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Abstract approved
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In the first chapter of this thesis, several methods are used to solve an n-th order linear ordinary differential equation with constant coefficients together with n known initial values. The first method is the standard elementary method where the general solution of the differential system is found as a sum of two solutions u and v where u is the solution of the homogeneous part of the ordinary differential equation and v is any particular solution of the non-homogeneous differential equation. The method is not strong enough to find a particular solution for every function that might be given as the non-homogeneous term of the ordinary differential equation and so we try a more powerful approach for finding v; hence the Lagrange's method of variation of parameters. Following this, the method of

Laplace transforms is employed to solve the differential system.

In the second chapter the n-th order linear ordinary differential equation is converted into a Volterra integral equation of second kind and in the next chapter, the idea of the resolvent kernel of an integral equation is introduced with some proofs of the existence and convergence of the resolvent kernel of the integral equation. The method of solving the Volterra integral equation by iteration is briefly discussed.

The fourth chapter is devoted to solving the Volterra integral equation with convolution type kernel by the method of E. T. Whittaker, but the method is found to be very involved, and as a result, a method suggested by G. C. Evans (1911) is employed in calculating the resolvent kernels for kernels made up of sums of two exponential functions (the method of iteration was applied to the same problem but it was tedious--it took about 20 pages of writing) and finally the method provides an easier way for calculating the resolvent kernel of the Volterra integral equation corresponding to an n-th order linear ordinary differential equation with constant coefficients.

Solution by the Method of G. C. Evans of the Volterra Integral Equation Corresponding to the Initial Value Problem for a Non-Homogeneous Linear Differential Equation with Constant Coefficients

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SOLUTION BY THE METHOD OF G.C. EVANS OF THE VOLTERRA INTEGRAL EQUATION CORRESPONDING TO THE INITIAL VALUE PROBLEM FOR A NON-HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

I. SOLUTION OF THE n-th ORDER NON-HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

1.1. Solution by the Standard Elementary Method

The solution--if there is one--to the initial value problem of the form:

$$(Dy)(x) \equiv a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \ldots + a_0 y(x) = f(x). \qquad (1)$$

$$y(0) = y_0, \quad y^{(1)}(0) = y_0^1, \ldots, y^{(n-1)}(0) = y_0^{n-1}. \qquad (2)$$

where $y^{(i)}(x)$, $i = 1, 2, \ldots, n$, denote the i-th derivative of $y(x)$ with respect to x and the a_j's, $j = 0, 1, 2, \ldots, n$ are real functions of x, together with the boundary conditions expressed in (2), can be obtained in several ways. In this section, we shall discuss its solution by the standard elementary method.

For this method, one finds the "general solution" of the linear differential equation as the sum of two functions $u(x)$ and $v(x)$ where $u(x)$ is the "general solution" (meaning that it contains n assignable parameters) of the homogeneous equation

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \ldots + a_1 y^{(1)}(x) + a_0 y(x) = 0 . \qquad (3)$$

while $v(x)$ is any particular solution of the non-homogeneous equation (1).

We see that expressing the "general solution" of the linear differential equation as the sum of two functions $u(x)$ and $v(x)$, i.e.,

$$y(x) = u(x) + v(x)$$

makes sense since, if one has the problem $(Dy)(x) = f(x)$ together with initial values, it follows from the expression for the general solution that

$$\begin{aligned}(Dy)(x) &= (D(u+v))(x) \\ &= (Du)(x) + (Dv)(x) \\ &= 0 + f(x) .\end{aligned}$$

We shall now assume that the coefficients $a_0, a_1, \ldots, a_n$ are constants and that the homogeneous equation in (3) has a solution of the form:

$$u(x) = e^{sx} . \qquad (4)$$

If we differentiate equation (4) n times we obtain

$$\left.\begin{aligned} u^{(1)}(x) &= se^{sx} \\ u^{(2)}(x) &= s^2e^{sx} \\ &\vdots \\ u^{(n)}(x) &= s^ne^{sx} \;. \end{aligned}\right\} \tag{5}$$

Substituting the set of functions in (4) and (5) into equation (3) yields:

$$a_ns^ne^{sx} + a_{n-1}s^{n-1}e^{sx} + \ldots + a_1se^{sx} + a_0e^{sx} = 0$$

from which

$$e^{sx}[a_ns^n+a_{n-1}s^{n-1}+\ldots+a_1s+a_0] = 0 \;. \tag{5.1}$$

The exponential e^{sx} never vanishes and so the polynomial $P(s)$:

$$P(s) = a_ns^n + a_{n-1}s^{n-1} + \ldots + a_1s + a_0$$

must be zero, so that we have the equation for s

$$P(s) = 0 \;. \tag{6}$$

Equation (6) has n solutions in the complex field, which may be real or non-real. If non-real, they occur in complex conjugate pairs. The n solutions may all be distinct or some of them may be repeated. The methods for solving equation (6) will not be discussed here since they may be found in many books dealing with solutions of algebraic equations. For our purpose in this section, we shall assume <u>first</u> that the roots are real and distinct and shall denote them by $m_1, m_2, \ldots, m_n$. The general solution of the homogeneous equation (3) can therefore be written thus:

$$u(x) = A_1 e^{m_1 x} + A_2 e^{m_2 x} + \ldots + A_n e^{m_n x} , \tag{7}$$

where the A_i's, $i = 1, 2, \ldots, n$, are n assignable parameters.

If some of the real roots of equation (6) are <u>repeated</u>, then the function (7) will no longer be the general solution of the homogeneous equation (3).

We recall that if s is a repeated root of the second order linear differential equation

$$a_2 y^{(2)}(x) + a_1 y^{(1)}(x) + a_0 y(x) = 0,$$

then two linearly independent solutions are e^{sx} and xe^{sx}. Similarly if the root s_1, say, is repeated k times for the n-th order differential equation, then

$$e^{s_1 x}, xe^{s_1 x}, x^2 e^{s_1 x}, \ldots, x^{k-1} e^{s_1 x}$$

and all linearly independent solutions of the homogeneous equation (3). More generally, suppose that equation (6) has m distinct real roots $s_1, s_2, \ldots, s_m$, $m \leqq n$, where s_j has <u>multiplicity</u> n_j, so that

$1 \leqq n_j \leqq n, \quad \sum_{j=1}^{n} n_j = n,$ and

$$P(s) = \prod_{j=1}^{m} (s-s_j)^{n_j},$$

then

$$u(x) = \sum_{i=1}^{m} e^{s_i x} \sum_{j=1}^{n_i} A_{ij} x^{j-1}.$$

Also if a non-real complex root $\lambda + i\mu$ occurs with multiplicity k, its complex conjugate $\lambda - i\mu$ also occurs k times. There will therefore be $2k$ complex-valued roots of (6) and corresponding to these, we can find $2k$ real-valued solutions of (3) by noting that the real and imaginary parts of

$$e^{(\lambda \pm i\mu)}, xe^{(\lambda \pm i\mu)x}, \ldots, x^{k-1}e^{(\lambda \pm i\mu)x}$$

are all linearly independent solutions and that

$$e^{i\mu x} = \cos \mu x + i \sin \mu x .$$

Hence the real valued solutions are

$$\begin{array}{ll} e^{\lambda x} \cos \mu x, & e^{\lambda x} \sin \mu x, \\ xe^{\lambda x} \cos \mu x, & xe^{\lambda x} \sin \mu x, \\ \vdots & \\ x^{k-1} e^{\lambda x} \cos \mu x, & x^{k-1} e^{\lambda x} \sin \mu x . \end{array}$$

Thus one has for $u(x)$ a "general solution" which may contain real exponentials, real exponentials times polynomials (in the repeated case), and real sines and cosines, possibly also multiplied by polynomials.

The real trouble comes in finding a $v(x)$. This can be done in only a few cases by the method of "undetermined coefficients." Such cases include those where $f(x)$ is a polynomial, or a linear combination of exponentials or a linear combination of sines and cosines, possibly multiplied by powers of x or by exponential functions, etc.

This limitation on the function $f(x)$ is a serious shortcoming of the "method of undetermined coefficients," and other methods have been devised which will handle more general problems. One such will be illustrated now by an example. It presents the solution $y(x)$ in terms of an integral involving $f(x)$.

Example. Consider the special differential problem of order 2 in which $f(x)$ is not specified:

$$y''(x) + 5y'(x) + 6y(x) = f(x) \ . \qquad \text{(i)}$$

$$y(0) = y_0, \quad y'(0) = y'_0 \ . \qquad \text{(ii)}$$

For the homogeneous equation

$$y''(x) + 5y'(x) + 6y(x) = 0,$$

let us try the solution

$$y = e^{\lambda x}.$$

Then

$$\lambda^2 + 5\lambda + 6 = 0,$$

$$(\lambda+3)(\lambda+2) = 0,$$

and so

$$\lambda_1 = -3 \quad \text{and} \quad \lambda_2 = -2.$$

Therefore the general solution of the non-homogeneous equation is

$$\begin{aligned} y(x) &= u(x) + v(x) \\ &= A_1 e^{-3x} + A_2 e^{-2x} + v(x). \end{aligned}$$

Rewriting the non-homogeneous equation (i) using the operator notation $y' = Dy$ and $y'' = D^2 y$ we have

$$(D^2+5D+6)y(x) = f(x)$$

or

$$(D+3)(D+2)y(x) = f(x)$$

and if we let

$$(D+2)y(x) = g(x) \tag{iii}$$

we have

$$(D+3)g(x) = f(x)$$

or

$$g'(x) + 3g(x) = f(x),$$

whence

$$(g(x)e^{3x})' = f(x),$$

and so

$$g(x)e^{3x} = \int_0^x f(s)ds + A.$$

From this

$$g(x) = e^{-3x}\int_0^x f(s)ds + Ae^{-3x}$$

where A is the constant of integration. Substituting this back in (iii) we obtain

$$(D+2)y(x) = e^{-3x}\int_0^x f(s)ds + Ae^{-3x}$$

and so

$$y(x)e^{2x} = \int_0^x \left[e^{-3t}\int_0^t f(s)ds\right]dt + \frac{Ae^{-3x}}{-3} + B$$

where B is the second constant of integration. Thus

$$y(x) = \int_0^x e^{-2x}\left\{\int_s^x e^{-3t}dt\right\} f(s)ds - \frac{Ae^{-5x}}{3} + Be^{-2x}$$

$$= \int_0^x e^{-2x}\left(-\frac{e^{-3x} - e^{-3s}}{3}\right) f(s)ds - \frac{Ae^{-5x}}{3} + Be^{-2x}.$$

Thus

$$y(x) = \int_0^x \left(\frac{e^{-2x-3s}}{3} - \frac{e^{-5x}}{3}\right)f(s)ds - \frac{Ae^{-5x}}{3} + Be^{-2x}. \qquad \text{(iv)}$$

Substituting the initial values from (ii) we have for $x = 0$;

$$y(0) = \frac{-A}{3} + B = y_0 \quad . \tag{v}$$

Differentiating equation (4) with respect to x we obtain

$$y'(x) = \int_0^x \frac{\partial}{\partial x}[(\frac{e^{-2x-3s}}{3} - \frac{e^{-5x}}{3})f(s)]ds + [(\frac{e^{-2x-3x}}{3} - \frac{e^{-5x}}{3})f(x)]$$

$$+ \frac{5}{3}Ae^{-5x} - 2Be^{-2x} .$$

Putting the value $x = 0$ yields:

$$y'(0) = \frac{5}{3}A - 2B = y_0' \quad . \tag{vi}$$

Multiplying equation (v) by 2 we have

$$\frac{-2A}{3} + 2B = 2y_0 \tag{vii}$$

and adding (vi) and (vii) yields

$$A = 2y_0 + y_0' ,$$

and putting this in (vii) we have

$$2B = 2y_0 + \frac{2}{3}(2y_0+y_0') ,$$

from which

$$B = \frac{5}{3}y_0 + \frac{1}{3}y_0' .$$

Equation (iv) therefore becomes

$$y(x) = \int_0^x \left(\frac{e^{-2x-3s}}{3} - \frac{e^{-5x}}{3}\right) f(s)ds$$

$$- \frac{1}{3}(2y_0+y_0')e^{-5x} + \left(\frac{5}{3}y_0+\frac{1}{3}y_0'\right)e^{-2x} .$$

This is the solution of the differential problem in terms of an integral involving the unspecified function $f(x)$.

