FINITE NON-PROJECTIVE GEOMETRIES WITHOUT THE AXIOM OF PARALLELS

by

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I wish to acknowledge my God's hand in this work.

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CHAPTER I

The Axioms and Some General Theorems

In this paper, "point", "line", and "incident upon" are undefined terms. The phrases, "point is on a line", "point is incident upon a line", "line is incident upon a point", "line is on a point", are all to be considered synonymous.

Definition 1: A line l is on a line k (at a point P), or a line l intersects a line k at P, if and only if there is a point P such that P is on both l and k. It may then be said that l and k have P in common.

Definition 2: A line l is parallel to a line k, or not on k, if and only if there is no point P such that P is on both l and k.

Axiom I: If P and Q are distinct points, there is exactly one line l on both P and Q (l may then be called PQ).

Axiom II: If l is a line, there is at least one point not on l.

Axiom III: If l is a line and P is a point not on l, there are exactly m distinct lines on P ($m \geq 2$), each parallel to l.

Axiom IV: There is at least one line with n points on it, $n \geq 2$.

Definition 3: The entire set of points and lines whose existence is postulated by these axioms (for given m and n) is a geometry.

Other work done with an axiom system containing the Axiom III here stated is not known to this investigator, but similar systems have been studied (2, p. 130-132; 3, p. 40-42). As here stated, the axioms may or may not be consistent, depending on the values assigned to n and m. For n=2, their consistency for any m is established by the following model: interpreting line as side or diagonal of a convex polygon of 5 or more vertices, and point as vertex of the same, it is easily seen that the axioms are satisfied by such a polygon with its diagonals.

It will be shown later that the axioms are inconsistent when m=2, n>2. For n=3, m=4, the following model satisfies the axioms: Points are the fifteen distinct elements, 1, 2, 3, ..., 15, and lines are triples of distinct elements. Then the following set of lines (with their points) satisfies the axioms (lines being here represented as columns of three elements):

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
4	5	6	7	8	9	10	11	12	13	14	15	1	2	3
5	6	7	8	9	10	11	12	13	14	15	1	2	3	4

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	4	5	6	7	8	9	10	11	12	13	14	15	1	2
9	10	11	12	13	14	15	1	2	3	4	5	6	7	8

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15

This result, with numbers replaced by letters, is found in an article by Abraham Barshop (1, p. 15).

Four General Theorems.

In the course of the proof of the first of these theorems, l denotes a line with exactly n distinct points on it, $n \geq 2$. The existence of at least one such line is specified by Axiom IV.

Theorem I: If P is any point, P has exactly n + m distinct lines on it, and if k is any line, k has exactly n distinct points on it.

The proof requires several lemmas.

Lemma a: If P is a point not on the line l, then P has exactly m + n distinct lines on it.

If P_i ($i = 1, 2, 3, \ldots, n$) is a point of l, then certainly a point P not on l is distinct from P_i, and the line PP_i exists. There are n distinct such lines. But there are m distinct lines on P which are parallel to l (and so are distinct from the lines PP_i). So there are at least m + n lines on P. A line must be either on l or parallel to l, so there are no more lines on P. Hence there are exactly m + n lines on P.

Lemma b: Every line has at least one point on it.

Assume some line j has no points on it. It is then parallel to l. But if P is a point not on l, P has m + n lines on it (each of which is necessarily parallel to j, since j has no points on it).

P cannot be on j; then P is a point not on j, such that there are $m + n$ lines on P, each of which is parallel to j. This is a contradiction of Axiom III. There is, therefore, at least one point on any line j.

Lemma c: Every line has at least two points on it.

Consider any line j: it may be l, or it may be on l at exactly one point, or it may be parallel to l. If j is l, the lemma is proved. If j is on l at exactly one point, Q, the proof goes as follows: there is a point P not on l. P is distinct from Q. PQ is a line, and if PQ is j, the lemma is proved. If j is not PQ, then P is not on j. But there are $m + n$ lines on P, only m of which are parallel to j, so n of the lines on P are on j. Suppose line q_i is on j at Q_i $(i=1, 2, 3, \ldots, n)$. These points are all distinct (if $Q_s = Q_t$, $s \neq t$, then $q_s = PQ_s = PQ_t = q_t$, a contradiction), so there are at least n points on j, $n \geq 2$, if j is on l at exactly one point.

