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Title: VIBRATIONS OF SUSPENDED CABLES

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The equations of motion of an elastic cable suspended in a viscous medium from two arbitrary points are derived. These are then linearized for the undamped, inextensible case, and it is found that the equations describing the in-plane motion of the cable are then independent of the equation describing the out-of-plane motion of the cable.

The linearized differential equations of motion of the cable contain coefficients which are irrational functions of the independent variable. A change of independent variable is made that transforms the equations of motion to a form with polynomial coefficients. Solution of these equations, using the method of Frobenius, yields both the in-plane and out-of-plane natural frequency ratios of the cable. The first six natural frequency ratios for both in-plane and out-of-plane motion are presented for a variety of cable geometries.

Using the equations developed for the normal mode motion, a technique is presented for determination of the displacements and

tensions throughout a cable when one end is subjected to a prescribed tangential displacement of known frequency. Maximum cable tensions are presented as a function of the forcing frequency for a variety of cable geometries.

Vibrations of Suspended Cables

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NOMENCLATURE

a	catenary size parameter; $S_0/\rho g$
a_k	coefficients for series expansions for the in-plane tangential displacement where $\lvert\xi\rvert \leq 1$
A_j	integration constants for in-plane series expansions for $U(\chi)$
b	horizontal distance between cable support points
b_k	coefficients for series expansions for the in-plane tangential displacement for independent variables χ and ψ
B_j	integration constants for in-plane series expansions for $U(\psi)$
C_j	integration constants for in-plane series expansions for $U(\xi)$
$D(\omega)$	boundary value determinant
D_j	integration constants for in-plane series expansions for $U(\xi)$ where $-1 \leq \xi < 0$
e_j	coefficients for series expansions for the out-of-plane displacement where $\lvert\xi\rvert \leq 2$
E_j	integration constants for out-of-plane series expansions for $P(\xi)$ where $-2 \leq \xi \leq 2$
f_1, f_2	linear drag coefficients for fluid moving tangent and normal to the cable
f_k	coefficients for series expansions for out-of-plane displacements for independent variable χ
F_j	integration constants for out-of-plane series expansions for $P(\chi)$
g	gravitational acceleration

h	vertical distance between cable support points
H_j	integration constants for out-of-plane series expansions for $P(\psi)$
i	imaginary unit; $\sqrt{-1}$
k	elastic proportionality constant for the cable
KE	kinetic energy of the cable
ℓ	length of the cable
L	the Lagrangian; $KE - V_g$
M_j	integration constants for out-of-plane series expansions for $P(\xi)$ where $-2 < \xi < 0$
N_j	nondimensional out-of-plane displacement series expansions where $\lvert\xi\rvert \leq 2$
$p(\xi)$	nondimensional out-of-plane displacement expansion where $-2 < \xi < 2$
P	nondimensional out-of-plane displacement
$[PL(\xi_{LM})]$	notation for matching matrix for $P(\xi)$ where $\xi < 0$
$[PR(\xi_{RM})]$	notation for matching matrix for $P(\xi)$ where $\xi > 0$
$q(\chi), q(\psi)$	nondimensional out-of-plane displacement expansions for $\lvert\xi\rvert > 0$
$[QL(-\xi_{LM}-2)]$	notation for matching matrix for $P(\psi)$
$[QR(\xi_{RM}-2)]$	notation for matching matrix for $P(\chi)$
Q_u, Q_v, Q_w	force per unit length of the cable due to fluid drag in the u, v, and w directions
r	indicial roots
$\lfloor R(\xi) \rfloor$	boundary value matrix for the out-of-plane displacement

R_u, R_v, R_w	force per unit length of the cable due to straining of the cable in the u, v, and w directions
s	position of a point on the cable
s_0	arc length from the origin to the apex of the cable when hanging in its equilibrium configuration
S	tension of the cable in its equilibrium configuration
S_0	tension of the cable at its apex when hanging in its equilibrium configuration
t	time
T	nondimensional, position dependent tension
u	in-plane displacement of a point on the cable in a direction tangent to the equilibrium cable configuration
U	nondimensional, position dependent in-plane tangential displacement
$[u(\xi_1, \xi_2, \omega_f)]$	forced frequency boundary value matrix
$\{U(\xi_2)\}$	forced frequency displacement matrix
v	in-plane displacement of a point on the cable in a direction normal to the equilibrium cable configuration
V	nondimensional, position dependent in-plane normal displacement
V_g	potential energy of the cable in its displaced configuration
V_{0g}	potential energy of the cable in its equilibrium configuration
w	out-of-plane displacement of a point on the cable normal to the plane of the equilibrium cable configuration

W	nondimensional, position dependent out-of-plane displacement
W_f	virtual work performed on the cable from a surrounding viscous medium
W_s	virtual work performed on the cable due to the straining of the cable
x, y, z	Cartesian coordinates of a point on the equilibrium cable configuration
x_0, y_0	constants of integration
$y(\xi)$	nondimensional, in-plane tangential displacement where $0 \leq \xi \leq 1$
y_j	nondimensional, in-plane series expansions for the tangential displacement where $\lvert\xi\rvert \leq 1$
$[YL(\xi_{LM})]$	notation for matching matrix for $U(\xi)$ where $\xi < 0$
$[YR(\xi_{RM}-2)]$	notation for matching matrix for $U(\xi)$ where $\xi > 0$
$z(\chi), z(\psi)$	nondimensional, in-plane tangential displacement where $\lvert\xi\rvert > 0$
$[ZL(-\xi_{LM}-2)]$	notation for matching matrix for $U(\psi)$
$[ZR(\xi_{RM}-2)]$	notation for matching matrix for $U(\chi)$

Greek

α	angle which the equilibrium cable makes with the horizontal plane at a point
β	$\bar{\alpha} - \alpha$
$\bar{\gamma}$	angle between the vertical plane of the equilibrium cable configuration and the vertical plane of the displaced cable configuration at a point
δ	nondimensional, real part of λ
$\delta(\sigma)$	Dirac Delta Function

$\bar{\epsilon}$	strain of the cable in its displaced configuration
η_1, η_2	nondimensional drag coefficients; $f_1/\rho(\sqrt{a/g})$, $f_2/\rho(\sqrt{a/g})$
θ	nondimensional time; $\sqrt{g/a}\,t$
κ	nondimensional flexibility coefficient; S_0/k
λ	nondimensional, complex natural frequency of oscillation; $\delta + i\omega$
ξ	nondimensional position of a point along the cable; $\alpha/\lvert\alpha\rvert(\sec\alpha - 1)$
ξ_{LM}	negative matching point for series solutions
ξ_{RM}	positive matching point for series solutions
ξ_p	an arbitrary point along the cagle
ξ_1, ξ_2	end points of the cable
ρ	mass per unit length of the cable
σ	nondimensional arc length; $S-S_0/a$
τ	nondimensional tension; $k\bar{\epsilon}/S_0$
χ	independent variable change for $\xi > 0$; $\xi - 2$
ψ	independent variable change for $\xi < 0$; $-\xi - 2$
ω	nondimensional, imaginary part of λ
ω_f	nondimensional forcing frequency

<u>Additional Symbology</u>

prefix : δ	δ(variable); denotes a virtual variation
overline — :	$\overline{\text{(variable)}}$; denotes cable is in a displaced configuration
subscript t:	(variable)_t; denotes a time derivative

subscript s:	$(\text{variable})_s$;	denotes a partial derivative with respect to arc length
slash ':	$(\text{variable})'$;	denotes a partial derivative with respect to nondimensional arc length σ
dot $^{\bullet}$:	$(\dot{\text{variable}})$;	denotes a derivative with respect to nondimensional time θ

VIBRATIONS OF SUSPENDED CABLES

I. INTRODUCTION

I.1. Background

In the late 1600's, James Bernoulli proposed the problem of determining the shape of a heavy chain suspended from two fixed points. James and John Bernoulli, Leibnitz, and Huyghens solved the problem and published their results in 1691. Not satisfied with having solved the most simple case, Bernoulli studied and solved the suspended chain problem for nonhomogeneous chains and extensible chains. In 1746, D'Alembert derived and gave a general solution form for the equation of motion of a taut, elastic string. This was the one dimensional wave equation.

In 1851, Rohrs [1] investigated the problem of determining the motion of a suspended chain that is nearly horizontal. Routh [2] derived the equations of motion of an inextensible string under the action of any impressed forces. Routh derived two forms of the equations of motion. One of these forms consisted of a set of four, first order, coupled differential equations in the dependent variables u, v, ϕ, and T and independent variables t and s. Variables u and v are normal and tangential velocities, ϕ is the angle which the cable makes with the horizontal, T is the cable tension,

t is time, and s is position along the cable. The other form, which he called the "intrinsic form of the motion equation," consisted of a single, second order differential equation of dependent variable ϕ and independent variables t and α in which α is the equilibrium angle of the cable with the horizontal and $\phi(\alpha, t)$ is the increase in this angle. Routh proceeded to solve his intrinsic equation of motion for the case of a nonhomogeneous cable which hangs in the shape of a cycloid. However, he was unable to obtain the solution for the case of the homogeneous cable, which hangs in the form of a common catenary.

Interest in suspension bridges and the collapse of the Tacoma Narrows Suspension Bridge prompted Pugsley [3] to study the dynamics of suspension bridges and hence to the fundamental study of the natural frequencies of a single, homogeneous, suspended chain. By considering that the oscillations in the chain arose from the propagation of transverse waves along its length, Pugsley derived a semi-empirical equation which provided fairly good correlation with some experimental work he did in predicting the natural frequencies of the suspended chain. Pugsley did not attempt to solve either of Routh's equations of motion for a suspended chain.

Using Routh's four, first order equations, Saxon and Cahn [4] made a small amplitude approximation and then expressed Routh's four equations in the form of a single, fourth order equation. Saxon and Cahn constructed an asymptotic solution for their fourth order equation and evaluated the first five natural frequencies for a number of different cable geometries.

Using Routh's intrinsic equation, Goodey [5] made another approximate solution for determining the frequency of the first two natural modes of motion of the cable for various cable geometries. It is of interest to note that all the work done by Routh, Pugsley, Saxon and Cahn, and Goodey was only usable for symmetric cables; in other words the end points of the cables had to be at the same level. No one had managed to obtain a closed form for the solution of Routh's equations for the catenary.

In the late 1960's interest was generated in the study of cable dynamics in conjunction with the mooring of large ocean research buoys at Oregon State University. Several research buoys had broken loose from their mooring lines and had been damaged or lost. It was believed that possibly the buoys had been oscillating near the resonant frequencies of the mooring lines and hence the lines were broken due to large induced tensions. A survey of the literature showed that there were no explicit solutions for determining the natural frequencies of a cable and further that there had been no work done with unsymmetric cables.

A serious impediment to analyses of motions about catenary equilibrium configurations has been that the coefficients in the differential equations are irrational functions of the space variable s or α. Smith and Thompson [6, 7] found that by writing the differential equation for tangential displacement in terms of the space variable $\xi = \frac{\alpha}{|\alpha|}(\sec \alpha - 1)$, the coefficients become polynomials. This then permits construction of solutions by the method of Frobenius. However, because the resulting series converges only for $|\alpha| < \pi/3$, their results are limited to relatively shallow catenaries.

Lamotte [8] solved Smith and Thompson's equations for the case where the cable was subject to linear damping. Again the solution was restricted to "shallow" catenaries. All work done up to this time was restricted to two-dimensional or in-plane motion of the cable. Rathje [9] derived the undamped three-dimensional equations of motion for the cable. He showed that the out-of-plane motion equation was independent of the in-plane motion equation in their linearized forms.

I. 2. Present Investigation

In this paper, the works of Smith, Thompson, Lamotte, and Rathje are tied together and results generated for a wider range of equilibrium configurations. The equations of motion are derived in a very general form. Linear damping of the cable from the surrounding medium along with a consideration of the elastic properties of the cable is taken into account when the three-dimensional equations of motion are derived. Using two power series expansions, the in-plane normal mode motion of the cable is described for a greater range of cable geometries than was first achieved by Thompson. The out-of-plane equation is solved for the first time for the normal mode motion and the natural frequency ratios of oscillation are predicted. Using the theory developed for the normal mode motion, a method is developed for determining the displacements and tensions throughout a cable when one end is subjected to a prescribed tangential displacement of known frequency. Maximum cable tensions are presented as a function of the forcing frequency for a variety of different cable geometries.

II. THE EQUATIONS OF MOTION AND THE SOLUTION METHOD
(Refer to Appendix D)

II. 1. Derivation of the Equations of Motion

The derivation of the equations of motion of a suspended cable have been presented by Routh [2], Rohrs [1], and Thompson [6] by direct application of Newton's second law of motion; and by Smith and Thompson [7] through the use of Hamilton's principle for two-dimensional cable systems. The derivation of the equations of motion presented here will be based on application of Hamilton's principle and the use of the Euler-Lagrange equations. This approach leads to the desired equations of motion somewhat more directly than does direct application of Newton's second law of motion.

Consider a perfectly flexible, elastic cable suspended in a viscous medium from two end points. In order to completely describe the general configuration of the cable, it is necessary to define three noncoplanar components of displacement. Let the apex of the cable in its equilibrium configuration be the origin of the coordinate system. The distance s along the cable defines an arbitrary point on the cable, and the angle between a horizontal plane and the tangent line to the cable at any point s, will be denoted as α.

When the cable is displaced from its equilibrium configuration, any point s on the cable may be displaced to a new position. In general, this new position will have components both in the plane and out of the plane of the equilibrium configuration of the cable. The displacement in the plane of the cable will be defined by a component u in the direction tangent to the equilibrium curve and by a component v which is normal to u. The component u is positive in the direction of increasing s and the component v is positive when directed toward the local center of curvature of the equilibrium configuration of the cable. The angle $\bar{\alpha}$ will represent the angle between the tangent to the cable in the displaced configuration and the horizontal plane. The component of displacement perpendicular to the plane of the equilibrium curve will be denoted as w. Positive w will be defined such that displacements u, v, and w would constitute a right handed system. $\bar{\gamma}$ is the angle between the plane of the equilibrium curve and the tangent line of the displaced cable as shown in Figure 2.1.

Reference to Figure 2.1 shows that the rectangular Cartesian coordinates $(\bar{x}, \bar{y}, \bar{z})$ of a displaced point which has coordinates (x, y, z) in the undisturbed configuration are given by:

$$\bar{x}(s,t) = x + u \cos \alpha - v \sin \alpha$$

$$\bar{y}(s,t) = y + u \sin \alpha + v \cos \alpha$$

$$\bar{z}(s,t) = z + w$$

Differentiation of the above equations with respect to s yields:

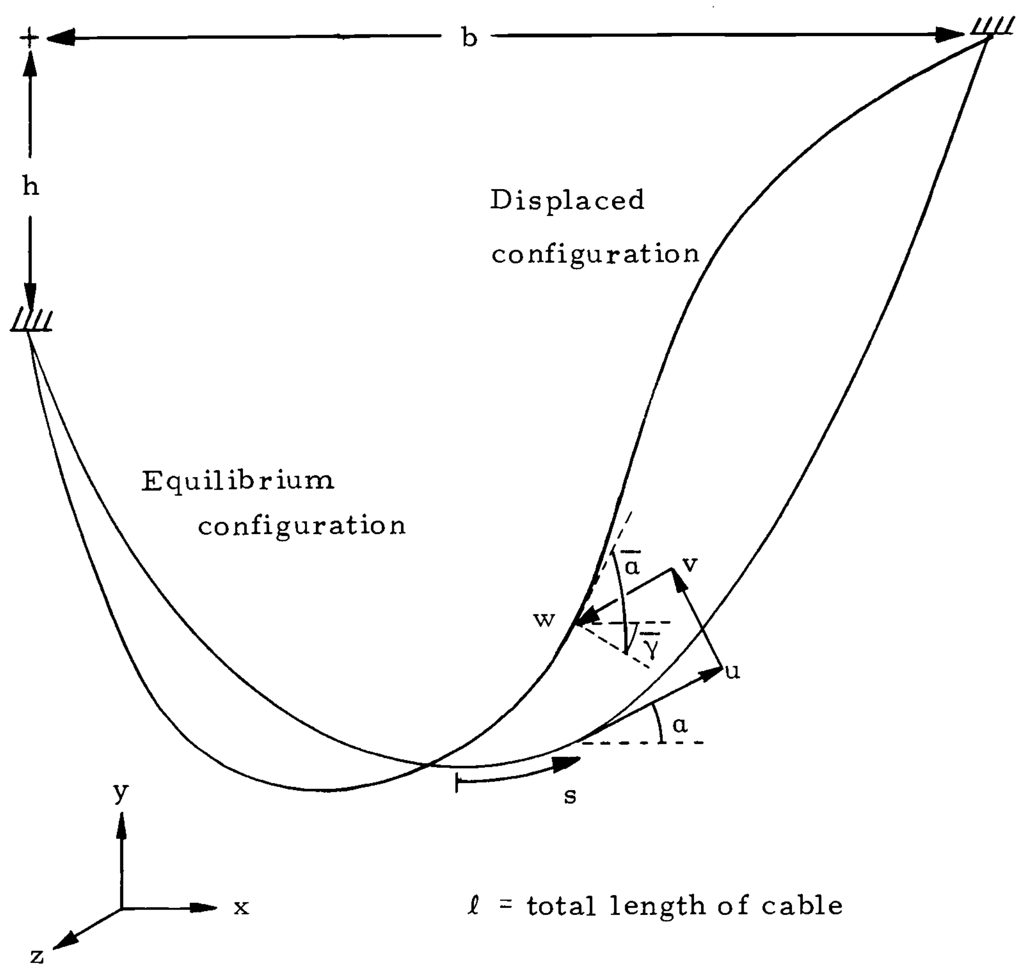

Figure 2. 1. Displaced configuration of hanging cable showing displacement vectors and angles.

$$\frac{d\bar{x}}{ds} = (1+u_s - v\alpha_s)\cos\alpha - (v_s + u\alpha_s)\sin\alpha \tag{2.1a}$$

$$\frac{d\bar{y}}{ds} = (1+u_s - v\alpha_s)\sin\alpha + (v_s + u\alpha_s)\cos\alpha \tag{2.1b}$$

$$\frac{d\bar{z}}{ds} = w_s \tag{2.1c}$$

where the subscript indicates differentiation with respect to arc length s.

Consideration of the geometry of a typical displaced cable element, shown in Figure 2.2, yields:

$$\frac{d\bar{x}}{ds} = \frac{d\bar{s}}{ds}\frac{d\bar{x}}{d\bar{s}} = \frac{d\bar{s}}{ds}\cos\bar{\alpha}\cos\bar{\gamma} \tag{2.2a}$$

$$\frac{d\bar{y}}{ds} = \frac{d\bar{s}}{ds}\frac{d\bar{y}}{d\bar{s}} = \frac{d\bar{s}}{ds}\sin\bar{\alpha} \tag{2.2b}$$

$$\frac{d\bar{z}}{dx} = \frac{d\bar{s}}{ds}\frac{d\bar{z}}{d\bar{s}} = \frac{d\bar{s}}{ds}\cos\bar{\alpha}\sin\bar{\gamma} \tag{2.2c}$$

Equating expressions 2.1 and 2.2 yields:

$$\frac{d\bar{s}}{ds}\cos\bar{\alpha}\cos\bar{\gamma} = (1+u_s - v\alpha_s)\cos\alpha - (v_s + u\alpha_s)\sin\alpha \tag{2.3a}$$

$$\frac{d\bar{s}}{ds}\sin\bar{\alpha} = (1+u_s - v\alpha_s)\sin\alpha + (v_s + u\alpha_s)\cos\alpha \tag{2.3b}$$

$$\frac{d\bar{s}}{ds}\cos\bar{\alpha}\sin\bar{\gamma} = w_s \tag{2.3c}$$

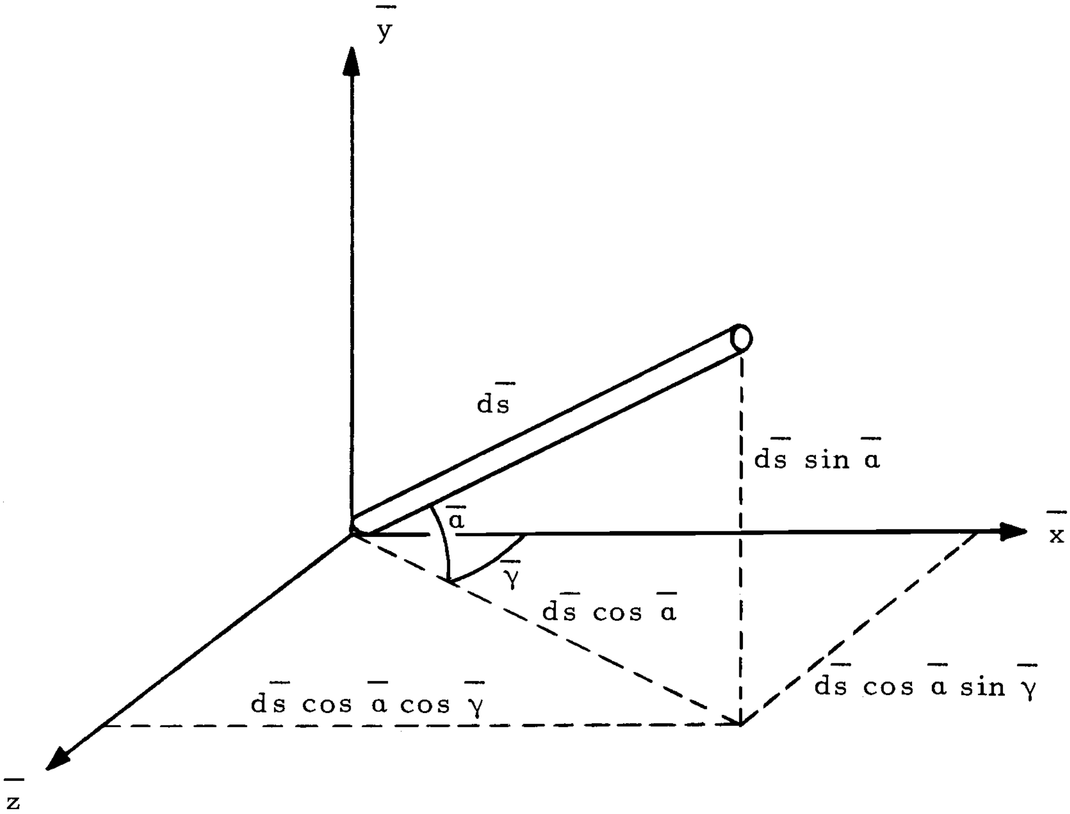

Figure 2.2. Geometry of an element of the displaced cable.

