defined as a square array of integers with the property that the elements of each row, of each column, and of each of the principal diagonals have equal sums (cf. E. 791). As shown by the preceding proofs, \( D/S = 1 \) for all third order squares of this broad type.

An \( n \)th order square, for which \( D/S = 1 \), may have this property spoiled without loss of its magical nature by adding the proper integer to each of \( n \) properly chosen elements. Let the coordinates of an element be determined by its row and column. Then, if \( n \) is even, the square may be spoiled by adding an integer \( x \) to the \( n \) elements \((1, 1), (2, n), (3, 2), (4, 3), \ldots, (n, n-1)\).

If \( n > 3 \) is odd, the square may be spoiled by adding \( x \) to the \( n \) elements \((1, 1), (2, n), (3, n-1), (4, 2), (5, 3), \ldots, (n, n-2)\). For example, consider the squares:

\[
\begin{array}{cccc}
15 & 6 & 9 & 4 \\
2 & 13 & 8 & 11 \\
7 & 12 & 1 & 14 \\
10 & 3 & 16 & 5 \\
\end{array}
\quad
\begin{array}{cccc}
17 & 24 & 1 & 8 & 15 \\
23 & 5 & 7 & 14 & 16 \\
4 & 6 & 13 & 20 & 22 \\
10 & 12 & 19 & 21 & 3 \\
11 & 18 & 25 & 2 & 9 \\
\end{array}
\]
The fourth order square has \( D/S = -96 \). When \( x \) is added to each of the underlined elements, the derived square is still magic, but now

\[
D/S = (x + 16)(x^2 + 4x - 24)/4,
\]

which is an integer only if \( x \) is even. The fifth order square has \( D/S = 15600 \). When \( x \) is added to each of the underlined elements, we find

\[
D/S = (x^4 + 25x^3 + 60x^2 - 3475x - 78000)/5,
\]

which is an integer only when \( x \) is a multiple of 5. It may be conjectured that if \( x \) is prime to \( n \), \( D/S \), is not an integer.

More frequently, an \( n^{th} \) order "magic square" is defined as above with the restriction that its elements are the first \( n^2 \) integers. Occasionally this restriction is weakened to include any \( n^2 \) integers in arithmetic progression. Under this latter condition \( D/S \) may be examined more profitably. Let the difference in the arithmetic progression be \( d \), and let \( Y \) be the sum of the elements of any row or column. Then

\[
S = nY = (n^2/2)[2a + (n^2 - 1)d],
\]

whence

\[
Y = (n/2)[2a + (n^2 - 1)d] = n[a+(d/2)(n^2 - 1)].
\]

If \( n \) is odd, \( n^2 - 1 \) is even, and \( Y = nk \), where \( k \) is an integer. If \( n \) is even and \( d \) is even, then again \( Y = nk \). In the
determinant add the elements of the first \( n - 1 \) columns to those of the \( n \)th column. In the derived determinant add the elements of the first \( n - 1 \) rows to those of the \( n \)th row. Every element of the \( n \)th row and \( n \)th column is now \( Y(=nk) \) except the common element of this row and column, which is \( nY \). Hence \( Y \) and \( n \) may be factored out, so that \( D = nYD' \) and \( D/S = D' \), which is an integer since the elements of the determinant \( D' \) are integers. It will be observed that no use has been made of the sum of the diagonals in the proof.

The case where \( n \) is even and \( d \) is odd offers more difficulty, for then \( Y = nk/2 \), where \( k \) is odd, so that

\[
D = \begin{vmatrix}
\cdots & \cdots & \cdots & Y \\
\cdots & \cdots & \cdots & Y \\
\cdots & \cdots & \cdots & Y \\
\cdots & \cdots & \cdots & Y \end{vmatrix} = \begin{vmatrix}
\cdots & \cdots & \cdots & 1 \\
\cdots & \cdots & \cdots & 1 \\
\cdots & \cdots & \cdots & 1 \\
Y & Y & Y & nY \end{vmatrix} = YnD'/2.
\]

Then, in order that \( D/S \) may be an integer, \( D' \) must be even.

In the evaluation of these determinants the rows may be interchanged at random, as may be the columns, since only the absolute value is of concern. Since \( n \) is even and \( d \) is odd, there will be \( n^2/2 \) even integers (e) and \( n^2/2 \) odd integers (o) in the array. If in any pair of columns (or rows) the odd and even integers occurs in the same sequence we shall say that the pair is "matched," e.g.,
If in any pair of columns (or rows) the odd and even integers occur completely out of sequence we shall say that the pair is "mismatched," e.g.,

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & 8 & 9 & 10 & 11 & 12
\end{array}
\]

If in a pair of columns (or rows) reference is made to two integers in the same row (or column) they will be called a "couple."

If the pair under consideration are rows, interchange the columns and rows. If a pair of columns are matched or matched except for a single couple, rearrange the columns so that neither of the pair is the \( n \)th column. Rearrange the rows so that the exceptional couple falls in the \( n \)th row. Convert \( D \) into \( YnD'/2 \).

In \( D' \) add the elements of one of the pair to the elements of the other of the pair, which will then contain all even elements, so \( D' \) is even. (All even-ordered magic squares which I have examined fall in this category.)

If the pair is mismatched or mismatched except for a single couple, prepare the determinant as in the case of the matched pair. Convert \( D \) into \( YnD'/2 \). In \( D' \) successively add each column of the pair to the \( n \)th column, which will then contain all even elements, so \( D' \) is even.
It may be noted further that if \( n \) is of the form \( 4m \), then \( Y \) is even, so in each of the columns, rows, and diagonals the odd and even elements are each even in number. If \( n \) is of the form \( 4m + 2 \), then \( Y \) is odd, so each column, row, and diagonal contains an odd number of odd elements.

When \( n = 4 \), all elements of a particular row or column will be like or there will be two odd and two even elements. There are 6 permutations of \( \text{eeoo} \), and 15 combinations of distinct permutations taken 4 at a time. (No combination containing two like permutations need be considered since they constitute a matched pair for which \( D' \) is even.) Only three of these combinations contain an even number of odd and an even number of even elements in every column. Hence, by interchanging rows, all possible squares devoid of matched pairs may be converted into one of the following:

\[
\begin{array}{cccc}
\text{e} & \text{e} & \text{e} & \text{e} \\
\text{o} & \text{o} & \text{o} & \text{o} \\
\text{e} & \text{e} & \text{e} & \text{e} \\
\text{o} & \text{o} & \text{o} & \text{o} \\
\end{array}
\]

Each of these squares contains a mismatched pair, so \( D' \) is even and \( D/S = 1 \) for all fourth order squares composed of 16 elements in arithmetic progression. (This fact has been established in another way by this writer in a note on Determinants of fourth order magic squares, this MONTHLY, November, 1948.) Since
the sum of the elements of the diagonals has not been employed, the squares need be magical only in their columns and rows.

When \( n = 6 \), an arrangement is possible which may be magical in rows, columns, and diagonals, although I have no numerical example of this, namely:

\[
\begin{align*}
o & e & e & o & o & o \\
o & o & e & e & o & e \\
e & o & o & e & o & e \\
o & e & o & e & e & o \\
e & o & e & e & o & o \\
e & e & o & e & o & o \\
\end{align*}
\]

When this is converted into \( Ynd'/2 \) and \( D' \) is expanded by minors, \( D' \) is found to be odd. Hence this approach will not confirm the integer nature of \( D/S \) for magic squares with \( n > 4 \) and even and \( d \) odd without calling upon some of the other properties of these magic squares.

**Problem 16.** [AMM, Vol. 56, p. 409, Prob. E834]

Show that

\[
F_n = \begin{pmatrix}
1 & -1 & 1 & -1 & 1 & -1 & \ldots \\
1 & 1 & 0 & 1 & 0 & 1 & \ldots \\
0 & 1 & 1 & 0 & 1 & 0 & \ldots \\
0 & 0 & 1 & 1 & 0 & 1 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{pmatrix}
\]
where $F_n$ is the $n$th term of the Fibonacci sequence $1, 1, 2, 3, 5, \ldots, x, y, x+y, \ldots$, and the determinant is of order $n-1$.

**Solution:** Denoting the above determinant by $D_n$, it is seen that $D_2 = 1$, $D_3 = 2$. It remains to be shown that

$$D_n = D_{n-1} + D_{n-2}, \quad n \geq 4.$$

In $D_n$ subtract the $(n-3)^{th}$ column from the $(n-1)^{th}$, the $(n-4)^{th}$ from the $(n-2)^{th}$, $\ldots$, the first from the third,

$$D_n = \begin{vmatrix} 1 & -1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & -1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & -1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{vmatrix}.$$ 

By expanding this determinant with reference to the first row, the result is the desired conclusion.

**Problem 17.** [AMM, Vol. 57, p. 636-637, Prob. 4331]

Let the equation $x^n + p_1 x^{n-1} + \ldots + p_n = 0$, possess a root $x_1$, whose modulus exceeds that of every other root of the equation. Prove that

$$\lim_{k \to \infty} \frac{\Delta_k}{\Delta_{k-1}} = -x_1,$$ 

where $\Delta_k$ is the $k$-rowed $(k > n)$ determinant.
Solution I: Developing $\Delta_k$ by the first column we get two determinants; keeping the first and developing the second by its first column, and so on, we get finally

$$\Delta_k = \sum_{j=1}^{n} (-1)^{j-1} p_j \Delta_{k-j},$$

a recursion formula expressing $\Delta_k$ as a function of the $n$ preceding determinants. (If $k \leq 2n$, certain $\Delta_n$ appear which are not defined in the proposal. A consistent definition, however, is obvious.) We may write the above system as

$$\sum_{j=0}^{n} (-1)^{j} p_j \Delta_{k-j} = 0, \quad p_0 = 1.$$

But this is a classic problem studied by Daniel Bernoulli
(Commentarii Acad. Sc. Petropol, III, 1732). The ratio of two successive terms of (1) tends to a limit which is that root $X_1$ (if one exists) whose modulus exceeds that of every other root of the characteristic equation of (1), namely

$$
(2) \quad \sum (-1)^j p_j X^{n-j} = 0 .
$$

Since the equation has for its roots the negatives of those of the given equation, we have as desired

$$
\lim \frac{\Delta_k}{\Delta_{k-1}} = X_1 = -x_1 .
$$

**Solution II:** Let $\frac{p_1 + 2p_2 x + \cdots + np_n x^{n-1}}{1 + p_1 x + p_2 x^2 + \cdots + p_n x^n} = \sum_{k=0}^{\infty} a_k x^k$.

Cross multiplication and the equating of like powers of $x$ gives a system of linear equations whose solution is

$$
a_{k-1} = (-1)^{k-1} \Delta_k .
$$

On the other hand, if we let

$$
f(x) = 1 + p_1 x + p_2 x^2 + \cdots + p_n x^n = \prod_{i=1}^{n} (1-x_ix) ,
$$
then \( \frac{f'(x)}{f(x)} = - \sum_{i=1}^{n} \frac{x_i}{1-x_i x} = - \sum_{i=1}^{n} \sum_{j=1}^{\infty} x_i^{j+1} x^j = - \sum_{j=0}^{\infty} x_j \sum_{i=1}^{n} x_i^{j+1} \).

Thus \( a_{k-1} = - \sum_{i=1}^{n} x_i^k \). Hence \( \Delta_k = (-1)^k \sum_{i=1}^{n} x_i^k \) and

\[
\lim \frac{\Delta_k}{\Delta_{k-1}} = - \lim \frac{\sum_{i=1}^{n} x_i^k}{\sum_{i=1}^{n} x_i^{k-1}} = -x_1^k.
\]

\[\text{Problem 18. [AMM, Vol. 58, p. 637, Prob. 4383]}\]

Let \( n \) symbols be ordered in two different ways, and let \( a_{ij} \) denote the number of symbols common to the first \( i \) in the first ordering and the first \( j \) in the second ordering. Prove that the determinant \( |a_{ij}| \) is 1 if the transition from the first ordering to the second is effected by an even permutation, and -1 if it is effected by an odd permutation.

\[\text{Solution: Let the symbols in the first ordering be numbered} 1, 2, \ldots, n, \text{ and in the second ordering} p_1, p_2, \ldots, p_n. \text{ Then, after defining} a_{0j} = a_{i0} = 0 \text{ for convenience, one can readily verify the following equalities which are true for} i, j = 1, 2, \ldots, n:\]
If $i > p_i$, then $a_{i,j} - a_{i-1,j} = a_{i-1,j} - a_{i-1,j-1} = 1$.

If $i < p_i$, then $a_{i,j} - a_{i-1,j} = a_{i-1,j} - a_{i-1,j-1} = 0$.

If $i = p_j$, then $a_{i,j} - a_{i-1,j} = 1$, and $a_{i-1,j} - a_{i-1,j-1} = 0$.

Hence the expression $a_{i,j} - a_{i-1,j} - a_{i-1,j} + a_{i-1,j-1}$ is 1 if $i = p_j$ and 0 otherwise. Thus if we subtract the $(n-1)^{st}$ row of matrix $(a_{ij})$ from the $n^{th}$, the $(n-2)^{nd}$ from the $(n-1)^{st}$, ..., the $1^{st}$ from the $2^{nd}$, and do the same to the columns, we obtain the matrix $(\delta_i p_j)$. This latter matrix is just the permutation matrix corresponding to the permutation $i \rightarrow p_i$, and has determinant 1 or -1 according as the permutation is even or odd.

A compact expression for the relationship between $(a_{ij})$ and $(\delta_i p_j)$ can be obtained as follows. The operations performed on the rows of $(a_{ij})$ are equivalent to left multiplication by the matrix which has 1's in the main diagonal, -1's in the sub main diagonal, and 0's elsewhere. This matrix has as inverse the matrix $M_{\text{sub}}$ which has 1's below the main diagonal and 0's elsewhere. A corresponding result holds for the columns of $(a_{ij})$: one need only replace the words left, sub, below by right, super, above. The final result is $(a_{ij}) = M_{\text{sub}} (\delta_i p_j) M_{\text{sup}}$. 

Find the element of likeness in: (a) simplifying a fraction, (b) powdering the nose, (c) building new steps on the church, (d) keeping emeritus professors on campus, and (e) putting B, C, D in the determinant

\[
\begin{vmatrix}
1 & a & a^2 & a^3 \\
3 & 1 & a & a^2 \\
B & a^3 & 1 & a \\
C & D & a^3 & 1 \\
\end{vmatrix}
\]

Solution: The value \((1-a^4)^3\) of the determinant is independent of the values B, C, D. Hence operation (e) does not change the value of the determinant, but merely changes its appearance. Thus the element of likeness in (a), (b), (c), (d), and (e) is only that the appearance of the principal entity is changed. The same element appears also in: (f) changing the name of a rose, (g) writing a decimal integer in the scale of 12, (h) gilding a lily, (i) whitewashing a politician, and (j) granting an honorary degree.

Problem 20. [AMM, Vol. 61, p. 647-648, Prob. 4544]

The elements of a determinant are arbitrary integers. Determine the probability that the value of the determinant is odd.
Solution: Let $P_n$ be the required probability where $n$, the order of the determinant, is fixed. Then $P_n$ is also the probability that a determinant of elements 0 and 1 shall have its value $\equiv 1 \pmod{2}$, and we now consider only values $\pmod{2}$ of $n$-rowed determinants with elements 0 or 1.

The probability that the first row shall be $(0, 0, \ldots, 0)$ is $2^{-n}$, and the probability that such a determinant has value 1 is 0. If the first row is $(1, 0, 0, \ldots, 0)$, the probability of the value 1 is $P_{n-1}$.

If the first row is $(1, x_2, \ldots, x_n)$, there is an elementary transformation reducing each such determinant to one with first row $(1, 0, 0, \ldots, 0)$, and this transformation is its own inverse $\pmod{2}$.

Thus there exists a 1-1 value preserving correspondence between the set of all determinants with the one first row and the set of all those with the other first row. Since the order of the columns is immaterial, the probability that a determinant with given first row has value 1 is $P_{n-1}$, unless the first row is $(0, 0, \ldots, 0)$.

Hence $P_n = (1 - 2^{-n}) P_{n-1} = (1 - 2^{-1})(1 - 2^{-2}) \ldots (1 - 2^{-n})$, since $p_1$ is 1/2.


Let $D_1, D_2, D_3, \ldots$ be the determinants
Find the value of $D_n$ for any positive integer $n$.

**Solution:** $D_n$ may be described as follows: The first row consists of $n+1$ ones followed by $n-2$ zeros. The next $n-2$ rows are cyclic permutations of this row which introduce successively $1, 2, \ldots, n-2$ zeros on the left. The $n$th row consists of integers from 1 to $n$ followed by $n-1$ zeros. The remaining $n-1$ rows are the corresponding cyclic permutations of the $n$th row.

To evaluate $D_n$, from row $k+n-1$ subtract the sum of the rows indexed from $k$ to $k+n-2$ inclusive, as $k$ assumes successively the values $1, 2, \ldots, n$. This will produce a triangular matrix. The first $n$ elements of the main diagonal will be 1; the last $n-1$ elements will be $n+1$. Hence $D_n$ has value $(n+1)^{n-1}$.

**Problem 22.** [AMM, Vol. 65, p. 780-781, Prob. 4771]

Muir (Contributions to the History of Determinants 1900-1920, London and Glasgow) reproduces the assertion of Vog t that the
determinant

\[ D_n = \begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ 2 & 2^2 & 2^n & \ldots & 2^{n+1} \\ n & n^2 & n^n & \ldots & n^{n+1} \\ \frac{n}{2} & \frac{n}{2} & \frac{n}{n+1} & \ldots & \frac{n}{n+1} \end{vmatrix} \]

vanishes for \( n \geq 2 \) and \( n \) even. Prove the truth of this assertion. Also show that \( D_n \neq 0 \) for \( n \) odd.

Solution: We show that the associated linear equations

\[ x_0 r + x_1 r^2 + \ldots + x_n r^{n+1}, \quad 1 \leq r \leq n, \]

\[ x_0 \frac{n}{2} + x_1 \frac{n}{3} + \ldots + x_n \frac{n}{n+2} = 0, \]

have nontrivial solutions for \( n \geq 2 \) and even, and only the trivial solution for \( n \) odd.

Writing \( x_0 z + x_1 z^2 + \ldots + x_n z^{n+1} = f(z) \), these equations read:

\[ f(r) = 0, \quad 1 \leq r \leq n; \quad \int_0^n f(z) \, dz = 0, \]

so that \( f(z) = x_n z(z-1) \ldots (z-n) \). This assertion is equivalent to the fact that
\[ I_n = \int_0^n \left( \frac{z}{n+1} \right) \, dz = 0 \text{ for } n \geq 2 \text{ and even,} \]

and \( I_n \neq 0 \) for \( n \) odd.

Now \( \left( \frac{z}{n+1} \right) = (-1)^{n+1} \left( \frac{n-z}{n+1} \right) \), so that \( I_n = -I_n \) if \( n \) is even.

If \( n \) is odd, we use \( \left( \frac{z}{n+1} \right) + \left( \frac{z}{n+2} \right) = \left( \frac{z+1}{n+2} \right) \), so that

\[
I_n = \int_0^n \left( \frac{z+1}{n+2} \right) \, dz - \int_0^n \left( \frac{z}{n+2} \right) \, dz
= \left( \int_1^{n+1} - \int_0^n \right) \left( \frac{z}{n+2} \right) \, dz
= - \int_0^1 \left( \frac{z}{n+2} \right) \, dz + \int_n^{n+1} \left( \frac{z}{n+2} \right) \, dz
= -2 \int_0^1 \left( \frac{z}{n+2} \right) \, dz, \text{ in view of } \left( \frac{z}{n+2} \right) = -\left( \frac{n+1-z}{n+2} \right).
\]

Thus \( I_n \) is negative.


Evaluate the \( n^{th} \) order determinant \( D = |a_{ik}| \) for two cases: (1) \( a_{ik} = 0 \) when \( i + k \) is even, (2) \( a_{ik} = 0 \) when \( i + k \) is odd.

Solution: Let \( r \) run through the even integers and \( s \)
through the odd integers in the sequence 1, 2, ..., n.

(1) Using the Laplace transformation on the even rows of D, we observe that each row in the expansion has as a factor a determinant with a column of zeros if n is odd, and there is only one possible nonzero term if n is even. Hence

\[ D = (-1)^{n(n+1)/2} |a_{rs}| |a_{sr}|, \quad n \text{ even}; \quad D = 0, \quad n \text{ odd}. \]

(2) In the Laplace expansion on the odd rows of D, all terms but one vanish identically, and for \( n > 1 \), \( D = |a_{rr}| |a_{ss}| \). If \( n = 1 \), \( D = a_{11} \).


At most how many different values may an \( n \)th order determinant have, if its elements are a given set of \( n^2 \) different, non-zero real numbers?

Solution: There are \( (n^2)! \) distinct \( n \times n \) matrices which may be formed from \( n^2 \) algebraically independent elements. We divide the matrices into equivalence classes of matrices having the same determinant. Each such equivalence class has at least \( (n!)^2 \) elements, as there is a set \( T \) of \( n \) \( n^2 \) distinct transformations obtained by combinations of row and column permutation and transposition leaving the determinant of any \( n \times n \) matrix invariant. On the other hand, the elements being algebraically independent, the
value of $|(\theta_{ij})|$ completely determines the elements lying in the same row and column as $\theta_{ij}$ as those elements which do not multiply $\theta_{ij}$ in $|(\theta_{ij})|$. If $|(\theta_{ij})| = |(\phi_{ij})|$ for two of our $n!^2$ matrices, then we may bring the element $\theta_{11}$ to the $(1,1)$ position by transforming $(\phi_{ij})$ by an element of $T$. Then, at worst by a transposition, we bring the elements of the first row and column of $(\theta_{ij})$ into the first row and column respectively of the transformed $(\phi_{ij})$ matrix. Continuing in this way, we transform $(\phi_{ij})$ into $(\overline{\phi}_{ij})$ by elements of $T$ until the first row and column, with the possible exception of the last two elements of the first column, are placed exactly as those of $(\theta_{ij})$. Then, since $|(\overline{\phi}_{ij})| = |(\phi_{ij})|$, it follows by the above that $(\overline{\phi}_{ij}) = (\theta_{ij})$ and thus, that $(\phi_{ij})$ is one of the $n!^2$ matrices obtained from $(\theta_{ij})$ by transformations of $T$. Then the number of distinct values that the determinants of our matrices have is the number of distinct matrices divided by the number of matrices in each equivalence class, i.e., $(n^2)!/n!^2$.

Problem 25. [AMM, Vol. 70, p. 676-677, Prob. 5031]

Let $A$ be an $n \times n$ matrix with $n$ elements equal to 1 and the rest 0. Let $S_r$ be the sum of all $\binom{n}{r}$ principal $r \times r$ subdeterminants of $A$. Show that $S_r = \binom{n}{r}$ for some $r$. 
1 \leq r < n, if and only if \ A = I, the \ n \times n \ identity matrix.

**Solution:** It is clear that \ A = I, implies \ S_r = (n \choose r) \ for each \ r. On the other hand, let \ S_r = (n \choose r) \ for some \ r. \ S_r \ is of the form

\[ S_r = \sum_{i_1, i_2, \ldots, i_r} \pm a_{i_1j_1} a_{i_2j_2} \cdots a_{i_rj_r} \]

where the sum is extended over the \ r-element subsets \ \{i_1, i_2, \ldots, i_r\} \ of the set of indices \ 1 \leq i \leq n \ and over all permutations \ \{j_1, j_2, \ldots, j_r\} \ of these subsets.

Since \ S_r = (n \choose r), there must be exactly \ (n \choose r) \ nonzero terms, each equal to 1. This implies that each product of \ r \ nonzero entries of \ A \ must appear as a term in \ S_r. \ It follows that no row or column of \ A \ can have more than one nonzero entry. Therefore \ A \ is a permutation matrix.

Any principal \ r \times r \ subdeterminant of a permutation matrix will be 1, 0, or -1. Since \ S_r = (n \choose r), it follows that each must equal 1. If \ a_{ij} = 1, \ j \neq i, \ then any principal \ r \times r \ subdeterminant containing row \ i \ and omitting column \ j \ equals 0.

Therefore \ A = I.

Find the value of the determinant of order \( n \) which is formed in the following way: The elements of the diagonals running in the direction of the main diagonal and beginning with \( a_{21}, a_{11}, a_{13}, a_{15}, \ldots, a_{1,2k-1}, \ldots \) are all unity, and all other elements are zero.

Solution: Let \( A_n \) denote the value of the determinant of order \( n \). Then \( A_1 = A_2 = 1 \). For \( n > 2 \), we expand \( A_n \) with respect to its first column, yielding \( A_n = A_{n-1} - B_{n-1} \). Similar expansion of \( B_{n-1} \) gives \( B_{n-1} = -A_{n-2} \). Thus \( A_n = A_{n-1} + A_{n-2} \), the \( n \)th Fibonacci number.

Problem 27. [AMM, Vol. 72, p. 903, Prob. E1823]

Show that the determinant

\[
\begin{vmatrix}
(a_0 + b_0)^n & (a_0 + b_1)^n & \cdots & (a_0 + b_n)^n \\
(a_1 + b_0)^n & (a_1 + b_1)^n & \cdots & (a_1 + b_n)^n \\
\vdots & \vdots & \ddots & \vdots \\
(a_n + b_0)^n & (a_n + b_1)^n & \cdots & (a_n + b_n)^n
\end{vmatrix}
\]

has value \((-1)^{n(n-1)/2} \prod_{j=0}^{n} \frac{n}{j} \prod_{j<i}^{n} (a_i - a_j)(b_i - b_j)\).
Solution: No solution to this problem has yet been published.


Let \( n_j, j = 1, 2, \ldots, r \) be natural numbers with \( n = \sum_{j=1}^{r} n_j \).

For each \( i, \ 1 \leq i \leq n \), let

\[
    i = \sum_{j=1}^{s_i-1} n_j + \ell_i, \ 1 \leq \ell_i \leq n_{s_i}.
\]

Also let \( a_{ik} = \binom{k-1}{\ell_i-1} x_{s_i}^{k-\ell_i} \), where \( x_1, x_2, \ldots, x_r \) are indeterminate. Prove that

\[
    \det(a_{ik})_{i=1,2,\ldots,n} = \prod_{k=1}^{n} (x_t - x_s)^{n_s n_t}.
\]

Solution: For \( n_j = 1, j = 1, 2, \ldots, r = n \) \( \det(a_{ik}) \) is the Vandermonde determinant, and the proposition holds. For

\[
    n_1, n_2, \ldots, n_q = n = n_{q+1} = \ldots = n_r = 1 \quad \text{with} \quad 1 < q < r,
\]

assume that

\[
    \det(a_{ik})_{i=1,2,\ldots,n} = \prod_{k=1}^{n} (x_t - x_s)^{n_s n_t} \prod_{1 \leq s \leq q-1} (x_t - x_s)^{n_s n_{q-1}} \prod_{q < t < r} (x_t - x_s)^{n_s n_{q-1}} \prod_{1 \leq s < t < r} (x_t - x_s)^{n_s n_{q-1}}
\]

holds. We take the \( (n_{q-1})^{th} \) partial derivative of both sides with respect to \( x_q \) and set \( x_q' = \frac{q}{q-1} \):
\[
\frac{\partial}{\partial x_q} \left\{ \det(a_{ik}) \right\} \bigg|_{x_q = x_{q-1}} = n_{q-1}! \cdot \det(a^*_{ik}),
\]

with \( a^*_{ik} = a_{ik} \) for \( i \neq 1 + \sum_{j=1}^{n} n_j, k = 1, 2, \ldots, n, \)

and \( a^*_{ik} = (n_{q-1})^x q_{q-1} \) for \( i = 1 + \sum_{j=1}^{n} n_j \).

On the other hand,

\[
\frac{\partial}{\partial x_q} \left[ \prod_{1 \leq s \leq q-1} (x_q - x_s)^{n_s} \cdot \prod_{q < t < r} (x_t - x_s) \cdot \prod_{1 \leq s < q-2} (x_q - x_s) \right]_{x_q = x_{q-1}} = n_{q-1}! \cdot \prod_{1 \leq s \leq q-2} (x_q - x_s)^{n_s} \cdot \prod_{1 \leq s < q-2} (x_q - x_s)^{n_s} \cdot \prod_{q < t < r} (x_q - x_s)^{n_s} \cdot \prod_{1 \leq s < q-2} (x_q - x_s)^{n_s} \cdot n_{q-1} \cdot n_{q-1} - 1
\]

\[
\cdot \prod_{q < t < r} (x_q - x_s)^{n_s} \cdot \prod_{1 \leq s < q-2} (x_q - x_s)^{n_s} = n_{q-1}! \prod_{1 \leq s < q-2} (x_q - x_s)^{n_s} \cdot \prod_{q < t < r} (x_q - x_s)^{n_s} \cdot \prod_{1 \leq s < q-2} (x_q - x_s)^{n_s} \cdot n_{q-1} \cdot n_{q-1} - 1
\]

\[
= n_{q-1}! \cdot \prod_{1 \leq s < q-2} (x_q - x_s)^{n_s} \cdot \prod_{q < t < r} (x_q - x_s)^{n_s} \cdot \prod_{1 \leq s < q-2} (x_q - x_s)^{n_s} \cdot n_{q-1} \cdot n_{q-1} - 1
\]
with \( n_{q-1}^* = n_{q-1} + 1 \) and \( n_s = n_s^* \) otherwise. With this we have the equation

\[
\det (a_{ik})_{i=1,2,\ldots,n} = \prod_{1 \leq s < t \leq r} (x_t - x_s)_{s,t \neq q}^{n_s^* n_t^*}
\]

that is, we have established the truth of the assertion for

\( n_{q-1}^* = n_{q-1} + 1 \) and \( n_s = n_s^* \) otherwise from the assumption that it held for an arbitrary set of numbers, \( n_1, n_2, \ldots, n_{q-1}, n_q = n_{q+1} = n_r = 1 \). Thus the statement holds generally for an arbitrary set \( n_1, n_2, \ldots, n_r \).

**Problem 29.** [MM, Vol. 27, p. 226-227, Prob. Q104]

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<td>1</td>
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<tr>
<td>1</td>
<td>1 + x</td>
<td>1 + x</td>
<td>1 + x</td>
</tr>
</tbody>
</table>

**Solution:** When \( x = 0 \), the four rows are identical, so \( x^3 \) is a factor. The sum of each of the rows is \( 4 + 2x \) which is therefore a factor. Thus \( D = Ax^3(x+2) \), where \( A \) is a constant. Next, \( x^4 \) appears in only two terms, each positive, so \( A = 1 + 1 \). Hence, \( D = 2x^3(x+2) \).
Let \( D = \begin{vmatrix} \sum_{i=1}^{2n} x_i & \sum_{i=1}^{2n-1} x_i & \cdots & \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{2n-1} x_i & \sum_{i=1}^{2n-2} x_i & \cdots & \sum_{i=1}^{n-1} x_i & \sum_{i=1}^{n-1} y_i \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n-1} x_i & \cdots & (n+1) \sum y_i \\ x & x & \cdots & 1 & y \end{vmatrix} \)

where the sums range from 0 to \( n \). If \( x_i \neq x_j \), prove that a necessary and sufficient condition that \( D = 0 \) is that

\[
\begin{vmatrix} x_0^n & x_0^{n-1} & \cdots & 1 & y_0 \\ x_1^n & x_1^{n-1} & \cdots & 1 & y_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_n^n & x_n^{n-1} & \cdots & 1 & y_n \\ x & x & \cdots & 1 & y \end{vmatrix} = 0.
\]

**Solution:** Call the second determinant defined in the statement of the problem \( E(x, y, y_0, y_1, \ldots, y_n) \). Using matrix multiplication, we may verify that \( E'(0, 1, 0, 0, \ldots, 0)E(x, y, y_0, y_1, \ldots, y_n) = D \), where \( E' \) is the transpose of \( E \) (with \( x_i, y_1 \) replaced by zeros, \( i = 0, 1, \ldots, n \), and \( y \) replaced by 1). Since

\[
E' = \begin{vmatrix} x_0^n & x_0^{n-1} & \cdots & 1 & y_0 \\ x_1^n & x_1^{n-1} & \cdots & 1 & y_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_n^n & x_n^{n-1} & \cdots & 1 & y_n \\ x & x & \cdots & 1 & y \end{vmatrix} = 0.
\]
E' \( (0, 1, 0, 0, \ldots, 0) \) expands to a Vandermonde determinant that is not zero under the condition \( x_i \neq x_j, \) \( D = 0 \) if and only if \( E(x, y, y_0, \ldots, y_n) = 0. \)

A second solution is given by noting that the second determinant represents the \( n \)th degree polynomial passing through \( n + 1 \) points.

**Problem 31. [MM, Vol. 38, p. 56, Prob. 553]**

Prove that the determinant of the \( n \times n \) magic square formed from the numbers 1 to \( n^2 \) is divisible by \( n \) if \( n \) is odd and by \( n^2 + 1 \) if \( n \) is even.

**Solution:** The sum of the entries of the determinant is \( n^2(n^2 + 1)/2, \) so that each column has a sum \( s = n(n^2 + 1)/2. \)

We pre-multiply the matrix of the determinant by the square matrix \( M, \) whose first row and main diagonal entries are all unity, and all other entries are zero. This is a unitary matrix so the determinant of the product is the determinant of the original matrix. This product has the value \( s \) everywhere in the first row. The other entries are unchanged by the multiplication; hence the determinant is divisible by \( s. \) If \( n \) is odd, then \( (n^2 + 1)/2 \) is an integer, and the determinant is divisible by \( n. \) If \( n \) is even, \( n/2 \) is an even integer, and the determinant is divisible by \( n^2 + 1. \)
It is not necessary to hypothesize a magic square, for it is sufficient that either the column sums be equal or the row sums be equal. In the latter case we post-multiply by the transpose of the same matrix.

Problem 32. [SSM, Vol. 8, p. 512, Prob. 99(Alg)]

Express \((x - a_1)(x - a_2)\ldots(x - a_n)\) as a determinant of order \(n + 1\) with monomial elements.

Solution: The determinant is

\[
\begin{vmatrix}
x & b_1 & b_2 & b_3 & \ldots & b_{n-1} & 1 \\
a_1 & x & c_1 & c_2 & \ldots & c_{n-2} & 1 \\
a_1 & a_2 & x & d_1 & \ldots & d_{n-3} & 1 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
a_1 & a_2 & a_3 & a_4 & \ldots & x & 1 \\
a_1 & a_2 & a_3 & a_4 & \ldots & a_n & 1 \\
\end{vmatrix}
\]

where \(b_1, b_2, \ldots, b_{n-1}; c_1, c_2, \ldots, c_{n-2}; \) etc., are any quantities. If we multiply the last column by \(a_1\) and subtract from the first, \(x - a_1\) is seen to be a factor. Similarly \(x - a_2, \ldots, x - a_n\) are found to be factors.
Problem 33. [SSM, Vol. 39, p. 185-186, Prob. 1576]

Prove that

\[
\begin{vmatrix}
  x & 1 & 0 & 0 & \ldots \\
  1 & x & 2 & 0 & \ldots \\
  1 & 1 & x & 3 & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \ddots
\end{vmatrix}
= \begin{vmatrix}
  x & 1 & 1 & 1 & \ldots \\
  1 & x & 1 & 1 & \ldots \\
  1 & 1 & x & 1 & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \ddots
\end{vmatrix},
\]

the determinants being of the same order.

Solution: Let the determinants be of order \( n \). In the first determinant, subtract the elements of the second column from those of the first column, the third from the second, and so on. Then add elements of the first row to those of the second, the new elements of the second row to the third, and so on. The result is

\[
\begin{vmatrix}
  x-1 & 1 & 0 & 0 & \ldots & 0 \\
  0 & x-1 & 2 & 0 & \ldots & 0 \\
  0 & 0 & x-1 & 3 & \ldots & 0 \\
  0 & 0 & 0 & x-1 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & 0 & \ldots & x+n-1
\end{vmatrix} = (x-1)^{n-1}(x+n-1).
\]

Subtract the elements of the \( n \)th column of the second determinant from the elements of each of the other columns. Then add all other rows to the \( n \)th row. This gives
| x-1 0 0 0 ... 1 |
| 0 x-1 0 0 ... 1 |
| 0 0 x-1 0 ... 1 |
| 0 0 0 x-1 ... 1 |
| ... ... ... ... ... |
| 0 0 0 0 0 ... x+n-1 |

= \( (x-1)^{n-1}(x+n-1) \).

Problem 34. [SSM, Vol. 48, p. 149-150, Prob. 2058]

Show by skill rather than strength that

\[
\begin{vmatrix}
  a & a+p & a+2p \\
  b+2q & b+q & b \\
  c & c+r & c+2r \\
\end{vmatrix} = 0.
\]

Solution: From twice the second column subtract the sum of the first and third columns:

\[
\begin{vmatrix}
  a & 0 & a+2p \\
  b+2q & 0 & b \\
  c & 0 & c+2r \\
\end{vmatrix} = 0.
\]

Problem 35. [SSM, Vol. 63, p. 524, Prob. 2878]

Arrange the nine positive digits in a square array whose determinant equals a non-trivial palindromic cube.
Solution: It has been shown [Problem E1135, AMM, Vol. 62, p. 257-258, (See Prob. 135, p. 214, this paper)] that the maximum value that a third order determinant with elements 1, 2, ..., 9 may have is 412. The only positive palindromic cube less than 412, other than the trivial 0, 1, and 8, is $343 = 7^3$.

$$\begin{vmatrix} 1 & 6 & 4 \\ 7 & 2 & 9 \\ 8 & 5 & 3 \end{vmatrix}$$

is one determinant with a value of 343. It is one of a family of 72 $[2(3!)^2]$ with an absolute value of 343. The other members of the family are obtained by permuting the rows, permuting the columns, and interchanging rows and columns starting with the determinant shown. Half of the family are positive and half are negative. No other of the 9! arrays of the nine positive digits has a determinant equal in absolute value to 343. Some other members of the family are:

$$\begin{vmatrix} 6 & 4 & 1 \\ 2 & 9 & 7 \\ 5 & 3 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 7 & 8 \\ 6 & 2 & 5 \\ 4 & 9 & 3 \end{vmatrix} = \begin{vmatrix} 6 & 2 & 5 \\ 8 & 5 & 3 \\ 1 & 6 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 1 & 7 \\ 5 & 6 & 2 \\ 3 & 4 & 9 \end{vmatrix}$$

Problem 36. [SSM, Vol. 65, p. 105, Prob. 2958]

Find a general form in integers for the second order determinants which may be evaluated in three ways: (a) in the usual
manner, or by summing the products of the elements of (b) the rows, or (c) the columns.

Solution: Let \[ \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = ac + bd = ab + cd. \] From the last two members \( a(c-b) = d(c-b), \) so that if \( c \neq b, \) then \( a = d. \) From the first two members we get \( c = a(a-b)/(a+b). \) Let \( a = mb \) where \( m \) is to be determined. Then \( c = bm(m-1)/(m+1). \) The numerator of \( c \) is always even and the denominator may be odd. Let \( m = 2k-1, \) then \( c = b(2k-1)(k-1)/k. \) Neither \( k-1 \) or \( 2k-1 \) is always a multiple of \( k; \) so let \( b = k, \) and \( c = (k-1)2k-1). \) The determinant is

\[
\begin{vmatrix} k(2k-1) & k \\ (k-1)(2k-1) & k(2k-1) \end{vmatrix} = k(2k-1)(2k^2 - 2k + 1).
\]


Evaluate the following determinant whose terms are powers of certain Fibonacci numbers \( u_1 = 1, u_2 = 1, u_3 = 2, u_4 = 3, u_5 = 5, \) etc., where any number is the sum of the two preceding numbers):
64

<table>
<thead>
<tr>
<th>5³</th>
<th>5²</th>
<th>3</th>
<th>5</th>
<th>3²</th>
<th>3³</th>
</tr>
</thead>
<tbody>
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<td>8</td>
<td>5²</td>
<td>5³</td>
</tr>
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<td>8³</td>
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<tr>
<td>21³</td>
<td>21²</td>
<td>13</td>
<td>21</td>
<td>13²</td>
<td>13³</td>
</tr>
<tr>
<td>34³</td>
<td>34²</td>
<td>21</td>
<td>34</td>
<td>21²</td>
<td>21³</td>
</tr>
<tr>
<td>55³</td>
<td>55²</td>
<td>34</td>
<td>55</td>
<td>34²</td>
<td>34³</td>
</tr>
</tbody>
</table>

Solution: Taking advantage of the law of formation of the Fibonacci series, \( u_{n+2} = u_{n+1} + u_n \), the third and fourth columns may be converted largely to zeros. After expanding by minors of these columns, the fourth order determinant obtained may be evaluated in the conventional fashion to obtain

\[
D = -2^7 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13^3 = -5 \cdot 13^3 \cdot 21 \cdot 144 = -u_5^3 u_7 u_8^3 u_{12}
\]

Alternants


Evaluate

\[
\begin{vmatrix}
1 & 2^2 & \ldots & n^2 \\
1 & 2^4 & \ldots & n^4 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 2^n & \ldots & n^{2n}
\end{vmatrix}
\]
Solution: Multiply the first and second rows, respectively, by the coefficients of $x^2$ and $x^4$ in the polynomial $x^2(x^2-1)(x^2-2^2)$, and add the products to the third row. The first two elements of this row then reduce to zero, and the $k^{th}$ ($k > 2$) element becomes $\frac{5!}{k} \left( \frac{k+2}{5} \right) k^2$. By an extension of this process, all elements below the principal diagonal may be reduced to zero. The value of the determinant is then found to be $n!3!5!\ldots(2n-1)!$.


If $n > 2$ and $\epsilon$ is a primitive root of $\epsilon^n = 1$, show that

$$\begin{vmatrix} \epsilon & \epsilon^2 & \epsilon^3 & \ldots & \epsilon^{n-1} \\ 2 & 4 & 6 & \ldots & \epsilon^{2(n-1)} \\ \epsilon & \epsilon & \epsilon & \ldots & \epsilon \\ \epsilon & \epsilon & 9 & \ldots & \epsilon^{3(n-1)} \\ \epsilon & \epsilon & \epsilon & \ldots & \epsilon^{n-1} \\ \epsilon^2 & \epsilon^{2(n-1)} & \epsilon^{3(n-1)} & \ldots & \epsilon^{(n-1)(n-1)} \end{vmatrix} = (-1)^{(n-1)(n-2)/2} \frac{(n-2)!}{n(n-2)/2}.$$ 

Solution: It is clear that the given determinant equals

$$\begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ \epsilon & \epsilon^2 & \epsilon^3 & \ldots & \epsilon^{n-1} \\ \epsilon & \epsilon & \epsilon & \ldots & \epsilon \\ \epsilon^2 & \epsilon^4 & \epsilon^6 & \ldots & \epsilon^{2(n-1)} \\ \epsilon & \epsilon & \epsilon & \ldots & \epsilon^{(n-1)(n-1)} \end{vmatrix}.$$
The coefficient of this determinant being $\epsilon^{n(n-1)/2} = 1$, we have

the original determinant equal to the well-known difference-product

$\frac{1}{n!} \xi^2(\epsilon, \epsilon^2, \epsilon^3, \ldots, \epsilon^{n-1})$. But

$$
\begin{vmatrix}
S_0 & S_1 & S_2 & \ldots & S_{n-2} \\
S_1 & S_2 & S_3 & \ldots & S_{n-1} \\
S_2 & S_3 & S_4 & \ldots & S_n \\
& \ldots & \ldots & \ldots & \ldots \\
S_{n-2} & S_{n-1} & S_n & \ldots & S_{2(n-2)}
\end{vmatrix},
$$

where

$S_i = (\epsilon^i) + (\epsilon^2)^i + \ldots + (\epsilon^{n-1})^i$ [Scott and Matthews, Theory of Determinants, 2nd Ed., p. 152]. Making use of Newton's formulae for the sums of powers of roots, we have for the equation $x^n - 1 = 0$,

$S_0 = \epsilon^0 + (\epsilon^2)^0 + \ldots + (\epsilon^{n-1})^0 = -\epsilon^0 + n = n - 1$,

$S_1 = \epsilon + \epsilon^2 + \ldots + \epsilon^{n-1} = -\epsilon = -1$,

$S_2 = (\epsilon)^2 + (\epsilon^2)^2 + \ldots + (\epsilon^{n-1})^2 = -\epsilon^2 = -1$,

$S_n = n - 1$,

$S_{n+1} = -1$. 
Therefore

\[ \zeta(\epsilon, \epsilon, \epsilon, \ldots, \epsilon^{n-1}) = \begin{vmatrix}
    n-1 & -1 & -1 & \ldots & -1 & -1 \\
    -1 & -1 & -1 & \ldots & -1 & -1 \\
    -1 & -1 & -1 & \ldots & -1 & n-1 \\
    -1 & -1 & -1 & \ldots & n-1 & -1 \\
    \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
    -1 & -1 & -1 & n-1 & \ldots & -1 & -1 \\
    -1 & -1 & n-1 & -1 & \ldots & -1 & -1
\end{vmatrix} \]

Subtracting the second row from the first, the third, the fourth, etc., we have

\[ \zeta(\epsilon, \epsilon, \epsilon, \ldots, \epsilon^{n-1}) = \begin{vmatrix}
    n & 0 & 0 & 0 & \ldots & 0 & 0 \\
    -1 & -1 & -1 & -1 & \ldots & -1 & -1 \\
    0 & 0 & 0 & 0 & \ldots & 0 & n \\
    0 & 0 & 0 & 0 & \ldots & n & 0 \\
    \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
    0 & 0 & 0 & n & \ldots & 0 & 0 \\
    0 & 0 & n & 0 & \ldots & 0 & 0
\end{vmatrix} = (-1)^{(n-1)(n-2)/2} n^{n-2}. \]

By extracting the square root of both sides, the proposition is established.
Problem 40. [AMM, Vol. 40, p. 243-246, Prob. 3516]

Prove that

\[
\begin{vmatrix}
1 & a_1 & a_1^2 & \ldots & a_1^{n-1} & a_1^n \\
1 & a_2 & a_2^2 & \ldots & a_2^{n-1} & a_2^n \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & a_{n+1} & a_{n+1}^2 & \ldots & a_{n+1}^{n-1} & a_{n+1}^n \\
\end{vmatrix}
\]

\[
\text{limit} \frac{r(r-1)\ldots(r-n+1)}{n!} a_{r-n}.
\]

Solution: Let the equation

\[(1) \quad c_0 + c_1 x + \ldots + c_n x^n = x^r \quad (r > n + 1)\]

have as \( n + 1 \) of its roots \( a_i \) \(( i = 1, \ldots, n+1)\). We can thus obtain the system

\[(2) \quad c_0 + c_1 a_i + \ldots + c_n a_i^n = a_i^r, \quad i = 1, \ldots, n+1. \]

Solving (2) for \( c_n \) we find
\[ c_n = f(a_1, a_2, \ldots, a_{n+1}) = \frac{1}{1 - \frac{a_n}{a_{n+1}} \cdot \frac{a_{n+1}}{a_n}}. \]

As \( a_i \to a \), \( c_n \) approaches \( \lim_{a_i \to a} f(a_1, a_2, \ldots, a_{n+1}), i = (1, 2, \ldots, n+1) \). Evidently this limit is the coefficient of \( x^n \) in (1) if the \( n + 1 \) roots \( a_i \) all equal \( a \). If (1) has \( n + 1 \) equal roots, \( a \), the equations obtained after differentiating (1) \( n \) times will be satisfied by \( x = a \). Thus \( n! \cdot c_n = r(r-1) \ldots (r-n+1)a^{r-n} \), which proves the theorem.

A comment on the proof and a generalization of it are also given.

**Problem 41.** [AMM, Vol. 45, p. 257-258, Prob. 3783]

Show that

\[
\begin{vmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & 2^2 & 3^2 & \ldots & n^2 \\
1 & 2^3 & 3^3 & \ldots & n^3 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
1 & 2^n & 3^n & \ldots & n^n \\
\end{vmatrix}
= 1!2!3! \ldots n![\binom{n}{1} - \frac{1}{2}\binom{n}{2} + \frac{1}{3}\binom{n}{3} - \ldots + (-1)^{n+1}\frac{1}{n}\binom{n}{n}] .
\]
Solution I: Consider the determinant of the \( n^{th} \) order differing from the one in the problem only in the exponent which is \( l + i - 2 \) for the \( i^{th} \) row, where \( l \) is a positive integer and \( i = 2, 3, \ldots, n \).