If j is parallel to l, j has at least one point R on it by lemma b, and l has at least two distinct points S and T. R is distinct from S and T, so there is a line RS, with T not on RS. Then there are m lines on T, each distinct from l, each of which is parallel to RS. If they are all parallel to j, as is l, there are at least $m + 1$ lines on T, each parallel to j, a contradiction of Axiom III. This shows that at least one of the parallels to RS is on j. Suppose it is on j at

U. Then $U \neq R$, since U is on a line parallel to RS. So in this case, and hence in all three cases, j has at least two distinct points on it.

Lemma d: If j and k are any two lines, there is a point not on j and not on k.

There is a point P not on line j. If P is not on k, the lemma is proved. If P is on k, then there are at least two distinct lines, a and b, on P, each parallel to j. At least one of these, b, suppose, is distinct from k. But there is another point Q on b, $Q \neq P$, according to lemma c. Then Q is not on j since b is parallel to j. Also, Q is not on k, because P is on k, and if Q were on k, b=PQ=k, a contradiction since b is distinct from k.

So in either case, there is a point not on j and not on k.

The lemma is a trivial result, of course, if $j = k$.

The theorem is now easily demonstrated. If k is any line, there is a point P not on k, not on l. Since P is not on l, there are exactly $m + n$ distinct lines on P. But only m of these lines are parallel to k, so n of the lines on P are on k, at distinct points R_i ($i=1, 2, 3, \ldots, n$). If there is a point K on k, distinct from those points, then the line PK is distinct from each of the lines PR_i and from the m lines on P parallel to k, so there are $m + n + 1$ lines on P, a contradiction. Hence there is no such point K; there are exactly n points on k.

This proves the second part of the theorem. To complete the proof of the first part, a demonstration will be made of the fact that if P is any point on l, then P has exactly m + n lines on it.

If P is on l, there is a point Q on l, Q $\neq$ P, and a point R not on l. P, then, is not on line QR. Then QR has n distinct points S_i (i=1, 2, 3, ..., n) (renaming Q and R as some S_j, S_k) on it, as previously shown, so there are n distinct lines PS_i on P. But there are also m distinct parallels to QR (S_jS_k) on P, all distinct from the lines PS_i; so there are at least m + n lines on P. If there is another line, f, on P, it must be on QR, and so is identical to some line PS_u, say. This shows that there are exactly m + n lines on P.

Then if P is any point (on l or not), there are exactly m lines on it. This completes the proof of Theorem I.

<u>Theorem II</u>: There are exactly (n + m)(n - 1) + 1 points in a geometry.

Proof: Consider a point P with the n + m lines on it. On each of those lines there are n - 1 points, excluding P. The points of a given line on P are distinct from other points on the same line and from points (not P) on other lines on P. So there are n - 1 points (not P) on each of the n + m lines, a total of at least (n + m)(n - 1) + 1 points, including P. If Q is any point not P, then Q is on line PQ, a line on P, and hence is counted as shown.

So there are no other points than those on lines on P. This proves that the number of points in a geometry is exactly $(n + m)(n - 1) + 1$.

Definition 4: The lines of a given set of lines are mutually parallel if and only if no pair of lines (distinct) in the set has a point in common, and there are at least two lines in the set.

Theorem III: If k is the number of lines in a set of mutually parallel lines then $k \leq \frac{(n + m)(n - 1) + 1}{n}$.

Proof: On k parallel lines there are kn distinct points, a number which cannot exceed the number of points in the geometry, so $kn \leq (n + m)(n - 1) + 1$, or $k \leq \frac{(n + m)(n - 1) + 1}{n}$.

It is seen from this theorem, then, that $k \leq n + m - 2$ (since $\frac{(n + m)(n - 1) + 1}{n} = n + m - 1 - \frac{m - 1}{n}$, and k is an integer).

Theorem IV: If P is a point not on any line of a set of k mutually parallel lines, and on P there are s distinct lines ($s \leq m$), each of which is parallel to each line of the set of mutually parallel lines, then $k \leq (n + m - s) \frac{(n - 1)}{n}$.