Elimination of $\bar{\alpha}$ and $\bar{\gamma}$ from the above set of equations yields:

$$\frac{d\bar{s}}{ds} = \sqrt{w_s^2 + (1+u_s - v\alpha_s)^2 + (v_s + u\alpha_s)^2} \tag{2.4}$$

The longitudinal strain in a typical element of the cable is defined by:

$$\bar{\epsilon} = \frac{d\bar{s} - ds}{ds}$$

Use of Equation 2.4 in this definition gives the strain-displacement relationship:

$$\bar{\epsilon} = \sqrt{w_s^2 + (1+u_s - v\alpha_s)^2 + (v_s + u\alpha_s)^2} - 1 \tag{2.5}$$

The potential energy density associated with the gravitational field is related to the displacement components by:

$$\frac{\partial V_g}{\partial s} = \frac{\partial V_{0g}}{\partial s} + \rho g(u \sin \alpha + v \cos \alpha) \tag{2.6}$$

The term $\partial V_{0g}/\partial s$ represents the potential energy per unit length of the equilibrium cable relative to some arbitrary, fixed reference system. The term $\rho g(u \sin \alpha + u \cos \alpha)$ is the contribution to the potential energy per unit length of the cable as it moves to a disturbed configuration in the gravitational field.

The kinetic energy per unit length of the cable is given by:

$$\frac{\partial KE}{\partial s} = \frac{\rho}{2}\left(u_t^2 + v_t^2 + w_t^2\right) \tag{2.7}$$

where the subscript indicates differentiation with respect to time t.

For a body moving at low speed through a viscous medium, the drag force on the body may be considered to be proportional to the speed and acts in a direction to oppose the motion of the body. Reference to Figure 2.3 shows that the components of the velocity parallel and perpendicular to the element in the disturbed configuration are related to the components parallel and perpendicular to the element in the equilibrium configuration by:

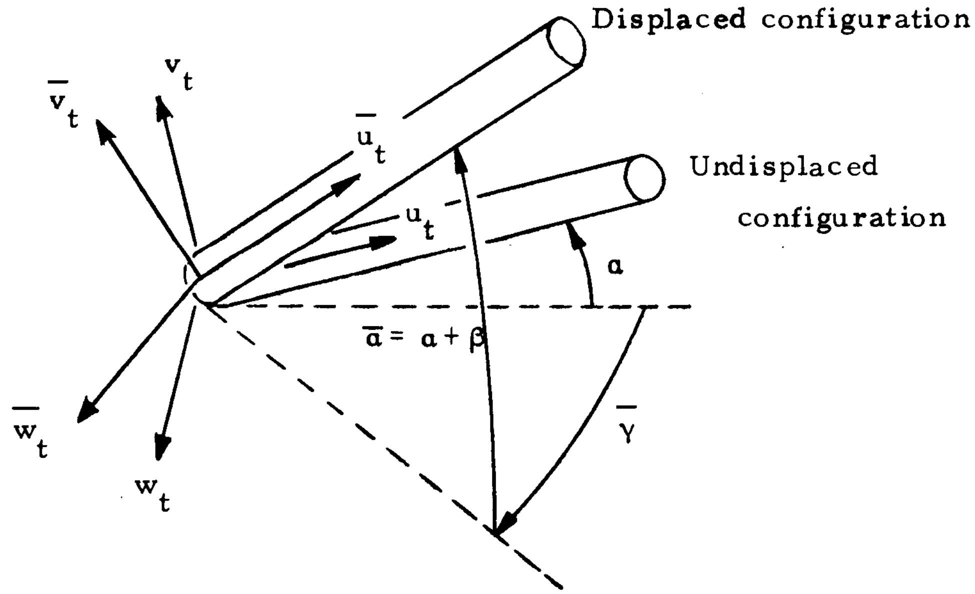

Figure 2.3. Displaced configuration of hanging cable showing velocity vectors.

$$\bar{u}_t = [u_t\cos\beta + v_t\sin\beta][\cos\bar{\gamma} + (1-\cos\bar{\gamma})\sin^2\bar{\alpha}]$$
$$+ [-u_t\sin\beta + v_t\cos\beta][1-\cos\bar{\gamma}]\sin\bar{\alpha}\cos\bar{\alpha} + w_t\sin\bar{\gamma}\cos\bar{\alpha} \qquad (2.8a)$$

$$\bar{v}_t = [u_t\cos\beta + v_t\sin\beta][1-\cos\bar{\gamma}]\sin\bar{\alpha}\cos\bar{\alpha}$$
$$+ [-u_t\sin\beta + v_t\cos\beta][\cos\bar{\gamma} + (1-\cos\bar{\gamma})\cos^2\bar{\alpha}] - w_t\sin\bar{\gamma}\sin\bar{\alpha} \qquad (2.8b)$$

$$\bar{w}_t = -[u_t\cos\beta + v_t\sin\beta][\sin\bar{\gamma}\cos\bar{\alpha}]$$
$$+ [-u_t\sin\beta + v_t\cos\beta][\sin\bar{\gamma}\sin\bar{\alpha}] + w_t\cos\bar{\gamma} \qquad (2.8c)$$

The increment of virtual work performed on the cable due to the viscous drag force as the cable is displaced a small amount $\delta\bar{u}$, $\delta\bar{v}$, and $\delta\bar{w}$ is given by:

$$\delta W_f = -\int_{s_1}^{s_2} (f_1\bar{u}_t\delta\bar{u} + f_2\bar{v}_t\delta\bar{v} + f_2\bar{w}_t\delta\bar{w})ds \qquad (2.9a)$$

where f_1 and f_2 are the linear drag coefficients relating the force on the cable to the velocity of the cable in the tangential and normal directions respectively. The virtual displacements $\delta\bar{u}$, $\delta\bar{v}$, and $\delta\bar{w}$ are given in terms of the equilibrium virtual displacements δu, δv, and δw by the use of Equations 2.8, where the velocity terms u_t, v_t, w_t, $\bar{u}_t$, $\bar{v}_t$, and $\bar{w}_t$ are replaced by the virtual displacements δu, δv, δw, $\delta\bar{u}$, $\delta\bar{v}$, and $\delta\bar{w}$. Equation 2.9a may be reduced to the form:

$$\delta W_f = \int_{s_1}^{s_2} (Q_u \delta u + Q_v \delta v + Q_w \delta w) ds \qquad (2.9b)$$

When the cable is in a displaced configuration, the tension at any position is denoted by $\bar{S}$. The increment of work of the internal forces on the cable as it is strained between two states is given by:

$$\delta W_s = -\int_{s_1}^{s_2} \bar{S} \delta \bar{\epsilon} ds \qquad (2.10)$$

where $\delta\bar{\epsilon}$ is an incremental change in the strain which, by use of Equation 2.5 may be expressed as:

$$\delta\bar{\epsilon} = \frac{(1+u_s - a_s v)(\delta u_s - a_s \delta v) + (v_s + a_s u)(\delta v_s + a_s \delta u) + w_s \delta w_s}{\sqrt{w_s^2 + (1+u_s - v a_s)^2 + (v_s + u a_s)^2}} \qquad (2.11)$$

Substitution of Equation 2.11 into Equation 2.10 yields:

$$\delta W_s = \int_{s_1}^{s_2} \left[\left\{ \frac{-\bar{S} a_s (v_s + a_s u)}{\bar{\epsilon}+1} + \left[\bar{S}\left(\frac{1+u_s - a_s v}{\bar{\epsilon}+1}\right)\right]_s \right\} \delta u \right.$$
$$+ \left\{ \frac{\bar{S} a_s (1+u_s - a_s v)}{\bar{\epsilon}+1} + \left[\bar{S}\left(\frac{v_s + a_s u}{\bar{\epsilon}+1}\right)\right]_s \right\} \delta v$$
$$\left. + \left\{ \left[\frac{\bar{S} w_s}{\bar{\epsilon}+1}\right]_s \right\} \delta w \right] ds$$
$$- \left[\frac{\bar{S}}{\bar{\epsilon}+1} \left\{ (1+u_s - a_s v)\delta u + (v_s + a_s u)\delta v + w_s \delta w \right\} \right]_{s_1}^{s_2} \qquad (2.12)$$

or

$$\delta W_s = \int_{s_1}^{s_2} [R_u \delta u + R_v \delta v + R_w \delta w] ds + \text{boundary terms} \qquad (2.13)$$

In terms of previously defined quantities, the Lagrangian for a cable fixed at end points s_1 and s_2 is given by:

$$L = L(t, s, u, v, w, u_s, v_s, w_s, u_t, v_t, w_t) = KE - V_g$$

$$= \int_{s_1}^{s_2} \left[\frac{\rho}{2}(u_t^2 + v_t^2 + w_t^2) - \frac{\partial V_{0g}}{\partial s} - \rho g(u \sin \alpha + v \cos \alpha) \right] ds \qquad (2.14)$$

Hamilton's principle states that for arbitrary variations of the path between two instants t_1 and t_2, then:

$$\int_{t_1}^{t_2} [\delta L + \delta W_f + \delta W_s] dt = 0 \qquad (2.15)$$

The first variation of the integral, $\int_{t_1}^{t_2} \delta L dt$, is defined as:

$$\int_{t_1}^{t_2} \delta L dt = \int_{t_1}^{t_2} \int_{s_1}^{s_2} \left\{ \frac{\partial L_s}{\partial u} \delta u + \frac{\partial L_s}{\partial v} \delta v + \frac{\partial L_s}{\partial w} \delta w + \frac{\partial L_s}{\partial u_t} \delta u_t + \frac{\partial L_s}{\partial v_t} \delta v_t + \frac{\partial L_s}{\partial w_t} \delta w_t + \frac{\partial L_s}{\partial u_s} \delta u_s + \frac{\partial L_s}{\partial v_s} \delta v_s + \frac{\partial L_s}{\partial w_s} \delta w_s \right\} ds dt$$

After integration by parts, the above equation may be written as:

$$\int_{t_1}^{t_2} \delta L dt = \int_{t_1}^{t_2}\int_{s_1}^{s_2}\left[\left\{\frac{\partial L_s}{\partial u} - \frac{\partial}{\partial t}\left(\frac{\partial L_s}{\partial u_t}\right) - \frac{\partial}{\partial s}\left(\frac{\partial L_s}{\partial u_s}\right)\right\}\delta u\right.$$

$$+\left\{\frac{\partial L_s}{\partial v} - \frac{\partial}{\partial t}\left(\frac{\partial L_s}{\partial v_t}\right) - \frac{\partial}{\partial s}\left(\frac{\partial L_s}{\partial v_s}\right)\right\}\delta v$$

$$\left.+\left\{\frac{\partial L}{\partial w} - \frac{\partial}{\partial t}\left(\frac{\partial L_s}{\partial w_t}\right) - \frac{\partial}{\partial s}\left(\frac{\partial L_s}{\partial w_s}\right)\right\}\delta w\right] ds dt$$

$$+\int_{s_1}^{s_2}\left[\frac{\partial L_s}{\partial u_t}\delta u + \frac{\partial L_s}{\partial v_t}\delta v + \frac{\partial L_s}{\partial w_t}\delta w\right]_{t_1}^{t_2} ds$$

$$+\int_{t_1}^{t_2}\left[\frac{\partial L_s}{\partial u_s}\delta u + \frac{\partial L_s}{\partial v_s}\delta v + \frac{\partial L_s}{\partial w_s}\delta w\right]_{s_1}^{s_2} dt \qquad (2.16)$$

Substitution of Equations 2.9b, 2.13, and 2.16 into Equation 2.15 yields:

$$\int_{t_1}^{t_2}\int_{s_1}^{s_2}\left[\left\{\frac{\partial L_s}{\partial u} - \frac{\partial}{\partial t}\left(\frac{\partial L_s}{\partial u_t}\right) - \frac{\partial}{\partial s}\left(\frac{\partial L_s}{\partial u_s}\right) + Q_u + R_u\right\}\delta u\right.$$

$$+\left\{\frac{\partial L_s}{\partial v} - \frac{\partial}{\partial t}\left(\frac{\partial L_s}{\partial v_t}\right) - \frac{\partial}{\partial s}\left(\frac{\partial L_s}{\partial v_s}\right) + Q_v + R_v\right\}\delta v$$

$$\left.+\left\{\frac{\partial L_s}{\partial w} - \frac{\partial}{\partial t}\left(\frac{\partial L_s}{\partial w_t}\right) - \frac{\partial}{\partial s}\left(\frac{\partial L_s}{\partial w_s}\right) + Q_w + R_w\right\}\delta w\right] ds dt$$

$$+ \text{boundary terms} = 0 \qquad (2.17)$$

The above integral must be valid for arbitrary values of δu, δv, and δw; hence a necessary condition that the integral always equal zero is that:

$$\frac{\partial L_s}{\partial u} - \frac{\partial}{\partial t}\left(\frac{\partial L_s}{\partial u_t}\right) - \frac{\partial}{\partial s}\left(\frac{\partial L_s}{\partial u_s}\right) + Q_u + R_u = 0 \tag{2.18a}$$

$$\frac{\partial L_s}{\partial v} - \frac{\partial}{\partial t}\left(\frac{\partial L_s}{\partial v_t}\right) - \frac{\partial}{\partial s}\left(\frac{\partial L_s}{\partial v_s}\right) + Q_v + R_v = 0 \tag{2.18b}$$

$$\frac{\partial L_s}{\partial w} - \frac{\partial}{\partial t}\left(\frac{\partial L_s}{\partial w_t}\right) - \frac{\partial}{\partial s}\left(\frac{\partial L_s}{\partial w_s}\right) + Q_w + R_w = 0 \tag{2.18c}$$

Equations 2.18 are the Euler-Lagrange equations which are associated with Hamilton's principle and represent the three independent equations of motion of the suspended cable system.

Substitution of L_s into Equations 2.18 yields:

$$\rho u_{tt} + \rho g \sin \alpha + \frac{\bar{S}\alpha_s(v_s + \alpha_s u)}{\bar{\epsilon}+1} - [\bar{S}(\frac{1+u_s - \alpha_s v}{\bar{\epsilon}+1})]_s + Q_u = 0 \tag{2.19a}$$

$$\rho v_{tt} + \rho g \cos \alpha + \frac{\bar{S}\alpha_s(1+u_s - \alpha_s v)}{\bar{\epsilon}+1} - [\bar{S}(\frac{v_s + \alpha_s u}{\bar{\epsilon}+1})]_s + Q_v = 0 \tag{2.19b}$$

$$\rho w_{tt} - [\frac{\bar{S}w_s}{\bar{\epsilon}+1}]_s + Q_w = 0 \tag{2.19c}$$

The above are the general equations of motion for a suspended cable subject to motion in three dimensions. A constituitive equation for the cable giving the tension as a function of the strain $\bar{\epsilon}$ and the strain rate $\bar{\epsilon}_t$,

$$\bar{S} = \bar{S}(\bar{\epsilon}, \bar{\epsilon}_t) \qquad (2.20)$$

will complete the set of equations that model the cable. The five equations, 2.5, 2.19, and 2.20 will govern the variables u, v, w, $\bar{S}$, and $\bar{\epsilon}$.

II.2. The Equilibrium Configuration of the Cable

When the cable assumes its equilibrium configuration, then $u = v = w = \bar{\gamma} = \beta = f_1 = f_2 = 0$, and Equations 2.19 reduce to:

$$\rho g \sin \alpha - S_s = 0 \qquad (2.21a)$$

$$\rho g \cos \alpha - S\alpha_s = 0 \qquad (2.21b)$$

Integration of the above equations yields:

$$S \sin \alpha = \rho g(s - s_0) \qquad (2.22a)$$

$$S = S_0 \sec \alpha \qquad (2.22b)$$

where s_0 and S_0 are constants of integration representing, respectively, the arc length and tension at the apex of the cable. The above two equations may be rearranged in the more useful form, giving the catenary shape and tension ratio as:

$$\tan \alpha = \frac{s - s_0}{a} \tag{2.23}$$

$$\frac{S}{S_0} = \sec \alpha = \sqrt{1 + \tan^2 \alpha} = \sqrt{1 + \left(\frac{s - s_0}{a}\right)^2} \tag{2.24}$$

where:

$$a = \frac{S_0}{\rho g}$$

Since the apex of the cable was designated as the origin of the coordinate system, then the constant s_0 is zero. Substitution of the constant a into Equation 2.23 for $s - s_0$ will yield:

$$\tan \alpha = 1$$

This shows that the tension ratio a is also equal to the arc length from the apex of the curve up to the point where $\alpha = \pi/4$.

The curve defined by Equations 2.23 and 2.24 is often expressed in retangular Cartesian coordinates. This form may be obtained by substituting the relationship:

$$\frac{dx}{ds} = \cos \alpha$$

and Equation 2.23 into the identity,

$$\cos \alpha = \pm \sqrt{\frac{1}{1 + \tan^2 \alpha}}$$

and integrating. The result is:

$$\frac{s-s_0}{a} = \sinh(\frac{x-x_0}{a}) \qquad (2.25a)$$

Similarly, combination of

$$\frac{dy}{ds} = \sin \alpha$$

and Equations 2.22 leads to:

$$\frac{y-y_0}{a} = \sqrt{1 + (\frac{s-s_0}{a})^2} \qquad (2.25b)$$

Combination of Equations 2.25a and 2.25b along with the identity

$$\cosh^2 z - \sinh^2 z = 1$$

yields:

$$\frac{y-y_0}{a} = \cosh(\frac{x-x_0}{a}) \qquad (2.26)$$

The above represents the equation of the common catenary of parameter a in Cartesian coordinates.

II.3. Linearization and Nondimensionalization of the Equations of Motion

From an engineering and scientific standpoint, nondimensionalization of the equations of motion will generalize the equations for use with all possible cable parameters. Linearization of the equations of motion will provide sufficient accuracy in describing the motion of the cable provided that the oscillations of the cable about the equilibrium position are small. Linearization simplifies the equations of motion to an extent that they may be solved using standard techniques for the solution of differential equations. The equations of motion, 2.19, will first be nondimensionalized; then these nondimensionalized equations will be linearized.

The remainder of the analysis will be for a linearly elastic cable, and the special case of this in which the cable is inextensible. For this case, the form of the tension-strain equation, 2.20, is:

$$\bar{S}(s,t) = S(s) + k\bar{\epsilon}(s,t) \tag{2.27}$$

where $S(s)$ is the equilibrium tension and k is the linear proportionality constant relating the tension difference $\bar{S} - S$ to the strain $\bar{\epsilon}$.