Still more general is J. L. Lagrange's method of variation of parameters, which will be described in the next section. It applies also to the case of variable coefficients, provided one can find a set of fundamental solutions of the homogeneous equation. If the coefficients $\{a_i(x)\}_0^n$ can be expanded in power series, a whole new approach is possible (method of Frobenius), but it will not be discussed in this thesis.

1.2. Lagrange's Solution by Variation of Parameters

The method discussed in Section 1.1 can be used to determine the general solution of any non-homogeneous equation with constant coefficients provided the non-homogeneous term is of suitable form. In this section we shall discuss a more powerful approach for finding the particular solution $v(x)$ since, as we pointed out in the last section, the method of undetermined coefficients can be applied in only a few cases. The method is known as Lagrange's method for finding the particular solution of the non-homogeneous system of the form:

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \ldots + a_0 y(x) = f(x) \qquad (*)$$

with

$$y(0) = y_0, y^{(1)}(0) = y_0', \ldots, y^{(n-1)}(0) = y_0^{n-1} .$$

For this method it is essential that n linearly independent solutions of the homogeneous differential equation

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \ldots + a_0 y(x) = 0 \qquad (1)$$

discussed in Section 1.1 be known. Let them be $u_1(x), u_2(x), \ldots, u_n(x)$ so that the general solution of equation (1) is

$$u(x) = A_1 u_1(x) + A_2 u_2(x) + \ldots + A_n u_n(x) . \qquad (2)$$

The method of variation of parameters involves replacing the constants $A_1, A_2, \ldots, A_n$ by functions $P_1(x), P_2(x), \ldots, P_n(x)$ of x respectively. Thus we shall <u>assume</u> that the solution of equation (1) is of the form:

$$u(x) = P_1(x)u_1(x) + P_2(x)u_2(x) + \ldots + P_n(x)u_n(x)$$

or

$$u(x) = \sum_{i=1}^{n} P_i(x)u_i(x) \quad . \qquad (3)$$

The problem then reduces to that of finding n suitable functions $P_1(x), P_2(x), \ldots, P_n(x)$ such that the function $u(x)$ defined by (3) satisfies equation (1). To do this we need a set of n independent conditions, and $(n-1)$ of them could be selected rather arbitrarily so as to simplify the calculations. The last condition arises from the fact that equation (3) must satisfy equation (1).

Differentiating equation (3) we obtain:

$$u^{(1)}(x) = \sum_{i=1}^{n} P_i(x)u_i^{(1)}(x) + \sum_{i=1}^{n} P_i^{(1)}(x)u_i(x) .$$

Since we can select $(n-1)$ conditions arbitrarily, suppose we let the first be

$$\sum_{i=1}^{n} P_i^{(1)}(x)u_i(x) = 0 ,$$

thus

$$u^{(1)}(x) = \sum_{i=1}^{n} P_i(x)u_i^{(1)}(x) \quad . \tag{4}$$

Differentiating equation (4) we have:

$$u^{(2)}(x) = \sum_{i=1}^{n} P_i^{(1)}(x)u_i^{(1)}(x) + \sum_{i=1}^{n} P_i(x)u_i^{(2)}(x) ,$$

and again suppose we set

$$\sum_{i=1}^{n} P_i^{(1)}(x)u_i^{(1)}(x) = 0 ,$$

to obtain

$$u^{(2)}(x) = \sum_{i=1}^{n} P_i(x)u_i^{(2)}(x) \quad . \tag{5}$$

We could continue the process of differentiation and each time if we set the term involving $P_i^{(1)}$'s $(i = 1, 2, \ldots, n-1)$ to zero, after n differentiations we would have

$$u^{(n)}(x) = \sum_{i=1}^{n} P_i(x)u_i^{(n)}(x) + \sum_{i=1}^{n} P_i^{(1)}(x)u_i^{(n-1)}(x) \quad . \tag{6}$$

We shall now substitute all the expressions for the n derivatives of $u(x)$ into equation (*) to obtain:

$$a_n \sum_{i=1}^{n} P_i(x)u_i^{(n)}(x) + a_n \sum_{i=1}^{n} P_i^{(1)}(x)u_i^{(n-1)}(x)$$

$$+ a_{n-1} \sum_{i=1}^{n} P_i(x)u_i^{(n-1)}(x) + a_{n-2} \sum_{i=1}^{n} P_i(x)u_i^{(n-2)}(x)$$

$$+ \ldots + a_1 \sum_{i=1}^{n} P_i(x)u_i^{(1)}(x) + a_0 \sum_{i=1}^{n} P_i(x)u_i(x) = f(x) \; .$$

Factorizing out each of the $P_i(x)$'s we shall obtain:

$$P_1(x)\{a_n u_1^{(n)}(x)+a_{n-1}u_1^{(n-1)}(x)+\ldots+a_1 u_1^{(1)}(x)+a_0 u_1(x)\}$$

$$+ P_2(x)\{a_n u_2^{(n)}(x)+a_{n-1}u_2^{(n-1)}(x)+\ldots+a_2 u_2^{(1)}(x)+a_0 u_2(x)\}$$

$$+ \ldots + P_n(x)\{a_n u_n^{(n)}(x)+a_{n-1}u_n^{(n-1)}(x)+\ldots+a_1 u_1^{(1)}(x)+a_0 u_n(x)\}$$

$$+ a_n \sum_{i=1}^{n} P_i^{(1)}(x)u_i^{(n-1)}(x) = f(x) \qquad (7)$$

We see that the expressions in the braces now vanish because each $u_i(x)$ is a solution of equation (1).

Thus,

$$a_n \sum_{i=1}^{n} P_i^{(1)}(x)u_i^{(n-1)}(x) = f(x) \qquad (8)$$

is our n-th condition. The first (n-1) conditions can be summarized as:

$$\sum_{i=1}^{n} P_i^{(1)}(x)u_i^{(k)}(x) = 0, \quad k = 0, 1, 2, \ldots, (n-2) ,$$

and from these conditions, the functions $P_1(x), \ldots, P_n(x)$ can be determined as

$$P_i(x) = \int_0^x \frac{A_{in}(t)f(t)}{a_n W(t)}\, dt ,$$

$a_n \neq 0$, $i = 1, 2, \ldots, n$; where $A_{in}(t)$ is the cofactor of the element in the nth row and ith column of $W(t)$ and $W(t) \neq 0$ is the

Wronskian of the independent functions $u_i(x)$ whose linear combination is $u(x)$ in (2). $W(t)$ is defined by the determinant

$$W(t) = \begin{vmatrix} u_1 & u_2 & \cdots & u_n \\ u_1^{(1)} & u_2^{(1)} & \cdots & u_n^{(1)} \\ u_1^{(2)} & u_2^{(2)} & \cdots & u_n^{(2)} \\ \vdots & & & \vdots \\ u_1^{(n-1)} & u_2^{(n-1)} & \cdots & u_n^{(n-1)} \end{vmatrix} . \tag{9}$$

The particular solution $v(x)$ is thus given by

$$v(x) = \int_0^x \frac{\sum_{i=1}^{n} u_i(x) A_{in}(t)}{a_n W(t)} f(t)dt \; ;$$

$$a_n \neq 0, \quad W(t) \neq 0, \quad i = 1, 2, \ldots, n.$$

The general solution of the non-homogeneous equation (*) is therefore given by

$$\begin{aligned} y(x) &= u(x) + v(x) \\ &= A_1 u_1(x) + \ldots + A_n u_n(x) + \int_0^x \frac{\sum_{i=1}^{n} u_i(x) A_{in}(t)}{a_n W(t)} f(t)dt \, , \end{aligned}$$

$a_n \neq 0$, $W(t) \neq 0$ is defined by (9) and $A_{in}(t)$ is the cofactor of the element in the nth row and ith column of $W(t)$. The independent functions $u_i(x)$, $i = 1, 2, \ldots, n$, are the independent solutions of the homogeneous equation (1), and the A_i's, $i = 1, 2, \ldots, n$ are n assignable parameters.

1.3. Solution by the Method of Laplace Transforms

Definition: The Laplace transform of a piecewise continuous function $f(t)$ for all $t \geq 0$ will be denoted by $L\{f(t)\}$ and is defined by the following expression:

$$L\{f(t)\} = (Lf)(s) = \int_0^\infty e^{-st} f(t)dt \quad . \tag{1}$$

It is evident from (1) that $L\{f(t)\}$ is a function of a parameter s. This definition is valid whenever the improper integral converges and for our discussion in this section we shall assume that s is real.

To establish that the Laplace transform of $f(t)$ as defined by equation (1) does in fact exist, the following sufficient conditions have to be imposed on the function $f(t)$. We shall suppose that

(a) f is piecewise continuous on the interval $0 \leq t \leq \bar{x}$ for all $\bar{x} > 0$,

(b) $|f(t)| \leq Ce^{pt}$ for all $t \geq N$, where C, p, N are all real constants and $C > 0$, $N > 0$, and $s > p$.

(*)

The problem therefore reduces to that of showing that the integral in equation (1) converges for all values of the parameter $s > p$. This we can show by simply splitting the integral in two parts, thus:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

and so

$$L\{f(t)\} = \int_0^N e^{-st} f(t)dt + \int_N^\infty e^{-st} f(t)dt \qquad (2)$$

for some fixed real constant $N > 0$.

Condition (a) in (*) imposed on $f(t)$ shows clearly that the first integral $\int_0^N e^{-st} f(t)dt$ exists and so our problem is narrowed down to that of showing that $\int_N^\infty e^{-st} f(t)dt$ converges. Observing the function $e^{-st} f(t)$ and applying condition (b) imposed on $f(t)$ we have $|e^{-st} f(t)| \leq Ce^{-st} e^{pt}$ from which we see that $|e^{-st} f(t)| \leq Ce^{(p-s)t}$. We know that $\int_N^\infty e^{(p-s)t} dt$ converges for $s > p$, C, p, N, being real constants, and $N > 0$, $C > 0$. Thus the Laplace transform of $f(t)$ defined by the improper integral in equation (1) does in fact exist.