Proof: There are kn distinct points on the lines of the set, and $s(n - 1) + 1$ distinct points on the given lines on P, all distinct from the points of the set of mutually parallel lines. Hence there are $kn + s(n - 1) + 1$ points so described, a number which may not exceed the number of points in the geometry:

$$kn + s(n - 1) + 1 \leq (n + m)(n - 1) + 1$$

$$kn \leq (n + m - s)(n - 1),$$ from which the result follows:

It is also easily shown, then, that $k \leq n + m - s - 1$:

$$k \leq (n + m - s)\frac{(n-1)}{n} = n + m - s - \frac{n+m-s}{n} = n+m-s-1-\frac{m-s}{n} \leq n+m-s-1$$

This concludes the study of the general theorems.

CHAPTER II

Some Methods of Approach for the case m = 2

The example of Barshop's Article (1, p. 15) initiates some optimism regarding the existence of some of these geometries for $n > 2$. Accordingly, this investigator examined two specific cases, namely, $n = 3$ and $n = 4$, with only those theorems at hand which have been previously stated in this paper. The primary idea was to establish the structure of a particular geometry as thoroughly as possible with deductive processes, dividing the problem into cases when it seemed necessary, making further deductions, then more cases, etc. The eventual objectives, of course, were (1) to obtain a model, or (2) to exhaust all possibilities with contradictions.

Specifically, the structuring of a geometry began with an appeal to Theorem II. With $m = 2$ and $n = 3$, say, the number of points in the geometry is $(3 + 2)(3 - 1) + 1 = 11$ points. These points all may be considered to be on the five lines incident upon a given point (in this case, $n + m = 5$, the number of lines on a point according to Theorem I). Subsequently, the investigation required two trivial theorems, which follow with brief proofs, as an introduction to a specimen of work from the proof of the non-existence of the case n = 3. These theorems were used extensively in the study of the cases n = 3 and n = 4.

Let one point of the geometry be named O.

Definition 5: A line is a ray if and only if it is on O.

Theorem a: If a line, not a ray, is on two distinct rays, it is on n - 2 other rays (not at O).

This follows from the fact that there are n points on the given line and if the line is on two rays, it is on them at two distinct points (since it is not a ray itself), and hence has n - 2 points not on those rays; but those points must be on other rays since there are no points not on rays. Finally, those n - 2 points are on distinct rays, since if two such points were on one ray, the given line is that ray, a contradiction.

Theorem b: If l_1, l_2, l_3, ..., l_{n+1} are distinct lines on a point P, and A_1, A_2, A_3, ... A_{n-1} are distinct points not P, not on the given lines, then those n-1 points are all collinear with P.

Suppose A_i is not on A_jP. Then A_iP is distinct from l_k (k = 1, 2, 3, ..., n + 1) as is A_jP; so A_iP, A_jP, and the lines l_k represent n + 3 distinct lines on P, a contradiction since there are exactly n + 2 lines on any point. So A_i is on A_jP for any A_i (i = 1, 2, 3, ..., n - 1), so all the points A_i are collinear with P. Q. E. D.

Then for the case n = 3, there are 5 rays, say on the point O. Let them be named 1, 2, 3, 4, 5. On 1, there is a point not O, say A, and on 2 there is a point not O, say B. Then line AB exists, and is on a third ray, by Theorem a. No generality is lost if it be considered that AB is on C on 3 ($C \neq O$), so line ABC exists (here the convention is adopted of naming a line by any two or more of the points on it).

Also on 5 there is a point not O, say E; line AE exists, and is on one of the rays 2, 3, 4. There are actually only two cases involved, however, since AE being on 2 is symmetrical to AE being on 3. The case that AE is on 4 will be demonstrated here, the other case being the case that AE is on 2 or 3.