The dimensionless variables necessary to describe the equations of motion are:

$$\tau = \frac{k\bar{\epsilon}}{S_0}, \quad (2.28a)$$

$$\kappa = \frac{S_0}{k}, \quad (2.28b)$$

$$\theta = \sqrt{\frac{g}{a}}\, t, \quad (2.28c)$$

$$\sigma = \frac{s - s_0}{a}, \quad (2.28d)$$

$$\eta_1 = \frac{f_1}{\rho}\sqrt{\frac{a}{g}}, \quad (2.28e)$$

$$\eta_2 = \frac{f_2}{\rho}\sqrt{\frac{a}{g}}. \quad (2.28f)$$

τ is a nondimensional tension difference, κ is a nondimensional elastic flexibility coefficient, θ is a nondimensional time, σ is a nondimensional arc length, and η_1 and η_2 are nondimensional linear drag coefficients. Substitution of the equilibrium equations, 2.21a and 2.21b, the tension-strain equation, 2.27; and the dimensionless variables given above, into the equations of motion, 2.19; will yield the nondimensional equations of motion.

Linearization of the nondimensional equations of motion is accomplished by noting that $\tau \ll 1$, $\frac{u}{a} \ll 1$, $\frac{v}{a} \ll 1$, and $\frac{w}{a} \ll 1$ and then neglecting products of any of the above terms. Linearization of the nondimensional equations of motion yields:

$$\frac{\ddot{u}}{a} + \frac{1}{a}\left[\frac{v'}{(1+\sigma^2)^{1/2}} + \frac{u}{(1+\sigma^2)^{3/2}}\right] - \tau' + \eta_1 \frac{\dot{u}}{a} = 0 \tag{2.29a}$$

$$\frac{\ddot{v}}{a} - \frac{\tau}{1+\sigma^2} - \frac{v''}{a}(1+\sigma^2)^{1/2} - \frac{v'}{a}\frac{\sigma}{(1+\sigma^2)^{1/2}} + \frac{u}{a}\frac{\sigma}{(1+\sigma^2)^{3/2}}$$

$$- \frac{u'}{a}\frac{1}{(1+\sigma^2)^{1/2}} + \eta_2 \frac{\dot{v}}{a} = 0 \tag{2.29b}$$

$$\frac{\ddot{w}}{a} - \frac{w''}{a}(1+\sigma^2)^{1/2} - \frac{w'}{a}\frac{\sigma}{(1+\sigma^2)^{1/2}} + \eta_2 \frac{\dot{w}}{a} = 0 \tag{2.29c}$$

The slash, $(\ )'$, represents differentiation with respect to σ; and the dot, $(\dot{\ })$, represents differentiation with respect to θ. The above three equations contain four unknowns; u, v, w, and τ. Equation 2.27 is the necessary fourth independent equation which, along with Equations 2.29 will govern the motion of the perfectly flexible, linearly elastic cable suspended in a viscous medium in a gravitational field. Nondimensionalization and linearization of Equation 2.27 yields:

$$\kappa\tau = \frac{1}{a}\left(u' - \frac{v}{1+\sigma^2}\right) \tag{2.30}$$

Solution of Equations 2.29 and 2.30 will yield the displacements and tension as a function of position along the cable and time. It is of interest to note that the linearized equation governing the out-of-plane motion of the cable, Equation 2.29c, is independent of Equations 2.29a and 2.29b governing the in-plane motion of the cable. This is not the case for the nonlinear equations of motion. Examination of the nonlinear equations of motion, Equations 2.19, shows that the equations are coupled through the strain $\bar{\epsilon}$ and the damping forces Q_u, Q_v, and Q_w.

II.4. Solution of the Equations of Motion

II.4.1. Normal Mode Motion and the Variable Change

The linearized equations describing the motion of the cable may be expected to yield solutions representing normal mode motion. A solution of the form:

$$\begin{Bmatrix} \tau(\sigma, \theta) \\ \frac{1}{a} u(\sigma, \theta) \\ \frac{1}{a} v(\sigma, \theta) \end{Bmatrix} = \begin{Bmatrix} T(\sigma) \\ U(\sigma) \\ V(\sigma) \end{Bmatrix} \exp(\lambda\theta) \tag{2.31}$$

will satisfy Equations 2.29a, 2.29b, and 2.30; whereas a solution of the form:

$$w(\sigma, \theta) = W(\sigma) \exp(\lambda\theta) \tag{2.32}$$

will serve to satisfy Equation 2.29c. The characteristic root $\lambda = \delta + i\omega$ contains the oscillation frequency ratio ω as its imaginary part.

Substitution of Equation 2.31 into Equations 2.29a, 2.29b, and 2.30 leads to:

$$\lambda^2 U + \frac{V'}{(1+\sigma^2)^{1/2}} + \frac{U}{(1+\sigma^2)^{3/2}} - T' + \lambda\eta_1 U = 0 \qquad (2.33a)$$

$$\lambda^2 V - \frac{T}{1+\sigma^2} - \frac{V'\sigma}{(1+\sigma^2)^{1/2}} + \frac{U\sigma}{(1+\sigma^2)^{3/2}} - V''(1+\sigma^2)^{1/2}$$

$$- \frac{U'}{(1+\sigma^2)^{1/2}} + \lambda\eta_2 V = 0 \qquad (2.33b)$$

$$\kappa T = U' - \frac{V}{1+\sigma^2} \qquad (2.34)$$

II.4.1.a. Undamped Motion of an Inextensible Cable. It would be of interest to initially study the motion of an undamped, inextensible cable; for this purpose the linear damping coefficients η_1 and η_2, and the linear flexibility coefficient κ, will be set equal to zero. Also, since undamped oscillations are expected, let $\lambda = i\omega$. For the undamped, inextensible cable, T and V may be eliminated from the above set of equations to yield a single fourth order equation governing the tangential displacement of the cable U:

$$(1+\sigma^2)^{5/2}U'''' + 10\sigma(1+\sigma^2)^{3/2}U'''$$

$$+ [3(3+8\sigma^2)(1+\sigma^2)^{1/2}+\omega^2(1+\sigma^2)^2]U''$$

$$+ 4\sigma(1+\sigma^2)^{1/2}(3+\omega^2(1+\sigma^2)^{1/2})U' + \omega^2 U = 0 \qquad (2.35)$$

The normal component of displacement V, and the dynamic contribution to the cable tension T may be expressed in terms of the tangential displacement U, as follows:

$$V = (1+\sigma^2)U' \tag{2.36}$$

$$T = \sigma(1+\sigma^2)^{-1/2}U - [(3+4\sigma^2)(1+\sigma^2)^{1/2}+\omega^2(1+\sigma^2)^2]U' - [(1+\sigma^2)^{5/2}U'']' \tag{2.37}$$

Substitution of Equation 2.32 into Equation 2.29c will yield a single independent equation governing the out-of-plane displacement of the cable as follows:

$$(1+\sigma^2)^{1/2}W'' + \sigma(1+\alpha^2)^{-1/2}W' + \omega^2 W = 0 \tag{2.38}$$

A well known method for the solution of linear differential equations having polynomial coefficients is to develop a power series representation for the dependent variable by substituting the series into the given equation. The coefficients in the series may then be evaluated by matching the coefficients of like powers of the independent variable. However, due to the irrational nature of the coefficients in Equations 2.35 and 2.38, it is necessary to transform the two equations into a form in which the coefficients are expressible as integer powers of the independent variable. A change of independent variable which reduces Equations 2.35 and 2.38 to ones with polynomial coefficients is:

$$\xi = \frac{\sigma}{|\sigma|}(\sqrt{1+\sigma^2} - 1) = \frac{\alpha}{|\alpha|}(\sec \alpha - 1) \qquad (2.39)$$

It is necessary to include the factor $\frac{\sigma}{|\sigma|}$ in the above equation to assure that there is a one-to-one correspondence between ξ and σ in the region where σ is negative. The inverse transformation is:

$$\sigma = \frac{\xi}{|\xi|}\sqrt{(1+|\xi|)^2 - 1} \qquad (2.40)$$

Transformation of Equations 2.35 and 2.38 may be accomplished with the aid of the chain rule along with the following four derivatives:

$$\frac{d\xi}{d\sigma} = \frac{|\sigma|}{\sqrt{1+\sigma^2}}$$

$$\frac{d^2\xi}{d\sigma^2} = \frac{\sigma}{|\sigma|(1+\sigma^2)^{3/2}}$$

$$\frac{d^3\xi}{d\sigma^3} = -\frac{3|\sigma|}{(1+\sigma^2)^{5/2}} + 2\delta(\sigma)$$

$$\frac{d^4\xi}{d\sigma^4} = -\frac{3|\sigma|(1-4\sigma^2)}{|\sigma|(1+\sigma^2)^{7/2}} + 2\frac{d\delta(\sigma)}{d\sigma}$$

where $\delta(\sigma)$ is the Dirac delta function.

Substitution of the change of variable and the necessary derivatives of the variable change into Equation 2.35 yields the differential equation with polynomial coefficients governing the tangential displacement of the cable:

$$\xi^2(1+|\xi|)(2+|\xi|)^2 \frac{d^4U}{d\xi^4} + 2|\xi|(2+|\xi|)(3+10|\xi|+5\xi^2) \frac{\xi}{|\xi|} \frac{d^3U}{d|\xi|^3}$$

$$+ \Big[3(1+|\xi|)(1+16|\xi|+8\xi^2)+\omega^2|\xi|(1+|\xi|)^2(2+|\xi|)$$

$$+ \frac{8}{\sqrt{2}}(1+|\xi|)^4 \sqrt{2|\xi|+\xi^2}\, \delta\Big(\frac{\xi}{\sqrt{|\xi|}}\Big)\Big] \frac{d^2U}{d\xi^2}$$

$$+ \Big[6(1+4|\xi|+2\xi^2)+\omega^2(1+|\xi|)(1+8|\xi|+4\xi^2)$$

$$+ \frac{20}{\sqrt{2}}(1+|\xi|)^3 \sqrt{2|\xi|+\xi^2}\, \delta\Big(\frac{\xi}{\sqrt{|\xi|}}\Big)$$

$$+ (1+|\xi|)^4 \frac{\xi}{|\xi|} \delta'\Big(\frac{\xi}{\sqrt{|\xi|}}\Big)\Big] \frac{dU}{d|\xi|} - \omega^2 U = 0 \tag{2.41}$$

For positive values of ξ, then the above equation reduces to:

$$\begin{aligned}\xi^2(1+\xi)(2+\xi)^2 \frac{d^4y}{d\xi^4} &+ 2\xi(2+\xi)(3+10\xi+5\xi^2) \frac{d^3y}{d\xi^3} \\ &+ [3(1+\xi)(1+16\xi+8\xi^2)+\omega^2\xi(1+\xi)^2(2+\xi)] \frac{d^2y}{d\xi^2} \\ &+ [6(1+4\xi+2\xi^2)+\omega^2(1+\xi)(1+8\xi+4\xi^2)] \frac{dy}{d\xi} - \omega^2 y = 0\end{aligned} \tag{2.42}$$

Substitution of the change of variable along with the necessary derivatives into Equation 2.38 yields the following equation with polynomial coefficients governing the out-of-plane motion of the cable:

$$(2|\xi|+\xi^2) \frac{d^2W}{d\xi^2} + \frac{\xi}{|\xi|}(1+|\xi|) \frac{dW}{d|\xi|} + (1+|\xi|)\omega^2 W = 0 \tag{2.43}$$

For positive values of ξ, Equation 2.43 reduces to:

$$(2\xi+\xi^2)\frac{d^2p}{d\xi^2} + (1+\xi)\frac{dp}{d\xi} + (1+\xi)\omega^2 p = 0 \qquad (2.44)$$

II.4.2. In-Plane Normal Mode Motions

Instead of pursuing a parallel development for the solution of the in-plane and out-of-plane motion equations simultaneously, each case will be analyzed separately. Equation 2.42 describing the in-plane motion of the cable is an ordinary, linear, fourth order differential equation with polynomial coefficients and hence lends itself to a power series solution of the form developed by Frobenius. The singular points of Equation 2.42 are given by the roots of the coefficient of $d^4y/d\xi^4$. Examination of Equation 2.42 will reveal that the singular points occur at $\xi = -2, -1,$ and 0. A series solution expansion about any of the singular points will yield at least one solution which will have a radius of convergence equal to the distance to the next nearest singular point. Regardless of the singular point about which the series solution is written, the radius of convergence will always equal 1.

A power series expansion about the singular point $\xi = 0$ will have the form:

$$y(\xi) = \sum_{k=0}^{\infty} a_k \xi^{k+r}, \qquad (a_0 \neq 0) \qquad (2.45a)$$

The first four derivatives of $y(\xi)$ are given by:

$$\frac{dy}{d\xi} = \sum_{k=0}^{\infty} (k+r)a_k \xi^{k+r-1} \tag{2.45b}$$

$$\frac{d^2y}{d\xi^2} = \sum_{k=0}^{\infty} (k+r)(k+r-1)a_k \xi^{k+r-2} \tag{2.45c}$$

$$\frac{d^3y}{d\xi^3} = \sum_{k=0}^{\infty} (k+r)(k+r-1)(k+r-2)a_k \xi^{k+r-3} \tag{2.45d}$$

$$\frac{d^4y}{d\xi^4} = \sum_{k=0}^{\infty} (k+r)(k+r-1)(k+r-2)(k+r-3)a_k \xi^{k+r-4} \tag{2.45e}$$

Substitution of Equations 2.45 into Equation 2.42 and equating the coefficient of the lowest power of ξ to zero yields the following indicial equation:

$$r(r-1)(2r-1)(2r-3)a_0 = 0 \tag{2.46}$$

Solution of the above equation yields the four characteristic exponents $r = 0,\ 1/2,\ 1,\ 3/2$. In general, if any pair of the characteristic exponents differ by an integer, then the solutions obtained may not be linearly independent. However analysis as outlined in Section 16.3 of Ince [10] indicates that the four series solutions obtained using the above characteristic exponents will yield independent solutions to

Equation 2.42. This will be verified shortly when the actual solutions are determined.

The complete solution to Equation 2.42 may be expressed as a linear combination of the four series solutions, each corresponding to one of the four roots obtained above:

$$U(\xi) = \sum_{j=0}^{3} C_j y_j(\xi) \tag{2.47}$$

where

$$y_j = y_{2r} = \sum_{k=0}^{\infty} a_k(r)\xi^{k+r} \tag{2.48}$$

The four arbitrary constants $a_0(r)$ will be assigned values of:

$$a_0(r) = 1, \qquad (r = 0, 1/2, 1, 3/2) \tag{2.49}$$

The remaining coefficients, a_k $(k = 1 \rightarrow \infty)$, may be evaluated using the following recursion relationship which arises when Equations 2.45 are substituted into Equation 2.42:

$$a_k = -\frac{2(k+r-2)(k+r-3)(4k+4r+7)+(51+2\omega^2)(k+r)-3(32+\omega^2)}{(k+r)[4(k+r)(k+r-2)+3]}\,a_{k-1}$$
$$-\frac{(k+r-2)[5(k+r-3)(k+r-4)(k+r+3)+(72+5\omega^2)(k+r)-6(32+\omega^2)]-\omega^2}{(k+r)(k+r-1)[4(k+r)(k+r-2)+3]}\,a_{k-2}$$
$$-\frac{(k+r-3)[(k+r-4)(k+r-5)(k+r+4)+4(6+\omega^2)(k+r)-4(21+\omega^2)]}{(k+r)(k+r-1)[4(k+r)(k+r-2)+3]}\,a_{k-3}$$

$$- \frac{\omega^2 (k+r-4)}{(k+r)[4(k+r)(k+r-2)+3]} a_{k-4} \tag{2.50}$$

All coefficients in the above equation with negative subscripts are zero. Substitution of Equations 2.49 and 2.50 into Equation 2.48 and finally substitution of Equation 2.48 into 2.47 will yield the solution for the tangential displacement of the cable for positive values of ξ.

Substitution of

$$\xi = \sqrt{1+\sigma^2} - 1 = \frac{\sigma^2}{2} \left(1 - \frac{\sigma^2}{4} + \frac{\sigma^4}{8} - \ldots \right)$$

into the series given by Equations 2.48, 2.49, and 2.50 yields the following lead terms in the four y_j series:

$$y_0 = 1 - \frac{1+\omega^2}{2} \sigma^2 + \ldots \tag{2.51a}$$

$$y_1 = \frac{1}{\sqrt{2}} - \frac{5+2\omega^2}{12\sqrt{2}} \sigma^3 + \ldots \tag{2.51b}$$

$$y_2 = \frac{\sigma^2}{2} - \frac{6+\omega^2}{24} \sigma^4 + \ldots \tag{2.51c}$$

$$y_3 = \frac{\sigma^3}{2\sqrt{2}} - \frac{23+2\omega^2}{40\sqrt{2}} \sigma^5 + \ldots \tag{2.51d}$$

Equation 2.47 is a solution to Equation 2.41 for positive ξ. For negative values of ξ, then a general solution to Equation 2.41 would be of the form:

$$U(-\xi) = \sum_{j=0}^{3} D_j y_j(-\xi) \tag{2.52}$$

At the origin, there must be continuity of the tangential and normal displacements and of the tension; from these conditions it is possible to determine the constants D_j in terms of the constants C_j. A glance at Equations 2.36 and 2.37 will reveal that the normal displacement along with the tension are proportional to the zeroth through the third derivatives of the tangential displacement. Hence it would be expected that if the zeroth through the third derivatives are continuous at the origin, then the displacements and tension will also be continuous at the origin; this requirement may be expressed as follows:

$$U(\sigma) = U(-\sigma) \qquad (\sigma = 0) \tag{2.53a}$$

$$U'(\sigma) = U'(-\sigma) \qquad (\sigma = 0) \tag{2.53b}$$

$$U''(\sigma) = U''(-\sigma) \qquad (\sigma = 0) \tag{2.53c}$$

$$U'''(\sigma) = U'''(-\sigma) \qquad (\sigma = 0) \tag{2.53d}$$

Substitution of Equations 2.47 and 2.52 (written in terms of the variable σ instead of ξ) into Equations 2.53 yields:

$$\begin{bmatrix} y_0(\sigma) & y_1(\sigma) & y_2(\sigma) & y_3(\sigma) \\ y_0'(\sigma) & y_1'(\sigma) & y_2'(\sigma) & y_3'(\sigma) \\ y_0''(\sigma) & y_1''(\sigma) & y_2''(\sigma) & y_3''(\sigma) \\ y_0'''(\sigma) & y_1'''(\sigma) & y_2'''(\sigma) & y_3'''(\sigma) \end{bmatrix} \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{Bmatrix}$$

$$= \begin{bmatrix} y_0(-\sigma) & y_1(-\sigma) & y_2(-\sigma) & y_3(-\sigma) \\ -y_0'(-\sigma) & -y_1'(-\sigma) & -y_2'(-\sigma) & -y_3'(-\sigma) \\ y_0''(-\sigma) & y_1''(-\sigma) & y_2''(-\sigma) & y_3''(-\sigma) \\ -y_0'''(-\sigma) & -y_1'''(-\sigma) & -y_2'''(-\sigma) & -y_3'''(-\sigma) \end{bmatrix} \begin{Bmatrix} D_0 \\ D_1 \\ D_2 \\ D_3 \end{Bmatrix} \qquad (2.54)$$

Differentiation and substitution of Equations 2.51 into the above set of equations and subsequent evaluation at the origin $(\sigma = 0)$ yields the following relations between the constants C_j and D_j:

$$D_0 = C_0, \quad D_1 = -C_1, \quad D_2 = C_2, \quad D_3 = -C_3$$

Now the solution for the tangential displacement, valid for both positive and negative ξ, may be expressed as:

$$U(\xi) = \sum_{j=0}^{3} C_j u_j(\xi) \qquad (2.55)$$

where

$$u_j(\xi) = \left(\frac{\xi}{|\xi|}\right)^j y_j(|\xi|) \qquad (2.56)$$

Equation 2.56 is valid for values of ξ in the range, $-1 < \xi < 1$. However as ξ approaches ± 1 then it is necessary to increase the number of terms in the series expansion to assure that convergence of the series is attained. Hence it is impractical to assume that Equation 2.56 is applicable for ξ close to ± 1. A glance at Equation 2.39 will reveal that for $\xi = \pm 1$ then the angle α will have a value of $\pi/3$. For many applications, it may be desirable to be able to describe the cable displacements and tensions for values of α greater than $\pi/3$. Hence it will be necessary to write another series solution to Equation 2.41 which will converge for values of $|\xi| \geq 1$.

