An Effect of the Thermobaric Nonlinearity of the Equation of State: A Mechanism for Sustaining Solitary Rossby Waves

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ABSTRACT

The thermobaric nonlinearity in the equation of state for seawater density—namely, the dependence of thermal expansibility on pressure—coupled with spatial variation of the oceanic temperature–salinity (θ–s) relation generates a nonlinear behavior in the buoyant force that can counter the linear dispersion of baroclinic Rossby waves and produce solitary waves. A Korteweg–deVries equation is derived in which the coefficient of the nonlinear term depends on the thermobaric parameter and the spatial gradient of the anomaly of the θ–s relation. Quantitative estimates can be made of the magnitude of the effect in terms of these parameters. For example, given first-baroclinic-mode spatial variations of order 0.1 psu (1000 km)$^{-1}$ or 0.7°C (1000 km)$^{-1}$, from a θ–s relation with a density ratio of 2, a solitary Rossby wave of maximum vertical displacement of approximately 100 m and horizontal scale of approximately 30 baroclinic Rossby radii of deformation can be generated.

1. Introduction

Because of the thermobaric nonlinearity of the seawater equation of state (EOS)—the dependence of thermal expansion coefficient on pressure (Müller and Willebrand 1986; McDougall 1987; Garwood et al. 1994; Akitomo 1999)—the aim of devising a quasi-material corrected density (or ''appropriate density'') that completely represents the effects of stratification on buoyancy is fundamentally flawed; the task is impossible (de Szoeke 2000). The shortcomings of potential density, for example, have been understood for a long time (Ekman 1934; Lynn and Reid 1968). Various empirical attempts to construct an appropriate density for use in descriptive physical oceanography have nonetheless been made: examples include patched potential density (Reid and Lynn 1971), neutral density (McDougall 1984; Jackett and McDougall 1997; Eden and Willebrand 1999), and orthobaric density (de Szoeke et al. 2000). All of these examples rely on making use of the oceanic temperature–salinity (θ–s) relation in some way. If the θ–s relation were globally uniform, they would presumably all reduce to the same thing and would all become aliases of the same appropriate density. Because the θ–s relation is not globally uniform, they all display a similar property; namely, they do not give material surfaces (quite apart from the irreversible effects of diffusion), because water leaks reversibly across them in proportion to the local deviation from the θ–s relation, with the proportionality factor dependent on the thermobaric coefficient. [This is so even when, as with patched potential density and neutral density, they are renormalized from region to region to take account of local water properties, because by so doing the density corrections introduce rents or tears in the isopleth surfaces of the variable, through which water will pass (de Szoeke et al. 2000).]

Is this difficulty with devising dynamically appropriate densities for seawater merely an inconvenient nuisance? Or is there an irreducible dynamical consequence associated with the impossibility of a material appropriate density? The goal of this paper is to argue for the latter alternative by demonstrating a nonlinear dynamical effect on Rossby waves that can only come about because of the thermobaric character of the EOS. This work will add to the arguments of earlier workers who found dynamical effects of thermobaricity (Müller and Willebrand 1986; Straub 1999). To facilitate the demonstration, we first describe, in section 2, an orthobaric specific volume (sp. vol.) function derived from an EOS that, though simplified, exhibits a nontrivial thermobaric effect and from an assumed θ–s relation with constant density ratio. Next, using this orthobaric sp. vol. as an independent variable and assuming a rest state of the ocean with density-compensated θ–s variations from the global relation (section 2), we will obtain the intermediate geostrophic limit of the vorticity equation (Williams and Yamagata 1984), which takes into account the effect of reversible thermobaric mass flux across orthobaric isopycnals associated with spatial θ–s

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TABLE 1. Thermodynamic parameters.

Parameter	Description	Value
α_0	Reference specific volume	9.731×10^{-4} m^3 kg^{-1}
ρ_0	Reference density ($=\alpha_0^{-1}$)	1027.7 kg m^{-3}
T_0	Reference temperature	5°C
S_0	Reference salinity	35 psu
α_1	First thermal expansion coefficient	1.067×10^{-4} °C^{-1}
α_2	Second thermal expansion coefficient	1.041×10^{-5} °C^{-2}
β_s	Haline contraction coefficient	0.754×10^{-3} psu^{-1}
γ	Thermobaric parameter	1.86×10^{-8} Pa^{-1}
H_α	Akitomo–Garwood depth ($\gamma^{-1}/\rho_0 g$)	5318 m
c_0	Reference sound speed	1466 m s^{-1}
α_0/c_0^2	Reference compressibility	4.53×10^{-10} Pa^{-1}
γ_c	Compressibility pressure coefficient	2.98×10^{-9} Pa^{-1}

variations (section 3). We will obtain from this development a nonlinear evolution equation, which has the form of the Korteweg–de Vries (KdV) equation, for the amplitude of long baroclinic Rossby waves (section 4).

Nonlinear effects on planetary waves have received considerable attention. Redekopp (1977) studied nonlinear Rossby waves propagating along zonal shear flows and obtained amplitude evolution equations of KdV or modified KdV type, showing that isolated soliton disturbances are possible (Whitham 1974; Drazin 1983). Anderson and Killworth (1979) examined the nonlinear steepening of baroclinic Rossby waves brought about by the thickening and thinning of internal layers by the divergence of the wave motion and the consequent alteration of propagation speeds that depend on these thicknesses. Williams and Yamagata (1984) extended this treatment to include effects of wave dispersion. Flierl (1979; also Flierl et al. 1980; McWilliams 1980) obtained solutions for nonlinear baroclinic solitary waves with cylindrically symmetric form propagating in mean shear flows. The novelty in the present paper is that the nonlinearity comes, not from the inertial self-advection of the disturbance or from the internal vortex stretching, but from the thermobaric nonlinearity of the EOS. The EOS nonlinearity is not enough by itself, but must be coupled with spatial variations from the mean θ–s relation (section 5).

2. Orthobaric specific volume; density-compensated rest state

By assuming simple forms for the seawater EOS and the ocean θ–s relation, we will be able to obtain a useful closed form for orthobaric sp. vol. The EOS we propose is

$$\alpha' \equiv \frac{\alpha}{\alpha_0} - 1 + \frac{\alpha_0}{c_0^2}p\left(1 - \frac{1}{2}\gamma_c p\right)$$
$$= (1 + \gamma p)\alpha_1(\Theta - T_0) + \frac{1}{2}\alpha_2(\Theta - T_0)^2 - \beta_s s, \tag{2.1}$$

where α, p, Θ, and $s = S - S_0$ are sp. vol., pressure, potential temperature (in degrees Celsius), and salinity;[1] α' is sp. vol. anomaly, normalized by α_0; α_0, T_0, S_0, and c_0 are reference sp. vol., temperature, salinity, and sound speed; and β_s, α_1, α_2, and γ are haline contraction coefficient, first and second thermal expansion coefficients, and thermobaric coefficient. Of particular significance are the nonlinearities controlled by the parameters γ and α_2. Numerical values for the constants in (2.1) are given in Table 1. The form of (2.1) is invariant to arbitrary choices of T_0, S_0, and c_0 and arbitrary constant shifts of p, provided the constants α_0, c_0, γ_c, β_s, α_1, α_2, and γ are suitably transformed. The Akitomo–Garwood pressure scale, a measure of the thermobaric effect (Akitomo 1999; Garwood et al. 1994), given by

$$p_{AG} = \frac{\partial\alpha'}{\partial\theta}\bigg/\frac{\partial^2\alpha'}{\partial p\partial\theta} = \gamma^{-1}(1 + \gamma p + \alpha_2\theta/\alpha_1),$$

(writing $\theta = \Theta - T_0$) is shown in Fig. 1 as a function of p and Θ, along with the same parameter calculated from the empirical international EOS (Fofonoff 1985). The parameters γ and $\alpha_2/\alpha_1\gamma$ (intercept and slope) were chosen to best fit the Akitomo-scale contour $p_{AG} = 70$ MPa. Correspondence is very good, except in the high-temperature (>10°C,) low-pressure (<15 MPa) part of the ocean. Figure 2 shows θ–s scatterplots at four pressures using the Atlantic World Ocean Circulation Experiment (WOCE) data, along with superimposed isopycnals from the prototype EOS and their deviations from the international EOS. The deviations are generally far less than 0.1 kg m^{-3}, except near the surface.

The neutral differential is given by

$$\bar{d}N = d\alpha' - \gamma\alpha_1\theta dp = \left(1 + \gamma p + \frac{\alpha_2}{\alpha_1^2}\alpha_1\theta\right)\alpha_1 d\theta - \beta_s ds, \tag{2.2}$$

where $-\gamma\alpha_1\theta$ is the departure, caused by temperature

[1] An "equation of state" is more correctly written in terms of in situ temperature, rather than potential temperature. However, the latter usage is convenient in oceanography.

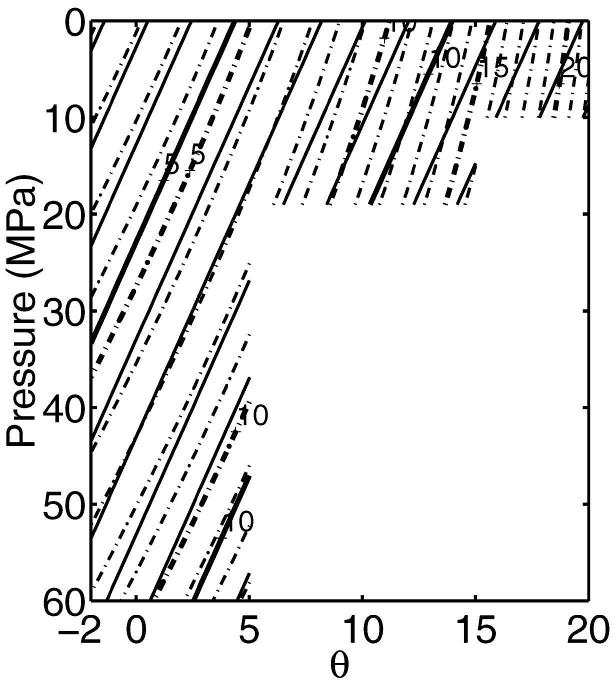

FIG. 1. The Akitomo–Garwood pressure scale p_{AG} for the international equation of state (dash–dotted line) and for the prototype EOS (2.1) (solid line) in units of 10^7 pascals (≡1000 m). Contours are not shown in ranges of p and θ that are not encountered in the ocean.

variation, of adiabatic compressibility from $(\alpha_0/c_0^2)(1 - \gamma_c p)$. The second equality of (2.2) follows from the EOS in (2.1). The neutral differential is not an exact differential in the same sense that the heat transfer in the differential form of the first law of thermodynamics is not: the neutral ''variable'' N, like heat transfer, is not a thermodynamic state variable.