The Laplace transforms of various functions $f(t)$ could be found by equation (1) provided $f(t)$ meets the conditions (a) and (b). For the purpose of this thesis, attention is focussed on the use of Laplace transforms to solve an nth order non-homogeneous linear differential equation of the form

$$\left.\begin{aligned} &a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \ldots + a_1 y^{(1)}(x) + a_0 y(x) = f(x) \\ \text{with} & \\ &y^{(n)}(0) = y_0^n, y^{(n-1)}(0) = y_0^{n-1}, \ldots, y(0) = y_0 \ . \end{aligned}\right\} \quad (3)$$

and we shall assume that $f(x)$ satisfies the two conditions in (*). We shall assume that the coefficient a_i's, $i = 0, 1, 2, \ldots, n$ of $y^{(i)}(x)$ in equation (3) are constants.

A very important result about the transform of the nth derivative of $f(t)$ will be applied, and the result is stated thus:

Assume that the function f, and its derivatives $f^{(1)}, f^{(2)}, \ldots, f^{(n-1)}$ are continuous and $f^{(n)}$ is piecewise continuous on any interval $0 \leq t \leq \bar{x}$ and that $f^{(1)}, f^{(2)}, \ldots, f^{(n-1)}$ all satisfy the conditions (a) and (b) of (*), namely that

$$|f^{(i)}(t)| \leq Ce^{pt} \quad \text{for} \quad t \geq N$$

where C, p, N are all real constants. Then the Laplace transform of $f^{(n)}(t)$ exists for $s > p$ and is expressed as:

$$L\{f^{(n)}(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0). \quad (4)$$

The proof for this could be found in many books on differential equations [8, p. 228].

We shall now apply the method of Laplace transform to solve our nth order linear differential equation with constant coefficients, i.e.,

$$\left.\begin{aligned} &a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \ldots + a_0 y(x) = f(x) \\ &y^{(n)}(0) = y_0^n, y^{(n-1)}(0) = y_0^{n-1}, \ldots, y(0) = y_0 \ . \end{aligned}\right\} \quad (5)$$

Knowing that if k is a constant the $L\{ky(x)\} = kL\{y(x)\}$ and that if $P(x)$ and $Q(x)$ are two functions such that $P(x) = Q(x)$, then by the uniqueness of Laplace transforms $L\{P(x)\} = L\{Q(x)\}$, we shall now take the Laplace transform of each term of equation (5), applying formula (4) for each derivative, knowing that for two functions y_1 and y_2, $L\{y_1 + y_2\} = L\{y_1\} + L\{y_2\}$, we obtain

$$a_n L\{y^{(n)}(x)\} + a_{n-1} L\{y^{(n-1)}(x)\} + \ldots + a_0 L\{y(x)\} = L\{f(x)\} ,$$

and so

$$\begin{aligned} &a_n[s^n L\{y(x)\} - s^{n-1} y(0) - s^{n-2} y^{(1)}(0) - \ldots - sy^{(n-2)}(0) - y^{(n-1)}(0)] \\ &+ a_{n-1}[s^{n-1} L\{y(x) - s^{n-2} y(0) - s^{n-3} y^{(1)}(0) - \ldots - sy^{(n-3)}(0) - y^{(n-2)}(0)] \\ &+ a_{n-2}[s^{n-2} L\{y(x)\} - s^{n-3} y(0) - s^{n-4} y^{(1)}(0) - \ldots - sy^{(n-4)}(0) - y^{(n-3)}(0)] \\ &+ \ldots + a_2[s^2 L\{y(x)\} - sy(0) - y^{(1)}(0)] + a_1[sL\{y(x)\} - y(0)] \\ &+ a_0[L\{y(x)\}] = L\{f(x)\} . \end{aligned}$$

We shall rewrite this, factorizing out the $L\{y(x)\}$ to obtain:

$$[a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0] L\{y(x)\}$$

$$- [a_n s^{n-1} + a_{n-1} s^{n-2} + \ldots + a_1] y(0)$$

$$- [a_n s^{n-2} + a_{n-1} s^{n-3} + \ldots + a_2] y^{(1)}(0)$$

$$- \ldots - [a_n s + a_{n-1}] y^{(n-2)}(0) - [a_n] y^{(n-1)}(0) = L\{f(x)\} ,$$

and solving for $L\{y(x)\}$ we have:

$$L\{y(x)\} = \frac{[P_1(s)]y(0) + P_2(s) y^{(1)}(0) + \ldots + a_n y^{(n-1)}(0) + L\{f(x)\}}{[a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0]} \tag{6}$$

where

$$P_1(s) = [a_n s^{n-1} + a_{n-1} s^{n-2} + \ldots + a_1]$$

$$P_2(s) = [a_n s^{n-2} + a_{n-1} s^{n-3} + \ldots + a_2]$$

etc.

Recalling that the Laplace transform of $f(x)$ is given by $L\{f(x)\} = \int_0^\infty e^{-sx} f(x) dx$, we see that $L\{f(x)\}$ is a function of s, and we shall denote it by $Q(s)$ say. We also note that the a_i's and the contributions of the derivatives of $y(x)$ evaluated at zero, are all constants. The expression for $L\{y(x)\}$ is therefore strictly a function of s as we expect from the definition of Laplace transform. We already pointed out the linearity of Laplace operator and its inverse, and so the rest of the problem is that of rearranging the right hand side of the equation into sums of functions of s, such that the inverse transform of each term is recognizable from tables of Laplace transforms. A method often employed to simplify the right hand side of (6) is the method of partial fractions, and if this is done, the inverse transform of each fraction could be found from tables of Laplace transforms [9, 10]. As an example let us suppose that after simplifying the right hand side of (6) we obtained

$$L\{y(x)\} = \frac{1}{s-a} - \frac{a}{s^2+a^2} + \frac{s}{s^2-a^2} .$$

From tables of Laplace transforms, the inverse transform which we shall denote by L^{-1} of the right hand side are respectively

$$L^{-1}[\frac{1}{s-a}] = e^{at}, \quad s > a$$

$$L^{-1}[\frac{a}{s^2+a^2}] = \sin(at), \quad s > 0$$

$$L^{-1}[\frac{s}{s^2-a^2}] = \cosh(at), \quad s > |a|$$

and so, again by the uniqueness of Laplace transforms

$$L^{-1}\{y(x)\} = L^{-1}[\frac{1}{s-a}] - L^{-1}[\frac{a}{s^2+a^2}] + L^{-1}[\frac{s}{s^2-a^2}]$$

from which

$$y(x) = e^{at} - \sin(at) + \cosh(at).$$

The solution of our non-homogeneous ordinary differential equation with constant coefficients would therefore be

$$y(x) = L^{-1}\ \frac{[P_1(s)]y_0+[P_2(s)]y_0'+\ldots+a_n y_0^{n-1}+Q(s)}{[a_n s^n+a_{n-1}s^{n-1}+\ldots+a_1 s+a_0]}\ .$$

It should be noted that the denominator in the solution is exactly the auxiliary equation obtained in equation (6) of Section 1.1. This will always be the case, and there shall always be the need to find the roots of this auxiliary equation regardless of which method is employed to solve the non-homogeneous linear differential equation with constant coefficients.

II. CONVERSION OF n-th ORDER NON-HOMOGENEOUS DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS INTO LINEAR INTEGRAL EQUATION OF VOLTERRA TYPE

2.1. Classification of Linear Integral Equation

The phrase "linear integral equation" is a broad term that encompasses large families of integral equations. To narrow this down, we need to specify what "TYPE" of integral equation, and furthermore, any type of integral equation could be any one of many "KINDS," and so we talk of an integral equation being of any one of 1st, 2nd or 3rd kind.

An integral equation whose limits of integration are not constant but variable such as:

$$a(x)y(x) = b(x) + \int_0^x k(x,s)y(s)ds$$

is known as an integral equation of VOLTERRA TYPE. If the limits of integration are fixed such as:

$$a(x)y(x) = b(x) + \int_0^1 k(x,s)y(s)ds,$$

it is known as the FREDHOLM TYPE of integral equation.

The nature of the coefficient $a(x)$ determines what "KIND" the equation is. If $a(x) = 0$ then we have an equation of the 1st kind.

If $a(x)$ is always positive or always negative we have an integral equation of the 2nd kind, and a 3rd kind is obtained if $a(x)$ vanishes at one or more isolated points.

If, in the integral equation, $b(x) = 0$, we say that the equation is homogeneous. This chapter will be devoted to transformation of the n-th order linear non-homogeneous differential equation with constant coefficients into an integral equation of VOLTERRA TYPE.

2.2. Transformation of n-th Order Non-Homogeneous Linear Differential Equation with Constant Coefficients into Integral Equation of Volterra Type

The n-th order non-homogeneous linear differential equations with constant coefficients

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \ldots + a_1 y^{(1)}(x) + a_0 y(x) = F(x)$$

together with (1)

$$y^{(n-1)}(0) = y_0^{n-1}, y^{(n-2)}(0) = y_0^{n-2}, \ldots, y(0) = y_0$$

can be transformed into a Volterra integral equation of the form

$$u(x) = f(x) + \int_0^x k(x,t)u(t)dt \qquad (2)$$

by the following procedure. Suppose we let

$$y^{(n)}(x) = u(x) \quad . \qquad (3)$$

We shall further define

$$(D^{-1}f)(x) = \int_0^x f(s)ds, \quad (D^{-2}f)(x) = (D^{-1}(D^{-1}f))(x),$$

etc. Performing successive integrations of $u(x)$ we therefore have

$$\left.\begin{aligned} D^{-1}u(x) &= \int_0^x u(t)dt, \\ D^{-2}u(x) &= (D^{-1}(D^{-1}u))(x) = D^{-1}\left\{\int_0^x u(t)dt\right\} = \int_0^x (x-t)u(t)dt \\ &\vdots \\ D^{-n}u(x) &= (D^{-1}(D^{-(n-1)}u))(x) = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} u(t)dt . \end{aligned}\right\} \quad (4)$$

Furthermore integrating both sides of equation (3) we obtain

$$y^{(n-1)}(x) = D^{-1}u(x) + y_0^{n-1} ,$$

where y_0^{n-1} is the constant of integration obtained from $y^{(n-1)}(0) = y_0^{n-1}$ in equation (1). Integrating successively and each time employing the boundary conditions in (1) we obtain:

$$\begin{aligned} y^{(n-2)}(x) &= y_0^{(n-1)}x + y_0^{n-2} + D^{-2}u(x), \\ y^{(n-3)}(x) &= y_0^{n-1}\frac{x^2}{2!} + y_0^{n-2}x + y_0^{n-3} + D^{-3}u(x), \\ &\vdots \\ y(x) &= y_0^{n-1}\frac{x^{n-1}}{(n-1)!} + y_0^{n-2}\frac{x^{n-2}}{(n-2)!} + \ldots + y_0'x + y_0 + D^{-n}u(x). \end{aligned}$$