In order to extend the range for which the u_j may be practically calculated, it will be desirable to make the following change of variable: $\chi = \xi - 2$. Substitution of this change of variable into Equation 2.42 yields:

$$(\chi+2)^2(\chi+3)(\chi+4)^2 \frac{d^4z}{d\chi^4} + 2(\chi+2)(\chi+4)[23+10\chi+5(\chi^2+4\chi+4)] \frac{d^3z}{d\chi^3}$$

$$+ [3(\chi+3)\{33+16\chi+8(\chi^2+4\chi+4)\} + \omega^2(\chi+2)(\chi+3)^2(\chi+4)] \frac{d^2z}{d\chi^3} \qquad (2.57)$$

$$+ [6\{9+4\chi+2(\chi^2+4\chi+4)\} + \omega^2(\chi+3)\{17+8\chi+4(\chi^2+4\chi+4)\}] \frac{dz}{d\chi} - \omega^2 z = 0$$

Equation 2.57 is a linear, fourth order differential equation with polynomial coefficients and as such lends itself to the use of a power series type solution. A power series expansion about the ordinary point $\chi = 0$ $(\xi = 2)$ will be of the form:

$$z(\chi) = \sum_{k=0}^{\infty} b_k \chi^{k+r} \tag{2.58}$$

The first four derivatives of $z(\chi)$ will have the same form as do Equations 2.45. Substitution of Equation 2.58 and its first four derivatives into Equation 2.57 and then equating the coefficient of the lowest power of χ to zero yields the following indicial equation:

$$192(r-3)(r-2)(r-1)(r)b_0 = 0 \tag{2.59}$$

Solution of the above equation yields the four characteristic exponents; $r = 0, 1, 2, 3$. Since the power series expansion is about an ordinary point rather than a singular point, then the solutions obtained will be linearly independent even though the characteristic exponents differ by integer values.

The complete solution to Equation 2.57 will be expressed as a linear combination of the four series solutions, each corresponding to one of the four independent indicial roots:

$$U(\chi) = \sum_{j=0}^{3} A_j z_j(\chi) \tag{2.60}$$

where,

$$z_j = z_r = \sum_{k=0}^{\infty} b_k(r)\chi^{k+r} \qquad (2.61)$$

The four arbitrary constants $b_0(r)$ will be assigned values of:

$$b_0(r) = 1, \qquad (r = 0, 1, 2, 3) \qquad (2.62)$$

The remaining coefficients, b_k $(k = 1 \to \infty)$, may be evaluated using the following recursion relationship which arises when Equation 2.58 is substituted into Equation 2.57:

$$\begin{aligned}
b_k = &- \frac{[352(k+r-4)+688]}{192(k+r)} b_{k-1} \\
&- \frac{[252(k+r-5)(k+r-4)+996(k+r-4)+585+72\omega^2]}{192(k+r-1)(k+r)} b_{k-2} \\
&- \frac{\begin{array}{l}[88(k+r-6)(k+r-5)(k+r-4)+526(k+r-5)(k+r-4) \\ \quad + (627+102\omega^2)(k+r-4)+102+99\omega^2]\end{array}}{192(k+r-2)(k+r-1)(k+r)} b_{k-3} \\
&- \frac{\begin{array}{l}[15(k+r-7)(k+r-6)(k+r-5)(k+r-4)+120(k+r-6)(k+r-5)(k+r-4) \\ \quad + (216+53\omega^2)(k+r-5)(k+r-4)+(72+105\omega^2)(k+r-4)-\omega^2]\end{array}}{192(k+r-3)(k+r-2)(k+r-1)(k+r)} b_{k-4} \\
&- \frac{\begin{array}{l}[(k+r-8)(k+r-7)(k+r-6)(k+r-5)+10(k+r-7)(k+r-6)(k+r-5) \\ \quad + (24+12\omega^2)(k+r-6)(k+r-5)+(12+36\omega^2)(k+r-5)]\end{array}}{192(k+r-3)(k+r-2)(k+r-1)(k+r)} b_{k-5} \\
&- \frac{[\omega^2(k+r-7)(k+r-6)+4\omega^2(k+r-6)]}{192(k+r-3)(k+r-2)(k+r-1)(k+r)} b_{k-6} \qquad (2.63)
\end{aligned}$$

All coefficients in the above equation with negative subscripts are zero. Substitution of Equations 2.62 and 2.63 into Equation 2.61, then subsequent substitution of Equation 2.61 into Equation 2.60 will yield the solution which describes the tangential displacement of the cable for positive χ.

The series expansion 2.61 for $U(\chi)$ will have a radius of convergence from the ordinary point $\chi = 0$ to the next nearest singular point, i.e., $\chi = -2$. In terms of the variable ξ, Equation 2.61 will have a region of convergence of $0 < \xi < 4$. The value of α which may be attained for $\xi = 4$ will be 78°.

It will be desirable to express the constants A_j (from Equation 2.60) in terms of the constants C_j (from Equation 2.47). This may be accomplished by matching the series solutions for $U(\xi)$ and $U(\chi)$ at some common point where both solutions converge. As was previously done with the series matching at the origin, matching of the zeroth through the third derivatives of $U(\xi)$ and $U(\chi)$ at some mutual point would assure that both displacements and tension are continuous at the matching point. Let the point at which the two series are to be matched be denoted as ξ_{RM}; then it follows that:

$$\begin{bmatrix} y_0(\xi_{RM}) & y_1(\xi_{RM}) & y_2(\xi_{RM}) & y_3(\xi_{RM}) \\ y_0'(\xi_{RM}) & y_1'(\xi_{RM}) & y_2'(\xi_{RM}) & y_3'(\xi_{RM}) \\ y_0''(\xi_{RM}) & y_1''(\xi_{RM}) & y_2''(\xi_{RM}) & y_3''(\xi_{RM}) \\ y_0'''(\xi_{RM}) & y_1'''(\xi_{RM}) & y_2'''(\xi_{RM}) & y_3'''(\xi_{RM}) \end{bmatrix} \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{Bmatrix}$$

$$= \begin{bmatrix} z_0(\xi_{RM}-2) & z_1(\xi_{RM}-2) & z_2(\xi_{RM}-2) & z_3(\xi_{RM}-2) \\ z_0'(\xi_{RM}-2) & z_1'(\xi_{RM}-2) & z_2'(\xi_{RM}-2) & z_3'(\xi_{RM}-2) \\ z_0''(\xi_{RM}-2) & z_1''(\xi_{RM}-2) & z_2''(\xi_{RM}-2) & z_3''(\xi_{RM}-2) \\ z_0'''(\xi_{RM}-2) & z_1'''(\xi_{RM}-2) & z_2'''(\xi_{RM}-2) & z_3'''(\xi_{RM}-2) \end{bmatrix} \begin{Bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{Bmatrix} \tag{2.64}$$

or in shorter notation:

$$[YR(\xi_{RM})]\{C\} = [ZR(\xi_{RM}-2)]\{A\} \tag{2.65}$$

Solution of the above matrix equation for $\{A\}$ yields:

$$\{A\} = [ZR(\xi_{RM}-2)]^{-1}[YR(\xi_{RM})]\{C\} \tag{2.66}$$

Substitution of $\{A\}$, Equation 2.66, into Equation 2.60 will yield an equation describing the tangential displacement of the cable for values of ξ in the range $0 < \xi < 4$ where the $\{A\}$ is given in terms of the $\{C\}$.

Through use of Equations 2.55 and 2.60, the tangential displacement can be determined for all values of ξ in the range of $-1 < \xi < 4$. Now it would be desirable to further extend the range of

ξ for which the tangential displacement may be determined, especially for $\xi < -1$. Due to the inherent symmetry of the cable, replacement of ξ with $-\xi$ in the defining equation for χ, i.e., $\chi = \xi - 2$, would provide a variable change that should extend the solution range for the tangential displacement of the cable. Hence we let:

$$\psi = -\xi - 2 \quad (\text{for} \quad \xi < 0)$$

or

$$\psi = |\xi| - 2.$$

Substitution of the above change of variable into Equation 2.42 yields an equation identical to Equation 2.57 except that the independent variable is ψ instead of χ. Solution of the equation for $U(\psi)$ is identical to solution of Equation 2.57 except that the independent variable is ψ, hence

$$U(\psi) = \sum_{j=0}^{3} B_j z_j(\psi) \tag{2.67}$$

where,

$$z_j = z_r = \sum_{k=0}^{\infty} b_k(r)\psi^{k+r} \tag{2.68}$$

The constants, b_j, are given by Equations 2.62 and 2.63.

As was done with positive ξ and χ, it will be desirable to match the solutions for negative ξ and ψ at some mutual point and hence determine the B_j in terms of the C_j. Let the point at which the series solutions for $U(-\xi)$ and $U(\psi)$ are to be matched be denoted as ξ_{LM}. Again, matching of the zeroth through the third derivatives at ξ_{LM} would assure that displacements and tension will be continuous at ξ_{LM} and hence it will be possible to solve for the B_j in terms of the C_j. It follows that:

$$\begin{bmatrix} y_0(\xi_{LM}) & -y_1(\xi_{LM}) & y_2(\xi_{LM}) & -y_3(\xi_{LM}) \\ -y_0'(\xi_{LM}) & y_1'(\xi_{LM}) & -y_2'(\xi_{LM}) & y_3'(\xi_{LM}) \\ y_0''(\xi_{LM}) & -y_1''(\xi_{LM}) & y_2''(\xi_{LM}) & -y_3''(\xi_{LM}) \\ -y_0'''(\xi_{LM}) & y_1'''(\xi_{LM}) & -y_2'''(\xi_{LM}) & y_3'''(\xi_{LM}) \end{bmatrix} \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{Bmatrix} \tag{2.69}$$

$$= \begin{bmatrix} z_0(-\xi_{LM}-2) & z_1(-\xi_{LM}-2) & z_2(-\xi_{LM}-2) & z_3(-\xi_{LM}-2) \\ -z_0'(-\xi_{LM}-2) & -z_1'(-\xi_{LM}-2) & -z_2'(-\xi_{LM}-2) & -z_3'(-\xi_{LM}-2) \\ z_0''(-\xi_{LM}-2) & z_1''(-\xi_{LM}-2) & z_2''(-\xi_{LM}-2) & z_3''(-\xi_{LM}-2) \\ -z_0'''(-\xi_{LM}-2) & -z_1'''(-\xi_{LM}-2) & -z_2'''(-\xi_{LM}-2) & -z_3'''(-\xi_{LM}-2) \end{bmatrix} \begin{Bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{Bmatrix}$$

or in shorter notation:

$$[YL(\xi_{LM})]\{C\} = [ZL(-\xi_{LM}-2)]\{B\} \tag{2.70}$$

Solution of the above matrix equation for $\{B\}$ yields:

$$\{B\} = [ZL(-\xi_{LM}-2)]^{-1}[YL(\xi_{LM})]\{C\} \tag{2.71}$$

Substitution of $\{B\}$ into Equation 2.67 will yield the equation describing the tangential displacement of the cable for values of ξ in the range $-4 < \xi < 0$, where the $\{B\}$ is given in terms of the $\{C\}$. Through the use of Equations 2.55, 2.60, and 2.67, the tangential displacement of the cable may now be determined for any value of ξ in the range $-4 < \xi < 4$.

Consider the cable to be fixed at two points ξ_1 and ξ_2. At these two points of fixity, the tangential and normal displacements of the cable will be zero. The tangential displacement will be given by either Equation 2.55, 2.60, or 2.67 and the normal displacement will be given by Equation 2.36 or written in terms of the derivative of U with respect to ξ by:

$$V = |\sigma| \sqrt{1+\sigma^2}\, \frac{dU}{d\xi} \tag{2.72}$$

Hence the boundary conditions are:

$$U(\xi_1) = \frac{dU}{d\xi}\Big|_{\xi_1} = U(\xi_2) = \frac{dU}{d\xi}\Big|_{\xi_2} = 0 \tag{2.73}$$

Written in matrix notation, Equation 2.73 will have the general form:

$$\begin{bmatrix} u_0(\xi_1) & u_1(\xi_1) & u_2(\xi_1) & u_3(\xi_1) \\ u_0'(\xi_1) & u_1'(\xi_1) & u_2'(\xi_1) & u_3'(\xi_1) \\ u_0(\xi_2) & u_1(\xi_2) & u_2(\xi_2) & u_3(\xi_2) \\ u_0'(\xi_2) & u_1'(\xi_2) & u_2'(\xi_2) & u_3'(\xi_2) \end{bmatrix} \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{2.74}$$

where the prime $(\ )'$ denotes differentiation with respect to ξ and

$$\lfloor u(\xi) \rfloor = \lfloor y(|\xi|) \rfloor \left(\frac{\xi}{|\xi|}\right)^j , \quad (\xi_{LM} < \xi < \xi_{RM})$$

$$\lfloor u(\xi) \rfloor = \lfloor z(\xi-2) \rfloor [ZR(\xi_{RM}-2)]^{-1} [YR(\xi_{RM})], \quad (\xi_{RM} \leq \xi < 4)$$

$$\lfloor u(\xi) \rfloor = \lfloor z(-\xi-2) \rfloor [ZL(-\xi_{LM}-2)]^{-1} [YL(\xi_{LM})], \quad (-4 < \xi \leq \xi_{LM})$$

A necessary and sufficient condition for the existence of a non-trivial $\{C\}$ satisfying 2.74 is that the determinant D of the square matrix in 2.74 vanish. Remembering that the $u(\xi)$'s are functions of the nondimensional frequency ratios ω's; then the values of ω which render zero values for the determinant $D(\omega)$ are the ratios of the natural frequencies of in-plane oscillation of the cable to the parameter $\sqrt{g/a}$:

$$\text{Natural frequency ratio} = \sqrt{g/a}\,\omega \qquad (2.75)$$

A trial-and-error solution for the roots of $D(\omega)$ will yield the desired natural frequency ratios.

II.4.3. In-Plane Forced Motion of the Cable

The necessary theory to describe the normal mode motion of the suspended cable has been derived in the previous two sections. Now it would be of interest to develop a means of determining the displacements and tensions for any point along a cable when one end is held fixed and the other end is subjected to a prescribed tangential displacement at a specified frequency.

Any periodic function may be expressed as the superposition of a series of sine and cosine terms through the use of a Fourier series. Thus if the displacements and tensions in a cable can be determined for a single, sinusoidally varying input displacement of frequency ratio ω_f, then through use of a Fourier series it will be possible to determine the tensions and displacements for any arbitrary periodic input. Consider the cable to be fixed at point ξ_1 and that at position ξ_2 the cable is subjected to a tangential displacement of $U(\xi_2)$ oscillating sinusoidally at a frequency ratio ω_f. For this analysis, the normal displacement at ξ_2 will be restrained to zero displacement. If one point on the cable is subjected to a forced oscillation of frequency ratio ω_f, then a steady state solution in which all points along the cable oscillate at this frequency exists. Now matrix Equation 2.74 may be written as:

$$\begin{bmatrix} u_0(\xi_1,\omega_f) & u_1(\xi_1,\omega_f) & u_2(\xi_1,\omega_f) & u_3(\xi_1,\omega_f) \\ u_0'(\xi_1,\omega_f) & u_1'(\xi_1,\omega_f) & u_2'(\xi_1,\omega_f) & u_3'(\xi_1,\omega_f) \\ u_0(\xi_2,\omega_f) & u_1(\xi_2,\omega_f) & u_2(\xi_2,\omega_f) & u_3(\xi_2,\omega_f) \\ u_0'(\xi_2,\omega_f) & u_1'(\xi_2,\omega_f) & u_2'(\xi_2,\omega_f) & u_3'(\xi_2,\omega_f) \end{bmatrix} \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ U(\xi_2) \\ 0 \end{Bmatrix} \tag{2.76}$$

or in shorter notation:

$$[u(\xi_1, \xi_2, \omega_f)]\{C\} = \{U(\xi_2)\} \tag{2.77}$$

A glance at matrix Equation 2.76 will reveal that it contains four equations in four unknowns, the constants C_j. Solution of 2.77 for the unknown constants C_j yields:

$$\{C\} = [u(\xi_1, \xi_2, \omega_f)]^{-1}\{U(\xi_2)\} \tag{2.78}$$

The above equation will yield the C_j constants providing the forcing frequency is not equal to one of the natural frequencies of the cable.

Now that the constants C_j have been determined, it will be a simple matter to determine the tangential displacement at any point along the cable ξ_p, from the relationship:

$$U(\xi_p) = \lfloor u_0(\xi_p)\ u_1(\xi_p)\ u_2(\xi_p)\ u_3(\xi_p) \rfloor \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{Bmatrix} \tag{2.79}$$

where the $u_j(\xi_p)$ have been previously defined in Equation 2.74 and the constants C_j by Equation 2.78. The normal displacement V and tension T in the cable are given in terms of σ by Equations 2.36 and 2.37; or written in terms of ξ, by:

$$V(\xi_p) = (|\xi_p| + \xi_p^2)\sqrt{\frac{2}{|\xi_p|} + 1}\,\frac{dU}{d\xi}\Big|_{\xi_p} \tag{2.80}$$

$$\begin{aligned}
T(\xi_p) = {} & \frac{\xi_p\sqrt{\frac{2}{|\xi_p|} + 1}}{1+|\xi_p|}\,U(\xi_p) \\
& - \Bigg\{\left[5+4\xi_p^2\left(\frac{2}{|\xi_p|} + 1\right)\right]\left[|\xi_p|\sqrt{\frac{2}{|\xi_p|} + 1}\right] \\
& + \omega_f^2|\xi_p|\sqrt{\frac{2}{|\xi_p|} + 1}\,(1+|\xi_p|)^3 + 2\delta\left(\xi_p\sqrt{\frac{2}{|\xi_p|} + 1}\right)(1+|\xi_p|)^5\Bigg\}\frac{dU}{d\xi}\Big|_{\xi_p} \\
& - \left\{5\xi_p^3\left(\frac{2}{|\xi_p|} + 1\right)^{3/2}(1+|\xi_p|) + 3\xi_p\sqrt{\frac{2}{|\xi_p|} + 1}\,(1+|\xi_p|)\right\}\frac{d^2U}{d\xi^2}\Big|_{\xi_p} \\
& - \left\{|\xi_p|^3\left(\frac{2}{|\xi_p|} + 1\right)^{3/2}(1+|\xi_p|)^2\right\}\frac{d^3U}{d\xi^3}\Big|_{\xi_p}
\end{aligned} \tag{2.81}$$

II.4.4. Out-of-Plane Normal Mode Motions

The technique for the solution of out-of-plane normal mode motion is identical to the technique in Section II.4.2 for in-plane normal mode motion. Hence the analysis in this section will not contain as much detail as was given in Section II.4.2. Equation 2.44, describing the out-of-plane motion of the cable, is an ordinary, linear, second order differential equation with polynomial coefficients. Thus Equation 2.44 lends itself to a power series solution of the form developed by Frobenius. The singular points of Equation 2.44 are given by the roots of the coefficient of $d^2p/d\xi^2$ and occur at $\xi = 0$ and -2.

A power series expansion about the singular point $\xi = 0$ will have the form:

$$p(\xi) = \sum_{k=0}^{\infty} e_k \xi^{k+r}, \quad e_0 \neq 0 \tag{2.82}$$

Substitution of Equation 2.82 into Equation 2.44 will yield the following indicial equation:

$$(2r^2 - r)e_0 = 0 \tag{2.83}$$

Solution of the above equation yields the two characteristic exponents $r = 0, 1/2$. Since these exponents do not differ by an integer, the solutions obtained will be linearly independent. The complete solution to Equation 2.44 will have the form:

$$P(\xi) = \sum_{j=0}^{1} E_j p_j(\xi) \tag{2.84}$$

where

$$p_j(\xi) = p_{2r}(\xi) = \sum_{k=0}^{\infty} e_k(r) \xi^{k+r}, \quad (r = 0, 1/2) \tag{2.85}$$

The two arbitrary constants $e_0(r)$ will be assigned values of:

$$e_0(r) = 1, \quad (r = 0, 1/2) \tag{2.86}$$

The remaining coefficients, e_k $(k = 1 \rightarrow \infty)$, are given by the following recursion relationship:

$$e_k = -\frac{(k+r-1)^2+\omega^2}{2(k+r)^2-(k+r)} e_{k-1} - \frac{\omega^2}{2(k+r)^2-(k+r)} e_{k-2} \tag{2.87}$$

All coefficients in the above equation with negative subscripts are zero.