Subtract the first column from each of the following. This results in a determinant of order \( n-1 \) which may be expanded in the form

\[
(1) \quad \sum (-1)^{r+1} \frac{(n!)}{r^l} V(1, 2, \ldots, r-1, r+1, \ldots, n),
\]

where \( V \) is the Vandermonde determinant \( V(1, 2, \ldots, r-1, r+1, \ldots, n) \)

\[
\begin{vmatrix}
1 & 1 & \ldots & 1 \\
1 & 2 & \ldots & r-1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 2^{n-2} & \ldots & (r-1)n-2 \\
1 & 2^{n-2} & \ldots & (r+1)n-2 \\
& & \ldots & \ldots \\
1 & 2^{n-2} & \ldots & (r-1)n-2 \\
1 & 2^{n-2} & \ldots & (r+2)n-2 \\
1 & 2^{n-2} & \ldots & (r+1)n-2 \\
\end{vmatrix} = \frac{(n-1)! (n-2)! \ldots 2!}{(r-1)! (n-r)!}
\]

For \( r = 1 \), we have \( V(2, 3, \ldots, n) \). The determinant then reduces to

\[
1! 2! \ldots (n-1)! (n!) \sum (-1)^{r+1} \frac{n}{r^l} (n) \frac{1}{r^l}.
\]

For \( l = 2 \), we obtain the special case of the problem.

Solution II: Let \( F(x) \) denote the polynomial which is obtained from the determinant in the problem by replacing its first column by \( 1, x, x^2, \ldots, x^n \). The given determinant is then \( F(1) \).
We have first

(1) \( F(0) = 1!2!\ldots n! n \) by reductions similar to those used for Vandermonde determinants. Since \( F(x) \) vanishes for \( x = 2, 3, \ldots, n \), we have

(2) \( F(x) = A(x-2)(x-3)\ldots(x-n)(x-B), \) where the two constants \( A \) and \( B \) will now be determined. Taking the logarithmic derivative of the two members of (2), then setting \( x = 0 \), and observing that \( F'(0) = 0 \), we have

\[
(3) \quad \frac{F'(0)}{F(0)} = - \sum_{i=2}^{n} \frac{1}{i} - \frac{1}{B} = 0.
\]

Set \( x = 0 \) in (2) and we have in turn

\[
AB = (-1)^n1!2!\ldots(n-1)!n,
\]

\[
A = (-1)^{n+1} 1!2!\ldots(n-1)!n \sum_{i=2}^{n} \frac{1}{i},
\]

\[
F(x) = (-1)^{n+1} (n-1)!n(x-2)(x-3)\ldots(x-n) \left[ x \sum_{i=2}^{n} \frac{1}{i} + 1 \right],
\]

\[
F(1) = 1!2!\ldots(n-1)!n! \sum_{i=1}^{n} \frac{1}{i}.
\]

Since \( \sum_{i=1}^{n} \frac{1}{i} = \sum_{i=1}^{n} (-1)^{i+1} (\binom{n}{i} \frac{1}{i} , \) this completes the proof.
Problem 42. [AMM, Vol. 52, p. 49-50, Prob. 4098]

Evaluate the determinant

\[
\begin{vmatrix}
  x_1 & x_2 & x_3 & \ldots & x_n \\
  2 & 2 & 2 & \ldots & 2 \\
  4 & 4 & 4 & \ldots & 4 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  2^{n-1} & 2^{n-1} & 2^{n-1} & \ldots & 2^{n-1} \\
  x_1 & x_2 & x_3 & \ldots & x_n 
\end{vmatrix}
\]

Solution: In 1841 Jacobi established the following factorization of the general simple alternate:

(1) \[ |x_1^{a_1}, x_2^{a_2}, x_3^{a_3}, \ldots, x_n^{a_n}| = \zeta^{\frac{1}{2}}(x_1 x_2 \ldots x_n) |H_{a_1-n+1}, H_{a_1-n+2}, \ldots, H_{a_1}|, \]

where \( \zeta^{\frac{1}{2}}(x_1 x_2 \ldots x_n) \) is Sylvester's notation for the continued product \((x_1 - x_2)(x_1 - x_3) \ldots (x_1 - x_n)(x_2 - x_3) \ldots (x_2 - x_n) \ldots (x_{n-1} - x_n)\)

and where \( H_j \) is the complete homogenous polynomial with unit coefficients of degree \( j \) in \( x_1, x_2, \ldots, x_n \).

Using this theorem, the determinant of the problem can be factored as

\[ |x_1^2, x_2^4, \ldots, x_n^{2^{n-1}}| = (x_1 x_2 \ldots x_n) \zeta^{\frac{1}{2}}(x_1 x_2 \ldots x_n) D, \]
where $D = \begin{vmatrix} H_{-n+1} & H_{-n+2} & \cdots & H_{-3} & H_{-2} & H_{-1} & H_{0} \\ H_{-n+2} & H_{-n+3} & \cdots & H_{-2} & H_{-1} & H_{0} & 1 \\ H_{-n+4} & H_{-n+5} & \cdots & H_{0} & H_{1} & H_{2} & H_{3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ H_{-n+2n-1} & H_{-n+2n-1+1} & \cdots & H_{2n-1} & H_{2n-2} & \cdots & H_{2n-2} \end{vmatrix}$.

Since $H_{-j} = 0$ and $H_{0} = 1$, $D$ reduces to

\[(2) \quad \begin{vmatrix} H_{-n+4} & H_{-n+5} & \cdots & H_{0} & H_{1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ H_{-n+2n-1} & H_{-n+2n-1+1} & \cdots & H_{2n-4} & H_{2n-3} \end{vmatrix}.
\]

Denoting the determinant in the problem by $\Delta_n$, and the determinant in (2) by $H$, we have $\Delta_n = -(x_1 x_2 \cdots x_n) \zeta^{\frac{1}{2}} (x_1 x_2 \cdots x_n) H$, where $H$ is a determinant of order $n-2$. It seems difficult to improve much upon this partial expansion of $\Delta_n$.

**Problem 43. [AMM, Vol. 52, p. 461-462, Prob. E660]**

Let $z_0, z_1, \ldots, z_k$ be $k+1$ different complex numbers, all contained in the circle $|z| \leq r$. Let
\[ B_{kp} = \begin{vmatrix} 1 & z_0 & 2 & \ldots & z_0 & k+1 & z_0 \\ 1 & z_1 & 2 & \ldots & z_1 & k+1 & z_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & z_k & 2 & \ldots & z_k & k+1 & z_k \end{vmatrix} \]

Prove \[ \left| \frac{B_{kp}}{B_{k0}} \right| \leq \left( \frac{k+p}{p} \right)^p. \]

**Solution I:** \[ B_{k0} \] is the well known Vandermonde determinant, whose value is
\[ \prod_{i > j} (z_i - z_j). \]

Expand \[ B_{kp} \] by elements of the last column to get
\[ B_{kp} = (-1)^k \left[ z_0^{k+p} \prod_{i > j} (z_1 - z_j) - z_1^{k+p} \prod_{i > j} (z_i - z_j) + \ldots \right]. \]

Thus
\[ \frac{B_{kp}}{B_{k0}} = \frac{z_0^{k+p}}{\prod_{i \neq 0} (z_i - z_0)} + \frac{z_1^{k+p}}{\prod_{i \neq 1} (z_i - z_1)} + \ldots + \frac{z_k^{k+p}}{\prod_{i \neq k} (z_i - z_k)}. \]

Let \[ f(z) = \prod_{i=0}^{k} (z - z_i). \] Then
\[
\frac{B_{kp}}{B_{k0}} = \sum_{i=0}^{k} \frac{z_i^{k+p}}{F'(z_i)} = \sum_{i=0}^{k} \left[ \text{residue at } z = z_i \text{ of } \frac{z^{k+p}}{F(z)} \right]
\]

\[
= \frac{1}{2\pi i} \int_{|z|=R>r} \frac{z^{k+p}}{f(z)} \, dz = \frac{1}{2\pi i} \int \frac{z^{p-1} \, dz}{\prod_{i=0}^{k} (1 - \frac{z_i}{z})}
\]

\[
= \frac{1}{2\pi i} \int z^{p-1} \left( 1 + \frac{z_0}{z} + \frac{z_1^2}{z^2} + \ldots \right) \, dz
\]

\[
= \sum \alpha_0 z_0^{\alpha_1} z_1^{\alpha_2} \ldots z_k^{\alpha_k},
\]

where the sum is taken over all sets of non-negative integers \(\alpha_i\) such that \(\sum \alpha_i = p\). The desired result now follows, since \(\binom{k+p}{p}\) is the coefficient of \(z^p\) in the expansion of \((1 - z)^{-(k+1)}\).

**Solution II:** It is well known that \(B_{kp} = B_{k0} H_p\), where \(H_p\) is the complete homogeneous polynomial of the \(p\)th degree in \(z_0, z_1, \ldots, z_k\). The number of terms in \(H_p\) is the number of ways of selecting \(p\) objects from \(k+1\) with repetitions allowed, which is \(\binom{k+p}{p}\). Hence

\[
\left| \frac{B_{kp}}{B_{k0}} \right| \leq \binom{k+p}{p} r^p.
\]
Problem 44. [AMM, Vol. 56, p. 692-693, Prob. E857]

Evaluate the \( s \times s \) determinant whose element in the 
\((i+1)^{\text{st}}\) row and \((j+1)^{\text{st}}\) column is \( d^{m+j}x^{n+1} \) / \(dx^{m+j}\).

**Solution:** We suppose that \( n \geq m \), since \( n < m \) makes the value of the determinant zero. Let \( \Delta(s) \) be the value of the \( s \times s \) determinant and let \( D \) represent \( d/dx \).

It is seen that a factor \( (n+i)/(n+i-m)! \) can be removed from the \((i+1)^{\text{st}}\) row. Let

\[
k = \frac{s-1}{\prod_{i=0}^{s-1} (n+i)/(n+i-m)!}.
\]

Then

\[
\Delta(s) = k \begin{vmatrix}
x^{n-m} & D x^{n-m} & D^2 x^{n-m} & \cdots \\
x^{n-m+1} & D x^{n-m+1} & D^2 x^{n-m+1} & \cdots \\
x^{n-m+2} & D x^{n-m+2} & D^2 x^{n-m+2} & \cdots \\
x^{n-m+s-1} & D x^{n-m+s-1} & D^2 x^{n-m+s-1} & \cdots \\
\end{vmatrix} = k \delta(s), \text{ say.}
\]

In \( \delta(s) \), multiply the \((s-1)^{\text{st}}\) row by \(-x\) and add it to the \(s^{\text{th}}\) row, multiply the \((s-2)^{\text{nd}}\) row by \(-x\) and add it to the \((s-1)^{\text{st}}\) row, etc. The result is
\[
\delta(s) = \begin{vmatrix}
\frac{x^{n-m}}{D_x^{n-m}} & \frac{D_x^{2}x^{n-m}}{D_x^{3}x^{n-m}} & \frac{D_x^{3}x^{n-m}}{D_x^{4}x^{n-m}} & \cdots \\
0 & \frac{x^{n-m}}{D_x^{n-m-1}} & \frac{D_x^{2}x^{n-m}}{D_x^{3}x^{n-m}} & \frac{D_x^{3}x^{n-m}}{D_x^{4}x^{n-m}} & \cdots \\
0 & \frac{x^{n-m+1}}{D_x^{n-m+1}} & \frac{D_x^{2}x^{n-m+1}}{D_x^{3}x^{n-m+1}} & \frac{D_x^{3}x^{n-m+1}}{D_x^{4}x^{n-m+1}} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \frac{x^{n-m+s-2}}{D_x^{n-m+s-2}} & \frac{D_x^{2}x^{n-m+s-2}}{D_x^{3}x^{n-m+s-2}} & \frac{D_x^{3}x^{n-m+s-2}}{D_x^{4}x^{n-m+s-2}} & \cdots & \vdots \\
0 & \frac{x^{n-m+s-2}}{D_x^{n-m+s-2}} & \frac{D_x^{2}x^{n-m+s-2}}{D_x^{3}x^{n-m+s-2}} & \frac{D_x^{3}x^{n-m+s-2}}{D_x^{4}x^{n-m+s-2}} & \cdots & \vdots \\
\end{vmatrix} = x^{n-m}(s-1)!
\]

We see that \( \delta(1) = x^{n-m} \), and hence,

\[
\delta(s) = x^{s(n-m)} \sum_{i=0}^{s-1} \frac{s-1}{i!}.
\]

Therefore

\[
\Delta(s) = x^{s(n-m)} \sum_{i=0}^{s-1} \frac{(n-1)!i!}{(n-m+i)!}.
\]


If \( k > 0 \), \( f_i(x) \) are independent, \( f_i^{(n)}(x) \) exist and are well-behaved, find

\[
\lim_{h \to 0} h^{-k} \begin{vmatrix}
f_1(x) & f_2(x) & \cdots & f_n(x) \\
f_1(x+h) & f_2(x+h) & \cdots & f_n(x+h) \\
\vdots & \vdots & \ddots & \vdots \\
f_1(x+2h) & f_2(x+2h) & \cdots & f_n(x+2h) \\
\vdots & \vdots & \ddots & \vdots \\
f_1(x+(n-1)h) & f_2(x+(n-1)h) & \cdots & f_n(x+(n-1)h) \\
\end{vmatrix}.
\]
Solution: By adding to each row multiples of the rows above it, as suggested by the difference operator

\[
\Delta^m f(x) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} f[x + (m-j)h],
\]

we may write the given expression in the form

\[
\lim_{h \to 0} h^{(n-1)n/2-k} \begin{vmatrix}
  f_1(x) & f_2(x) & \cdots & f_n(x) \\
  \frac{\Delta f_1(x)}{h} & \frac{\Delta f_2(x)}{h} & \cdots & \frac{\Delta f_n(x)}{h} \\
  \frac{\Delta^2 f_1(x)}{2h} & \frac{\Delta^2 f_2(x)}{2h} & \cdots & \frac{\Delta^2 f_n(x)}{2h} \\
  \cdots & \cdots & \cdots & \cdots \\
  \frac{\Delta^{n-1} f_1(x)}{h^{n-1}} & \frac{\Delta^{n-1} f_2(x)}{h^{n-1}} & \cdots & \frac{\Delta^{n-1} f_n(x)}{h^{n-1}}
\end{vmatrix}
\]

The limit of the determinant is the well known Wronskian, which, according to the conditions under consideration, is finite and different from zero. It follows that the limit of the expression is the Wronskian for \( k = n(n-1)/2 \), equals zero for \( k < n(n-1)/2 \), and does not exist for \( k > n(n-1)/2 \).

An allied problem is that of showing that the determinant
This may be done by showing that \( \frac{dH}{dx} = 0 \). The value of the determinant, found by taking \( x = 0 \), is 
\[
1!2!3! \ldots \; n! \; h^{n(n-1)/2}.
\]

Problem 46. [AMM, Vol. 63, p. 588-589, Prob. 4645]

Let \( 0 \leq r \leq p-1 \), where \( p \) is prime, and let \( a_0, a_1, \ldots, a_r \) be integers not divisible by \( p \). Define the determinant of order \( r+1 \),
\[
\Delta_r = \begin{vmatrix} a_i^j \end{vmatrix}_{i,j = 0, 1, \ldots, r}.
\]

Show that 
\[
\Delta_r \equiv 0 \pmod{p^{r(r+1)(r+2)/6}}.
\]

Solution: We shall prove the following result. Let \( m \geq 0 \) and define
\[
\Delta_r^{(m)} = \begin{vmatrix} a_i^{p^{m+j}} \end{vmatrix}_{i,j = 0, 1, \ldots, r}.
\]

Then we show that 
\[
(1) \quad \Delta_r^{(m)} \equiv 0 \pmod{p^e}, \quad \text{where} \; e = m(r+1)/2 + r(r+1)(r+2)/6.
\]
Define \( \Delta p^n = \frac{a_p^{n+1} - a_p^n}{p^{n+1}} \), \( \Delta a_p^n = \frac{a a_p^{n+1} - a a_p^n}{p^{n+1}} \), and generally,

\[ \Delta s+1 a_p^n = \frac{\Delta s a_p^{n+1} - \Delta s a_p^n}{p^{n+1}}. \]

In \( \Delta_r^{(m)} \) subtract the \( j \)th column from the \( (j+1) \)th for \( j = 0, 1, \ldots, r-1 \). Then

\[ \Delta_r^{(m)} = p \begin{vmatrix} m \cr \Delta a_i^m \cr \Delta a_i^m \end{vmatrix} \left| \begin{array}{cccc} a_p^m & \Delta a_p^m & \Delta a_i^m & \Delta a_i^m \end{array} \right|, \]

\[ (i = 0, 1, \ldots, r). \]

Again subtracting the \( j \)th column from the \( (j+1) \)th, for \( j = 1, 2, \ldots, r-1 \), we get

\[ \Delta_r^{(m)} = p^c \begin{vmatrix} m \cr \Delta a_i^m \cr \Delta a_i^m \end{vmatrix} \left| \begin{array}{cccc} a_p^m & \Delta a_p^m & \Delta a_i^m & \Delta a_i^m \end{array} \right|, \]

where \( c = m(2r-1) + r(r+1)/2 + r(r-1)/2 \). Continuing in this way we finally get

\[ (2) \Delta_r^{(m)} = p^e \begin{vmatrix} m \cr \Delta a_i^m \end{vmatrix} \left| \begin{array}{c} \Delta a_i^m \end{array} \right|, \]

\[ (i, j = 0, 1, \ldots, r), \]

where \( e = m \{ r + (r-1) + \ldots + 1 \} + \{ r(r+1)/2 + (r-1)r/2 + \ldots + 1 \} \)

\[ = mr(r+1)/2 + r(r+1)(r+2)/6. \]

Now Schur (Sitzungsberichte der Preussischen Akademie der
Wissenschaften, 1933, p. 145-151) has proved that for all \((a, p) = 1, r < p\), the numbers \(\Delta^r a^p\) are integral for all \(n > 0\). Consequently (2) implies (1).

Problem 47. [AMM, Vol. 69, p. 929-930, Prob. 4979]

If \(z_1, z_2, z_3\) are distinct numbers of modulus 1, and

\[
\begin{vmatrix}
1 & 1 & 1 \\
z_1^m & z_2^m & z_3^m \\
z_1^n & z_2^n & z_3^n
\end{vmatrix} = 0,
\]

then either two rows or two columns of the determinant are identical.

Solution: The example \(m = 1, n = 3; z_1 = -1, z_2 = \omega, z_3 = -\omega^2, \omega^2 + \omega + 1 = 0\), shows that the conclusion is not correct as stated. It is correct however, if identical is replaced by proportional.

The vanishing of the determinant implies the existence of three numbers \(\alpha, \beta, \gamma\) not all zero such that

\[
\begin{cases}
\alpha + \beta + \gamma = 0 \\
\alpha z_1^m + \beta z_2^m + \gamma z_3^m = 0 \\
\alpha z_1^n + \beta z_2^n + \gamma z_3^n = 0
\end{cases}
\]
If \( a = 0 \), then \( \beta = -\gamma \), and the second and third columns are identical. Similarly if \( \beta = 0 \) or \( \gamma = 0 \). We may accordingly suppose that \( \alpha = -1, \beta + \gamma = 1, \beta \gamma \neq 0 \). Then (1) reduces to

\[
(2) \quad z_1^m = \beta z_2^m + (1-\beta)z_3^m, \quad z_1^n = \beta z_2^n + (1-\beta)z_3^n,
\]

so that

\[
(3) \quad z_1^m - z_3^m = \beta (z_2^m - z_3^m), \quad z_1^n - z_3^n = \beta (z_2^n - z_3^n).
\]

Remembering that \( |z_1| = 1 \) and taking conjugates, we get

\[
(4) \quad z_2^m (z_1^m - z_3^m) = \bar{\beta} z_1^m (z_2^m - z_3^m),
\]

\[
\bar{z}_2^n (z_1^n - z_3^n) = \bar{\beta} z_1^n (z_2^n - z_3^n).
\]

If \( z_2^m - z_3^m = 0 \), then \( z_1^m - z_3^m = 0 \), and the second row is proportional to the first. Similarly if \( z_2^n - z_3^n = 0 \). Thus comparison of (3) and (4) yields \( z_2^m \beta = \bar{\beta} z_1^m \), \( z_2^n \beta = \bar{\beta} z_1^n \), so that

\[ z_1^{m-n} = z_2^{m-n}. \]

In exactly the same way, if we had solved (2) for \( 1-\beta \), we would have obtained \( z_1^{m-n} = z_3^{m-n} \). Hence if we put

\[ \lambda = z_1^{m-n} = z_2^{m-n} = z_3^{m-n}, \]

it is evident that the second and third rows are proportional.

**Problem 48. [AMM, Vol. 70, p. 448, Prob. 5015]**

Suppose \( f(x) \) is continuous on \([a, b]\) and \( n \) times
differentiable on \((a, b)\). Suppose \(a = x_0 < x_1 < \ldots < x_n = b\).

Show that

\[
\begin{vmatrix}
1 & 1 & \ldots & 1 \\
x_0 & x_1 & \ldots & x_n \\
\ldots & \ldots & \ldots & \ldots \\
x_0^{n-1} & x_1^{n-1} & \ldots & x_n^{n-1} \\
f(x_0) & f(x_1) & \ldots & f(x_n)
\end{vmatrix}
= \frac{1}{n!} f^{(n)}(\xi) \prod_{i>j} (x_i - x_j)
\]

where \(a < \xi < b\).

Solution: Let \(A\) be the given determinant and consider

\[
F(x) =
\begin{vmatrix}
1 & 1 & 1 & \ldots & 1 \\
x & x_0 & x_1 & \ldots & x_n \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
x^{n-1} & x_0^{n-1} & x_1^{n-1} & \ldots & x_n^{n-1} \\
f(x) & f(x_0) & f(x_1) & \ldots & f(x_n)
\end{vmatrix}
\]

It is clear that \(F(x_0) = F(x_1) = \ldots = F(x_n) = 0\). Hence by repeated application of Rolle's Theorem, \(F^{(n)}(\xi) = 0\) for \(a < \xi < b\), where \(F^{(n)}(\xi)\) is the determinant (1) with its first column replaced by \(0, 0, \ldots, 0, n!\), \(f^{(n)}(\xi)\). By expansion,

\[
F^{(n)}(\xi) = (-1)^n n! A - f^{(n)}(\xi) \cdot V,
\]

where \(V\) is the familiar Vandermonde determinant whose value is \(\prod_{i>j} (x_i - x_j)\).
The desired result follows immediately.

Problem 49. [AMM, Vol. 72, p. 671, Prob. E1707]

Prove that the \( n \)th order determinant \( |A_{rs}(x)| \), where
\[
A_{rs}(x) = x^{(r-1)(r-s)},
\]
has value
\[
|A_{rs}(x)| = \prod_{j=1}^{n-1} (1 - x^j)^{n-j}.
\]

Solution: The assertion is clear for \( x = 0 \). For \( x \neq 0 \), extraction of the common factor \( x^{(1-r)(n-r)} \) from the \( r \)th row gives
\[
|A_{rs}(x)| = x^{-\sigma} V_n(x^{n-1}, x^{n-2}, \ldots, x, 1) \quad \text{with}
\]
\[
\sigma = \sum_{r=1}^{n} (r-1)(n-r) \quad \text{and} \quad V_n(x_1, x_2, \ldots, x_n)
\]
is the Vandermonde determinant \( |x_{i-1}^{j-1}| \) of order \( n \). Thus
\[
V_n(x^{n-1}, x^{n-2}, \ldots, x, 1) = \prod_{s>t} (x^{n-s} - x^{-t}) = \prod_{s>t} x^{n-s}(1-x^{s-t})
\]
\[
= x^{p \sum_{t=1}^{n-1} n-t} \prod_{j-1}^{n} (1-x^j) = x^{p \sum_{j=1}^{n-1} n-j} \prod_{j=1}^{n} (1-x^j)^{n-j},
\]
where \( p = \sum_{t=1}^{n} \sum_{s=t+1}^{n} (n-s) = \sum_{s=1}^{n} \sum_{t=1}^{s-1} (n-s) = \sigma \), and the result follows.

**Problem 50.** [AMM, Vol. 73, p. 415, Prob. E1761]

Let \( A \) be an \( n \times n \) matrix with \( a_{ij} = j^{i-1} \). Prove that 
\[
\det A = (n-1)(n-2)^{2} \cdots 2^{n-2}.
\]

**Solution:** This is the Vandermonde determinant

\[
V(x_1, x_2, \ldots, x_n) \text{ with } x_j = j \text{ and, hence, has the value}
\]

\[
\prod_{i \geq j} (i-j) = \prod_{k=2}^{n-1} k^{n-k}.
\]

**Problem 51.** [DMVJ, Vol. 33, Pt. 2, p. 28, Prob. 9]

Let \( \Delta_{2p+1} \) and \( \Delta_1 \) be defined as follows:

\[
\Delta_{2p+1} = \begin{vmatrix}
1 & 2 & \cdots & n \\
1 & 2^3 & \cdots & n^3 \\
\cdots & \cdots & \cdots & \cdots \\
1^{2p-1} & 2^{2p-1} & \cdots & n^{2p-1} \\
1^{2p+3} & 2^{2p+3} & \cdots & n^{2p+3} \\
\cdots & \cdots & \cdots & \cdots \\
1^{2n+1} & 2^{2n+1} & \cdots & n^{2n+1}
\end{vmatrix}
\]

\[
\Delta_1 = \begin{vmatrix}
1^3 & 2^3 & 3^3 & \cdots & n^3 \\
1^5 & 2^5 & 3^5 & \cdots & n^5 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
1^{2n+1} & \cdots & \cdots & 2n+1
\end{vmatrix}.
\]
where \( n \) is a positive integer and \( p \) is a positive integer \( \leq n \).

Show that

\[
\lim_{n \to \infty} \frac{\Delta_{2p+1}}{\Delta_1} = \frac{\Pi^{2p}}{(2p+1)!}.
\]

Solution: Let \( x^\lambda = \lambda^2, \ (\lambda = 1, 2, \ldots, n) \). Then \( \Delta_1 = n!3V \)

where

\[
V = \begin{vmatrix}
1 & 1 & \cdots & 1 \\
x_1 & x_2 & \cdots & x_n \\
x_1 & x_2 & \cdots & x_n \\
\vdots & \vdots & \ddots & \vdots \\
x^1_1 & x^1_2 & \cdots & x^1_n \\
\end{vmatrix}.
\]

Moreover,

\[
\Delta_{2p+1} = n!(-1)^{p-1}.
\]

\[
\begin{vmatrix}
x_1 & x_2 & \cdots & x_n \\
2 & 2 & \cdots & 2 \\
x_1 & x_2 & \cdots & x_n \\
\vdots & \vdots & \ddots & \vdots \\
x^{p-1}_1 & x^{p-1}_2 & \cdots & x^{p-1}_n \\
1 & 1 & \cdots & 1 \\
x^{p+1}_1 & x^{p+1}_2 & \cdots & x^{p+1}_n \\
\vdots & \vdots & \ddots & \vdots \\
x^n_1 & x^n_2 & \cdots & x^n_n \\
\end{vmatrix}.
\]
Now let \( x_1, x_2, \ldots, x_n \) be the roots of the equation

\[
Q_n(z) = 1 - f_{1,n} z + f_{2,n} z^2 - f_{3,n} z^3 + \cdots + (-1)^n f_{n,n} z^n = 0
\]

\( z = x_1, x_2, \ldots, x_n \). In this latter determinant we can replace the elements of the \( p \)th row by the expressions

\[
f_{1,n} x_v - f_{2,n} x_v^2 + f_{3,n} x_v^3 - \cdots + (-1)^n f_{n,n} x_v^n (v = 1, 2, \ldots, n).
\]

Multiply the first row by \(-f_{1,n}\), the second row by \(f_{2,n}\), etc., and add all these products to the \( p \)th row. We then have

\[
\Delta_{2p+1} = n!^3 f_{p,n} V,
\]

where \( V \) is the Vandermonde determinant defined above. Consequently,

\[
\frac{\Delta_{2p+1}}{\Delta_1} = \frac{n!^3 f_{p,n} V}{n!^3 V} = f_{p,n}.
\]

Now \( Q_n(z) = \prod_{v=1}^{n} \left(1 - \frac{z}{x_v}\right) = \sum_{\lambda=1}^{n} (-1)^\lambda f_{\lambda,n} z^\lambda \). But since

\[
\sum_{v=1}^{\infty} \frac{1}{x_v}
\]

is absolutely convergent, we see, by application of the Weierstrass Product Theorem, that \( Q_n(z) \) converges uniformly toward

\[
Q(z) = \prod_{v=1}^{\infty} \left(1 - \frac{z}{x_v}\right) = \frac{\sin \pi \sqrt{z}}{\pi \sqrt{z}}
\]

in the interior of each circle \(|z| = R\). Thus \( Q(z) \) itself represents an analytic function,
\[ Q(z) = 1 - \frac{2}{3!} z + \frac{4}{5!} z^2 - \ldots + \frac{(-1)^\lambda \Pi \lambda^2}{(2\lambda+1)!} z^\lambda = \sum_{\lambda=1}^{\infty} a_\lambda z^\lambda. \]

Consequently the series \( Q_1(z) + (Q_2(z) - Q_1(z)) + \ldots + (Q_n(z) - Q_{n-1}(z)) + \ldots \)
converges uniformly toward

\[ \sum_{\lambda=1}^{\infty} a_\lambda z^\lambda \]

for each \( z, |z| < R. \)

According to the Weierstrass double sequence theorem

\[ (-1)^P \frac{a_p}{p} = f_{p,1} + \sum_{v=2}^{\infty} (f_{p,v} - f_{p,v-1}), \] that is, \( \frac{\Pi 2p}{(2p+1)!} = \lim_{n \to \infty} f_{p,n}. \)

Thus, \( \frac{\Delta_{2p+1}}{\Delta_1} = \frac{\Pi 2p}{(2p+1)!} \), which was to be proved.

**Problem 52. [MM, Vol. 23, p. 208-209, Prob. 39]**

Let \( A_{ij} = [(i+1)^{j+1} - 1]/(i+1)!; \ i, j = 1, 2, \ldots, n. \) Show that the \( n^{th} \) order determinant \( |A_{ij}| = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n+1}. \)

Solution: If we take common factors from the rows and perform the operations \( \text{col} \ k - \text{col} \ (k-1), \ k = 2, 3, \ldots, n, \) we have
The last two determinants are alternants [Muir and Metzler, A Treatise on the Theory of Determinants, Longmans, Green and Co. (1933), p. 321-363]. The first is expressible as a difference product which is obviously \(2!3\ldots(n-1)!\); the second is the same difference product multiplied by a symmetric function of degree \(n-1\).

On reducing, we have

\[
\begin{vmatrix} |A_{ij}| \end{vmatrix} = 1 + [3 \cdot 4 \cdot 5 \ldots (n-1) + 2 \cdot 4 \cdot 5 \ldots (n-1) + \ldots + 2 \cdot 3 \cdot 4 \ldots n]/(n+1)! = \sum_{k=1}^{n+1} \frac{1}{k}.
\]
Problem 53.  [SSM, Vol. 38, p. 709-710, Prob. 1545]

For any integer \( a \), show that

\[
f(a, x) = 1!2! \ldots n!x^{\frac{n(n+1)}{2}}
\]

where \( f(a, x) = \)

\[
\begin{vmatrix}
1 & 1 & \ldots & 1 \\
a & a+x & \ldots & a+nx \\
a^2 & (a+x)^2 & \ldots & (a+nx)^2 \\
\vdots & \vdots & \ddots & \vdots \\
a^n & (a+x)^n & \ldots & (a+nx)^n
\end{vmatrix}
\]

\[
= a \begin{vmatrix}
1 & 1 & \ldots & 1 \\
x & 2x & \ldots & nx \\
x(a+x) & 2x(a+2x) & \ldots & nx(a+nx) \\
\vdots & \vdots & \ddots & \vdots \\
x(a+x)^{n-1} & 2x(a+2x)^{n-1} & \ldots & nx(a+nx)^{n-1}
\end{vmatrix}
\]

Solution: Let \( f(a, x, n) \) be the symbol for the given determinant. Beginning at the bottom row and working upward to the second row subtract, from each element, \( a \) times the element which stands above it. This changes the appearance but not the value.

Consequently \( f(a, x, n) \) is equal to

\[
\begin{vmatrix}
1 & 1 & \ldots & 1 \\
0 & x & 2x & \ldots & nx \\
0 & x(a+x) & 2x(a+2x) & \ldots & nx(a+nx) \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
0 & x(a+x)^{n-1} & 2x(a+2x)^{n-1} & \ldots & nx(a+nx)^{n-1}
\end{vmatrix}
\]

Inspection of this shows that

(1) \( f(a, x, n) = x \cdot 2x \cdot 3x \ldots nx f(a+x, x, n-1) \)

\[
= x^n n! f(a+x, x, n-1).
\]
Similarly,

\( f(a+x, x, n-1) = x^{n-1} (n-1)! f(a+2x, x, n-2); \)

(2) \( f(a+nx-2x, x, 2) = x^2 2! f(a+nx-x, x, 1). \)

From the definition we have by expansion

\[
\begin{vmatrix}
1 & 1 \\
a + nx - x & a + nx
\end{vmatrix} = x(1!) .
\]

The required result follows from the multiplication of these \( n \) equations: \( f(a, x, n) = x^{n(n+1)/2} (1!)(2!) \ldots (n!) \).

**Problem 54.** [SSM, Vol. 40, p. 585, Prob. 1655]

\[
\begin{vmatrix}
a & a & a^4 - 1 \\
a & b & b^4 - 1 \\
c & c & c^4 - 1
\end{vmatrix}
\]

If \( a, b, c \) are different and if \( b b^3 b^4 - 1 = 0, \)

show that \( abc(ab + bc + ca) = a + b + c. \)

**Solution:**

\[
\begin{vmatrix}
a & a & a - 1 \\
b & b & b^4 - 1 \\
c & c & c^4 - 1
\end{vmatrix}
\]

\[
\begin{vmatrix}
a & a & a^4 \\
 b & b & b^4 \\
c & c & c^4
\end{vmatrix}
\]

\[
\begin{vmatrix}
a & a & a^4 \\
 b & b & b^4 \\
c & c & c^4
\end{vmatrix}
\]

\[
\begin{vmatrix}
a & a & a - 1 \\
 b & b & b^4 - 1 \\
c & c & c^4 - 1
\end{vmatrix}
\]

or \( b b^3 b^4 = b^3 1 \),
Factoring \((a-b)\) and \((b-c)\) from the first and second row respectively from each determinant, and reducing, we get:

\[
\begin{vmatrix}
1 & a^2 & a^3 \\
b & b^2 & b^3 \\
c & c^2 & c^3 \\
\end{vmatrix}
= 
\begin{vmatrix}
a & a & 1 \\
b & b & 1 \\
c & c & 1 \\
\end{vmatrix}
\]

or

\[
\begin{vmatrix}
0 & a^2 - b^2 & a^3 - b^3 \\
b^2 - c^2 & b^3 - c^3 \\
c^2 & c^3 \\
\end{vmatrix}
= 
\begin{vmatrix}
a - b & a^3 - b^3 & 0 \\
b - c & b^3 - c^3 & 0 \\
c & c^3 & 1 \\
\end{vmatrix}
\]

Expanding, collecting terms, and factoring, we obtain:

\[
abc \left[ b(c+a) + ac \right] (c-a) = (b + c + a)(c-a)
\]

or

\[
abc (ab + bc + ca) = a + b + c.
\]

Symmetric Determinants

**Problem 55.** [AMM, Vol. 15, p. 144, Prob. 153 (Num. Theory)]

Let \((k, \ell)\) represent the greatest common divisor of \(k\) and \(\ell\), and let \(\phi(k)\) represent the number of integers prime to \(k\) and not greater than \(k\). Show that
Solution: This problem was reproposed [AMM, Vol. 18, p. 45]. No solution to this problem has been published. The result is not difficult to establish, however. From Problem 62 (p. 106), with $\lambda=1$ we have

\[
\begin{vmatrix}
(1, 1) & (1, 2) & (1, 3) & \ldots & (1, n) \\
(2, 1) & (2, 2) & (2, 3) & \ldots & (2, n) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(n, 1) & (n, 2) & (n, 3) & \ldots & (n, n)
\end{vmatrix} = \phi(1) \phi(2) \ldots \phi(n).
\]

It is also known that $\phi(k) = k(1- \frac{1}{p_1})(1- \frac{1}{p_2}) \ldots (1- \frac{1}{p_j})$, where the prime decomposition of $k$ is

\[
\prod p_1^{a_1} p_2^{a_2} \ldots p_j^{a_j}.
\]

Considering $\phi(k)$ for arbitrary integer $k$, we note that $1 - \frac{1}{p_i}$ will be a factor of $\phi(k)$ if $p_i$ is a factor of $k$. Hence, in the product $\phi(1)\phi(2) \ldots \phi(n)$, $1 - \frac{1}{p_i}$ will be a factor once for each $k \leq n$ which is divisible by $p_i$. In other words, $1 - \frac{1}{p_i}$ will be a factor of the product $\left[ \frac{n}{p_i} \right]$ times. We can now conclude that

\[
\phi(1) \phi(2) \ldots \phi(n) = n!(1- \frac{1}{2})(1- \frac{1}{3}) \ldots (1- \frac{1}{5}) \ldots,
\]
from which the desired result is obvious.

Problem 56. [AMM, Vol. 16, p. 11-12, Prob. 303 (Alg)]

Evaluate the determinant

$$\Delta = \begin{vmatrix}
D_1 & x_1x_2 & x_1x_3 & \ldots & x_1x_n \\
x_1x_2 & D_2 & x_2x_3 & \ldots & x_2x_n \\
x_1x_3 & x_2x_3 & D_3 & \ldots & x_3x_n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x_1x_n & x_2x_n & x_3x_n & \ldots & D_n
\end{vmatrix}$$

Solution:

$$\Delta = x_1x_2\ldots x_n\begin{vmatrix}
\frac{D_1}{x_1} & x_2 & x_3 & \ldots & x_n \\
x_1 & \frac{D_2}{x_2} & x_3 & \ldots & x_n \\
x_1 & x_2 & \frac{D_3}{x_3} & \ldots & x_n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x_1 & x_2 & x_3 & \ldots & \frac{D_n}{x_n}
\end{vmatrix}$$

$$= \prod_{i=1}^{n} x_i \begin{vmatrix}
(D_1-x_1^2)/x_1 & 0 & 0 & \ldots & (x_n^2-D_n)/x_n \\
0 & (D_2-x_2^2)/x_2 & 0 & \ldots & (x_n^2-D_n)/x_n \\
0 & 0 & (D_3-x_3^2)/x_3 & \ldots & (x_n^2-D_n)/x_n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x_1 & x_2 & x_3 & \ldots & D_n/x_n
\end{vmatrix}$$
by subtracting the last row from each row;

\[
= \prod_{r=1}^{n} \frac{x_r}{r} \prod_{r=1}^{n} \frac{D_r - x_r^2}{x_r}
\]

\[
\begin{vmatrix}
1 & 0 & 0 & \ldots & -1 \\
0 & 1 & 0 & \ldots & -1 \\
0 & 0 & 1 & \ldots & -1 \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
x_1^2 & x_2^2 & x_3^2 & \ldots & D_n \\
D_1 - x_1^2 & D_2 - x_2^2 & D_3 - x_3^2 & \ldots & \frac{D_n}{x_n}
\end{vmatrix}
\]

by adding the last column to the first. Since the first row contains \(n-1\) zeros, this reduces the determinant to one of the \((n-1)^{st}\) order. Adding the last column to the first, we reduce to one of the \((n-2)^{nd}\) order. Repeat the process \(n-1\) times in all. Then,
\[
\Delta = \prod_{r=1}^{n} (D_r - x_r^2) \left[ \frac{D_n}{D_n - x_r^2} + \frac{x_1^2}{D_n - x_1^2} + \frac{x_2^2}{D_n - x_2^2} + \ldots + \frac{x_{n-1}^2}{D_n - x_{n-1}^2} \right]
\]

\[
= \prod_{r=1}^{n} (D_r - x_r^2) \left[ \frac{x_1^2}{D_1 - x_1^2} + \frac{x_2^2}{D_2 - x_2^2} + \ldots + \frac{x_{n-1}^2}{D_{n-1} - x_{n-1}^2} + \frac{x_n^2}{D_n - x_n^2} + 1 \right]
\]

Problem 57. [AMM, Vol. 32, p. 142-143, Prob. 3086]

Prove that the \( n \)th order determinant

\[
\begin{vmatrix}
  x_1^2 + \lambda & x_1 x_2 & x_1 x_3 & \ldots & x_1 x_n \\
  x_1 x_2 & x_2^2 + \lambda & x_2 x_3 & \ldots & x_2 x_n \\
  x_1 x_3 & x_2 x_3 & x_3^2 + \lambda & \ldots & x_3 x_n \\
  \ldots & \ldots & \ldots & \ldots & \ldots \\
  x_1 x_n & x_2 x_n & x_3 x_n & \ldots & x_n^2 + \lambda
\end{vmatrix}
\]

is divisible by \( \lambda^{n-1} \) and find the other factor.

Solution: Let \( A \) denote the matrix whose constituents are \( (A)_{ij} = x_i x_j \) \((i, j = 1, 2, \ldots, n)\). Expanding the given determinant in powers of \( \lambda \), we have \( |A + \lambda| = \lambda^n + \psi_1 \lambda^{n-1} + \ldots + \psi_{n-1} \lambda + \psi_n \),
where the \( \psi \)'s are functions of the \( x \)'s.

If \( S_1, S_2, \ldots, S_n \) denote the sums of the powers of the roots of the equation \( |A-\lambda| = 0 \), then

\[
S_1 = \sum_{i=1}^{n} (A)^i_{ii} = \sum_{i=1}^{n} x_i^2,
\]
\[
S_2 = \sum_{i=1}^{n} (A^2)^i_{ii} = \sum_{i,j=1}^{n} (A)^i_{ij}(A)^j_{ji} = \sum_{i,j=1}^{n} x_i x_j = S_1^2,
\]
\[
S_3 = \sum_{i=1}^{n} (A^3)^i_{ii} = \sum_{i,j=1}^{n} (A)^i_{ij}(A^2)^j_{ji} = \sum_{i,j,h=1}^{n} x_i x_j x_h = S_1^3,
\]
and similarly in general for \( k = 1, 2, \ldots, n \),

\[
S_k = \sum_{i=1}^{n} (A^k)^i_{ii} = \sum_{i,j=1}^{n} (A)^i_{ij}(A^{k-1})^j_{ji} = \cdots \sum_{h_1, h_2, \ldots, h_k=1}^{n} x_{h_1} x_{h_2} \cdots x_{h_k} = S_1^k.
\]

Substituting these values in Newton's formulae connecting the sums of the powers of the roots with the coefficients, we see that

\[ \psi_1 = S_1, \quad \psi_2 = \psi_3 = \cdots \psi_n = 0. \]

\[ \therefore \quad |A+\lambda| = \lambda^n + \psi_1 \lambda^{n-1} = \lambda^{n-1}(\lambda + S_1) = \lambda^{n-1}(\lambda + \sum_{x=1}^{n} x_i^2). \]
Problem 58. [AMM, Vol. 38, p. 355, Prob. 3468]

Evaluate the \( n \)th order determinant whose elements are given by 

\[
a_{11} = a_{1j} = a_{il} = 1; \quad a_{ij} = a_{i-1,j} + a_{i,j-1}, \quad 1 < i, j \leq n.
\]

Solution: Denote by \( D_n \) the determinant in question. Subtract each row of \( D_n \) from the row following it, beginning the process with the last pair of rows. After \( n-1 \) subtractions the element \( a_{ij} \) is replaced by \( a_{i,j-1} \) and all the elements in the first column, except \( a_{11} = 1 \), become zeros. Now subtract each column from the one following it, beginning with the last pair. After \( n-1 \) subtractions the element \( a_{i,j-1} \) is replaced by \( a_{i-1,j-1} \). The result of the two operations is to replace \( a_{ij} \) by \( a_{i-1,j-1} \) and to reduce each element in the first row and in the first column, except \( a_{11} = 1 \), to zero. Hence \( D_n = D_{n-1} \), and consequently

\[
D_n = D_{n-1} = D_{n-2} = \ldots = D_2 = D_1 = 1.
\]

This same problem reappears as Prob. 3873 [AMM, Vol. 47, p. 325], and occurs in two slightly different forms in this paper.

[See Problem 66, p. 112 and Problem 67, p. 113.]
Problem 59. [AMM, Vol. 42, p. 400-401, Prob. 3667]

Show that $(-1)^{n-1}(n-1)2^{n-2}$ is the value of the $n$-rowed determinant for which $a_{ij} = |i-j|$.

**Solution:** The matrix of the given set of elements is a particular case of the matrix $A$ for which $a_{ij} = a + |i-j|h$. The value of this determinant is easily determined by subtracting the elements of the $(j+1)^{th}$ column from the corresponding elements of the $j^{th}$ column, leaving the $n^{th}$ column unchanged. Repeat the process on the new determinant, leaving the last two columns unchanged. After adding the last row to the first, we find that

$$|A| = (-1)^{n-1}2^{n-2}h^{n-1}[2a + (n-1)h].$$

Since, however, $A$ is symmetric, the following proof may be of greater interest; for it reduces the quadratic form associated with $A$ to a very simple form. Let $B$ denote the matrix with the elements

$$b_{nn} = \frac{1}{2}, \quad b_{ii} = 1, \quad b_{i,i+1} = -1, \quad i < n;$$

and with zeros for all other elements. Let $B'$ denote the transposed matrix of $B$, i.e., the rows of $B'$ are the columns of $B$. Then a simple calculation shows that $BAB' = C$, where $C$ is a diagonal matrix, i.e., all of its elements are zeros except
those in the principal diagonal, and these in order are \(-2h, \ldots, -2h, a + \frac{1}{2}(n-1)h\). Since the determinants of B and B' are each unity, we obtain again result (1). This shows also that the quadratic form having the matrix A is reduced by a unimodular transformation to the form

\[-2h \left[ x_1^2 + x_2^2 + \ldots + x_{n-1}^2 \right] + \left[ a + \frac{1}{2}(n-1)h \right] x_n^2.\]

Problem 60. [AMM, Vol. 43, p. 52-56, Prob. 3687]

If \(S(i, j)\) denotes the sum of the divisors common to both i and j, show that

\[
\begin{vmatrix}
S(1, 1) & S(1, 2) & S(1, 3) & \ldots & S(1, n) \\
S(2, 1) & S(2, 2) & S(2, 3) & \ldots & S(2, n) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
S(n, 1) & S(n, 2) & S(n, 3) & \ldots & S(n, n)
\end{vmatrix} = n! .
\]

Solution I: Lemma: Let \( n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t} \), then

\[
S(r, n) = \sum_{k=1}^{t} S\left( r, \frac{n}{p_k} \right) + \sum_{k, \ell = 1}^{t} S\left( r, \frac{n}{p_k p_\ell} \right) - \sum_{k, \ell, m = 1}^{t} S\left( r, \frac{n}{p_k p_\ell p_m} \right) + \ldots = 0 \text{ when } r < n,
\]

\[
= n \text{ when } r = n.
\]

(1)
The proof of the Lemma follows from the following theorem

(See P. Bachmann, *Niedere Zahlentheorie*, 1st Part, p. 96):

If two functions \( f(n) \), \( \psi(n) \) exist such that for every integer

\[
n = p_1^{a_1} p_2^{a_2} p_3^{a_3} \ldots, \quad \text{and all its divisors } \, 1, d, d', \ldots, n,
\]

then the function \( \psi(n) \) can be expressed by

\[
(2) \quad f(n) = \psi(1) + \psi(d) + \psi(d') + \ldots + \psi(n),
\]

then the function \( \psi(n) \) can be expressed by

\[
(3) \quad \psi(n) = f(n) - \sum f\left(\frac{n}{p_1}\right) + \sum f\left(\frac{n}{p_1p_2}\right) - \sum f\left(\frac{n}{p_1p_2p_3}\right) + \ldots,
\]

where the first summation extends over all different prime factors

of \( n \), the second over all combinations of two, and so on.

Setting \( f(n) = S(r, n) \), \( r \) being fixed, and

\[
\psi(k) = \begin{cases} 
  k, & \text{if } k \text{ divides } r, \\
  0, & \text{if } k \text{ does not divide } r,
\end{cases}
\]

proves the Lemma.

Proof of the Theorem: From the last column of the determinant subtract each of the columns numbered \( n/p_k \) where \( p_k \) runs through all different primes dividing \( n \), then add all columns numbered \( n/p_k p_{k'} \), subtract all numbered \( n/p_k p_{k'} p_m \), etc.