Suppose AE is on 4 at point D. Then line AED exists. There is another point on 2, not O, not B. Let it be G. Then line AG exists, and is on a third ray; it is on one of 3, 4, or 5. Again, 4 and 5 are symmetrical, so there are only two cases. Considering the case that AG is on 4, it is immediately obvious that it is not on O or D, so it is on the third point, say I, of 4 (note Figure 1). Then the line AGI exists. Then if the third point of 5 is J and third point of 3 is H, line AJH exists (by Theorem b, since there are four lines on A, and H and J are not on any of them). At this

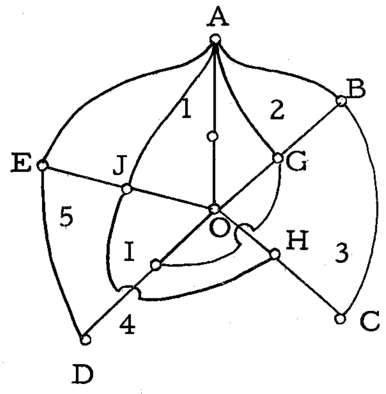

Figure 1.

stage five lines have been described on A, so there are no other lines on A.

Let the third point of 1 be F. Line DC exists, and is on one of 1, 2, or 5 (a symmetry again exists between 2 and 5). Proceeding with the case that DC is on 5, it is readily seen that DC is on J (If DC is on O, then DCO = DIO, so I and C are identical, a contradiction since neither is O, and they are on distinct rays. The same type of argument arises on the assumption that DC is on E--so DC is not on E or O, thus is on J). So line DCJ exists. Then it is noted that AJH and DCJ are two lines on J which are parallel to 2 (see Figure 2). Hence any other line on J must be on 2. Obviously, line JI exists and is distinct from AJH and DCJ, so it is on 2. It is

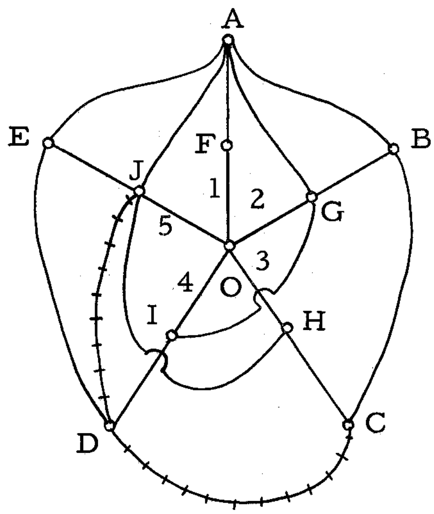

Figure 2

not on O (J could not be on IO), or on G (then J would be on IG=IGA, which is not possible), so JI is on B, i. e., line JIB exists. There are now exhibited 4 lines on J and points F and G are on none of them, so line JFG exists by theorem b. Then there are no more lines on J.

There are two lines on C which are parallel to AGI, namely CDJ and CHO (3). So any other line on C is on AGI. Line CF exists and is distinct from the two lines CDJ and CHO, so it is on AGI. But it is not on A or G (see Figure 3), so it must be on I. Thus, line CFI exists. Four lines, then, have been described, which are all on C; G and E are not on them, so line CGE exists. Similarly, line EIH exists (four lines on I), so EFB exists (four

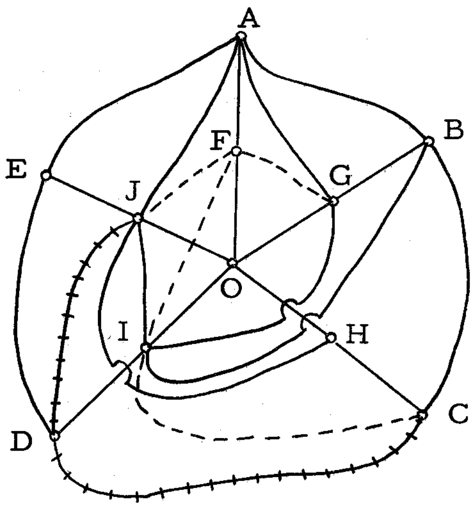

Figure 3

lines on E); then BHD exists (four lines on B), and therefore line HGF exists (four lines on H). But HGF=GF=JFG, so H=J, a contradiction (refer to Figure 4). Hence this case does not exist.