Substitution of

$$\xi = \sqrt{1+\sigma^2} - 1 = \frac{\sigma^2}{2}\left(1 - \frac{\sigma^2}{4} + \frac{\sigma^4}{8} - \ldots\right)$$

into the series defined by Equations 2.85, 2.86 and 2.87 yields the following lead terms in the two p_j series:

$$p_0 = 1 - \frac{\omega^2}{2}\sigma^2 + \ldots \tag{2.88a}$$

$$p_1 = \frac{\sigma}{\sqrt{2}} - \frac{1+4\omega^2}{12}\sigma^3 + \ldots \tag{2.88b}$$

Equation 2.84 is a solution to Equation 2.44 for positive ξ. For negative values of ξ, then a general solution to Equation 2.44 would be of the form:

$$P(-\xi) = \sum_{j=0}^{1} M_j p_j(-\xi) \qquad (2.89)$$

Continuity of the zeroth and first derivatives at the origin will insure that the displacement and slope at the origin are continuous; this requirement may be expressed as:

$$P(\sigma) = P(-\sigma) \qquad (\sigma = 0) \qquad (2.90a)$$

$$P'(\sigma) = P'(-\sigma) \qquad (\sigma = 0) \qquad (2.90b)$$

Substitution of Equations 2.84 and 2.89 (written in terms of the variable σ) into Equations 2.90 yields:

$$\begin{bmatrix} p_0(\sigma) & p_1(\sigma) \\ p_0'(\sigma) & p_1'(\sigma) \end{bmatrix} \begin{Bmatrix} E_0 \\ E_1 \end{Bmatrix} = \begin{bmatrix} p_0(-\sigma) & p_1(-\sigma) \\ p_0'(-\sigma) & p_1'(-\sigma) \end{bmatrix} \begin{Bmatrix} M_0 \\ M_1 \end{Bmatrix} \qquad (2.91)$$

Differentiation and substitution of Equations 2.88 into the above matrix equation and subsequent evaluation at the origin $(\sigma = 0)$ yields the following relations between the constants E_j and M_j:

$$M_0 = E_0, \quad M_1 = -E_1$$

Now the solution for the out-of-plane displacement, valid for both positive and negative ξ, may be expressed as:

$$P(\xi) = \sum_{j=0}^{1} E_j N_j(\xi) \tag{2.92}$$

where

$$N_j(\xi) = \left(\frac{\xi}{|\xi|}\right)^j p_j(|\xi|) \tag{2.93}$$

Equation 2.93 is valid for values of ξ in the range, $-2 < \xi < 2$. The maximum angle α for which the above equation is valid is $\alpha = 70.5°$. For many applications, it may be desirable to determine the out-of-plane displacements for values of α greater than 70.5°. Hence it will be necessary to write another series solution to Equation 2.43 which will converge for values of $|\xi| \geq 2$.

Substitution of the variable change

$$\chi = \xi - 2$$

into Equation 2.44 yields:

$$(\chi+2)(\chi+4)\frac{d^2q}{d\chi^2} + (\chi+3)\frac{dq}{d\chi} + (\chi+3)\omega^2 q = 0 \tag{2.94}$$

A power series expansion about the ordinary point $\chi = 0$ $(\xi = 2)$ will be of the form:

$$q(\chi) = \sum_{k=0}^{\infty} f_k \chi^{k+r} \tag{2.95}$$

Substitution of this power series into Equation 2.94 will yield the following indicial equation,

$$8r(r-1)f_0 = 0 \tag{2.96}$$

which yields characteristic exponents $r = 0, 1$.

The complete solution to Equation 2.94 will have the form:

$$P(\chi) = \sum_{j=0}^{1} F_j q_j(\chi) \tag{2.97}$$

where

$$q_j(\chi) = q_r(\chi) = \sum_{k=0}^{\infty} f_k(r)\chi^{k+r}, \quad (r = 0, 1) \tag{2.98}$$

The two arbitrary constants $f_0(r)$ will be assigned values of:

$$f_0(r) = 1, \quad (r = 0, 1) \tag{2.99}$$

The remaining coefficients, f_k $(k = 0 \rightarrow \infty)$, are given by the following recursion relationship:

$$\begin{aligned} f_k = &- \frac{3(2k+2r-3)}{8(k+r)} f_{k-1} \\ &- \frac{(k+r-2)(k+r-3)+(k+r-2)+3\omega^2}{8(k+r)(k+r-1)} f_{k-2} \\ &- \frac{\omega^2}{8(k+r)(k+r-1)} f_{k-3} \end{aligned} \tag{2.100}$$

All coefficients in the above equation with negative subscripts are zero. In terms of the variable ξ, Equation 2.98 will have a region of convergence of $0 < \xi < 4$. The value of α which may be attained for $\xi = 4$ is 78°.

As was done for the in-plane motion case, it will be desirable to express the constants F_j in terms of the constants E_j. This may be accomplished by matching the zeroth and first derivatives at some mutual point where both series expansions converge. If both series expansions converge at a point ξ_{RM}, then:

$$\begin{bmatrix} p_0(\xi_{RM}) & p_1(\xi_{RM}) \\ p_0'(\xi_{RM}) & p_1'(\xi_{RM}) \end{bmatrix} \begin{Bmatrix} E_0 \\ E_1 \end{Bmatrix} = \begin{bmatrix} q_0(\xi_{RM}-2) & q_1(\xi_{RM}-2) \\ q_0'(\xi_{RM}-2) & q_1'(\xi_{RM}-2) \end{bmatrix} \begin{Bmatrix} F_0 \\ F_1 \end{Bmatrix} \tag{2.101}$$

or in shorter notation:

$$[PR(\xi_{RM})]\{E\} = [QR(\xi_{RM}-2)]\{F\} \tag{2.102}$$

Solution of the above matrix equation for $\{F\}$ yields:

$$\{F\} = [QR(\xi_{RM}-2)]^{-1}[PR(\xi_{RM})]\{E\} \tag{2.103}$$

Equations 2.92 and 2.97 will yield solutions for the out-of-plane displacement of a cable in the range $-2 < \xi < 4$. The solution range can be further extended by replacement of ξ with $-\xi$ in the defining equation for χ, i.e., $\chi = \xi - 2$. This variable change would allow solutions to be determined for the range $-4 < \xi < 0$. Substitution of the variable change:

$$\psi = -\xi - 2 \qquad (\text{for } \xi < 0)$$

or

$$\psi = |\xi| - 2$$

into Equation 2.44 yields an equation identical to Equation 2.94 except that the independent variable is ψ instead of χ. Solution of this equation yields:

$$P(\psi) = \sum_{j=0}^{1} H_j q_j(\psi) \tag{2.104}$$

where,

$$q_j(\psi) = q_r(\psi) = \sum_{k=0}^{\infty} f_k(r)\psi^{k+r} \tag{2.105}$$

By matching Equations 2.92 and 2.104 at a point where both equations are valid, it will be possible to determine the constants H_j in terms of the constants E_j. Let ξ_{LM} be a point at which both equations are valid, then:

$$\begin{bmatrix} p_0(\xi_{LM}) & -p_1(\xi_{LM}) \\ -p_0'(\xi_{LM}) & p_1'(\xi_{LM}) \end{bmatrix} \begin{Bmatrix} E_0 \\ E_1 \end{Bmatrix} = \begin{bmatrix} q_0(-\xi_{LM}-2) & q_1(-\xi_{LM}-2) \\ -q_0'(-\xi_{LM}-2) & -q_1'(-\xi_{LM}-2) \end{bmatrix} \begin{Bmatrix} H_0 \\ H_1 \end{Bmatrix} \tag{2.106}$$

or in shorter notation:

$$[PL(\xi_{LM})]\{E\} = [QL(-\xi_{LM}-2)]\{H\} \tag{2.107}$$

Solution of the above matrix equation for $\{H\}$ yields:

$$\{H\} = [QL(-\xi_{LM}-2)]^{-1}[PL(\xi_{LM})]\{E\} \tag{2.108}$$

Through the use of Equations 2.92, 2.97, and 2.104, the out-of-plane displacement of the cable may be determined for any value of ξ in the range $-4 < \xi < 4$.

Consider the cable to be fixed at two points ξ_1 and ξ_2. At these two points of fixity, the out-of-plane displacement will be zero. Hence the boundary conditions are:

$$P(\xi_1) = P(\xi_2) = 0 \tag{2.109}$$

Written in matrix notation, Equation 2.109 will have the form:

$$\begin{bmatrix} R_0(\xi_1) & R_1(\xi_1) \\ R_0(\xi_2) & R_1(\xi_2) \end{bmatrix} \begin{Bmatrix} E_0 \\ E_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{2.110}$$

where

$$\lfloor R(\xi) \rfloor = \lfloor p(|\xi|) \rfloor \left(\frac{\xi}{|\xi|}\right)^j , \quad (\xi_{LM} < \xi < \xi_{RM})$$

$$\lfloor R(\xi) \rfloor = \lfloor q(\xi-2) \rfloor [QR(\xi_{RM}-2)]^{-1}[PR(\xi_{RM})], \quad (\xi_{RM} \leq \xi < 4)$$

$$\lfloor R(\xi) \rfloor = \lfloor q(-\xi-2) \rfloor [QL(-\xi_{LM}-2)]^{-1}[PL(\xi_{LM})], \quad (-4 < \xi \leq \xi_{LM})$$

A necessary and sufficient condition for the existence of a non-trivial $\{E\}$ satisfying Equation 2.110 is that the determinant D of the square matrix in 2.110 vanish. Remembering that the $R(\xi)$'s are functions of the nondimensional frequency ratios ω's; then the values of ω which render zero values for the determinant $D(\omega)$ are the ratios of the natural frequencies of out-of-plane oscillation of the cable to the parameter $\sqrt{g/a}$:

$$\text{Natural frequency ratio} = \sqrt{\frac{g}{a}}\,\omega \tag{2.111}$$

A trial-and-error solution for the roots of $D(\omega)$ will yield the desired natural frequency ratios.

III. RESULTS

The theory, equations, and solutions necessary to describe the general catenary geometry, the in-plane and out-of-plane normal mode motion, and in-plane forced motion have been developed in Section II of this paper. This section will cover each of the categories listed above by giving numerical results based on the previously derived theory and solutions. For the general catenary geometry, relationships between the dimensionless variables ξ_1 and ξ_2 and dimensionless ratios involving the horizontal and vertical distance between supports and the cable length will be presented. A method of relating the catenary size parameter a to the other cable parameters will be presented. The convergence region of the solutions for in-plane and out-of-plane normal mode motion will also be investigated. For the in-plane and out-of-plane normal mode motion, curves of the natural frequency ratios as a function of specific cable geometries are given. The entity of interest for the case of forced motion of the cable is the maximum tension in the cable. Hence curves of the maximum tension in the cable as functions of the forcing frequency and cable geometry are presented.

The solutions which describe the cable geometry and motions contain many series expansions and trial and error procedures, hence it is imperative that the computer be used to obtain the necessary results. Presentation of the results will include a very brief description of any computer programs used to obtain the particular results along with a listing of the program and calculated numerical data. Details pertaining to the computer programs may be obtained by study of the program listings.

III. 1. Cable Geometry and Solution Convergence Region

The equations and solutions for the cable motion are written in terms of the nondimensional variables ξ_1 and ξ_2. For convenience in using the results obtained, it is desirable to express the variables ξ_1 and ξ_2 in terms of the horizontal and vertical distances between the support points and the cable length. Let ℓ be the total length of the cable, b be the horizontal distance between the support points, and h be the vertical height difference between the support points, as shown in Figure 2.1. The nondimensional cable length from the apex of the cable to a position ξ is given by Equation 2.40. The total length of the line will be given by:

$$\ell = s_2 - s_1 = a(\sigma_2 - \sigma_1) \tag{3.1}$$

Substitution of Equation 2.25a into the above equation yields the following expression for the line length ℓ:

$$\ell = s_2 - s_1 = a[\sinh(\frac{x_2}{a}) - \sinh(\frac{x_1}{a})] \qquad (3.2)$$

The height difference h may be determined using Equation 2.26 as follows:

$$h = y_2 - y_1 = a[\cosh(\frac{x_2}{a}) - \cosh(\frac{x_1}{a})] \qquad (3.3)$$

The horizontal difference b is given by the following:

$$b = x_2 - x_1 \qquad (3.4)$$

Elimination of s, y, and x from Equations 3.2, 3.3, and 3.4 yields a transcendental equation which defines the ratio of a/b for each value of the parameter $b/\sqrt{\ell^2-h^2}$ (Routh [2]).

$$\frac{\frac{b}{2a}}{\sinh \frac{b}{2a}} = \frac{b}{\sqrt{\ell^2-h^2}} \qquad (3.5)$$

Data Table A-1 and the corresponding Figure A-2 in Appendix A give values of $b/\sqrt{\ell^2-h^2}$ and a/b satisfying Equation (3.5).

Using the parameters $b/\sqrt{\ell^2-h^2}$ and h/b as inputs, then unique values of ξ_1 and ξ_2 may be determined with the aid of Equation 3.5 along with the other geometric relationships developed for the cable. Transformation of the cable parameters ℓ, b, and h to the boundary ξ's is necessary as the equations describing the motion of the cable are written in terms of ξ.

Using ξ_2 and h/b as inputs, then it is also possible to determine ξ_1 and $b/\sqrt{\ell^2-h^2}$ with the aid of the catenary equations. This will be useful in determining the values of h/b and $b/\sqrt{\ell^2-h^2}$ for which the series expansions are valid. The series solutions for both the in-plane and out-of-plane motion of the cable, as developed in Section II of this paper, have a convergence region of $-4 < \xi < 4$. If ξ_2 is assigned a value of 4 and h/b is given, then determination of $b/\sqrt{\ell^2-h^2}$ will provide the minimum value of $b/\sqrt{\ell^2-h^2}$ for which the solutions are valid.

Appendix A-3 is a listing of the program which calculates ξ_1 and the minimum value of $b/\sqrt{\ell^2-h^2}$ for which the motion equations are theoretically valid given ξ_2 and h/b as inputs. Data Table A-4 gives the values of h/b and computer generated values of $b/\sqrt{\ell^2-h^2}$ for which the solutions are valid. This relationship is also presented in Figure 3.1.

III.2. Natural Frequency Ratios for Normal Mode Motion

Calculation of the natural frequency ratios for the cable is the first and most difficult step in the sequence of determining the normal mode motion of the cable. The next step after calculation of the natural frequency ratios, would be to determine the eigen-vectors associated with each natural frequency ratio. Determination of the

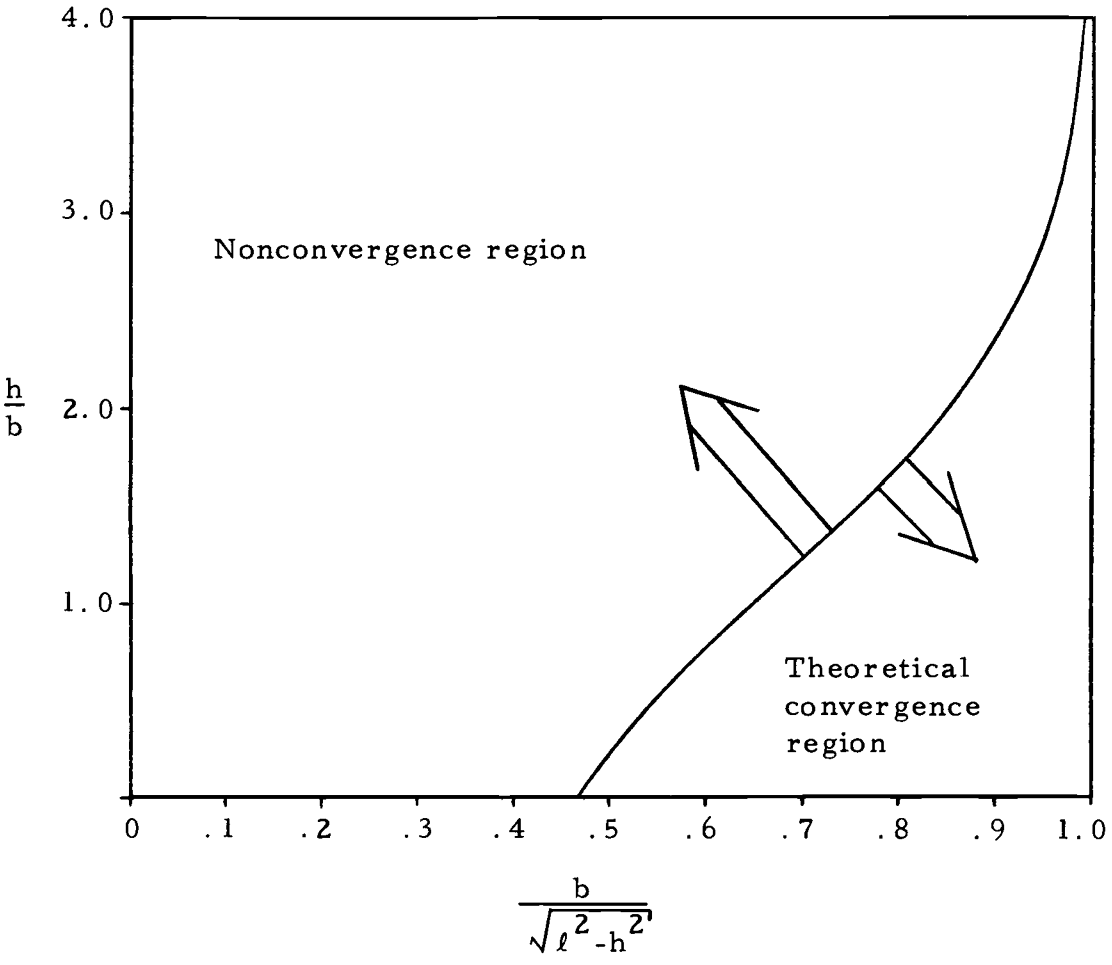

Figure 3.1. Normal mode motion solution convergence region versus sag and support parameters.

natural frequency ratios and their corresponding eigen-vectors are the necessary entities to completely describe the dynamic motion of the free vibrations in the cable.

The natural frequency ratios for the in-plane and out-of-plane normal mode motion may be determined by using Equations 2.74 and 2.110 respectively. The natural frequency ratios are the roots which yield zero values for the determinants in Equations 2.74 and 2.110 subject to given boundary values ξ_1 and ξ_2. Due to the nature of the determinants in Equations 2.74 and 2.110, the computer was used in determining the natural frequency ratios. The computer program consists of three sections. The first part determines the boundary ξ's given the cable length ℓ and support positions b and h. The second part evaluates the determinants given by Equation 2.74 or 2.110 for any arbitrary value of the natural frequency ratio ω. This part of the program includes the construction of all necessary series and matching considerations necessary to define each term in the determinants. The last part of the program uses a Newton-Raphson root finding procedure with a numerical approximation for the derivative to determine the values of ω which yield zero values for the determinants defined by Equations 2.74 or 2.110.

Appendices B-1 and B-2 are listings of the computer programs used to evaluate the natural frequency ratios for the in-plane and out-of-plane motion of the cable. Data tables B-3 through B-4 give natural frequency ratios for various combinations of h/b and $b/(\sqrt{\ell^2 - h^2})$ for both the in-plane and out-of-plane motion of the cable. These results are depicted graphically in Figures 3.2 to 3.11.

Figure 3.12 is identical to Figure 3.2 however natural frequencies from the works of Pugsley [3], Saxon and Cahn [4], and Goodey [5] for the in-plane motion are shown. The lowest or fundamental mode of vibration of the cable in the transverse direction may be approximated with the aid of Rayleigh's Principle. Rayleigh's Principle states that in the fundamental mode of vibration of an elastic system, the distribution of kinetic and potential energies is such as to make the frequency a minimum. The derivation of the fundamental natural frequency of a cable in transverse vibration using Rayleigh's Principle is given in Appendix B-5. Calculated values of these fundamental frequencies are shown on Figure 3.13.