In special circumstances, an integrating factor may be found to replace (2.2) by an exact differential. For example, suppose there is a specific empirical θ–s relation, say a linear relation such as

$$\alpha_1\theta = R\beta_s s, \tag{2.3}$$

where the constant R is called the density ratio. Then, using this equation to eliminate $\beta_s s$ from the EOS, which in turn may be used to write $\alpha_1\theta$ as a function of α' and p, one may obtain an integral of the expression after the first equality of (2.2). More simply, though, in the present example, one merely substitutes (2.3) in the second form of (2.2), obtaining

$$\bar{d}N = \left(1 + m_0\gamma p + m_0\frac{\alpha_2}{\alpha_1^2}\alpha_1\theta\right)\alpha_1 d\theta/m_0 = \phi d\nu', \tag{2.4}$$

where $m_0 = R/(R - 1)$. In the second equality, ν' has been written for $\alpha_1\theta/m_0$, which is related to α' by (2.1):

$$\alpha' = (1 + m_0\gamma p)\nu' + \frac{1}{2}\sigma\nu'^2, \tag{2.5a}$$

where

$$\sigma = m_0^2\alpha_2/\alpha_1^2.$$

The integrating factor in (2.4) is

$$\phi = \left(\frac{\partial\alpha'}{\partial\nu'}\right)_p = 1 + m_0\gamma p + \sigma\nu'. \tag{2.5b}$$

Solving (2.5a),

$$\nu' = \frac{2\alpha'}{(1 + m_0\gamma p) + [(1 + m_0\gamma p)^2 + 2\sigma\alpha']^{1/2}}. \tag{2.5c}$$

The differential $d\nu'$ is exact; ν' is a state variable called the orthobaric sp. vol., normalized by α_0; $\nu = \alpha_0\nu'$ is dimensional orthobaric sp. vol. When $\alpha_2 = 0$, $\nu' = \alpha'/(1 + m_0\gamma p)$ and $\phi = 1 + m_0\gamma p$.

In converse, suppose (2.5a) is used to define $\nu'(p, \alpha')$ and employed to eliminate $d\alpha'$ from (2.2):

$$\begin{aligned}\bar{d}N &= d\alpha' - \gamma m_0\nu' dp - \gamma(\alpha_1\theta - m_0\nu')dp \\ &= \phi d\nu' - \gamma\alpha_1\Delta\theta dp.\end{aligned} \tag{2.6}$$

Here $\alpha_1\Delta\theta = \alpha_1\theta - m_0\nu'$, a graphical interpretation of which is furnished in Fig. 3. The reference θ–s relation can be represented parametrically as a function of ν':

$$\beta_s s_0(\nu') = (m_0 - 1)\nu' \quad \text{and} \quad \alpha_1\theta_0(\nu') = m_0\nu'. \tag{2.7}$$

Notice how the temperature difference $\Delta\theta$ between actual temperature θ and the reference temperature on the θ–s relation, $m_0\nu'/\alpha_1$, multiplied by the thermobaric parameter γ, figures in the coefficient of the pressure differential in (2.6). Equations (2.2) and (2.6) give an evolution equation for ν':

$$\dot{\nu}' = \phi^{-1}\gamma\alpha_1\Delta\theta\dot{p} + \phi^{-1}\dot{N}. \tag{2.8a}$$

Here

$$\dot{N} = \left(1 + \gamma p + \frac{\alpha_2}{\alpha_1^2}\alpha_1\theta\right)\alpha_1\nabla\cdot(\mathbf{K}\nabla\theta) - \beta_s\nabla\cdot(\mathbf{K}\nabla s) \tag{2.8b}$$

is the sum of irreversible transports of buoyancy, $\mathbf{K}$ being a turbulent diffusivity tensor. Alternatively $\dot{N}$ may be written

$$\begin{aligned}\dot{N} = \nabla\cdot\left\{\mathbf{K}\left[\left(1 + \gamma p + \frac{\alpha_2}{\alpha_1^2}\alpha_1\theta\right)\alpha_1\nabla\theta - \beta_s\nabla s\right]\right\} \\ - \gamma\alpha_1\nabla p\cdot\mathbf{K}\nabla\theta - \alpha_2\nabla\theta\cdot\mathbf{K}\nabla\theta,\end{aligned} \tag{2.8c}$$

in which the first term is an eddy transport divergence of buoyancy and the second and third terms are respectively thermobaric buoyancy production (either sign is possible) and cabbeling buoyancy destruction (strictly nonpositive) (McDougall 1987; Davis 1994). While the term appearing in (2.8c), representing thermodynamically irreversible eddy heat flux up or down the pressure gradient, is the classic thermobaric buoyancy term, another distinct *reversible* thermobaric contribution arises

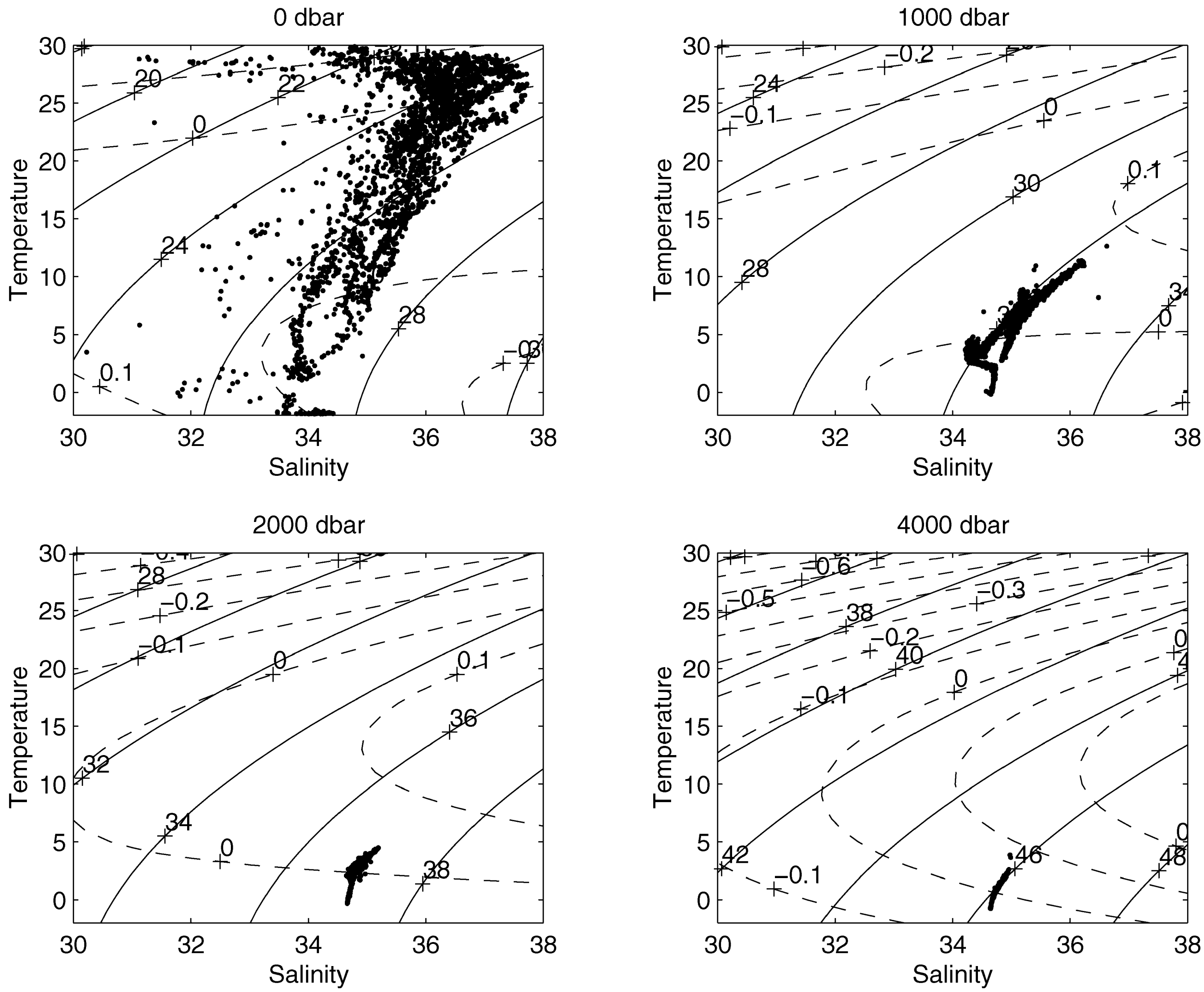

FIG. 2. Scatterplots of θ–s at 0, 10, 20, and 40 MPa from the Atlantic WOCE dataset, with isopycnal contours from the prototype EOS (2.1), and its deviation from the international EOS.

from the first term of (2.8a). This novel term, coming about from the use of orthobaric sp. vol. as coordinate, and highlighting the ocean's deviation from the standard θ–s relation, is the crux of this paper. We shall neglect $\dot{N}$ in the remainder of this paper, though, for completeness, we have written the explicit form (2.8c) for it here.