Substituting the values of each $D^{-i}u(x)$, $i = 1, 2, \ldots, n$ from (4) into the last set of equations yields:

$$y^{(n-1)}(x) = y_0^{n-1} + \int_0^x u(t)dt$$

$$y^{(n-2)}(x) = y_0^{n-1}x + y_0^{n-2} + \int_0^x (x-t)u(t)dt$$

$$\vdots$$

$$y(x) = y_0^{n-1}\frac{x^{n-1}}{(n-1)!} + y_0^{n-2}\frac{x^{n-2}}{(n-2)!} + \ldots + y_0'x + y_0 + \int_0^x \frac{(x-t)^{n-1}}{(n-1)!}u(t)dt ,$$

from which we further see that

$$a_{n-1}y^{(n-1)}(x) = a_{n-1}y_0^{n-1} + a_{n-1}\int_0^x u(t)dt$$

$$a_{n-2}y^{(n-2)}(x) = a_{n-2}y_0^{n-1}x + a_{n-2}y_0^{n-2} + a_{n-2}\int_0^x (x-t)u(t)dt$$

$$\vdots$$

$$a_0y(x) = a_0y_0^{n-1}\frac{x^{n-1}}{(n-1)!} + a_0y_0^{n-2}\frac{x^{n-2}}{(n-2)!} + \ldots + a_0y_0'x + a_0y_0 + a_0\int_0^x \frac{(x-t)^{n-1}}{(n-1)!}u(t)dt .$$

Adding both sides of these equations we have

$$a_{n-1}y^{(n-1)}(x) + a_{n-2}y^{(n-2)}(x) + \dots + a_1y^{(1)}(x) + a_0y(x)$$

$$= a_{n-1}y_0^{n-1} + a_{n-2}(y_0^{n-1}x + y_0^{n-2}) + \dots + a_0\left(y_0^{n-1}\frac{x^{n-1}}{(n-1)!} + \dots + y_0'x + y_0\right)$$

$$+ \int_0^x \sum_{i=0}^{n-1} a_{n-i-1}\frac{(x-t)^i}{(i)!} \quad u(t)dt \ .$$

We can simplify this as

$$a_{n-1}y^{(n-1)}(x) + a_{n-2}y^{(n-2)}(x) + \dots + a_1y^{(1)}(x) + a_0y(x)$$

$$= f(x) + \int_0^x k(x,t)u(t)dt \tag{5}$$

where

$$f(x) = a_{n-1}y_0^{n-1} + a_{n-2}(y_0^{n-1}x + y_0^{n-2}) + \dots + a_0\left(y_0^{n-1}\frac{x^{n-1}}{(n-1)!} + \dots + y_0\right) \tag{6}$$

and

$$k(x,t) = \sum_{i=1}^{n-1} a_{n-i-1}\frac{(x-t)^i}{(i)!} \quad . \tag{7}$$

However, we already set $y^{(n)}(x) = u(x)$ and so

$$a_n y^{(n)}(x) = a_n u(x) \quad . \tag{8}$$

Adding equations (5) and (8) we have

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \dots + a_1 y^{(1)}(x) + a_0 y(x)$$

$$= a_n u(x) + f(x) + \int_0^x k(x, t)u(t)dt$$

$$= F(x)$$

from equation (1). Thus

$$a_n u(x) + f(x) + \int_0^x k(x, t)u(t)dt = F(x) ,$$

and since the a_i's are constant coefficients, suppose we divide through by $-a_n$ to obtain

$$u(x) = \left(\frac{-f(x)+F(x)}{a_n}\right) + \int_0^x \frac{k(x, t)}{-a_n} u(t)dt .$$

This can be further written as

$$u(x) = \mathbb{F}(x) + \int_0^x K(x, t)u(t)dt \qquad (9)$$

where $\mathbb{F}(x) = \frac{1}{a_n}(F(x)-f(x))$.

Equation (9) is a Volterra integral equation with kernel

$$K(x, t) = -\sum_{i=0}^{n-1} \frac{a_{n-i-1}}{a_n} \frac{(x-t)^i}{(i)!} . \qquad (10)$$

We have therefore converted the original non-homogeneous linear differential equation with constant coefficient into a Volterra integral equation expressed in (9) with its kernel $K(x, t)$ given by equation (10). If the integral equation has a solution, it will furnish one for the differential problem.

III. SOLUTION OF THE VOLTERRA INTEGRAL EQUATION BY THE METHOD OF ITERATION

In Chapter II, the n-th order non-homogeneous linear differential equation

$$\left.\begin{aligned} &a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \ldots + a_1 y^{(1)}(x) + a_0 y(x) = F(x) \\ &\text{together with} \\ &y^{(n-1)}(0) = y_0^{n-1}, y^{(n-2)}(0) = y_0^{n-2}, \ldots, y(0) = y_0 \end{aligned}\right\} \quad (1)$$

was converted into a Volterra integral equation of the form

$$y(x) = f(x) + \lambda \int_0^x k(x, t) y(t) dt \quad (2)$$

$0 \leq x < \infty$, where $f(x)$ is a function of x and $k(x, t)$ is a function of x and t, and as a matter of convenience, we are introducing the parameter λ which we shall assume real. In this chapter we shall look into the method of solving equation (2) (and hence solving the differential equation in (1)) by the method of iteration.

Substitution for $y(t)$ in equation (2) gives

$$y(x) = f(x) + \lambda \int_0^x k(x,t)\left\{ f(t)+\lambda \int_0^t k(t,s)y(s)ds \right\} dt,$$

$$= f(x) + \lambda \int_0^x k(x,t)f(t)dt + \lambda^2 \int_0^x k(x,t)\left\{ \int_0^t k(t,s)y(s)ds \right\} dt,$$

$$= f(x) + \lambda \int_0^x k(x,t)f(t)dt + \lambda^2 \int_0^x \left(\int_s^x (k(x,t)k(t,s))dt \right) y(s)ds,$$

$$= f(x) + \lambda \int_0^x k(x,t)f(t)dt + \lambda^2 \int_0^x k^{(2)}(x,s)y(s)ds,$$

where we shall define $k^{(2)}(x, s)$ by

$$k^{(2)}(x,s) = \int_s^x k(x,t)k(t,s)dt \ .$$

The equation therefore becomes, on substituting $s = t$, $ds = dt$,

$$y(x) = f(x) + \lambda \int_0^x k(x,t)f(t)dt + \lambda^2 \int_0^x k^{(2)}(x,t)y(t)dt \ . \qquad (3)$$

The process of repeated substitution of expression for $y(t)$ as done in the last step is known as "ITERATION." We shall iterate again substituting value for $y(t)$ in equation (3) to obtain:

$$y(x) = f(x) + \lambda \int_0^x k(x,t)f(t)dt + \lambda^2 \int_0^x k^{(2)}(x,t) \left\{ f(t) + \lambda \int_0^t k(t,s)y(s)ds \right\} dt$$

$$= f(x) + \lambda \int_0^x k^{(1)}(x,t)f(t)dt + \lambda^2 \int_0^x k^{(2)}(x,t)f(t)dt$$

$$+ \lambda^3 \int_0^x k^{(2)}(x,t) \left\{ \int_0^t k(t,s)y(s)ds \right\} dt$$

$$= f(x) + \lambda \int_0^x k(x,t)f(t)dt + \lambda^2 \int_0^x k^{(2)}(x,t)f(t)dt$$

$$+ \lambda^3 \int_0^x \left(\int_s^x k^{(2)}(x,t)k(t,s)dt \right) y(s)ds$$

$$= f(x) + \lambda \int_0^x k(x,t)f(t)dt + \lambda^2 \int_0^x k^{(2)}(x,t)f(t)dt$$

$$+ \lambda^3 \int_0^x k^{(3)}(x,s)y(s)ds$$

where

$$k^{(3)}(x,s) = \int_s^x k^{(2)}(x,t)k(t,s)dt \quad . \qquad (3.1)$$

If we let $s = t$, $ds = dt$, we have

$$y(x) = f(x) + \lambda \int_0^x k(x,t)f(t)dt + \lambda^2 \int_0^x k^{(2)}(x,t)f(t)dt$$

$$+ \lambda^3 \int_0^x k^{(3)}(x,t)y(t)dt \quad . \qquad (4)$$

This process of iteration could be carried on, and after n substitutions we would have

$$y(x) = f(x) + \lambda \int_0^x k(x, t)f(t) + \lambda^2 \int_0^x k^{(2)}(x, t)f(t)dt$$
$$+ \ldots + \lambda^n \int_0^x k^{(n)}(x, t)f(t)dt + \lambda^{n+1} \int_0^x k^{(n+1)}(x, s)y(s)ds \tag{5}$$

where

$$k^{(n+1)}(x, s) = \int_s^x k^{(n)}(x, t)k(t, s)dt$$

and

$$k^{(1)}(x, t) = k(x, t) \quad . \tag{6}$$

Simplifying equation (5) we have:

$$y(x) = f(x) + \sum_{i=1}^{n} \int_0^x \lambda^i k^{(i)}(x, t)f(t)dt + \lambda^{n+1} \int_0^x k^{(n+1)}(x, t)y(t)dt \,. \tag{7}$$

We see that equation (7) satisfies our original Volterra equation (2) since the former has been deduced from the latter. However, observation of equation (7) suggests that we may be generating an infinite series for $y(x)$ of the form

$$y(x) = f(x) + \sum_{i=1}^{\infty} \int_0^x \lambda^i k^{(i)}(x, t)f(t)dt \quad . \tag{8}$$

This is the case, but to establish this it is necessary to show that the solution of this equation converges to some useable function and satisfies equation (2). In addition we ought to show that this solution is unique and that the iterated kernels as defined by equation (6) actually do exist.

We saw from equation (3.1) above that

$$k^{(3)}(x, s) = \int_s^x k^{(2)}(x, t)k(t, s)dt \ .$$

We shall assume that the integrand is uniformly continuous in $0 \leq t < X$ for $X < \infty$, so that the expression may be Riemann-integrable for all $x \geq 0$. It will also be a continuous function of x on $0 \leq x < X$ for each t. This shows that $k^{(3)}(x, s)$ will exist and will have the same continuity properties. The argument is also true for $k^{(n)}(x, s)$ for $n > 2$. We therefore see that if $f(x)$ is a continuous function on $0 \leq x < X$, then all the terms of the series expressed in equation (8) exist and are continuous functions of x.

Under our continuity assumption for $k(x, t)$ and $f(x)$ in the above paragraph, each of the sequence of partial sum $\{y_n(x)\}$ given by

$$\{y_n(x)\} = f(x) + \sum_{i=1}^{n} \int_0^x \lambda^{(i)} k^{(i)}(x, t)f(t)dt \tag{9}$$

$n = 1, 2, \ldots,$ is continuous on $0 \leq x < \infty$. The problem thus reduces to that of showing that $\{y_n(x)\}$ is uniformly convergent on the closed interval $0 \leq x \leq X$ for some $X < \infty$, since this being so implies that the limit function must also be continuous. We shall employ Cauchy's theorem stating that for $\{y_n(x)\}$ to converge uniformly on $[0, X]$ it is necessary and sufficient that, given any $\epsilon > 0$, there exists an integer $N(\epsilon)$ such that if $n > N(\epsilon)$ then $|y_{n+p}(x)-y_n(x)| < \epsilon$ for every positive integer p and every x in $[0, X]$.