III.3. Tensions and Displacements for In-Plane Forced Motion

In order to determine the tension in a cable when one end of the cable is subjected to any arbitrary periodic motion, then a Fourier series expansion may be utilized. Through Fourier series analysis, then any arbitrary periodic input motion may be approximated by the superposition of a series of sinusoidally varying input motions made up of different frequencies. Thus it is necessary to first express the tension in the cable when one end is subjected to an arbitrary

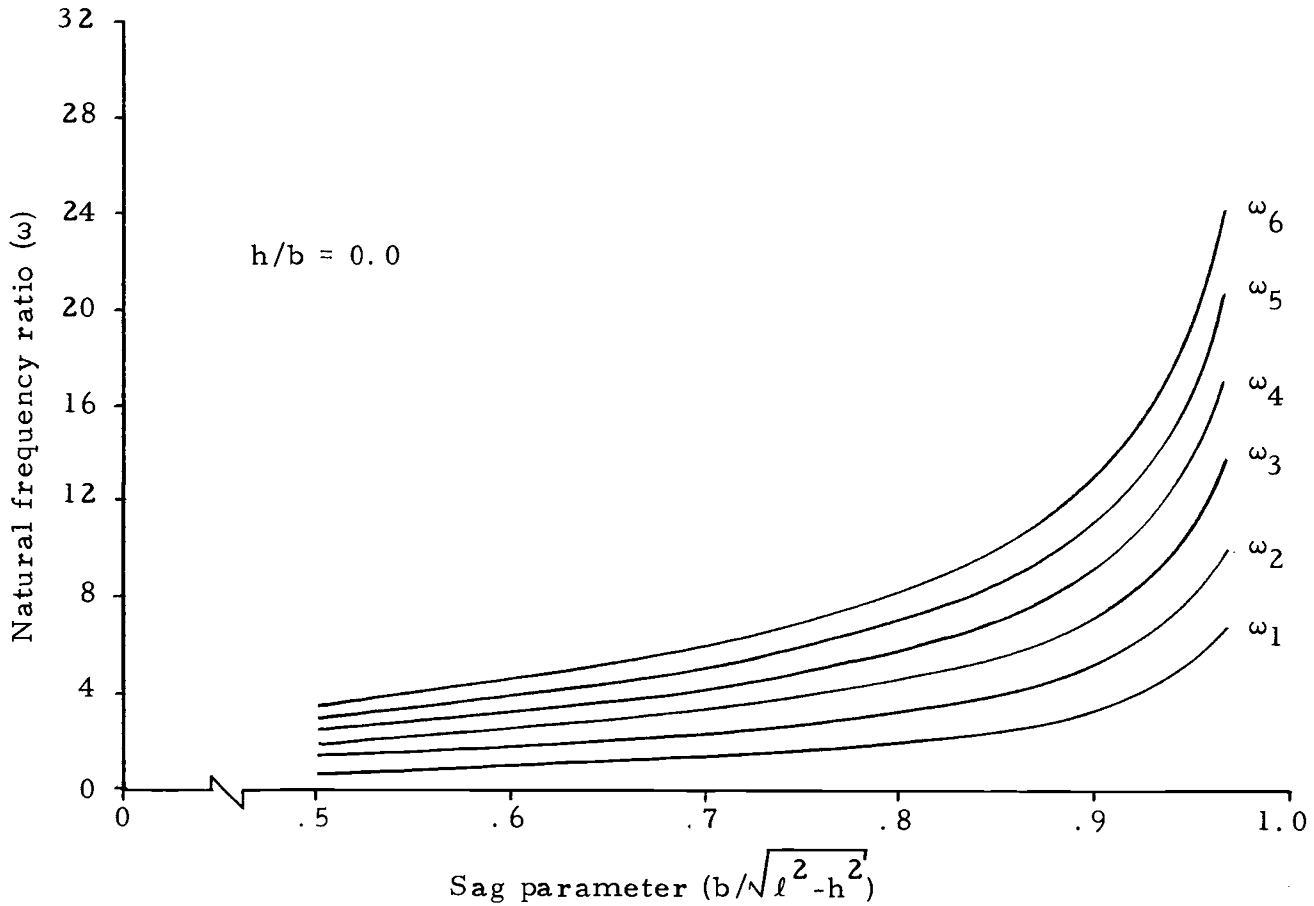

Figure 3.2. Natural frequency ratios for in-plane normal mode motion versus sag parameter for $h/b = 0.0$.

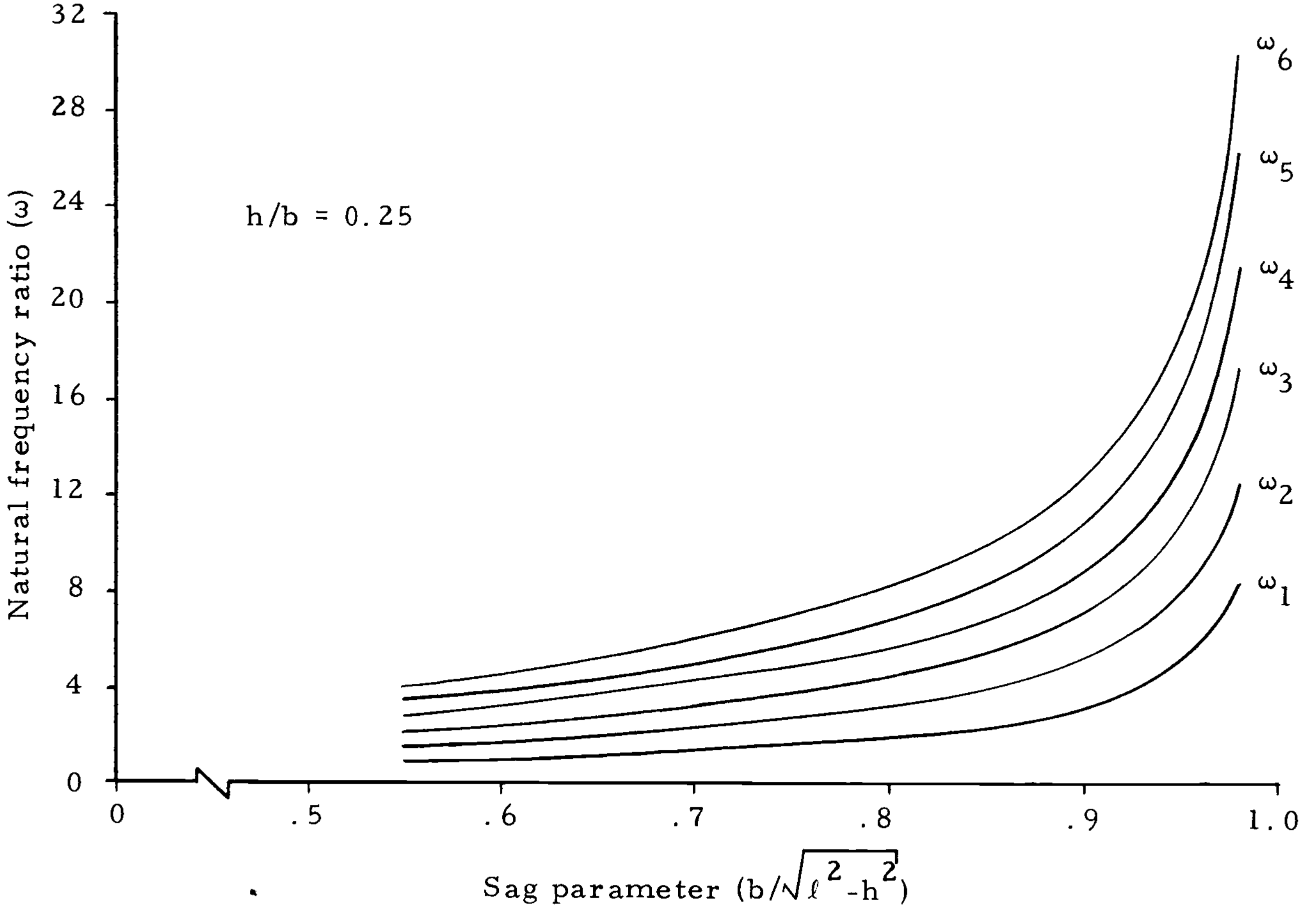

Figure 3.3. Natural frequency ratios for in-plane normal mode motion versus sag parameter for $h/b = 0.25$.

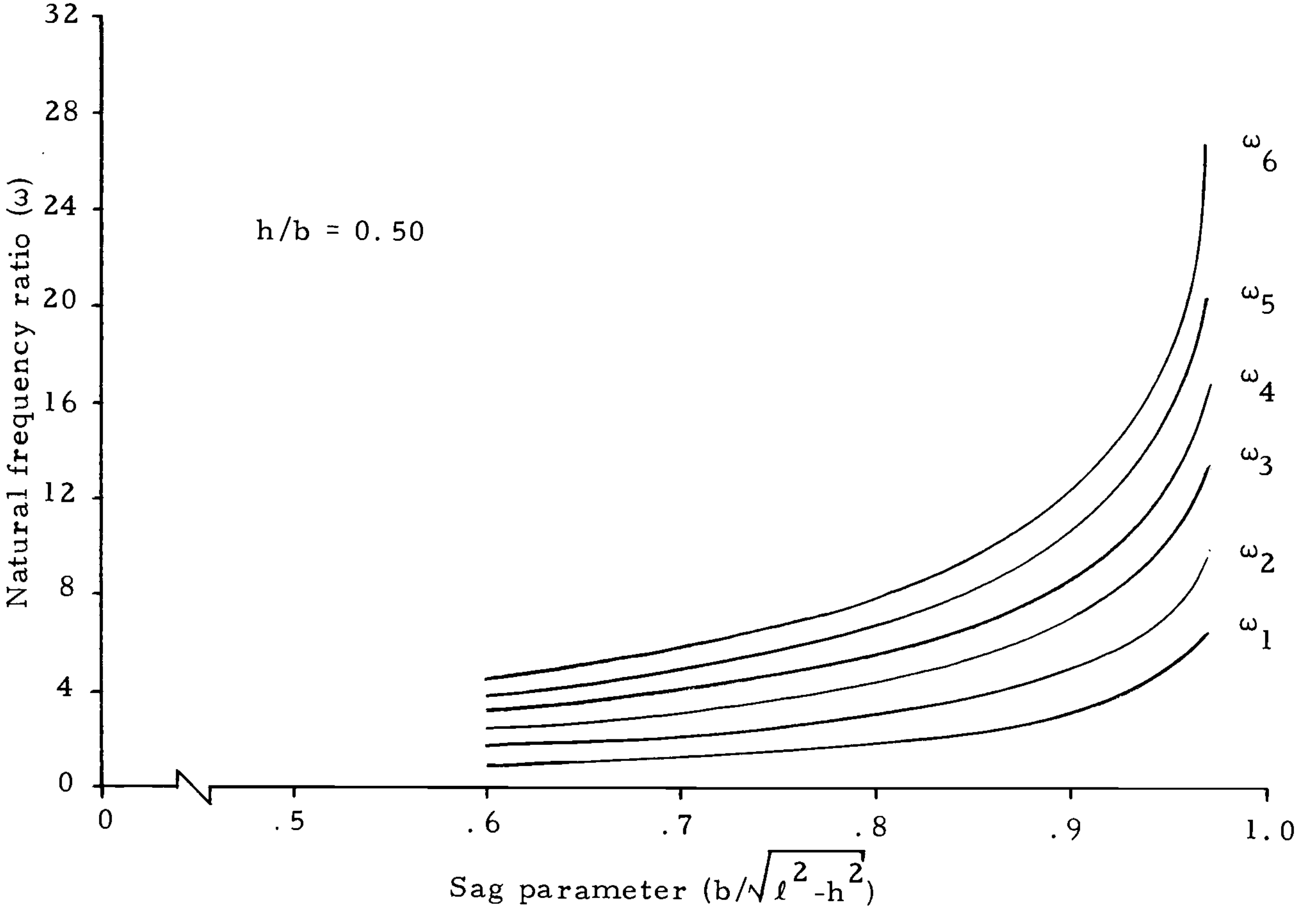

Figure 3.4. Natural frequency ratios for in-plane normal mode motion versus sag parameter for h/b = 0.50.

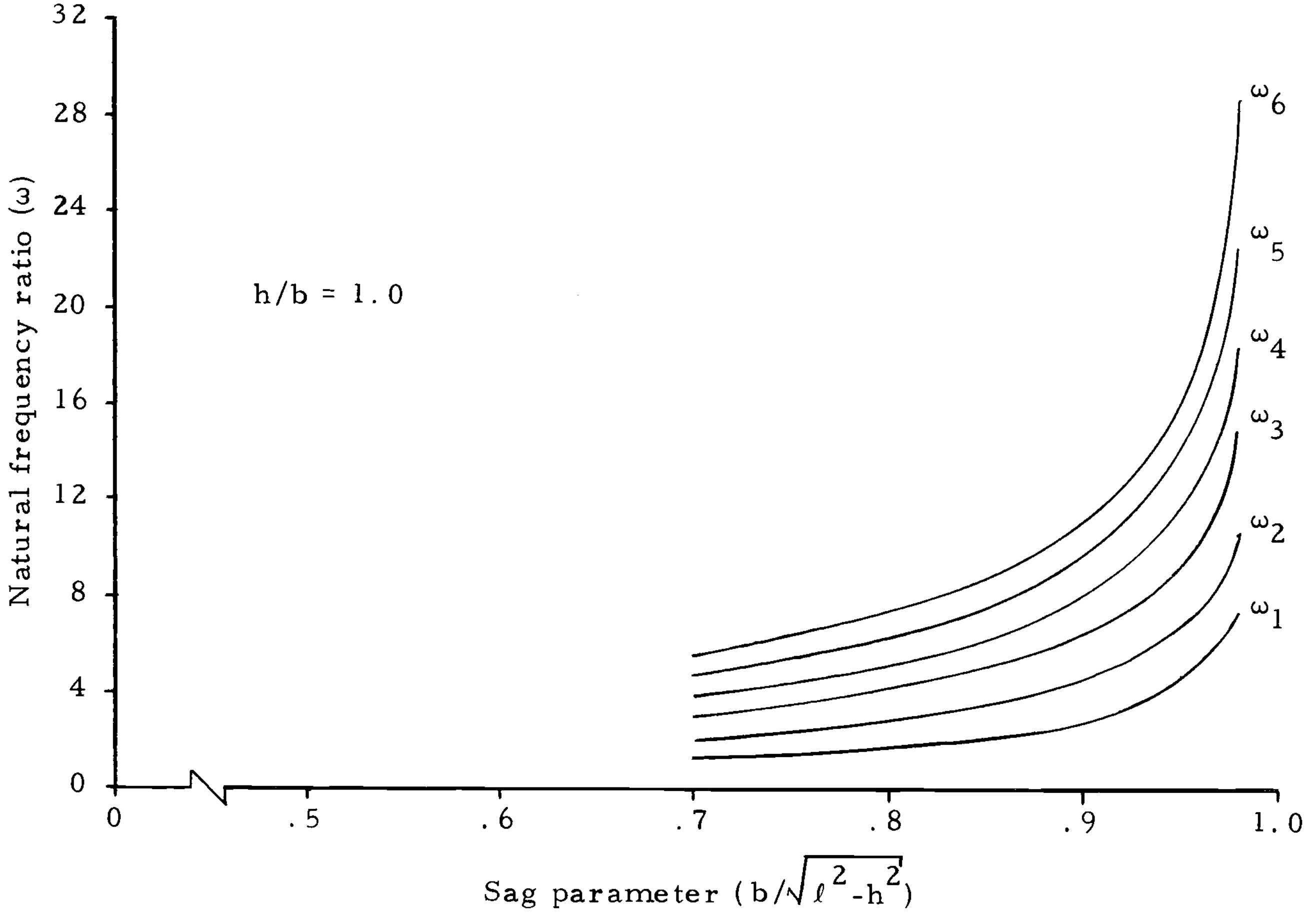

Figure 3.5. Natural frequency ratios for in-plane normal mode motion versus sag parameter for h/b = 1.0.

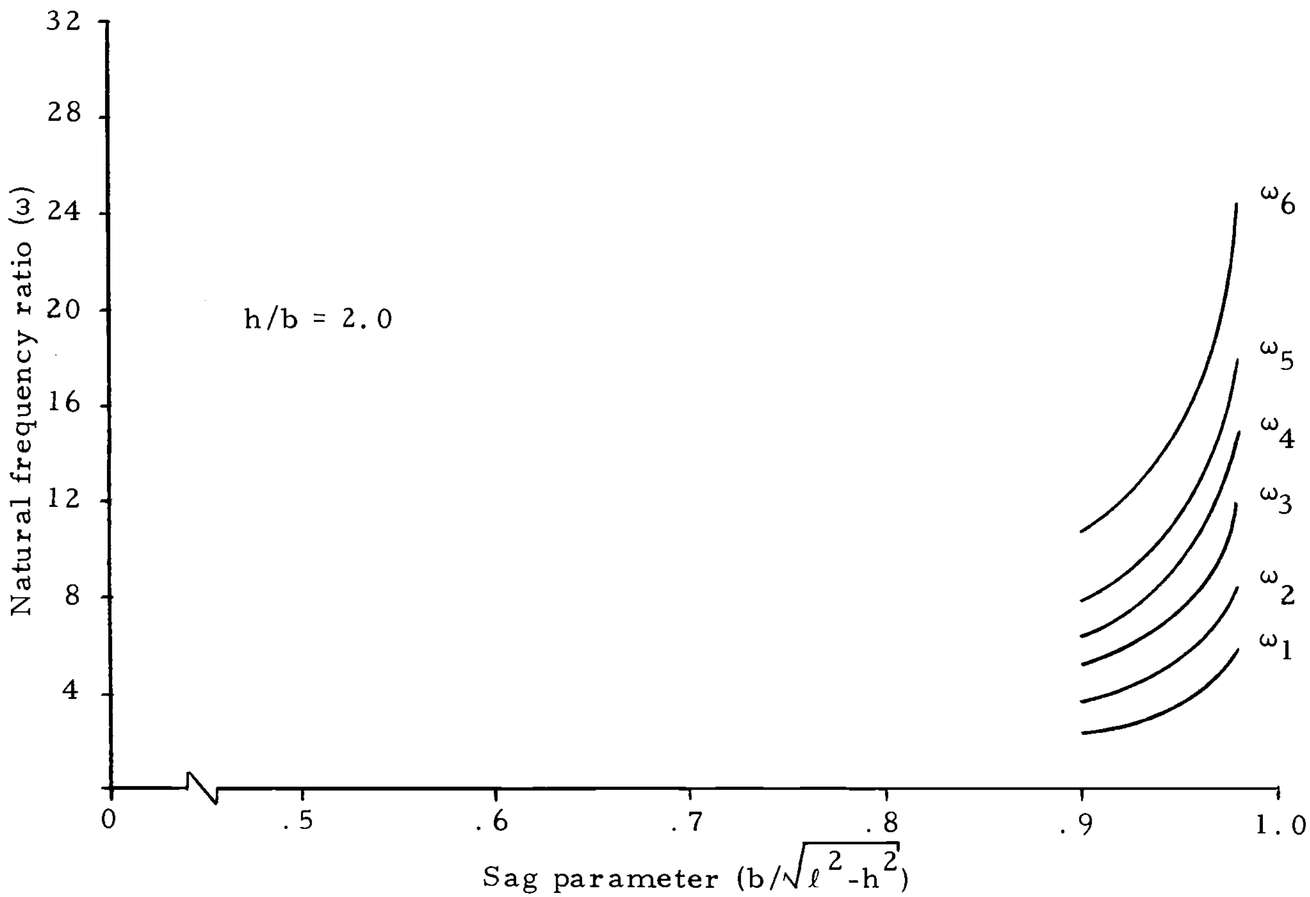

Figure 3.6. Natural frequency ratios for in-plane normal mode motion versus sag parameter for $h/b = 2.0$.