By the above Lemma the elements in the last column will then be

\( 0, 0, 0, \ldots, n \).
Expanding the determinant by means of the elements of the last column we obtain a determinant of order $n-1$ of the form of the given determinant multiplied by $n$.

By induction we may then conclude that the original determinant has the value $n!$.

**Solution II:** We observe that:

(a) The determinant is symmetric.

(b) Since the common factors of $k$ and $mk + h$, $h < k$, are the same as those for $k$ and $h$, the first $k$ elements in the $k$th row (or column) keep repeating.

The first column consists of 1's. Subtract this column from each of the others, then the first row from each of the others. The resulting determinant is still symmetric. The leading element is 1, the aligned elements 0. Each element $(i, j)$ is now the sum of all factors of $i$ and $j$, excluding 1. The principal minor of order 2 is now

$$\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix},$$

the second row of which keeps repeating horizontally and the second column vertically.

We next subtract the second column from all others of order $2k$. Now all the elements not in column 2 contain all the common
factors of \( i \) and \( j \), except 1 and 2. Now subtract the second row from all others of order \( 2k \). By (b), all the other elements in the first 2 rows or columns are 0. The resulting determinant is still symmetric and has the principal third-order minor

\[
\begin{vmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 3
\end{vmatrix},
\]

the third row of which keeps repeating horizontally.

Similarly, we next subtract column 3 from all others of order \( 3k \). The elements of these columns now exclude the factors 1, 2, 3. Subtracting the third row from all others of order \( 3k \), we obtain a symmetric determinant with

\[
\begin{vmatrix}
1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 0 & 0 & 4
\end{vmatrix},
\]

as a principal minor. By (b), all other elements of the first three rows or columns are 0, and the fourth row keeps repeating horizontally.

Suppose we have reduced the determinant to the form where the principal minor of order \( k \) is
with all other elements of the same rows or columns 0, and such that all the remaining elements \((i, j)\) have the possible common factors of \(i\) and \(j\) from 1 to \(k\) excluded. Then the \((k+1)^{\text{st}}\) element of the principal diagonal must be \(k+1\), since all other possible smaller factors have been excluded. Hence the \((k+1)^{\text{st}}\) row or column consists of the number \(k+1\) in every \((k+1)^{\text{st}}\) position, the other elements being 0. If we subtract the \((k+1)^{\text{st}}\) column from all others of order \(m(k+1)\), the elements of these columns now exclude the factor \(k+1\). Since no elements of the other columns have this factor, all elements not in the \((k+1)^{\text{st}}\) column now have all possible factors from 1 to \(k+1\) subtracted. Subtracting now the \((k+1)^{\text{st}}\) row from all the others of order \(m(k+1)\), we have a symmetric determinant of the same form as previously, with order \(k+1\) instead of \(k\).

The determinant thus ultimately reduces to one where each element \((i, i)\) is \(i\) and all others 0. Hence the determinant has the value \(n!\).
Problem 61. [AMM, Vol. 50, p. 131-134, Prob. 4023]

Find an expression for the determinant of order $2n$

$$\begin{vmatrix} \theta I_n & A_n \\ A_n & \theta I_n \end{vmatrix},$$

where $\theta I_n$ is a square matrix of order $n$ having the variable $\theta$ in the principal diagonal and zeros elsewhere, and $A_n$ is a square symmetric matrix of order $n$ with $a$ for each principal diagonal element, unity for the elements in the two parallels immediately above and below the principal diagonal, and zeros elsewhere.

**Solution:** By matrix multiplication we find that

$$\begin{bmatrix} I_n & 0 \\ -\theta^{-1}A_n & I_n \end{bmatrix} \begin{bmatrix} \theta I_n & A_n \\ A_n & \theta I_n \end{bmatrix} = \begin{bmatrix} \theta I_n & A_n \\ 0 & \theta I_n - \theta^{-1}A_n^2 \end{bmatrix},$$

and, by taking determinants of both sides of this matrix equation,

$$\Delta = \begin{vmatrix} \theta I_n & A_n \\ A_n & \theta I_n \end{vmatrix} = \left| \theta^2 I_n - A_n^2 \right| = \left| \theta I_n + A_n \right| \left| \theta I_n - A_n \right|$$

$$= (-1)^n \left| A_n + \theta I_n \right| \left| A_n - \theta I_n \right| = (-1)^n f(\theta)f(-\theta),$$

where $(-1)^n f(\theta)$ is the characteristic polynomial of $A_n$. Therefore $\Delta$ can easily be found if its characteristic polynomial can be
calculated. In particular, if \( A_n \) has the value given in this problem, and if

\[
(2) \quad P_n(a) = |A_n|, \\
(3) \quad |A_n - \theta I_n| = P_n(a - \theta).
\]

The determinant \( P_n(a) \) is a perpetuant, and its value can be calculated from the recursion formula

\[
P_n(a) = aP_{n-1}(a) - P_{n-2}(a),
\]

with the initial conditions \( P_2(a) = a^2 - 1, \ P_1(a) = a \). It is now easy to show by induction that

\[
P_n(a) = \sum_{i=0}^{[n/2]} (-1)^i \binom{n-i}{i} a^{n-2i}.
\]

Hence, from (1) and (3), \( \Delta = (-1)^n P_n(a - \theta)P_n(a + \theta) \), where \( P_n(a) \) is defined by (4).

A generalization of this problem is also discussed.

Problem 62. [AMM, Vol. 52, p. 165-166, Prob. 4101]

Show that

\[
D(n) = \begin{vmatrix} (1,1)^\lambda & (1,2)^\lambda & \cdots & (1,n)^\lambda \\ (2,1)^\lambda & (2,2)^\lambda & \cdots & (2,n)^\lambda \\ \vdots & \vdots & \ddots & \vdots \\ (n,1)^\lambda & (n,2)^\lambda & \cdots & (n,n)^\lambda \end{vmatrix} = (n!)^\lambda \left(1 - \frac{1}{2^\lambda}\right) \left(1 - \frac{1}{3^\lambda}\right) \left(1 - \frac{1}{5^\lambda}\right) \cdots,
\]

where

\[
\left(\frac{P_2}{n^2}\right) = \frac{P_3}{n^3}, \quad \left(\frac{P_4}{n^4}\right) = \frac{P_5}{n^5}, \quad \cdots,
\]
where \((i, j)\) means the greatest common divisor of the integers \(i, j\).

Solution: We can generalize the problem by considering

\[
D_n = \left| f \left( (i, j) \right) \right|,
\]

where \(f(x)\) is defined for all positive integral values of \(x\). To evaluate \(D(n)\), we define \(\psi(k)\) by the equations

\[
f(\ell) = \sum_{k \mid \ell} \psi(k), \tag{2}
\]

and we define \(a_{k\ell}\) to be 1 if \(\ell\) divides \(k\), zero otherwise.

We then have

\[
\sum_{\ell} a_{r\ell} a_{s\ell} \psi(\ell) = \sum_{\ell \mid (r, s)} \psi(\ell) = f( (r, s) ),
\]

and we can write \(D(n) = \left| a_{r\ell} \right| \left| a_{s\ell} \psi(\ell) \right|\), where the determinants on the right are of order \(n\). Since \(a_{r\ell} = 0\) if \(r < \ell\) and \(a_{\ell\ell} = 1\), we have \(\left| a_{r\ell} \right| = 1\) and

\[
\left| a_{s\ell} \psi(\ell) \right| = \prod_{\ell=1}^{n} \psi(\ell),
\]

and hence,

\[
D(n) = \prod_{\ell=1}^{n} \psi(\ell).
\]

Now inverting (2) by the Möbius inversion formula we have
\[ \psi(\ell) = \sum_{k \mid \ell} \mu(k) f\left( \frac{\ell}{k} \right), \]

and hence, the formula

\[ D(n) = \prod_{\ell=1}^n \left[ \sum_{k \mid \ell} \mu(k) f\left( \frac{\ell}{k} \right) \right]. \]

For the case \( f(x) = x^\lambda \), we have

\[ \psi(\ell) = \sum_{k \mid \ell} \mu(k) \left( \frac{\ell}{k} \right)^\lambda = \ell^\lambda \sum_{k \mid \ell} \frac{\mu(k)}{k^\lambda} = \ell^\lambda \prod_{p \mid \ell} \left( 1 - \frac{1}{\lambda} \right), \]

and hence,

\[ D(n) = \prod_{\ell=1}^n \ell^\lambda \prod_{\ell \mid n} \left( 1 - \frac{1}{\lambda} \right) \prod_{p \mid \ell} \left( 1 - \frac{1}{\lambda} \right) \]

\[ = (n!)^\lambda \prod_{p \mid n} \prod_{1 \leq \ell \leq n} \left( 1 - \frac{1}{\lambda} \right) \left[ \frac{n}{p} \right] \]

which is the required result.

Formula (3) can be used to obtain interesting evaluations of (1) in other cases. For example if \( f(x) = \delta(x) \), the sum of the divisors of \( x \), it is easily seen that

\[ \sum_{k \mid \ell} \mu(k) \delta\left( \frac{\ell}{k} \right) = \ell, \]
and hence the evaluation \( D(n) = n! \).

It is also possible to invert the process and determine \( f(x) \) to obtain a desired \( D(n) \). From (3) we have

\[
\sum_{k \mid \ell} \mu(k) f\left( \frac{\ell}{k} \right) = \frac{D(n)}{D(n-1)}.
\]

Inverting this by the Möbius formula we obtain

\[
f(n) = \sum_{k \mid n} \frac{D(k)}{D(k-1)}.
\]

As an example, if we wish to obtain \( D(n) = a^n \), we take

\[
f(n) = \sum_{k \mid n} a = a \tau(n),
\]

where \( \tau(n) \) is the number of divisors of \( n \).

Problem 63. [AMM, Vol. 61, p. 260, Prob. E1081]

Let \( f(x_1, x_2, \ldots, x_n) \) be the \( n \)th order determinant \( |a_{ij}| \) with \( a_{ii} = x_i \) and \( a_{ij} = 1 \) for \( i \neq j \). Clearly, \( f(x_1, \ldots, x_n) \) is symmetric in \( (x_1, x_2, \ldots, x_n) \). Find its representation in terms of the elementary symmetric functions in \( x_1, \ldots, x_n \).

Solution: Perform the operation \( \text{col. } k - \text{col. } n \) for \( k = 1, 2, \ldots, n-1 \), and take out the factors \( x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1} \).
Now add the first $n-1$ rows to the last row to get a triangular determinant with diagonal elements

$$1, 1, 1, \ldots, 1 + \sum_{i=1}^{n} 1/(x_i - 1).$$

Hence the determinant can be written in the form

$$f(x_1, \ldots, x_n) = \prod_{i=1}^{n} (x_i - 1) \left[ 1 + \sum_{i=1}^{n} \frac{1}{x_i + 1} \right]$$

$$= p_n = p_{n-2} + 2p_{n-3} - 3p_{n-4} + \ldots + (-1)^{n-1}(n-1)$$

$$= \sum_{i=0}^{n} (-1)^{i-1} (i-1) p_{n-i} , (p_0 = 1),$$

as required.

**Problem 64. [AMM, Vol. 67, p. 476-477, Prob. E1389]**

Evaluate the $n \times n$ determinant $A_n$ whose $(i,j)^{th}$ entry is $a_{|i-j|}$.

**Solution I:** Expand the determinant $A_n$ by elements of the first row. The minor of the first element is $A_{n-1}$; the minor of the second element is $aA_{n-1}$; the minors of the remaining elements of the first row are all zero. Hence $A_n = (1-a^2)A_{n-1}$. Since
Solution II: By subtracting a times the second row from the first row, we see that \( A_n = (1 - a^2)A_{n-1} \). Since \( A_1 = 1 \), it follows that \( A_n = (1 - a^2)^{n-1} \).

Solution III: By multiplying the \((i-1)^{st}\) row by a and subtracting it from the \(i^{th}\) row, \( i = n, n-1, \ldots, 2 \), \( A \) is reduced to the triangular form with \( a_{11} = 1, \ a_{ii} = 1 - a^2 \) when \( i \neq 1 \). Therefore \( A_n = (1 - a^2)^{n-1} \).


Show that for any integer \( n \), one can construct a symmetric fourth order determinant whose elements are ten consecutive integers and whose value is \( n \).

Solution: The determinant

\[
\begin{vmatrix}
n+1 & n-8 & n & n-7 \\
n-8 & n-1 & n-6 & n-2 \\
n & n-6 & n-5 & n-3 \\
n-7 & n-2 & n-3 & n-4 \\
\end{vmatrix}
\]

is composed of ten consecutive integers \( n-8, n-7, \ldots, n, n+1 \), is symmetric, and has value \( n \).
This problem appears in the unpublished notes of the late Fenton S. Stancliff, who studied extensively the effect of adding a constant to each element of a matrix. The problem has many essentially different solutions. If we are restricted to use only two consecutive integers, we have the following two matrices,

\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1
\end{bmatrix}
\]

for each of which the determinant value decreases by \( k \) when \( k \) is added to each element.


Evaluate the determinant

\[
\begin{vmatrix}
0 & 1 & \ldots & n \\
1 & 2 & \ldots & n+1 \\
\vdots & \vdots & \ddots & \vdots \\
0 & n+1 & \ldots & 2n
\end{vmatrix}
\]

Solution: Subtracting the \( m \)th column from the \( (m+1) \)th, \((m = 1, 2, \ldots, n-1)\), and then subtracting the \( m \)th row from the
\[(m+1)^{th}, \ m = 1, 2, \ldots, n-1, \ \text{we get 0's in the first row and first column except for } a_{11} = 1, \ \text{and the rest of } D_n \ \text{identical to } D_{n-1}. \ \text{Therefore } D_n = D_{n-1} = \ldots = D_1 = 1.\]

**Problem 67.** [AMM, Vol. 71, p. 917, Prob. E1645]

Let \( r \) be a nonnegative integer and let \( A = (a_{ij}) \) be an \( nxn \) matrix where \( a_{ij} = (i+j+r-2)!/(i-1)!(j+r-1)! \). Show that \( |A| = 1. \)

**Solution:** Problems 3468 [See Prob. 58, p. 98 of this paper] and E1600 [See Prob. 66, p. 112, this paper] amount to the statement that

\[
\begin{vmatrix}
(r+s) \\
\vdots \\
(s)
\end{vmatrix} = 1, \ 0 \leq r, s \leq n, \ n \ a \ nonnegative \ integer.
\]

A more general statement is that any determinant "hung" from the \( h \)\(^{th} \) one at either side of the Pascal triangle is unity,

\[
F(n,h) = \begin{vmatrix}
(r+s+h) \\
\vdots \\
(s)
\end{vmatrix} = 1, \ h \ a \ nonnegative \ integer.
\]

Replace each row of \( F(n,h) \) by the difference between itself and the preceding row; do likewise with the columns of the resulting determinant, arriving at
\[
\begin{pmatrix}
  (r+s+h) \\ s
\end{pmatrix} \rightarrow
\begin{pmatrix}
  (r+s+h) \\ s
\end{pmatrix} - \begin{pmatrix}
  (r+s+h-1) \\ s
\end{pmatrix} + \begin{pmatrix}
  (r+s+h-1) \\ s-1
\end{pmatrix}
\]

That is \( F(n, h) = F(n-1, h) \) with \( F(0, h) = 1 \).

Problem 68. [AMM, Vol. 72, p. 1030, Prob. 5334]

If \( b = (b_{ij}) \) is a symmetric square matrix with \( b_{ii} = 1 \),

\[
\sum_{j \neq 1} |b_{ij}| \leq 1 \quad \text{for each } i, \quad \text{then } \det (b) \leq 1.
\]

Solution: No solution to this problem has yet been published.

Problem 69. [DMVJ, Vol. 39, Pt. 2, p. 3-4, Prob. 66]

Let \( A \) be a symmetric matrix of \( n^2 \) elements, and let \( B \) be an arbitrary matrix of \( n^2 \) elements. Let \( \Delta \) denote the determinant of the matrix

\[
\begin{vmatrix}
  A & B \\ B & A
\end{vmatrix},
\]

which has order \( 2n \). Prove each of the following:

(1) If \( B \) is also symmetric, \( \Delta \) can be expressed as the
product of two \( n \)th order determinants, one with binomial elements \( a_{ik} + b_{ik} \), the other \( a_{ik} - b_{ik} \).

(2) If \( B \) is skew-symmetric, then \( \Delta \) can be expressed as the square of an \( n \)th order determinant with elements \( a_{ik} + b_{ik} \).

(3) (1) holds if \( B \) is symmetric (\( A \) arbitrary).

(4) If \( B \) is skew-symmetric, then \( \Delta \) is the square of the complete rational function of the elements of \( A \), but in general \( \Delta \) is perhaps not the square of an \( n \)th order determinant as in (2).

Solution: (1) Let \( A \) and \( B \) be arbitrary matrices, and \( E \) the unit matrix of order \( n \). Then

\[
\begin{bmatrix}
E & 0 \\
-E & E
\end{bmatrix}
\begin{bmatrix}
A & B \\
B & A
\end{bmatrix}
\begin{bmatrix}
E & 0 \\
E & E
\end{bmatrix}
= \begin{bmatrix}
A + B & B \\
0 & A - B
\end{bmatrix}.
\]

Thus \( \Delta = |A + B| \cdot |A - B| \). Thus (1) holds not only for symmetric, but also for arbitrary matrices.

(2) Let \( X' \) be the transpose of \( X \). Then \( A = A' \), \( B = -B' \), and \((A-B)' = A' - B' = A + B \). Hence \( \Delta = |A + B|^2 \).

(3) If \( A \) is arbitrary and \( B \) is symmetric, then (3) follows from (1).

(4) If \( A = A' \) and \( B = -B' \), then
and from
\[
\begin{bmatrix}
E & 0 \\
-iE & E
\end{bmatrix}
\begin{bmatrix}
A & B \\
-B & A
\end{bmatrix}
\begin{bmatrix}
E & 0 \\
iE & E
\end{bmatrix}
= \begin{bmatrix}
A + iB & B \\
0 & A - iB
\end{bmatrix},
\]

it follows that \( \Delta = |A + iB| \cdot |A - iB| \). But
\[(A - iB)' = A' - iB' = A + iB, \quad \text{so} \quad |A - iB| = |A + iB|, \quad \text{and we have}
\]
\[\Delta = |A + iB|^2.\]
Thus we can, even in this case, always express \( \Delta \) as the square of an \( n \)th order determinant, in which the elements have the form \( a_{jk} + ib_{jk} \). The determinant \( |A + iB| \), since it coincides with the determinant \( |A - iB| \), is a complete rational function of the elements of \( A \) and \( B \) having real coefficients.

**Problem 70.** [DMVJ, Vol. 44, Pt. 2, p. 76-78, Prob. 157]

If \( a_{ik} \) is the number of common factors of \( i \) and \( k \), prove
\[
\begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix} = 1.
\]

**Solution I:** Theorem. Let \( T(i + a, j + a) \) be the number of
common factors of \( i + a \) and \( j + a \) which are \( \geq a \), where \( a \) is a fixed integer \( > 0 \). Show that the determinant \( |T(i+j, j+a)| = 1 \), for \( i, j = 1, 2, \ldots, n \).

Proof. Let \( k(r, s) = 0 \), for \( r \neq s \), \( r \) or \( s \leq a \),

\[ = 1, \quad \text{for} \quad r = s \leq a, \]

\[ = T(r, s) \quad \text{for} \quad r > a, \quad s > a. \]

Then \( \det |k(r, s)| = \det |T(i + a, j + a)| \),

\[ r, s = 1, 2, \ldots, n + a; \quad i, j = 1, 2, \ldots, n. \]

Let

\[ b_{rs} = 1 \quad \text{for} \quad r, s > a, \quad r \equiv 0 \pmod{s}, \]

\[ b_{rr} = 1 \quad \text{for} \quad r \leq a, \]

\[ b = 0 \quad \text{otherwise.} \]

Now \( k(r, s) = b_{r1}s_1 + b_{r2}s_2 + \ldots + b_{r,n+a}s, n+a \),

since \( b_{ri}b_{si} = 1 \quad \text{for} \quad r, s > a, \quad i \) a common factor of \( r \) and \( s > a, \)

\[ = 1 \quad \text{for} \quad r = s = i \leq a \]

\[ = 0 \quad \text{otherwise.} \]

Hence \( \det |T(i+a, j+a)| = \det |k(r, s)| = \det |b_{rs}|^2. \)

But \( b_{rs} = 0 \) for \( s > r \), so \( \det |b_{rs}| = b_{11}b_{22} \ldots b_{n+a, n+a} = 1. \)

Thus \( \det |T(i+a, j+a)| = 1, \quad i, j = 1, 2, \ldots, n. \)

Finally, for \( a = 0 \), \( T(i,j) \) is the number of common factors of \( i \) and \( j \), and it follows that \( \det |T(i,j)| = 1. \)

**Solution II:** Let \( (i, k; j) \) represent the number of common factors of \( i \) and \( k \) which are greater than \( j \). We can then write
the elements of the given matrix as \( (i, k; 0) \). For \( h = 1, 2, \ldots, j \), subtract the row following row \( h \) from all following rows whose number is a multiple of \( h \). After \( j \) steps, the matrix consists of the elements \( (i, k; i-1) \) for \( i = 1, 2, \ldots, j \) and \( (i, k; j) \) for \( i > j \).

This is obviously true for \( j = 1 \). If \( 1 < j < n \), assume the proportion established for \( j - 1 \) steps. It remains to be shown that it also holds for \( j \) steps.

After \( j-1 \) steps, the \( j^{th} \) row consists of the elements \((j, k; j-1)\), that is, ones or zeros according as \( k \) is divisible by \( j \) or not. If we subtract this row from all those following whose number \( i \) is a multiple of \( j \), the \( (i,k)^{th} \) element is decreased by one if \( i \) and \( k \) contain the same factors of \( j \); otherwise the element remains unchanged. In other words, \( (i, k; j) \) becomes \( (i, k; j-1) \) for \( i > j \).

After \( j \) steps the first \( j \) columns contains ones in the main diagonal positions and zeros below. The determinant of the matrix is thus equal to the \( (n-j) \)-rowed determinant of the numbers \( (i, k; j) \ i, k = j+1, j+2, \ldots, n \). Moreover this equality holds for \( j = 0, 1, 2, \ldots, n-1 \). The determinants are thus equal, and their common value is \( 1 \), as is easily seen from the latter.

A third proof is also given.
Problem 71. [MM, Vol. 24, p. 110-111, Prob. 61]

Find the value of the determinant \( D \) in which \( a = \sin \theta \), \( b = \cos \theta \).

\[
\begin{vmatrix}
3 & 2b & ab & b^2 & ab & b \\
2b & 3 & 2b & ab & b^2 & ab \\
ab & b^2 & a & b & a & b \\
b^2 & ab & b & a & b & a \\
ab & b^2 & a & b & 0 & 0 \\
b^2 & ab & b & a & 0 & 0
\end{vmatrix}
\]

Solution I: Let \( P \) denote the matrix \( \begin{bmatrix} a & b \\ b & a \end{bmatrix} \). Then the given matrix may be written in the form

\[
\begin{vmatrix}
a^2 \cdot P & b \cdot P & b \cdot P \\
b \cdot P & 1 \cdot P & 1 \cdot P \\
b \cdot P & 1 \cdot P & 0 \cdot P
\end{vmatrix}
= P \times \begin{vmatrix} a^2 & b & b \\
& b & 1 & 1 \\
& b & 1 & 0
\end{vmatrix}
= P \times Q.
\]

The determinant of this product, \( |P \times Q| = |P|^3 \cdot |Q|^2 \). [See C. C. MacDuffee, An Introduction to Abstract Algebra, (1940), p. 248].

Hence

\[
D = |P|^3 \cdot |Q|^2 = (a^2 - b^2)^3 (b^2 - a^2)^2 = -(b^2 - a^2)^5
\]

\[
= -(\cos^2 \theta - \sin^2 \theta)^5 = -\cos^5 2\theta.
\]
Solution II: We perform the operations \( \text{col. 1} - \text{b \ col. 3} \), 
\( \text{col. 2} - \text{b \ col. 4} \), \( \text{col. 3} - \text{col. 5} \), \( \text{col. 4} - \text{col. 6} \), \( \text{col. 5} - \text{col. 6} \), and take common factors from the columns. Next, we perform the operations 
\( \text{row 1} + \text{row 2} \), \( \text{row 3} + \text{row 4} \), \( \text{row 6} + \text{row 5} \), followed by the 
operations \( \text{col. 1} - \text{col. 2} \), \( \text{col. 3} - \text{col. 4} \), \( \text{col. 6} - \text{b \ col. 2} \), and take common factors from the rows and columns. Thus, we have

\[
D = (a^2 - b^2, 2) (a-b)
\]

\[
\begin{vmatrix}
  a & b & 0 & 0 & b & b^2 \\
  b & a & 0 & 0 & -b & ab \\
  0 & 0 & 0 & 0 & 1 & b \\
  0 & 0 & 0 & 0 & -1 & a \\
  0 & 0 & a & b & 0 & 0 \\
  0 & 0 & b & a & 0 & 0 \\
\end{vmatrix}
\]

\[
= (a^2 - b^2)^5
\]

\[
\begin{vmatrix}
  1 & b & 0 & 0 & b & 0 \\
  0 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 1 \\
  0 & 0 & 0 & 0 & -1 & a \\
  0 & 0 & 1 & b & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 \\
\end{vmatrix}
\]

Obviously the last determinant equals \( 1 \). Thus,

\[
D = (a^2 - b^2)^5 = -(b^2 - a^2)^5 = -(\cos^2 \theta - \sin^2 \theta)^5 = -\cos^5 2\theta.
\]

Evaluate the determinant \( \Delta = \begin{vmatrix} a^2 + d^2 & ab + de & ac + df \\ ab + de & b^2 + e^2 & bc + ef \\ ac + df & bc + ef & c^2 + f^2 \end{vmatrix} \).

Solution: It is clear that \( \Delta \) is the square of the determinant

\[ \begin{vmatrix} a & d & 0 \\ b & e & 0 \\ c & f & 0 \end{vmatrix} \]. Hence \( \Delta = 0 \).

Problem 73. [MM, Vol. 37, p. 280-281, Prob. 540]

Show that if the \( n! \) terms in the expansion of an \( n \) th order symmetric determinant with positive elements \( a_{ij} \) have the same absolute value, then there exists a set of numbers \( b_1, b_2, \ldots, b_n \) such that \( a_{ij} = b_i b_j \), \( i, j = 1, 2, \ldots, n \).

Solution: Let \( (p_1, p_2, \ldots, p_{n-2}, r, s) \) and \( (i_1, i_2, \ldots, i_{n-2}, r, s) \) be two permutations on the integers \( (1, 2, \ldots, n) \). Then

\[ a_{p_1 i_1} a_{p_2 i_2} \cdots a_{p_{n-2} i_{n-2}} a_{rs} a_{ss} = a_{p_1 i_1} a_{p_2 i_2} \cdots a_{p_{n-2} i_{n-2}} a_{rs} a_{sr} \]

so that \( a_{rs} = a_{rr} a_{ss} \). Thus we can take \( b_i = \sqrt{(a_{ii})} \), \( i = 1, 2, \ldots, n \). Conversely, if \( b_i = \sqrt{(a_{ii})} \), then we have

\[ a_{1, i_1} a_{2, i_2} \cdots a_{n-1, r} a_{ns} = (b_1 b_2 \cdots b_n)^2. \]
Problem 74. [SSM, Vol. 45, p. 581-582, Prob. 1924] ²

Resolve into linear factors, not necessarily real, the determinant

\[
D = \begin{vmatrix}
  a & b & c & d \\
  b & a & d & c \\
  c & d & a & b \\
  d & c & b & c \\
\end{vmatrix}.
\]

Solution: (1) By adding together all rows, we see that \((a+b+c+d)\) is a factor of the determinant. (2) By adding together the first and third rows and subtracting from the result the sum of the second and fourth rows, we see that \((a-b+c-d)\) is also a factor. (3) By adding together the first and fourth rows and subtracting from the result the sum of the second and third rows, we see that \((a-b-c+d)\) is also a factor. (4) By adding together the first and second rows and subtracting from the result the sum of the third and fourth rows, we see that \((a+b-c-d)\) is a factor. (5) The remaining factor is numerical, and from a comparison of the terms involving \(n^4\) on each side, is seen to be 1. (6) Hence

\[
D = (a+b+c+d)(a-b+c-d)(a-b-c+d)(a+b-c-d).
\]

Orthosymmetric Determinants

Problem 75. [AMM, Vol. 17, p. 116-117, Prob. 336 (Alg)]

Evaluate the determinant

\[
D = \begin{vmatrix}
    a_1^2 & a_2^2 & a_3^2 & \ldots & a_n^2 \\
    a_2^2 & a_3^2 & a_4^2 & \ldots & a_{n+1}^2 \\
    a_3^2 & a_4^2 & a_5^2 & \ldots & a_{n+2}^2 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    a_n^2 & a_{n+1}^2 & a_{n+2}^2 & \ldots & a_{2n-1}^2 \\
\end{vmatrix}
\]

Solution: Subtract each row from the one which follows it, beginning with the \((n-1)^{st}\). Repeat this operation, stopping at the second row. Keep repeating this operation, leaving out a row each time, until all the rows have been omitted; then if

\[
\Delta_a^2 = \Delta_{a_{s+1}}^2 - \Delta_{a_s}^2,
\]

we get

\[
D = \begin{vmatrix}
    a_1^2 & a_2^2 & a_3^2 & \ldots & a_n^2 \\
    \Delta a_1^2 & \Delta a_2^2 & \Delta a_3^2 & \ldots & \Delta a_n^2 \\
    \Delta^2 a_1^2 & \Delta^2 a_2^2 & \Delta^2 a_3^2 & \ldots & \Delta^2 a_n^2 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    \Delta^{n-1} a_1^2 & \Delta^{n-1} a_2^2 & \Delta^{n-1} a_3^2 & \ldots & \Delta^{n-1} a_n^2 \\
\end{vmatrix}
\]

Repeating the same series of operations on the columns, we get
If $a_2^2$ is a function of $r$ of the $p^{th}$ degree in $r$, whose highest term has coefficient one, the quantities $a_1^2, a_2^2, a_3^2, \ldots$ form an arithmetic series of the $p^{th}$ order.

If $p = n-1$, all the elements below the second diagonal vanish, while all those in it are equal to $(n-1)!$, and

$$D = (-1)^{n(n-1)/2} [(n-1)!]^n.$$ If $p < n-1$, $D = 0$.

**Problem 76.** [AMM, Vol. 25, p. 27-28, Prob. 258 (Num. Theory)]

Find a recursion formula in terms of the binomial coefficients for $a_n$, where the $a's$ are defined by the condition that the orthosymmetric determinants

$$\begin{vmatrix}
a_0 & a_1 & a_2 & \cdots & a_n-1 \\
a_1 & a_2 & a_3 & \cdots & a_n \\
a_2 & a_3 & a_4 & \cdots & a_n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
& & & & a_n \\
\end{vmatrix} \quad \text{and} \quad \begin{vmatrix}
a_1 & a_2 & a_3 & \cdots & a_n \\
a_2 & a_3 & a_4 & \cdots & a_n \\
a_3 & a_4 & a_5 & \cdots & a_n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
& & & & a_n \\
\end{vmatrix}$$
are each equal to unity for every positive integer \( n \).

**Solution:** Though the solution does not directly involve binomial coefficients, yet by finding the value of \( a_n \), it may be considered to dispose of the problem sufficiently.

The given conditions show that \( a_0 = a_1 = 1 \), and that the other \( a \)'s may be found in succession uniquely from the equations in which they appear with coefficient unity. The \( a \)'s being determinate, there exists a sequence \( x_0, x_1, \ldots \) such that

\[
(1) \quad a_n = a_{n-1} x_0 + a_{n-2} x_1 + \ldots + a_0 x_{n-1}, \quad n = 1, 2, \ldots,
\]

for the first \( n \) equations of (1) have a determinant equal to unity.

If we apply the substitution (1) to the last row of

\[
\begin{vmatrix}
  a_0 & a_1 & a_2 & \cdots & \cdots \\
  a_1 & a_2 & \cdots & \cdots & \cdots \\
  a_2 & \cdots & \cdots & \cdots & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots \\
  a_n & a_{n+1} & \cdots & \cdots & a_{2n}
\end{vmatrix}
\]

and simplify by means of the other rows, the last becomes

\[
0, a_0 x_n, a_0 x_{n+1} + a_1 x_n, a_0 x_{n+2} + a_1 x_{n+1} + a_2 x_n, \ldots.
\]

With similar treatment, the preceding row becomes

\[
0, a_0 x_{n-1}, a_0 x_n + a_1 x_{n-1}, \ldots,
\]

and so for all but the first row. On simplifying by columns we get, since \( a_0 = 1 \),
A like treatment of the other determinant gives

\[
\begin{vmatrix}
 x_1 & x_2 & x_3 & \cdots & \cdots \\
 x_2 & x_3 & \cdots & \cdots \\
 x_3 & \cdots & \cdots \\
 \cdots & \cdots & \cdots & \cdots & \cdots \\
 x_{2n-1} & \cdots & \cdots \\
\end{vmatrix} = 1.
\]

Hence \( x_2, x_3, \ldots \) are defined in terms of \( x_0, x_1 \) in the same way as \( a_2, a_3, \ldots \) are defined in terms of \( a_0, a_1 \). Also, \( a_0 = a_1 = 1, \ a_2 = 2. \) Hence \( x_0 = x_1 = 1, \ x_2 = 2, \) by direct calculation. Hence \( x_n = a_n, \) and (1) becomes

\[
(2) \quad a_n = a_{n-1} a_0 + a_{n-2} a_1 + \ldots + a_0 a_{n-1}, \ n = 1, 2, \ldots.
\]

To calculate \( a_n, \) we infer from (2) that the coefficients of \( t^n \) in \( u = a_0 + a_1 t + a_2 t^2 + \ldots \) is equal to the coefficient of \( t^{n-1} \) in \( u^2, \) when \( n = 1, 2, \ldots. \) Hence \( (u-1)/t = u^2. \) Hence

\[
u = \frac{1 - \sqrt{1-4t}}{2t}, \text{ the minus sign being necessary to make } u \text{ a series of positive powers of } t. \text{ Hence the coefficient of } a_n \text{ of } t^n
in \( u \) equals \( \frac{2n}{n(n+1)} \).

Problem 77. [AMM, Vol. 30, p. 196-198]

Evaluate the \( n \)th order determinant \( D_n = \left| \frac{1}{1/(r+s-1)!} \right| \).

**Solution:** Although it belongs to the class of determinants called orthosymmetric, and specially studied by Hankel, the principal theorem relating to this class of determinants is not directly effective.

If the elements of each column of \( D_n \) be multiplied by the reciprocal of the last element in the column, and compensation be made by means of appropriate multipliers for the determinant, we may write

\[
D_n = \frac{1}{(n!)(n+1)! \ldots (2n-1)!} \left| a_{rs} \right| \quad (r, s = 1, \ldots, n)
\]

where

\[
a_{rs} = \frac{(n+s-1)!}{(r+s-1)!}.
\]

Every element of the last row of \( |a_{rs}| \) is equal to one. Hence it may be reduced to a determinant in which all elements of the last row are zero with the exception of the last one, by subtracting the second column from the first, the third from the second, and in general the \( (s+1) \)th column from the \( s \)th column for \( s=1,\ldots,n-1 \).
By this transformation, the element in the \( r \)th row and \( s \)th column becomes

\[
\frac{(n + s - 1)!}{(r + s - 1)!} - \frac{(n + s)!}{(r + s)!} = -(n - r) \frac{(n + s - 1)!}{(r + s)!} \quad (r, s = 1, 2, \ldots, n-1),
\]

and by taking the factor \(-(n-r)\) out of the \( r \)th row for \( r = 1, \ldots, n-1 \), we may reduce the determinant \( |a_{rs}| \) as follows:

\[
|a_{rs}| = (-1)^{n-1} (n-1)! |A_{rs}^{(1)}| \quad \text{where} \quad a_{rs}^{(1)} = \frac{(n+s-1)!}{(r+s)!},
\]

\((r, s = 1, \ldots, n) \quad (r, s = 1, \ldots, n-1)\)

Each element of the last row of \( |a_{rs}^{(1)}| \) is equal to one. Hence a transformation similar to that used on \( |a_{rs}| \) will suffice to reduce its order by one. We find

\[
|a_{rs}^{(1)}| = (-1)^{n-2} (n-2)! |a_{rs}^{(2)}|, \quad \text{where} \quad a_{rs}^{(2)} = \frac{(n+s-1)!}{(r+s+1)!},
\]

\((r, s = 1, \ldots, n-1) \quad r, s = 1, \ldots, n-2\)

Again the elements of the last row of \( |a_{rs}^{(2)}| \) are equal to one.

The process of reduction of order can be repeated successively, the \( j \)th repetition giving the reduction formula

\[
|a_{rs}^{(j-1)}| = (-1)^{n-j} (n-j)! |a_{rs}^{(j)}|, \quad \text{where} \quad a_{rs}^{(j)} = \frac{(n+s-1)!}{(r+s+j-1)!},
\]

\((r, s = 1, \ldots, n-j+1) \quad (r, s = 1, \ldots, n-j)\)

After \( n-1 \) reductions we obtain the determinant of a single element,
Assembling the results of successive reductions we have

\[
D_n = (-1)^{n(n-1)/2} \frac{1!2!3!\ldots(n-1)!}{n!(n+1)\ldots(2n-1)!}.
\]

A modification is suggested which converts the elements to binomial coefficients. This suggestion is treated in more detail in an article by J.J. Nassau [AMM, Vol. 31, p. 341-342].

Problem 78. [AMM, Vol. 35, p. 95-96, Prob. 3249]

If \( i!f_i = \frac{d^i f(x)}{dx^i} \) and

\[
D = \begin{vmatrix}
    f_r & f_{r+1} & \cdots & f_{r+n-1} \\
    f_{r+1} & f_{r+2} & \cdots & f_{r+n} \\
    \vdots & \vdots & \ddots & \vdots \\
    f_{r+n-1} & f_{r+n} & \cdots & f_{r+2n-2}
\end{vmatrix},
\]

prove that \( \frac{dD}{dx} = (r + 2n -1)D' \), where \( D' \) is the determinant \( D \) with the subscripts of the last row increased by unity.

**Solution:** Set \( a_{ij} = f_{r+i+j-2} \) (\( j = 1, 2, \ldots, n; \ i = 1, 2, \ldots, n+1 \)).
Then
\[ D = \sum_{j=1}^{n} a_{ij} A_{ij} \quad (i = 1, 2, \ldots, n), \]

where \( A_{ij} \) denotes the cofactor of \( a_{ij} \) in \( D \); also

\[ \frac{da_{ij}}{dx} = (r+i+j-1) a_{i+1,j}. \]

Hence
\[ \frac{dD}{dx} = \sum_{i=1}^{n} \left[ \sum_{j=1}^{n} (r+i+j-1) a_{i+1,j} A_{ij} \right]. \]

Subtract from each bracket above

\[ (r+2i-1) \sum_{i=1}^{n} a_{i+1,j} A_{ij} = 0, \quad i = 1, 2, \ldots, n-1 \]
\[ = (r+2n-1)D', \quad i = n, \]

and there results

\[ \frac{dD}{dx} = \sum_{i=1}^{n} \sum_{j=1}^{n} (j-i)a_{i+1,j} A_{ij} + (r+2n-1)D' \]
\[ = (r+2n-1)D', \]

for to each term in the double sum there is a corresponding term

\( (i-j)a_{i+1,j} A_{ji}, \) except where \( i = j, \) and the two cancel since

\[ a_{i+1,j} = a_{j+1,i} \quad \text{and} \quad A_{ij} = A_{ji}. \]
Show that the determinant
\[
\begin{vmatrix}
(x-1)/1 & (x^2-1)/2 & \cdots & (x^n-1)/n \\
(x^2-1)/2 & (x^3-1)/3 & \cdots & (x^{n+1}-1)/(n+1) \\
\vdots & \vdots & \ddots & \vdots \\
(x^n-1)/n & (x^{n+1}-1)/(n+1) & \cdots & (x^{2n-1}-1)/(2n-1)
\end{vmatrix}
\]
is a constant times \((x-1)^2\).

Solution: We shall first prove the following:

**Theorem:** Let \(f_1(x), f_2(x), \ldots, f_n(x), g_1(x), g_2(x), \ldots, g_n(x)\) be \(2n\) polynomials, and let
\[
\phi_{ij}(x) = \int_a^x f_i(u) g_j(u) \, du.
\]
Then the determinant \(|\phi_{ij}(x)|\) vanishes at \(x = a\) with a multiplicity of at least \(n^2\).

**Proof:** We have to show that, if \(F(x)\) denotes the given determinant, \(F^{(k)}(a) = 0\) for \(0 \leq k < n^2\). Using the well-known method of differentiating a determinant (by rows, say), we obtain
\[
(1) \quad F^{(k)}(x) = \sum [k!/(k_1!k_2!\ldots k_n!)] \left| \phi_{ij}^{(k)}(x) \right|.
\]
where the sum is taken over all sets of nonnegative integers $k_1, k_2, \ldots, k_n$ with $k_1 + k_2 + \ldots + k_n = k$. Let $F(k)(x) = G_k(x) + H_k(x)$, where $G_k(x)$ consists of those terms in (1) in which at least one of the $k_i$ is zero, and $H_k(x)$ of those in which all $k_i$ are positive. Since $\phi_{ij}(a) = 0$, we have at once $G_k(a) = 0$ (which completes the proof when $0 \leq k < n$). We shall now show that $H_k(x) = 0$ identically in $x$ and may henceforth omit the variable $x$ in our notation.

Since $\phi'_{ij} = f_i g_j$, we have for $i, j = 1, 2, \ldots, n$ and $k_i \geq 1$,

$$
(2) \quad \phi_{ij} = \sum_{(k_i)} \left[ \frac{(s_i + t_j)!(s_i t_j)!}{(s_i)! t_j!} \right] f_i^s g_j^t,
$$

where the sum is taken over all nonnegative values of $s_i$ and $t_j$ with $s_i + t_j = k_i - 1$. Substituting (2) for the elements of $|\phi_{ij}|$ in each term of (1) that enters into $H_k'$, we obtain a determinant each element of whose $i$th row is written as a sum of $k_i$ terms. Thus we can write this determinant as a sum of $k_1 k_2 k_3 \ldots k_n$ determinants and arrive, after further simplifications, at the following expression for $H_k$:

$$
(3) \sum \frac{k!}{(s_1 + t_1 + 1) \ldots (s_n + t_n + 1)} \cdot \frac{(s_1)! \ldots (s_n)!}{s_1! \ldots s_n! t_1! \ldots t_n!} \cdot \left| f_i \right| \cdot \left| g_j \right|
$$

where the sum is taken over all sets of nonnegative integers
\[ s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_n \text{ with } \sum_i (s_i + t_i) = k - n. \]

Those terms in (3) in which there are at least two equal \( t_i \) vanish. In the remaining terms (in which the \( t_i \) are distinct) at least two \( s_i \) are equal to one another, since otherwise

\[ k - n = \sum_i (s_i + t_i) \geq n(n-1), \]

which contradicts \( k < n^2 \). These remaining terms of (3) can now be paired according to the following principle: In any such term of (3), let \( m \) be the smallest index for which \( s_m \) equals one of the other \( s_i \), and let \( r(m) \) be the smallest index for which \( s_r = s_m \). Then we associate with this term the one with the same values of \( s_i \) and \( t_i \), except that the values of \( t_m \) and \( t_r \) are interchanged. It is easily seen that this constitutes actually a pairing of the remaining terms of (3), and that the sum of the members of each such pair vanishes. This completes the proof of the theorem.

We now put \( f_i(x) = g_i(x) = x^{i-1}, \ i = 1, 2, \ldots, n, \) and \( a = 1. \)

We obtain

\[ \phi_{ij} = (x^{i+j-1} - 1)/(i+j-1). \]
and hence, by the theorem, the determinant given in this problem is divisible by \((x-1)^n\). It is easily verified that each term in the expansion of this determinant is a polynomial of degree \(n^2\), and thus we obtain

\[
|(x^{i+j-1} - 1)/(i+j-1)| = A_n (x-1)^{n^2},
\]

where \(A_n = |1/(i+j-1)|\) depends on \(n\) only. The value of \(A_n\) is given by \(A_1 = 1,\)

\[
A_n = \prod_{r=1}^{n-1} (2r+1)^{-1}(C_{2r, r})^{-2}, \quad n = 2, 3, \ldots
\]

Problem 80. [AMM, Vol. 58, p. 273-274, Prob. 4362]

Evaluate

\[
\begin{vmatrix}
1 & 1 & 1 & \cdots & 1 \\
2 & 3 & 4 & \cdots & n \\
3 & 4 & 5 & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 2 & \cdots & n
\end{vmatrix}
\]

Solution I: Subtract the \(n\)th column from the 1st, 2nd, \ldots, \((n-1)\)th columns in turn, and factor out common factors of the rows and of the columns. Now subtract the \(n\)th row from the 1st, 2nd, \ldots, \((n-1)\)th rows in turn and again factor out common factors of
the rows and of the columns. This gives the recurrence formula

\[ F(n) = \frac{[(n-1)!]^4}{(2n-1)!(2n-2)!} F(n-1), \]

where \( F(n) \) is the given determinant. Thus we obtain

\[ F(n) = \frac{[1!2!3!\ldots(n-1)!]^4}{1!2!3!\ldots(2n-1)!} \]

The same procedure will give the evaluation of a determinant whose elements \( a_{ij} \) are reciprocals of the \((i+j-1)\)th terms of any arithmetic progression. The determinant has the value

\[ F(n) = \left[1!2!3!\ldots(n-1)!\right]^2 d^{n(n-1)} \prod, \]

where \( \prod \) is the product of all the elements of the determinant.

**Solution II:** Consider the determinant \(|A|\), (the Cauchy double alternant)

\[ |A| = \left|\left(a_i - b_i\right)^{-1}\right|. \]

Clearing the fractions by multiplying each row by the continued product of the denominators of that row, we have

\[ |A| = \frac{1}{\prod_{i,j} (a_i - b_j)} \left|C\right|, \]

where the \((i,j)\)th element of \( C \) is \( \left[\prod_j (a_i - b_j)\right]/(a_i - b_j) \).

Then \( C \) is a polynomial of order \( n(n-1) \) in the \( a_i, b_j \). It
vanishes when \( a_i = a_j \) and when \( b_i = b_j \) \( (i \neq j) \). Hence it is
divisible by the two difference products \( \Delta(a_1, a_2, \ldots, a_n) \) and
\( \Delta(b_1, b_2, \ldots, b_n) \). Since their combined degree is \( n(n-1) \), the
remaining factor is numerical. To find it, put \( a_i = b_i \) when all
terms except those in the leading diagonal vanish. The continued
product of the diagonal elements gives the difference product twice.
However, with each factor \( (a_i - a_j) \) occurs also \( (a_j - a_i) \), so that
the numerical term is seen to be \( (-1)^{n(n-1)/2} \). Then

\[
|A| = \frac{|C|}{\prod_{i,j} (a_i - b_j)} = \frac{(-1)^{n(n-1)/2} \Delta(a_1, \ldots, a_n) \Delta(b_1, \ldots, b_n)}{\prod_{i,j} (a_i - b_j)}.
\]

The value of the proposed determinant may be obtained by setting
\( a_i = n+1, n+2, \ldots, 2n \) and \( b_j = n, n-1, \ldots, 2, 1 \).

**Problem 81.** [AMM, Vol. 58, p. 568-569, Prob. E955]

Evaluate the \( n \) th order determinant \( |a_{ij}| \), where

\[
a_{ij} = \frac{1}{(i+j-1)!}.
\]

**Solution:** We shall solve a slight generalization of the given
problem. Let \( I_{n,k} \) be the \( n \) th order determinant \( |a_{ij}| \), where
\( a_{ij} = 1/(i+j+2k-1)! \) for \( k = 0, 1, 2, \ldots \). Multiply the \( i \) th row
by \( i + 2k \) and subtract from the \( (i-1) \) th row for \( i = n, n-1, \ldots, 2 \).
Factor $1-j$ from the $j^{th}$ column for $j = 2, 3, ..., n$. Then

$$I_{n,k} = [(-1)^{n-1}(n-1)!/(n+2k)!] I_{n-1,k+1}$$

$$= (-1)^{n(n-1)/2} \prod_{r=1}^{n} [(r-1)!/(n+2k+r-1)!].$$

**Problem 82.** [AMM, Vol. 60, p. 552-553, Prob. E1053]

Evaluate the determinant

$$| x \quad x-1 \quad \ldots \quad x-k |$$
$$| x-1 \quad x \quad \ldots \quad x-k+1 |$$
$$| \ldots \quad \ldots \quad \ldots \quad \ldots |$$
$$| x-k \quad x-k+1 \quad \ldots \quad x |$$

**Solution:** Subtract the second row from the first, the third from the second, ..., the $(k+1)^{th}$ from the $k^{th}$. Now add the first column to the second, third, ..., $(k+1)^{th}$ to get a triangular determinant with diagonal elements $1, 2, 2, \ldots, 2, 2x-k$. Hence the value is $2^{k-1}(2x-k)$.