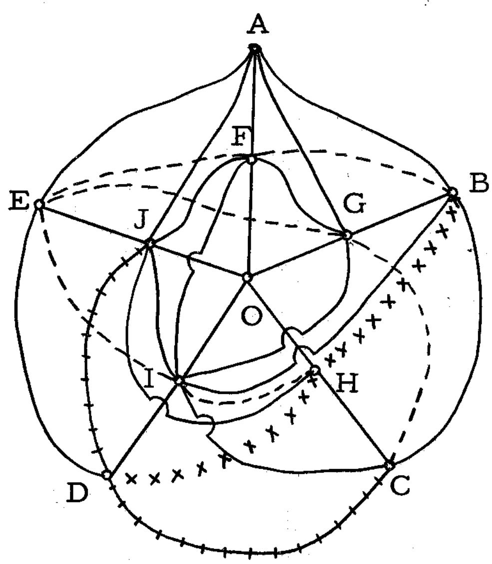

Figure 4

This shows the follow-through of one case to its end, and demonstrates those techniques of proof which were used in all the cases. In the subsequent investigation of the geometry in which $n = 3$, all cases ended in a contradiction, proving that this geometry does not exist. The equivalent of approximately eight cases such as the one just demonstrated were required for the entire proof.

After much effort and only trivial results in the direction of more general theorems (for $m = 2$, but for any number n), the investigator returned again to a specific geometry, this time with $m = 2$, $n = 4$. This study was begun with full expectation that the result would require much more effort and persistence than was the case for the geometry in which $n = 3$. But it seemed necessary to ferret out a better intuitive grasp of the geometries, so the step was made. The return to a specific geometry failed to accomplish this intuitional result immediately, and the work involved was three times the amount expected, but in the process, it was shown that this geometry also fails to exist. The proof involved following over 100 cases through (to a contradiction in each case); nearly 200 pages of cases, diagrams, and deductions were required. The techniques were exactly the same as those used in the example taken from the proof of the non-existence of the geometry in which $n = 3$.

The investigation then turned to more general theorems with occasional observations of the case $n = 5$. The discovery of Theorem V (stated in the next chapter) came about in a short time, and its result reduced the number of cases to be considered substantially. It was possible to apply it to show the non-existence of cases $n = 5$, 6, and 7 quite readily. With it, the proof for the case $n = 4$ was reduced to four pages. All of these proofs, however, were immediately overshadowed by the theorem to which they pointed. The remainder of the paper is devoted to that result.

CHAPTER III

Proof of the Inconsistency of the Axioms When $n>2$ and $m=2$

In this chapter, it will be assumed that $n>2$ and $m=2$. Then there are exactly $(n + 2)(n - 1) + 1 = n^2 + n - 1$ points in the geometry.

Definition 6: A set of lines (and the points on those lines) is an (x, y) configuration if and only if the lines x and y are in the set, x is parallel to y, and each of the lines (not x) parallel to y, on points of x, are in the set, and no other lines or points are in the set. This set may be called merely a configuration when it is clear which lines are involved. The lines (not x) parallel to y will be called bars of the configuration, or bars.

There exists a line l, a point not on l, and a line r on that point, such that r is parallel to l. Furthermore, at each point P_i ($i = 1, 2, 3, \ldots, n$) of r there is another line such that it is also parallel to l (according to Axiom III). At each point P_i of r, let such a line be a_i. Then r, l, and the lines a_i are an (r, l) configuration; this shows that a configuration exists. This (r, l) configuration will be referred to in the following theorems and lemmas.

Lemma e: In a configuration, there are at least two bars which have a point in common.

Proof: Suppose that each bar of some (r, 1) configuration is parallel to every other bar. Then there are n bars with n distinct points each, and there are n points on 1, a total of $n^2 + n$ distinct points, a contradiction since there are only $n^2 + n - 1$ points in the geometry. Hence at least two bars have a point in common.

Theorem V: In a configuration, any given bar intersects at most one other bar.