When orthobaric sp. vol. $\nu = \alpha_0 \nu'$ is used as a vertical coordinate, the substantial rate of change operator is written

$$\frac{D}{Dt} = \left.\frac{\partial}{\partial t}\right|_{\nu} + \mathbf{u}\cdot\nabla_{\nu} + \dot{\nu}\frac{\partial}{\partial \nu}. \tag{2.9}$$

For example,

$$\dot{p} = \frac{Dp}{Dt} = \left.\frac{\partial p}{\partial t}\right|_{\nu} + \mathbf{u}\cdot\nabla_{\nu} p + \dot{\nu} p_{\nu}, \tag{2.10}$$

so that (2.8a) may be written

$$p_{\nu}\dot{\nu} = (\psi - 1)\left(\left.\frac{\partial p}{\partial t}\right|_{\nu} + \mathbf{u}\cdot\nabla_{\nu} p\right). \tag{2.11a}$$

Here

$$\psi = (1 - \phi^{-1}\gamma\alpha_1\Delta\theta\alpha_0 p_{\nu})^{-1} \tag{2.11b}$$

is the buoyancy gain factor. It is also the ratio n_a^2/n^2, where n_a^2 is the apparent buoyancy frequency squared,

$$n_a^2 = \frac{-\phi g^2}{\alpha^2 p_{\nu}}, \tag{2.12}$$

which differs from true buoyancy frequency squared,

$$n^2 = -\frac{g^2}{\alpha^2 p_{\alpha}} - \frac{g^2}{c^2}, \tag{2.13}$$

where g is gravitational acceleration. In the following when we use (2.12) or (2.13) we shall replace α by

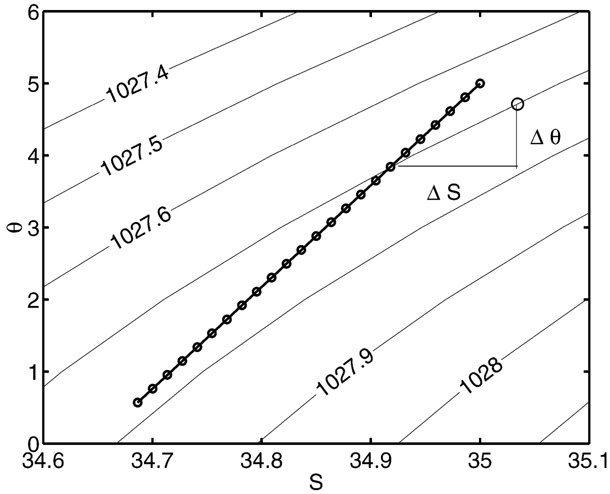

FIG. 3. Schematic θ–s diagram at fixed pressure p showing the standard relation with normalized slope $R = m_0/(m_0 - 1)$, and deviations $\Delta\theta$ and Δs of an observation point, measured along an orthobaric or in situ isopycnal (which coincide at fixed pressure).

constant α_0 (a minor approximation) without further comment. Equation (2.11a) is the thermobaric mass flux across orthobaric isopycnals caused by the deviation of a water mass from a reference θ–s relation (de Szoeke et al. 2000). (It would be zero if $\gamma = 0$ or $\Delta\theta = 0$.)

Rest state: Constant buoyancy frequency

A motionless rest state can be described in which both orthobaric and in situ isopycnals are level surfaces while isotherms and isohalines slope across level surfaces and across each other because of the thermobaric effect. If constant apparent buoyancy frequency $n_a = n_0$ is assumed, (2.12) for the orthobaric sp. vol. profile can be integrated, using (2.5b), to give

$$p = P(\nu) = \frac{1}{m_0\gamma}\left[\left(1 - \sigma\frac{\alpha_0 n_0^2}{g^2\gamma m_0}\right)(-1 + e^{-g^2\gamma m_0\nu/\alpha_0^2 n_0^2}) - \sigma\frac{\nu}{\alpha_0}\right]. \tag{2.14}$$

The argument of the exponential can be shown to be small, so that this is very well approximated by $P(\nu) \cong -g^2\nu/\alpha_0^2 n_0^2$, that is, linear in ν. [For an arbitrary buoyancy frequency profile, e.g., $n_a = n_0 e^{-p/p_0}$, the counterpart of (2.14) may be readily obtained.]

For an ocean at rest, one may suppose that this relation between pressure and orthobaric sp. vol. holds everywhere. From (2.5a), sp. vol. anomaly is easily obtained, being a function only of ν' and p. In particular, it can be obtained for the rest state [(2.14)]. Also, the separate temperature and salinity profiles are easily obtained from (2.7). If these profiles pertain everywhere, then $n_a^2 = n_0^2$ and $\psi = 1$. However, suppose there are meridional deviations (say) from these salinity and temperature profiles; that is,

$$\beta_s\bar{s} = (m_0 - 1)\nu/\alpha_0 + \beta_s s_2(y, \nu) \quad \text{and} \tag{2.15a}$$

$$\alpha_1\bar{\theta} = m_0\nu/\alpha_0 + \alpha_1\theta_2(y, \nu), \tag{2.15b}$$

where $s_2 = \theta_2 = 0$ at $y = 0$.

The rest state will be maintained if the deviations are density compensated; that is,

$$\alpha_1[1 + \gamma P(\nu) + \sigma\nu/(m_0\alpha_0)]\theta_2 + \frac{1}{2}\alpha_2\theta_2^2 = \beta_s s_2, \tag{2.16}$$

so that (2.14) applies everywhere. However, then the local θ–s relation, and the local density ratio, defined in terms of local contributions of temperature and salinity to density, will vary with position y. Only at $y = 0$ will it match the constant R of the reference θ–s relation.

Consider in particular a temperature anomaly of the form

$$\theta_2 = \theta_2' y Z^{(s)}(\nu), \tag{2.17a}$$

with corresponding salinity anomaly given by (2.16), and with

$$Z^{(s)}(\nu) = \cos[\pi P(\nu)/P_B]. \tag{2.17b}$$

The buoyancy gain factor $\overline{\psi}$ for the rest state is given by (2.11b), upon setting $p = P(\nu)$, given by (2.14), and using (2.17):

$$\overline{\psi}(y, \nu)^{-1} = 1 - \phi(P)^{-1}\gamma\alpha_1\theta_2(y, \nu)P_\nu\alpha_0 = 1 + byZ^{(s)}(\nu), \tag{2.18}$$

where

$$b = \gamma g^2\alpha_1\theta_2'/\alpha_0 n_0^2.$$

This may differ significantly from 1, on a meridional length scale b^{-1}. The true buoyancy frequency is $n^2 = \overline{\psi}^{-1}n_0^2$, which we see is not constant, but varies meridionally and vertically, even though the ocean is at rest. This is because of the meridional variation of the θ–s relation and the thermobaric nonlinearity of the EOS, measured by the parameter γ.

In Fig. 4, meridional sections are shown of θ, s, in situ and orthobaric sp. vol., and potential sp. vol. that give a rest state for the parameter settings listed in Tables 1 and 2. As the foregoing discussion made clear, the temperature and salinity sections were engineered to give level in situ and orthobaric sp. vol. surfaces, though with meridional variation from the central standard θ–s relation (Fig. 4g). Potential sp. vol. surfaces (Figs. 4e,f) are markedly not level, irrespective of choice of reference pressure, even showing an inversion—this despite a less than 20% variation of squared buoyancy frequency from n_0^2 (Fig. 4h). This behavior of potential density comes about from its reliance on the expansion coefficient at the reference pressure, rather than allowing for its variation with pressure (Ekman 1934). Orthobaric sp. vol. surface labels show only the small variations due principally to θ and s, rather than the far larger contributions from pressure that dominate in situ sp. vol.

3. The model equations

A vorticity equation will be obtained for motion perturbations of a resting ocean with a spatially variable θ–s relation, an effect measured by nonunit buoyancy gain factor. We begin with the time-dependent equations of motion on the beta plane, written in terms of an arbitrary function $\nu(p, \alpha)$ instead of z as vertical independent variable (de Szoeke et al. 2000):

$$u_t + uu_x + \upsilon u_y - f\upsilon = -M_x, \tag{3.1}$$

$$\upsilon_t + y\upsilon_x + \upsilon\upsilon_y + fu = -M_y, \tag{3.2}$$

$$\Pi(p, \nu) = M_\nu, \tag{3.3}$$

$$p_{\nu t} + (up_\nu)_x + (\upsilon p_\nu)_y + (\dot{\nu} p_\nu)_\nu = 0, \quad \text{and} \tag{3.4}$$

$$\theta_t + u\theta_x + \upsilon\theta_y + \dot{\nu}\theta_\nu = 0. \tag{3.5}$$

Here x, y, and t are east, north, and time coordinates; u and υ are horizontal velocity components; $f = f_0 + \beta y$ is Coriolis parameter; p is pressure; s is salinity; M is the Montgomery function; and Π is the Exner function, the latter two respectively defined as

$$M = \int_0^p \alpha(r, \nu)\, dr + \Phi \quad \text{and} \tag{3.6a}$$

$$\Pi = \int_0^p \partial\alpha(r, \nu)/\partial\nu\, dr, \tag{3.6b}$$

where $\Phi = gz$ is the geopotential. Friction and diabatic effects are neglected, so that the diapycnal pseudovelocity $\dot{\nu}$ is due solely to reversible thermobaric effects. The associated diapycnal mass flux is given by (2.11a):

$$\dot{\nu} p_\nu = (\psi - 1)(p_t + up_x + \upsilon p_y). \tag{3.7}$$

Here ψ would be the buoyancy gain factor, defined by (2.11b), if $\nu/\alpha_0 = \nu'$ were the normalized orthobaric sp. vol. given by (2.5). In more general terms, de Szoeke (2000) showed how ψ can be defined for arbitrary $\nu(p, \alpha)$.

By cross-differentiating (3.1) and (3.2), one may obtain the vorticity equation

$$(\partial_t + u\partial_x + \upsilon_y)(\upsilon_x - u_y) + (f + \upsilon_x - u_y)(u_x + \upsilon_y) + \beta\upsilon = 0. \tag{3.8}$$

Assuming an approximate geostrophic balance in (3.1) and (3.2) and replacing the Coriolis parameter by the constant f_0 as follows:

$$-f_0\upsilon = -M_x \quad \text{and} \tag{3.9a}$$

$$f_0 u = -M_y, \tag{3.9b}$$

one may calculate the relative vorticity in (3.8),

$$\upsilon_x - u_y = f_0^{-1}\nabla^2 M. \tag{3.9c}$$

The thickness equation (3.4) may be written, using (3.7), as

$$p_{\nu t} + up_{\nu x} + \upsilon p_{\nu y} + p_\nu(u_x + \upsilon_y) + \partial_\nu[(\psi - 1)(p_t + up_x + \upsilon p_y)] = 0. \tag{3.10}$$

From (3.3), $\Pi_\xi = \Pi_p p_\xi = M_{\nu\xi}$, where partial differentiation with respect to ξ stands for t, x, or y differentiation, holding ν constant. Also, from (2.5b) and (3.6b), $\phi = \partial\Pi/\partial p$. Hence, using (3.9),

$$up_{\nu x} + \upsilon p_{\nu y} = \partial_\nu(up_x + \upsilon p_y) + f_0^{-1}\phi^{-1}(M_{y\nu}M_{\nu x} - M_{x\nu}M_{\nu y}),$$

so that (3.10) may be written

$$\partial_\nu[\psi(p_t + up_x + \upsilon p_y)] + p_\nu(u_x + \upsilon_y) = 0. \tag{3.11}$$

Eliminating $u_x + \upsilon_y$ between (3.8) and (3.11), one obtains

$$f_0^{-1}\frac{D_g}{Dt}\nabla^2 M - (f + f_0^{-1}\nabla^2 M)p_\nu^{-1}\partial_\nu\left(\psi\phi^{-1}\frac{D_g}{Dt}M_\nu\right) + f_0^{-1}\beta M_x = 0, \tag{3.12}$$

where

$$\frac{D_g}{Dt} = \partial_t + f_0^{-1}J(M, \cdot\,) \tag{3.13}$$

is the geostrophic advective rate-of-change operator. Boundary conditions on (3.12) are that perturbation pressures vanish at top and bottom boundaries:

$$\phi p' = M_\nu = 0 \quad \text{at } \nu = \nu_1 \quad \text{and} \quad \nu_2. \tag{3.14}$$

Equation (3.12) is a form of the intermediate geostrophic approximation to the vorticity balance (Williams and Yamagata 1984), here written relative to x, y, ν coordinates. The first term is the rate of change of the component of relative vorticity *perpendicular to* ν *surfaces* (de Szoeke 2000); the second term is internal vortex stretching due to thickening and thinning of ν layers; the last term is advection of planetary vorticity.