**Problem 83.** [AMM, Vol. 72, p. 91-92, Prob. 5165]

Let $p_k$ and $q_k$, $k = 1, 2, \ldots$ be terms of two arithmetic progressions. Evaluate the determinant whose element in the $i^{th}$ row and $j^{th}$ column is $p_{i+j}/q_{i+j}$.
Solution: The terms of the determinant have the form 
\[ [a + c(i+j)]/[b + d(i+j)] \]. If each element is multiplied by \( d/c \), the given determinant is seen to be \( (c/d)^n \) times the determinant

\[
D_n(a, \beta) = \det \begin{vmatrix} \frac{a + i + j}{\beta + i + j} \end{vmatrix}, \quad i, j = 1, 2, \ldots, n, 
\]

where \( a = a/c \) and \( \beta = b/d \).

In \( D_n \) subtract row \( n \) from row \( i \) (\( i = 1, 2, \ldots, n-1 \)). Factor \((a-\beta)(n-i)\) from row \( i \) (\( i = 1, 2, \ldots, n-1 \)). Factor 
\((\beta + n + j)^{-1}\) from column \( j \) (\( j = 1, 2, \ldots, n \)). Next subtract column \( n \) from column \( j \) (\( j = 1, 2, \ldots, n-1 \)). Factor \((n-j)\) from column \( j \) (\( j = 1, 2, \ldots, n-1 \)). Factor 
\((\beta + i + n)^{-1}\) from row \( i \) (\( i = 1, \ldots, n-1 \)). Now subtract \((a + 2n)^{-1}\) times row \( n \) from row \( i \) (\( i = 1, \ldots, n-1 \)). We now have

\[
D_n(a, \beta) = (n-1)!^2(a-\beta)^{n-1} \prod_{i=1}^{n-1} (\beta + i + n)^{-1} \prod_{j=1}^{n} (\beta + n + j)^{-1} F_n, 
\]

\[
F_n = \begin{vmatrix} (a + \beta + 2n) + i + j \cdot 0 \\
(a + 2n)(\beta + i + j) & \cdot \\
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & 0 \\
-1 \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot 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\[ D(n, \beta) = \sum_{i,j=1}^{n-1} (\beta + i j)^{-1} \prod_{k=1}^{n-1} \left[ (\beta + n-1) \delta + (n+1) \right] \]

**Circulants**

**Problem 84.** [AMM, Vol. 25, p. 257-258, Prob. 2740]

Establish the identity (the determinant is of order \( n \)):

\[
\begin{vmatrix}
\alpha & \alpha & \ldots & \alpha \\
\alpha & \alpha - \lambda & \ldots & \alpha \\
\vdots & \vdots & \ddots & \vdots \\
\alpha & \alpha & \ldots & \alpha - \lambda
\end{vmatrix}
= (-1)^{n-1} \lambda^{-1} (n \alpha - \lambda) .
\]

**Solution:** We proceed to establish this relation by mathematical induction. The relation is obviously true for \( n = 2 \). Assume it is true for a determinant of order \( n \) as written above. Multiply both sides of this equality by \( \frac{-\lambda (n \alpha + \alpha - \lambda)}{n \alpha - \lambda} \). The right hand member of the resulting expression is \( (-1)^{n-1} \lambda^{-1} [(n+1) \alpha - \lambda] \), and is of the form demanded by our theorem in the case of \( n+1 \). The left hand member of the resulting expression may be written so that its determinant is of order \( n+1 \).
Hence the \(-\lambda\) of our multiplier appears as the element of the first row and first column of the determinant of order \(n+1\).

Add the second, third, and each of the following rows in turn to the first row, obtaining a new determinant exactly the same as (1) except that each element of the first row is \(na - \lambda\). Multiply the factor preceding this determinant into the elements of the first row, obtaining a third determinant which is again exactly like (1) except that each element of the first row is \(na + a - \lambda\). Subtract the second, third, and each of the following rows from the first, thus obtaining a determinant of order \(n+1\) with precisely the form demanded by our theorem if it is to be true in the case of \(n+1\). Thus if the relation is true for \(n\), it is true for \(n+1\), and the induction is complete.
Problem 85. [AMM, Vol. 27, p. 235-236, Prob. 2774]

Evaluate the circulants

\[
\Delta = \begin{vmatrix}
1 & 2 & 3 & \ldots & n-1 & n \\
n & 1 & 2 & \ldots & n-2 & n-1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
2 & 3 & 4 & \ldots & n & 1 \\
\end{vmatrix}
\quad \text{and} \quad
\Delta' = \begin{vmatrix}
a_1 & a_2 & a_3 & \ldots & a_{n-1} & a_n \\
a_n & a_1 & a_2 & \ldots & a_{n-2} & a_{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a_2 & a_3 & a_4 & \ldots & a_n & a_1 \\
\end{vmatrix}
\]

where in the latter, \(a_1, a_2, \ldots, a_n\) form an arithmetic progression.

Solution I: Let \(S_n = \frac{n(n+1)}{2}\) be the sum of the first \(n\) positive integers. Add to the elements of the last row of \(\Delta\) the sum of the corresponding elements of all the preceding rows. We obtain

\[
\Delta = S_n \begin{vmatrix}
1 & 2 & 3 & \ldots & n-2 & n-1 & n \\
n & 1 & 2 & \ldots & n-3 & n-2 & n-1 \\
n-1 & n & 1 & \ldots & n-4 & n-3 & n-2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
4 & 5 & 6 & \ldots & 1 & 2 & 3 \\
3 & 4 & 5 & \ldots & n & 1 & 2 \\
1 & 1 & 1 & \ldots & 1 & 1 & 1 \\
\end{vmatrix}
\]

If we subtract from the elements of the first \(n-1\) columns of this determinant the corresponding elements of the last column, we obtain a determinant which we can easily reduce to order \(n-1\).

Now subtract from the first \(n-2\) rows of this new determinant the corresponding elements of the last row. We obtain, after a little
reduction, 

\[
\Delta = -S_n \begin{vmatrix} -n & -n & \cdots & -n \\ 0 & -n & \cdots & -n \\ 0 & 0 & \cdots & -n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -n \end{vmatrix} = (-n)^{n-1} \frac{n+1}{2},
\]

the determinant being of order \( n-2 \).

Now consider \( \Delta' \). Let \( a_k = a_1 + k(n-1)r \), \( k = 1, 2, \ldots, n \), and we have

\[
\Delta' = \begin{vmatrix} a_1 & a_1+r & a_1+2r & \cdots & a_1+(n-2)r & a_1+(n-1)r \\ a_1+(n-1)r & a_1 & a_1+r & \cdots & a_1+(n-3)r & a_1+(n-2)r \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ a_1+r & a_1+2r & a_1+3r & \cdots & a_1+(n-1)r & a_1 \end{vmatrix}.
\]

The same reasoning in this case will lead us to

\[
\Delta' = (-n)^{n-1} r^{n-1} (a_1 + \frac{n-1}{2} r).
\]

**Solution II:** The reduction of these special circulants can be made to depend upon the known fact that the general circulant reduces to the product \( f(w_1)f(w_2)f(w_3) \cdots f(w_n) \), where the \( w \)'s are the roots of \( x^n - 1 = 0 \), and \( f(x) = a_1 + a_2x + a_3x^2 + \cdots + a_nx^{n-1} \) (Ces\'aro, *Elementares Lehrbuch der algebraischen Analysis*, p. 25).
If we set \( a_k = a_1 + (k - 1)r \), then

\[
f(w) = a_1 (1 + w + w^2 + \ldots + w^{n-1}) + r[w + 2w^2 + 3w^3 + \ldots + (n-1)w^{n-1}],
\]

\[
= 0 + \frac{r n}{w-1}, \text{ if } w \neq 1
\]

\[
= a_1 n + \frac{r(n-1)n}{2} = n[a_1 + \frac{n-1}{2} r], \text{ if } w = w_1 = 1,
\]

whence

\[
\Delta' = \frac{r^{n-1} n \left[ a_1 + \frac{n-1}{2} r \right]}{(w_2 - 1)(w_3 - 1) \ldots (w_{n-1})} = (-1)^{n-1} r^{n-1} n^{n-1} \left[ a_1 + \frac{n-1}{2} r \right].
\]

**Problem 86.** [AMM, Vol. 45, p. 50, Prob. E280]

Prove that the determinant

\[
\begin{vmatrix}
a & a+d & a+2d & \ldots & a+(n-1)d \\
a+d & a+2d & a+3d & \ldots & a \\
a+2d & a+3d & a+4d & \ldots & a+d \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a+(n-1)d & a & a+d & \ldots & a+(n-2)d
\end{vmatrix}
\]

has the value \( d^{n-1} 2^n S_n (-1)^{n(n-1)/2} \), where \( S_n \) is the sum of the elements forming the arithmetic progression in the first column.

**Solution:** The determinant \( D \), whose value is to be found, is a circulant; and as such has the value

\[
D = (-1)^{n(n-1)/2} b_1 b_2 \ldots b_n,
\]

where \( b_i = a + (a+d)w_i + (a+2d)w_i^2 + \ldots + [a + (n-1)d]w_i^{n-1} \), and \( w_1 = 1, w_2, w_3, \ldots, w_n \) are the \( n \)th roots of unity [See Muir,
The Theory of Determinants, Vol. 2, p. 406]. But \( b_1 = S_n \), and if \( i \neq 1 \), we have

\[
b_i = w_i d \left[ 1 + 2w_i + 3w_i^2 + \ldots + (n-1)w_i^{n-2} \right] + (a + aw_i + aw_i^2 + \ldots + aw_i^{n-1})
\]

\[
= w_i d \left[ 1 + 2w_i + 3w_i^2 + \ldots + (n-1)w_i^{n-2} \right]
\]

\[
= w_i d \left[ \frac{1-nw_i^{n-1} + (n-1)w_i^n}{(1-w_i)^2} \right]
\]

\[
= \frac{dn(w_i - 1)}{(1 - w_i)^2} = - \frac{nd}{1-w_i}.
\]

Since \( \prod_{i=2}^{n} (1 - w_i) = n \), we have

\[
b_1 b_2 b_3 \ldots b_n = S_n (-1)^{n(n-1)/2} d^{n-1} n^{n-2} S_n,
\]

and therefore,

\[
D = (-1)^{n(n-1)/2} d^{n-1} n^{n-2} S_n.
\]

Problem 87. [AMM, Vol. 50, p. 271-273, Prob. 3994]

If \( a_{11} = a_{22} = a_{33} = \sum_{k} \binom{k}{j} (n-1)^{k-j}, j \equiv 0 \pmod{3}; \)

\( a_{12} = a_{23} = a_{31} = \sum_{k} \binom{k}{j} (n-1)^{k-j}, j \equiv 1 \pmod{3}; \)

\( a_{13} = a_{32} = a_{21} = \sum_{k} \binom{k}{j} (n-1)^{k-j}, j \equiv 2 \pmod{3}; \)
Show that the determinant $|a_{ij}| = [(n-1)^3 + 1]^k$.

**Solution:** The determinant $|a_{ij}|$ is a circulant and as such has value

$$
(1) \quad \frac{3}{\prod_{i=1}^{3}} (a_{11}^i + w_i a_{12} + w_i^2 a_{13}),
$$

where $w_1, w_2, w_3$ are the cube roots of unity. Moreover,

$$
(x^3 + 1)^k = \frac{3}{\prod_{i=1}^{3}} (x + w_i)^k = \frac{3}{\prod_{i=1}^{3}} \left[ \sum_{j=1}^{3} w_j^i (x + w_i)^k x^{k-j} \right] = \frac{3}{\prod_{i=1}^{3}} (a_{11}^i + w_i a_{12} + w_i^2 a_{13}), \quad \text{and if } x = n-1, \\
= |a_{ij}| \quad \text{by (1)}.
$$

The proof of the following generalization is exactly similar to the above. Let $|a_{rs}|$ be circulant of odd order $n$, where

$$
(2) \quad a_{rs} = \sum_{j} \binom{k}{j} x^{k-j}, \quad j \equiv s-r \pmod{n}.
$$

Then $|a_{rs}| = (x^n + 1)^k$. The corresponding theorem for $n$ even or odd is the following: Let $|b_{rs}|$ be a circulant of order $n$ where

$$
(3) \quad b_{rs} = \sum_{j} (-1)^j \binom{k}{j} y^{k-j}, \quad j \equiv s-r \pmod{n}.
$$
Then \( |b_{rs}| = (y^n - 1)^k \).

If \( y = -x \) and \( n \) is odd, it follows from (2) and (3) that \( b_{rs} = (-1)^k a_{rs} \); and therefore that \( |b_{rs}| = (-1)^k |a_{rs}| \). Since \( (y^n - 1)^k = (-1)^k (x^n + 1)^k \), the theorem for odd \( n \) may be deduced from this last one.

Problem 88. [AMM, Vol. 64, p. 748-749, Prob. 4719]

Let \( p \) be a prime \( > 2 \). Show that the determinant of order \( p - 1 \),

\[ \Delta_p = \left| \left( \frac{r-s}{p} \right) \right| (r, s = 0, 1, \ldots, p-2), \]

where \( \left( \frac{r}{p} \right) \) is the Legendre symbol, satisfies \( \Delta_p = p^{(p-3)/2} \).

Solution: The well-known formula for a circulant

\[ \left| \begin{array}{cccc}
    x_r & x_{r+1} & \cdots & x_{r+p-1} \\
    x_{r+1} & x_r & \cdots & x_{r+p-2} \\
    \vdots & \vdots & \ddots & \vdots \\
    x_{r+p-2} & x_{r+p-3} & \cdots & x_r \\
\end{array} \right| = p^{-1} \sum_{r=0}^{p-1} \sum_{s=0}^{p-1} \epsilon^{rs} x_r \]  

yields, when \( \sum x_r = 0 \),

\[ (1) \left| \begin{array}{cccc}
    x_r & x_{r+1} & \cdots & x_{r+p-1} \\
    x_{r+1} & x_r & \cdots & x_{r+p-2} \\
    \vdots & \vdots & \ddots & \vdots \\
    x_{r+p-2} & x_{r+p-3} & \cdots & x_r \\
\end{array} \right| = \frac{1}{p} \sum_{s=1}^{p-1} \sum_{r=0}^{p-1} \epsilon^{rs} x_r. \]

Now take \( x_r = \left( \frac{r}{p} \right) \); then (1) becomes
Next we recall that the Gauss sum \( G(s) \) satisfies

\[
G(s) = \prod_{r=1}^{p-1} \left( \frac{r}{p} \right) e^{rs}, \quad G(s) = \left( \frac{s}{p} \right) G(1), \quad G^2(s) = \left( \frac{-1}{p} \right) p.
\]

Thus (2) gives

\[
\Delta_p = \frac{1}{p} \sum_{r=1}^{p-1} \left( \frac{r}{p} \right) e^{rs}.
\]

\[
= \frac{1}{p} \left( -1 \right)^{(p-1)/2} \left[ \left( \frac{-1}{p} \right) p \right]^{(p-1)/2}
\]

\[
= \frac{1}{p} \left( -1 \right)^{(p-1)/2} \left( \frac{-1}{p} \right) p^{(p-1)/2} = p^{(p-3)/2}.
\]

Problem 89. [AMM, Vol. 65, p. 44-45, Prob. E1268]

Evaluate the determinant \( D_n \) which has \( (1, 2, \ldots, n) \) as first row, \( (2, 3, \ldots, n, 1) \) as second row, etc.

Solution: It is just as easy to evaluate the determinant which has \( (a_1, a+d, \ldots, a+(n-1)d) \) as first row, \( (a+d, a+2d, \ldots, a) \) as second row, etc. Take \( n > 2 \). By adding all subsequent rows to the first row, we obtain \( na + n(n-1)d/2 \) as a factor. Removing this and performing the column operations \( \text{col } r \to \text{col } n, \)
r=1, 2, ..., n-1, we obtain $d^{n-1}$ as a factor. Now we can readily reduce the order of the determinant by one. In the reduced determinant, we perform the column operations $\text{col } r + (n-r) \text{ col } 1$, $r = 2, ..., n-1$, to obtain a determinant having only zeros below the secondary diagonal, only ones in the first column, and all remaining elements equal to $n$. This is readily evaluated so as to yield for a final result $(-1)^{n(n-1)/2}(nd)^{n-1} [a+(n-1)d/2]$, which proves to be true also for $n = 1$, $n = 2$, and which for $a = d = 1$, becomes $(-1)^{n(n-1)/2} n^{-1} (n+1)/2$. Such determinants are discussed by Muir, 


Problem 90. [DMVJ, Vol. 41, Pt. 2, p. 36-40]

Let $A_1, A_2, ..., A_p$ be square matrices each with $n^2$ elements, and let $a$ be arbitrary. From these matrices, with the help of $a$, we can construct the following determinant $\Delta$ of degree $pn$:

$$
\Delta = \begin{vmatrix}
A_1 & A_2 & A_3 & \cdots & A_{p-1} & A_p \\
aA_p & A_1 & A_2 & \cdots & A_{p-2} & A_{p-1} \\
& \cdots & \cdots & \cdots & \cdots & \cdots \\
aA_2 & aA_3 & aA_4 & \cdots & aA_{p} & A_1
\end{vmatrix}
$$

Further, let $a_1, ..., a_p$ be the $p$ roots of the equation
\[
x^p - a = 0, \quad \text{and, for arbitrary } x, \\
M(x) = A_1 + xA_2 + \ldots + x^{p-1}A_p.
\]

Show that 
\[
\Delta = |M(a_1)| \cdot |M(a_2)| \ldots |M(a_p)|,
\]

where \( |M(a_k)| \) is the \( n \)th order determinant of the matrix \( M(a_k) \).

Solution I: For \( a = 0 \), the truth of the proposition is evident.

For \( a=1 \), we find the solution in Muir and Metzler, _A Treatise on the Theory of Determinants_, p. 487. The method presented there can be carried out without difficulty for an arbitrary \( a \neq 0 \), although the following procedure is probably simplest.

Let \( E \) be the \( n \)th order identity matrix, and let

\[
V = \begin{vmatrix}
1 & 1 & \ldots & 1 \\
\alpha_1 & \alpha_2 & \ldots & \alpha_p \\
\ldots & \ldots & \ldots & \ldots \\
\alpha_1^{p-1} & \alpha_2^{p-1} & \ldots & \alpha_p^{p-1}
\end{vmatrix},
\]

the Vandermonde matrix of the values \( \alpha_1, \alpha_2, \ldots, \alpha_p \). We now construct the Kronecker product

\[
W = V \times E = \begin{vmatrix}
E & E & \ldots & E \\
\alpha_1E & \alpha_2E & \ldots & \alpha_pE \\
\ldots & \ldots & \ldots & \ldots \\
\alpha_1^{p-1}E & \alpha_2^{p-1}E & \ldots & \alpha_p^{p-1}E
\end{vmatrix},
\]
and set \( A = |W|, \ B = |V| \). Thus,

\[
(1) \quad A = B^n.
\]

By suitable changes of the rows and columns, we can change \( V \times E \) into the form \( E \times V \).

Then

\[
\Delta \cdot A = \begin{vmatrix}
M(a_1) & M(a_2) & \ldots & M(a_p) \\
a_1 M(a_1) & a_2 M(a_2) & \ldots & a_p M(a_p) \\
\vdots & \vdots & \ddots & \vdots \\
a_1^{p-1} M(a_1) & a_2^{p-1} M(a_2) & \ldots & a_p^{p-1} M(a_p)
\end{vmatrix}
\]

\[
= A \begin{vmatrix}
M(a_1) & 0 & \ldots & 0 \\
0 & M(a_2) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & M(a_p)
\end{vmatrix}
\]

Hence,

\[
\Delta \cdot A = A \cdot \prod_{v=1}^{P} |M(a_v)|.
\]

For \( a \neq 0 \), the equation \( x^P - a = 0 \) has distinct roots, so \( B \neq 0 \).

By (1), \( A \neq 0 \) also. Thus we have the desired result,

\[
\Delta = \prod_{v=1}^{P} |M(a_v)|.
\]
Solution II: Let \( \alpha \) be an arbitrary root of the equation \( x^p - a = 0 \). Multiply the \((n+1)^{st}\) column by \( \alpha \), the \((2n+1)^{st}\) by \( \alpha^2 \), ..., the \([(p-1)n+1]\) by \( \alpha^{p-1} \), and add all resulting products to the first column. Likewise, multiply the \((n+2)^{nd}\) column by \( \alpha \), the \((2n+2)^{nd}\) by \( \alpha^2 \), ..., the \([(p-1)n+2]\) by \( \alpha^{p-1} \), and add the resulting products to the second column. Continuing in such a manner, we finally multiply the \(2n^{th}\) column by \( \alpha \), the \(3n^{th}\) column by \( \alpha^2 \), ..., the \(pn^{th}\) column by \( \alpha^{p-1} \), and add the resulting products to the \(n^{th}\) column. We see immediately that \( \Delta \) has the following form:

\[
\Delta = \begin{vmatrix}
M(\alpha) & \ldots & \ldots \\
\alpha M(\alpha) & \ldots & \ldots \\
\alpha^2 M(\alpha) & \ldots & \ldots \\
\ldots & \ldots & \ldots \\
\alpha^{p-1} M(\alpha) & \ldots & \ldots 
\end{vmatrix}
\]

Expanding by means of Laplace's formula according to the first \( n \) columns, or more simply, by similar changes on the rows, by which we reduce the determinant to one in which the elements of the first \( n \) columns and rows are the same as those of the matrix \( M(\alpha) \) (while all other elements of the same columns vanish), we see that \( \Delta \) is divisible by \( |M(\alpha)| \). Consequently, \( \Delta \) is divisible by the product \( \Delta_1 = |M(\alpha)| \cdot |M(\alpha_2)| \ldots |M(\alpha_p)| \).
Considering $\Delta$ as a complete rational function of independent variables, it follows immediately that $\Delta$ and $\Delta_1$ differ at most by a constant. By a comparison of leading terms, this is seen to be 1.

The most general case of this problem can always be reduced easily to the case $a=1$ by multiplication of the rows and columns by powers of one of the roots of $x^p-a=0$.

Two other solutions are given. Two subsequent solutions were also published [DMVJ, Vol. 42, Pt. 2, p. 6-12].

**Problem 91.** [SSM, Vol. 44, p. 480-481, Prob. 1864]

If $a+b+c = 0$, show that

\[
\begin{vmatrix}
bc & ca & ab \\
ab & bc & ca \\
ca & ab & bc
\end{vmatrix}
\]

is a perfect cube.

Solution: Let

\[
(1)\ N = \begin{vmatrix}
a & c \\
-a-b & b
\end{vmatrix} = \begin{vmatrix}
b & a \\
-b-c & c
\end{vmatrix} = \begin{vmatrix}
c & b \\
-c-a & a
\end{vmatrix} = ab+bc+ca.
\]

Using the given relation, $a+b+c = 0$, replace the binomial in each determinant by its equal, and

\[
(2)\ N = \begin{vmatrix}
a & c \\
c & b
\end{vmatrix} = \begin{vmatrix}
b & a \\
a & c
\end{vmatrix} = \begin{vmatrix}
c & b \\
a & c
\end{vmatrix} = ab+bc+ca.
\]
Let the given determinant be $D$. Add the second and third rows of $D$ to the first row, replace $ab+bc+ca$ by $N$, and

$$D = \begin{vmatrix} N & N & N \\ ab & bc & ca \\ ca & ab & bc \end{vmatrix}.$$  

(3)

Expand (3) by minors of the first row, remove the common factors from the columns of each minor, and

$$D = Nbc \begin{vmatrix} c & a \\ a & b \end{vmatrix} + Nca \begin{vmatrix} a & b \\ b & c \end{vmatrix} + Nab \begin{vmatrix} b & c \\ c & a \end{vmatrix}.$$  

(4)

Replace each determinant of (4) by $N$, and

$$D = N^2 (bc + ca + ab).$$  

(5)

Replace $bc + ca + ab$ by $N$, and

$$D = N^3.$$  

(6)

Problem 92. [SSM, Vol. 45, p. 581-582, Prob. 1925]

Resolve into linear factors, not necessarily real, the determinant

$$D = \begin{vmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{vmatrix}.$$
Solution: 1. By adding together all rows, we see that 
\((a+b+c+d)\) is a factor. 2. By adding together the first and third rows and subtracting from the result the sum of the second and fourth rows, we see that \((a-b+c-d)\) is also a factor. 3. Expanding \(D\) by minors and dividing this expansion by 
\((a+b+c+d)(a-b+c-d)\) gives \(a^2-2ac+c^2+b^2-2bd+d^2=(a-c)^2+(b-d)^2\).

4. Factoring, we get \((a+ib-c-id)(a-ib-c+id)\). 5. Hence the linear factors of \(D\) are 
\((a+b+c+d)(a-b+c-d)(a+ib-c-id)(a-ib-c+id)\).

**Continuants**

**Problem 93.** [AMM, Vol. 12, p. 134, Prob. 230 (Alg)]

Find the value of the \(n^{th}\) order determinant

\[
\begin{vmatrix}
5 & 2 & 0 & 0 & 0 & \ldots \\
2 & 5 & 2 & 0 & 0 & \ldots \\
0 & 2 & 5 & 2 & 0 & \ldots \\
0 & 0 & 2 & 5 & 2 & \ldots \\
0 & 0 & 0 & 2 & 5 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{vmatrix}
\]

**Solution:** Denote the determinant by \(u_n\). Then 
\[
u_n = 5u_{n-1} - 4u_{n-2}.
\]
Let \(S = u_1 + u_2x + u_3x^2 + \ldots\). Then
\[(1 - 5x + 4x^2)S = u_1 + (u_2 - 5u_1)x = 5 - 4x.\]

\[S = \frac{5 - 4x}{1 - 5x + 4x^2} = \frac{1}{3} \left[ \frac{16}{1 - 4x} - \frac{1}{1 - x} \right].\]

Then \( u_n \) equals the coefficient of \( x^{n-1} \) in \( S \); i.e.,

\[u_n = \frac{1}{3} \left[ 4^{n+1} - 1 \right].\]

**Problem 94.** [AMM, Vol. 18, p. 61-63, Prob. 349 (Alg)]

Show that the determinant of the \( n \)th order,

\[
\begin{vmatrix}
C & -1 & 0 & 0 & 0 & \ldots \\
-1 & C & -1 & 0 & 0 & \ldots \\
0 & -1 & C & -1 & 0 & \ldots \\
0 & 0 & -1 & C & -1 & \ldots \\
0 & 0 & 0 & -1 & C & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{vmatrix}
\]

has the value

\[D_n = C^n + \sum_{r=1}^{n} (-1)^r \frac{(n-r)(n-r-1)\ldots(n-2r+1)}{r!} C^{n-2r}.\]

**Solution I:** If the determinant is expanded by elements of the first row or column, we get \( D_n = C \cdot D_{n-1} - D_{n-2} \) where \( D_{n-1} \) and \( D_{n-2} \) are determinants similar to \( D_n \) but having, respectively, one and two fewer rows and columns. Compare
Put \( a = n\theta \) and \( b = \theta \). Then (2) becomes
\[
\sin(n+1)\theta + \sin(n-1)\theta = 2 \sin n\theta \cos \theta.
\]

Now put
\[
(3) \quad C = 2 \cos \theta
\]
and \( D_n = c \sin(n+1)\theta \), where \( c \) is independent of \( n \). To determine its value, we note for \( n = 1 \), \( D_1 = C = 2 \cos \theta = \frac{\sin 2\theta}{\sin \theta} \).

\[
\therefore \quad (4) \quad c = \csc \theta.
\]

As a result of (3) and (4) we can write \( D_n = \frac{\sin(n+1)\theta}{\sin \theta} \).

But from Crystal's *Algebra*, Vol. 2, p. 252, we have
\[
\frac{\sin(n+1)\theta}{\sin \theta} = (2 \cos \theta)^2
\]
\[
\sum_{r=1}^{n} (-1)^{r-1} \frac{(n-r)(n-r-1) \ldots (n-2r+1)}{r!} (2 \cos \theta)^{n-2r}.
\]

Using (5) and (3), the desired result is obtained.

**Solution II:** Expanding in terms of the first column, we have
the following relation connecting three determinants of the kind
here considered whose orders are \( n, n-1, n-2 \): \( D_n = C D_{n-1} - D_{n-2} \).

Forming some of the successive values of \( D \), we find:
\[D_1 = C\]
\[D_2 = C^2 - 1\]
\[D_3 = C^3 - 2C\]
\[D_4 = C^4 = 3C^2 + 1\]
\[D_5 = C^5 - 4C^3 + 3C\]
\[D_6 = C^6 - 5C^4 + 6C^2 - 1\]
\[D_7 = C^7 = 6C^5 + 10C^3 - 4C\]
\[D_8 = C^8 - 7C^6 + 15C^4 - 10C^2 + 1.\]

It is clear that starting with \(D_n\) and reading the terms diagonally, we have the expansion of \((C-1)^n\). For instance, starting with \(D_4\), we have \(C^4 - 4C^3 + 6C^2 - 4C + 1\). Hence reading horizontally the 1, 2, 3, 4, etc. terms of \(D_n\) will be the 1, 2, 3, 4, etc. terms in the expansion of \((C-1)^n\), \((C-1)^{n-1}\), \((C-1)^{n-2}\), \((C-1)^{n-3}\), etc., respectively. The \(r^{th}\) term will be the \(r^{th}\) term in the expansion of \((C-1)^{n-r+1}\). Hence

\[D_n = C_n - (n-1)C^{n-2} + \frac{(n-2)(n-3)}{2!} C^{n-4} - \frac{(n-2)(n-3)(n-4)}{3!} C^{n-6} + \ldots\]

\[= C^n + \sum_{r=1}^{n} (-1)^r \frac{(n-4)(n-r-1)\ldots(n-2r+1)}{r!} C^{n-2r}.\]
Problem 95. [AMM, Vol. 52, p. 343-344, Prob. E648]

Show that, when \( x = 2 \cos \frac{\pi}{(n+1)} \), the \( n \)-rowed determinant

\[
\begin{vmatrix}
  x & 1 & 0 & 0 & \ldots & 0 & 0 \\
  1 & x & 1 & 0 & \ldots & 0 & 0 \\
  0 & 1 & x & 1 & \ldots & 0 & 0 \\
  0 & 0 & 1 & x & \ldots & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & 0 & \ldots & x & 1 \\
  0 & 0 & 0 & 0 & \ldots & 1 & x \\
\end{vmatrix} = 0.
\]

**Solution I:** Set \( x = p + p^{-1} \), and let \( P_n \) denote the \( n \)-rowed determinant. Then \( P_1 = x = p + p^{-1} \), \( P_2 = x^2 - 1 = p^2 + 1 + p^{-2} \). Since \( P_n = xP_{n-1} - P_{n-2} \) and \( p^{n+1} - p^{-n+1} = (p + p^{-1})(p^n - p^{-n}) - (p^{n-1} - p^{-n+1}) \), we can prove by induction that

\[
P_n = p^n + p^{n-2} + \ldots + p^{-n+2} + p^{-n} = (p^{n+1} - p^{-n-1})/(p-p^{-1}).
\]

Set \( p = e^{i\theta} \), so that \( x = 2 \cos \theta \). Then \( P_n = \frac{\sin(n+1)\theta}{\sin\theta} \), which vanishes when \( \theta = \pi/(n+1) \).

**Solution II:** If \( x = 2 \cos \theta \), we have (identically)

\[
x \sin \theta - \sin 2\theta = 0
\]

\[
\sin \theta - x \sin 2\theta + \sin 3\theta - \sin 2\theta + x \sin 3\theta - \sin 4\theta = 0
\]

\[
\pm \sin(n-1)\theta + x \sin n\theta \pm \sin(n+1)\theta = 0.
\]

If \( \theta = \pi/(n+1) \), the last term of the last equation vanishes, so we
can formally eliminate \( \sin \theta \): \( \sin 2\theta ; \ldots ; \sin n\theta \) from the \( n \) equations (with that term omitted), obtaining the desired result immediately.

Problem 96. [AMM, Vol. 66, p. 593-594, Prob. E1349]

Consider the \( n \times n \) matrix \( \mathbf{A} \) where \( a_{11} = \cos \theta \),
\[ a_{ii} = 2 \cos \theta \quad (i = 2, \ldots, n), \quad a_{i,i+1} = a_{i+1,i} = 1 \quad (i = 1, \ldots, n-1), \]
and all other elements are zero. Show that \( |a_{ij}| = \cos^n \theta \).

Solution: Denote the determinant by \( D_n \). Clearly, \( D_1 = \cos \theta \) and \( D_2 = \cos 2\theta \). Assume \( D_{k-1} = \cos (k-1)\theta \) and \( D_k = \cos k\theta \), and expand \( D_{k+1} \) by minors with respect to the last row to obtain

\[
D_{k+1} = 2D_k \cos \theta - D_{k-1} = 2 \cos \theta \cos k\theta - \cos (k-1)\theta
= \cos \theta \cos k\theta - \sin \theta \sin k\theta = \cos (k+1)\theta .
\]

The desired result now follows by mathematical induction.

Problem 97. [AMM, Vol. 67, p. 191, from Prob. 4845]

Show that the \( m \)th order determinant
\[
\begin{vmatrix}
2 \cos \theta & -1 & 0 & \ldots & 0 \\
-1 & 2 \cos \theta & -1 & \ldots & 0 \\
0 & -1 & 2 \cos \theta & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 2 \cos \theta
\end{vmatrix} = \frac{\sin (m+1)\theta}{\sin \theta} .
\]
Solution: Denote the $m$th order determinant by $D_m$. Expanding by minors of the first row or first column, it is clear that

$$D_m = 2 \cos \theta D_{m-1} - D_{m-2}.$$  
We proceed by induction. Suppose

$$D_k = \frac{\sin(k+1)\theta}{\sin \theta} \quad \text{for} \quad k = 1, 2, \ldots, m-1.$$

Then

$$D_m = 2 \cos \theta \frac{\sin m\theta}{\sin \theta} - \frac{\sin(m-1)\theta}{\sin \theta}$$

$$= \frac{2 \cos \theta \sin m\theta - \sin(m-1)\theta}{\sin \theta}.$$

From elementary trigonometry we know that

$$2 \sin a \cos b = \sin(a+b) + \sin(a-b).$$

Setting $a = m\theta$, $b = \theta$, we have

$$2 \cos \theta \sin m\theta = \sin(m+1)\theta + \sin(m-1)\theta.$$

It follows that

$$D_m = \frac{\sin(m+1)\theta}{\sin \theta}.\quad \text{Since}\quad D_1 = \frac{\sin 2\theta}{\sin \theta},$$

the induction is complete.

Problem 98. [AMM, Vol. 73, p. 201-202, Prob. E1754]

Let $p + q = 1$, and let $a_m$ denote the value of the $m$th order determinant

$$
\begin{vmatrix}
1 & -q & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
-p & 1 & -q & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & -p & 1 & -q & 0 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \ldots & -p & 1 & -q \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & -p & 1
\end{vmatrix}
$$
with \( a_0 = a_1 = 1 \). Show that \( a_m = \sum_{k=0}^{m} p^{m-k} q^k \).

Solution: It is known that the \( n \)th order continuant

\[
A_n = \det (a_{ij}), \quad \text{with} \quad a_{ii} = c, \quad a_{i,i+1} = a, \quad a_{i-1,i} = b, \quad \text{and} \quad a_{ij} = 0 \quad \text{otherwise},
\]

has value

\[
\frac{(n+1)c^n}{2^{n+1}} \quad \text{when} \quad c^2 = 4ab, \quad \text{and} \quad \frac{(c+\sqrt{c^2-4ab})^{n+1}(c-\sqrt{c^2-4ab})^{n+1}}{2^{n+1}\sqrt{c^2-4ab}}
\]

if \( c^2 \neq 4ab \). To see this, we expand \( A_n \) by minors according to elements of the last row or column to get

\[
A_n = cA_{n-1} - abA_{n-2}
\]

for \( n = 2, 3, 4, \ldots \), and then apply the usual techniques for solving second order linear difference equations. The required result follows upon taking \( c = 1, b = -p, \) and \( a = p - 1 = -q \).

Equations and Identities


The product of two numbers, each the sum of four squares, is the sum of eight squares.
Solution: \((a^2 + b^2 + c^2 + d^2)(a^2 + \beta^2 + \gamma^2 + \delta^2)\)

\[= [(a+bi)(a-bi) - (c+di)(-c-di)] [(a+\beta i)(a-\beta i) - (\gamma + di)(-\gamma - di)] \]

\[
\begin{vmatrix}
    a + b\sqrt{-1} & -c + d\sqrt{-1} & \times & a + \beta\sqrt{-1} & -\gamma + \delta\sqrt{-1} \\
    c + d\sqrt{-1} & a - b\sqrt{-1} & & \gamma + \delta\sqrt{-1} & a - \beta\sqrt{-1}
\end{vmatrix}
\]

\[
\begin{vmatrix}
    a + b\sqrt{-1} & -c + \sqrt{-1} & 0 & a + \beta\sqrt{-1} & 0 & -\gamma + \delta\sqrt{-1} \\
    c + d\sqrt{-1} & a - \sqrt{-1} & 0 & \times & (-1) & \gamma + \delta\sqrt{-1} & 0 & a - \beta\sqrt{-1} \\
    0 & 0 & 1 & 0 & 1 & 0
\end{vmatrix}
\]

\[
\begin{vmatrix}
    a^2 - b^2 + (a\beta + ba)\sqrt{-1} & a\gamma - b\delta + (a\delta + b\gamma)\sqrt{-1} & -c + d\sqrt{-1} \\
    c\alpha - d\beta + (c\beta + da)\sqrt{-1} & c\gamma - d\delta + (c\delta + d\gamma)\sqrt{-1} & a - b\sqrt{-1} \\
    -\gamma + \delta\sqrt{-1} & a - \beta\sqrt{-1} & 0
\end{vmatrix}
\]

\[
\begin{vmatrix}
    c\alpha - d\beta + (c\beta + da)\sqrt{-1} & c\gamma - d\delta + (c\delta + d\gamma)\sqrt{-1} \\
    -c\gamma + d\delta + (c\delta + d\gamma)\sqrt{-1} & c\alpha - d\beta - (c\beta + da)\sqrt{-1}
\end{vmatrix}
\]

\[
\begin{vmatrix}
    a^2 - b^2 + (a\beta + ba)\sqrt{-1} & a\gamma - b\delta + (a\delta + b\gamma)\sqrt{-1} \\
    - a\gamma + b\delta + (a\delta + b\gamma)\sqrt{-1} & a\alpha - b\beta - (a\beta + ba)\sqrt{-1}
\end{vmatrix}
\]

\[
= (ca - db)^2 + (c\beta + da)^2 + (c\gamma - d\delta)^2 + (c\delta + d\gamma)^2
\]

\[
+ (a\alpha - b\beta)^2 + (a\beta + ba)^2 + (a\gamma - b\delta)^2 + (a\delta + b\gamma)^2.
\]

Euler's Theorem is an easy corollary of this result, and conversely.
Problem 100. [AMM, Vol. 9, p. 268-269, Prob. 116 (Misc)]

Prove that

| a b c a_1 | a b c a_1 | a b c a_1 |
| b d e a_2 | b d e a_2 | b d e a_2 |
| c e f a_3 | c e f a_3 | c e f a_3 |
| a_1 a_2 a_3 0 | a_2 a_3 0 | a_1 a_2 a_3 0 |
| \beta_1 \beta_2 \beta_3 0 | \beta_1 \beta_2 \beta_3 0 | \gamma_1 \gamma_2 \gamma_3 0 |

\[
-|a_1 \beta_2 \gamma_3|^2 = \begin{vmatrix} a b c \beta_1 \beta_2 \beta_3 \gamma_1 \gamma_2 \gamma_3 \\ a b c \beta_1 \beta_2 \beta_3 \gamma_1 \gamma_2 \gamma_3 \\ b d e \beta_2 \gamma_2 \gamma_3 \\ b d e \beta_2 \gamma_2 \gamma_3 \\ c e f \beta_3 \gamma_3 \\ c e f \beta_3 \gamma_3 \\ a_1 a_2 a_3 0 \end{vmatrix}
\]

Solution: Let \( x, y, z \) be the minors with respect to \( a_1, a_2, a_3 \) in the first row; \( u, v, w \), the minors with respect to \( \beta_1, \beta_2, \beta_3 \) in the second row; and \( r, s, t \) the minors with respect to \( \gamma_1, \gamma_2, \gamma_3 \) in the third row of the determinant \( \Delta \) on the right hand side of the equation. Then
Let \( A, B, C, \text{ etc.} \), be minors with respect to \( a, b, c, \text{ etc.} \). Then

\[
\begin{vmatrix}
a_{1} x - a_{2} y + a_{3} z \\
\beta_{1} x - \beta_{2} y + \beta_{3} z \\
\gamma_{1} x - \gamma_{2} y + \gamma_{3} z
\end{vmatrix}
\]

\[
\Delta = - \begin{vmatrix}
a_{1} u - a_{2} v + a_{3} w \\
\beta_{1} u - \beta_{2} v + \beta_{3} w \\
\gamma_{1} u - \gamma_{2} v + \gamma_{3} w
\end{vmatrix}
\]

\[
\begin{vmatrix}
a_{1} r - a_{2} s + a_{3} t \\
\beta_{1} r - \beta_{2} s + \beta_{3} t \\
\gamma_{1} r - \gamma_{2} s + \gamma_{3} t
\end{vmatrix}
\]

\[
\begin{vmatrix}
a_{1} x y z \\
\beta_{1} u v w \\
\gamma_{1} r s t
\end{vmatrix} = (a_{1} \beta_{2} \gamma_{3}) \begin{vmatrix}
x y z \\
u v w \\ s t
\end{vmatrix}
\]

\[
\begin{vmatrix}
| b c a_{1} \\
| d e a_{2} \\
| e f a_{3}
\end{vmatrix} = (a_{1} \beta_{2} \gamma_{3}) \begin{vmatrix}
| b c \beta_{1} \\
| d e \beta_{2} \\
| e f \beta_{3}
\end{vmatrix}.
\]

\[
\begin{vmatrix}
| b c \gamma_{1} \\
| d e \gamma_{2} \\
| e f \gamma_{3}
\end{vmatrix} = (a_{1} \beta_{2} \gamma_{3}) \begin{vmatrix}
| b c \gamma_{1} \\
| d e \gamma_{2} \\
| e f \gamma_{3}
\end{vmatrix}.
\]

\[
\begin{vmatrix}
a_{1} x - a_{2} y + a_{3} z \\
\beta_{1} x - \beta_{2} y + \beta_{3} z \\
\gamma_{1} x - \gamma_{2} y + \gamma_{3} z
\end{vmatrix}
\]

\[
\Delta = - \begin{vmatrix}
a_{1} u - a_{2} v + a_{3} w \\
\beta_{1} u - \beta_{2} v + \beta_{3} w \\
\gamma_{1} u - \gamma_{2} v + \gamma_{3} w
\end{vmatrix}
\]

\[
\begin{vmatrix}
a_{1} r - a_{2} s + a_{3} t \\
\beta_{1} r - \beta_{2} s + \beta_{3} t \\
\gamma_{1} r - \gamma_{2} s + \gamma_{3} t
\end{vmatrix}
\]

Let \( A, B, C, \text{ etc.} \), be minors with respect to \( a, b, c, \text{ etc.} \). Then

\[
\begin{vmatrix}
a_{1} x y z \\
\beta_{1} u v w \\
\gamma_{1} r s t
\end{vmatrix} = (a_{1} \beta_{2} \gamma_{3}) \begin{vmatrix}
x y z \\
u v w \\ s t
\end{vmatrix}
\]

\[
\begin{vmatrix}
| b c a_{1} \\
| d e a_{2} \\
| e f a_{3}
\end{vmatrix} = (a_{1} \beta_{2} \gamma_{3}) \begin{vmatrix}
| b c \beta_{1} \\
| d e \beta_{2} \\
| e f \beta_{3}
\end{vmatrix}.
\]

\[
\begin{vmatrix}
| b c \gamma_{1} \\
| d e \gamma_{2} \\
| e f \gamma_{3}
\end{vmatrix} = (a_{1} \beta_{2} \gamma_{3}) \begin{vmatrix}
| b c \gamma_{1} \\
| d e \gamma_{2} \\
| e f \gamma_{3}
\end{vmatrix}.
\]
\[ \Delta = (a_1 b_2 c_3)^2 \]

Problem 101. [AMM, Vol. 16, p. 31, Prob. 306 (Alg)]

Prove directly without finding the terms of both determinants:

\[
\begin{vmatrix}
1 & a & a^2 \\
1 & b & b^2 \\
1 & c & cc' \\
1 & d & dd'
\end{vmatrix}
= (a-b) \begin{vmatrix}
1 & ab & a+b \\
1 & cd' & c+d' \\
1 & c'd & c'+d
\end{vmatrix}.
\]

Solution: Subtracting the first row from each of the others:

\[
\begin{vmatrix}
1 & a & a^2 \\
0 & b-a & b^2-a^2 \\
0 & c-a & cc'-a^2 \\
0 & d-a & dd'-a^2
\end{vmatrix}
= (a-b) \begin{vmatrix}
1 & l & b+a \\
1 & d-a & dd'-a^2 \\
1 & c-a & cc'-a^2
\end{vmatrix}
= \Delta.
\]
Multiply the first row by \( a \) and add it to the next two rows. Then

\[
\begin{vmatrix}
1 & 1 & b + a \\
\Delta = (a-b) & d & d' + ba \\
& c & c' + ba \\
\end{vmatrix}
\]

\[
= (a-b) [d'cc' + d'ba - c'dd' - c'ba - ddc' - dab + cdd' + abc + bdc' \\
+ adc' - bcd' - acd']
\]

\[
= (a-b) [ab(c + d' - c' - d) + c'd'(c' + d - a - b) - c'd(d' + c - a - b)]
\]

\[
= (a-b) [ab(c+d') - ab(c'+d) = cd'(c+d) - cd'(a+b) - c'd(d'+c) + c'd(a+b)],
\]

which can be written

\[
\Delta = (a - b) \begin{bmatrix}
ab & a+b \\
cd' & c+d'
\end{bmatrix} - \begin{bmatrix}
ab & a+b \\
c'd & c'+d'
\end{bmatrix} + \begin{bmatrix}
c'd & c+d' \\
c'd & c'+d'
\end{bmatrix},
\]

which by Cor. 1, page 34, of Hanus' Elements of Determinants, is

\[
\Delta = (a - b) \begin{bmatrix}
1 & ab & a + b \\
1 & cd' & c + d' \\
1 & c'd & c'+d
\end{bmatrix}.
\]

Problem 102. [AMM, Vol. 23, p. 123-124, Prob. 444 (Alg)]

If \( A, B, C \) are angles of a plane triangle, prove that
\[
\begin{vmatrix}
\cot A & \cot B & \cot C \\
1 & 1 & 1 \\
\cos^2 A & \cos^2 B & \cos^2 C \\
\end{vmatrix} = 0.
\]

**Solution:** Transforming trigonometrically and rearranging, the determinant becomes

\[
\frac{-1}{4 \sin A \sin B \sin C}
\]

\[
\begin{vmatrix}
2 \cos A \sin B \sin C & 2 \cos B \sin C \sin A & 2 \cos C \sin A \sin B \\
1 & 1 & 1 \\
2 \sin^2 A & 2 \sin^2 B & 2 \sin^2 C \\
\end{vmatrix}
\]

By the formula \(2 \sin A \sin B \cos C = \sin^2 A + \sin^2 B - \sin^2 C\), when \(A + B + C = 180^\circ\), this reduces to

\[
\frac{-1}{4 \sin A \sin B \sin C}
\]

\[
\begin{vmatrix}
\sin^2 B + \sin^2 C - \sin^2 A & \sin^2 A + \sin^2 C - \sin^2 B & \sin^2 A + \sin^2 B - \sin^2 C \\
1 & 1 & 1 \\
2 \sin^2 A & 2 \sin^2 B & 2 \sin^2 C \\
\end{vmatrix}
\]

\[= \frac{-1}{4 \sin A \sin B \sin C}
\]

\[
\begin{vmatrix}
\sin^2 B + \sin^2 C + \sin^2 A & \sin^2 A + \sin^2 B + \sin^2 C & \sin^2 A + \sin^2 B + \sin^2 C \\
1 & 1 & 1 \\
2 \sin^2 A & 2 \sin^2 B & 2 \sin^2 A \\
\end{vmatrix}
\]
But this determinant equals zero since two rows are alike after
dividing out \( \sin^2 A + \sin^2 B + \sin^2 C \). It is to be noticed that the
above determinant is equal to zero for any values of \( A, B, C \)
whatever, provided only that two of them are alike.