However we see that from equation (9),

$$y_{n+p}(x) - y_n(x) = \sum_{i=n+1}^{n+p} \int_0^x \lambda^i k^{(i)}(x,t)f(t)dt \quad , \tag{10}$$

Since $f(t)$ was assumed continuous, this means that $|f(t)|$ has some maximum value which we shall denote by $F(X)$. Further, let us assume that $k(x, t)$ is bounded on the interval $0 \leq t \leq x \leq X$ and that $|k(x,t)| \leqq M(X)$, where $M(X) < \infty$ for $0 < X < \infty$. Substituting these in (10) leads to

$$|y_{n+p}(x)-y_n(x)| \leqq F(X) \sum_{i=n+1}^{n+p} \int_0^x \lambda^i |k^{(i)}(x,t)| dt \tag{11}$$

at each point x in the interval $[0, X]$. We note that

$$\int_0^x |k(x,t)|\,dt \leqq M(X)\int_0^x dt = M(X)\ x\ ,$$

and

$$\int_0^x |k^2(x,t)|\,dt = \int_0^x \left| \int_t^x k(x,s)k(s,t)ds \right| dt$$

$$\leqq M^2(X)\int_0^x (x-t)dt = \frac{M^2(X)}{2!}x^2$$

Using equation (6), it could be shown by the process of induction that

$$\int_0^x |k^{(n)}(x,t)|\,dt \leq M^n(X)\frac{x^n}{n!}\ , \qquad n = 1, 2, \ldots\ .$$

We shall put all these values in (11) to obtain

$$|y_{n+p}(x)-y_n(x)| \leq F(X)\sum_{i=n+1}^{n+p} \frac{\lambda^i M^i(X)x^i}{(i)!} \tag{12}$$

and it is valid for all x in $[0,X]$. However we note that the sum on the right hand side of (12) is a segment of the Maclaurin series for the exponential $e^{[M(X)\,x\,\lambda]}$, which we know from Calculus converges uniformly on each bounded interval. Therefore given any $\epsilon > 0$, any $N(\epsilon)$ can be found so that for $n > N(\epsilon)$ and $p > 0$, an integer,

$$|y_{n+p}(x)-y_n(x)| < \epsilon \quad \text{for all} \quad x \quad \text{in} \quad [0,X]\ .$$

This shows uniform convergence for the partial sum $\{y_n(x)\}$ on the interval $[0, X]$, to a continuous limit function.

The uniqueness of the solution of the Volterra equation can be shown simply by assuming that there exist two distinct continuous solutions and that the difference between them is denoted by $w(x)$. We know that this difference ought to satisfy the homogeneous Volterra equation

$$w(x) = \lambda \int_0^x k(x, t) w(t) dt$$

since if the two solutions are denoted by $y_1(x)$ and $y_2(x)$ we have

$$y_1(x) = f(x) + \lambda \int_0^x k(x, t) y_1(t) dt$$

and

$$y_2(x) = f(x) + \lambda \int_0^x k(x, t) y_2(t) dt \, .$$

Subtracting one equation from the other we have:

$$y_1(x) - y_2(x) = w(x) = \lambda \int_0^x k(x, t)[y_1(t) - y_2(t)] dt$$

from which

$$w(x) = \lambda \int_0^x k(x, t) w(t) dt \, .$$

Iterating this n-times yields

$$w(x) = \lambda^n \int_0^x k^{(n)}(x, t)w(t)dt, \quad n = 2, 3, \ldots,$$

and using the same notation for bounds on $k^{(n)}(x, t)$ as done in showing convergence above, we have

$$|w(x)| \leqq \frac{M^n(X)\, x^n \lambda^n}{n!} \left\{ \max_{0 \leq t \leq x} |w(t)| \right\}.$$

We notice that the right hand side is just a term of the Maclaurin series for exponential $(M(X)\, x\lambda)$ multiplied by $\max|w(t)|$. We also know that each term of a convergent series approaches zero, and since $\exp(M(X)\, x\lambda)$ converges, the single term $\frac{M^n(X)\lambda^n x^n}{n!}$ must approach zero, hence

$$\frac{M^n(X)x^n \lambda^n}{n!} |\max w(t)| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$

This means that $w(x) = 0$ on $[0, X]$, and we thus show that there is only one solution to the Volterra integral equation defined by equation (2).

Having thus shown that the iterated kernels defined by equation (6) do exist, and are uniformly convergent on the interval $[0, X]$, and that the solution of the Volterra integral equation is unique, we shall from now on accept that

$$y(x) = f(x) + \sum_{i=1}^{\infty} \lambda^i k^{(i)}(x, t)f(t)dt \tag{13}$$

where $k^{(i)}(x, t)$ is the iterated kernel of equation (2).

3.1. The Resolvent Kernel

It was shown preceding this section that the solution of the Volterra integral equation, which was deduced from our n-th order non-homogeneous linear differential equation with constant coefficients, was

$$y(x) = f(x) + \sum_{i=1}^{\infty} \int_0^x \lambda^i k^{(i)}(x, t)f(t)dt \tag{1}$$

and, of course since we have shown that the infinite series expressed in equation (5) of the preceding section converges uniformly, we can from now interchange the summation and the integration signs. Thus equation (1) becomes:

$$y(x) = f(x) + \int_0^x \left(\sum_{i=1}^{\infty} \lambda^i k^{(i)}(x, t) \right) f(t)dt \tag{2}$$

and we shall further simplify it as

$$y(x) = f(x) + \lambda \int_0^x R(x, t; \lambda) f(t)dt \tag{3}$$

where

$$R(x, t;\lambda) = \sum_{i=1}^{\infty} \lambda^{i-1} k^{(i)}(x, t) \quad . \tag{4}$$

$R(x, t;\lambda)$ is called "The Resolvent Kernel" for the kernel $k(x, t)$. As we shall see in Chapter V, the resolvent kernel plays an important role in finding solutions to some Volterra integral equations that would have been otherwise involved or too complicated to solve. First we need to establish that this resolvent kernel $R(x, t;\lambda)$ actually satisfies the Volterra integral equation as defined in Chapter II.

3.2. The Resolvent Kernel $R(x, t;\lambda)$ is a Solution of the Volterra Integral Equation

$$R(x, t;\lambda) = k(x, t) + \lambda \int_0^x k(x, \zeta) R(\zeta, t;\lambda) d\zeta \quad . \tag{5}$$

Proof: Let us consider an integral of the form

$$\int_t^x k(x, \zeta) R(\zeta, t;\lambda) d\zeta = \int_t^x k(x, \zeta) \sum_{i=1}^{\infty} \lambda^{i-1} k^{(i)}(\zeta, t) d\zeta \; .$$

The uniform convergence of the series $\sum_{i=1}^{\infty} \lambda^{i-1} k^{(i)}(\zeta, t)$ enables us to interchange the summation and the integral signs so that

$$\int_t^x k(x,\zeta)R(\zeta,t;\lambda)d\zeta = \sum_{i=1}^{\infty} \lambda^{i-1} \int_t^x k^{(1)}(x,\zeta)k^{(i)}(\zeta,t)d\zeta$$

$$= \sum_{i=1}^{\infty} \lambda^{i-1} k^{(i+1)}(x,t) ,$$

applying equation (6) of the first section of Chapter III, i.e.,

$$k^{(n+1)}(x,t) = \int_t^x k^{(1)}(x,\zeta)k^{(n)}(\zeta,t)d\zeta .$$

Thus

$$\int_t^x k(x,\zeta)R(\zeta,t;\lambda)d\zeta = \sum_{i=1}^{\infty} \lambda^{i-1} k^{(i+1)}(x,t)$$

$$= \sum_{i=2}^{\infty} \lambda^{i-2} k^{(i)}(x,t)$$

and so

$$\lambda \int_t^x k(x,\zeta)R(\zeta,t;\lambda)d\zeta = \sum_{i=2}^{\infty} \lambda^{i-1} k^{(i)}(x,t) ,$$

$$= \left\{ \sum_{i=2}^{\infty} \lambda^{i-1} k^{(i)}(x,t) + k^{(1)}(x,t) \right\} - k^{(1)}(x,t)$$

$$= R(x,t;\lambda) - k^{(1)}(x,t) ,$$

since

$$R(x,t;\lambda) = \sum_{i=2}^{\infty} \lambda^{i-1} k^{(i)}(x,t) + k^{(1)}(x,t) = \sum_{i=1}^{\infty} \lambda^{i-1} k^{(i)}(x,t),$$

and $k^{(1)}(x, t) = k(x, t)$. Rearranging, the integral finally becomes

$$R(x, t;\lambda) = k(x, t) + \lambda \int_t^x k(x, \zeta)R(\zeta, t;\lambda)d\zeta \quad . \tag{6}$$

Equation (6) is of the same form as our original Volterra equation if we replace $y(x)$ by $R(x, t;\lambda)$ and $f(x)$ by $k(x, t)$. We shall further impose that $k(x, t) \equiv 0$ for $x < t$ to take care of the lower limit in equation (6). From now on, and especially in Chapter V, we shall use this version of the Volterra integral equation (6) involving the resolvent kernel $R(x, t;\lambda)$ since it is evident that $R(x, t;\lambda)$ is a solution of the original Volterra integral equation and hence a solution of our original initial value problem from which the Volterra integral equation was deduced.

IV. SOLUTION OF INTEGRAL EQUATION BY WHITTAKER'S METHOD

In this chapter we shall briefly discuss E. T. Whittaker's numerical method of solving integral equations of the form:

$$y(x) + \int_0^x y(s)k(x-s)ds = f(x) \tag{1}$$

where the kernel $k(x, s)$ is of special form known as a "convolution-type" kernel. We shall represent the kernel $k(x)$ as a sum of n exponential functions of x, thus

$$k(x) = Q_1 e^{q_1 x} + Q_2 e^{q_2 x} + \ldots + Q_n e^{q_n x} \tag{2}$$

where the coefficients $Q_1, Q_2, \ldots, Q_n$ of the exponentials and the coefficients $q_1, q_2, \ldots, q_n$ of x in the exponentials are constants. Using this expression for the kernel, we shall show that a solution of the integral equation (1) is of the form:

$$y(x) = f(x) - \int_0^x K(x-s)f(s)ds \tag{3}$$

where

$$K(x) = P_1 e^{p_1 x} + P_2 e^{p_2 x} + \ldots + P_n e^{p_n x} \tag{4}$$

another sum of n exponential functions where $P_1, \ldots, P_n$ and

$p_1, \ldots, p_n$ are to be determined in equation (3). We note that the integral equation given by (3) has a unique solution since it is of Volterra type. Furthermore it was shown in Section 3.2 that the resolvent kernel $K(x)$ satisfies the Volterra integral equation

$$K(x) + \int_0^x K(x)k(x-s)ds = k(x) \quad . \qquad (*)$$

If we put back values for $k(x)$ and $K(x)$ as defined by equations (2) and (4) respectively we obtain:

$$P_1e^{p_1x} + P_2e^{p_2x} + \ldots + P_ne^{p_nx}$$

$$+\int_0^x \left\{P_1e^{p_1s} + P_2e^{p_2s} + \ldots + P_ne^{p_ns}\right\}\left\{Q_1e^{q_1(x-s)} + \ldots + Q_ne^{q_n(x-s)}\right\} ds$$

$$= Q_1e^{q_1x} + Q_2e^{q_2x} + \ldots + Q_ne^{q_nx} .$$

Expanding this by multiplying out the product under the integral sign we have:

$$P_1e^{p_1x} + P_2e^{p_2x} + \ldots + P_ne^{p_nx}$$

$$+\int_0^x \left\{\left[P_1Q_1e^{q_1x+p_1s-q_1s} + P_1Q_2e^{q_2x+p_1s-q_2s} + \ldots + P_1Q_ne^{q_nx\ p_1s-q_ns}\right]\right.$$