Though it is our intention to take advantage of the properties of the particular orthobaric sp. vol. function introduced in section 2, the use of the independent variable ν hitherto in this section is general; it is only required that it be a monotonic function of p and α.

To make the vorticity equation in (3.12) look a little more familiar, we may take a special case, $\nu = p$; in more general terms, let $\nu = p - c_1^2/\alpha$ before taking the limit $c_1 \to 0$. De Szoeke (2000) shows that in this case

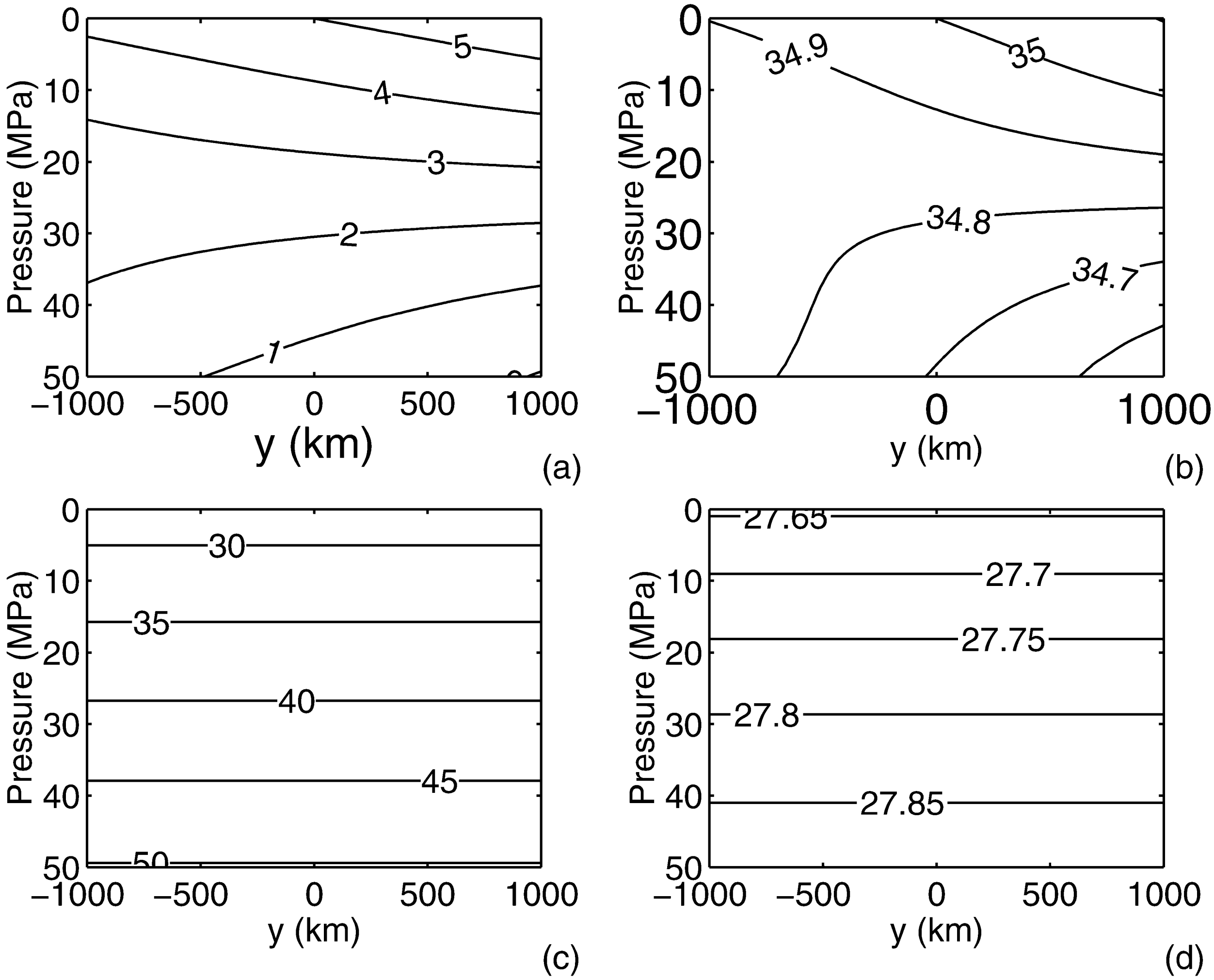

FIG. 4. Meridional sections across an idealized ocean in a rest state. (a), (b) $\overline{\theta}$ and $\overline{s}$, respectively, calculated from (2.15)–(2.17), so that meridional gradients are density-compensated, giving level (c) in situ and (d) orthobaric isopycnals. Orthobaric isopycnal values lack the strong pressure variation of in situ isopycnals (units: kg m^{-3}). (e), (f) Potential density referenced to 0 and 20 MPa, respectively.

TABLE 2. Geophysical parameters.

Parameter	Description	Value
f_0	Reference Coriolis parameter	10^{-4} s^{-1}
β	Coriolis parameter gradient	2×10^{-11} m^{-1} s^{-1}
H_B	Mean ocean depth	5 km
P_B	Mean ocean depth (in pressure units)	50 MPa
n_0	Reference buoyancy frequency	0.8×10^{-3} s^{-1} ($\equiv$0.46 cph)
λ_1	Baroclinic Rossby radius of deformation ($=n_0H_B/f_0\pi$)	12.7 km
c_1	Long Rossby wave speed ($=-\beta\lambda_1^2$)	-0.32 cm s^{-1}
R	Mean density ratio [$=m_0/(m_0-1)$]	2.0
θ'_2	Meridional gradient of mean temperature	7.07×10^{-7} °C m^{-1}
b	Coefficient of meridional variation of buoyancy gain factor ($=g^2\gamma\alpha_1\theta'_2/\alpha_0 n_0^2$)	2.2×10^{-7} m^{-1}
$a_2^{(1)}$	Principal thermobaric contribution to nonlinear coefficient of KdV equation ($=b/4f_0\lambda_1^2$)	3.35×10^{-12} m^{-3} s
$\beta/a_2^{(1)}$	Scale for first-mode solitary Rossby wave amplitude (Montgomery potential units)	6 m^2 s^{-2}
$\dfrac{\pi}{n_0^2H_B}\dfrac{\beta}{a_2^{(1)}}$	Scale for first-mode solitary Rossby wave amplitude (isopycnal displacement units)	6000 m

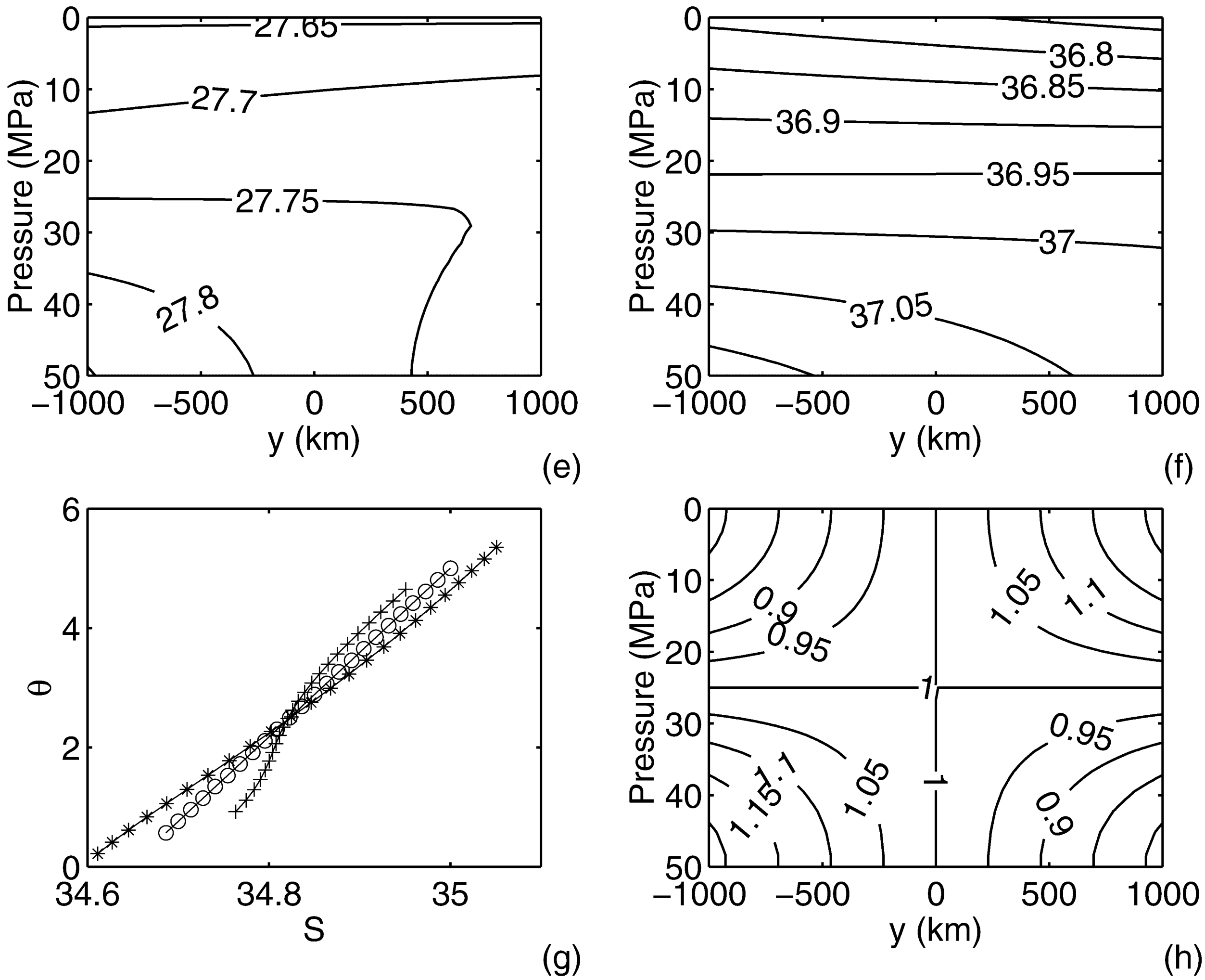

FIG. 4. (*Continued*) Potential isopycnals are not level for the rest state and may show inversions even though stratification is stable. (g) The θ–s diagrams at $y = -500$ (+), 0 (○), and 500 (*) km. The standard θ–s relation is taken from $y = 0$, where $R = 2$ (or $m_0 = 2$). (h) The buoyancy gain factor ψ, the ratio of the reference squared buoyancy frequency to the local squared buoyancy frequency.