Problem 103. [AMM, Vol. 24, p. 428-429, Prob. 479 (Alg)]

Prove or disprove:

\[
\begin{vmatrix}
x - v - z & 2 \\
y - v - z & 2 \\
- y - z v & + \\
- z y - x & \\
- v y & \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
x y - z & 2 \\
x y - z & 2 \\
- y x & + \\
- z v & + \\
- x v & \\
- x v & \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
x y - z & 2 \\
- y x & \\
- z v & \\
- x v & \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
x - y - z & v \\
y - x & \\
z v & \\
v - z y & \\
\end{vmatrix}^{2}
\]

\[
\frac{1}{(x^2 + y^2 + z^2 + v^2)^{-1}}
\]

Solution: Let \( Q \) be the symbol for \( x^2 + y^2 + z^2 + v^2 \), and
represent the determinants in the order they appear by \( Z, V, X, Y, H \),
so that the equation is \( (Z^2 + V^2 + X^2 + Y^2) \cdot H^2 = Q^{-1} \).

Application of the rule for multiplication of determinants gives:

\[
Z^2 = \begin{vmatrix}
Q - y^2 & -yx & -yv \\
-xy & Q - x^2 & -xv \\
-vy & -vx & Q - v^2 \\
\end{vmatrix}
\]
Dividing the first, second and third columns by \( y, x, v \), respectively, and multiplying the first, second and third rows, respectively, by the same symbols gives:

\[
Z^2 = \begin{vmatrix}
Q - y^2 & -y^2 & -y^2 \\
-x^2 & Q - x^2 & -x^2 \\
-v^2 & -v^2 & Q - v^2
\end{vmatrix},
\]

which easily reduces to \( Z^2 = Q^2(Q - v^2 - y^2 - x^2) = Q^2z^2 \). In exactly the same way, \( V^2 = Q^2v^2, X^2 = Q^2x^2, Y^2 = Q^2y^2 \).

Hence \( (Z^2 + V^2 + X^2 + Y^2) = Q^2(z^2 + v^2 + x^2 + y^2) = Q^3 \). Direct application of the rule for multiplication of determinants gives for \( H^2 \) a determinant in which each element in the leading diagonal is \( Q \), and all other entries are zero; thus \( H^2 = Q^4 \). These values set in the equation prove it to be correct.

**Problem 104.** [AMM, Vol. 25, p. 120, Prob. 488 (Alg)]

\[
\begin{vmatrix}
a_1 & 1 & a_2 \\
1 & 1/a_4 & 1 \\
a_2 & 1 & a_3
\end{vmatrix} = 0,
\]

where \( a_k = \frac{\sin(k\theta + \alpha)}{\sin k\theta} \), and \( \theta \) and \( \alpha \) have any values which do not make a denominator zero.
Solution: It is easily shown that the following equations hold for all values of $\theta$:

$$\cot \theta - 2 \cot 2\theta + \cot 3\theta = \frac{2 \cos 2\theta}{\sin 4\theta + \sin 2\theta}$$

$$\cot \theta \cot 3\theta - \cot^2 \theta = \frac{2 \cos 2\theta \cot 4\theta}{\sin 4\theta + \sin 2\theta}$$

Thus, $\cot \theta \cot 3\theta - \cot^2 2\theta = \cot 4\theta (\cot \theta - 2 \cot 2\theta + \cot 3\theta)$.

Now $a_k$ may be written in the form $a_k = \cos \alpha + \sin \alpha \cot k\theta$.

Thence we have $a_1 - 2a_2 + a_3 = \sin \alpha (\cot \theta - 2 \cot 2\theta + \cot 3\theta)$, and

$$a_1 a_3 - a_2^2 = \sin \alpha \cos \alpha (\cot \theta - 2 \cot 2\theta + \cot 3\theta)$$

$$+ \sin^2 \alpha (\cot \theta \cot 3\theta - \cot^2 2\theta)$$

$$= \sin \alpha (\cot \theta - 2 \cot 2\theta + \cot 3\theta)(\cos \alpha + \sin \alpha \cot 4\theta)$$

$$= (a_1 - 2a_2 + a_3)a_4.$$ 

Hence $(a_1 a_3 - a_2^2)/a_4 + 2a_2 - a_1 - a_3 = 0$; that is,

$$\begin{vmatrix} a_1 & 1 & a_2 \\ 1 & 1/a_4 & 1 \end{vmatrix} = 0.$$
Problem 105. [AMM, Vol. 25, p. 304, Prob. 489 (Alg)]

Prove or disprove:

\[
\begin{vmatrix}
-x & -ay & -bu & abv \\
y & x & -bv & -bu \\
u & av & x & ay \\
-v & -u & y & -x \\
\end{vmatrix}^2
= \begin{vmatrix}
x & -x & -bu & abv \\
y & y & -bv & -bu \\
u & u & x & ay \\
v & -v & y & -x \\
\end{vmatrix}^2
\]

\[
\begin{vmatrix}
x & -ay & -x & abv \\
y & x & -bu & +b \\
u & av & u & ay \\
v & -u & -v & -x \\
\end{vmatrix}^2 + a
= \begin{vmatrix}
x & -ay & -bu & abv \\
y & x & -bv & y \\
u & av & x & u \\
v & -u & y & -v \\
\end{vmatrix}^2
\]

Solution: The quantities \( X, Y, U, V, W, \) defined by the equations

\[
xX - ayY - buU + abV = -xW,
\]
\[
yX + x Y - bvU - buV = yW,
\]
\[
uX + avY + xU + ayV = uW,
\]
\[
vX - uY + yU - xV = -vW,
\]

are proportional to the five determinants taken in order.

It can be easily shown that the last determinant in the left member of the proposed equation is equal to zero for all values of \( x, y, u, v, a, b. \) Hence the above equations may be written
in the form

\[(X + W)x - aYy - bUu = 0\]
\[Yx + (X-W)y - bUv = 0\]
\[Ux + (X-W)u + aYv = 0\]
\[Uy - Yu + (X+W)v = 0.\]

The quantities \(x, y, u, v\) can satisfy this system of homogeneous linear equations if, and only if, the determinant

\[
\begin{vmatrix}
X+W & -aY & -bU & 0 \\
Y & X-W & 0 & -bU \\
U & 0 & X-W & aY \\
0 & U & -Y & X+W
\end{vmatrix} = 0.
\]

It can be shown that the value of this determinant is \(K^2\), where \(K = (X^2 - W^2) + aY^2 + bU^2\). Hence, \(X^2 + aY^2 + bU^2 = W^2\), or since \(V = 0\), \(X^2 + aY^2 + bU^2 + abV^2 = W^2\). Thus the relation stated in the problem holds for all values of \(x, y, u, v, a, b\).
Problem 106. [AMM, Vol. 32, p. 52-53, Prob. 3065]

Prove the following identity:

$$\begin{vmatrix}
  a^{m+1} + c & b(bc^{m-1} + a) & b(a^{m-1} + c + a + c^m) \\
  (a + b) & b^{m+1} + a & (b + c) \\
  c(ab^{m-1} + a + b^m) & b^{m+1} + a & (b + c) \\
  a(ab^{m-1} + c) & a(bc^{m-1} + b + c^m) & c^{m+1} + b^2 (a + c)
\end{vmatrix}$$

$$\begin{vmatrix}
  a^{n+1} + c & b(bc^{n-1} + a) & b(a^{n-1} + c + a + c^n) \\
  (a + b) & b^{n+1} + a & (b + c) \\
  c(ab^{n-1} + a + b^n) & b^{n+1} + a & (b + c) \\
  a(ab^{n-1} + c) & a(bc^{n-1} + b + c^n) & c^{n+1} + b^n (c + a)
\end{vmatrix}$$

$$\begin{vmatrix}
  a^{m+n+2} + c & 2, 2 & 2, m+n-2, a+2 \\
  (a + b) & b(bc^{m+n-2} + a) & b(a^{m-n-2} + c) \\
  c(ab^{m+n} + a) & b^{m+n+2} + a & b(a^{m+n-2} + c) \\
  a(ab^{m+n-2} + c) & a(bc^{m+n} + b + c) & a(bc^{m+n} + b + c)
\end{vmatrix}$$

$$\begin{vmatrix}
  a^{m+n-2} + c+2 & 2, 2 & m+n+2, a+2 \\
  (a + b) & b(bc^{m+n-2} + c+2) & b(a^{m-n-2} + a) \\
  c(ab^{m+n} + a) & b^{m+n+2} + a & b(a^{m+n-2} + a) \\
  a(ab^{m+n-2} + c) & a(bc^{m+n} + b + c) & a(bc^{m+n} + b + c)
\end{vmatrix}$$

Solution: Let $\Delta = D(a, \beta, k; p, s)$ be the determinant obtained by replacing in the first determinant on the left $a, b, c$ by $a, \beta, \gamma$, where $a \beta \gamma = k$; and the indices $m, m \pm 1, 1$ by $p, p \pm s, s$. Then if $k = abc$, the problem may be stated:

(1) Prove $D(a, b, k; m, 1) \cdot D(a, b, k; n, 1) = 8D(a, b, k; m+n, 2)$.

If $k = 0$, $\Delta = 0$, so assume $\Delta \neq 0$. In $\Delta$, multiply the elements
of the first, second, and third columns by \( \beta^s, \gamma^s, \alpha^s \), respectively, and divide the corresponding rows by \( \beta^s, \gamma^s, \alpha^s \), respectively.

Next subtract the elements of the second and third columns from the corresponding elements of the first. Then add one-half of the elements of the first column to the corresponding elements of the second and third. Removal of the common factors from the elements of the columns gives

\[
(2) \quad \Delta \equiv -2k^s \begin{vmatrix} a^p & \gamma^p & a^p + \gamma^p \\ a^p & a^p + \beta^p & \beta^p \\ \beta^p + \gamma^p & \gamma^p & \beta^p \end{vmatrix} = 8k^{p+s}.
\]

From (2) and from the first sentence of the above paragraph we obtain the following:

\[
(3) \quad \prod_{i=1}^{n} D(a_i, \beta_i, k; q_i, t_i) \equiv 8^{n-1} D(a', \beta', k; \sum_{i=1}^{n} q_i, \sum_{i=1}^{n} t_i).
\]

(1) is a special case of (3), and the identity is established.
Problem 107 [AMM, Vol. 33, p. 527-528, Prob. 3154 (3150)]

Prove

\[
\begin{vmatrix}
  b^2+c^2 & 0 & -cd & be & 0 \\
  0 & a^2+d^2 & 0 & ab & ad \\
  -cd & 0 & d^2+e^2 & 0 & ce \\
  be & ab & 0 & b^2+e^2 & 0 \\
  0 & ad & ce & 0 & a^2+c^2 \\
\end{vmatrix}
\]

\[= \begin{vmatrix}
  b^4+c^4 & 0 & -c^2d^2 & b^2e^2 & 0 \\
  0 & a^4+d^4 & 0 & a^2b^2 & a^2d^2 \\
  -c^2d^2 & 0 & d^4+e^4 & 0 & c^2e^2 \\
  b^2e^2 & a^2b^2 & 0 & b^4+e^4 & 0 \\
  0 & a^2d^2 & c^2e^2 & 0 & a^4+c^4 \\
\end{vmatrix}
\]

**Solution:** If we square the determinant

\[
\begin{vmatrix}
  0 & b & \pm c & 0 & 0 \\
  a & 0 & 0 & d & 0 \\
  0 & 0 & d & 0 & e \\
  b & e & 0 & 0 & 0 \\
  0 & 0 & 0 & a & c \\
\end{vmatrix}
\]
we obtain the first or second determinant of the problem, according as \(-c\) or \(c\) is taken. Expanding this according to minors of the first two columns, we obtain at once \(c(b^2d^2 + a^2e^2)\). Replacing each letter in this equality by its square (choosing the upper sign) and squaring, we obtain the value of the third determinant. Thus, the identity is established.

Problem 108. [AMM, Vol. 35, p. 324, Prob. 2807]

Establish the identity

\[
\begin{vmatrix}
-x & bcy & -acu & -abv \\
y & x & -av & au \\
u & bv & x & by \\
v & cu & cy & -x \\
\end{vmatrix}^2 =
\begin{vmatrix}
x & bcy & -x & -abv \\
-y & x & y & au \\
u & bv & u & by \\
-v & cu & -v & -x \\
\end{vmatrix}^2
\]

\[
\begin{vmatrix}
x & bcy & -acu & -x \\
-y & x & -av & y \\
u & bv & x & u \\
-v & cu & cy & -v \\
\end{vmatrix}^2 =
\begin{vmatrix}
x & bcy & -acu & -abv \\
-y & x & -av & au \\
u & bv & x & by \\
-v & cu & cy & -x \\
\end{vmatrix}^2
\]

Solution: Lagrange's like producing quadrinomial is

\[
(x^2 + bcy^2 + acu^2 + abv^2)(p^2 + bcq^2 + acr^2 + abs^2) = (P^2 + bcQ^2 + acR^2 + abS^2),
\]
where \[ P = xp - bcyq - acur - abvs, \]
\[ Q = -yp - xq - avr + aus, \]
\[ R = up - bvq + xr + bys, \]
and \[ S = -vp - cuq + cyr - xs. \]

Now assume for \( p, q, r, \) and \( s \) such values as render \( P = -x, Q = y, R = u, \) and \( S = -v. \) Substitute these values of \( p, q, r, s \) in the like producing quadrinomial, divide by the first factor, clear of fractions, and obtain the given identity since \( q \equiv 0. \)

**Problem 109.** [AMM, Vol. 37, p. 39, Prob. 3349]

If \(| fmn|\) denotes the determinant whose columns are the \( l^{th}, m^{th}, n^{th} \) columns of the array

\[
\begin{bmatrix}
y & x & -av & au \\
u & bv & x & -by \\
v & -cu & cy & x \\
\end{bmatrix},
\]

prove that \( (x^2 + bcy^2 + acu^2 + abv^2)^3 = |124|^2 + bc|134|^2 + ac|124|^2 + ab|123|^2. \)

**Solution:** On expanding the given determinants, \( x^2 + bcy^2 + acu^2 + abv^2 \) is found to be their common factor. If we denote this factor by \( D, \) the given equation becomes

\[ D^3 = x^2 D^2 + bcy^2 D^2 + acu^2 D^2 + abv^2 D^2, \] an identity.
Problem 110. [AMM, Vol. 37, p. 197-199, Prob. 3309]

If \((1, 2, 3, 4, 5, 6)\) denotes the determinant of the 6th order whose \(i\)th row is \(x_i^2, x_i y_i, y_i^2, x_i, y_i, 1\), and if

\[
(i, j, k) = \begin{vmatrix} x_i & y_i & 1 \\ y_j & y_j & 1 \\ y_k & y_k & 1 \end{vmatrix}
\]

show that

\[
(1, 2, 3, 4, 5, 6) = (6, 1, 2)(2, 3, 4)(4, 5, 6)(1, 3, 5) - (1, 2, 3)(3, 4, 5)(5, 6, 1)(2, 4, 6)
\]

is an identity in the twelve independent variables \(x_i, y_i\).

Solution: Denote by \(\phi\) the polynomial \((1, 2, 3, 4, 5, 6)\) and by \(\psi\) the other polynomial on the right. If the set \((x_1, y_1)\) is replaced by \((x_2, y_2), (x_3, y_3), (x_5, y_5), (x_6, y_6)\), \(\psi\) vanishes since each of its two parts vanish. If \((x_1, y_1)\) is replaced by \((x_4, y_4)\), the two parts of \(\psi\) are equal and cancel, and \(\psi\) again vanishes. If a cyclic interchange is made in the subscripts of \(\psi\) so that \(1 2 3 4 5 6\) is replaced by \(2 3 4 5 6 1\), then \(\psi\) merely changes sign. Hence, when the subscript 2 is replaced in turn by each of the remaining five subscripts, \(\psi\) vanishes, etc. Thus \(\psi\) vanishes when the subscript \(i\) is replaced by a different subscript \(j\).

This is obviously true of \(\phi\).

The polynomial \(\phi\) has only one term of the form
$x_1^2x_2y_2^2y_3^2x_4y_5$, and this comes from the principal diagonal. We shall examine \( \psi \) for the presence of this term, considering the two parts separately. There are four determinant factors in each part, and, since the total degree of the term considered is eight, each factor must yield a term of the second degree for the product. Marking the letters \( x_3, x_5, x_6, y_1, y_4, y_6 \) which must be rejected, we see that \((6 \ 1 \ 2)\) yields one and only one term which may be used, \( x_1y_2 \); similarly \((4 \ 5 \ 6)\) yields \( x_4y_5 \); then in \((1 \ 3 \ 5)\) we must reject \( y_5 \) since we already have it, and there is left \( x_1y_3 \); and finally in \((2 \ 3 \ 4)\) we must reject every term but \( x_2y_3 \). Thus the first part yields this term only once with the coefficient unity. In the second part \((5 \ 6 \ 1)\) yields only \( x_1y_5 \).

Then \((3 \ 4 \ 5)\) yields only \(-x_4y_3\), since we already have \( y_5 \).

In \((2 \ 4 \ 6)\) the only term which may be used is \(-x_4y_2\), but this would give in the product \( x_4^2 \). Hence the second part does not contain the term in question, and it follows that \( \phi \) and \( \psi \) have this same term with the coefficient unity. Also neither \( \phi \) nor \( \psi \) can be identically zero as polynomials in the twelve independent variables which they contain.

Consider the different determinants \((i \ j \ k)\) by taking all possible combinations of \(i, j, k\) so that no two subscripts are the same in the same determinant. Denote the product of all these determinants by \( \pi \). No single factor of \( \pi \) is identically zero, and
hence $r$ is not identically zero; also the product of $\phi \pi$ is not identically zero, and we can find a set of particular values for the twelve variables such that $\phi \pi$ is not zero for these values. Since $\phi \pi$ is continuous in the twelve variables, we can determine a region $R$ for these variables such that $\phi \pi$ does not vanish for any values of the variables selected from this region $R$. Select a definite set of values from $R$ for all the variables except $x_1, y_1$. Then for these special fixed values $\phi$ is a polynomial of the second degree in $x_1, y_1$ not identically zero. We shall show that it is irreducible. When $x_1, y_1$ assume the values already selected $x_2, y_2; x_3, y_3; \ldots; x_6, y_6$, $\phi$ vanishes. Hence, if $\phi$ consists of two linear factors in $x_1, y_1$, one factor at least must vanish for three of these sets of points, say $x_2, y_2; x_3, y_3; x_4, y_4$; and then (2 3 4) would be zero. But this is impossible since these values are taken from $R$. Hence $\phi$ is irreducible. Consider $\psi$ as a polynomial in $x_1, y_1$ with the same constant values for the other variables that appear in $\phi$. The two polynomials are of the second degree in $x_1, y_1$; one of them is irreducible, and each vanishes for five distinct sets of values, the values of the constants. Hence we must have $\phi = c_1 \psi$, where $c_1$ is independent of $x_1, y_1$, and $\phi$ and $\psi$ are regarded as polynomials in $x_1, y_1$.

Now regard $x_2, y_2$ as variables in the two polynomials, while the remaining variables assume fixed values from the region
R. Then as before we conclude that \( \phi \equiv c_2 \psi \), where \( c_2 \) is independent of \( x_2 y_2 \). If in the two equations \( \phi \equiv c_1 \psi \), \( \phi \equiv c_2 \psi \) we suppose that the twelve variables have the same values which are taken from \( R \), then \( (c_2 - c_1)\psi = 0 \). But since \( \phi \) does not vanish for these values neither does \( \psi \), and hence \( c_2 = c_1 \). This shows that neither \( c_1 \) nor \( c_2 \) depends upon \( (x_1, y_1) \) or \( (x_2, y_2) \), at least for the region \( R \). Reasoning in the same way, we show that \( c_1 = c_2 = c_3 \), and each is independent of the variables with the subscripts \( 1, 2, 3 \); and so on until finally we have \( \phi \equiv c \psi \), where \( c \) is an absolute constant, at least for \( R \). But since \( \phi \) and \( \psi \) have each the same term with the same coefficient unity, we have \( c = 1 \), and then \( \phi = \psi \). Hence the corresponding coefficients in \( \phi \) and \( \psi \) are the same for the region \( R \), and we have therefore \( \phi \equiv \psi \) without any restriction upon the range of the twelve variables.

As an application of this result, suppose that one of the second order determinants is zero, say \( (1 2 3) \), then the determinant \( \phi \) breaks up into the product of four determinants each of the second order.

If \( \phi(x, y, z) = 0 \) and \( \psi(x, y, z) = 0 \), prove that

\[
\begin{vmatrix}
\frac{d^2x}{ds^2} & \frac{d^2y}{ds^2} & \frac{d^2z}{ds^2} \\
\phi_x & \phi_y & \phi_z \\
\psi_x & \psi_y & \psi_z
\end{vmatrix} = 0
\]

under suitable conditions.

**Solution:** Let the surfaces \( \phi = 0 \) and \( \psi = 0 \) determine a curve \( c \) and let \( s \) be the distance measured on \( c \) from a fixed point on the curve. Then \( dx/ds, dy/ds, dz/ds \) are the direction cosines of a tangent to \( c \), and \( (\frac{dx}{ds})^2 + (\frac{dy}{ds})^2 + (\frac{dz}{ds})^2 = 1 \). Differentiating with respect to \( s \), we obtain

\[
(1) \quad \frac{d^2x}{ds^2} \frac{dx}{ds} + \frac{d^2y}{ds^2} \frac{dy}{ds} + \frac{d^2z}{ds^2} \frac{dz}{ds} = 0.
\]

In addition we have

\[
(2) \quad \phi_x \frac{dx}{ds} + \phi_y \frac{dy}{ds} + \phi_z \frac{dz}{ds} = 0
\]

\[
(3) \quad \psi_x \frac{dx}{ds} + \psi_y \frac{dy}{ds} + \psi_z \frac{dz}{ds} = 0.
\]

Eliminating \( dx/ds, dy/ds, \) and \( dz/ds \), and multiplying by \( ds^2 \), we obtain the stated relationship.
Problem 112. [AMM, Vol. 44, p. 401, Prob. 3747]

Find the single condition that all the roots of the secular equation

\[
\begin{vmatrix}
  a_{11} - x & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} - x & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nn} - x
\end{vmatrix} = 0
\]

should be equal, the \( a_i \)'s being real and \( a_{ji} = a_{ij} \), and hence determine the cases in which all the roots are equal.

Solution I: If the given equation is written \( |A-xI| = 0 \), where \( A \) is symmetric and has a unique characteristic value, \( a \), then \( a \) is real and the traces of the matrices \( B = A-aI, B^2, \ldots, B^n \) are all zero; for the characteristic equation satisfied by \( B \) is \( B^n = 0 \). If we let \( B = (b_{ij}), (i,j = 1,2,\ldots,n) \), where \( b_{ij} = b_{ji} = a_{ij} \), if \( i \neq j \) and \( b_{ii} = a_{ii} - a \), then the trace of \( B^2 \) is

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}^2 = 0 .
\]

Since \( b_{ij}, (i,j = 1,2,\ldots,n) \), are all real, this equation can be satisfied if and only if \( B = 0 \). That is \( A = aI \) is the only
real symmetric matrix having all characteristic roots equal to \( a \).

**Solution II:** All the roots of the secular equation are real; but for any such equation, \( x^n + a_1 x^{n-1} + \ldots + a_n = 0 \), with roots \( a_1, a_2, \ldots, a_n \), the condition for all the roots to be equal is

\[
\sum (a_i - a_j)^2 = 0, \text{ or } (n-1)a_1^2 - 2na_2 = 0.
\]

Applying this to the secular equation, we get as our condition

\[
(n-1) \left( \sum a_{ii} \right)^2 - 2n \sum \left( a_{ij} \right)(a_{jj} - a_{ij}^2) = 0,
\]

\[
(n-1) \sum a_{ii}^2 - 2 \sum a_{ii}a_{jj} + 2n \sum a_{ij}^2 = 0,
\]

\[
\sum (a_{ii} - a_{jj})^2 + 2n \sum a_{ij}^2 = 0.
\]

Hence all the roots will be equal if and only if \( a_{ij} = 0 \), \( i \neq j \), and \( a_{ii} = a_{jj} \).

**Editorial Note.** A solution may be obtained also from the following theorem:

If \( A \) is a symmetric square matrix of \( n^2 \) real elements with characteristic roots \( c_1, c_2, \ldots, c_n \), there exists a real orthogonal matrix \( S \) with determinant of unit value such that the
transform of $A$ by $S$ is a matrix all of whose elements are zero except those in the principal diagonal, and they are the characteristic roots.


Let $X_i, Y_i, Z_i$ be the cofactors of $x_i, y_i, z_i$ in the general third order determinant $D = \begin{vmatrix} x_1 y_2 z_3 \end{vmatrix}$. Prove that

$$\begin{vmatrix} X_2 X_3 & Y_2 Y_3 & Z_2 Z_3 \\ X_3 X_1 & Y_3 Y_1 & Z_3 Z_1 \\ X_1 X_2 & Y_1 Y_2 & Z_1 Z_2 \end{vmatrix} = -D^2 \begin{vmatrix} x_2 x_3 & y_2 y_3 & z_2 z_3 \\ x_3 x_1 & y_3 y_1 & z_3 z_1 \\ x_1 x_2 & y_1 y_2 & z_1 z_2 \end{vmatrix}.$$

Solution: Multiplying the respective rows of the determinant

$$\begin{vmatrix} x_2 x_3 & y_2 y_3 & z_2 z_3 \\ x_3 x_1 & y_3 y_1 & z_3 z_1 \\ x_1 x_2 & y_1 y_2 & z_1 z_2 \end{vmatrix}$$

by $x_1, x_2, x_3$ and then subtracting the first row from the second and third, we find that it is equal to

$$\frac{1}{x_1 x_2 x_3} \begin{vmatrix} x_1 x_2 x_3 & x_1 y_2 y_3 & x_1 z_2 z_3 \\ 0 & -y_3 Z_3 & z_3 Y_3 \\ 0 & y_2 Z_2 & -z_2 Y_2 \end{vmatrix} = y_3 z_2 Y_2 Z_3 - y_2 z_3 Y_3 Z_2.$$
By a well-known result on determinants (e.g., Bôcher, Introduction to Higher Algebra, p. 33) the cofactors of the terms of the adjoint of \( D \) are equal to the corresponding terms of \( D \) multiplied by the value of \( D \). Hence, from the result already obtained, it follows that

\[
\begin{vmatrix}
X_2X_3 & Y_2Y_3 & Z_2Z_3 \\
X_3X_1 & Y_3Y_1 & Z_3Z_1 \\
X_1X_2 & Y_1Y_2 & Z_1Z_2 \\
\end{vmatrix} = D^2(Y_3Z_2y_2y_3 - y_2Z_3y_3z_2)
\]

\[
= -D^2 \begin{vmatrix}
x_2x_3 & y_2y_3 & z_2z_3 \\
x_3x_1 & y_3y_1 & z_3z_1 \\
x_1x_2 & y_1y_2 & z_1z_2 \\
\end{vmatrix}
\]

**Problem 114.** [AMM, Vol. 55, p. 366-367, Prob. E789]

If \( y = \tan x \), show that

\[
\frac{d^n y}{\cos^{n+1} x} = \begin{vmatrix}
\cos x & 0 & \ldots & \sin x \\
\cos(x+\pi/2) & \cos x & \ldots & \sin(x+\pi/2) \\
\cos(x+\pi) & 2 \cos(x+\pi/2) & \ldots & \sin(x+\pi) \\
\ldots & \ldots & \ldots & \ldots \\
\cos(x+n\pi/2) & n\cos[x+(n-1)\pi/2] & \ldots & \sin(x+n\pi/2) \\
\end{vmatrix}
\]
Solution: Let \( w = \sin x, z = \cos x \), so that \( w = yz \). If we differentiate \( r \) times, we have by the Leibnitz rule,

\[
w^{(r)} = \sum_{k=0}^{r} \binom{r}{k} y^{(k)} z^{(r-k)}, \quad r = 0, 1, 2, \ldots, n.
\]

Regard this as a system of \( n+1 \) linear equations in the quantities \( y^{(k)}, k = 0, 1, 2, \ldots, n \). The determinant of this system, being triangular and having \( z^{(0)} = \cos x \) down the main diagonal, has the value \( \cos^{n+1} x \). Hence, solving for \( y^{(n)} \), we find

\[
y^{(n)} \cos^{(n+1)} x = \begin{vmatrix}
z^{(0)} & \cdots & \cdots & \cdots & \cdots & w^{(0)} \\
z^{(1)} & z^{(0)} & \cdots & \cdots & \cdots & w^{(1)} \\
z^{(2)} & (2)z^{(1)} & z^{(0)} & \cdots & \cdots & w^{(2)} \\
z^{(3)} & (3)z^{(2)} & (3)z^{(1)} & z^{(0)} & \cdots & w^{(3)} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
z^{(n)} & (n)z^{(n-1)} & \cdots & \cdots & \cdots & w^{(n)} \\
\end{vmatrix}
\]

We now observe that \( z^{(k)} = \cos(x + k\pi/2), \quad w^{(k)} = \sin(x + k\pi/2) \), and the proof is complete.


Let \( L_i, i = 1, 2, \ldots, 5 \), be five lines in space. Denote the angle between \( L_i \) and \( L_j \) by \( (ij) \). Prove that the determinant
\[
\left| \sin^2 \left( \frac{(ij)}{2} \right) \right| = 0, \quad (i, j = 1, 2, \ldots, 5).
\]

Solution: We have \( \sin^2 \left( \frac{(ij)}{2} \right) = \frac{1 - \cos (ij)}{2} \)

\[
= \frac{1}{2} (1 - x_i x_j - y_i y_j - z_i z_j),
\]
where \( x_i, y_i, z_i \) are the direction cosines of \( L_i \). Thus,

\[
|A| = | \sin^2 \left( \frac{(ij)}{2} \right) | = \frac{1}{32} \left| 1 - x_i x_j - y_i y_j - z_i z_j \right|,
\]

which can be expressed as the product of the two singular determinants:

\[
\begin{vmatrix}
1 & x_1 & y_1 & z_1 & 0 \\
1 & x_2 & y_2 & z_2 & 0 \\
1 & x_3 & y_3 & z_3 & 0 \\
1 & x_4 & y_4 & z_4 & 0 \\
1 & x_5 & y_5 & z_5 & 0
\end{vmatrix}
\times
\begin{vmatrix}
1 & 1 & 1 & 1 & 1 \\
-x_1 & -x_2 & -x_3 & -x_4 & -x_5 \\
-y_1 & -y_2 & -y_3 & -y_4 & -y_5 \\
-z_1 & -z_2 & -z_3 & -z_4 & -z_5 \\
0 & 0 & 0 & 0 & 0
\end{vmatrix}
\]

and hence, \( |A| = 0 \).


Let \( A \) and \( B \) be two \( n \)th order determinants, and let \( C \) be a third determinant whose \( (i, j) \)th element is the
determinant $A$ with its $i$th column replaced by the $j$th column of $B$. Show that $C = A^{n-1}B$.

**Solution:** Let $a, b$ be the matrices whose determinants are $A, B$, respectively, and let $c$ be a matrix with determinant $C$ described above. Let $A_{ij}$ be the cofactor of the $(i, j)^{th}$ element of $a$ and let $\text{adj } a$ be the matrix whose $(i, j)^{th}$ element is $A_{ji}$. Then $c = (\text{adj } a)b$. Taking determinants, and noting Cauchy's result that the determinant of $\text{adj } a$ is $A^{n-1}$, the desired result follows.

It is interesting to note the following generalization. If $k$ distinct columns of $B$ should be substituted for $k$ distinct columns of $A$ in all possible ways, and the resulting determinant arranged lexicographically to form $c$, then

$$C = A^{n-1}B^{k-1}.$$ 

This result and several others appear in Aitken, *Determinants and Matrices*, (1956), p. 101-103.
Problem 117. [AMM, Vol. 72, p. 795, Prob. 5312]

Prove (or disprove) the equation

\[
\begin{vmatrix}
1 & 1 & \ldots & 1 \\
\frac{1}{\partial x_1} & \frac{\partial}{\partial x_2} & \ldots & \frac{\partial}{\partial x_n} \\
\ldots & \ldots & \ldots & \ldots \\
\frac{\partial^{n-1}}{\partial x_1^{n-1}} & \frac{\partial^{n-1}}{\partial x_2^{n-1}} & \ldots & \frac{\partial^{n-1}}{\partial x_n^{n-1}}
\end{vmatrix}
= 1!2!\ldots n!.
\]

Solution: No solution to this problem has yet been published.

Problem 118. [AMM, Vol. 73, p. 310, Prob. E1870]

(1) Prove that the third order determinant \((a_{ij})\) vanishes identically where

- \(a_{11} = (A+B)DE - (D+E)AB\),
- \(a_{12} = AB - DE\),
- \(a_{13} = A+B-D-E\),
- \(a_{21} = (B+C)EF - (E+G)BC\),
- \(a_{22} = BC - EF\),
- \(a_{23} = B+C-E-F\),
- \(a_{31} = (C+D)FA - (F+A)CD\),
- \(a_{32} = CD - FA\),
- \(a_{33} = C+D-F-A\);

(2) the same where

- \(a_{11} = \sin(A+B) \cos(D-E) - \sin(D+E) \cos(A-B)\),
- \(a_{12} = \cos(A+B) \cos(D-E) - \cos(D+E) \cos(A-B)\),
- \(a_{13} = \cos(A+B+D+E)\),
- \(a_{23} = \cos(B+C+E+F)\),
- \(a_{33} = \cos(C+D+E+A)\).
\[ a_{21} = \sin(B+C) \cos(E+F) - \sin(E+F) \cos(B-C), \]
\[ a_{22} = \cos(B+C) \cos(E-F) - \cos(E+F) \cos(B-C), \]
\[ a_{31} = \sin(C+D) \cos(F-A) - \sin(F+A) \cos(C-D), \]
\[ a_{32} = \cos(C+D) \cos(F-A) - \cos(F+A) \cos(C-D). \]

**Solution:** No solution to this problem has yet been published.

**Problem 119.** [DMVJ, Vol. 42, Pt. 2, p. 117-120, Prob. 67]

In order to compute the discriminant of an equation of the form \( f(x) = x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n = 0 \) \((a_n \neq 0)\), we proceed as follows: Establish the "reduction formula"

\[ x^{n-1+\alpha} \equiv c_{\alpha 1} x^{n-1} + c_{\alpha 2} x^{n-2} + \ldots + c_{\alpha n} \pmod{f(x)}, \]

for \( \alpha = 1, 2, \ldots, n \), and construct the determinant \( \Delta \) from the \( n^2 \) values

\[ (\alpha + \beta - 1) c_{\alpha \beta} \quad (\alpha, \beta = 1, 2, \ldots, n). \]

Show that

\[ \Delta = (-1)^{n-1} a_n D. \]

**Solution I:** It is sufficient to prove the assertion as an identity involving the roots \( x_x \) of the equation \( f(x) = 0 \), where the \( x_x \) are all different. Let

\[ f_\alpha (x) = x^{n-1+\alpha} - c_{\alpha 1} x^{n-1} - \ldots - c_{\alpha n} = f(x) g_\alpha (x), \]
where \( g_a(x) \) is of degree \( a - 1 \) with leading coefficient 1. Multiplying \( \Delta \) by the Vandermonde determinant \( V \) composed of the \( x_v \):

\[
\Delta V = \left| (a + \beta - 1) c_{a \beta} \right| x_v^{n-\beta} = \left| \sum_{\beta=1}^{n} (a + \beta - 1) c_{a \beta} x_v^{n-\beta} \right|
\]

\[
= \left| x_v^f (x_v) - (n-I+a) f_a (x_v) \right| = \left| x_v f'(x_v) g_a (x_v) \right|
\]

\[
= (-1)^n a n \prod_{v=1}^{n} f'(x_v) g_a (x_v) = (-1)^n a n (-1)^{n(n-1)/2} D(-1)^{n(n-1)/2} V.
\]

Division by \( V \) now gives

\[
\Delta = (-1)^n a n D.
\]

**Solution II:** Designate the right side of the equation

\[
x^{n-1+a} = c_1 x^{n-1} + c_2 x^{n-2} + \ldots + c_n
\]

by \( \phi_a(x) \) and construct the expression

\[
\psi_a(x) = (n + a - I) \psi_a(x) - x \frac{\phi_a(x)}{dx}
\]

\[
= a c_1 x^{n-1} + (a+1)c_2 x^{n-2} + \ldots + (a + n - I)c_n.
\]

If \( w \) is a multiple root of the equation \( f(x) = 0 \), then
We have also in this case

\[ \psi_a(w) = a C_{a1} w^{n-1} + (a+1) C_{a2} w^{n-2} + \ldots + (a+n-1) C_{an} = 0 \]

\[ (a = 1, 2, \ldots, n) . \]

If the equation \( f(x) = 0 \) also has a multiple root \( w \), the homogeneous system of equations with determinant \( |(a + \beta - 1) C_{a\beta}| \) has the non trivial solution set \( 1, w, \ldots, w^{n-1} \). That is, the determinant \( |(a + \beta - 1) C_{a\beta}| \) vanishes if \( f(x) = 0 \) has a multiple root.

This determinant is a symmetric function of the roots of the equation. The element \( (a + \beta - 1) C_{a\beta} \) is of degree \( a + \beta - 1 \), and the determinant itself is of degree \( n^2 \) in the roots. Since all coefficients \( C_{an} \) are divisible by \( C_{an} = -a_n \), \( \Delta \) contains the factor \( a_n \). The symmetric function \( \frac{\Delta}{a_n} \) is thus of degree \( n(n-1) \), and vanishes if the equation has a multiple root. Moreover, \( \frac{\Delta}{a_n} \) corresponds to the discriminant \( D \), up to a constant factor, which is found to be \((-1)^n\) by looking at the special case \( x^n - 1 = 0 \).

A third solution is also given.
Problem 120. [DMVJ, Vol. 44, Pt. 2, p. 22-26, Prob. 146]

Let the values \( C_{a\beta} \) be defined for the polynomial
\[ f(x) = x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n \]
by means of the reduction formula
\[ x^{n-1} + a \equiv C_{a1} x^{n-1} + C_{a2} x^{n-2} + \ldots + C_{an}, \quad (\text{mod } f(x)), \quad a = 1, 2, \ldots, n. \]

Let \( d_{a\beta} \) be similarly defined for a second polynomial of the same degree,
\[ g(x) = x^n + b_1 x^{n-1} + \ldots + b_{n-1} x + b_n. \]

Then \( \Delta \), the determinant of the \( n^2 \) differences \( C_{a\beta} - d_{a\beta} \) \((a, \beta = 1, 2, \ldots, n)\),
and the resultant \( R(f, g) \) of \( f(x) \) and \( g(x) \) are related as follows:
\[ \Delta = (1)^{n(n-1)/2} R(f, g). \]

[See DMVJ, Vol. 42, Pt. 2, p. 117-120, Prob. 128, or Prob. 119, p. 191, this paper,]

**Solution I:** It is sufficient to prove the assertion as an identity involving the roots \( x_v \) of the equation \( f(x) = 0 \) and the roots \( y_v \)
of \( g(x) = 0 \) for equations without multiple roots. Let \( V(x) \) and \( V(y) \)
be the Vandermonde determinants formed from the \( x_v \)'s and the \( y_v \)'s. Then
\[
R(f, g)V(x)V(y) = \frac{1}{\chi, \rho < \lambda} (x - y_{\sigma}) \frac{1}{\rho < \sigma} (y - y_{\sigma}^{-1})
\]
\[
= \begin{vmatrix}
2n-a_v & n-a_v \\
2n-a_v & n-a_v \\
\end{vmatrix}
\begin{vmatrix}
x_v^{n-a_v} & y_v^{n-a_v} \\
2n-a_v & n-a_v \\
\end{vmatrix}.
and after application of the reduction formula,

\[
\begin{vmatrix}
(c_{n-a+1,1} x_v^{n-1} + c_{n-a+1,2} x_v^{n-2} + \ldots + c_{n-a+1,n}) (x_v^{n-a}) \\
(d_{n-a+1,1} y_v^{n-1} + d_{n-a+1,2} y_v^{n-2} + \ldots + d_{n-a+1,n}) (y_v^{n-a})
\end{vmatrix}
\]

We now subtract \(d_{n-a+1,\rho}\) times the \((n+\rho)\)th column from the \(a\)th column \((a, \rho = 1, 2, \ldots, n)\). Thus we have

\[
\begin{vmatrix}
\begin{pmatrix}c_{n-a+1,1} - d_{n-a+1,1} \end{pmatrix} x_v^{n-1} + \ldots + \begin{pmatrix}c_{n-a+1,n} - d_{n-a+1,n} \end{pmatrix} y_v^{n-1}
\end{vmatrix}
\]

\[
= \left| \sum_{\beta=1}^{n} \left( c_{n-a+1,\beta} - d_{n-a+1,\beta} \right) x_v^{n-\beta} \right| x_v^{n-a} |y_v^{n-a}|
\]

\[
= \left| c_{n-a+1,\beta} - d_{n-a+1,\beta} \right| x_v^{n-\beta} |x_v^{n-a}|
\]

\[
\frac{n(n-1)}{2} = (-1)^{n-1} \left| \begin{pmatrix}c_{a\beta} - d_{a\beta} \end{pmatrix} \right| V(x) V(y).
\]

Division by \(V(x) V(y)\) completes the proof.

**Solution II:** Let \(x_1, x_2, \ldots, x_n\) be the roots of \(f(x) = 0\), and further

\[
x^{n-1+a} - c_{a1} x^{n-1} - c_{a2} x^{n-2} - \ldots - c_{an} = f(x) \cdot f_a(x), \quad (a=1, 2, \ldots, n)
\]

\[
x^{n-1+a} - d_{a1} x^{n-1} - d_{a2} x^{n-2} - \ldots - d_{an} = g(x) \cdot g_a(x);
\]
where \( f_\alpha(x) \) and \( g_\alpha(x) \) are polynomials of degree \( \alpha - 1 \) with leading coefficient one. We construct the product of \( \Delta \) and the Vandermonde determinant \( V \) of the \( x_\gamma \):

\[
\Delta \cdot V = \left| c_{\alpha, \beta} - d_{\alpha, \beta} \right| \cdot \left| x_{\gamma}^{n-\beta} \right| = \left| \sum_{\beta=1}^{n} (c_{\alpha, \beta} x_{\gamma}^{n-\beta} - d_{\alpha, \beta} x_{\gamma}^{n-\beta}) \right|
\]

\[
= \left| g(x_{\gamma}) g_\alpha(x_{\gamma}) - f(x_{\gamma}) f_\alpha(x_{\gamma}) \right| = \left| g(x_{\gamma}) \cdot g_\alpha(x_{\gamma}) \right|
\]

\[
= \prod_{\delta=1}^{n} g(x_{\delta}) \cdot \left| g_\alpha(x_{\gamma}) \right| = (-1)^{\frac{n(n-1)}{2}} R(f, g) \cdot V.
\]

Since \( V \neq 0 \), we have the identity

\[
\Delta = (-1)^{n(n-1)/2} R(f, g).
\]

A third solution is also given.


\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

Put \( a = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \) and \( b = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \); further,
\[
\begin{vmatrix}
 a_{12}b_{13} - a_{13}b_{12} & a_{22}b_{23} - a_{23}b_{22} & a_{32}b_{33} - a_{33}b_{32} \\
 a_{13}b_{11} - a_{11}b_{13} & a_{23}b_{21} - a_{21}b_{23} & a_{33}b_{31} - a_{31}b_{33} \\
 a_{11}b_{12} - a_{12}b_{11} & a_{21}b_{22} - a_{22}b_{21} & a_{31}b_{32} - a_{32}b_{31}
\end{vmatrix}
\]

and

\[
\begin{vmatrix}
 A_{12}B_{13} - A_{13}B_{12} & A_{22}B_{23} - A_{23}B_{22} & A_{32}B_{33} - A_{33}B_{32} \\
 A_{13}B_{11} - A_{11}B_{13} & A_{23}B_{21} - A_{21}B_{23} & A_{33}B_{31} - A_{31}B_{33} \\
 A_{11}B_{12} - A_{12}B_{11} & A_{21}B_{22} - A_{22}B_{21} & A_{31}B_{32} - A_{32}B_{31}
\end{vmatrix}
\]

where \( A_{ik} \) and \( B_{ik} \) are the cofactors of \( a_{ik} \) and \( b_{ik} \), respectively. If \( a \neq 0 \), \( b \neq 0 \), show that \( D = abd \).

**Solution:** Multiply \( b \) and \( d \), using the well known theorem on multiplication of determinants, and substitute the values of \( B_{ik} \) as they occur. The result is:

\[
bd = \begin{vmatrix}
 0 & -a_{21}B_{31} - a_{22}B_{32} - a_{23}B_{33} & C_1 \\
 a_{11}B_{31} + a_{12}B_{32} + a_{13}B_{33} & 0 & C_2 \\
 -a_{11}B_{21} - a_{12}B_{22} - a_{13}B_{23} & a_{21}B_{11} + a_{22}B_{12} + a_{23}B_{13} & 0
\end{vmatrix}
\]

in which

\[
C_1 = a_{31}B_{21} + a_{32}B_{22} + a_{33}B_{23}
\]

and

\[
C_2 = -a_{31}B_{11} - a_{32}B_{12} - a_{33}B_{13}
\]
Let $A$ be the determinant obtained from $a$ by replacing $a_{ik}$ by $A_{ik}$. Noting that the effect of interchanging the $A$'s and $B$'s in each element of $D$ is to change the sign of $D$, one may write immediately the product of $A$ and $-D$ by analogy with the above.

It is: $$-AD =$$

$$
\begin{vmatrix}
0 & -B_{21}a_{31} - B_{22}a_{32} - B_{23}a_{33} & D_1 \\
B_{11}a_{31} + B_{12}a_{32} + B_{13}a_{33} & 0 & D_2 \\
-B_{11}a_{21} - B_{12}a_{22} - B_{13}a_{23} & B_{21}a_{11} + B_{22}a_{12} + B_{23}a_{13} & 0
\end{vmatrix}

$$
in which $$D_1 = B_{31}a_{21} + B_{32}a_{22} + B_{33}a_{23}$$
and $$D_2 = -B_{31}a_{11} - B_{32}a_{12} - B_{33}a_{13},$$

where $a_{ik}$ is the cofactor of $A_{ik}$ in $A$. It is easy to show that $A = a^2$ and that $a_{ik} = a \cdot a_{ik}$. Thus one finds $$-A \cdot D = a^2 \cdot D =$$

$$
\begin{vmatrix}
0 & a_{31}B_{21} - a_{32}B_{22} - a_{33}B_{23} & E_1 \\
a^3 & a_{31}B_{11} + a_{32}B_{12} + a_{33}B_{13} & 0 & E_2 \\
-a_{21}B_{11} - a_{22}B_{12} - a_{23}B_{13} & a_{11}B_{21} + a_{12}B_{22} + a_{13}B_{23} & 0
\end{vmatrix}

$$
in which $$E_1 = a_{21}B_{31} + a_{22}B_{32} + a_{23}B_{33}$$
and $$E_2 = -a_{11}B_{31} - a_{12}B_{32} - a_{13}B_{33}.$$
Noting the correspondences of elements, one sees easily that the determinants given in the expressions for \( \text{bd} \) and \(-a^2D\) are numerically equal but of opposite sign. Therefore
\[-a^2D = a^3(-\text{bd}) \text{ or } D = \text{abd}.\]

**Problem 122. [MM, Vol. 15, p. 206-207, Prob. 368]**

If \( A, B, C \) and \( a, b, c \) are respectively the angles and lengths of the opposite sides of a triangle, then

\[
\begin{vmatrix}
\cos^2 A & \cos^2 B & \cos^2 C \\
\cos A & \cos B & \cos C \\
a & b & c
\end{vmatrix} = 0.
\]

**Solution:** If the determinant \( D \) is expanded in terms of elements of the second row, we get for the first of the three terms

\[
P_A = -bc \cos A (\cos^2 B - \cos^2 C).
\]

By simple trigonometric relations this term becomes

\[
P_A = bc \sin A \cos A \sin(B-C),
\]

and upon the use of the law of sines, can be written as

\[
P_A = abc \cos A \sin(B-C).
\]

By cyclic change of the symbols involved, the other two terms of the expansion, \( P_B \) and \( P_C \), are:

\[
P_B = abc \cos B \sin(C-A), \text{ and }
\]
\[
P_C = abc \cos C \sin(A-B).
\]

Summing up the three terms and performing the expansion indicated, it appears at once that

\[
D = 0.
\]
Problem 123. [MM, Vol. 18, p. 89, Prob. 498]

Prove

| cosB cosC | cosA cosC | cosA cosB |
| cosA      | cosB      | cosC      |
| bc        | ac        | ab        |

= 0, where the letters represent angles and sides of any triangle.