$$+\left[P_2Q_1e^{q_1x+p_2s-q_1s} + P_2Q_2e^{q_2x+p_2x-q_2s} + \ldots + P_2Q_ne^{q_nx+p_2x-q_ns}\right]+$$

$$+\dots+\left[P_nQ_1e^{q_1x+p_ns-q_1s}+P_nQ_2e^{q_2x+p_ns-q_2s}+\dots+P_nQ_ne^{q_nx+p_ns-q_ns}\right]\Big\}ds$$

$$= Q_1e^{q_1x} + Q_2e^{q_2x} + \dots + Q_ne^{q_nx} .$$

Performing the integrations we obtain:

$$P_1e^{p_1x} + P_2e^{p_2x} + \dots + P_ne^{p_nx}$$

$$+\Bigg\{\Bigg[\frac{P_1Q_1}{(p_1-q_1)}e^{q_1x+(p_1-q_1)s}\Bigg|_{s=0}^{s=x} + \frac{P_1Q_2}{(p_1-q_2)}e^{q_2x+(p_1-q_2)s}\Bigg|_{x=0}^{s=x}$$

$$+\dots+\frac{P_1Q_n}{(p_1-q_n)}e^{q_nx+(p_1-q_n)s}\Bigg|_{x=0}^{s=x}\Bigg]$$

$$+\Bigg[\frac{P_2Q_1}{(p_2-q_1)}e^{q_1x+(p_2-q_1)s}\Bigg|_{s=0}^{s=x} + \frac{P_2Q_2}{(p_2-q_2)}e^{q_2x+(p_2-q_2)s}\Bigg|_{s=0}^{s=x}$$

$$+\dots+\frac{P_2Q_n}{(p_2-q_n)}e^{q_nx+(p_2-q_n)s}\Bigg|_{s=0}^{s=x}\Bigg]$$

$$+\dots+\Bigg[\frac{P_nQ_1}{(p_n-q_1)}e^{q_1x+(p_n-q_1)s}\Bigg|_{x=0}^{s=x}+\dots+\frac{P_nQ_n}{(p_n-q_n)}e^{q_nx+(p_n-p_n)s}\Bigg|_{s=0}^{s=x}\Bigg]\Bigg\}$$

$$= Q_1e^{q_1x} + Q_2e^{q_2x} + \dots + Q_ne^{q_nx} .$$

Evaluating the definite integrals we have:

$$P_1 e^{p_1 x} + P_2 e^{p_2 x} + \dots + P_n e^{p_n x}$$

$$+\left\{\left[\frac{P_1Q_1}{(p_1-q_1)}(e^{p_1x}-e^{q_1x})+\frac{P_1Q_2}{(p_1-q_2)}(e^{p_1x}-e^{q_2x})+\dots+\frac{P_1Q_n}{(p_1-q_n)}(e^{p_1x}-e^{q_nx})\right]\right.$$

$$+\left[\frac{P_2Q_1}{(p_2-q_1)}(e^{p_2x}-e^{q_1x})+\frac{P_2Q_2}{(p_2-q_2)}(e^{p_2x}-e^{q_2x})+\dots+\frac{P_2Q_n}{(p_2-q_n)}(e^{p_2x}-e^{q_nx})\right]$$

$$\left.+\dots+\left[\frac{P_nQ_1}{(p_n-q_1)}(e^{p_nx}-e^{q_1x})+\frac{P_nQ_2}{(p_n-q_2)}(e^{p_nx}-e^{q_2x})+\dots+\frac{P_nQ_n}{(p_n-q_n)}(e^{p_nx}-e^{q_nx})\right]\right\}$$

$$= Q_1 e^{q_1 x} + Q_2 e^{q_2 x} + \dots + Q_n e^{q_n x} \,. \qquad (9)$$

Equating coefficients of $e^{p_1 x}$ on both sides of the equation yields:

$$P_1 + \frac{P_1Q_1}{(p_1-q_1)} + \frac{P_1Q_2}{(p_1-q_2)} + \dots + \frac{P_1Q_n}{(p_1-q_n)} = 0 \,,$$

and so

$$P_1\left(1+\frac{Q_1}{(p_1-q_1)} + \frac{Q_2}{(p_1-q_2)} + \dots + \frac{Q_n}{(p_1-q_n)}\right) = 0 \,.$$

We do not want P_1 to vanish otherwise we shall be contradicting our assumption that the function $K(x)$ defined by equation (4) consists of n exponential functions. Therefore

$$1+\frac{Q_1}{(p_1-q_1)} + \frac{Q_2}{(p_1-q_2)} + \dots + \frac{Q_n}{(p_1-q_n)} = 0 \,, \qquad (10)$$

By similar reasoning, after equating coefficients of $e^{p_2 x}$ in equation (9) we would obtain:

$$1 + \frac{Q_1}{(p_2 - q_1)} + \frac{Q_2}{(p_2 - q_2)} + \ldots + \frac{Q_n}{(p_2 - q_n)} = 0 \quad . \tag{11}$$

The process could be carried out n-times and each time equating coefficients of $e^{p_i x}$, $i = 1, 2, \ldots, n,$ in equation (9) and we would obtain:

$$1 + \frac{Q_1}{(p_i - q_1)} + \frac{Q_2}{(p_i - q_2)} + \ldots + \frac{Q_n}{(p_i - q_n)} = 0 \tag{12}$$

$i = 1, 2, 3, \ldots, n.$ Thus, we see that $p_1, p_2, \ldots, p_n$ are the roots of the algebraic equation in x

$$1 + \frac{Q_1}{(x - q_1)} + \frac{Q_2}{(x - q_2)} + \ldots + \frac{Q_n}{(x - q_n)} = 0 \tag{13}$$

and from this, we can determine the values of $p_1, p_2, \ldots, p_n$.

Returning to equation (9) and equating coefficients of $e^{q_1 x}$ on both sides of the equation we obtain:

$$-\left(\frac{P_1 Q_1}{(p_1 - q_1)} + \frac{P_2 Q_1}{(p_2 - q_1)} + \ldots + \frac{P_n Q_1}{(p_n - q_1)}\right) = Q_1 \ ,$$

from which

$$Q_1\left(1 + \frac{P_1}{(p_1 - q_1)} + \frac{P_2}{(p_2 - q_1)} + \ldots + \frac{P_n}{(p_n - q_1)}\right) = 0.$$

By similar reasoning done in the forgoing paragraphs we do not want Q_1 to vanish, hence

$$1 + \frac{P_1}{(p_1 - q_1)} + \frac{P_2}{(p_2 - q_1)} + \ldots + \frac{P_n}{(p_n - q_1)} = 0 \ . \tag{14}$$

Equating coefficients of $e^{q_2 x}$ on both sides of (9) also yields

$$1 + \frac{P_1}{(p_1 - q_2)} + \frac{P_2}{(p_2 - q_2)} + \ldots + \frac{P_n}{(p_n - q_2)} = 0 \tag{15}$$

and equating the coefficients of a general exponential $e^{q_j x}$ we have:

$$1 + \frac{P_1}{(p_1 - q_j)} + \frac{P_2}{(p_2 - q_j)} + \ldots + \frac{P_n}{(p_n - q_j)} = 0 \tag{16}$$

$j = 1, 2, \ldots, n$. Again we see that $q_1, q_2, \ldots, q_n$ are the roots of the algebraic equation in z:

$$1 + \frac{P_1}{(p_1 - z)} + \frac{P_2}{(p_2 - z)} + \ldots + \frac{P_n}{(p_n - z)} = 0 \tag{17}$$

and this will enable us to determine their values.

If $p_1, p_2, \ldots, p_n$, and $q_1, q_2, \ldots, q_n$ are known they could be used in the set of $2n$ equations given by equation (12) and (16) to determine our coefficients $P_1, P_2, \ldots, P_n$ of the function $K(x)$ that we are constructing. Hence if $p_1, p_2, \ldots, p_n$ and

$P_1, P_2, \ldots, P_n$ are determined, they could be put back in our expression for $K(x)$ and we thus see that the resulting expression for $K(x)$ given by equation (4) will satisfy the integral equation (*).

The problem therefore reduces to that of finding the value for $K(x)$ and this can be done by eliminating $P_1, P_2, \ldots, P_n$ determinantly from the n-equations defined by (16) together with (4). From these equations we have:

$$K(x) \begin{vmatrix} \frac{1}{(p_1-q_1)} & \frac{1}{(p_2-q_1)} & \cdots & \frac{1}{(p_n-q_1)} \\ \frac{1}{(p_1-q_2)} & \frac{1}{(p_2-q_2)} & \cdots & \frac{1}{(p_n-q_2)} \\ \vdots & & & \\ \frac{1}{(p_1-q_n)} & \frac{1}{(p_2-q_n)} & \cdots & \frac{1}{(p_n-q_n)} \end{vmatrix}$$

$$= -e^{p_1 x} \begin{vmatrix} 1 & \frac{1}{(p_2-q_1)} & \cdots & \frac{1}{(p_n-q_1)} \\ 1 & \frac{1}{(p_2-q_2)} & \cdots & \frac{1}{(p_n-q_2)} \\ & \vdots & & \\ 1 & \frac{1}{(p_2-q_n)} & \cdots & \frac{1}{(p_n-q_n)} \end{vmatrix} - e^{p_2 x} \begin{vmatrix} \frac{1}{(p_1-q_1)} & 1 & \cdots & \frac{1}{(p_n-q_1)} \\ \frac{1}{(p_1-q_2)} & 1 & \cdots & \frac{1}{(p_n-q_2)} \\ \vdots & & & \\ \frac{1}{(p_1-q_n)} & 1 & \cdots & \frac{1}{(p_n-q_n)} \end{vmatrix}$$

$$- \cdots$$

and factorizing these determinants [5, p. 380] we have

$$K(x) = -\frac{(p_1-q_1)(p_1-q_2)\cdots(p_1-q_n)}{(p_1-p_2)(p_1-p_3)\cdots(p_1-p_n)}e^{p_1x} - \frac{(p_2-q_1)(p_2-q_2)\cdots(p_2-q_n)}{(p_2-p_1)(p_2-p_3)\cdots(p_2-p_n)}e^{p_2x}$$

$$- \cdots - \frac{(p_n-q_1)(p_n-q_2)\cdots(p_n-q_n)}{(p_n-p_1)(p_n-p_2)\cdots(p_n-p_{n-1})}e^{p_nx} \quad . \qquad (18)$$

With this value of $K(x)$, equation (*) and hence equation (3) is a solution of our original integral equation defined by (1).

The method, as a whole is very much involved and so in the next chapter we shall discuss a method proposed by G. C. Evans in 1911 [2] for finding the resolvent kernels (and hence solving the integral equation containing the kernel) for kernels that satisfy differential equations together with known initial conditions. The method will therefore be applicable in treating a kernel of Whittaker's exponential-sum type since it will always satisfy a linear ordinary differential equation with constant coefficients.