$$\Pi = -\alpha - \frac{\alpha c_1^2}{\alpha p - c_1^2} \to -\alpha,$$

$$M = c_1^2 \ln\left(\frac{p - \nu}{\nu}\right) + \Phi \to \Phi,$$

$$\psi = 1 + \frac{g^2}{n^2}\left(\frac{1}{c^2} - \frac{1}{c_1^2}\right), \quad \text{and}$$

$$\phi = \frac{\alpha^2}{c_1^2}. \tag{3.15}$$

The last two expressions are singular as $c_1^2 \to 0$, though their ratio, which occurs in (3.12), is bounded:

$$\psi\phi^{-1} \to -\alpha^{-2}g^2n^{-2}.$$

Hence (3.12) becomes

$$f_0^{-1}\frac{D_g}{Dt}\nabla^2\Phi + (f + f_0^{-1}\nabla^2\Phi)\partial_p\left(\alpha^{-2}g^2n^{-2}\frac{D_g}{Dt}\Phi_p\right) + f_0^{-1}\beta\Phi_x = 0. \tag{3.16}$$

Making the quasigeostrophic (QG) approximation $f_0^{-1}\nabla^2\Phi \ll f \cong f_0$, replacing $\alpha_0^{-1}g\partial_p = -\partial_z$, and making the approximation $\alpha^2/\alpha_0^2 \cong 1$, one recovers Straub's (1999) form of the QG vorticity equation in z coordinates with thermobaric effect included. It should be stressed that n^2, buoyancy frequency squared, is yet a function of all four space and time variables, not of p alone. Because γ is nonzero, it is not possible to obtain a conserved potential vorticity form, unless a θ–s relation is universally satisfied. For then, and only then, if ν were chosen with reference to that relation, buoyancy gain factor ψ would be 1 everywhere and, because

$$\frac{D_g p_\nu}{Dt} = \frac{\partial}{\partial \nu}\frac{D_g p}{Dt} = \frac{\partial}{\partial \nu}\left(\phi^{-1}\frac{D_g}{Dt}M_\nu\right),$$

(3.12) would become

$$\frac{D_g}{Dt}\left(\frac{f + f_0^{-1}\nabla^2 M}{p_\nu}\right) = 0.$$

This equation expresses the conservation of what one might call intermediate geostrophic potential vorticity.

While the vorticity balance (3.12) relative to a general coordinate of the form $\nu(p, \alpha)$ reduces to (3.16) when $\nu \to p$ is taken, the reverse does not seem to be the case; that is, it is not obvious how to arrive at (3.12) by starting from (3.16). Hence, while Straub's (1999) conclusions about thermobaric effects, drawn from (3.16), may be considered to follow also from (3.12), it may be possible to derive further insights from the more general (3.12), perhaps by specializing to other coordinates $\nu(p, \alpha)$. Indeed the choice of an appropriate orthobaric sp. vol. for ν, which tends to make the buoyancy gain factor ψ close to 1 (contrast $\nu \to p$, which corresponds to $\psi \to -\infty$!), and diapycnal mass flux $\dot{\nu}p_\nu$ small, will be shown to have particular merit.

4. Perturbation analysis

We shall obtain solutions of the extended vorticity equation in (3.12) by conducting a perturbation around the rest state described in section 2. Suppose

$$M = \overline{M}(\nu) + M', \qquad p = P(\nu) + p', \tag{4.1}$$

$$\theta = \overline{\theta}(y, \nu) + \theta', \quad \text{and} \quad s = \overline{s}(y, \nu) + s', \tag{4.2}$$

where $\overline{\theta}$ and $\overline{s}$ are given by (2.15), and the fluctuating primed variables introduced here are functions of x, y, ν, and t. The subsidiary variables ϕ and ψ defined by (2.5b) and (2.11b) must also be decomposed into means and fluctuations and are given by

$$\begin{aligned}
\overline{\phi} &= 1 + m_0\gamma P(\nu) + \sigma\nu/\alpha_0,\\
\phi' &= m_0\gamma p',\\
\overline{\psi} &= (1 - \overline{\phi}^{-1}\gamma\alpha_1\theta_2\alpha_0 P_\nu)^{-1}, \quad \text{and}\\
\psi' &= \overline{\psi}^{-2}(-\overline{\phi}^{-2}\phi'\gamma\alpha_1\theta_2\alpha_0 P_\nu + \overline{\phi}^{-1}\gamma\alpha_1\theta'\alpha_0 P_\nu\\
&\quad + \overline{\phi}^{-1}\gamma\alpha_1\theta_2\alpha_0 p'_\nu),
\end{aligned} \tag{4.3}$$

where $\alpha_1\theta_2 = \alpha_1\overline{\theta} - m_0\nu/\alpha_0$. The linearized hydrostatic balance (3.3) gives

$$p' = \overline{\phi}^{-1}M'_\nu, \tag{4.4}$$

and the linearized diapycnal mass flux (3.7) is

$$\dot{\nu}P_\nu = (\overline{\psi} - 1)p'_t. \tag{4.5}$$

An equation for the fluctuating temperature is obtained by substituting (4.2) into (3.5), using (3.9), (4.4), and (4.5) and neglecting quadratic terms,

$$\theta'_t + f_0^{-1}M'_x\overline{\theta}_y + (\overline{\psi} - 1)(\overline{\phi}P_\nu)^{-1}M'_{\nu t}\overline{\theta}_\nu = 0. \tag{4.6}$$

This will be required to evaluate ψ', in which θ' appears.

Once θ' and p' are determined from M', obtained in the manner to be shown below, the total pressure p and temperature θ can be obtained from (4.1) and (4.2). These are given as functions of ν. To obtain salinity, one calculates sp. vol. anomaly from (2.5a), $\alpha' = (1 + m_0\gamma p)\nu/\alpha_0 + (1/2)\sigma(\nu/\alpha_0)^2$, then inverts the EOS (2.1) for s:

$$\beta_s s = -\alpha' + (1 + \gamma p)\alpha_1\theta + \frac{1}{2}\alpha_2\theta^2. \tag{4.7}$$

This can be partitioned into $\overline{s}(y, \nu)$ and s' if required. It is not necessary to solve a prognostic equation for salinity.

To anticipate perturbations propagating zonally with a constant speed c, not yet known, we change to coordinates moving with this speed:

$$x' = x - ct \quad \text{and} \tag{4.8a}$$

$$t' = t. \tag{4.8b}$$

Hence partial derivatives transform according to

$$\partial_t = -c\partial_{x'} + \partial_{t'}, \quad \text{and} \quad \partial_x = \partial_{x'}. \tag{4.9}$$

The premise of this transformation is that the temporal evolution of a perturbation waveform, and its self-advection, is slow following its propagation:

$$\partial_{t'} + f_0^{-1}J(M, \cdot) \ll c\partial_{x'}. \tag{4.10a}$$

Also, it is assumed that fluctuating perturbations are small in comparison with means:

$$p' \ll P, \qquad \psi' \ll \overline{\psi}, \quad \text{and} \quad \phi' \ll \overline{\phi}. \tag{4.10b}$$

Thus, (4.6), by dropping primes on M', x', and t' and using these ordering assumptions, may be written

$$-c\theta'_x + f_0^{-1}M_x\overline{\theta}_y - c(\overline{\psi} - 1)(\overline{\phi}P_\nu)^{-1}M_{\nu x}\overline{\theta}_\nu = 0. \tag{4.11}$$

Integrated with respect to x, supposing that both θ' and $M \to 0$ as $|x| \to \infty$, this gives

$$\alpha_1\theta' = (f_0 c)^{-1}\alpha_1\overline{\theta}_y M - (\overline{\psi} - 1)(\overline{\phi}P_\nu)^{-1}\alpha_1\overline{\theta}_\nu M_\nu. \tag{4.12}$$

Here we may substitute from (2.15b) and (2.17) for $\overline{\theta}(y, \nu)$.

Equation (3.12) similarly becomes

$$\begin{aligned}
-c(\overline{\psi}\,\overline{\phi}^{-1}M_{\nu x})_\nu - \beta f_0^{-2}P_\nu M_x
&= -(\overline{\psi}\,\overline{\phi}^{-1}M_{\nu t})_\nu + \beta f_0^{-2}(\overline{\phi}^{-1}M_\nu)_\nu M_x\\
&\quad + f_0^{-2}P_\nu[-c\partial_x + f_0^{-1}J(M, \ \cdot)]\nabla^2 M\\
&\quad + c[(\psi'\overline{\phi}^{-1} - \overline{\psi}\,\overline{\phi}^{-2}\phi')M_{\nu x}]_\nu\\
&\quad + cf_0^{-1}\beta y\partial_\nu(\overline{\psi}\,\overline{\phi}^{-1}M_{\nu x})\\
&\quad - f_0^{-1}\partial_\nu[\overline{\psi}\,\overline{\phi}^{-1}J(M, M_\nu)].
\end{aligned} \tag{4.13}$$

Terms that have been written on the right appear because they are small in comparison with those on the left on account of the criteria enunciated by (4.10); some terms have been omitted altogether because they are small in comparison with those retained on the right. The rate of change of relative vorticity—the third term on the right—has been held to be small in comparison with internal vortex stretching and planetary vorticity advection on the left.