Solution: Upon substitution of \( a = 2R \sin A \), \( b = 2R \sin B \), and \( c = 2R \sin C \) in the third row, the determinant becomes

\[
\Delta = 4R^2
\]

\[
\begin{vmatrix}
\cos B & \cos C & \cos A & \cos C & \cos A & \cos B \\
\cos A & \cos B & \cos C & \\
\sin B & \sin C & \sin A & \sin C & \sin A & \sin B
\end{vmatrix}
\]

If the elements of the first row be subtracted from the corresponding elements of the third row, the third row becomes

\[
-\cos(B+C) & -\cos(A+C) & -\cos(A+B) 
\]

As \( A + B + C = 180^\circ \), the last two rows are now identical, hence \( \Delta = 0 \).

Problem 124. [MM, Vol. 18, p. 89-90, Prob. 499]

If \( A, B, C \) are the angles of a triangle, show that

\[
2 + 2 \cos A \cos B \cos C = \sin^2 A + \sin^2 B + \sin^2 C, \quad \text{and hence show that}
\]

the determinant
\[
\begin{vmatrix}
0 & y + x \cos C & z + x \cos B \\
x + y \cos C & 0 & z + y \cos A \\
x + z \cos B & y + z \cos A & 0
\end{vmatrix}
\]

= \((x \sin A + y \sin B + z \sin C)(yz \sin A + xz \sin B + xy \sin C)\)
identically in \(x, y, z\).

**Solution:** The preliminary statement may be shown as follows:

\[
sin^2 A + \sin^2 B + \sin^2 C = \frac{1}{2} (1 - \cos 2A + 1 - \cos 2B) + 1 - \cos^2 C
\]

= \(2 - \frac{1}{2} (\cos 2A + \cos 2B) - \cos^2 (A+B)\)

= \(2 - \cos(A+B) \cos(A-B) - \cos^2 (A+B)\)

= \(2 - \cos(A+B) [\cos(A-B) + \cos(A+B)]\)

= \(2 - \cos(A+B) \cdot 2 \cos A \cos B\),

which is the proposed relation since \(\cos C = -\cos(A+B)\).

The expansion of the determinant is

\[
xyz(2 + 2 \cos A \cos B \cos C) + x(y^2 + z^2)(\cos B \cos C + \cos A)
\]

\[+ y(z^2 + x^2)(\cos C \cos A + \cos B) + z(x^2 + y^2)(\cos A \cos B + \cos C).\]

The product of the two trinomials is

\[
xyz(\sin^2 A + \sin^2 B + \sin^2 C) + x(y^2 + z^2) \sin B \sin C
\]

\[+ y(z^2 + x^2) \sin C \sin A + z(x^2 + y^2) \sin A \sin B.\]

The first pair of coefficients is equal by the proof given above.

Since \(\cos B \cos C + \cos A = \cos B \cos C - \cos (B+C) = \sin B \sin C\),
the second pair is equal. Similarly, the other pairs are equal and the identity is established.

**Problem 125.** [MM, Vol. 32, p. 171-172, Prob. Q237]

\[
\begin{vmatrix}
  x - 2 & x - 3 & x - 4 \\
  x + 1 & x - 1 & x - 3 \\
  x - 4 & x - 7 & x - 10
\end{vmatrix} = 0.
\]

**Solution:** If \( x = 1 \), the first and third rows are proportional, and the determinant vanishes. Also, if \( x = 5 \), the first and second rows are proportional; if \( x = -11 \), the second and third rows are proportional; and if \( x = -3 \), the third row is proportional to the sum of the first two. Thus, the polynomial expansion has four distinct zeros, and since its degree is not greater than 3, it must be identically zero.

Three alternate solutions similar to the one above have also been published [MM, Vol. 36, p. 77-78].

**Problem 126.** [MM, Vol. 34, p. 176-177, Prob. 413]

If \( a, b, c, d, e, f \) are consecutive terms of the Fibonacci Series \( 1, 1, 2, 3, 5, 8, \ldots \), prove that
Solution: If \( F_1, F_2, F_3, \ldots \) are successive terms of the Fibonacci sequence, then

\[
(1) \quad F_n + F_{n+1} = F_{n+2} \\
(2) \quad F_{n-1} F_{n+1} - F_n^2 = (-1)^n.
\]

In each determinant add col. 1 and col. 2 to col. 3; then add row 2 to row 1, use (1), and subtract row 3 from row 1. Then the left hand side becomes

\[
\begin{vmatrix}
0 & 0 & x \\
b & c & x \\
c & d & x \\
\end{vmatrix} = x^2 \cdot (bd-c^2) \cdot x(c-e-d^2)
\]

\[
\begin{vmatrix}
0 & 0 & x \\
0 & 0 & x \\
0 & 0 & x \\
\end{vmatrix} = x \cdot (bd-c^2) \cdot x(c-e-d^2)
\]

while the right hand side yields

\[
\begin{vmatrix}
0 & 0 & x \\
d & e & x \\
e & f & x \\
\end{vmatrix} = x^2 (df-e^2) = x^2 (-1)^{n+2} = (-1)^n x^2. \quad \text{Therefore},
\]

\[
\begin{vmatrix}
a & b & x-a-b \\
b & c & x-b-c \\
c & d & x-c-d \\
\end{vmatrix} = \frac{x}{x} \begin{vmatrix} c & d & x^2-c-d \end{vmatrix},
\]

\[
\begin{vmatrix}
b & c & x-b-c \\
c & d & x-c-d \\
d & e & x-d-e \\
\end{vmatrix} = \frac{x}{x} \begin{vmatrix} d & e & x^2-d-e \end{vmatrix},
\]

\[
\begin{vmatrix}
c & d & x-c-d \\
d & e & x-d-e \\
e & f & x^2-e-f \\
\end{vmatrix} = \frac{x}{x} \begin{vmatrix} e & f & x^2-e-f \end{vmatrix}
\]
depending upon which term of the Fibonacci sequence we start with.

For instance, if we identify \( a, b, c, d, e, f \) with \( 1, 1, 2, 3, 5, 8 \); i.e., \( a = F_1 \), then we have \((-x)(x) = +x^2\), whereas if we identify \( a, b, c, d, e, f \) with \( 1, 2, 3, 5, 8, 13 \), i.e., \( a = F_2 \), we have \( x(-x) = -(x^2) \).


Show that

\[
\begin{vmatrix}
  x & y & z & w \\
  a & b & c & d \\
  d & c & b & a \\
  w & z & y & x \\
\end{vmatrix}
\]

\[
= \begin{vmatrix}
  x+w & y+z & z & w \\
  a+d & b+c & d & d \\
  d+a & c+b & b & a \\
  w+x & z+y & y & x \\
\end{vmatrix}
\]

Solution: Multiply column 4 by +1 and add to column 1; multiply column 3 by +1 and add to column 2, to obtain

\[
\begin{vmatrix}
  x & y & z & w \\
  a & b & c & d \\
  d & c & b & a \\
  w & z & y & x \\
\end{vmatrix}
\]

\[
= \begin{vmatrix}
  x+w & y+z & z & w \\
  a+d & b+c & d & d \\
  d+a & c+b & b & a \\
  w+x & z+y & y & x \\
\end{vmatrix}
\]

Multiply row 1 by -1 and add to row 4; multiply row 2 by -1 and add to row 3, to obtain
which by Laplace's Development equals

\[
\begin{vmatrix}
  x+w & y+z & z & w \\
  a+d & b+c & c & d \\
  d+a & c+b & b & a \\
  w+x & z+y & y & x \\
\end{vmatrix}
= \begin{vmatrix}
  x+w & y+z & z & w \\
  a+d & b+c & c & d \\
  0 & 0 & b-c & a-d \\
  0 & 0 & y-z & x-w \\
\end{vmatrix},
\]

By suitable interchanges of rows and columns in the second determinant we get the desired result.


Solve for \( x \):

\[
\begin{vmatrix}
  2 & 2 & 2 & 2 \\
  x-a & x-b & x-c & 2 \\
  (x-a)^3 & (x-b)^3 & (x-c)^3 & 1 \\
  (x+a)^3 & (x+b)^3 & (x+c)^3 & 1 \\
\end{vmatrix} = 0.
\]

Solution: Add the third row to the second and then remove the factor \( 2x \) from the elements of the second row. We then obtain

\[
\begin{vmatrix}
  2 & 2 & 2 & 2 \\
  x-a & x-b & x-c & 2 \\
  2x & x+3a & x+3b & x+3c \\
  (x+a)^3 & (x+b)^3 & (x+c)^3 & 1 \\
\end{vmatrix} = 0.
\]

Adding three times the first row to the second and removing the factor \( 4x^2 \) from the elements of the second row, we have
Now subtract the second column from the first, and the third column from the second; after removing factors of \((a-b)\) from the first column and \((b-c)\) from the second column, we obtain

\[
\begin{vmatrix}
8x^3 \\
\underbrace{\begin{array}{ccc}
2 & 2 & 2 \\
\hline
x-a & x-b & x-c \\
\end{array}}
\end{vmatrix}
= 0.
\]

Expanding according to elements of the second row,

\[
\begin{vmatrix}
8x^3 \\
\underbrace{\begin{array}{ccc}
-(a+b) & -(b+c) \\
\hline
0 & 0 & 1 \\
3x^2+3(a+b)x+a^2+ab+b^2 & 3x^2+3(b+c)x+b^2+bc+c^2 \\
\end{array}}
\end{vmatrix}
= 0, \text{ or,}
\]

upon expanding and combining like terms

\[
8x^3 \{3x^2(a-c) - (a-c) \cdot (bc+ca+ab)\} = 0
\]

\[
8x^3 \{3x^2 - (bc+ca+ab)\} = 0
\]

\[
\therefore \quad x = 0, \quad \pm \sqrt{\frac{bc+ca+ab}{3}}
\]
Problem 129. [SSM, Vol. 48, p. 745-746, Prob. 2096]

If \( a + \beta + \psi = 0 \), show that

\[
\begin{vmatrix}
\cos a & \cos \beta & \cos \psi \\
\csc a & \csc \beta & \csc \psi \\
\sec a & \sec \beta & \sec \psi
\end{vmatrix} = 0.
\]

Solution: It is easily shown by elementary trigonometry that if \( x + y + z = 180^\circ \), then

\[(1) \quad (\sec x)(\csc y)(\cos z) = \tan x - \cot y.\]

Since the given determinant is symmetrical in the three variables, it is at once seen that if formula (1) is used to replace each of the six triple products in the expansion of the determinant, the sum of the resulting twelve terms is zero.

Note: The trilinear coordinates of the circumcenter, centroid, and ortho center of the triangle of reference are respectively the rows of the determinant, and since the points lie on the Euler line, the determinant must equal zero.

Problem 130. [SSM, Vol. 55, p. 750-751, Prob. 2468]

Solve the equation

\[
\begin{vmatrix}
4x & 6x+2 & 8x+1 \\
6x+2 & 9x+3 & 12x \\
8x+1 & 12x & 16x+2
\end{vmatrix} = 0.
\]
Solution: Two transformations are first made on the determinant in the equation. The first column is multiplied by \((-2)\) and added to the last column. Then the first column is multiplied by \((-\frac{3}{2})\) and added to the second column.

\[
\begin{vmatrix}
4x & 2 & 1 \\
6x+2 & 0 & -4 \\
8x+1 & -\frac{3}{2} & 0
\end{vmatrix} = 0.
\]

The expansion then becomes

Expanding the determinant yields \(-64x-89x-3-24x = 0\). Hence,

\[x = -\frac{11}{97} .\]


Prove \(|a_1 b_2 c_3|^2 = |A_1 B_2 C_3|\), where

\[
|a_1 b_2 b_3| = \begin{vmatrix}
a_1 & b_1 & c_1 \\
a_2 & b_2 & c_2 \\
a_3 & b_3 & c_3
\end{vmatrix}
\]

and \(A_1, A_2, A_3\), etc. are the minors of \(a_1, a_2, a_3\), etc.

Solution: Let \(|a_1 b_2 c_3| = D\). Consider the product

\[
\begin{vmatrix}
a_1 & b_1 & c_1 \\
a_2 & b_2 & c_2 \\
a_3 & b_3 & c_3
\end{vmatrix} \begin{vmatrix}
A_1 & A_2 & A_3 \\
-B_1 & -B_2 & -B_3 \\
C_1 & C_2 & C_3
\end{vmatrix}.
\]
The elements of the product $a_i A_j - b_i B_j + c_i C_j$ will be $+D$ or $-D$ if $i = j$, and will be 0 if $i \neq j$. When $i \neq j$ the element is equivalent to the expansion of a determinant with two rows the same. The product determinant becomes

$$\begin{vmatrix} D & 0 & 0 \\ 0 & -D & 0 \\ 0 & 0 & D \end{vmatrix} = -D^3.$$ 

Then the second member of the product above must equal $-D^2$.

Removal of the $-1$ factor and interchanging rows and columns will show that $|A_1 B_2 C_3| = |a_1 b_2 c_3|^2$.

**Inequalities**

**Problem 132.** [AMM, Vol. 36, p. 237-238, Prob. 3315]

Show that

$$x^2 + y^2 + z^2 - yz - zx - xy + u^2 + v^2 + w^2 - vw - wu - uv \geq \sqrt{3}$$

for all real values of the variables. Also find the conditions for which the equality holds.

**Solution:** Let $A = (u, x), B = (v, y), C = (w, z)$ be three points determining a triangle for which sides will be denoted by $a, b, c$, with area $D$. Then we are to show that
\[(a^2 + b^2 + c^2) \sqrt{3}/4 \geq 3D.\]

Of the triangles having a given perimeter \(a+b+c=3p\), the equilateral triangle has the maximum area \(\sqrt{3}p^2/4\). Also, \(a^2+b^2+c^2 > 3p^2\), unless \(a=b=c=p\), in which case the two members are equal.

Hence the equality holds when triangle ABC is equilateral; otherwise the inequality holds.


Prove that a real determinant of order 6, with elements numerically not exceeding unity, cannot have a value greater than 160.

**Solution:** It is sufficient to prove the conclusion for determinants of order 6 having all elements equal to \(\pm 1\). For, denoting the elements of the determinant by \(a_{ij}\), if \(-1 < a_{ij} < 1\), and the cofactor \(A_{ij}\) is positive, replace \(a_{ij}\) by 1; but if \(A_{ij}\) is negative, replace \(a_{ij}\) by -1; and if \(A_{ij} = 0\), use either value. In any case, the determinant is either increased or unaltered by such changes, and we can keep on in this way until every element is replaced by \(\pm 1\).

Now, any determinant of order 6 with all elements \(\pm 1\) is divisible by \(2^5\). For, on reducing in the elementary way to order 5, every element becomes \(\pm 1 \cdot 1 \pm 1 \cdot 1\), which is even, and 2
factors out of each row.

Consider the Laplace expansion by means of the 20 minor determinants of order 3 in the first three rows. Each such minor determinant either has two columns proportional (and so is zero), or can be gotten by permuting rows or columns or by changing their signs from

\[
\begin{vmatrix}
1 & 1 & 1 \\
1 & 1 & -1 \\
1 & -1 & 1 \\
\end{vmatrix} = -4, \quad \begin{vmatrix}
1 & 1 & 1 \\
1 & 1 & -1 \\
1 & -1 & -1 \\
\end{vmatrix} = -4.
\]

Let \( a, b, c, d \) denote distinct non-proportional columns of three elements \( \pm 1 \). Then, for example, if the first three rows of the determinant have the form \( \text{aaaabc} \), only four determinants \( \left| abc \right| \) do not vanish, and the Laplace expansion will not exceed \( 4 \cdot 4 \cdot 4 = 64 \). Similarly, in the case of \( \text{aaabbc} \), there are six nonzero minor determinants, and the expansion cannot exceed 96. The case \( \text{aaabcd} \) permits at most \( (1+3 \cdot 3)16 = 160 \). The case \( \text{aabbcc} \) gives 128.

There remains only the possibility \( \text{aabbcd} \). Apparently we might then have \( (4+2 \cdot 4)16 = 192 \). This implies, however, that the last three rows are in some order \( \text{xxyyzw} \), and that the complementary minor of each non-zero determinant such as \( abc \) is a non-zero determinant such as \( xzw \). Unless \( xx \) is under \( cd \), a non-zero determinant \( abc \) or \( acd \) or \( bcd \) will be multiplied by one
of the zero determinants having two columns \( xx \). And if \( xx \) is under \( cd \), then \( yy \) is not under \( cd \), and a similar remark applies. Hence an expansion equal to 192 does not exist.

An example with determinant equal to 160 is

\[
\begin{vmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & 1 \\
1 & -1 & -1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 \\
\end{vmatrix}
\]

An extended comment concerning the state of knowledge about this type of determinant is also given. The reader is referred to an article entitled "Determinants whose Elements are 0 and 1" [AMM, Vol. 53, p. 427-434].

**Problem 134. [AMM, Vol 58, p. 565, Prob. E949]**

Show that if \( D_n \) is a determinant whose elements are \( a_{ij} \), \( i, j = 1, 2, \ldots, n \), and for all \( i \),

\[
a_{ii} \geq \frac{1}{2} \sum_{k=1}^{n} |a_{ik}|,
\]

then

\[
D_n \geq a_{11} \prod_{i=2}^{i-1} (a_{ii} - \sum_{k=1}^{i-1} |a_{ik}|).
\]
Solution: Olga Taussky, in "A Recurring Theorem on Determinants" [AMM, Vol. 56, p. 672-675], has shown that \( D_n \geq 0 \). Now

\[
D_n = \begin{vmatrix}
    a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
    0 & a_{22} & a_{23} & \cdots & a_{2n} \\
    a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}
\end{vmatrix}
+ \begin{vmatrix}
    a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
    a_{21} & a_{21} & 0 & \cdots & 0 \\
    a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}
\end{vmatrix}
\]

By Taussky's result, the second determinant in the sum is non-negative, and \( D_n \) is therefore not less than the first determinant in sum. By applying this process to the remaining \( n-2 \) rows, one obtains

\[
D_n \geq \begin{vmatrix}
    a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
    0 & a_{22} & a_{23} & \cdots & a_{2n} \\
    0 & 0 & a_{33} & \cdots & a_{3n} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & a_{nn} - \sum_{k=1}^{n-1} |a_{nk}|
\end{vmatrix}
\]

\[
= a_{11} \prod_{i=2}^{n} \left( a_{ii} - \sum_{k=1}^{i-1} |a_{ik}| \right).
\]

If a third order determinant has elements 1, 2, ..., 9, what is the maximum value it may have?

Solution: Let $P$ be the sum of the three positive terms of the determinant and $-N$ the sum of the three negative terms. The maximum value of $P$ is $9 \cdot 8 \cdot 7 + 6 \cdot 5 \cdot 4 + 3 \cdot 2 \cdot 1 = 630$, and the minimum value of $N$ is $9 \cdot 8 \cdot 1 + 7 \cdot 5 \cdot 2 + 6 \cdot 4 \cdot 3 = 214$. The minimum combination for $N$ consistent with $P$ is $9 \cdot 6 \cdot 1 + 8 \cdot 5 \cdot 2 + 7 \cdot 4 \cdot 3 = 218$. Any change in $P$ would result in lowering the sum by more than 4. Therefore 412 is the maximum value for the determinant, and one form of the determinant is

\[
\begin{vmatrix}
9 & 4 & 2 \\
3 & 8 & 6 \\
5 & 1 & 7
\end{vmatrix}
\]

Problem 136. [AMM, Vol. 69, p. 63-64, Prob. 4935]

Prove that if $c$ is a rational number such that the equation $x^m = c$ is irreducible over the rational field, then the determinant
for all sets of \( m \) rational numbers \( a_0, a_1, \ldots, a_{m-1} \) which are not all zero.

**Solution I:** Consider the matrix

\[
M = \begin{bmatrix}
I & C \\
O & A
\end{bmatrix},
\]

where \( I \) is the \((m-1)\times(m-1)\) identity, \( O \) is the \(m\times(m-1)\) zero matrix, \( C \) is the \((m-1)\times m\) matrix with \(-c\) in the \((i,i+1)\) positions and zeros elsewhere, and \( A \) is the given \(m\times m\) array.

At once \( \det(M) = \det(A) \).

For all integers \( r, s \) satisfying \( 1 \leq s \leq r \leq m-1 \), multiply the \( r \)\(^{th}\) row of \( M \) by \( a_{m-s} \) and add to the \((r+m-s)\)\(^{th}\) row.

The determinant of the result, while unchanged in value, is recognized as the eliminant of \( x^m - c \) and \( a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \ldots + a_1x + a_0 \). Since \( x^m - c \) is irreducible over the rationals, it will have no factor in common with a rational polynomial of lesser degree, and thus the eliminant will not vanish (unless, trivially, \( a_{m-1} = \ldots = a_0 = 0 \)).
Solution II: Let \( \mathbb{Q} \) denote the field of rationals and \( \mathbb{K} \) the extension \( \mathbb{K} = \mathbb{Q}(\theta) \) where \( \theta^m = c \), so that \( [\mathbb{K} : \mathbb{Q}] = m \). If 
\[
a = a_0 + a_1\theta + \ldots + a_{m-1}\theta^{m-1} \neq 0,
\]
then the given determinant is, up to the sign, nothing but the norm \( N_{\mathbb{K}/\mathbb{Q}}(a) \), which is nonzero. The simplest way to see this: The mapping \( \beta \mapsto a\beta \) is a non singular linear transformation over \( \mathbb{Q} \) on \( \mathbb{K} \) onto \( \mathbb{K} \). The norm is merely its determinant, here computed with respect to the linear basis \( 1, \theta, \theta^2, \ldots, \theta^{m-1} \).

Problem 137. [AMM, Vol. 70, p. 1011-1012, Prob. E1561]

Find an explicit example of a positive \( n \)th order determinant whose elements are 1's and -1's and which has minimum possible value.

Solution I: Add the first row to each of the remaining rows of the determinant. Since a 2 can be factored out of each of the rows, the minimum positive value must be at least \( 2^{n-1} \). A determinant with this value is 
\[
|a_{ij}|; \quad a_{ij} = 1, \quad i < j; \quad a_{ij} = -1, \quad i > j.
\]

Solution II: An \( n \)th order determinant of 1's and -1's can be regarded as \( n! \) times the content of an \( n \)-dimensional simplex with one vertex at the origin, the others being chosen from the vertex set of the \( n \)-cube with side two units long and center at the origin.
This simplex will have minimum content when the vertices (other than the origin) are chosen so as to lie on the star of a single vertex in the one dimensional skeleton of one of the \((n-1)\)-dimensional faces of the \(n\)-cube. Thus, for example, the minimum positive value is realized by the determinant of the \(nxn\) matrix with 1's in the first column and along the main diagonal, and -1's elsewhere. This value is \(2^{n-1}\).


Let an \(nxn\) matrix have positive entries along the main diagonal and negative entries elsewhere. Assume that it is normalized so that the sum of each column is one. Prove that its determinant is greater than one.

Solution: Use induction. Let the \(nxn\) matrix be \(A=(a_{ij})\). The result is immediately true for \(n=2\); assume that it is true for \((n-1)x(n-1)\) matrices with the required properties. Then, using \(a_{11}\) as a pivotal element, eliminate column 1 of \(A\) to give:

\[
a_{ij} \rightarrow a_{ij} - a_{1j} + a_{i1}/a_{11} = b_{ij} \quad (i,j = 2,\ldots,n).
\]

Note that \(b_{ij} < 0, \quad i \neq j,\) and
Thus \( b_{ii} \geq 1 \) \((i = 2, 3, \ldots, n)\), and \( (b_{ij}) \) is an \((n-1) \times (n-1)\) matrix having the required properties except for normalization.

Hence

\[
|A| = a_{11} \prod_{j=2}^{n} c_j |B|
\]

where \( |B| \) is the determinant of the matrix which is obtained from \( (b_{ij}) \) by dividing each \( j^{th} \) column by \( c_j \) \((j = 2, 3, \ldots, n)\), and is thus normalized. Since \( a_{11} > 1 \), the induction is complete.

Recurrents and Determinants
Whose Elements are Binomial Coefficients

Problem 139. [AMM, Vol. 9, p. 11-13, Prob. 129 (Alg)]

Prove that

\[
\begin{vmatrix}
1 & -(\binom{m-1}{0}) & -(\binom{m-1}{1}) & \cdots & -(\binom{m-1}{m-2}) \\
1 & 1 & -(\binom{m-2}{0}) & \cdots & -(\binom{m-2}{m-3}) \\
1 & 0 & 1 & \cdots & -(\binom{m-3}{m-4}) \\
& \cdots & \cdots & \cdots & \cdots \\
1 & 0 & 0 & \cdots & 1 & -(\binom{1}{0}) \\
1 & 0 & 0 & \cdots & 0 & 1
\end{vmatrix}
\]

\[
e = \sum_{n=1}^{\infty} \frac{n^m}{n!}.
\]
Solution: Let \( f(x) = e^{e^x} = 1 + e^x + \frac{e^{2x}}{2!} + \ldots + \frac{e^{rx}}{r!} + \ldots \)

\[ = 1 + \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \right) + \frac{1}{2!} \left( 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \ldots \right) \]

\[ + \frac{1}{3!} \left( 1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \ldots \right) + \ldots + \frac{1}{r!} \left( 1 + rx + \frac{(rx)^2}{2!} + \frac{(rx)^3}{3!} + \ldots \right) + \ldots \]

\[ = e + \sum_{r=1}^{\infty} a_r x^r, \quad \text{where} \quad a_r = \frac{1}{r!} \sum_{k=1}^{\infty} \frac{k^r}{k!} \cdot \]

But \( f(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \ldots \).

(1) \quad \therefore \quad f^{(m)}(0) = \sum_{k=1}^{\infty} \frac{k^m}{k!} \cdot \]

Let \( y = e^{e^x} \), then \( \log y = e^x \) and \( \frac{dy}{dx} = ye^x \). By Leibnitz's Formula,

(2) \quad f^{(m)}(0) = f^{(m-1)}(0) + (m-1)f^{(m-2)}(0) + \frac{(m-1)(m-2)}{2!} f^{(m-3)}(0) \]

\[ + \ldots + (m-1)f'(0) + f(0) = \sum_{k=1}^{\infty} \frac{k^m}{k!} \quad [\text{by (1)}] . \]

On letting \( x_m \) represent \( f^{(m)}(0) \), etc.,
\[ x^m - x^{m-1} - c_{m-1}, x^{m-2} - \ldots - c_{m-2}, 1^{x_1} = f(0) = e \]

\[ x^{m-1} - x^{m-2} - c_{m-2}, x^{m-3} - \ldots - c_{m-3}, 1^{x_1} = e \]

\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]

\[ x_2 - x_1 = e \]

\[ x_1 = e \cdot \]

\[
\begin{bmatrix}
  e & -\binom{m-1}{0} & -\binom{m-1}{1} & \ldots & -\binom{m-1}{m-2} \\
  e & 1 & -\binom{m-2}{0} & \ldots & -\binom{m-2}{m-3} \\
  e & 0 & 1 & \ldots & -\binom{m-3}{m-4} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  e & 0 & 0 & \ldots & -\binom{1}{0} \\
  e & 0 & 0 & \ldots & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
  1 & -\binom{m-1}{0} & -\binom{m-1}{1} & \ldots & -\binom{m-1}{m-2} \\
  1 & 1 & -\binom{m-2}{0} & \ldots & -\binom{m-2}{m-3} \\
  1 & 0 & 1 & \ldots & -\binom{m-3}{m-4} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  1 & 0 & 0 & \ldots & -\binom{1}{0} \\
  0 & 0 & 0 & \ldots & 1 \\
\end{bmatrix}
\]

\[ e = \sum_{k=1}^{\infty} \frac{m}{k!} \cdot \]
Problem 140. [AMM, Vol. 15, p. 35-36, Prob. 288 (Alg)]

Evaluate the determinant which arises in finding the inverse of the transformation, with binomial coefficients,

\[ T: \xi_i = \sum_{j=1}^{g-1} (j_i)x_j, \quad (i = 1, \ldots, g-1). \]

**Solution:** Denote by \( D_{n,m} \) the minor of the element in the \((n+1)^{\text{th}}\) row and \((m+1)^{\text{th}}\) column. Evidently \( D_{n,n} = 1, D_{n,m} = 0 \) for \( n < m \). For \( n = m + k, \quad k > 0 \),

\[
D_{m+k,m} = \begin{vmatrix}
(m+1) & (m+2) & (m+3) & \ldots & (m+k) \\
1 & (m+2) & (m+3) & \ldots & \(m+k\) \\
0 & 1 & (m+1) & \ldots & \(m+k\) \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & (m+k) \\
\end{vmatrix}.
\]

Studnicka has evaluated a similar determinant:

\[
\begin{vmatrix}
(m+1) & (m+1) & (m+1) & \ldots & (m+1) \\
1 & (m+1) & (m+1) & \ldots & \(m+1\) \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & (m+1) \\
\end{vmatrix} = \binom{m+k}{k}.
\]
Multiply the first \((j-1)\) columns of the latter respectively by \((j-1), (j-2), (j-3), \ldots, (j-1)\), and add the products to the \(j\)th column. Then the \(i\)th element of the new \(j\)th column is (for \(s = j-r\)),

\[
\sum_{r=i-1}^{j} \binom{m+1}{r-i+1} (j-1) = \sum_{s=0}^{j-i+1} \binom{m+1}{j-i+1-s} (j-1) = \binom{m+j}{j+i+1}.
\]

Taking \(j = k, k-1, \ldots, 2\) in turn, we obtain \(D_{m+k, m}\). Hence \(D_{m+k, m} = \binom{m+k}{k} , D_{nm} = \binom{n}{m}\) if \(n > m\). Hence the inverse of transformation \(T\) is

\[
T^{-1}: x_i = \sum_{j=1}^{g-1} (-1)^{i+j} \binom{j}{i} \xi_j \quad (i = 0, 1, \ldots, g-1).
\]

Since the product of two transformations is the identity, we have

\[
\sum_{j=i}^{\ell} (-1)^{\ell+j} \binom{\ell}{j} \binom{j}{i} = \delta_{\ell+i} \binom{\ell}{i} (\delta_{ii} = 1, \delta_{ii} = 0 \text{ if } \ell \neq i).
\]

Conversely, from this well-known formula (cf. Netto, p. 225, (43)) follows the evaluation of the determinant \(D\).
Problem 141. [AMM, Vol. 23, p. 122-123, Prob. 443 (Alg)]

If \( p_r \) denotes the sum of all the different \( r \)-factor products that can be formed from the first \( n \) natural numbers

\[ (p_r = 0 \text{ for } r > n), \text{ and if} \]

\[
\begin{pmatrix}
\begin{array}{cccc}
p_1 & 1 & 0 & \ldots & 0 \\
2p_2 & p_1 & 1 & \ldots & 0 \\
3p_3 & p_2 & p_1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p_s & p_{s-1} & p_{s-2} & \ldots & p_1 \\
\end{array}
\end{pmatrix}
\]

\( D_s = \)

show that

\[
\sum_{i=0}^{k} (-1)^i \binom{k}{i} D_{2k-i} = 0, \quad k, n = 1, 2, 3, \ldots,
\]

where \( c_i = \frac{2k+1-i}{1+i} \) when \( i \) is even and \( 2n+1 \) when \( i \) is odd, and \( \binom{k}{i} \) is the coefficient of \( x^i \) in \((1+x)^k\).

**Solution:** The roots of the equation

\[ x_n - p_1 x^{n-1} + p_2 x^{n-2} - \ldots + (-1)^n p_n = 0 \]

are the natural numbers \( 1, 2, \ldots, n \). Solving Newton's formulae

[Cajori, *Theory of Equations*, p. 85-86] for the sums of like powers of the roots, we obtain

\[ 1^s + 2^s + 3^s + \ldots + n^s = D_s, n, s = 1, 2, 3, \ldots \]
A relation between the \( D_i \)'s of odd subscript has been published [Stern, Crelle's Journal, Vol. 84, p. 216-218] which is equivalent to

\[
(1) \quad \sum_{i=0}^{k/2} \binom{k}{2i+1} D_{2k-1-2i} = 2^{k-1} D_1, \quad k, n = 1, 2, 3, \ldots ,
\]

and the following relation [Proceedings of the Indiana Academy of Sciences, 1914, p. 440] exists among \( D_i \)'s of even subscript,

\[
(2) \quad \sum_{i=0}^{2k-2i-1} \frac{2k+1-2i}{1+2i} \binom{k}{2i} D_{2k-2i} = (2n+1)2^{k-1} D_1, \quad k, n = 1, 2, 3, \ldots .
\]

These formulae, in which \( I(k/2) \) denotes the integral part of \( k/2 \), are readily established by induction. Multiplying (1) by \((2n+1)\) and subtracting the result from (2), we get the formula sought.

**Problem 142.** [AMM, Vol. 32, p. 204-205, Prob. 3081]

Prove

\[
\begin{vmatrix}
\begin{array}{cccccc}
\binom{2}{1} & \binom{4}{2} & \binom{6}{3} & \cdots & \binom{2m-2}{m-1} & \binom{2m}{m} \\
1 & \binom{4}{1} & \binom{6}{2} & \cdots & \binom{2m-2}{m-2} & \binom{2m}{m-1} \\
0 & 1 & \binom{6}{1} & \cdots & \binom{2m-2}{m-3} & \binom{2m}{m-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & \binom{2m}{1}
\end{array}
\end{vmatrix} = 2.
\]
Solution: From each element of the second row subtract half the corresponding element of the first row (the elements of the first row are even integers), and the determinant becomes

| 2 4 6 ... (2m-2k) ... (2m-2) (2m) |
|---|---|---|---|---|---|
| 0 1 5 ... (2m-2k-1) ... (2m-3) (2m-1) |
| 0 1 6 ... (2m-2k) ... (2m-2) (2m) |
| ... ... ... ... ... ... ... ... ... ... ... |
| 0 0 0 ... 0 ... 1 (2m) |

In this determinant subtract each element of the second row from the corresponding elements of the third row; in the resulting determinant subtract each element of the third row from the corresponding element of the fourth row; again in the resulting determinant subtract each element of the fourth row from the corresponding element of the fifth row; and so on, effecting the subtraction by using theorem \( \binom{n}{j} - \binom{n-1}{j} = \binom{n-1}{j-1} \). There will finally result the determinant
This determinant is equal to \(2\), since the cofactor of \(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\), its upper left hand element, is unity.

**Problem 143.** [AMM, Vol. 40, p. 364-369, Prob. 3517]

Let \( \Delta_N(a, \omega) \) denote the determinant

\[
\begin{vmatrix}
1 & -\omega & 0 & \ldots & 0 & 0 \\
-\omega & 1-a\omega & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1-a^{N-3}\omega & 0 \\
0 & 0 & \ldots & \ldots & 1-a^{N-2}\omega \\
0 & 0 & \ldots & \ldots & \ldots & \ldots & 1-a^{N-1}\omega \\
\end{vmatrix}
\]
Solution: The determinant $\Delta_N(a, \omega)$ of order $N$ is of the type in which the elements are of the form

\[
(i, j) = \begin{pmatrix} i \\ j-1 \end{pmatrix}, \quad 1 \leq i \leq N, \quad 1 \leq j \leq i,
\]

\[
(i, i+1) = a_{i, i+1}, \quad 1 \leq i \leq N - 1,
\]

\[
(i, j) = 0, \quad 1 \leq i \leq N - 2, \quad i + 2 \leq j \leq N
\]

where $i, j$ denote the row and column, respectively, of the element, and where the elements $a_{i, i+1}$ are arbitrary. These latter elements lie in a line parallel to the principal diagonal and just above it; call this line the $a$ parallel for brevity. Such a determinant has the property that its value is unaltered if the elements in the $a$ parallel are written in the reverse order; such a reversal may be obtained by rotating the $a$ parallel through $180^\circ$ about the secondary diagonal as an axis, while each other element remains fixed.

This may be proved as follows: Multiply the elements in the $j$ column by $(N-j+1)!(j-1)!$, and divide the elements in the $i$ row by $(N-i)!i!$. When this is done for all the columns and rows, it will be easily seen that the elements $a_{i, i+1}$ are unaltered, whereas any other element $(i, j)$ not zero is replaced by the element which was at $(N-j+1, N-i+1)$. In other words the elements, except those in the $a$ parallel, have been interchanged by a rotation through $180^\circ$ about the secondary diagonal. The value of the determinant has not been changed, since the multiplications and divisions
cancel in the end. Now rotate all the elements of the resulting determinant through 180° about the secondary diagonal. This operation does not alter the value of the determinant, and the result is that each element not in the a parallel has been restored to its original position, but the succession of the elements in the a parallel has been reversed in order. This completes the proof of the proposition.

Now consider a determinant \( D \) of this kind in which \( a_{i, i+1} = 1 - x^i \). To each element of the second column add the product of \( x - 1 \) times the corresponding element of the first column; to the elements of the third column add the products of \( x - 1 \) times the resulting elements of the second column; and continue in this same manner. In the end we have for the element \((i, i)\) in the \(i\)th row

\[
(i, i) = \sum_{k=1}^{i} \binom{i}{i-k} (x-1)^{k-1},
\]

\[
= \frac{1}{x-1} \left[ \sum_{k=0}^{i} \binom{i}{k} (x-1)^{k-1} \right] = \frac{x-1}{x-1}.
\]

In the same row we have zero as the result for \((i, i+1)\), and hence zeros for all following elements. Thus

\[
D = \prod_{i=1}^{N} (1 + x + x^2 + \ldots + x^{i-1}).
\]
Next consider $D'$, where the element $a_{i,i+1} = 1 - y^{N-i}$, $y = x^{-1}$. As shown above, $D'$ is unaltered if each $a_{i,i+1}$ is replaced by $a_{N-i,N-i+1}$ or if $1 - y^{N-i}$ is replaced by $1 - y^i$. When this is done, $D'$ is evaluated by replacing $x$ in $D$ by $y$. Thus we have finally the desired result

$$x^{N(N-1)/2}D' = D = \prod_{i=1}^{N} (1 + x + x^2 + \ldots + x^{i-1})$$

An alternate solution which is quite lengthy is also given.

**Problem 144. [AMM, Vol. 42, p. 53-56, Prob. 3645]**

Show that the value of the determinant formed by deleting the $k^{th}$ column from the array

$$\begin{vmatrix}
1 & 1 & 1 & 1 & 1 & \ldots & 1 & 1 \\
0 & C_1 & 2C_1 & 3C_1 & 4C_1 & \ldots & nC_1 & n+1C_1 \\
0 & 2C_2 & 3C_2 & 4C_2 & \ldots & nC_2 & n+1C_2 \\
0 & 0 & 3C_3 & 4C_3 & \ldots & nC_3 & n+1C_3 \\
0 & 0 & 0 & 4C_4 & \ldots & nC_4 & n+1C_4 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & nC_n & n+1C_n \\
\end{vmatrix}$$

is $n+1C_{k-1}$.
Solution I: From the binomial expansion of \((1-1)^{s-r}\) we have

\[
\sum_{t=0}^{s-r} (-1)^t \binom{s-r}{r} = 0, \quad r < s.
\]

Upon multiplying each term by \(s-1\binom{s-r}{r-1}\) and noting that

\[
\frac{(s-1)!}{(r-1)!} \frac{s-r}{r} \binom{s-r}{r} = \frac{(s-r)!}{r+t-1} \binom{r+t-1}{t} s-1 \binom{s-1}{r+t-1},
\]

we have

\[
\sum_{t=0}^{s-r} (-1)^t \binom{r+t-1}{t} s-1 \binom{s-1}{r+t-1} = 0, \quad r < s
\]

\[
= 1, \quad r = s.
\]

Let us transform the given matrix by taking the rows in order \((r = 1, 2, \ldots, n+1)\) and adding to the elements of each row the corresponding elements of the succeeding rows multiplied respectively by \(-r \binom{1}{r} + (r+1) \binom{2}{r} - (r+2) \binom{3}{r} \cdots\). By (1) we shall have zero for every element for which \(r < s \leq n+1\), where \(s\) represents the column and \(r\) the row in which the element is situated. If \(r = s \leq n+1\), the element is 1. In the last column \((s = n+2)\), for each element, the sum (1) lacks its last term. Hence the element in the \(r^{th}\) row of the last column is

\[
(-1)^{n-r+1} n+1 \binom{n-r+2}{n-r+1} = (-1)^{n+r+1} n+1 \binom{n+1}{r-1}.
\]

The matrix is now such that, if we delete the last column, there is
left a square matrix with unity in the principal diagonal and all other elements of this square matrix are zero.

These transformations will not affect the value of any determinant formed by the deletion of a column.

If now the \( k^{th} \) column \((k \neq n+2)\) is deleted, all elements of the \( k^{th} \) row of the resulting determinant will be zero, except the last, \((-1)^{n+k+1} C_{n+1,k-1}\). The cofactor of this last element is easily seen to be \((-1)^{n+k+1}\). Hence the expansion of the determinant according to elements of the \( k^{th} \) row is \( n+1 C_{n+1,k-1} \). If, however, the \((n+2)^{th}\) column is deleted, the resulting determinant has clearly the value 1, which equals \( n+1 C_{n+1} \).

**Solution II:** From the given array form a square matrix \( A \) of \( n+2 \) rows and columns by adding a last row whose elements are all zeros except \( n+1 C_{n+1} \) in the last column. This matrix is non-singular of determinant unity, and its inverse \( B \) has the same principal diagonal as \( A \) with zeros below this diagonal.

The remaining elements of \( B \) are those of \( A \) but with signs alternately - and + above the principal diagonal; i.e.,

\[ b_{ij} = (-1)^{i+j} \frac{C_{i-1,j-1}}{j-1} \]

**Proof:** If \( d_{ij} \) denotes the element in the \( i^{th} \) row and \( j^{th} \) column of \( AB \), we see immediately that \( d_{ij} = 0 \) if \( i > j \) and that \( d_{ii} = 1 \) \((i, j = 1, 2, \ldots n+2)\). If \( j = i+r \) where \( r \) is positive,
\[ d_{i+1,j+1} = (-1)^{i+j} \left[ C_i \cdot C_j - i+1 C_i \cdot C_{i+1} + \ldots + (-1)^r C_i \cdot C_j \right] \]

\[ = (-1)^{i+j} C_i \left[ 1 - C_1 + C_2 - \ldots + (-1)^r C_r \right] \]

\[ = (-1)^{i+j} C_i (1-1)^r \]

\[ = 0. \]

If \( a_{ij}, b_{ij} \) denote the elements in the \( i^{th} \) row and \( j^{th} \) column of the matrices \( A \) and \( B \) respectively, the cofactor of \( a_{n+2,k} \) is \( b_{k,n+2} = (-1)^{n+2+k} D_k \), where \( D_k \) is the determinant of the matrix obtained from \( A \) by removing the last row and the \( k^{th} \) column. Hence \( D_k = n+1 C_{k-1} \).

**Solution III:** Set \( n_r = 0 \) for \( n < r \). If we add a row to the array, it becomes the matrix \( E = \begin{vmatrix} e_{ij} \end{vmatrix}, e_{ij} = C_i \) \((i,j = 0, 1, \ldots, n+1)\) whose determinant is 1. If this matrix is regarded as operating on the polynomial basis \( (1, x, x^2, x^3, \ldots) \), its effect is to change \( x \) into \( x+1 \); the inverse is therefore

\[ E^{-1} = \begin{vmatrix} f_{ij} \end{vmatrix}, f_{ij} = (-1)^{i+j} e_{ij}, \] which changes \( x \) into \( x-1 \), and the minors required in the problem are, therefore, apart from sign, the elements in the last column of this matrix.

Several editorial comments on the various solutions to this problem are also given.
Problem 145. [AMM, Vol. 42, p. 582-583, Prob. 3679]

Prove that for any positive integer \( p \),

\[
\begin{array}{c|cccc}
(a+nh)^p-a_p & (\frac{p}{2})h^2 & (\frac{p}{3})h^3 & \ldots & (\frac{p}{p-1})h^{p-1} & h^p \\
(a+nh)^{p-1}-a^{p-1} & (\frac{p-1}{1})h & (\frac{p-1}{2})h^2 & \ldots & (\frac{p-1}{p-2})h^{p-2} & h^{p-1} \\
(a+nh)^{p-2}-a^{p-2} & 0 & (\frac{p-2}{1})h & \ldots & (\frac{p-2}{p-3})h^{p-3} & h^{p-2} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
(a+nh)^2-a^2 & 0 & 0 & \ldots & (\frac{2}{1})h & h^2 \\
(a+nh)-a & 0 & 0 & \ldots & 0 & h \\
\end{array}
\]

\[=p!h^p \sum_{k=0}^{n-1} (a+kh)^{p-1}.\]

Solution: Let

\[v_p = \sum_{k=0}^{n-1} (a+kh)^{p-1}.\]

(1) \[v_p = \sum_{k=0}^{n-1} (a+kh)^{p-1}.\]

Then

\[
\sum_{r=0}^{j} \binom{j}{r} v_{j+1-r} h^r = \sum_{r=0}^{j} \binom{j}{r} \sum_{k=0}^{n-1} (a+kh)^{j-r} h^r
\]

(2) \[= \sum_{k=0}^{n-1} \sum_{r=0}^{j} \binom{j}{r} (a+kh)^{j-r} h^r = \sum_{k=0}^{n-1} [a+(k-1)h]^j
\]

\[= v_{j+1} + (a+nh)^j - a^j, \text{ whence}
\]

(3) \[\sum_{r=1}^{j} \binom{j}{r} v_{j+1-r} h^r = (a+nh)^j - a^j.
\]
If we write (3) for all values of $j$ from one to $p$, we have $p$ equations in $p$ unknown $v$'s. The determinant of their coefficients clearly has the value $p!/h^p$. If $D$ be the given determinant, we have then $v_p = D/[p!h^p]$, from which the identity follows.