V. CALCULATION OF THE RESOLVENT KERNEL BY THE METHOD OF G.C. EVANS

The resolvent kernel $R(x, t;\lambda)$ of the Volterra integral equation

$$y(x) = f(x) + \lambda \int_0^x k(x, t)y(t)dt \qquad \text{(i)}$$

corresponding to the initial value problem

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \ldots + a_1 y^{(1)}(x) + a_0 y(x) = f(x)$$

$$y^{(n-1)}(0) = y_0^{n-1}, y^{(n-2)}(0) = y_0^{n-2}, \ldots, y(0) = y_0 ,$$

was found to be

$$R(x, t;\lambda) = \sum_{i=1}^{\infty} \lambda^{i-1} k^{(i)}(x, t) \qquad \text{(ii)}$$

where $k(x, t)$ is the kernel of the integral equation (i). We pointed out that different kernels $k(x, t)$ will produce different resolvent kernels according to equation (ii). The method of iteration could be employed to solve for the resolvent kernels for some simple kernels of the form

$$k(x, t) = 1$$

or

$$k(x, t) = k(x-t) = e^{m(x-t)} .$$

However the work involved in calculating resolvent kernels for kernels of the form

$$k(x,t) = k(x-t) = A_1 e^{m_1(x-t)} + A_2 e^{m_2(x-t)}$$

is formidable, and certainly for more involved kernels, finding the resolvent kernels by the method of iteration does not help. We hope that there is a way of finding the resolvent kernel by some other method and so for this chapter we shall discuss the method suggested by G.C. Evans in 1911. Some kernels $k(x,t)$ do in fact satisfy linear differential equations with known initial conditions, and it is to this family of kernels that we shall apply the method of G.C. Evans. Before we employ the method to solve our n-th order non-homogeneous linear differential equation with constant coefficients, we shall apply it to solve for the resolvent kernel for the kernel

$$k(x,t) = k(x-t) = A_1 e^{m_1(x-t)} + A_2 e^{m_2(x-t)}$$

where it was almost impossible to solve for it by the method of iteration.

We shall assume, without loss of generality that $m_1 > m_2$, and A_1, A_2, m_1, m_2 are real constants. This kernel $k(x,t)$ is in fact a solution of the linear differential system:

$$\left.\begin{aligned} &k''(x-t) - (m_1+m_2)k'(x-t) + m_1m_2k(x-t) = 0 \,, \\ &k(0) = A_1 + A_2 \,, \\ &k'(0) = m_1A_1 + m_2A_2 \quad . \end{aligned}\right\} \tag{1}$$

The general integral equation satisfied by the resolvent kernel $R(x,t;\lambda)$ which may be represented by $R(x,t;\lambda) = r(x-t;\lambda)$ as we saw in Chapter III is,

$$r(x-t;\lambda) = k(x-t) + \lambda \int_t^x k(x-u)r(u-t;\lambda)du \quad . \tag{2}$$

Differentiating twice with respect to x yields:

$$r_x(x-t;\lambda) = k_x(x-t) + \lambda \int_t^x k_x(x-u)r(u-t;\lambda)du + \lambda k(0)r(x-t;\lambda) \tag{2a}$$

and

$$r_{xx}(x-t;\lambda) = k_{xx}(x-t) + \lambda \int_t^x k_{xx}(x-u)r(u-t;\lambda)du$$

$$+ \lambda k_x(0)r(x-t;\lambda) + \lambda k(0)r_x(x-t;\lambda) \quad . \tag{3}$$

Substituting (1) in (3) gives

$$r_{xx}(x-t;\lambda) = \{(m_1+m_2)k'(x-t)-m_1m_2k(x-t)\}$$

$$+\lambda\int_t^x \{(m_1+m_2)k'(x-u)-m_1m_2k(x-u)\}r(u-t;\lambda)du$$

$$+ \lambda(m_1A_1+m_2A_2)r(x-t;\lambda) + \lambda(A_1+A_2)r_x(x-t;\lambda) \quad . \tag{4}$$

For simplicity we shall leave the $(x-t;\lambda)$ out of the resolvent kernels and $(x-t)$ out of the kernels $k(x-t)$. We therefore have:

$$r_{xx} = k_{xx} + \lambda \int_t^x \{(m_1+m_2)k_x - m_1 m_2 k\} r du$$
$$+ \lambda\{m_1 A_1 + m_2 A_2\} r + \lambda(A_1+A_2) r_x \quad . \tag{5}$$

Multiplying (2) by $(m_1 m_2)$ yields;

$$(m_1 m_2) r = (m_1 m_2) k + \lambda(m_1 m_2) \int_t^x k r du \quad . \tag{6}$$

Adding (5) and (6) leads to;

$$r_{xx} + (m_1 m_2) r = k_{xx} + \lambda \int_t^x (m_1+m_2) k_x r du + \lambda(m_1 A_1 + m_2 A_2) r$$
$$+ \lambda(A_1+A_2) r_x + (m_1 m_2) k \; . \tag{7}$$

From (2a), substituting (1) gives;

$$r_x = k_x + \lambda \int_t^x k_x(x-u) r(u-t;\lambda) du + \lambda(A_1+A_2) r \tag{8}$$

and multiplying (8) by (m_1+m_2) gives;

$$(m_1+m_2) r_x = (m_1+m_2) k_x + \lambda(m_1+m_2) \int_t^x k_x r du + \lambda(m_1+m_2)(A_1+A_2) r . \tag{9}$$

Subtracting (7)-(9) leads to;

$$r_{xx} + (m_1 m_2)r - (m_1+m_2)r_x$$

$$= k_{xx} + \lambda \int_t^x \{(m_1+m_2)k_x r\}du + \lambda(m_1 A_1 + m_2 A_2)r$$

$$+ \lambda(A_1+A_2)r_x + (m_1 m_2)k - (m_1+m_2)k_x - \lambda(m_1+m_2)\int_t^x k_x r du$$

$$- \lambda(m_1+m_2)(A_1+A_2)r$$

$$= \{k_{xx} - (m_1+m_2)k_x + (m_1 m_2)k\} + \lambda(A_1+A_2)r_x$$

$$+ \lambda\{(m_1 A_1 + m_2 A_2) - (m_1+m_2)(A_1+A_2)\}r$$

$$= \lambda(A_1+A_2)r_x + \lambda\{(m_1 A_1 + m_2 A_2) - (m_1+m_2)(A_1+A_2)\}r ,$$

Since $\{k_{xx} - (m_1+m_2)k_x + m_1 m_2 k\} = 0$ from (1), leading to the differential equation in r:

$$r_{xx} - \{(m_1+m_2) + \lambda(A_1+A_2)\}r_x$$
$$+ \{(m_1 m_2) - \lambda(m_1 A_1 + m_2 A_2) + \lambda(m_1+m_2)(A_1+A_2)\}r = 0,$$

and simplifying gives:

$$r_{xx} - \{(m_1+m_2) + \lambda(A_1+A_2)\}r_x + \{(m_1 m_2) + \lambda(m_1 A_2 + m_2 A_1)\}r = 0 \qquad (10)$$

which for simplicity may be written as:

$$r_{xx} - a r_x + b r = 0 \qquad (11)$$

where the constants

$$\left.\begin{aligned} a &= \{(m_1+m_2)+\lambda(A_1+A_2)\} \\ b &= \{(m_1m_2)+\lambda(m_1A_2+m_2A_1)\} . \end{aligned}\right\} \qquad (12)$$

In solving equation (11) assume that the solution is of the form

$$r = e^{s(x-t)} ,$$

and differentiating with respect to x, we obtain:

$$r_x = se^{s(x-t)} ,$$

and

$$r_{xx} = s^2 e^{s(x-t)} .$$

Substituting these in the differential equation (11) yields:

$$s^2 - as + b = 0,$$

with solution

$$s = \frac{[a \pm (a^2-4b)^{1/2}]}{2} .$$

Now to find the value of (a^2-4b) from (12), we have that:

$$a^2 - 4b = \{(m_1+m_2)+\lambda(A_1+A_2)\}^2 - 4\{(m_1m_2)+\lambda(m_1A_2+m_2A_1)\}$$

$$= (m_1^2+m_2^2+2m_1m_2) + \lambda^2(A_1+A_2)^2 + 2(m_1+m_2)\lambda(A_1+A_2) -$$

$$- 4(m_1 m_2) - 4\lambda(m_1 A_2 + m_2 A_1)$$

$$= [m_1^2 + m_2^2 - 2(m_1 m_2)] + \lambda^2 (A_1 + A_2)^2 + 2\lambda m_1 A_1 + 2\lambda m_1 A_2$$

$$+ 2\lambda m_2 A_1 + 2\lambda m_2 A_2 - 4\lambda m_1 A_2 - 4\lambda m_2 A_1$$

$$= (m_1 - m_2)^2 + \lambda^2 (A_1 + A_2)^2 + 2\lambda m_1 A_1 - 2\lambda m_1 A_2$$

$$- 2\lambda m_2 A_1 + 2\lambda m_2 A_2$$

$$= (m_1 - m_2)^2 + \lambda^2 (A_1 + A_2)^2 + 2\lambda A_1 (m_1 - m_2) - 2\lambda A_2 (m_1 - m_2)$$

$$= (m_1 - m_2)^2 + \lambda^2 (A_1 + A_2)^2 + 2(m_1 - m_2)\lambda(A_1 - A_2) .$$

Unfortunately, it does not come to a nice perfect square as hoped, but anyway, the two roots of equation (11) are:

$$s_1 = \frac{1}{2} [\{(m_1 + m_2) + \lambda(A_1 + A_2)\}$$
$$+ \{(m_1 - m_2)^2 + \lambda^2 (A_1 + A_2)^2 + 2\lambda(m_1 - m_2)(A_1 - A_2)\}^{1/2}] ,$$

and

$$s_2 = \frac{1}{2} [\{(m_1 + m_2) + \lambda(A_1 + A_2)\}$$
$$- \{(m_1 - m_2)^2 + \lambda^2 (A_1 + A_2)^2 + 2\lambda(m_1 - m_2)(A_1 - A_2)\}^{1/2}] .$$

So the general solution of (11), which is the resolvent kernel, is:

$$r(x-t; \lambda) = C_1 e^{s_1 (x-t)} + C_2 e^{s_2 (x-t)}$$

where C_1, C_2 are real constants. Therefore,

$$R(x,t;\lambda)$$
$$= C_1 e^{\frac{1}{2}[\{(m_1+m_2)+\lambda(A_1+A_2)\}+\{(m_1-m_2)^2+\lambda^2(A_1+A_2)^2+2\lambda(m_1-m_2)(A_1-A_2)\}^{1/2}](x-t)}$$
$$+C_2 e^{\frac{1}{2}[\{(m_1+m_2)+\lambda(A_1+A_2)\}-\{(m_1-m_2)^2+\lambda^2(A_1+A_2)^2+2\lambda(m_1-m_2)(A_1-A_2)\}^{1/2}](x-t)} .$$

To eliminate the constants C_1, C_2, we see from (2) that if $\lambda = 0$ then;

$$r(x-t;0) = k(x-t)$$
$$= A_1 e^{m_1(x-t)} + A_2 e^{m_2(x-t)} . \tag{14}$$

Also from (13) putting $\lambda = 0$ gives:

$$R(x-t;0) = C_1 e^{m_1(x-t)} + C_2 e^{m_2(x-t)} . \tag{15}$$

Comparing (14) and (15) shows that:

$$A_1 = C_1,$$

and

$$A_2 = C_2.$$

Thus the resolvent kernel is given by the expression

$$R(x,t;\lambda) = A_1 e^{\frac{1}{2}[\{(m_1+m_2)+\lambda(A_1+A_2)\}+\{(m_1-m_2)^2+\lambda^2(A_1+A_2)^2+2\lambda(m_1-m_2)(A_1-A_2)\}^{1/2}](x-t)}$$
$$+A_2 e^{\frac{1}{2}[\{(m_1+m_2)+\lambda(A_1+A_2)\}-\{(m_1-m_2)^2+\lambda^2(A_1+A_2)^2+2\lambda(m_1-m_2)(A_1-A_2)\}^{1/2}](x-t)} .$$

As mentioned before, calculating this resolvent kernel by the method of iteration was almost impossible because of the amount of work involved. The method of G. C. Evans has provided a way out.