Substituting an ordered expansion of M,

$$M = M^{(0)} + M^{(1)} + \cdots, \tag{4.14}$$

in which $M^{(n)} \ll M^{(n-1)}$, one obtains, at the lowest orders,

$$-c[\overline{\psi}\overline{\phi}^{-1}M^{(0)}_\nu]_\nu - \beta f_0^{-2}P_\nu M^{(0)} = 0 \quad \text{and} \tag{4.15}$$

$$\begin{aligned}\partial_x\{-c[\overline{\psi}\overline{\phi}^{-1}M^{(1)}_\nu]_\nu - \beta f_0^{-2}P_\nu M^{(1)}\} = &-[\overline{\psi}\overline{\phi}^{-1}M^{(0)}_{\nu t}]_\nu + \beta f_0^{-2}[\overline{\phi}^{-1}M^{(0)}_\nu]M^{(0)}_x \\ &+ c[(\psi'\overline{\phi}^{-1} - \overline{\psi}\overline{\phi}^{-2}\phi')M^{(0)}_{\nu x}]_\nu + f_0^{-2}P_\nu\{-c\partial_x \\ &+ f_0^{-1}J[M^{(0)}, \ \cdot\ ]\}\nabla^2 M^{(0)} + cf_0^{-1}\beta y\partial_\nu[\overline{\psi}\overline{\phi}^{-1}M^{(0)}_{\nu x}] \\ &- f_0^{-1}\partial_\nu\{\overline{\psi}\overline{\phi}^{-1}J[M^{(0)}, M^{(0)}_\nu]\}.\end{aligned} \tag{4.16}$$

The boundary conditions (3.14) apply at all orders of the expansion (4.14). Equation (4.15) is a classical Sturm–Liouville eigenvalue problem for c. It is also, in somewhat disguised form, the equation for the vertical baroclinic modes. Let

$$M^{(0)} = A(x, y, t)Z(\nu); \tag{4.17}$$

then $Z(\nu)$ satisfies (4.15), with boundary conditions

$$Z_\nu = 0 \quad \text{at } \nu = \nu_1 \text{ and } \nu_2. \tag{4.18}$$

Convert this equation by making $P = P(\nu)$ the independent variable, with $\partial_\nu = P_\nu\partial_P$:

$$-c\left(\frac{f_0^2 g^2}{\overline{n}^2\alpha_0^2}Z_P\right)_P + \beta Z = 0 \quad \text{with}$$
$$Z_P = 0 \quad \text{at } P = 0 \quad \text{and} \quad P_B, \tag{4.19}$$

where

$$\overline{n}^2 = -\overline{\psi}^{-1}\overline{\phi}g^2/P_\nu\alpha_0^2 = \overline{\psi}^{-1}n_0^2 \tag{4.20}$$

is the meridionally nonuniform buoyancy frequency calculated from the rest state. The gain factor $\overline{\psi}$ is a function of y [see (2.18)], though this is merely a parametric dependence in (4.19). The eigenvalue problem posed in the form (4.19) is standard and transparent to the definition of orthobaric specific volume ν, as it should be. It gives the vertical modes and phase speeds of the baroclinic planetary waves. For n_0 = a constant, for example, the solutions of (4.19), neglecting the deviation of $\overline{\psi}$ from 1, are

$$Z_m = \cos(m\pi P/P_B) \quad \text{and} \tag{4.21a}$$

$$c_m = -\frac{\beta P_B^2 n_0^2\alpha_0^2}{m^2\pi^2 f_0^2 g^2}, \tag{4.21b}$$

for $m = 1, 2, 3 \ldots$. Even for general $n_0(P)$, the vertical modes and phase speeds may be readily computed. For example, Chelton et al. (1998) have done this for $\overline{n}(P, x, y)$ profiles averaged over geographical degree-squares for the world's oceans, displaying maps of first-mode Rossby deformation radius, which is $\lambda_1 = (-c_1/\beta)^{1/2}$ in the present notation. If $\overline{\psi}$ varies meridionally, this will force a meridional dependence in (4.21). Yet if the vertical average of $\overline{\psi}$ is close to 1, the effect on the c_m, especially the low modes, which depend on the average of $\overline{n}$, will be slight.

Equation (4.16) describes corrections to the linear modes that are driven by several effects neglected at lowest order. Considering the various terms on the right side of (4.16), these are internal vortex stretching due to long-term temporal changes with respect to the moving reference frame, advection of planetary vorticity associated with internal thickness anomalies generated by low-order motions, internal vortex stretching induced by anomalies in the buoyancy gain factor, relative vorticity changes encountered in the moving frame, and self-advection of relative vorticity. The operator on the left side of (4.16) has the same form as (4.15). To ensure that the right side of (4.16) does not force secular resonances of the eigenvalues of (4.15), it must be orthogonal to the eigenmode $Z(\nu) = Z[P(\nu)]$. Formally, one multiplies both sides of (4.16) by $Z(\nu)$ and integrates from ν_1 to ν_2. Partial integration of terms on the left side of (4.16) and use of boundary conditions (3.14) show that this integral is zero. Hence the integral of the right side must also be zero. Substituting (4.17), one obtains

$$\begin{aligned}&(-A_t + cf_0^{-1}\beta yA_x)\int_{\nu_1}^{\nu_2}(\overline{\psi}\overline{\phi}^{-1}Z_\nu)_\nu Z\,d\nu \\ &+ AA_x\beta f_0^{-2}\int_{\nu_1}^{\nu_2}(\overline{\phi}^{-1}Z_\nu)_\nu Z^2\,d\nu + \nabla^2 A_x c f_0^{-2} \\ &\times \int_{\nu_1}^{\nu_2}(-P_\nu)Z^2\,d\nu \\ &- f_0^{-3}J(A, \nabla^2 A)\int_{\nu_1}^{\nu_2}(-P_\nu)Z^3\,d\nu \\ &+ A_x c\int_{\nu_1}^{\nu_2}[(\psi'\overline{\phi}^{-1} - \overline{\psi}\overline{\phi}^{-2}\phi')Z_\nu]_\nu Z\,d\nu = 0.\end{aligned} \tag{4.22}$$

{The internal vortex stretching self-advection makes no contribution because $J[M^{(0)}, M^{(0)}_\nu] = J(A, A)ZZ_\nu = 0$.} From (4.15), one can show that

$$\begin{aligned}\int_{\nu_1}^{\nu_2}(\overline{\psi\phi}^{-1}Z_\nu)_\nu Z\,d\nu &= \frac{\beta}{f_0^2 c}\int_{\nu_1}^{\nu_2}(-P_\nu)Z^2\,d\nu \\ &= \frac{\beta}{f_0^2 c}\int_0^{P_B}Z^2\,dP \equiv \frac{\beta}{f_0^2 c}N,\end{aligned} \tag{4.23}$$

which defines the normalizing constant N. The coefficient of AA_x in (4.22) may be written

$$\beta f_0^{-2}\int_{\nu_1}^{\nu_2}(\overline{\phi}^{-1}Z_\nu)_\nu Z^2\,d\nu = \beta f_0^{-2}\frac{g^2}{\alpha_0^2}\int_0^{P_B}\left(\frac{Z_P}{n_0^2}\right)_P Z^2\,dP$$
$$\equiv f_0^{-2}Na_1, \tag{4.24a}$$

where the second equality defines a_1. For the coefficient of $A_x c$, upon integrating by parts and substituting from (4.3), (4.4), and (4.12), one obtains at length

$$\int_{\nu_1}^{\nu_2}[(\psi'\overline{\phi}^{-1} - \overline{\psi}\overline{\phi}^{-2}\phi')Z_\nu]Z_\nu\,d\nu$$
$$= -\int_{\nu_1}^{\nu_2}(\psi'\overline{\phi}^{-1} - \overline{\psi}\overline{\phi}^{-2}\phi')Z_\nu^2\,d\nu$$
$$= ANc^{-1}f_0^{-2}a_2, \tag{4.24b}$$

where

$$a_2 = cf_0^2\int_{\nu_1}^{\nu_2}\left\langle\overline{\psi}^{-2}\left\{\overline{\phi}^{-3}m_0\gamma^2\overline{\phi}^{-1}Z_\nu\alpha_1\theta_2\alpha_0P_\nu - \overline{\phi}^{-2}\gamma\left[\alpha_1\overline{\theta}_y\frac{Z}{f_0c} - (\overline{\psi} - 1)(\overline{\phi}P_\nu)^{-1}\alpha_1\overline{\theta}_\nu Z_\nu\right]\alpha_0P_\nu - \overline{\phi}^{-2}\gamma\alpha_1\theta_2\alpha_0(\overline{\phi}^{-1}Z_\nu)_\nu\right\} + \overline{\psi\phi}^{-2}m_0\gamma\overline{\phi}^{-1}Z_\nu\right\rangle Z_\nu^2\,d\nu/N. \tag{4.24c}$$

The coefficient of $J(A, \nabla^2 A)$ is

$$-f_0^{-3}\int_{\nu_1}^{\nu_2}(-P_\nu)Z^3\,d\nu = -f_0^{-3}\int_0^{P_B}Z^3\,dP$$
$$\equiv -f_0^{-3}Na_3. \tag{4.24d}$$

Hence, (4.22) may be written

$$-\frac{\beta}{c}A_t + c\nabla^2A_x - a_3f_0^{-1}J(A, \nabla^2A)$$
$$+ (a_1 + a_2)AA_x + \beta^2yf_0^{-1}A_x = 0. \tag{4.25}$$

Equation (4.25) is identical in form and similar in physical interpretation to the intermediate geostrophic vorticity equation (Matsuura and Yamagata 1982; Williams and Yamagata 1984). The wave steepening term—the AA_x term in (4.25)—is composed of two contributions, one from the coefficient a_1, measuring the nonlinear effect of finite-amplitude changes in thickness on divergence of ageostrophic motion within isopycnals (Anderson and Killworth 1979), and a novel one from the coefficient a_2, which, by contrast, measures divergence in orthobaric isopycnals driven by the diapycnal motion associated with fluctuating variations s' and θ' away from the θ–s relation, coupled with the thermobaric effect. The βyA_x term in (4.25), representing the meridional variation of Rossby-wave propagation speed and included here for completeness and comparison with Williams and Yamagata (1984), will be neglected. If motions are elongated in the meridional direction, so that $\partial_y \ll \partial_x$, (4.25) becomes

$$-\frac{\beta}{c}A_t + cA_{xxx} + (a_1 + a_2)AA_x = 0, \tag{4.26}$$

which is the well-known Korteweg-de Vries equation (Whitham 1974; Drazin 1983).

For any of the eigenmodes given by (4.21), valid when n_0^2 = a constant, it may easily be verified that

$$a_1 = 0 \quad\text{and}\quad a_3 = 0. \tag{4.27}$$

These properties are merely a consequence of the symmetry of the modes when buoyancy frequency is constant in the vertical direction. If buoyancy frequency is not constant, this symmetry is broken and a_1 and a_3 may be nonzero.