Proof: Suppose that in some (r, 1) configuration a bar a_i intersects two other bars, a_j and a_k. Then on each point P_h (h = 1, 2, 3, ..., n, $h \neq i$) of r, there are two lines, b_h, b_h', say, each parallel to a_i (note Figure 5). Then b_j, b_j', b_k, and b_k' are on 1 (b_j, b_j' can certainly not be r, since r intersects a_i; similarly neither of them is a_j, since a_j intersects a_i. But r and a_j are two lines on P_j which are parallel to 1, so all other lines on P_j are on 1. Hence b_j, b_j' are on 1, and by a similar argument, b_k, b_k' are also on 1). Furthermore on the points P_f (f = 1, 2, 3, ..., n, $f \neq i, j, k$) of r, at least one of the lines b_f, b_f' is not the bar a_f, and is therefore on 1 (assume not; then there are three lines on P_f, each parallel to 1). There are at least n - 3 such lines on 1, each distinct from the four lines b_j, b_j', b_k, b_k'. Therefore, since there are at least n+1 of these lines on 1 and only n points on 1, it must be that some two

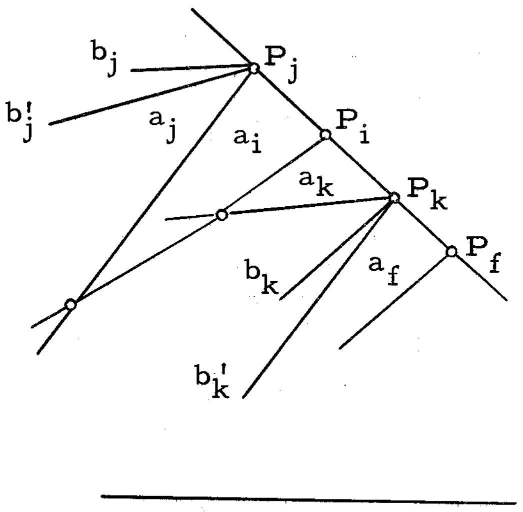

Figure 5

of these lines are on the same point of l, so there are at that point three lines parallel to a_i (the two lines are distinct from l, which is also, of course, parallel to a_i) contradicting Axiom III. The conclusion is that any bar a_i intersects at most one other bar.
Q. E. D.

So, if bars intersect, they intersect in pairs, each bar of an intersecting pair being parallel to all bars except that one which it intersects. Two such bars which intersect each other (and no others) will be referred to as a pair of intersecting bars (when the context indicates the configuration to which they belong).

Theorem VI. There is no configuration with exactly one pair of intersecting bars.

Proof: Consider some (r, l) configuration with bars a_i on the respective points P_i of r (i = 1, 2, 3, ..., n), with exactly one pair of those bars intersecting one another. In this configuration there are exactly $n^2 + n - 1$ points (the count is the same as that established in lemma e, $n^2 + n$, except that the point of intersection of the two bars as given by the theorem must only be counted once, deducting one from the total), the total number of points in the geometry, so all points are on the configuration. Since $n > 2$, and there are n bars, there is at least one bar, a_k, which is not one of the intersecting pair of bars. Let the points not P_k of a_k be R_1, R_2, R_3, ..., R_{n-1}. On each such point R_i there is a unique line r_i, $r_i \neq a_k$, r_i parallel to l. These lines, along with r, are actually bars of the (a_k, l) configuration, so by lemma e, at least two of them intersect at some point Q, Q not on l. But all points of the geometry are on the (r, l) configuration, and hence are on the bars of that configuration (a point of r is of necessity on some bar) or on l. Q is a point of the geometry, not on l, so it must be on some bar of the (r, l) configuration, not a_k, say a_h. But the two bars of the $(a_k, 1)$ configuration which intersect at Q are distinct from a_h since they each have a point of a_k on them, while a_h is parallel to a_k. Here, then, are three lines on Q, each parallel to l, giving the desired contradiction. Thus, there is no configuration with exactly one pair of intersecting bars. Q. E. D.

Theorem VII: If a configuration with k pairs of intersecting bars $(k>1)$ exists, where $2k<n$, then a configuration with at most k-1 pairs of intersecting bars exists.

Proof: Suppose there exists an (r, 1) configuration with k pairs of intersecting bars, and $2k<n$. Then since there are n bars, there is at least one, say a_j, which intersects no other bar. In this configuration there are $n^2 + n$ points all distinct except for the k intersections of bars -- so there are $n^2 + n - k$ points in the configuration. There are $n^2 + n - 1$ points in the geometry, so there are exactly k-1 points not in the configuration. Let those points be named $N_1, N_2, N_3, \ldots, N_{k-1}$. It will be shown that there is a configuration with bars intersecting at no other points than those k-1 points, hence with at most k-1 intersecting pairs of bars.