5.1. Solution of the Volterra Integral Equation Corresponding to the Initial Value Problem for a Non-Homogeneous Linear Ordinary Differential Equation

In Chapter II, we converted the initial value problem for a non-homogeneous linear differential equation with constant coefficients

$$\left.\begin{aligned} & a_n y^{(n)}(x) + a_{n-1}y^{(n-1)}(x) + \ldots + a_0 y(x) = f(x) \\ & y^{(n-1)}(0) = y_0^{n-1}, \ldots, y(0) = y_0 \end{aligned}\right\} \tag{1}$$

into a Volterra integral equation of the form:

$$y(x) = f(x) + \lambda \int_0^x k(x,t)y(t)dt \tag{2}$$

where we found the kernel of equation (2) to be

$$k(x,t) = -\sum_{i=0}^{n-1} \frac{a_{n-i-1}}{a_n} \frac{(x-t)^i}{(i)!} \qquad\qquad (3)$$

In Chapter III, we introduced the idea of the resolvent kernel $R(x,t;\lambda)$ for kernel $k(x,t)$ and we showed that the resolvent kernel actually satisfies the Volterra integral equation

$$R(x,t;\lambda) = k(x,t) + \lambda \int_t^x k(x,\zeta)R(\zeta,t;\lambda)d\zeta .$$

In other words, in solving for $y(x)$, the solution of our initial value problem, it is sufficient to solve for $R(x,t;\lambda)$ since equation (2) on page 30 is solved by equation (3) on page 40.

In solving (1), our problem therefore reduces to that of finding the resolvent kernel $R(x,t;\lambda)$, and, as mentioned in the preceding section, the method of iteration will do us no practical good. We shall find it by the method of G. C. Evans.

First, we need to find out what differential equation together with suitable boundary conditions is satisfied by the kernel given in equation (3). To do this, let us write out the terms of the kernel $k(x,t)$ and then differentiate n times. From equation (3),

$$k(x,t) = \sum_{i=0}^{n-1} \frac{a_{n-i-1}}{-a_n} \frac{(x-t)^i}{(i)!}$$

$$= -\frac{1}{a_n}\left[a_{n-1}+a_{n-2}\frac{(x-t)}{1!}+a_{n-3}\frac{(x-t)^2}{2!}+a_{n-4}\frac{(x-t)^3}{3!}+\ldots+a_0\frac{(x-t)^{n-1}}{(n-1)!}\right].$$

Differentiating with respect to x we have:

$$k^{(1)}(x-t) = -\frac{1}{a_n}\left[a_{n-2}+a_{n-3}(x-t)+a_{n-4}\frac{(x-t)^2}{2!}+\ldots+a_0\frac{(x-t)^{n-2}}{(n-2)!}\right],$$

and

$$k^{(2)}(x-t) = -\frac{1}{a_n}\left[a_{n-3}+a_{n-4}(x-t)+a_{n-5}\frac{(x-t)^2}{2!}+\ldots+a_0\frac{(x-t)^{n-3}}{(n-3)!}\right],$$

$$\vdots$$

$$k^{(n-1)}(x-t) = -\frac{a_0}{a_n},$$

from which

$$k^{(n)}(x-t) = 0 .$$

Our differential system, satisfied by $k(x-t)$ is therefore:

$$\left.\begin{aligned} k^{(n)}(x-t) &= 0 \\ k(0) &= -a_{n-1}/a_n \\ k^{(1)}(0) &= -a_{n-2}/a_n \\ k^{(2)}(0) &= -a_{n-3}/a_n \\ &\vdots \\ k^{(n-1)}(0) &= -a_0/a_n . \end{aligned}\right\} \qquad (4)$$

Having obtained these, we are now in a position to apply the method of G.C. Evans to solve for the resolvent kernel $R(x, t;\lambda)$ for the kernel $k(x-t)$. As we pointed out before, the resolvent kernel $R(x, t;\lambda)$ is a solution of the Volterra integral equation:

$$R(x, t;\lambda) = k(x, t) + \lambda \int_0^x k(x, u)R(u, t;\lambda)du \tag{5}$$

and since $k(x, t)$ is of the form $k(x-t)$ in equation (3) we may assume that the resolvent kernel $R(x, t;\lambda)$ is of the form $r(x-t;\lambda)$ so that rewriting equation (5) we have:

$$r(x-t;\lambda) = k(x-t) + \lambda \int_0^x k(x-u)r(u-t;\lambda)du \quad . \tag{6}$$

To simplify our writing suppose we just write r for $r(x-t;\lambda)$ and $r(u-t;\lambda)$ and k for $k(x-t)$ and $k(x-u)$, thus obtaining

$$r = k + \lambda \int_0^x krdu \quad . \tag{7}$$

Differentiating this with respect to x, we obtain

$$r^{(1)} = k^{(1)} + \lambda \int_0^x k^{(1)}rdu + \lambda k(0)r ,$$

and differentiating again leads to

$$r^{(2)} = k^{(2)} + \lambda \int_0^x k^{(2)} r du + \lambda k^{(1)}(0) r + \lambda k(0) r^{(1)} .$$

A third differentiation yields:

$$r^{(3)} = k^{(3)} + \lambda \int_0^x k^{(3)} r du + \lambda k^{(2)}(0) r + \lambda k^{(1)}(0) r^{(1)} + \lambda k(0) r^{(2)} .$$

We can see a general pattern building up, and after the i-th differentiation we would obtain:

$$r^{(i)} = k^{(i)} + \lambda \int_0^x k^{(i)} r du + \lambda \sum_{j=0}^{i-1} k^{(j)}(0) r^{(i-j-1)} .$$

In particular, after the n-th differentiation we have

$$r^{(n)} = k^{(n)} + \lambda \int_0^x k^{(n)} r du + \lambda \sum_{i=0}^{n-1} k^{(i)}(0) r^{(n-i-1)} . \tag{8}$$

Equation (8) can be further simplified since we already showed that $k^{(n)} = 0$ and so

$$k^{(n)} + \lambda \int_0^x k^{(n)} r du = 0 .$$

Equation (8) therefore becomes

$$r^{(n)} - \lambda \sum_{i=0}^{n-1} k^{(i)}(0) r^{(n-i-1)} = 0 \tag{9}$$

Writing out a few terms we have

$$r^{(n)} - \lambda k(0) r^{(n-1)} - \lambda k^{(1)}(0) r^{(n-2)} - \dots - \lambda k^{(n-1)}(0) r = 0 \quad . \tag{10}$$

However, from our expressions for the derivatives of $k(x-t)$ we see that

$$\left.\begin{aligned} k(0) &= -a_{n-1}/a_n \\ k^{(1)}(0) &= -a_{n-2}/a_n \\ k^{(2)}(0) &= -a_{n-3}/a_n \\ &\vdots \\ k^{(n-1)}(0) &= -a_0/a_n \quad . \end{aligned}\right\} \tag{11}$$

Equation (10) therefore becomes:

$$r^{(n)}(x-t;\lambda) + \lambda \frac{a_{n-1}}{a_n} r^{(n-1)}(x-t;\lambda) + \lambda \frac{a_{n-2}}{a_n} r^{(n-2)}(x-t;\lambda) + \dots + \lambda \frac{a_0}{a_n} r = 0 \, . \tag{12}$$

This is the differential equation satisfied by the resolvent kernel. We can solve this equation provided we have n boundary conditions, since the differential equation is of order n. To find the conditions we substitute $\lambda = 0$ in the general equation for the derivatives of r: i.e., in

$$r^{(i)} = k^{(i)} + \lambda \int_0^x k^{(i)} r\,du + \lambda \sum_{j=0}^{i-1} k^{(j)}(0) r^{(i-j-1)}$$

to obtain

$$r^{(i)}(x-t, 0) = k^{(i)}(x-t), \quad i = 0, 1, 2, \ldots, n-1.$$

We shall further substitute $x-t = 0$ to obtain the set of n boundary conditions:

$$r(0,0) = k(0) = -a_{n-1}/a_n \quad \text{from (11)},$$

and

$$r^{(1)}(0,0) = k^{(1)}(0) = -a_{n-2}/a_n \quad \text{from (11)},$$
$$\vdots$$
$$r^{(n-1)}(0,0) = k^{(n-1)}(0) = -a_0/a_n \quad \text{from (11)}.$$

Our problem therefore reduces to that of solving the homogeneous linear differential equation with constant coefficients, together with n boundary conditions:

$$\left.\begin{aligned} a_n r^{(n)}(x-t;\lambda) + \lambda a_{n-1} r^{(n-1)}(x-t;\lambda) + \ldots + \lambda a_0 r &= 0 \\ r(0,0) &= -a_{n-1}/a_n \\ r^{(1)}(0,0) &= -a_{n-2}/a_n \\ r^{(2)}(0,0) &= -a_{n-3}/a_n \\ &\vdots \\ r^{(n-1)}(0,0) &= -a_0/a_n \quad . \end{aligned}\right\} \quad (13)$$

Several methods for solving such equations were discussed in detail in Chapter I and so any of them could be readily applied. We already pointed out that the resolvent kernel $R(x-t;\lambda)$ is in fact a solution of the Volterra integral equation deduced from our original initial value problem; therefore once the resolvent kernel $R(x-t;\lambda)$ is determined, we essentially have the solution of our Volterra integral equation corresponding to our original initial value problem for a non-homogeneous linear differential equation with constant coefficients.

SUMMARY

Different methods have been employed to solve the initial value problem for a non-homogeneous linear differential equation with constant coefficients

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \dots + a_0 y(x) = f(x)$$

$$y(0) = y_0, y^{(1)}(0) = y_0', \dots, y^{(n-1)}(0) = y_0^{n-1} .$$

We saw that the method of "undetermined coefficients" has its limitations since it could be applied in only a few cases, and so we sought a more powerful method--the Lagrange's method of variation of parameters--for finding a particular solution $v(x)$ of the non-homogeneous differential equation.

The method of Laplace transform was also employed in solving the differential system. It is a nice method but we would almost always have to have a table of Laplace transforms handy every time we want to solve such differential systems.

We further converted our differential system into a Volterra integral equation and found that solving by the method of iteration was not the best thing to do because of the work involved. We tried Whittaker's numerical method but here again we saw that the method was very much involved.

The limitations encountered with these various methods led us into developing the idea of the resolvent kernel of the Volterra integral equation and calculating it by the method suggested by G. C. Evans in 1911, having shown that the resolvent kernel actually is a solution of the Volterra integral equation corresponding to the initial value problem for a non-homogeneous linear differential equation with constant coefficients.

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