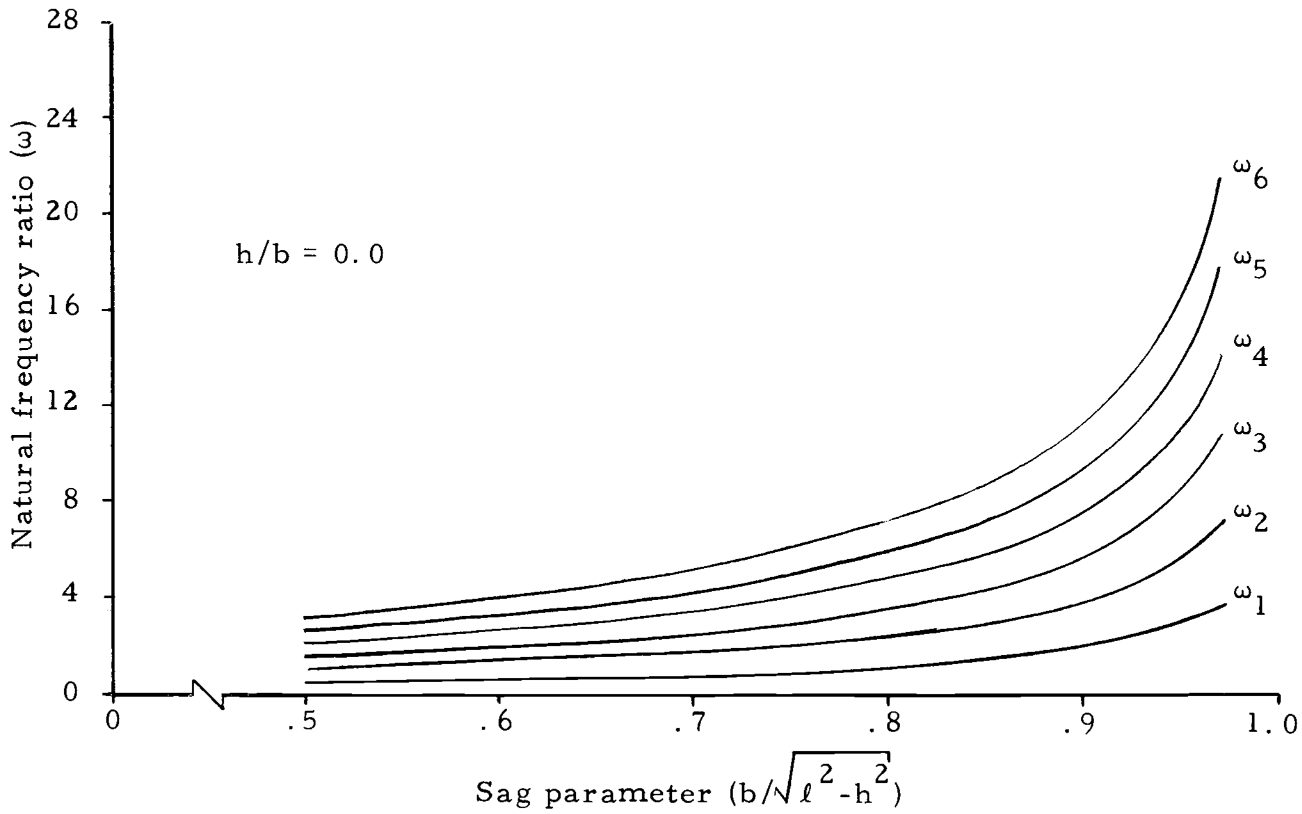

Figure 3.7. Natural frequency ratios for out-of-plane normal mode motion versus sag parameter for $h/b = 0.0$.

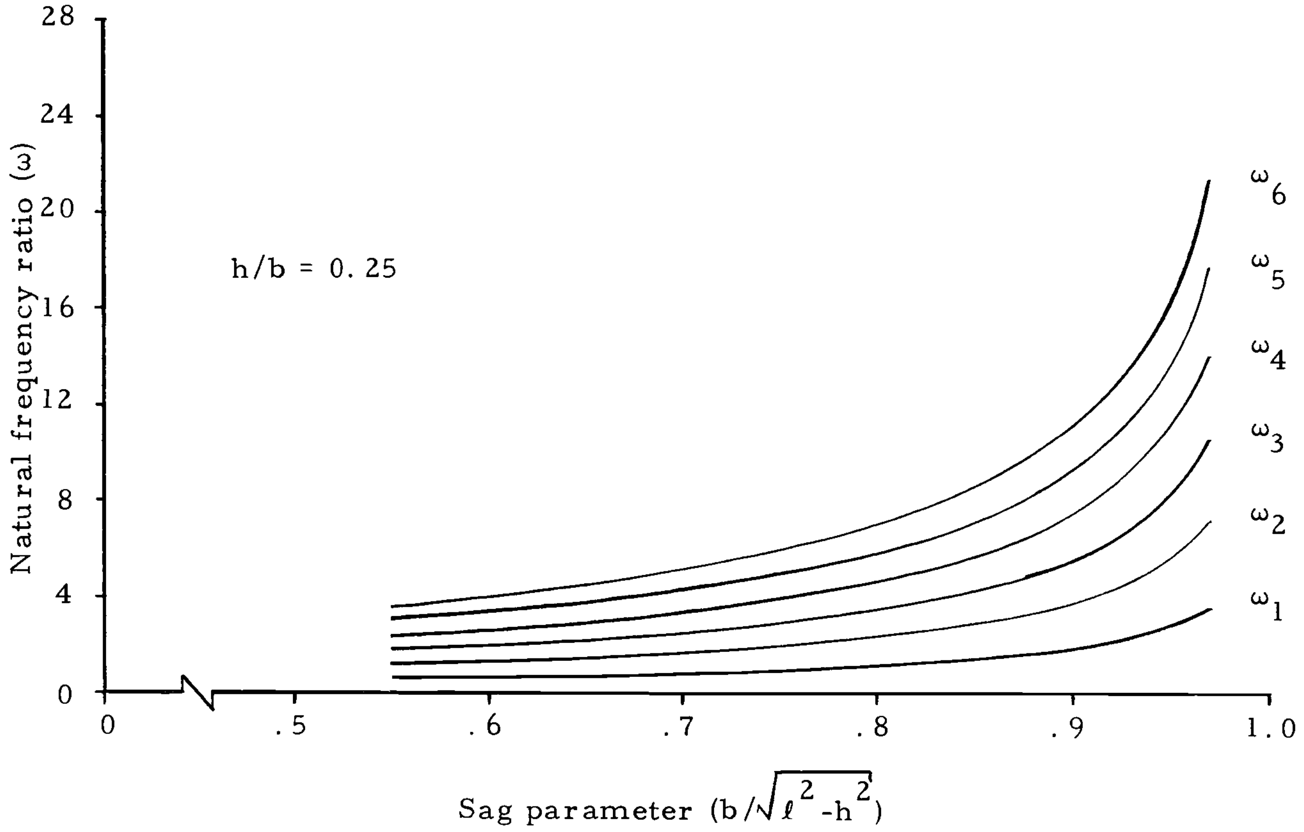

Figure 3. 8. Natural frequency ratios for out-of-plane normal mode motion versus sag parameter for h/b = 0. 25.

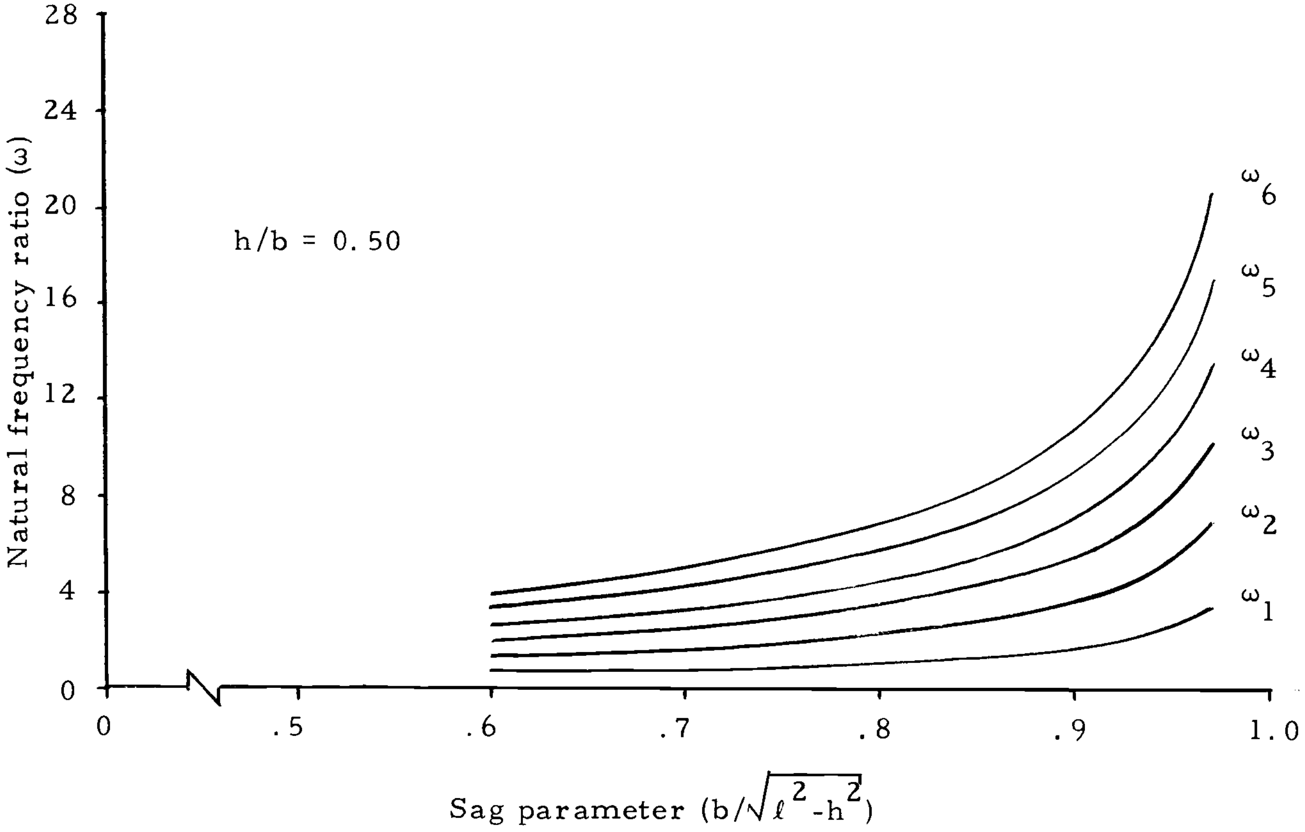

Figure 3.9. Natural frequency ratios for out-of-plane normal mode motion versus sag parameter for $h/b = 0.50$.

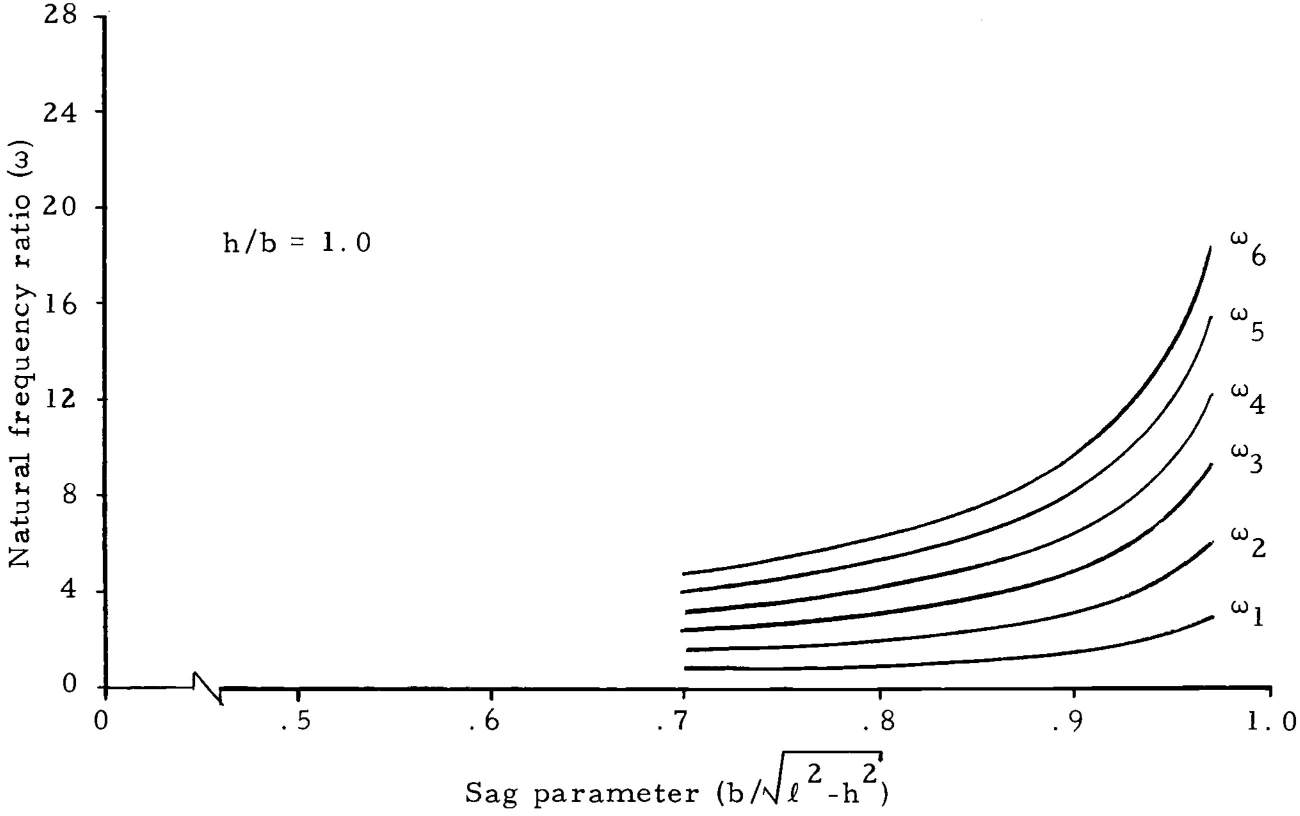

Figure 3.10. Natural frequency ratios for out-of-plane normal mode motion versus sag parameter for h/b = 1.0.

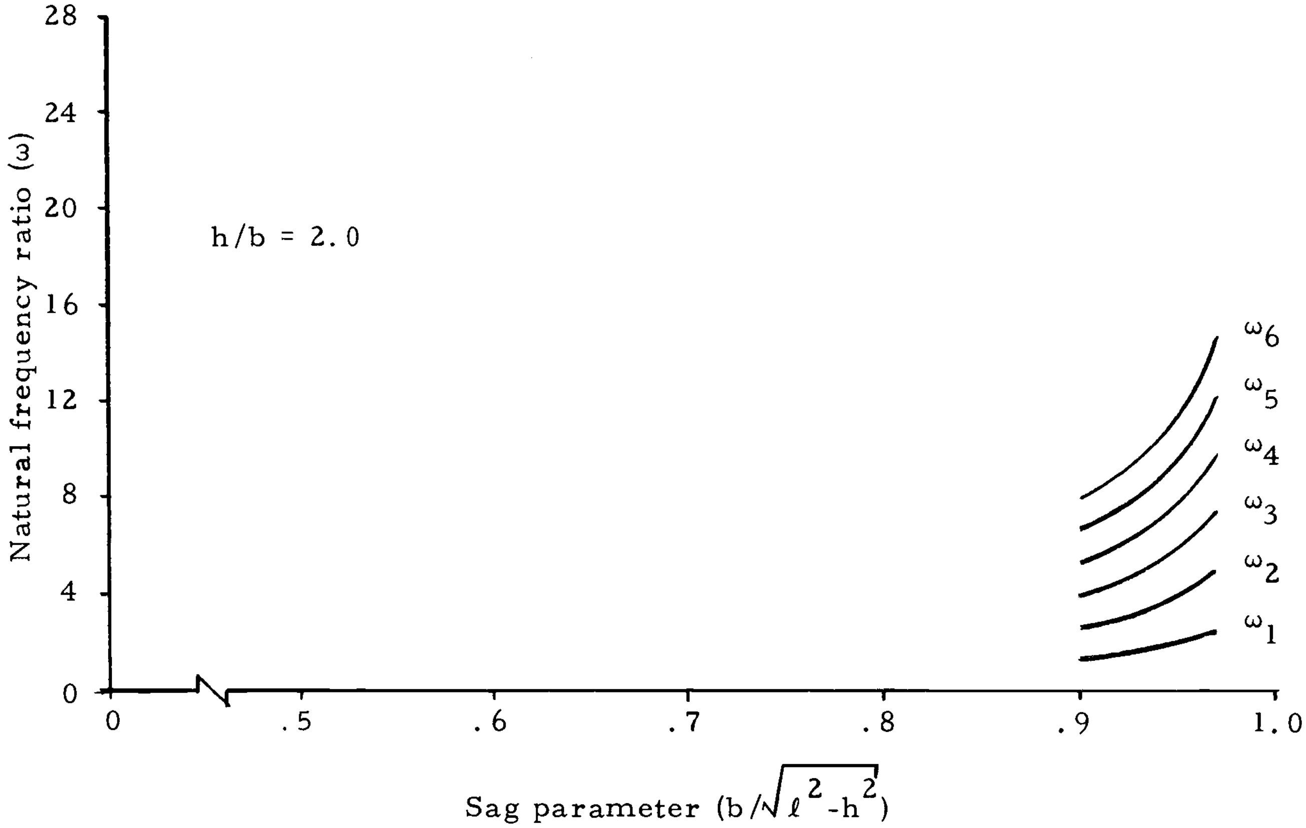

Figure 3.11. Natural frequency ratios for out-of-plane normal mode motion versus sag parameter for h/b = 2.0.

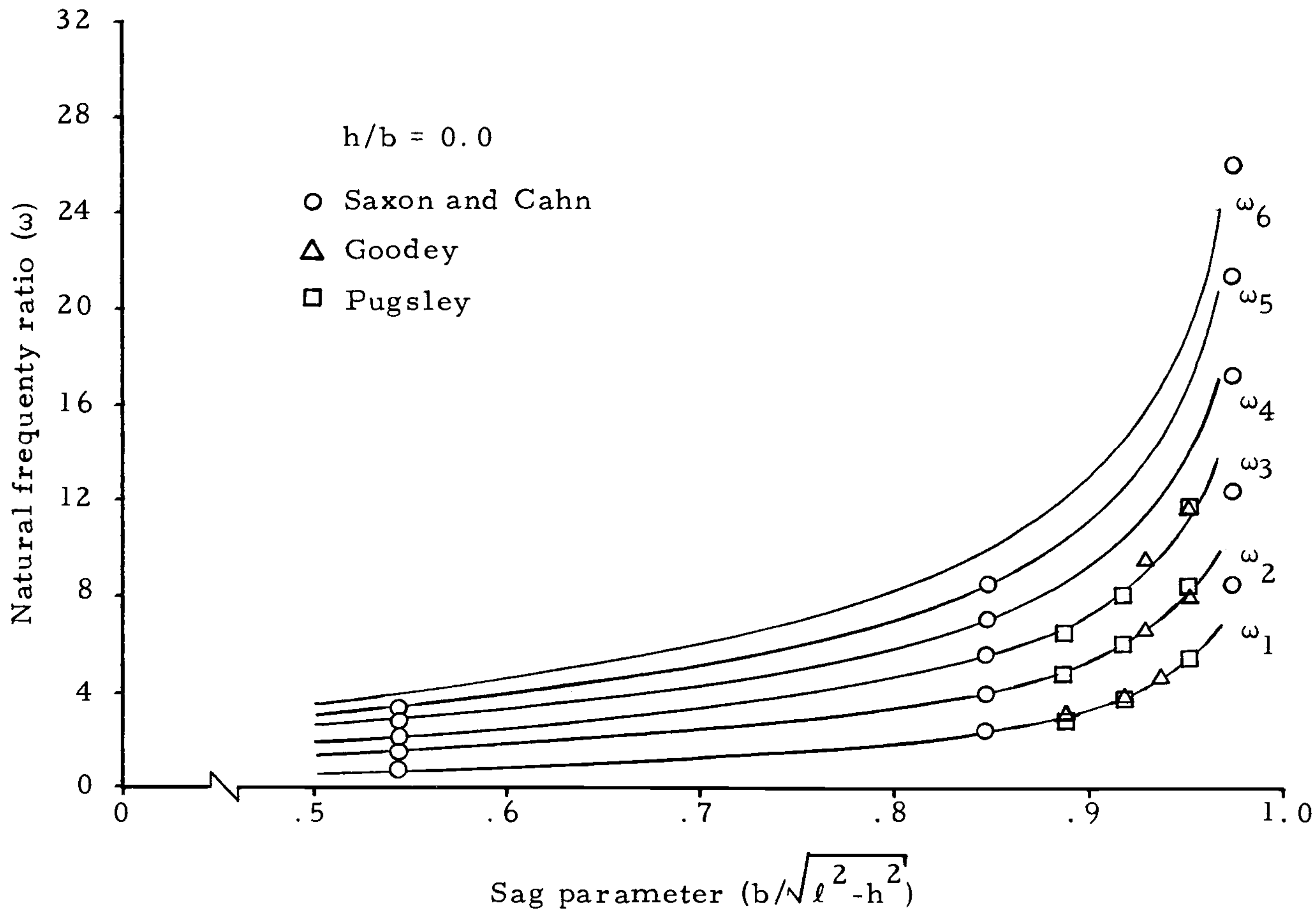

Figure 3.12. Natural frequency ratios for in-plane normal mode motion versus sag parameter for h/b = 0.0 including results of other investigations.