Assuming that the above-mentioned configuration with k intersecting pairs of bars exists, we have:

Lemma f: The (l, r) configuration exists, and the points of the (r, l) configuration are on it, and no other points are on it.

Proof: The lines l and r exist, and on points Q_i (i=1, 2, 3, ..., n) of l there exist lines b_i respectively, b_i not l, b_i parallel to r. Thus, the (l, r) configuration exists. Then it only remains to show that the points of the (l, r) configuration are all points of the (r, l) configuration, and that the points of the (r, l) configuration are all points of the (l, r) configuration.

This proof is again by contradiction. Suppose some point N of the (l, r) configuration is not on the (r, l) configuration. Then N is not on r or l, so it is on one of the bars, say b_u, of the (l, r) configuration. There exist two lines, a and b, on N, each parallel to l, at least one of which is on r (if neither is on r, then a, b, and b_u all are parallel to r, a contradiction. They are all distinct, since a and b are parallel to l and b_u is on l). Say a is on r at P_q. But N is not on the (r, l) configuration by assumption, so it is not on any bars of that configuration, or on r. Hence a is distinct from a_q, the bar of the (r, l) configuration on P_q, and a is distinct from r. Then at P_q the three lines a_q, r, and a all are parallel to l, a contradiction. All points of the (l, r) configuration are, therefore, on the (r, l) configuration. The proof that all points of the (r, l) configuration are on the (l, r) configuration is entirely similar to this previous proof. Hence, the lemma is established.

Next, it will be shown that there is a bar of the (l, r) configuration, b_j, which is parallel to a_j (the bar of the (r, l) configuration given as not intersecting any other bar of that configuration). To get this result, the assumption will be made that a_j intersects all bars b_i of the (l, r) configuration, with the hope of obtaining a contradiction. None of these intersections, then, occurs at P_j, the intersection of r and a_j, since the lines b_i are parallel to r. So there are only n-1 possible points on a_j at which the intersections with the n lines b_i can occur; this can only mean that at least one of these intersections occurs at a point, say F, which two of the bars, say b_q, b_r have in common. At F, however, there are two lines, s, t, parallel to l, both on r (assume, for instance, s is parallel to r; so are b_q and b_r, a contradiction). One of s, t is a_j (if not, here are three parallels--again--to l, a contradiction), but one is not--say t is not a_j. t is on r, and is parallel to l, so it is a bar of the (r, l) configuration which intersects a_j. But this contradicts the assumption that a_j intersects no other bars of that configuration. Hence, there is some bar, b_j, of the (l, r) configuration which is parallel to a_j.

Also, no other bar of the (r, l) configuration is parallel to b_j. This will be shown by the monotonous contradiction again. Assume a_f ($f \neq j$) is also parallel to b_j. The points of b_j are all on the (r, l) configuration, so all of its points except the one on l are on the lines a_i. There are n-1 such points, found on at most n-2 of the lines a_i (since b_j is not on two of those lines, a_j and a_f). Therefore, b_j has at least two points on one of these lines, say a_h. Then $b_j = a_h$, which is impossible since a_h is parallel to l, and b_j is not. So a_j is the only bar of the (r, l) configuration which is parallel to b_j.

Further, b_j intersects no other bar of the (l, r) configuration (assume b_j intersects bar b_m at G. Then there exist lines p and q on G, both parallel to l. They are both distinct from b_j and b_m, the two parallels to r at the point G, so both p and q are on r. Hence p and q are bars of the (r, l) configuration, and are parallel to a_j. Then p, q, and b_j are parallel to a_j, an absurd result).

Finally, then, at each point P_i $(i \neq j)$ of r, there is a unique line c_i, not r, parallel to b_j. If any two of these lines intersect, the intersection is not on r, but it is at some point of the geometry, and hence must be on one of the bars b_i $(i \neq j)$ or on one of the points $N_1, N_2, N_3, \ldots, N_{k-1}$. But the lines c_i may not intersect on any of the bars b_i, since the c_i are distinct from the b_i (c_i are on r, b_i are not) and there would be three parallels to b_j at such an intersection. Thus, these intersections may only be at the k-1 points $N_1, N_2, N_3, \ldots, N_{k-1}$, which are not on the (r, l) or (l, r) configurations. But the lines c_i are bars of the (r, b_j) configuration, a configuration with at most k-1 intersecting pairs of bars. Q. E. D.