The parameter a_1 measures a classic planetary wave-steepening effect (Anderson and Killworth 1979). This can be illustrated most advantageously in two layers. Consider two layers of fluid with thickness $H_1 - h(x, t)$ and $H_2 + h(x, t)$ and densities ρ_1 and ρ_2. The long-planetary-wave speed is given by

$$c_R = -\beta f_0^{-2}g'[(H_1 - h)^{-1} + (H_2 + h)^{-1}]^{-1}$$
$$\cong c_1[1 + (H_2^{-1} - H_1^{-1})h + 0(h^2)], \tag{4.28}$$

where $g' = g(\rho_2 - \rho_1)/\rho_0$ and $c_1 = -\beta f_0^{-2}g'(H_1^{-1} + H_2^{-1})^{-1}$ is the small-amplitude limit of the wave speed. By taking into account nonlinear effects of baroclinic vortex stretching, Anderson and Killworth (1979) showed that the wave equation for the baroclinic displacement $h(x - c_1t, t)$ is

$$h_t + c_1(H_2^{-1} - H_1^{-1})hh_{x'} = 0, \tag{4.29}$$

where $x' = x - c_1t$. The wave-steepening term vanishes for two symmetric layers, $H_1 = H_2$—analogous to the constant-n continuous ocean. [The Sturm-Liouville problem (4.19) with $n^2 = g^2(\rho_2 - \rho_1)\delta(P - \rho_0gH_1)$ yields c_1 as sole eigenvalue, with the two-layer baroclinic eigenmode

$$Z = H_1^{-1}\ (P < \rho_0gH_1) \quad\text{and}$$
$$Z = -H_2^{-1}\ (P > \rho_0gH_1). \tag{4.30}$$

By a simple calculation, this mode gives $a_1 = \beta^2 c_1^{-1} f_0^{-2}(H_1^{-1} - H_2^{-1})$, which is nonzero for $H_1 \neq H_2$. Keeping only the temporal tendency and wave-steepening terms from (4.26),

$$A_t + \beta f_0^{-2}(H_2^{-1} - H_1^{-1})AA_x \approx 0. \qquad (4.31)$$

This is the same equation as (4.29) when one identifies interface displacement h with $-(H_1^{-1} + H_2^{-1})A/g'$.]

Among the contributions to a_2, defined by (4.24c), we focus on one, proportional to $\overline{\theta}_y$, given by (2.15b) and (2.17),

$$a_2^{(1)} = f_0 N^{-1} \int_{\nu_1}^{\nu_2} \overline{\psi}^{-2}\overline{\phi}^{-2}\gamma\alpha_1\theta_2'\alpha_0(-P_\nu)Z^{(s)}ZZ_\nu^2\,d\nu$$

$$= f_0 b \frac{g^2}{\alpha_0^2 n_0^2} \frac{\displaystyle\int_0^{P_B} \overline{\psi}^{-2} Z^{(s)}ZZ_P^2\,dP}{\displaystyle\int_0^{P_B} Z^2\,dP}. \qquad (4.32)$$

Here b^{-1} is the meridional length scale—(2.18)—on which $\overline{\psi}$ varies. Replacing $\overline{\psi}$ approximately by 1 in the integrand of (4.32), however, and using (2.17b) for $Z^{(s)}$ (i.e., first baroclinic mode) as the vertical shape factor of the θ–s relation variation, as well as $Z = Z_1$, one may calculate this parameter to be

$$a_2^{(1)} \cong f_0 b \frac{g^2}{\alpha_0^2 n_0^2}\frac{1}{4}\left(\frac{\pi}{P_B}\right)^2 \cong \frac{f_0 b}{4n_0^2}\left(\frac{\pi}{H_B}\right)^2, \qquad (4.33)$$

where $H_B = \alpha_0 P_B/g$ is the ocean depth. The assumption of the vertical antisymmetry of $\overline{\theta}_y$ due to $Z^{(s)}(\nu)$ is crucial to (4.33). Were this factor symmetric, the integral in the numerator of (4.32) would vanish. It is essential that there be some vertically antisymmetrizing component of $\overline{\theta}_y$ for the parameter $a_2^{(1)}$, which controls the nonlinear modification of the first baroclinic mode, itself antisymmetric, not to vanish. For these very reasons of symmetry, the other parameter a_1 appearing in (4.26), defined by (4.24a), vanishes for constant buoyancy frequency, for the first baroclinic mode.

On the other hand, for the second baroclinic mode, $Z = Z_2$, which is symmetric, it is necessary for $Z^{(s)}$ to contain a symmetric contribution—for example, $Z^{(s)} = \sin(\pi P/P_B)$—to make a nonzero contribution to $a_2^{(1)}$. In this example (4.32) gives

$$a_2^{(1)} \cong -\frac{128}{105}\frac{f_0 b}{n_0^2}\left(\frac{\pi}{H_B}\right)^2. \qquad (4.34)$$

In Table 2 are listed the geophysical parameters, and some representative numerical values for them, that appear in the theory developed above. Among them is $a_2^{(1)}$, the nonlinear thermobaric coefficient that appears in the KdV equation, (4.26), for the first baroclinic mode, and the parameters from which it is calculated. The well-known solitary-wave solution satisfying (4.26) is

TABLE 3. Vertical and horizontal scales of solitary thermobaric Rossby waves.

ϵ	ϵ^2	$3\epsilon^2$	$z'_{\max}$ (m)	$(\Delta x/\lambda_1)_{0.5}$	$(\Delta x/\lambda_1)_{0.1}$
0.183	0.033	0.1	600	10	20
0.129	0.0167	0.05	300	14	28
0.058	0.0033	0.01	60	31	63

$$\frac{A(x,t)}{\beta/a_2^{(1)}} = -3\epsilon^2\,\mathrm{sech}^2\left\{\frac{1}{2}\epsilon\left[\frac{x - c_1(1+\epsilon^2)t}{\lambda_1}\right]\right\} \qquad (4.35)$$

(Whitham 1974). The dimensional amplitude scale (in Montgomery potential units) is $\beta/a_2^{(1)}$, and the length scale is λ_1, both listed in Table 1; the dimensionless parameter ϵ is arbitrary. A pressure perturbation associated with (4.35) is

$$p' = \phi^{-1}M_\nu = \phi^{-1}P_\nu\frac{\partial Z}{\partial P}A(x,t). \qquad (4.36)$$

Substituting from (4.21), one may write this as

$$z' = -\frac{\alpha_0 p'}{g} = -\frac{\pi}{n_0^2 H_B}\sin\left(\frac{\pi P}{P_B}\right)A(x,t), \qquad (4.37)$$

which is the displacement, associated with the solitary wave, of an orthobaric isopycnal from its rest position (in meters, positive upward). In Table 3 are shown the maximum displacements $z'_{\max} = 3\epsilon^2(\pi/n_0^2 H_B)[\beta/a_2^{(1)}]$ for several choices of ϵ, and the associated horizontal distances $\Delta x/\lambda_1$, measured from the center of the solitary wave in units of the Rossby deformation radius, where the solitary-wave displacement amplitudes are one-half and one-tenth of the maximum. Also shown is ϵ^2, the nonlinear wave propagation speed-up factor. This factor is very small, no more than a few percent, and is certainly not large enough to cause any significant speed-up. The first row of Table 3 corresponds to a 600-m wave with half-amplitude scale of 10 deformation radii. This is a very vigorous wave certainly, though not inconceivable. The third row shows a 60-m isopycnal-displacement wave with half-amplitude scale of 31 deformation radii. This is a very modest wave: What is interesting about Table 3 is that, at planetary wave scales of tens of deformation radii, even moderate-amplitude waves (>60-m displacement) evidently can show nonlinear inhibition of planetary wave dispersion owing to thermobaric effect.

The sense of the internal displacement associated with a solitary wave like (4.35) is important. If the parameter b is positive, that is, positively increasing mean temperature and salinity variation from the standard θ–s relation with latitude in the upper pycnocline (negatively in the lower pycnocline), the first-baroclinic-mode thermobaric solitary wave is an elevation of the pycnocline. If $b < 0$, that is, colder and fresher meridional variation from the standard θ–s relation in the upper water column, warmer and saltier in the deep, the first-mode thermobaric solitary wave is a depression of the pycnocline.

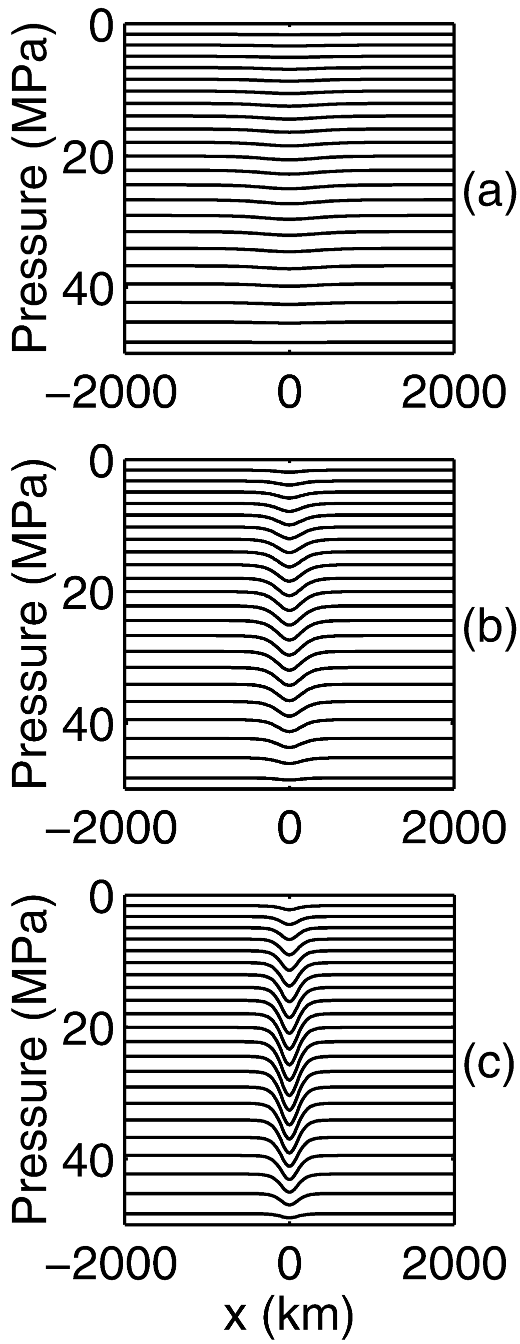

FIG. 5. Zonal sections of first-mode solitary waves in the pycnocline for various amplitude settings ϵ (Table 3): (a) 0.058, (b) 0.129, and (c) 0.183.

On the other hand, if the meridional deviation from the standard θ–s relation is negative throughout the pycnocline (colder and fresher at all depths, and increasing with latitude), it generates a second-mode (i.e., varicose) thermobaric solitary wave, in which the middle layers thicken while the upper and lower layers thin. Figure 5 shows zonal sections through the pycnocline of first-mode solitary waves of depression for the parameters of Table 3.