Problem 146. [AMM, Vol. 43, p. 381-382, Prob. 3710]

If the $C$'s represent binomial coefficients, show that

$$
\begin{vmatrix}
C_2^2 & C_3^3 & C_4^4 & \cdots & C_{n-1}^{n-1} & C_n^n & C_{n+1}^{n+1} \\
-(n-1) & C_2^3 & C_3^4 & \cdots & C_{n-2}^{n-1} & C_{n-1}^n & C_n^{n+1} \\
0 & -(n-2) & C_2^4 & \cdots & C_{n-3}^{n-1} & C_{n-2}^n & C_{n-1}^{n+1} \\
0 & 0 & -(n-3) & \cdots & C_{n-4}^{n-1} & C_{n-3}^n & C_{n-2}^{n+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -2 & C_2^n & C_3^{n+1} \\
0 & 0 & 0 & \cdots & 0 & -1 & C_2^{n+1}
\end{vmatrix} = (n!)^2.
$$

**Solution:** Let the given determinant be denoted by $D_n$. Then we may write (see page 235)

After adding the $(n+1+j)^{th}$ row to the $(n-j)^{th}$ row

$(j = 0, 1, \ldots, n-1)$, and expanding along the $n^{th}$ row, we have

(see page 236).
\[
\begin{array}{cccccccccccc}
C_2^2 & C_3^3 & C_4^4 & \ldots & C_{n-2}^{n-2} & C_{n-1}^{n-1} & C_n^n & C_{n+1}^{n+1} & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
-(n-1) & C_2^3 & C_3^4 & \ldots & C_{n-3}^{n-2} & C_{n-2}^{n-1} & C_{n-1}^n & C_{n+1}^{n+1} & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & -(n-2) & C_2^4 & \ldots & C_{n-4}^{n-2} & C_{n-3}^{n-1} & C_{n-2}^n & C_{n+1}^{n+1} & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & -(n-3) & \ldots & C_{n-5}^{n-2} & C_{n-4}^{n-1} & C_{n-3}^n & C_{n+1}^{n+1} & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & 0 & -2 & C_2^n & C_{n+1}^n & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & C_{n+1}^2 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 & -C_{n+1}^2 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 1 & 0 & 0 & -C_{n+1}^3 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 1 & 0 & 0 & 0 & -C_{n+1}^4 & 0 & 0 & 1 & \ldots & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 1 & 0 & \ldots & 0 & 0 & 0 & -C_{n+1}^{n-1} & 0 & 0 & 0 & \ldots & 1 & 0 & 0 \\
1 & 0 & 0 & \ldots & 0 & 0 & 0 & -C_{n+1}^n & 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & -C_{n+1}^{n+1} & 0 & 0 & 0 & \ldots & 0 & 0 & 1 \\
\end{array}
\]
\[ (1) \quad D_n = \begin{bmatrix}
   C_2^2 & C_3^3 & C_4^4 & \ldots & C_{n-2}^{n-2} & C_{n-1}^{n-1} & C_n^n & 0 & 0 & 0 & \ldots & 0 & 0 & 1 \\
   -(n-2) & C_2^3 & C_3^4 & \ldots & C_{n-3}^{n-2} & C_{n-2}^{n-1} & C_{n-1}^n & 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\
   0 & -(n-3) & C_2^4 & \ldots & C_{n-4}^{n-2} & C_{n-3}^{n-1} & C_{n-2}^n & 0 & 0 & 0 & \ldots & 1 & 0 & 0 \\
   0 & 0 & -(n-4) & \ldots & C_{n-5}^{n-2} & C_{n-4}^{n-1} & C_{n-3}^n & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
   \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
   0 & 0 & 0 & \ldots & -2 & C_{n-2}^{n-1} & C_n^n & 0 & 0 & 1 & \ldots & 0 & 0 & 0 \\
   0 & 0 & 0 & \ldots & 0 & -1 & C_2^n & 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\
   0 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 & -C_2^{n+1} & 0 & 0 & \ldots & 0 & 0 & 0 \\
   0 & 0 & 0 & \ldots & 0 & 0 & 1 & 0 & -C_3^{n+1} & 1 & 0 & \ldots & 0 & 0 & 0 \\
   0 & 0 & 0 & \ldots & 1 & 0 & 0 & 0 & -C_4^{n+1} & 0 & 1 & \ldots & 0 & 0 & 0 \\
   \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
   0 & 1 & 0 & \ldots & 0 & 0 & 0 & -C_{n-1}^{n+1} & 0 & 0 & \ldots & 1 & 0 & 0 \\
   1 & 0 & 0 & \ldots & 0 & 0 & 0 & -C_n^{n+1} & 0 & 0 & \ldots & 0 & 1 & 0 \\
   0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & -C_{n+1}^{n+1} & 0 & 0 & \ldots & 0 & 0 & 1 \\
\end{bmatrix} \]
Subtract the $(2n-1-j)^{\text{th}}$ row from the $(2n-2-j)^{\text{th}}$ row 

$(j = 0, 1, 2, \ldots, n-3)$ in turn, and then add the $(n+1)^{\text{st}}$ row to the $n^{\text{th}}$ row. Finally multiply the $j^{\text{th}}$ row by $(-1)^{n-j}$ $(j=1, 2, \ldots, n-1)$, and add to the $n^{\text{th}}$ row. The $n^{\text{th}}$ row is now

$$(2) \quad a_1, a_2, \ldots, a_{n-2}, a_{n-1}, -n, 0, 0, \ldots, 0, 0,$$

where the $a$'s, as we shall see later, need not be computed.

Adding the $(n+2+j)^{\text{th}}$ row to the $(n+1+j)^{\text{th}}$ row $(j=0, 1, \ldots, n-3)$ in turn, we obtain determinant (1), except for the $n^{\text{th}}$ row which is given by (2). Expanding this determinant along the $n^{\text{th}}$ row, we have, letting $\Delta_{a_j}^k$ denote the minor of $a_j$,

$$D_n = n^2 D_{n-1} + (-1)^n \sum_{j=1}^{n-1} (-1)^j a_j \Delta_{a_j}$$

$$= n^2 D_{n-1}$$

since $\Delta_{a_j}^k = 0$, as can be seen by subtracting from the $(2n-2-j)^{\text{th}}$ row the $(j+1)^{\text{st}}$ row $(j=0, 1, 2, \ldots, n-2)$.

Since $D_2 = 2^2$, the proof by induction is complete.

Problem 147. [AMM, Vol. 44, p. 482-483, Prob. 3748]
where the C's are binomial coefficients, show that

\[
\begin{vmatrix}
 a_1 & -a_2 & a_3 & \cdots & (-1)^{n-1}a_{n-2} & (-1)^n a_{n-1} & (-1)^{n+1} a_n \\
-a_0 & a_1 & -a_2 & \cdots & (-1)^{n-2}a_{n-3} & (-1)^{n-1} a_{n-2} & (-1)^n a_{n-1} \\
0 & -a_0 & a_1 & \cdots & (-1)^{n-3}a_{n-4} & (-1)^{n-2} a_{n-3} & (-1)^{n-1} a_{n-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -a_0 & a_1 & -a_2 \\
0 & 0 & 0 & \cdots & 0 & -a_0 & a_1 \\
\end{vmatrix}
= \frac{1}{n!n^n}.
\]

**Solution:** If we set \( n \alpha_t = A_{tn} \), it may be shown that

\[
(1) \quad A_{tn} - A_{t, n-1} = \frac{1}{n} A_{t-1, n}, \quad A_{0n} = 1, \quad A_{t1} = 1, \quad A_{1n} = \sum_{j=1}^{n} \frac{1}{j}.
\]

The determinant of the problem becomes \( D_{nn}/n^n \), where \( D_{nn} \) has in its \( j^{th} \) row

\[
(2) \quad 0, 0, \ldots, 0, -A_{0n}, A_{1n}, \ldots, (-1)^{n-j+2} A_{n-j+1, n}.
\]

The first row begins with \( A_{1n} \) and ends with \( (-1)^{n+1} A_{nn} \). Replace the \( j^{th} \) row \( R_j \) by \( R_j + R_{j+1}/n, \quad j = 1, 2, \ldots, n-1 \). From (1), it will be seen that, except for the last row, the new determinant is obtained from the original by replacing \( A_{tn} \) by \( A_{t, n-1} \).

By (1) the last row may be written

\[
(3) \quad 0, 0, \ldots, 0, -A_{0, n-1} A_{1, n} + \frac{1}{n}.
\]
and we have

\[(4) \quad D_{nn} = D_{n-1,n} + \frac{1}{n} D_{n-1,n-1},\]

where the second subscript is the order of the determinant. Since we may use the same reduction repeatedly, we have

\[D_{n-1,n} = D_{n-2,n} + \frac{1}{n-1} D_{n-2,n-1} = \sum_{j=1}^{n} b_j D_{j,j+1} = 0.\]

Since \( D_{22} = \frac{1}{2} \), we have \( D_{nn} = \frac{1}{n!} \), and this completes the proof.

(For more detail see the solution to Prob. 3701 [AMM, Vol. 43, p. 197].)


Prove that for any integer \( k \) the following determinant is zero:

\[
\begin{vmatrix}
\frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & \cdots & \frac{1}{(2k+1)!} & \frac{1}{(2k+2)!} \\
1 & \frac{1}{2!} & \frac{1}{3!} & \cdots & \frac{1}{(2k)!} & \frac{1}{(2k+1)!} \\
0 & 1 & \frac{1}{2!} & \cdots & \frac{1}{(2k-1)!} & \frac{1}{(2k)!} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \frac{1}{2!} & \frac{1}{3!} \\
0 & 0 & 0 & \cdots & 1 & \frac{1}{2!}
\end{vmatrix}
\]
Solution: Using a slightly different notation, let \( f(n) \) denote the determinant obtained by replacing in the one given \( (2k+1) \) by \( n \), \( f(0) = 1 \). Then the minor of the \( k \)th element of the first row is \( f(n-k) \), so that for \( n > 0 \), these determinants must satisfy the recursion formula:

\[
f(n) - \frac{1}{2!} f(n-1) + \frac{1}{3!} f(n-2) - \ldots + \frac{(-1)^n}{(n+1)!} f(0) = 0.
\]

The coefficient of \( f(n-k) \) in this recursion formula is seen to be the coefficient of \( x^k \) in the expansion of \( \frac{1}{(1-e^{-x})/x} \). Since this function is developable in a power series with constant term unity, its reciprocal also has a power series expansion (convergent for \( x < 2\pi \)) which is of the form

\[
x/(1-e^{-x}) = F(x) = \sum_{n=0}^{\infty} a_n x^n = 1 + \frac{1}{2} x + \frac{B_1}{2!} x^2 - \frac{B_2}{4!} x^4 + \frac{B_3}{6!} x^6 - \ldots,
\]

where \( B_n \) is the \( n \)th Bernoulli number (by one definition of these numbers), and where the coefficients \( a_n \) are determined by

\[
F(x) \cdot (1-e^{-x})/x = 1.
\]

We easily find that \( a_0 = 1, a_1 = \frac{1}{2} \), and that \( a_n \) is determined by the same recursion formula as \( f(n) \). Since this determination is unique, \( f(n) = a_n \). Furthermore, since

\[
F(x) - \frac{x}{2} = \frac{x}{2} \frac{1+e^{-x}}{1-e^{-x}} = \frac{x}{2} \operatorname{ctnh} \frac{x}{2}
\]
is an even function, the coefficients of the remaining odd powers of $x$ vanish.

Thus $f(2k+1) = 0$, $k$ a positive integer; and this proves the proposition. Also $f(2k) = (-1)^{k-1} B_k/(2k)!$.

Problem 149. [AMM, Vol. 48, p. 274-275, Prob. 3919]

Prove that

$$
\begin{vmatrix}
\frac{1}{1-x} & 1 & 0 & 0 & 0 & \ldots & 0 \\
\frac{x}{1-x^2} & \frac{x}{1-x} & 2 & 0 & 0 & \ldots & 0 \\
\frac{x}{1-x^3} & \frac{x^2}{1-x^2} & \frac{x}{1-x} & 3 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
\frac{x}{1-x^r} & \frac{x^{r-1}}{1-x^{r-1}} & \frac{x^{r-2}}{1-x^{r-2}} & \ldots & \frac{x}{1-x} & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0
\end{vmatrix}
= \frac{r!x^{r(r+1)/2}}{(1-x)(1-x^2)\ldots(1-x^r)}.
$$

Solution: If we write $f_k = x^k/(1-x^k)$, then the quantities $f_k$ satisfy the identity

$$(1) \quad f_k f_{r-k}/f_r = f_k + f_{r-k} + 1, \quad 0 < k < r.$$  

Denoting by $D_r$ the determinant obtained from the given one by dividing its successive rows by $1, 2, \ldots, r$, we are to prove that

$$(2) \quad D_r = f_1 f_2 \ldots f_r.$$
Expanding $D_r$ by minors of its last row, and multiplying by $r$, we have

\[ rD_r = f_r D_{r-1} - f_{2r} D_{r-2} + f_{3r} D_{r-3} - \ldots + (-1)^{r-1} f_r D_0, \]

where we set $D_0 = 1$, $D_{-1} = D_{-2} = \ldots = 0$.

We shall prove equation (2) by mathematical induction. Noting that $D_1 = f_1$, we assume $D_k = f_k D_{k-1}$ for $k < r$, and prove this for $k = r$. Multiplying (1) by $f_r D_{r-k-1}$, we have

\[ f_r D_{r-k} = f_r (f_r D_{r-k-1} + D_{r-k} + D_{r-k-1}), \]

if we use the induction hypothesis $D_{r-k} = f_{r-k} D_{r-k-1}$. Substituting from (4) into (3), we find that

\[
\begin{align*}
\left\{ \begin{array}{ll}
\left[ rD_r = f_r (r-1) D_{r-1} + f_r \sum_{k=1}^{r} (-1)^{k-1} D_{r-k} - f_r \sum_{k=1}^{r} (-1)^{k-2} D_{r-k-1} \\
= f_r (r-1) D_{r-1} + f_r D_{r-1} - f_r D_{r-1} \\
= rf_r D_{r-1} \right.
\end{array} \right.
\]

Factoring $r$, we have the conclusion of the induction.

An outline of an alternate proof, as sketched by the proposer, is also given.
Problem 150. [AMM, Vol. 51, p. 237, Prob. 4701]

Setting \[ \frac{1}{t+2} + \frac{1}{2t+1} + \frac{1}{3t+1} + \ldots + \frac{1}{nt+1} = b_t, \] show that

\[
\begin{vmatrix}
    b_0 & -b_1 & b_2 & \ldots & (-1)^{n-1}b_{n-1} \\
    -(n-1) & b_0 & -b_1 & \ldots & (-1)^{n-2}b_{n-2} \\
    0 & -(n-2) & b_0 & \ldots & (-1)^{n-3}b_{n-3} \\
    \ldots & \ldots & \ldots & \ldots & \ldots \\
    0 & 0 & 0 & \ldots & b_0 \\
\end{vmatrix} = 1.
\]

**Solution:** Consider the \( n \) numbers \( \beta_1, \beta_2, \beta_3, \ldots, \beta_n \); let \( S_k \) denote the sum of the \( k \)th powers; and \( p_k \) the elementary symmetric function involving \( k \) factors. Then by Newton's formula [See Dickson, *Elementary Theory of Equations*, p. 70], we have

\[
S_k = p_1 S_{k-1} + p_2 S_{k-2} - \ldots + (-1)^{k-1} p_{k-1} S_1 + (-1)^k p_k = 0,
\]

\( k = 1, 2, \ldots, n. \)

Solving these equations for \( p_n \) by Cramer's rule and making certain simplifications and rearrangements in the determinants, we have
We obtain the desired result by setting \( b_k = \frac{1}{k} \) and \( S_k = b_{k-1} \).

**Problem 151.** [AMM, Vol. 54, p. 473-474, Prob. E756]

Show that

\[
\begin{vmatrix}
 a-x & 1 & 0 & 0 & \cdots & 0 \\
\binom{a}{2} & a-x & 1 & 0 & \cdots & 0 \\
\binom{a}{3} & \binom{a}{2} & a-x & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\binom{a}{n} & \binom{a}{n-1} & \binom{a}{n-2} & \binom{a}{n-3} & \cdots & a-x \\
\end{vmatrix} = \binom{a+n-1}{n} - \binom{2a+n-2}{n-1}x + \binom{3a+n-3}{n-2}x - \cdots + (-1)^n x^n.
\]

**Solution:** We consider first the general problem: Being given

\( a_0, a_1, a_2, \ldots \), to find \( u_0, u_1, u_2, \ldots \) so that for the indeterminant

\( t, (a_0 + a_1 t + a_2 t^2 + \cdots) (u_0 + u_1 t + u_2 t^2 + \cdots) = 1 \). We have
\[ a_0 u_0 = 0 \]
\[ a_1 u_0 + a_0 u_1 = 0 \]
\[ a_2 u_0 + a_1 u_1 + a_0 u_2 = 0 \]
\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]
\[ a_n u_0 + a_{n-1} u_1 + a_{n-2} u_2 + \ldots + a_0 u_n = 0. \]

Hence, by Cramer's Rule,
\[
(-1)^n a_0 a_1 a_2 a_3 \ldots a_n a_{n-1} a_{n-2} a_{n-3} \ldots a_1 u_n = 0.
\]

Our problem is a special case in which \( a_0 = 1, \ a_1 = a-x, \) \( a_n = \binom{a}{n} \) for \( n \geq 2. \) In order to evaluate the determinant, we have to pick out \( u_n, \) the coefficient of \( t^n, \) in the expansion of
\[
\frac{1}{(1+t)^a - xt} = \frac{(1+x)^{-a}}{1-xt(1-t)^{-a}}
\]
\[
= (1+t)^{-a} + xt(1+t)^{-2a} + \frac{x^2}{2}(1+t)^{-3a} + \ldots.
\]
Thus \( u_n = \binom{-a}{n} + x\binom{-2a}{n-1} + \frac{x^2}{2}\binom{-3a}{n-2} + \ldots \). Observing that
\[
\binom{-a}{n} = \frac{(-a)(-a-1)\ldots(-a-n+1)}{n!} = (-1)^n \binom{a+n-1}{n}.
\]
we obtain the proposed equation.


Show that

\[
\begin{vmatrix}
\binom{r+1}{r} & \binom{r+1}{r+1} & 0 & 0 & \ldots & 0 \\
\binom{r+2}{2} & \binom{r+2}{r+1} & \binom{r+2}{r+2} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\binom{n}{r} & \binom{n}{r+1} & \binom{n}{r+2} & \binom{n}{r+3} & \ldots & \binom{n}{n-1}
\end{vmatrix}
= \binom{n}{r}.
\]

**Solution:** Let us give a proof by induction. We first note that

the identity holds for \( n = r+1 \) and \( n = r+2 \). On the assumption

that it is true for \( n = r+1, r+2, \ldots, k-1 \), we now show that it is

true for \( n = k \). Denote the determinant for \( n = k \) by \( D_k \). Ex-

panding \( D_k \) repeatedly with respect to elements of the last column,

we obtain

\[
D_k = k \binom{k-1}{r} - \frac{k(k-1)}{2!} \binom{k-2}{r} + \frac{k(k-1)(k-2)}{3!} \binom{k-3}{r} - \ldots
\]

\[
\pm \frac{k(k-1) \ldots (r+2)}{(k-r-1)!} \binom{r+1}{r} \mp \binom{k}{r}.
\]

Factoring out \( \binom{k}{r} (k-r) \) from all terms, \( (k-r-1)/2 \) from all but

the first, \( (k-r-2)/3 \) from all but the first two, etc., we obtain
\[ D_k = \binom{k}{r} (k-r) \left\{ 1 - \frac{k-r-1}{2} \right\} \left\{ 1 - \frac{k-r-2}{3} \right\} \left\{ 1 - \frac{k-r-3}{4} \right\} \ldots \left\{ 1 - \frac{2}{k-r-1} \right\} \right\} \ldots \right\} \right\} \right) \right) . \]

Successively simplifying the last bracket, the \( j \)th bracket is seen to be \( (k-r-j)/(k-r) \), and continuing,

\[ D_k = \binom{k}{r} (k-r) \left\{ 1 - \frac{k-r-1}{2} \cdot \frac{2}{k-r} \right\} = \binom{k}{r} . \]

The identity is now established for all integral \( n \geq r + 1 \).

**Problem 153.** [AMM, Vol. 58, p. 423-424, Prob. 4367]

Evaluate the determinant

\[ D_n = \begin{vmatrix} \binom{n}{r} & \binom{n}{r+1} & \binom{n}{r+2} & \cdots & \binom{n}{r+n} \\ \binom{n-1}{r} & \binom{n-1}{r+1} & \binom{n-1}{r+2} & \cdots & \binom{n-1}{r+n-1} \\ \binom{n-2}{r} & 0 & \binom{n-2}{r+1} & \cdots & \binom{n-2}{r+n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \binom{0}{r} \end{vmatrix} \]

of the \( (n+1) \)th order where \( \binom{n}{r} \) and \( \binom{n}{r} \) are the usual permutation and combination symbols.

**Solution I:** Noting that \( \binom{r}{0} = 1 \) for all \( r \), we obtain by
expanding \( D_n \) with respect to the elements of the first row,

\[
D_n = n \cdot P_n - \sum_{i=1}^{n} nC_i \cdot D_{n-i} \quad \text{or} \quad P_n = \sum_{i=0}^{n} nC_i \cdot D_{n-i}.
\]

Using this, it is readily proved by induction that

\[
D_n = \sum_{s=0}^{n} (-1)^{n-s} nP_s = n! \sum_{i=0}^{n} (-1)^i / i!.
\]

**Solution II:** Subtract the \((r+1)\)th row from the \(r\)th row

\((r = 1, 2, \ldots, n)\) to obtain a determinant whose last column is filled with zeros except the last element, which is 1. We may then delete its last column and last row to get

\[
\begin{vmatrix}
\Delta(n-1)! & n-1C_1 & n-1C_2 & \cdots & n-1C_{n-1} \\
\Delta(n-2)! & n-2C_0 & n-2C_1 & \cdots & n-2C_{n-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\Delta 0! & 0 & 0 & \cdots & 0C_0
\end{vmatrix},
\]

where \( \Delta r! = (r+1)! - r! \). This has the same form as \( D_{n-1} \) except for the first column which is symbolically multiplied by the difference operator \( \Delta \). We repeat the process \( n \) times to arrive at the one-rowed determinant.
\[ D_n = \Delta^n 0! = n! - \sum_{k=1}^{n} C_n^k (n-k)! \cdot \sum_{k=1}^{n} C_n^k (n-k+1) - \cdots + (-1)^n C_n^0 \]

\[ = n! \left( 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!} \right) . \]

This is the "recontre number" and is asymptotic to \( n!/e \).

A note concerning possible alternate solutions is also given.

**Problem 154.** [AMM, Vol. 58, p. 633-634, Prob. E958]

If \( a_n = \frac{1}{n!} \) show that

\[
\begin{vmatrix}
    a_1 & a_0 & 0 & 0 & \cdots & 0 \\
    a_2 & a_1 & a_0 & 0 & \cdots & 0 \\
    a_3 & a_2 & a_1 & a_0 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    a_n & \ldots & \ldots & \ldots & \ldots & a_1
\end{vmatrix} = a_n .
\]

**Solution I:** The result is obviously true for determinants of orders 1 and 2. If the result is assumed for all orders less than \( n \), then expansion by cofactors of elements of the first rows yields for the determinant \( D_n \) of order \( n \):

\[
D_n = a_{n-1} - a_{n-2} a_{n-3} + a_{n-3} a_{n-4} - \cdots + (-1)^{n+1} a_n
\]

\[
= \frac{1}{n!} \left[ \binom{n}{1} - \binom{n}{2} + \binom{n}{3} - \cdots + (-1)^{n+1} \right] = \frac{1}{n!} = a_n .
\]
Solution II: Subtract the first column from the second; in the resulting determinant subtract twice the second column from the third; in the resulting determinant subtract three times the third column from the fourth; and continue. We may show by induction that when the process is completed the resulting determinant is triangular with the \( j \)th element along the principal diagonal equal to \( 1/j \), from which the result follows.

Solution III: Let \( f(x) = \frac{1}{p(x)} \), where

\[
p(x) = \sum_{k=0}^{\infty} a_k x^k, \quad a_0 \neq 0.
\]

Then the power series expansion for \( f(x) \) is

\[
\sum_{k=0}^{\infty} b_k x^k,
\]

where \( b_n = (-1)^n D_n / b_0^{n+1} \), \( D_n \) being the determinant of the problem. Therefore \( D_n = (-1)^n b_n b_0^{n+1} \). Now consider the special case \( a_n = \frac{1}{n!} \). Then \( p(x) = e^x \), \( f(x) = e^{-x} \) and \( b_n = (-1)^n n! \). Hence \( D_n = \frac{1}{n!} = a_n \).

Evaluate

\[
\begin{vmatrix}
  b_1 & -1 & 0 & \cdots & 0 \\
  b_2 & b_1 & -2 & \cdots & 0 \\
  b_{n-1} & b_{n-2} & b_{n-3} & \cdots & 1-n \\
  b_n & b_{n-1} & b_{n-2} & \cdots & b_1 \\
\end{vmatrix}
\]

where (2) \( b_n = \frac{n^n}{n!} \).

**Solution:** In general, to obtain a relation between the coefficients of \( a_n \) and \( b_n \) in (1), without taking account of (2), we may set \( a_0 = 1, a_1 = b_1 \), and define the two functions \( y \) and \( z \) by the power series

\[
(3) \quad y = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}, \quad z = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!}.
\]

Upon expanding determinant (1) by cofactors of the last row, we obtain the recurrence relation

\[
(4) \quad n \frac{a_n}{n!} = a_0 b_n + \frac{a_1}{1!} b_{n-1} + \frac{a_2}{2!} b_{n-2} + \cdots + \frac{a_{n-1} b_1}{(n-1)!},
\]

which is equivalent to differential equation \( y' = yz' \), whose solution satisfying \( y = 1 \) when \( x = z = 0 \) is

\[
(5) \quad y = e^z.
\]
Hence, using the coefficients \( b_n \) defined in (2), our problem is to find the coefficient \( a_n \) of \( x^n/n! \) in the expansion of \( e^x \) when

\[
z = \sum_{k=1}^{\infty} \frac{(kx)^k}{k!}.
\]

If in (6) we make the substitution

\[
x = te^{-t},
\]

we find

\[
\frac{z}{t} = \sum_{k=1}^{\infty} \frac{(kt)^{k-1} e^{-kt}}{k!} = \sum_{m,k=1}^{\infty} (-1)^{m-k} \frac{m}{k} \frac{(kt)^{m-1}}{m!}
\]

\[
= 1 + \sum_{m=2}^{\infty} \frac{t^{m-1}}{m!} \left[ \sum_{k=0}^{m} (-1)^{m-k} \frac{m}{k} \frac{m-1}{k} \right].
\]

The expression in brackets vanishes for \( m > 1 \), since it is the coefficient of \( t^{m-1}/(m-1)! \) in the expansion of \( (e^t-1)^m \).

Hence, the right member of (8) reduces to 1, and we have \( z=t \), and from (7) \( x = ze^{-z} \). It follows from (5), (3), and (2) that

\[
y = \frac{e^z}{x} = \sum_{n=0}^{\infty} b_{n+1} \frac{x^n}{(n+1)} = \sum_{n=0}^{\infty} (n+1)^{n-1} \frac{x^n}{n!}.
\]

By comparing coefficients \( x^n/n! \) in (3) and (9) we obtain the
required formula for the determinant $a_n$ in (1), namely, 

$$a_n = (n+1)^{n-1}.$$ 

Several alternate solutions are discussed briefly by the editors.


Let $D$ be a determinant of order $n$ whose $i^{th}$ row, 
i = 1, 2, \ldots, n, \text{ is } a_1, a_{i-1}, \ldots, a_1, 1, 0, \ldots, 0. \text{ Show that}$

$$D = (-1)^n \sum \frac{(-1)(-2) \cdots (-\sum a_i)}{a_1! a_2! \cdots a_n!} a_1 a_2 \cdots a_n,$$

where the sum is taken over all non-negative integral solutions of 

$$a_1 + 2a_2 + \ldots + na_n = n.$$ 

Solution: The sum is the coefficient, for $x$ sufficiently

small, of $x^n$ in the expansion of $(1 + a_1 x + a_2 x^2 + \ldots + a_n x^n)^{-1}$,


$$(1 + a_1 x + a_2 x^2 + \ldots + a_n x^n)^{-1} = 1 + b_1 x + b_2 x^2 + \ldots.$$ 

Then 

$$(1 + a_1 x + a_2 x^2 + \ldots + a_n x^n)(1 + b_1 x + b_2 x^2 + \ldots) = 1,$$ 

and collecting coefficients we have
\[ a_1 + b_1 = 0 \]
\[ a_2 + a_1 b_1 + b_2 = 0 \]
\[ a_3 + a_2 b_1 + a_1 b_2 + b_3 = 0 \]
\[ \ldots \]
\[ a_n + a_{n-1} b + a_{n-2} b_2 + \ldots + b_n = 0. \]

Solving this system of linear equations in \( b_i \) for \( b_n \), we easily find \( b_n = (-1)^n D \).

A sidelight is that the number of solutions of \( a_1^2 + 2a_2^2 + \ldots + n^2 = n \) is the number of partitions \( P(n) \) of the integer \( n \), so that the number of terms in the expansion of the determinant \( D \) is \( P(n) \).


If \( S_k = 1^k + 2^k + \ldots + n^k \), show that

\[
\begin{vmatrix}
S_1 & 1 & 0 & 0 & \ldots & 0 \\
S_2 & S_1 & 2 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
S_{n-1} & S_{n-2} & \ldots & \ldots & (n-1) \\
S_n & S_{n-1} & \ldots & \ldots & S_1
\end{vmatrix} = (n!)^2.
\]

**Solution:** The following more general result may be of
interest. Let \( x_1, \ldots, x_n \) be arbitrary numbers, let
\[
S_k = x_1^k + x_2^k + \ldots + x_n^k,
\]
let \( D \) denote the given determinant
under the more general interpretation, and let \( p_1, \ldots, p_n \) denote
elementary symmetric functions of the \( x'\)'s. Then it is well known
that
\[
p_0 S_k - p_1 S_{k-1} + p_2 S_{k-2} - \ldots + (-1)^{k-1} p_{k-1} S_1 + (-1)^k k p_k = 0
\]
for \( k = 1, 2, \ldots, n \) and where \( p_0 = 1 \). We may consider this a
system of equations in the unknowns \( p_0, -p_1, \ldots, (-1)^{n-1} p_{n-1} \).
Solving for \( p_0 \) we get \( p_0 = n! p_n / D \), or \( D = n! p_n \).

Problem 158. [AMM, Vol. 68, p. 69-70, Prob. 4782]

Given a composite function \( F(x) = f(g(x)) \), denote the \( n^{\text{th}} \)
derivative of \( f(g) \) by \( D^{(n)} f \), and the derivatives of \( g(x) \) by
\( g', g'', \ldots, g^{(n)} \). Show that

\[
F^{(n)}(x) = \begin{vmatrix}
g' & g'' & g''' & g^{(4)} & \cdots & g^{(n)} \\
-1 & g'D & 2g''D & 3g'''D & \cdots & (n-1)g^{(n-1)}D \\
0 & -1 & g'D & 3g''D & \cdots & (n-2)g^{(n-2)}D \\
0 & 0 & -1 & g'D & \cdots & (n-3)g^{(n-3)}D \\
\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\cdots \\
0 & 0 & 0 & 0 & \cdots & g'D \\
\end{vmatrix}
\]

\( Df \).
Solution: Expand the determinant by minors, using the elements of the last row: \( F^{(n)}(x) = \)

\[
\begin{vmatrix}
g' & g'' & \ldots & g^{(n-2)} & g^{(n)} \\
-1 & g'D & \ldots & (n-3)g^{(n-3)}D & (n-1)g^{(n-1)}D \\
0 & -1 & \ldots & (n-3)g^{(n-4)}D & (n-1)g^{(n-2)}D \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & -1 & (n-2)g''D
\end{vmatrix}
\]

Continuing in like manner:

\[
F^{(n)}(x) = g'DF^{(n-1)}(x) + \left( \frac{n-1}{n-2} \right) g''DF^{(n-2)}(x) + \ldots
\]

\[
+ \left( \frac{n-1}{1} \right) g^{(n-1)}DF'(x) + g^{(n)}Df,
\]

where \( F^{(n-1)}(x), F^{(n-2)}(x), \ldots, F'(x) \) can be written as determinants. From \( F(x) = f[g(x)] \), we obtain at once

\[ F'(x) = (df/dg)g'(x) = g'Df, \]

which agrees with (2) for \( n=1 \).

The argument now proceeds by induction. Assume (2) to be true for all \( n < N \). Replace \( n \) in (2) by \( N \), differentiate both sides with respect to \( x \), and group like terms:
\[ F^{(N+1)}(x) = g'DF^{(N)}(x) + \binom{N-1}{N-2} g''DF^{(N-1)}(x) + \binom{N-1}{N-3} g'''DF^{(N-2)}(x) \]
\[ + g''DF^{(N-1)}(x) + \binom{N-1}{N-2} g'''DF^{(N-2)}(x) \]
\[ + \ldots + \binom{N-1}{1} g^{(N-1)}DF''(x) + \binom{N-1}{0} g^{(N)}DF'(x) \]
\[ + \ldots + \binom{N-1}{2} g^{(N-1)}DF''(x) + \binom{N-1}{1} g^{(N)}DF'(x) + g^{(N+1)}Df. \]

(3)

Making use of the fact,

\[
\binom{N-1}{N-k+1} + \binom{N-1}{N-k} = \binom{N}{N-k+1},
\]

and the fact (easily proved) that the order of differentiation with respect to \( g \) and differentiation with respect to \( x \) may be reversed, equation (3) becomes

\[ F^{(N+1)}(x) = g'DF^{(N)}(x) + \binom{N}{N-1} g''DF^{(N-1)}(x) + \binom{N}{N-2} g'''DF^{(N-1)}(x) \]
\[ + \ldots + \binom{N}{1} g^{(N-1)}DF''(x) + \binom{N}{1} g^{(N)}DF'(x) + g^{(N+1)}Df, \]

which is the same as (2) with \( n \) replaced by \( N+1 \). Thus (2) is proved and the equivalent relation (1) is established.
Problem 159. [AMM, Vol. 70, p. 899-900, Prob. 3172]

Making the abbreviation

$$D_n(a_1, a_2, \ldots, a_n) = \begin{vmatrix} a_1 & a_2 & a_3 & \ldots & a_n \\ 1 & a_1 & a_2 & \ldots & a_{n-1} \\ 0 & 1 & a_1 & \ldots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a_1 \end{vmatrix}$$

prove the following:

(a) If $$\delta_n = D_n \left( \frac{1}{2!}, \frac{1}{4!}, \frac{1}{6!}, \ldots, \frac{1}{(2n)!} \right)$$,

then $$\lim_{n \to \infty} \left| \frac{\delta_{n-1}}{\delta_n} \right| = \frac{\pi^2}{4}$$.

(b) The Bernoulli numbers, $$B_n$$, are given by the determinant

$$B_n = (2n)! D_n \left( \frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}, \ldots, \frac{1}{(2n+1)!} \right)$$.

(c) If $$D_i$$ denotes $$D_i(a_1, a_2, \ldots, a_i)$$ ($$i = 1, 2, \ldots, n$$), then we have $$D_n(-D_1, D_2, -D_3, \ldots, (-1)^n D_n) = (-1)^n a_n$$.

Solution: (a) We use the following theorem of R. P. Agnew, J. B. Rosser, and R. J. Walker [AMM, Vol. 49, p. 462-463]:

Theorem: Let $$f(z)$$ be an analytic function, and $$r$$ a
complex number, such that (i) \( f(0) = -1 \), (ii) \( r \) is a simple root of \( f(z) = 0 \), (iii) if \( s \) is a root of \( f(z) = 0 \) distinct from \( r \) then \(|s| > |r|\), (iv) \( f(z) \) has no singular points, except possible poles, in or on the circle \(|z| = |r|\). Let \(-1 + a_1 z + a_2 z^2 + \ldots\) be the Maclaurin expansion for \( f(z) \), and define \( A_0, A_1, \ldots \) by \( A_0 = 1, \)

\[
A_n = a_1 A_{n-1} + a_2 A_{n-2} + \ldots + a A_{n-1} \quad n > 0.
\]

Then

\[
r = \lim_{n \to \infty} \frac{A_n}{A_{n+1}}.
\]

If we examine the Maclaurin expansion for \(-\cos \sqrt{x}\), we obtain for the numbers \( A_n \) precisely the numbers \( \delta_n \). The result follows from the theorem, once we note that \( \pi^2/4 \) is the zero of \(-\cos \sqrt{x}\), which is smallest in absolute value.

(b) This is a well-known result. Its proof can be found in Muir's *History of the Theory of Determinants*, §762.

(c) Denote \( D_n (-D_1, D_2, -D_3, \ldots, (-1)^n D_n) \) by \( Y_n \). Expanding, we obtain

\( (*) \)

\[
Y_n = -D_1 Y_{n-1} - D_2 Y_{n-2} - D_3 Y_{n-3} - \ldots - D_n
\]

\( (\dagger) \)

\[
= -a_1 (-D_1 Y_{n-2} - D_2 Y_{n-3} - D_3 Y_{n-4} - \ldots )
\]

\[
- \quad (a_1 D_1 - a_2) Y_{n-2}
\]

\[
- (a_1 D_2 - a_2 D_1 + a_3) Y_{n-3}
\]

\[
- (a_1 D_3 - a_2 D_2 + a_3 D_1 - a_4) Y_{n-4}, \text{ etc.,}
\]
where the $i^{th}$ row of $(\dagger)$ corresponds to the $i^{th}$ term of $(\ast)$. Now the first row of $(\dagger)$ adds with the first term of each of the remaining rows to give zero. Then a similar expansion of $Y_{n-2}$ in the second row adds with the second terms of all the remaining rows to give zero. Continuing this process, we see that all that is left is $(-1)^n a_n$.

Problem 160. [AMM, Vol. 72, Prob. 5343]

Show that

$$
\sum_{t=1}^{r} \frac{r!}{t!} = (-1)^r

\begin{vmatrix}
0! & 1 & 0 & 0 & \ldots & 0 & 0 \\
1! & -\binom{1}{0} & \binom{1}{1} & 0 & \ldots & 0 & 0 \\
2! & \binom{2}{0} & -\binom{2}{1} & \binom{2}{2} & \ldots & 0 & 0 \\
(r-1)! & (-1)^{r-1} \binom{r-1}{0} & - & \ldots & -(-1)^{r-2} \binom{r-2}{r-1} & (r-1)^{r-1} \\
r! & (-1)^r \binom{r}{0} & (-1)^{r-1} \binom{r}{1} & - & \ldots & (r)^{r-2} & -(r)^{r-1}
\end{vmatrix}
$$

Solution: No solution to this problem has yet been published.

Problem 161. [DMVJ, Vol. 63, Pt. 2, p. 16-17, Prob. 377]

Show that for $n, k = 1, 2, 3, \ldots$, the $k^{th}$ order determinant in the upper left corner of the infinite matrix
\[
\begin{bmatrix}
  n^1 & 1 \\
  -n^2 & 1 & 2 \\
  n^3 & 1 & 3 & 3 \\
  -n^4 & 1 & 4 & 6 & 4 \\
    &    &    &   &
\end{bmatrix}
\]

has the value \( k! S_{k-1}(n) = k! \left( 1^{k-1} + 2^{k-1} + \ldots + n^{k-1} \right) \).

**Solution I:** Denote \( S_{k-1}(n) \) by the shorter \( S(n) \). Then, from the well-known fact

\[
S(n) = \sum_{v=1}^{n} v^{k-1} = \sum_{v=1}^{k} a_v n^v = \sum_{v=1}^{k} a_v (k)n^v,
\]

follows the Taylor expansion

\[
S(n-1) = \sum_{v=1}^{n-1} v^{k-1} = \sum_{v=0}^{k} (-1)^v \frac{S^{(v)}(n)}{v!}.
\]

With that we have

\[
S(n) - S(n-1) = n^{k-1} = -\sum_{v=1}^{k} (-1)^v \frac{S^{(v)}(n)}{v!} = -\sum_{v=1}^{k} \left[ (-1)^v \sum_{u=0}^{k} \binom{k}{u} a_u n^{u-v} \right].
\]

Comparing coefficients of the \( k \) powers \( n^0, \ldots, n^{k-1} \) leads to the system of equations.
\[(0^1, \ldots, 0^{k-1}, -(-1)^k) = (a_1, -a_2, a_3, \ldots, -(-1)^ka_k)\]

\[
\begin{pmatrix}
1 & & \\
1 & 2 & \\
& 1 & 3 & 3 \\
& 4 & 6 & 4 \\
& \cdots & \cdots & \cdots \\
1 & k & \cdots & k \\
\end{pmatrix}
\]

for \(k > 1\) (the assertion holds for \(k = 1\)).

The proof is completed by multiplying on the right with the inverse matrix, and then with

\[
\begin{pmatrix}
n^1 \\
n^2 \\
n^3 \\
\vdots \\
-(-n)^{k} \\
\end{pmatrix}
\]

In so doing we calculate, with the notation \((-1)^k = \pm 1\), in the usual manner, as follows: on the right, \(a_1n^1 + \ldots + a_kn^k\), and on the left,
It is not necessary here to know the numerical values of the elements of the inverse matrix.

Solution II: By subtracting equation (1) or (2),

\[ (1) \quad 2 \sum_{v=1}^{\infty} \binom{k}{2v} S_{k-2v}(n) = (n+1)^k - n^k - 1 \quad (k = 2, 3, 4, \ldots), \]

\[ (2) \quad 2 \sum_{v=1}^{\infty} \binom{k}{2v-1} S_{k-2v+1}(n) = (n+1)^k + n^k - 1 \quad (k = 1, 2, 3, \ldots), \]
\((S_0^{(n)} = n)\) from the well known equation

\[
\sum_{v=0}^{\infty} \binom{k}{v} S_v(n) = (n+1)^k - 1,
\]

it follows that

\[
\sum_{k=0}^{k-1} (-1)^v \binom{k}{v} S_v(n) = -(-n)^k \quad (k = 1, 2, 3, \ldots).
\]

Solving the last system of equations for \(S_v(n)\) by Cramer's Rule yields the desired conclusion.

**Problem 162.** [DMVJ, Vol. 63, Pt. 2, p. 17-18, Prob. 378]

It is evident that \(n = -1\) and \(n = 0\) are zeros \((k > 1)\) for the following two determinant forms of the Bernoulli polynomial

\[
S_{k-1}(n) = S_{k-1}:
\]

\[
S_{k-1}(n) = 1^{k-1} + 2^{k-1} + \ldots + n^{k-1}
\]

\[
(1) \quad \frac{1}{1!2!\ldots k!k} \begin{vmatrix}
(n+1)^k & 1^2 & 0^2 & \ldots & 1^k & 0^k \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
(n+k)^{-k} & k^{-2(k-1)} & 2^{-(k-1)} & \ldots & k^{-(k-1)} & k^{-1}
\end{vmatrix},
\]
The zero $n = -\frac{1}{2}$ of $S_2, S_4, S_6, \ldots$ is not evident however.

Find a determinant form for these latter $S_k$ which allows all three zeros to be easily recognized.

**Solution:** Let

\begin{align*}
(3) \quad & 2M_k(n) = 1 - n^k - (n+1)^k; \quad 2N_k(n) = 1 + n^k - (n+1)^k.
\end{align*}

then from the recursion formulas,

\begin{align*}
(4) \quad & 2 \sum_{v=1}^{\infty} \binom{k}{2v} S_{k-2v}(n) = (n+1)^k - n^k - 1 \quad (k = 2, 3, 4, \ldots),
\end{align*}

\begin{align*}
(5) \quad & 2 \sum_{v=1}^{\infty} \binom{k}{2v-1} S_{k-2v+1}(n) = (n+1)^k + n^k - 1 \quad (k = 1, 2, 3, \ldots),
\end{align*}

and Cramer's Rule, we get the $\binom{k+1}{2}$ rowed determinant forms
\[
S_k(n) = \begin{vmatrix}
M_1 & \binom{1}{0} & \binom{1}{0} \\
M_3 & \binom{3}{0} & \binom{3}{2} & -\binom{3}{2} \\
M_5 & \binom{5}{0} & \binom{5}{2} & \binom{5}{4} & -\binom{5}{4} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
M_{k+1} & \vdots & \vdots & \vdots & -\binom{k+1}{k}
\end{vmatrix}
\]

\( (k=2,4,6,\ldots) \)

\[
N_2 = \begin{vmatrix}
\binom{2}{0} & -\binom{2}{0} \\
N_4 & \binom{4}{0} & \binom{4}{2} & -\binom{4}{2} \\
N_6 & \binom{6}{0} & \binom{6}{2} & \binom{6}{4} & -\binom{6}{4} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
N_{k+2} & \vdots & \vdots & \vdots & -\binom{k+2}{k}
\end{vmatrix}
\]

By applying (3),

\[
M_{k-1} = N_k = -n = \text{constant}
\]

for \( k = 2,4,6,\ldots \) and \( n = 0, \ -\frac{1}{2}, \ -1 \). Thus, in these three cases, the first two columns of the determinants in the numerator of \( S_k \) are proportional. Thus, in two ways, \( S_k \) reveals the three zeros, \( n = 0, \ -\frac{1}{2}, \ -1 \), as required.
Problem 163. [MM, Vol. 31, p. 172-173, Prob. 311]

Prove the following relation:

\[
\begin{vmatrix}
1 & 1 & 0 & \ldots & 0 \\
1 & -\binom{3}{3} & \binom{3}{1} & \ldots & 0 \\
1 & \binom{5}{5} & -\binom{5}{3} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & (-1)^n\binom{2n+1}{2n+1} & (-1)^{n-1}\binom{2n+1}{2n-1} & \ldots & -(\frac{2n+1}{3})
\end{vmatrix}
\]

where \((2n+1)!! = 1 \cdot 3 \cdot 5 \cdots (2n+1)\) and \(B_{2n}\) are the coefficients in the expansion of \(\sinh n / \sin n\).

**Solution:** It is known that

\[
\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}
\]

and

\[
\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}
\]

define their quotient (not in the usual sense) by

\[
\sum_{n=0}^{\infty} \frac{B_{2n} x^{2n}}{(2n)!}
\]

Then we have by equating the corresponding coefficients,
Solving for $B_{2n}$ by means of Cramer's Rule, we get the required result by interchanging the last column of the determinant in the numerator to the first column and noting that the value of the denominator is precisely $(2n+1)!!$. If $B_{2n}$ are defined in the ordinary sense, we need a factor $(2n)!$ to the left of the given expression.

Problem 164. [MM, Vol. 17, p. 231-233, Prob. 473]

Prove the obvious generalization of the following relation:

$$\sum_{x=1}^{n} x^5 = \frac{n(n-1)}{6!}$$

where the portion of the determinant below the principal diagonal is identical with a portion of the Pascal triangle except for the
negative signs in alternate diagonals. The determinant is unchanged in value if all signs are made positive and \( n \) is replaced by \( n+1 \).

**Solution:** The relation suggested in the problem may be written

\[
\sum_{x=1}^{n} x^k = \frac{n(n+1)}{(k+1)!} \begin{vmatrix}
1 & C_0 & 0 & 0 & \cdots & 1 \\
-2 & C_0 & 2C_1 & 0 & \cdots & n \\
3 & 0 & -3C_1 & 3C_2 & \cdots & n^2 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
(-1)^{k+1} & kC_0 & (-1)^k C_1 & (-1)^{k-1} C_2 & \cdots & n^{k-1}
\end{vmatrix}
\]

where \( rC_3 \) is the binomial coefficient \( r!/(r-s)!s! \). It is known that the left member is equal to a polynomial of degree \( k+1 \) which has \( n(n+1) \) as a factor. Hence we set

\[(1) \quad x^k = n(n+1)f_{k-1}(n),\]

\[(2) \quad f_{k-1}(n) = a_0 + a_1 n + a_2 n^2 + \ldots + a_{k-1} n^{k-1},\]

and proceed as follows:

\[
k^n = \sum_{x=1}^{n} x^k = \sum_{x=1}^{n} x^k = n(n+1)f_{k-1}(n) - (n-1)nf_{k-1}(n-1)
\]

\[
= n \left\{ \left[ 1+1 \right] a_0 + \left[ (n+1)n - (n-1)^2 \right] a_1 + \left[ (n+1)n^2 - (n-1)^3 \right] a^2 + \ldots + \left[ (n+1)n^{k-1} - (n-1)^k \right] a_{k-1} \right\} .
\]
\[ n^{k-1} = [1+1]a_0 + [n^2 - (n-1)^2 + n]a_1 + \ldots \\
+ [n^k - (n-1)^k + n^{k-1}]a_{k-1}. \]

\[ (3) \quad n^{k-1} = a_4\{C_0 + 1\} + a_1 \{-2C_0 + [C_1 + 1]n\} \\
+ a_2 \{3C_0 - 3C_0n + [3C_2 + 1]n^2\} + \ldots \\
+ a_{k-1} \{(-1)^{k+1}C_0 + (-1)^kC_1n + (-1)^{k-1}C_2n^2 + \ldots \\
+ [kC_{k-1} + 1]n^{k-1}\}. \]

If we equate coefficients in this identity in \( n \), we have

\[ [1C_0 + 1]a_0 - 2C_0a_1 + 3C_0a_2 + \ldots + (-1)^{k+1}C_0a_{k-1} = 0, \]

\[ [2C_1 + 1]a_1 - 3C_1a_2 + \ldots + (-1)^kC_1a_{k-1} = 0, \]

\[ [3C_2 + 1]a_2 + \ldots + (-1)^{k-1}C_2a_{k-1} = 0, \]

\[ \ldots \ldots \ldots \ldots \ldots \ldots \]

\[ [kC_{k-1} + 1]a_{k-1} = 0. \]

Elimination of \( a_0, a_1, a_2, \ldots, a_{k-1} \) between these equations and

\[ (2) \] yields
\[ \begin{vmatrix} 1 & C_0^0 & -2C_0 & 3C_0 & \cdots & (-1)^{k+1}C_0 \\ 0 & 2C_1^0 & -3C_1 & \cdots & (-1)^{k}C_1 \\ 0 & 0 & 3C_2^0 & \cdots & (-1)^{k-1}C_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & C_{k-1}^0 -1 \\ 1 & n & n^2 & \cdots & n^{k-1} -f_{k-1}(n) \end{vmatrix} = 0 \]

or

\[ -(k+1)!f_{k-1}(n) + \begin{vmatrix} 1 & C_0^0 & -2C_0 & 3C_0 & \cdots & (-1)^{k+1}C_0 \\ 0 & 2C_1^0 & -3C_1 & \cdots & (-1)^{k}C_1 \\ 0 & 0 & 3C_2^0 & \cdots & (-1)^{k-1}C_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & n & n^2 & \cdots & n^{k-1} \end{vmatrix} = 0. \]

Except for an interchange of rows and columns, this is the desired result.