<u>Theorem VIII</u>: A configuration with k pairs of intersecting bars, where $2k<n$, exists.

Proof: Given any configuration with k pairs of intersecting bars, either $2k=n$ or $2k<n$. If $2k<n$, then the theorem is true. If $2k=n$, then n is even (and greater than 2), so $n\geq 4$. Hence there are at least 2 pairs of intersecting bars in the configuration, say a, b, c, and d, where a and b intersect, as do c and d, but of course a and b are parallel to each of c and d. Suppose c and d intersect at D_1 and a and b intersect at S_1, the other points of a being S_2, S_3, ..., S_n. At each point S_i $(i=2, 3, \ldots, n)$ there are two lines P_i, P_i', each parallel to b (see Figure 6). At least one of each of

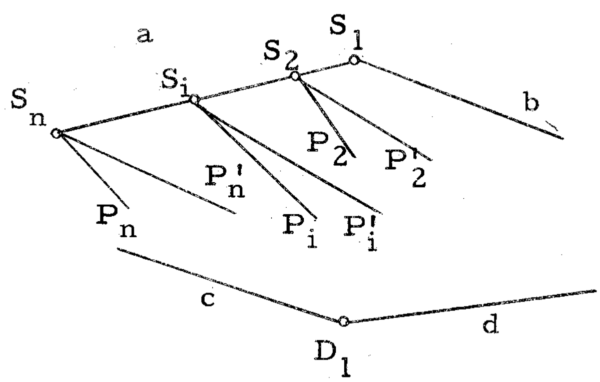

Figure 6

these pairs of lines is on d (if not, a and the two lines of the pair are three distinct parallels to d), say p_i is on d in each case, at point D_i. It is seen, then, that if $j \neq k$, $D_j \neq D_k$ (if $D_j = D_k$, then on D_j lines p_j, p_k, and d are all distinct and parallel to b, a contradiction). Also, $D_i \neq D_1$ for any i, since $D_i = D_1$ implies that p_i, c, and d are parallel to b at that point. Then the n-1 points D_i, together with the point D_1 are n distinct points of d; so there are no other points of d.

Next, suppose that some line p_k' is also on d. Certainly it is not on D_k; say it is on D_r, $r \neq k$. Then at that point, p_k', p_r, and d are distinct parallels to b, a contradiction. So p_k' is not on any such point D_r. But it is not on D_1, by a similar argument, so it is not on d. Therefore, the (a, d) configuration with bars b, p_2', p_3', ..., p_n' is a configuration in which at least one bar, b, intersects no other (since the lines p_k' are all parallel to b). Then if this configuration has s intersecting pairs of bars, it is seen that $2s < n$, since if $2s = n$, every bar is one of a pair of intersecting bars--and that does not occur in this configuration. Q. E. D.

Then since the number of pairs of intersecting bars is a positive integer, there must be a configuration for which this number is the least possible, yet greater than 1 (for a given n). Suppose such a configuration exists, with h pairs of intersecting bars. Then, by Theorem VIII, $2h < n$, so by Theorem VII, a configuration exists with at most $h-1$ pairs of intersecting bars, a contradiction to the assumption that h was the least such number. So no configuration exists with more than one pair of intersecting bars. But no configuration with one or no pairs of intersecting bars exists either, so no configurations exist, contradicting the simple proof given earlier that they do exist. Therefore, the axioms are inconsistent for the case $n > 2$, $m=2$.

EPILOGUE

It seems, perhaps, unusual that four simple axioms such as these are actually an inconsistent set of axioms. The fact that a small collection of assumed truths may collectively imply two contradictory results should arrest any mathematician. This paper may not be unique, perhaps, in the demonstration of an inconsistent set of axioms; but it becomes more than a mere exercise in implications if it only serves as a reminder of the importance of foundations.

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