Much mathematical machinery is available for dealing with far more general solutions of the KdV equations than the sech-squared solitary wave—for example, inverse scattering theory for nonlinear dispersion of a general initial state into individual solitons, or the merging of several individual solitary waves (Whitham 1974; Drazin 1983). This approach is beyond the scope of this exploratory investigation, but perhaps it suffices to note that an initial arbitrary isopycnal disturbance composed of wavenumbers in the range of tens to hundreds of deformation radii but with amplitudes far smaller than indicated in Table 3 will disperse linearly according to well-known wave-mechanical principles. Initial disturbances in this range of scales with larger amplitudes must evolve nonlinearly, however, with modifications caused by the temperature dependence of the compressibility of seawater and spatial nonuniformities in the θ–s relation.

Certainly the thermobaricity of the equation of state for seawater does not exhaust the possible nonlinear mechanisms that affect baroclinic wave propagation, though it seems a hitherto unexamined one. One might also have considered the effects of mean flow shear, for example, as in Redekopp (1977) and Flierl (1979). Yet despite neglecting such effects, it is surprising that modest vertical excursions of isopycnals ($\sim$100 m), coupled with the effect of the pressure dependence of thermal expansion, for which the natural scale of vertical variation is measured in thousands of meters, can produce such marked effects. Nor does the free solitary-wave solution shown here describe the complete range of possible dynamical influences of the thermobaric effect. Yet it provides an illustration of how the effect may quantitatively modify ocean phenomena long taken to be well understood.

5. Summary and discussion

It has been shown that the pressure dependence of the thermal expansion coefficient of seawater—the thermobaric effect—coupled with large-scale meridional variation of the oceanic θ–s relation can induce dynamical modifications of the propagation of planetary Rossby waves. The means for this demonstration has been the use of orthobaric density as the vertical coordinate in writing the intermediate geostrophic equations of motion. Orthobaric density is in situ density corrected for pressure by using a virtual compressibility, obtained by assuming a particular θ–s relation (de Szoeke et al. 2000). We gave an example in section 2 of orthobaric density calculated for a simple equation of state (containing a bilinear thermobaric term, so that $\partial^2\alpha/\partial p\partial\theta =$ a constant $\neq 0$), coupled with a linear θ–s relationship, illustrating in a closed form how thermobaricity and the θ–s relation (in the guise of the density ratio R) determine the pressure correction. If the assumed θ–s relation were exactly satisfied throughout the ocean, orthobaric density would be a quasi-material variable everywhere, like potential temperature or salinity (or potential density). If there are deviations from the θ–s relation, however, then, in addition to the irreversible mixing and diffusion effects that change any quasi-material variable, there is a reversible contribution to orthobaric density change from a remnant of the adiabatic compressive effect that is proportional to the thermobaric coefficient, the deviation from the θ–s relation, and the apparent vertical motion of orthobaric isopycnals.

The different effects of thermal expansion (and haline contraction) of seawater at different pressures and their anomalous influence on potential density have long been appreciated (Ekman 1934; Lynn and Reid 1968; McDougall 1987). Often these effects have been dismissed as a nuisance, to be overcome with better formulation of potential density, perhaps, rather than a fundamental consequence of the nature of seawater and the variation of θ–s properties. This paper is offered as a corrective to this view, by demonstrating that precisely the effect responsible for the anomalous properties of potential density can cause interesting and even unexpected consequences for an important class of baroclinic planetary motions in the ocean.

The effect is caused by the mass flux across fluctuating orthobaric isopycnals that occurs when water properties deviate from a uniform θ–s relation. This unavoidable diapycnal mass flux has an internal stretching effect on potential vorticity, defined as the projection of absolute vorticity onto the gradient of orthobaric density. It prevents the development of a potential vorticity conservation principle in a fluid of variable composition (variable salinity), whenever its local composition is not in a firm universal relation to its local entropy or potential temperature (i.e., nonhomentropic flow). Potential vorticity can be shown to be a Casimir invariant of single-component (i.e., fixed composition) Hamiltonian fluid mechanical systems viewed from an Eulerian perspective (Shepherd 1990; Salmon 1998; Müller 1995). This means that fluid parcels need extra labeling, provided by their potential vorticity, to supplement the Eulerian view, which otherwise does not distinguish fluid parcels' origins. When the fluid varies in composition, this invariance is broken (potential vorticity is *not* conserved) because the variation of composition does provide such information (unless it is superfluous, as when the composition is uniquely related to the entropy, i.e., when there is a unique θ–s relation).

A vorticity equation modified by thermobaricity was obtained that reduces to the intermediate geostrophic potential vorticity equation when salinity is related to potential temperature (or the thermobaric effect can be neglected). Otherwise this equation is linked nonlinearly to a conservation equation for temperature (or salinity) fluctuations from the θ–s relation. By carrying out a perturbation expansion around the horizontally homogeneous mean stratification, having included the thermobaric effect, a propagation equation was obtained for the amplitude of a baroclinic planetary wave disturbance. This equation turned out to be similar to the reduced-gravity form of the intermediate geostrophic vorticity equation (Williams and Yamagata 1984). In the limit of meridionally uniform disturbances, it reduces to the familiar Korteweg-de Vries equation, which admits solitary-wave solutions. The coefficient of the nonlinear term in the KdV equation is, first of all, proportional to the thermobaric coefficient. Beyond that, it depends on geographical deviation of the mean θ–s relation in some way from a horizontally homogeneous relation. To produce a significant effect, we assumed a meridional gradient of deviation from the mean θ–s relation that was of opposite sense in the upper ocean as in the deep ocean. An unexpected result was that even modest thermocline displacements ($>$60 m) of a few tens of Rossby deformation radii in horizontal scale are sufficiently affected by the thermobaric nonlinearity to exist as solitary waves. If the upper-ocean temperature and salinity deviation gradients are positive (increasing toward the Pole, with the opposite gradients at depth), the solitary planetary wave is a zonally propagating elevation of the thermocline; if negative, it is a depression.

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REFERENCES

Akitomo, K., 1999: Open-ocean deep convection due to thermobaricity. 1. Scaling argument. *J. Geophys. Res.*, **104,** 5225–5234.

Anderson, D. L. T., and P. D. Killworth, 1979: Non-linear propagation of long Rossby waves. *Deep-Sea Res.*, **26,** 1033–1050.

Chelton, D. B., R. A. de Szoeke, M. G. Schlax, K. El Naggar, and N. Siwertz, 1998: Geographical variability of the first baroclinic Rossby radius of deformation. *J. Phys. Oceanogr.,* **28,** 433–460.

Davis, R. E., 1994: Diapycnal mixing in the ocean: Equations for large-scale budgets. *J. Phys. Oceanogr.,* **24,** 777–800.

de Szoeke, R. A., 2000: Equations of motion using thermodynamic coordinates. *J. Phys. Oceanogr.,* **30,** 2814–2829.

——, S. R. Springer, and D. M. Oxilia, 2000: Orthobaric density: A thermodynamic variable for ocean circulation studies. *J. Phys. Oceanogr.,* **30,** 2830–2852.

Drazin, P. G., 1983: *Solitons.* Cambridge University Press, 136 pp.

Eden, C., and J. Willebrand, 1999: Neutral density revisited. *Deep-Sea Res.,* **46B,** 33–54.

Ekman, V. W., 1934: Review of G. Wüst: Das Bodenwasser und die Gliederung der Atlantischen Tiefsee (The bottom water and the layering of the deep Atlantic). *J. Cons. Perm. Int. Explor. Mer,* **9,** 102–104.

Flierl, G. R., 1979: Baroclinic solitary waves with radial symmetry. *Dyn. Atmos. Oceans,* **3,** 15–38.

——, V. D. Larichev, J. C. McWilliams, and G. M. Reznik, 1980: The dynamics of baroclinic and barotropic solitary eddies. *Dyn. Atmos. Oceans,* **5,** 1–41.

Fofonoff, N. P., 1985: Physical properties of seawater: A new salinity scale and equation of state for seawater. *J. Geophys. Res.,* **90,** 3332–3342.

Garwood, R. W., S. M. Isakari, and P. C. Gallacher, 1994: Thermobaric convection. *The Polar Oceans and Their Role in Shaping the Global Environment, Geophys. Monogr.,* No. 85, Amer. Geophys. Union, 199–209.

Jackett, D. R., and T. J. McDougall, 1997: A neutral density variable for the world's oceans. *J. Phys. Oceanogr.,* **27,** 237–263.

Lynn, R. J., and J. L. Reid, 1968: Characteristics and circulation of deep and abyssal waters. *Deep-Sea Res.,* **15,** 577–598.

Matsuura, T., and T. Yamagata, 1982: On the evolution of nonlinear planetary eddies larger than the radius of deformation. *J. Phys. Oceanogr.,* **12,** 440–456.

McDougall, T. J., 1984: The relative roles of diapycnal and isopycnal mixing on subsurface water mass conversion. *J. Phys. Oceanogr.,* **14,** 1577–1589.

——, 1987: Thermobaricity, cabbeling and water mass conversion. *J. Geophys. Res.,* **92,** 5448–5464.

McWilliams, J. C., 1980: Application of equivalent modons to atmospheric blocking. *Dyn. Atmos. Oceans,* **5,** 43–66.

Müller, P., 1995: Ertel's potential vorticity theorem in physical oceanography. *Rev. Geophys.,* **33,** 67–97.

——, and J. Willebrand, 1986: Compressibility effects in the thermohaline circulation: A manifestation of the temperature–salinity mode. *Deep-Sea Res.,* **33,** 559–571.

Redekopp, L. G., 1977: On the theory of solitary Rossby waves. *J. Fluid Mech.,* **82,** 725–745.

Reid, J. L., and R. J. Lynn, 1971: On the influence of the Norwegian–Greenland and Weddell Seas upon the bottom waters of the Indian and Pacific Oceans. *Deep-Sea Res.,* **18,** 1063–1088.

Salmon, R., 1998: *Lectures on Geophysical Fluid Dynamics.* Oxford University Press, 378 pp.

Shepherd, T. G., 1990: Symmetries, conservation laws, and Hamiltonian structure in geophysical fluid dynamics. *Advances in Geophysics,* Vol. 32, Academic Press, 287–338.

Straub, D. N., 1999: On thermobaric production of potential vorticity in the ocean. *Tellus,* **51A,** 314–325.

Whitham, G. B., 1974: *Linear and Nonlinear Waves.* John Wiley and Sons, 636 pp.

Williams, G. P., and T. Yamagata, 1984: Geostrophic regimes, intermediate solitary vortices and Jovian eddies. *J. Atmos. Sci.,* **41,** 453–478.