If we set

\[ f_{k-1}(n) = b_0 + b_1(n+1) + b_2(n+1)^2 + \ldots + b_{k-1}(n+1)^{k-1}, \]

the work is quite similar to the above, except that in the analogue of (3) all the signs are positive. As a result we would get

\[ (k+1)!f_{k-1}(n) \]

equal to the determinant in (4) with all the signs positive and \( n \) replaced by \( n + 1 \).
Jacobians

Problem 165. [AMM, Vol. 19, p. 57-58, Prob. 319 (Calc)]

Given \( u = \frac{yz}{x} \), \( v = \frac{zx}{y} \), \( w = \frac{xy}{z} \), prove that

\[
\begin{vmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\
\frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}
\end{vmatrix} = 4.
\]

Solution: If \( \Delta \) is the proposed determinant, we have:

\[
\Delta = \begin{vmatrix}
-\frac{yz}{x} & \frac{z}{x} & \frac{y}{x} \\
\frac{z}{y} & -\frac{xz}{2y} & \frac{x}{y} \\
\frac{y}{z} & -\frac{xz}{2} & \frac{-xy}{z}
\end{vmatrix} = xyz
\]

\[
\begin{vmatrix}
\frac{1}{x} & \frac{1}{y} & \frac{1}{z} \\
\frac{1}{x} & \frac{1}{y} & \frac{1}{z}
\end{vmatrix}
\]

\[
= xyz \begin{vmatrix}
0 & 0 & \frac{2}{z} \\
\frac{2}{x} & 0 & 0
\end{vmatrix} = 4.
\]

\[
\begin{vmatrix}
\frac{1}{x} & \frac{1}{y} & \frac{1}{z}
\end{vmatrix}
\]
Problem 166. [AMM, Vol. 49, p. 694-696, Prob. 4009]

If the roots of $x^n - c_1 x^{n-1} + \ldots + (-1)^n c_n$ are the variables $x_1, x_2, \ldots, x_n$, find the Jacobian of $c_n, c_{n-1}, \ldots, c_1$ with respect to $x_1, x_2, \ldots, x_n$.

**Solution I:** It seems better to find the Jacobian

$$\frac{\partial(c_1, \ldots, c_n)}{\partial(x_1, \ldots, x_n)}$$

and to note that

$$\frac{\partial(c_n, \ldots, c_1)}{\partial(x_1, \ldots, x_n)} = (-1)^\mu \frac{\partial(c_1, \ldots, c_n)}{\partial(x_1, \ldots, x_n)}$$

where $\mu = n(n-1)/2$. Direct computation reveals that

$$\frac{\partial(c_1, \ldots, c_n)}{\partial(x_1, \ldots, x_n)}$$

is the alternant

$$P(x_1, \ldots, x_n) = \pi(x_i - x_j) (j > i; i, j = 1, \ldots, n)$$

for $n = 2, 3$.

To establish this result for all $n$ we use induction. Hence, suppose that the result holds for $m = n-1$. Write $s_k(x_1, \ldots, x_n)$ for $c_k$, and $1$ for $s_0(x_1, \ldots, x_n)$, and note that

$$\frac{\partial}{\partial x_i} s_k(x_1, \ldots, x_n) = s_{k-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

In the Jacobian $\frac{\partial(c_1, \ldots, c_n)}{\partial(x_1, \ldots, x_n)}$ subtract the first column from each of the remaining, noticing that

$$s_{k-1}(x_2, \ldots, x_n) - s_{k-1}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$$

$$= (x_j - x_1) s_{k-2}(x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) (j, k - 2, \ldots, n)$$
Expand by minors of the first row, remove the factor $\pi(x_1-x_j)$

$(j = 2, \ldots, n)$, and we find that our Jacobian is

$$\frac{\partial (d_1, \ldots, d_{n-1})/\partial (x_2, \ldots, x_n)}{\prod_{j=2}^{n} (x_1-x_j)}, \quad d_i = s_i(x_2, \ldots, x_n).$$

Application of the hypothesis of induction shows immediately that our result holds for $m = n$, and the proof is complete.

**Solution II:** Let $c_r$ be the $r$th elementary symmetric function of $x_1, \ldots, x_n$. Let $c_r^{(i)}$ be the $r$th elementary symmetric function of $x_1, \ldots, x_i-1, x_{i+1}, \ldots, x_n$. Then

$$c_r^{(i)} = c_r - x_i c_{r-1}, \quad r = 1, \ldots, n-1,$$

where for convenience we put $c_0^{(i)} = 1$. From this we obtain

$$\frac{\partial c_r}{\partial x_i} = c_r^{(i)}, \quad r = 1, \ldots, n.$$

Hence

$$\begin{vmatrix} \frac{\partial c_r}{\partial x_i} \\ \frac{\partial c_r}{\partial x_1} \end{vmatrix} = \begin{vmatrix} c_0^{(i)} & c_1^{(i)} & c_2^{(i)} & \cdots & c_{n-1}^{(i)} \\ c_{n-1}^{(i)} & \cdots & c_2^{(i)} & c_1^{(i)} & c_0^{(i)} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & c_1^{(i)} & c_2^{(i)} - x_1 c_1^{(i)} & \cdots & c_{n-1}^{(i)} - x_1 c_{n-2}^{(i)} \\ c_{n-1}^{(i)} & \cdots & c_2^{(i)} & c_1^{(i)} & c_0^{(i)} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -x_i & -x_i c_1^{(i)} & \cdots & -x_i c_{n-2}^{(i)} \\ c_{n-1}^{(i)} & \cdots & c_2^{(i)} & c_1^{(i)} & c_0^{(i)} \end{vmatrix}$$

$$= (-1)^{n-1} x_1 \cdots x_n \begin{vmatrix} x_1^{-1} & c_1^{(i)} & \cdots & c_{n-2}^{(i)} \end{vmatrix}$$
\[= (-1)^{n-1} x_1 \cdots x_n | x_i^{-1} c_{i-1} - x_i c_{i-2} - \cdots - x_i c_{n-3} |\]

\[= (-1)^{n-1+n-2} x_1 \cdots x_n | x_i^{-2} x_{i+1} - x_i x_{i+2} - \cdots - 1 |\]

\[= (-1)^{n(n-1)/2} x_1 \cdots x_n | x_i^{-n+1} x_{i+1} - n+2 \cdots 1 |\]

\[= (-1)^{n(n-1)/2} | x_i x_{i+1}^2 \cdots x_i^{n-1} |\]

\[\prod_{i<j} (x_i - x_j),\]

\[\frac{\partial c_i}{\partial x_j} = MV, \quad \left| \frac{\partial c_i}{\partial x_j} \right| = (-1)^{n(n-1)/2} |V|, \quad \left| \frac{\partial c_{n-i+1}}{\partial x_j} \right| = |V|,
\]

and the required result is the Vandermonde determinant.

An editorial comment concerning alternate solutions is also given.

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Hessians

**Problem 167.** [AMM, Vol. 30, p. 41-42, Prob. 2908]

If \( f \) is a homogeneous polynomial in \( n \) variables and \( H \) is its Hessian determinant, prove that the Hessian of \( f^2 \) is \( cH f^n \), where \( c \) is a constant.

**Solution:** This is a special case of the general theorem that the Hessian of \( f^m \) is \( cH f^{n(m-1)} \), which may be proved as follows:
\[ H(f^m) = \left| \frac{\partial^2 f^m}{\partial x_i \partial x_j} \right| \quad (i, j = 1, 2, \ldots, n) \]

\[ = m \left| f^{m-1} \frac{\partial^2 f}{\partial x_i \partial x_j} + (m-1)f^{m-2} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} \right| \]

\[ (i, j = 1, 2, \ldots, n). \]

This determinant may be broken up into the sum of \( 2^n \) determinants, one of which is \( H(f) \) with every element multiplied by \( f^{m-1} \), and the others each the same as this determinant with the elements in the \( j \)th column replaced by \( (m-1)f^{m-2} \left( \frac{\partial f}{\partial x_i} \right) \left( \frac{\partial f}{\partial x_j} \right) \), for one or more values of \( j \). Of these last, all containing more than one such column will be zero; for suppose the \( j \)th and \( k \)th columns so replaced, then we may factor out \( \frac{\partial f}{\partial x_j} \) and \( \frac{\partial f}{\partial x_k} \) and leave these columns the same. Thus,

\[ H(f^m) = m^n \left\{ Hf^{n(m-1)} + (m-1)f^{n(m-1)-1} \sum_{i,j} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} C_{ij} \right\}, \]

where \( C_{ij} \) is the cofactor of \( \frac{\partial^2 f}{\partial x_i \partial x_j} \) in \( H \), and where the indices of summation run from 1 to \( n \).

By Euler's identity, if \( p \) is the order of \( f \) in the variables,

\[ \sum_j x_j \frac{\partial f}{\partial x_j} = pf, \quad \text{and} \quad \sum_k x_k \frac{\partial^2 f}{\partial x_k \partial x_i} = (p-1) \frac{\partial f}{\partial x_i}, \]

\( \frac{\partial f}{\partial x_i} \) being a homogeneous polynomial of order \( p-1 \) in the
variables. Thus

\[ \sum_i \frac{\partial f}{\partial x_i} C_{ij} = \frac{1}{p-1} \sum_{i,k} x_k \frac{\partial^2 f}{\partial x_i \partial x_k} C_{ij} = \frac{1}{p-1} x_j H, \]

since

\[ \sum_i \frac{\partial^2 f}{\partial x_i \partial x_j} C_{ij} = \begin{cases} 0 \text{ when } k \neq j, \\ H \text{ when } k = j. \end{cases} \]

Then

\[ \sum_{i,j} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} C_{ij} = \frac{1}{p-1} H \sum x_j \frac{\partial f}{\partial x_j} = \frac{p}{p-1} H f, \]

and

\[ H(f^m) = \frac{m^n (mp-1)}{p-1} H f^{n(m-1)}. \] In particular, if \( m = 2, \)

\[ H(f^2) = \frac{2^n (2p-1)}{p-1} H f^n. \]

**Problem 168.** [AMM, Vol. 39, p. 48, Prob. 3140]

Let \( f \) be any algebraic form of total degree \( m > 1 \) in \( n \) variables, and \( H(f) \) its Hessian. Let \( \phi \) be any analytic function. Prove that

\[ H[\phi(f)] = H(f) \left( \frac{\partial \phi}{\partial f} f + \frac{m f}{m-1} \frac{\partial^2 \phi}{\partial f^2} (\frac{\partial \phi}{\partial f} f)^{-1} \right). \]

In particular, when \( \phi(f) = f^r \), we have Mrs. Ballantine's generalization of Problem 2908 proposed by Prof. Dickson

[AMM, Vol. 30, p. 41-42, or see Problem 167, p. 275.]
Again, when \( \phi(f) = \log f \), we have \( H(f) = (1-m) f^n H[\log f] \), which is a generalization of Ex. 22 of Sir Thomas Muir's "Budget of Exercises on Determinants" [AMM, Vol. 29, p. 10-14].

Solution: For brevity of writing set \( \phi' = \frac{d\phi}{df}, \phi'' = \frac{d^2\phi}{df^2} \),
\[ f_i = \frac{\partial f}{\partial x_i}, \quad \text{and} \quad f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}. \] Then

\[
(1) \quad H[\phi(f)] = \left| \frac{\partial^2 \phi(f)}{\partial x_i \partial x_j} \right| = \left| \phi' f_{ij} + \phi'' f_i f_j \right|.
\]

This can be expanded into the sum of \( 2^n \) determinants. But since all of these which involve more than one column of elements of the type \( \phi'' f_{ij} \) are zero, we may write

\[
(2) \quad H[\phi(f)] = (\phi')^n H(f) + (\phi')^{n-1} \phi'' \sum_i \left| \begin{array}{cccc}
  f_{11} & \ldots & f_1 & \ldots & f_{1n} \\
  \ldots & \ldots & \ldots & \ldots & \ldots \\
  f_{ni} & \ldots & f_n & \ldots & f_{nn}
\end{array} \right|,
\]

where the \( i \)th column contains the first partial derivatives of \( f \), and the summation is for \( i = 1, 2, \ldots, n \). By Euler's theorem,
\[
f_j = \frac{1}{m-1} \sum_{k=1}^m x_k^j f_{jk'}
\]

and by use of this substitution in the \( i \)th column, the determinant in (2) expands into \( n \) determinants all of which but one is zero. The remaining one is \( (m-1)^{-1} x_i H(f) \). Therefore,
\[
H[\phi(f)] = (\phi')^n H(f) + (\phi')^{n-1} \phi'' (m-1)^{-1} H(f) \sum x_i f_i.
\]

Since \(\sum x_i f_i = mf\), we have the desired result:

\[
H[\phi(f)] = H(f) [(\phi')^n + m(m-1)^{-1} f \cdot (\phi')^{n-1} \phi''].
\]

**Problem 169.** [AMM, Vol. 41, p. 118, Prob. 3601]

Prove or disprove that \(H = [(n-1)^2 F \frac{F}{x} - n(n-1)\frac{FF}{xy}] / xy\), where \(F\) is a homogeneous function of \(x\) and \(y\) of order \(n\), and \(H\) is its Hessian; i.e., \(H = \frac{F}{x} \frac{F}{y} - (\frac{F}{xy})^2\).

**Solution:** Differentiating Euler's equation by homogeneous functions, \(xF_x + yF_y = nF\), partially, with respect to \(x\) and \(y\), we obtain

\[
x \frac{F}{xx} + y \frac{F}{xy} = (n-1) \frac{F}{x},
\]

\[
x \frac{F}{xy} + y \frac{F}{yy} = (n-1) F_y, \text{ respectively.}
\]

Consider these last equations as two simultaneous equations in \(x\) and \(y\), and solve for \(x\):

\[
x H = \begin{vmatrix}
(n-1)F_x & F_x \\
(n-1)F_y & F_y
\end{vmatrix}.
\]

Substituting for \(F_{yy}\) its value as given in the last equation above,
and expanding the determinant, we obtain

\[ H = (n-1)[(n-1)F^x F^y - F^x (x F^x + y F^y)] / xy. \]

But \( x F^x \) and \( y F^y = nF \), and therefore

\[ H = [(n-1)^2 F^x F^y - n(n-1) F^x F^y] / xy, \]

which was to be proved.

**Wronskians**

**Problem 170.** [AMM, Vol. 63, p. 726-727, Prob. E1213]

Find the Wronskian, \( W(n, k) \), of the set of functions \( x^n, x \ln x, x \ln^2 x, \ldots, x \ln^k x \).

**Solution:** Using Leibnitz's formula for the \( r \)th derivative of a product, we find

\[ D^r x^n \ln^s x = \sum_{j=0}^{k} \binom{r}{r-j} D^{r-j} x^n D^j \ln^s x, \]

where \( \binom{r}{r-j} = 0 \) if \( j > r \).

It now follows, from the familiar procedure for multiplying two determinants, that

\[ W(n, k) = \begin{vmatrix} a_{rj} & b_{js} \end{vmatrix} \quad \text{where} \quad |a_{rj}| = |(r-j) D^{r-j} x^n|, \]

and \( |b_{js}| = |D^j \ln^s x| \).
Since \( a_j = 0 \), for \( j > r \), and \( a_r = x^n \) for \( r = 0, 1, \ldots, k \), we now have

\[
(1) \quad W(n, k) = x^{n(k+1)} \left| D^i \ln x \right| = x^{n(k+1)} W(0, k).
\]

Examining \( W(0, k) \), it is readily found that

\[
(2) \quad W(0, k) = k! W(-1, k-1).
\]

But from (1), \( W(-1, k-1) = x^{-k} W(0, k-1) \), so that (2) implies

\[
(3) \quad W(0, k) = k! x^{-k} W(0, k-1).
\]

Since \( W(0, 0) = 1 \), repeated use of the recurrence formula (3), together with (1), gives

\[
W(n, k) = x^{(k+1)(2n-k)/2} \prod_{t=1}^{k} t!.
\]

It can be shown more generally that if \( u \) and \( v \) are differentiable functions of \( x \), then the Wronskian of the set

\[ u, uv, uv^2, \ldots, uv^k \]

is

\[
u^{k+1} (Dv)^{k(k+1)/2} \prod_{t=1}^{k} t!.
\]

Setting \( u = x^n \) and \( v = \ln x \), we obtain the above value for \( W(n, k) \).
Problem 171. [AMM, Vol. 50, p. 573-574, Prob. 4042]

Prove that if \( A \) is a fixed positive definite hermitian matrix and \( X \) is a variable non-negative hermitian matrix (rank = index), then the minimum value of the determinant \( |A + X| \) is \( |A| \), and is obtained if and only if \( X = 0 \).

**Solution:** There exists a square matrix \( P \) such that

\[
(1) \quad P^*AP = I. \quad \text{Then} \quad P^*(A+X)P = I + P^*XP.
\]

Since \( P^*XP \) is a non-negative hermitian, there exists a unitary matrix \( S_x \) (depending on \( X \)),

\[
(3) \quad S_x^*S_x = I, \quad \text{such that} \quad S_x^*(P^*XP)S_x = (y_{ij} \delta_{ij}),
\]

a diagonal matrix with \( y_i \) real and \( > 0 \). Applying \( S_x^* \) on the left and \( S_x \) on the right of (2) yields

\[
(4) \quad S_x^*P^*(A+X)PS_x = I + (y_i \delta_{ij}).
\]

Taking determinants in (1), (3), (4), we get

\[
|S_x^*| \cdot |S_x| \cdot |P^*| \cdot |P| \cdot |A+X| = \prod_{i} (1+y_i),
\]

\[
|P^*| \cdot |P| \cdot |A| = 1, \quad |S_x^*| \cdot |S_x| = 1,
\]
whence \[ |A + X| = |A| \prod_{i} (1 + y_i). \]

The minimum is obviously \(|A|\), and is obtained if and only if all \(y_i = 0\), that is, if and only if \(X = 0\).

**Problem 172. [AMM, Vol. 63, p. 193, Prob. 4618]**

\[ A = (a_{ik}) \text{ and } B = (b_{ik}) \] are two non-negative Hermitian matrices of order \(n\). \(C_n\) is a matrix whose elements are \(a_{ik}^n + b_{ik}^n\). If \(a, b, c\) are the determinants of \(A, B, C_n\), prove that \(c_n \geq (a + b)^n\).

**Solution:** Let \(E = (e_{ik}), F = (f_{ik})\) be non-negative definite Hermitian matrices. Then \(G = (e_{ik} + f_{ik})\) and \(H = (e_{ik} - f_{ik})\) are both non-negative definite Hermitian matrices and their determinants satisfy the inequalities

1. \( g \geq ef \)
2. \( h^{1/n} \geq e^{1/n} + f^{1/n} \).

(See *Journal London Math. Soc.*, 5(1930), 114-119, for references and proofs.) Let \( A = (a_{ik}), B = (b_{ik})\), and write

\[ A_m = (a_{ik}^m), B_m = (b_{ik}^m), C_m = A_m + B_m, m = 1, 2, \ldots \]

If the respective determinants are \(a_m, b_m, c_m\), then, by repeated application of (1), \(a_m \geq a^m, b_m \geq b^m\), and by (2),
\[ \frac{1}{n} \geq \frac{1}{m} + \frac{1}{n} \geq \frac{m}{n} + \frac{m}{n} \]. In particular, with \( m = n \), \( \frac{c_n}{n} \geq (a + b)^n \), the inequality of the problem.

**Applications**


Show that the coordinates of the center \( (X, Y) \) of the circle through the points \( P_i(x_i, y_i), \ i = 1, 2, 3 \), are given by

\[
X = \frac{D[x_i, r^2, 1]}{2D[x, y, 1]} \quad \text{and} \quad Y = \frac{D[x, r^2, 1]}{2D[x, y, 1]},
\]

and that the length \( t \) of the tangent to the circle from the origin is given by \( t^2 = -\frac{D[x, y, r^2]}{D[x, y, 1]} \), where

\[
\begin{align*}
 r_i^2 &= x_i^2 + y_i^2 \\
 D[x, y, z] &= \begin{vmatrix}
 x_1 & y_1 & z_1 \\
 x_2 & y_2 & z_2 \\
 x_3 & y_3 & z_3
\end{vmatrix}
\end{align*}
\]

**Solution:** The equation of the circle with center \( (X, Y) \) and radius \( R \) is given by

\[
(1) \quad x^2 + y^2 - 2Xx - 2Yy + X^2 + Y^2 - R^2 = 0.
\]

The equation of the circle through the points \( P_i(x_i, y_i) \) is
A comparison of equations (1) and (2) shows that, if the determinant be expanded by elements of the first row and the equation divided through by \( D[x, y, 1] \), the coefficients of \( x \) and \( y \) give

\[
X = D[r^2, y, 1] / 2D[x, y, 1] \quad \text{and} \quad Y = D[x, r^2, 1] / 2D[x, y, 1].
\]

Similarly, the square of the tangent from the origin, by the Pythagorean Theorem, is

\[
t^2 = x^2 + y^2 = -D[x, y, r^2] / D[x, y, 1].
\]

**Problem 174. [AMM, Vol. 44, p. 55-57, Prob. 3714]**

Prove that, if the functions \( x_i(t) \), \( i = 1, 2, 3 \) possess second derivatives, and, if

\[
(1) \quad (x_1 x'_1 - x_2 x'_1)^2 + (x_2 x'_3 - x_3 x'_2)^2 + (x_3 x'_1 - x_1 x'_3)^2 = 0,
\]

\[
(2) \quad x_1^2 + x_2^2 + x_3^2 \neq 0, \quad x_2 x'_3 - x_3 x'_2 \neq 0,
\]

then \( (x_1 x'_1 - x_2 x'_1) / (x_2 x'_3 - x_3 x'_2) \) is a constant, and \( (x_3 x'_1 - x_1 x'_3) / (x_2 x'_3 - x_3 x'_2) \) is a constant.
Solution: The problem arose in geometry, \( t \) being real and the functions \( x_1 \) being complex variables. That there exist functions satisfying the given conditions is shown by the examples

\[
x_1 = -it^3, \quad x_2 = t^2, \quad x_3 = t^3,
\]

and

\[
x_1 = mh(t) + i\sqrt{1+m^2}g(t), \quad x_2 = -g(t), \quad x_3 = h(t),
\]

where \( m \) is a constant, and \( g(t), h(t) \) are real functions possessing continuous derivatives.

To prove that \( (x_1x_2' - x_2x_1')/(x_2x_3' - x_3x_2') = \) constant, it is sufficient to show that the derivative of this fraction is zero. Differentiating this ratio we obtain

\[
\frac{(x_2x_3' - x_3x_2')(x_1x_2'' - x_2x_1'') - (x_1x_2' - x_2x_1')(x_2x_3'' - x_3x_2'')}{(x_2x_3' - x_3x_2')^2},
\]

which by a simple computation can be shown to be equal to

\[
\begin{vmatrix}
  x_1 & x_2 & x_3 \\
  x_1' & x_2' & x_3' \\
  x_1'' & x_2'' & x_3''
\end{vmatrix}
\]

We proceed to prove that the determinant \( \Delta = (x_1x_2'x_3'') \) vanishes identically. Let a repeated subscript indicate summation over 1, 2, 3; furthermore, let
Then by an identity of Lagrange and by our hypothesis

$$A = (x_1 x_2)(x_1' x_2') - (x_1' x_2')^2 = \sum_{i,j} (x_i x_j' - x_j x_i')^2 = 0.$$

By a theorem on determinants (O. Perron, *Algebra*, vol. 1, 1927, p. 117), we have also

$$A = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}^2 + \begin{vmatrix} x_1 & x_3 \\ x_1' & x_3' \end{vmatrix}^2 + \begin{vmatrix} x_2 & x_3 \\ x_2' & x_3' \end{vmatrix}^2 = 0.$$

Therefore

$$A' = \frac{dA}{dt} = 2 \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} + 2 \begin{vmatrix} x_1 & x_3 \\ x_1' & x_3' \end{vmatrix} + 2 \begin{vmatrix} x_2 & x_3 \\ x_2' & x_3' \end{vmatrix} = 0.$$

Evidently

$$\Delta^2 = \begin{vmatrix} x_1 x_2 & x_1 x_2' & x_1 x_2'' \\ x_1' x_2 & x_1' x_2' & x_1' x_2'' \\ x_2' x_2 & x_2' x_2' & x_2' x_2'' \end{vmatrix}.$$

Since $A = 0$, we have for $c_1$ and $c_2$ not both zero,
\[ c_1 x_1 x_i + c_2 x_1 x'_i = 0, \quad c_1 x'_i x_i + c_2 x'_i x'_i = 0. \]

Moreover, since \( A' = 0 \) and \( x x_i \neq 0 \), we have also
\[ c_1 x''_i x_i + c_2 x''_i x'_i = 0. \]

Therefore
\[
\begin{vmatrix}
  x'_i x_i & x'_i x'_i \\
  x''_i x_i & x''_i x'_i
\end{vmatrix} = 0,
\]

and consequently, since all the minors of the elements in the third column of \( \Delta^2 \) vanish, \( \Delta^2 = 0 \), which is what we wished to prove. In the same manner it can be proved that
\[
(x_3 x'_1 - x_1 x'_3)/(x_2 x'_3 - x_3 x'_2) = \text{constant}.
\]


If the sides of a triangle in a plane are \( a_i x + b_i y + c_i = 0 \), (i = 1, 2, 3), the area \( K \) of the triangle is given (numerically) by
\[
2K = \frac{1}{2} \begin{vmatrix}
  a_1 & b_1 & c_1 \\
  a_2 & b_2 & c_2 \\
  a_3 & b_3 & c_3
\end{vmatrix}.
\]
If \( A_i x + B_i y + C_i z + D_i = 0 \), \( i = 1, 2, 3, 4 \), are the equations of the planes of its faces, the volume \( V \) of a tetrahedron is given (numerically) by

\[
V = \frac{1}{6} \begin{vmatrix}
A_1 & B_1 & C_1 & D_1 \\
A_2 & B_2 & C_2 & D_2 \\
A_3 & B_3 & C_3 & D_3 \\
A_4 & B_4 & C_4 & D_4
\end{vmatrix}
\]

Comment: While this is not a problem in the usual sense, it is nevertheless an interesting application of determinants to coordinate geometry, and as such is worthy of mention in this paper.

The above results are stated in the *American Mathematical Monthly*, Vol. 44, p. 101-102, and are developed in a paper by Richard Schafter (Univ. of Buffalo), entitled "Derivation of Certain Formulas for finding the Area of a Triangle or the Volume of a Tetrahedron."

Problem 176. [AMM, Vol. 60, p. 186-187]

While not a problem in the usual sense, the following discussion illustrates an interesting application of determinants to
an elementary derivation of Cramer's Rule.

Consider the simultaneous equations

\[
\begin{align*}
    a_1 x + b_1 y + c_1 z &= d_1 \\
    a_2 x + b_2 y + c_2 z &= d_2 \\
    a_3 x + b_3 y + c_3 z &= d_3
\end{align*}
\]

(1)

Now

\[
\begin{bmatrix}
    a_1 & b_1 & c_1 \\
    a_2 & b_2 & c_2 \\
    a_3 & b_3 & c_3
\end{bmatrix}
\begin{bmatrix}
    x \\
    y \\
    z
\end{bmatrix}
= 
\begin{bmatrix}
    a_1 x + b_1 y + c_1 z \\
    a_2 x + b_2 y + c_2 z \\
    a_3 x + b_3 y + c_3 z
\end{bmatrix}
= 
\begin{bmatrix}
    b_1 & c_1 \\
    b_2 & c_2 \\
    b_3 & c_3
\end{bmatrix}
\begin{bmatrix}
    x \\
    y \\
    z
\end{bmatrix}
\]

by elementary transformations of a determinant. Hence if \( x \) is to satisfy system (1), it is necessary that

\[
\begin{bmatrix}
    a_1 & b_1 & c_1 \\
    a_2 & b_2 & c_2 \\
    a_3 & b_3 & c_3
\end{bmatrix}
\begin{bmatrix}
    d_1 \\
    d_2 \\
    d_3
\end{bmatrix}
= 
\begin{bmatrix}
    d_1 & b_1 & c_1 \\
    d_2 & b_2 & c_2 \\
    d_3 & b_3 & c_3
\end{bmatrix}
\]

, or \( x \) = \begin{bmatrix}
    d_1 \\
    d_2 \\
    d_3
\end{bmatrix}
\]

provided \( \Delta \neq 0 \), where \( \Delta \) is the determinant of the coefficient matrix of system (1). Similarly,

\[
\begin{bmatrix}
    a_1 & d_1 & c_1 \\
    a_2 & d_2 & c_2 \\
    a_3 & d_3 & c_3
\end{bmatrix}
\begin{bmatrix}
    y \\
    z \\
    x
\end{bmatrix}
= 
\begin{bmatrix}
    a_1 & d_1 \\
    a_2 & d_2 \\
    a_3 & d_3
\end{bmatrix}
\begin{bmatrix}
    y \\
    z \\
    x
\end{bmatrix}
\]

That these conditions are sufficient when \( \Delta \neq 0 \), can be established by substituting back into (1) which gives:
That this is true follows from
\[
\begin{vmatrix}
da_1 & b_1 & c_1 \\
d_2 & b_2 & c_2 \\
d_3 & b_3 & c_3 \\
\end{vmatrix}
= \begin{vmatrix}
a_1 & d_1 & c_1 \\
a_2 & d_2 & c_2 \\
a_3 & d_3 & c_3 \\
\end{vmatrix}
\]

since the top row is equivalent to one of the other rows.

The method can be extended immediately to \( n \) linear equations in \( n \) unknowns.

**Problem 177.** [AMM, Vol. 66, p. 67-68, Prob. 4778]

Given \( f(r) = a^r(b - \gamma) + \beta^r(\gamma - a) + \gamma^r(a - \beta) \) in which \( a, \beta, \gamma \) are nonzero and distinct. If \( n \) is a positive integer and if \( f(n+1) = 0 \), prove that

\[
f(n+2)f(n) = a^n\beta^n\gamma^n(\beta - \gamma)(\gamma - a)(a - \beta)f(-n).
\]

**Solution:** From the product rule for determinants, the following identity is evident:
Replacing each by its expansion, one obtains
\[
\begin{align*}
f(n+2) f(n) &= f^2(n+1) \\
&= (\beta \gamma (\beta -\gamma) + \gamma a (\gamma -a) + a \beta (a -\beta)) (\beta -\gamma) (\gamma -a) (a -\beta)
\end{align*}
\]
\[
= a \beta \gamma (\beta -\gamma) (\gamma -a) (a -\beta) f(-n).
\]
Setting \( f(n+1) = 0 \) gives the desired result. There need be no restriction on \( n \).

Problem 178. [AMM, Vol. 66, p. 241, Prob. 4791]

If \( a, \beta, \gamma \) are real numbers and \( a^3 + \beta^3 + \gamma^3 = 0 \), prove
\[
\sum \frac{1}{(\beta -\gamma)^2} \left[ \sum a^4 \right] \geq \left[ \sum a^2 \right]^3.
\]

Solution: Pet \( S_n = \sum a_n \). Then \( \Delta = (a -\beta)^2 (\beta -\gamma)^2 (\gamma -a)^2 \)

\[
\begin{vmatrix}
1 & 1 & 1 \\
\alpha & \beta & \gamma \\
\alpha^2 & \beta^2 & \gamma^2
\end{vmatrix}^2
= \begin{vmatrix}
3 & S_1 & S_2 \\
S_1 & S_2 & S_3 \\
S_2 & S_3 & S_4
\end{vmatrix}
= (3S_2 - S_1^2)S_4 - S_3 (3S_3 - S_1S_2) + S_2 (S_1S_3 - S_2^2).
Since \[ \sum (a - \beta)^2 = 2S_2 - 2 \sum a \beta = 3S_2 - S_1^2, \] it follows that

\[
\sum (a - \beta)^2 \cdot \sum a^4 - \left[ \sum a^2 \right]^3 = (3S_2 - S_1^2)S_4 - S_2^3
\]

\[ = \Delta + S_3^4 (3S_3 - 2S_1S_3). \]

Therefore when \( S_3 = 0, \) \[ \sum (a - \beta)^2 \cdot \sum a^4 - \left[ \sum a^2 \right]^3 = \Delta \geq 0. \]

Problem 179. [AMM, Vol. 73, p. 86-87, Prob. E1751]

Let \( A \) be an involutory matrix \( (A^{-1} = A) \). Prove that if every element of \( A \) is replaced by its cofactor, then the resulting matrix is also involutory.

Solution: The matrix obtained from \( A \) as in the statement of the problem is of course \( B = (\text{adj } A)^t \), and we prove the generalization: If \( A^n = I \) for some integer \( n \), then \( B^n = I \).

Proof: The conclusion is clear for the case \( n = 0 \). In the remaining cases, \( A \) is nonsingular, and so \( \text{adj } A = (\det A) A^{-1} = (\det A) A^{n-1} \). But then
\[ B^n = BB^{n-1} = \det(A^{n-1})^{\text{tr}} ((\text{adj} A)^{\text{tr}})^{n-1} \]

\[ = (\det A)((\text{adj} A)^{n-1}A^{n-1})^{\text{tr}} \]

\[ = (\det A) (((\text{adj} A)A)^{n-1})^{\text{tr}} \]

\[ = (\det A) (((\det I)^{n-1})^{\text{tr}} \]

\[ = (\det A)^n I, \]

where we have used the fact that \( A \) commutes with its adjoint.

The proof is completed by observing that \( 1 = \det I = \det A^n = (\det A)^n \).

Problem 180. [DMVJ, Vol. 49, Pt. 2, p. 4-6, Prob. 245]

A real orthogonal transformation in a \( 2n \)-dimensional Euclidean space, as is well known, can be composed of \( n \) rotations in absolutely orthogonal planes with angles of rotation \( \xi_1, \xi_2, \xi_3, \ldots, \xi_n \). In the case of \( 2n+1 \) dimensions, a complex orthogonal transformation can be composed of \( n \) rotations of the same type and a reflection. We get special orthogonal transformations of this type when we carry out \( m \) consecutive reflections on the hyperplanes, where \( m = 2n \) and \( m = 2n+1 \) are the respective dimensions. The cosine of the angle between the \( i^{\text{th}} \) and \( k^{\text{th}} \) hyperplanes is designated by \( c_{ik} \). Under what conditions are

\[ \pm \cos \xi_1/2, \pm \cos \xi_2/2, \ldots, \pm \cos \xi_n/2 \] (and 0 for \( m = 2n+1 \))

the roots of the equation
Solution: We associate the \( m \) hyperplanes with the points of a graph in such a way that two points are joined by a segment if the corresponding hyperplanes are not perpendicular to each other. We shall show that the above assertion always holds when the graph is a forest (a mutually unconnected group of trees); i.e., when the graph contains no closed sequence of line segments.

In the case of a forest, the product is conjugate in some order to the group \( R_1 R_2 \ldots R_m \), from which it is generated. In other cases we cannot expect that the \( \xi_v \) contain the \( m \) indices symmetrically.

If we expand determinant (1), each term corresponds to an element of the symmetric permutation group of order \( m \). Thus

\[
-C_{12}^2 x^{m-2} \quad \text{corresponds to the transposition } (1,2), \quad \text{and} \quad -C_{12} C_{31} C_{45} C_{56} C_{67} C_{74} x^{m-7} \quad \text{corresponds to } (1,2,3)(4,5,6,7).
\]

If the graph is a forest, \( C_{12} C_{31} C_{45} C_{56} C_{67} C_{74} = 0 \), and so on. Generally, all terms vanish from which the order of the corresponding permutation is greater than 2. In this case, consequently, (1) has the form

\[
\begin{vmatrix}
x & c_{12} & \cdots & c_{1m} \\
c_{21} & x & \cdots & c_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
c_{m1} & c_{m2} & \cdots & x \\
\end{vmatrix} = 0 ?
\]
\[(2) \quad x^m - \sum_{i<j} C^2_{ij} x^{m-2} + \sum_{i<j<k<\ell} (C^2_{ij} C^2_{k\ell} + C^2_{ik} C^2_{j\ell} + C^2_{i\ell} C^2_{jk}) x^{m-4} - \ldots = 0,\]

where the coefficients correspond to the involutory permutations 

\[(i, j), (i, j)(k, \ell), (i, k)(j, \ell), (i, \ell)(j, k), \ldots .\]

We normalize the \(i^{th}\) hyperplane \(a_i X + b_i Y + \ldots = 0\) so that \(a_i^2 + b_i^2 + \ldots = 1;\) consequently

\[(3) \quad a_i a_k + b_i b_k + \ldots = -c_{ik}.\]

The reflection with respect to \(aX + bY + \ldots = 0\) is

\[
\begin{cases}
X = X' - 2a(aX' + bY' + \ldots ), \\
y = Y' - 2b(aX' + bY' + \ldots ), \\
\ldots \ldots . \\
\end{cases}
\]

With the help of the hyperplanes, we establish oblique coordinates in which we let \(x_1 = aX + b_1 Y + \ldots .\) Transformation (4) now has the form \(x_1 = x_1' - 2(a, a + b_1 b + \ldots ) (aX' + bY' + \ldots ).\)

Consequently, \(R_k\) can be represented by

\[x_i = x_i' + 2c_{ik} x_{ik} \quad (c_{kk} = -1).\]

Then \(R_1 R_2 \ldots R_m\) is the transformation that takes \(x_i)\) into \(x_1^{(m)},\) whereby
\[ x_i = x_i' + 2c_{i1}x_1', \]
\[ x_{i}' = x_i'' + 2c_{i2}x_2'', \]
\[ \ldots \ldots \ldots \]
\[ x_i^{(m-1)} = x_i^{(m)} + 2c_{im}x_i^{(m)}. \]

After multiplying all except the first \( i \) by an arbitrary value \( q \), we add these equations (for a fixed \( i \)) to get

\[ x_i + qx_i^{(i)} = x_i^{(i)} + qx_i^{(m)} + 2 \sum_{k=1}^{i} c_{ik}x_k^{(k)} + 2q \sum_{k=i+1}^{m} c_{ik}x_k^{(k)}. \]

Consequently, since \( c_{ii} = -1 \),

\[ (q+1)x_i^{(i)} = 2 \sum_{k=1}^{i-1} c_{ik}x_k^{(k)} - 2q \sum_{k=i+1}^{m} c_{ik}x_k^{(k)} = qx_i^{(m)} - x_i. \]

The characteristic values of this transformation are the roots of the equation (for \( q \))

\[ qx_i^{(m)} = x_i, \]

which we get by elimination of \( x_i^{(k)} \) from (5). Since (6) holds for all values of \( q \), it follows from (5) and (7) that

\[ (q+1)x_i^{(i)} = 2 \sum_{k=1}^{i-1} c_{ik}x_k^{(k)} - 2q \sum_{k=i+1}^{m} c_{ik}x_k^{(k)} = 0 \quad (i=1, 2, \ldots, m). \]
Dividing by $2q^{\frac{1}{2}}$ and eliminating the $x_k^{(k)}$, we have

$$
\begin{vmatrix}
\frac{1}{2}(q^{\frac{1}{2}}+q^{-\frac{1}{2}}) & -c_{12}q^{\frac{1}{2}} & -c_{13}q^{\frac{1}{2}} & \ldots & -c_{1m}q^{\frac{1}{2}} \\
-c_{21}q^{-\frac{1}{2}} & \frac{1}{2}(q^{\frac{1}{2}}+q^{-\frac{1}{2}}) & -c_{23}q^{\frac{1}{2}} & \ldots & -c_{2m}q^{\frac{1}{2}} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
-c_{m1}q^{-\frac{1}{2}} & -c_{m2}q^{-\frac{1}{2}} & -c_{m3}q^{-\frac{1}{2}} & \ldots & \frac{1}{2}(q^{\frac{1}{2}}+q^{-\frac{1}{2}})
\end{vmatrix} = 0. 
$$

This is now the equation with roots

$$
e^{\pm i\xi_1}, \ e^{\pm i\xi_2}, \ldots, \ e^{\pm i\xi_n} \ \text{(except -1 for odd m)}. 
$$

Our problem is now to determine when this determinant is identical to

$$
\begin{vmatrix}
\frac{1}{2}(q^{\frac{1}{2}}+q^{-\frac{1}{2}}) & c_{12} & c_{13} & \ldots & c_{1m} \\
c_{21} & \frac{1}{2}(q^{\frac{1}{2}}+q^{-\frac{1}{2}}) & c_{23} & \ldots & c_{2m} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
c_{m1} & c_{m2} & c_{m3} & \ldots & \frac{1}{2}(q^{\frac{1}{2}}+q^{-\frac{1}{2}})
\end{vmatrix}. 
$$

A sufficient condition is the vanishing of all expressions $c_{ij} \ c_{jk} \ c_{k\ell} \ c_{ij} \ c_{jk} \ c_{k\ell} \ c_{i\ell}$, $\ldots$, which contain cycles of at least three indices. This is equivalent to saying that the graph represents a forest. Although this condition is not necessary (for $m > 3$), all special cases are too complicated to serve our interest. The interesting cases are those, naturally, in which the reflections generate
a finite group. In all these cases it is clear that we have a forest.


(1) If $A_1(x_1, y_1)$, $A_2(x_2, y_2)$, $A_3(x_3, y_3)$ are the vertices of a triangle and $a_i$ is the length of the side opposite $A_i$, show that the equation of the circumscribing circle is

$$a_1^2 x_1 y_1 + a_2^2 x_2 y_2 + a_3^2 x_3 y_3 = 0.$$ 

(2) Prove that there is one, and only one, ellipse circumscribing the triangle $A_1A_2A_3$ for which this triangle is an inscribed triangle of maximum area. Prove that the equation of this ellipse is

$$x_1 y_1 + x_2 y_2 + x_3 y_3 = 0.$$ 

(3) Show that an hyperbola having its center at the centroid of a triangle cannot pass through all three vertices of the triangle.
Solution:

Write \( L_1 = \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} \), \( L_2 = \begin{vmatrix} x_3 & y_3 & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} \), and let \( p_i, q_i, r_i \) be defined by \( L_i = p_i x + q_i y + r_i \). Let

\[
\begin{align*}
\text{and let } \quad & p_i, q_i, r_i \quad \text{be defined by } \quad L_i = p_i x + q_i y + r_i. \quad \text{Let} \\
& s = r_1 + r_2 + r_3, \quad \text{and we have } \quad a_i^2 = p_i^2 + q_i^2, \quad p_1 + p_2 + p_3 = q_1 + q_2 + q_3 = 0, \\
& \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} = \begin{vmatrix} p_2 & p_3 \\ q_2 & q_3 \end{vmatrix} = \begin{vmatrix} p_3 & p_1 \\ q_3 & q_1 \end{vmatrix} = s \neq 0. 
\end{align*}
\]

The equation of the family of conics through \( A_1, A_2, A_3 \) is

\[ F = K_1 L_2 L_3 + K_2 L_3 L_1 + K_3 L_1 L_2 = 0, \]

where \( K_1, K_2, K_3 \) are parameters.

The conic is a circle if the coefficient of \( xy \) is zero and the coefficient of \( x^2 \) is equal to the coefficient of \( y^2 \). That is, if

\[
\begin{align*}
(p_2 q_3 + p_3 q_2)K_1 + (p_3 q_1 + p_1 q_3)K_2 + (p_1 q_2 + p_2 q_1)K_3 &= 0, \\
(p_2 p_3 - q_2 q_3)K_1 + (p_3 p_1 - q_3 q_1)K_2 + (p_1 p_2 - q_1 q_2)K_3 &= 0.
\end{align*}
\]

We note that the successive terms in each of these equations are formed by cyclic permutation of the subscripts. Since

\[
\begin{align*}
\begin{vmatrix} p_3 q_1 + p_1 q_3 \\ p_3 p_1 - q_3 q_1 \end{vmatrix} + \begin{vmatrix} p_1 q_2 + p_2 q_1 \\ p_1 p_2 - q_1 q_2 \end{vmatrix} &= \begin{vmatrix} p_2 & p_3 \\ q_2 & q_3 \end{vmatrix} = a_i^2 s,
\end{align*}
\]
it follows that \( K_1 : K_2 : K_3 = a_1^2 : a_2^2 : a_3^2 \) and we have the first required result.

A triangle inscribed in an ellipse is an inscribed triangle of maximum area if its centroid is at the center of the ellipse. The center of any conic of the above family is the intersection of the two lines

\[
\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0.
\]

If we substitute the coordinates of the centroid of \( \triangle A_1A_2A_3 \) in the expressions above we get \( L_1 = L_2 = L_3 = s/3 \). The values of \( K_1, K_2, K_3 \) for the conic with center at the centroid of \( \triangle A_1A_2A_3 \) are given by the equations

\[
(p_2+p_3)K_1 + (p_3+p_1)K_2 + (p_1+p_2)K_3 = 0,
\]

\[
(q_2+q_3)K_1 + (q_3+q_1)K_2 + (q_1+q_2)K_3 = 0.
\]

Again the successive terms of these equations are formed by a cyclic permutation of subscripts. Since

\[
\begin{vmatrix}
  p_3 + p_1 & p_1 + p_2 \\
  q_3 + q_1 & q_1 + q_2
\end{vmatrix} = s,
\]

it follows that \( K_1 = K_2 = K_3 \) in this case, and we have the second required result.

We have seen that there is one, and only one, conic of the
family through \( A_1, A_2, A_3 \) with center at the centroid of the triangle. Its equation is 
\[ L_2 L_3 + L_3 L_1 + L_1 L_2 = 0, \]
and it is always an ellipse since

\[
(p_2 q_3 + p_3 q_2 + p_3 q_1 + p_1 q_3 + p_1 q_2 + p_2 q_1)^2
\]

\[-4(p_1 p_2 + p_2 p_3 + p_3 p_1)(q_1 q_2 + q_2 q_3 + q_3 q_1)\]

\[= (p_1 q_1 + p_2 q_2 + p_3 q_3)^2 - (p_1^2 + p_2^2 + p_3^2)(q_1^2 + q_2^2 + q_3^2)\]

\[= -\begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} - \begin{vmatrix} p_2 & p_3 \\ q_2 & q_3 \end{vmatrix} - \begin{vmatrix} p_3 & p_1 \\ q_3 & q_1 \end{vmatrix} = -3s^2.\]

This follows at once by squaring each of \( p_1 + p_2 + p_3 = 0, \)
\( q_1 + q_2 + q_3 = 0, \) and forming their products. Therefore, there is no hyperbola with center at the centroid of the triangle and passing through the three vertices.
CHAPTER IV

A DESCRIPTION OF EACH PROBLEM

In the following pages, each problem is described individually.

The purpose of this treatment is threefold. First, such a treatment is not available in the indexes of periodicals from which the problems are taken. Secondly, it is hoped that the analysis will facilitate any search for a particular problem or type of problem contained in this paper. Finally, relationships between various problems should be more evident here than in Chapter III, where problems related to a given topic may be included under different headings.

The name of the person who is credited with the solution of each problem follows its description. A (p) following a name indicates that the individual also proposed the problem.

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93 A continuant of 2's and 5's expanded by means of partial fractions [C. W. Greenwood (p)]

94 Two essentially different expansions—one trigonometric, the other in terms of binomial coefficients [Joseph A. Nyberg (p), S. G. Barton]

95 A continuant with diagonal elements

\[ 2 \cos \left( \frac{\pi}{n+1} \right) \] [J. B. Kelley, Editors (AMM)]

96 A modification of the previous problem

[D. A. Breault]

97 A continuant with diagonal element \(2 \cos \theta\)—closely related to Problem 95, p. 158

[Albert M. Liebetrau]

98 A continuant with elements 1, -p, -q, where \(p + q = 1\) [Douglas Lind]

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99 The product of two numbers, each the sum of four squares, is the sum of eight squares

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134 Concerning the maximum value of a determinant under certain conditions [Robert Oeder (p)]

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136 A non-zero determinant associated with \[x^m = c\] [D.C.B. Marsh, Harley Flanders]

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138 The lower limit of a normalized determinant with negative elements everywhere except in the main diagonal [Sidney Heller]
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