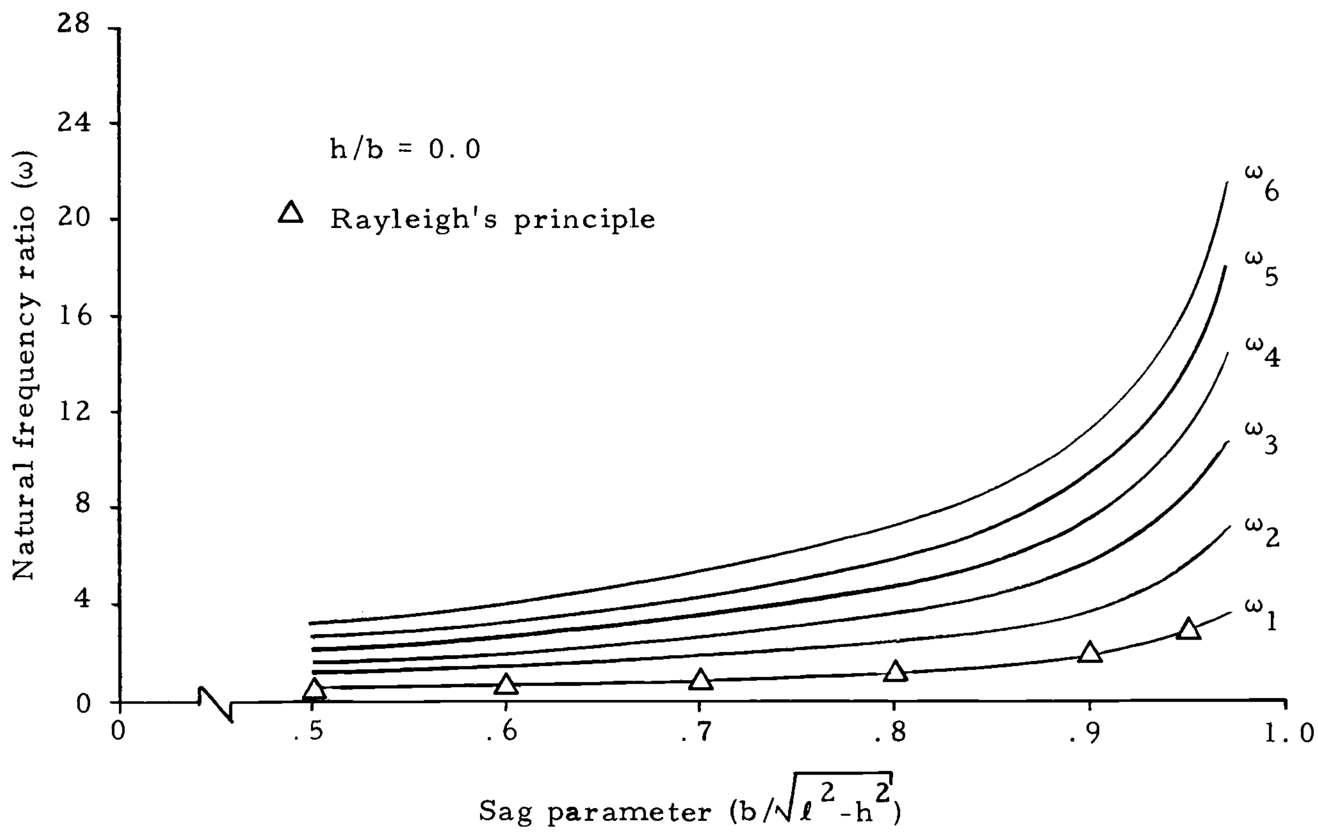

Figure 3.13. Natural frequency ratios for out-of-plane normal mode motion versus sag parameter for h/b = 0.0 including results obtained using Rayleigh's principle.

sinusoidally varying input displacement of frequency ω_f.

The displacements and tensions at any position along the cable for a prescribed input motion of frequency ω_f at one end of the cable are given by Equations 2.79, 2.80, and 2.81. Using the theory given in Section II.4.3 of this paper, a computer program was written which evaluated the tangential and normal displacements and tension in a cable at various points along the cable. Appendix C-1 is a listing of this program. Data Tables C-2 through C-3 give the cable displacements and tensions for various combinations of h/b, $b/\sqrt{\ell^2-h^2}$, and ω_f. Figures 3.14 and 3.15 graphically show the relation between the maximum dynamic tension in the cable and the forcing frequency for various combinations of h/b and $b/\sqrt{\ell^2-h^2}$. A glance at these figures shows that for forcing frequencies near the natural frequencies of the cable, then the maximum tension becomes large. For a forcing frequency equal to a natural frequency of the cable, then the maximum tension will be infinite. Figure 3.16 shows how the displacements and tensions vary along a cable for a typical cable geometry and forcing frequency. Figure 3.17 depicts the shape of a cable when it is oscillating near its lowest natural frequency. Figure 3.17(a) shows the maximum cable displacement when it is oscillating at slightly less than its lowest natural frequency and Figure 3.17(b) shows the maximum cable displacement when it is oscillating slightly higher than its lowest natural frequency. Since the cable is oscillating very near its lowest natural frequency in both of the above cases, then Figures 3.17(a) and (b) will very closely approximate the first mode shape of the in-plane oscillations of the cable.

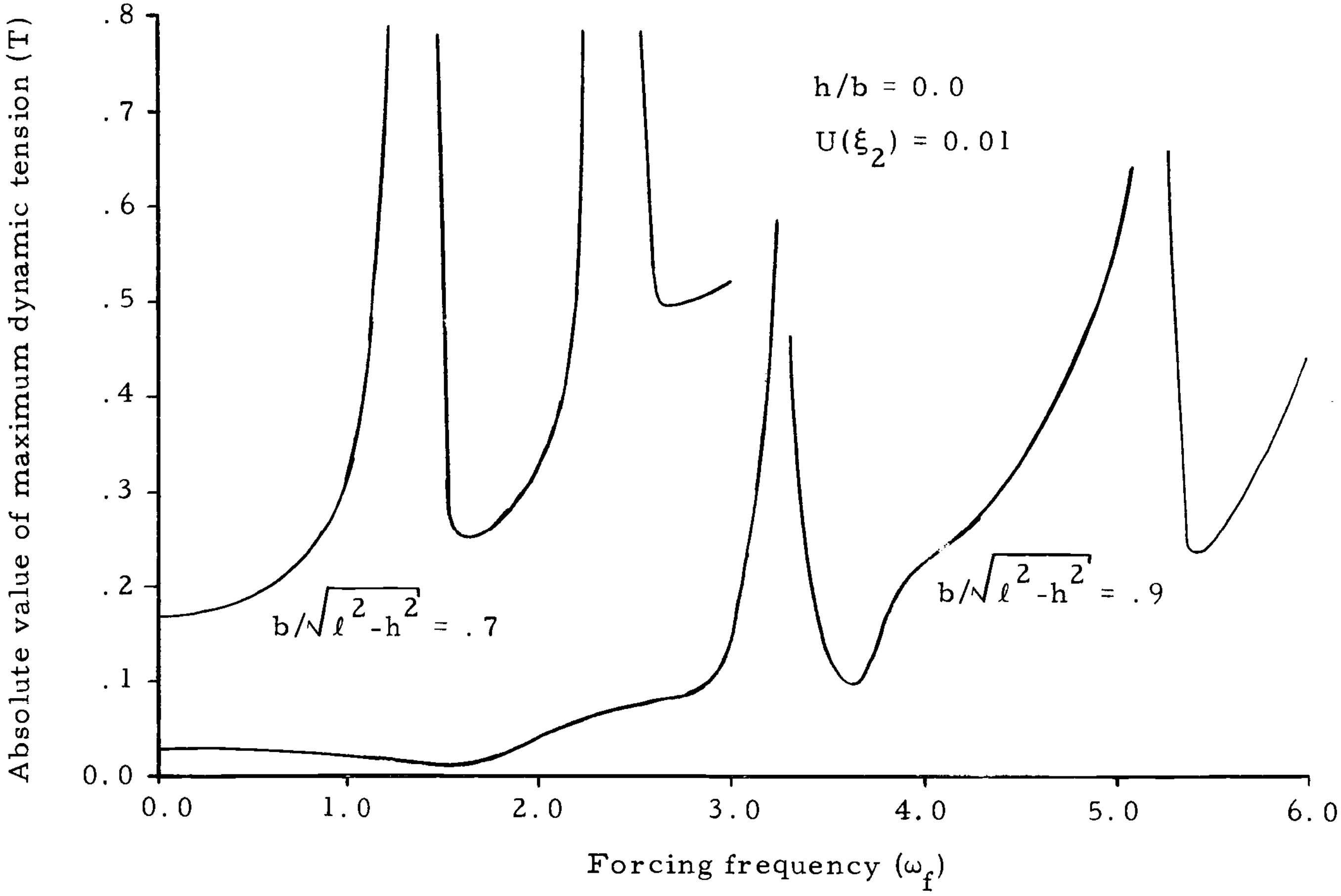

Figure 3.14. Maximum dynamic tension versus forcing frequency for various values of the sag parameter and $h/b = 0.0$.

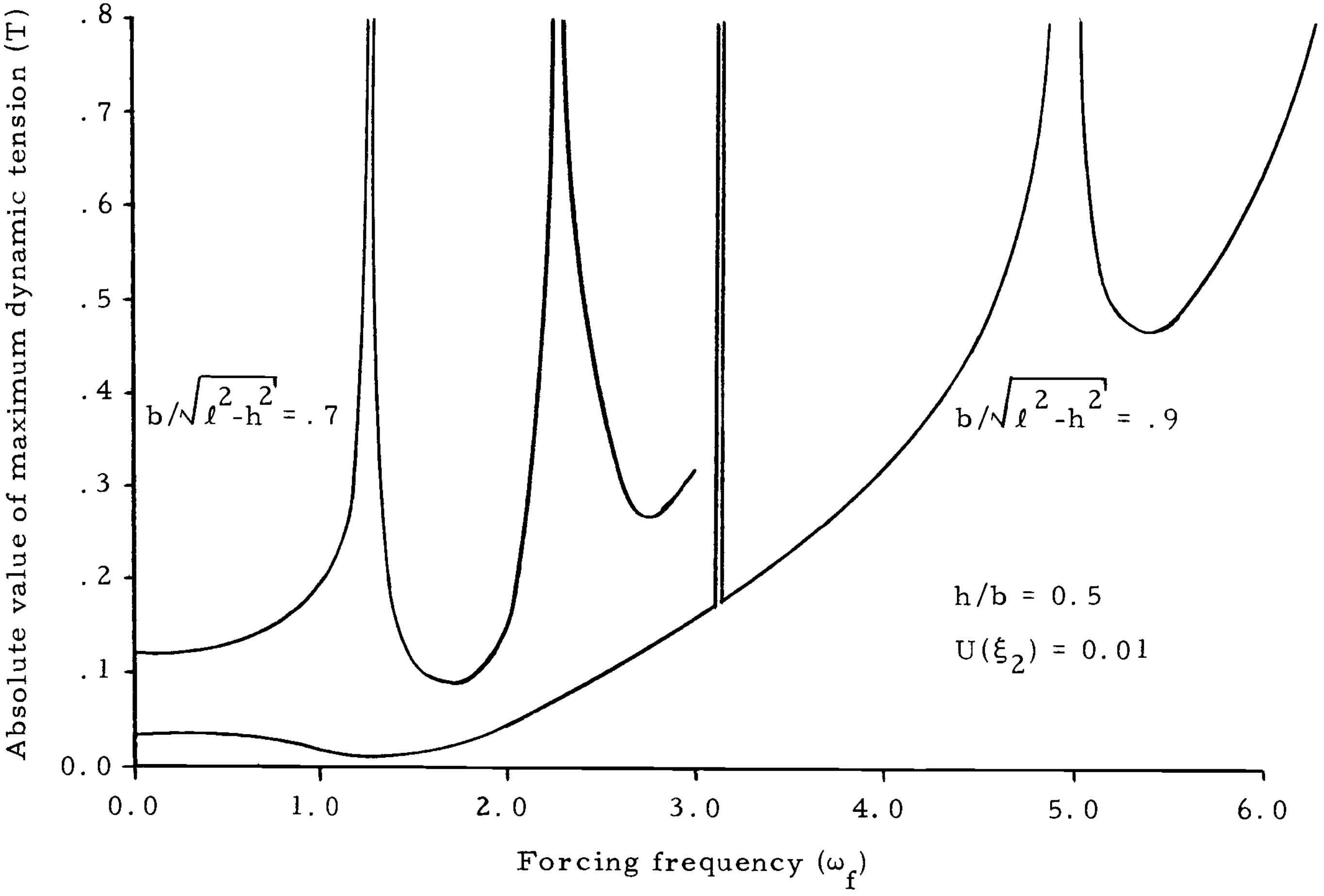

Figure 3.15. Maximum dynamic tension versus forcing frequency for various values of the sag parameter and h/b = 0.5.

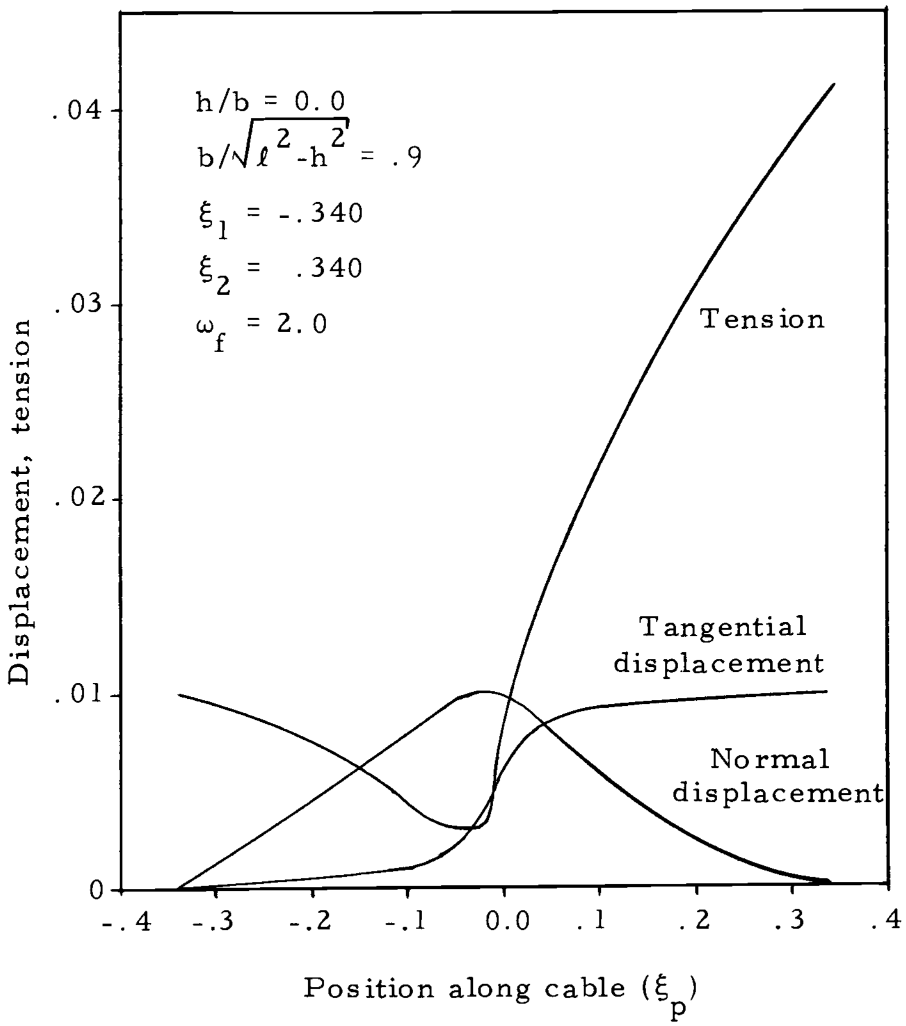

Figure 3.16. Normal and tangential displacements and dynamic tension versus position for $h/b = 0.0$, $b/\sqrt{\ell^2 - h^2} = .9$, and $\omega_f = 2.0$.

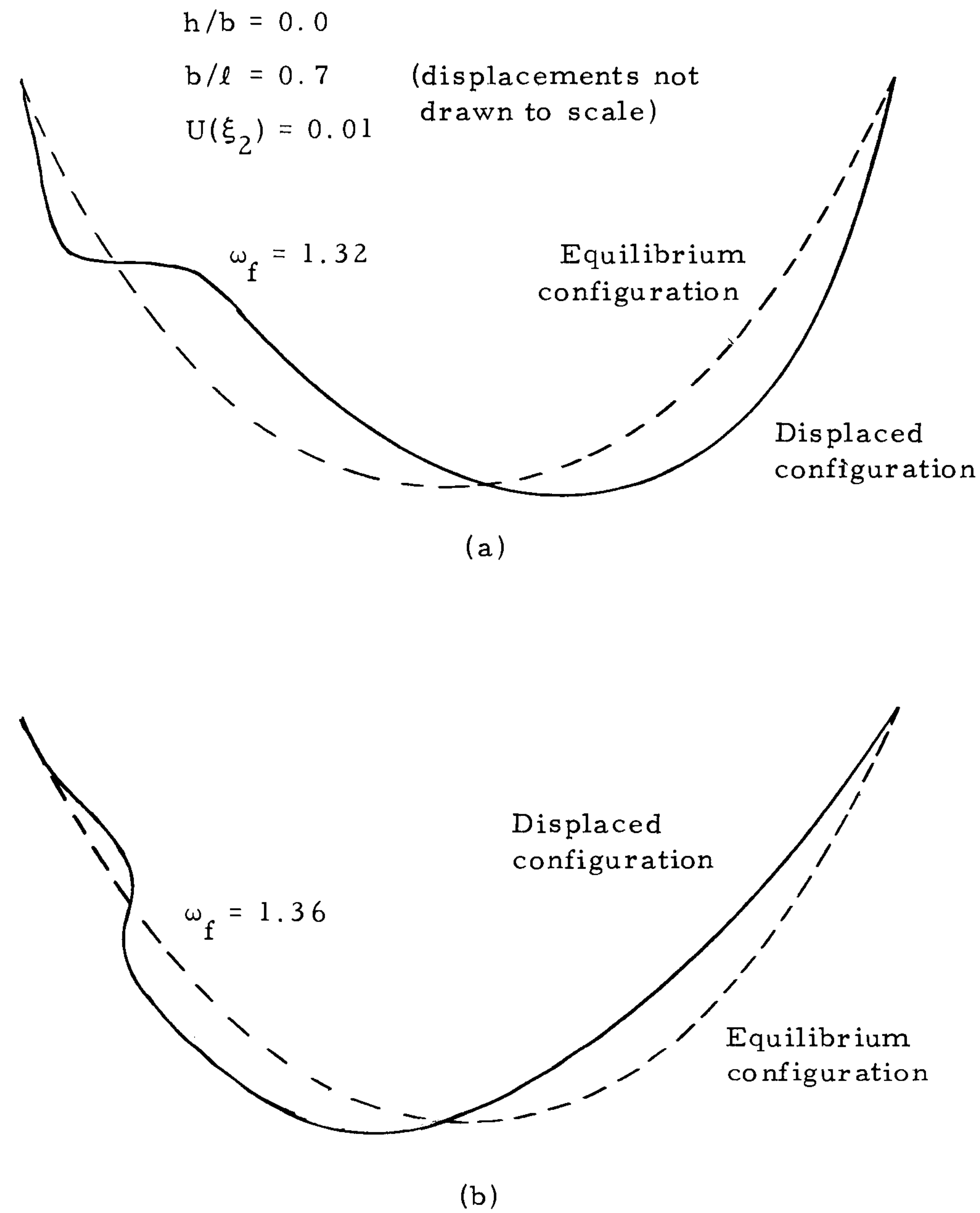

Figure 3.17. Cable displacement forced harmonically near its lowest natural frequency.

IV. CONCLUSIONS AND RECOMMENDATIONS

Prior to Smith and Thompson's analysis, all cable dynamics solutions had been approximations. The variable change which Smith and Thompson found was the breakthrough which allowed a closed form solution to the equations of motion of the catenary. This paper is an extension of the basic groundwork laid out by Smith and Thompson. The following list gives a brief description of the significant findings of this paper.

1. Whereas all previous work in cable dynamics had been concerned with the two dimensional, in-plane motion of the cable, this paper covered the three dimensional motion of the cable. Considered in the derivation of the equations of motion was the inclusion of a linear drag term which acted on the cable from the surrounding medium and the inclusion of elasticity (extensibility) of the cable.
2. Smith and Thompson's cable solution was limited to a range for the depth-to-span ratio of no greater than 0.76. Using Smith and Thompson's power series solution along with a second power series solution about a different point, it was possible to further extend the solution range for the motion equations. The solution range has been extended for depth-to span ratios of up to 1.09.

3. It was found that the equation governing the out-of-plane motion was independent of the equations governing the in-plane-motion of the cable in the linearized case. The same variable change which was used on the in-plane equation of motion to convert it to a form which contained polynomial coefficients was also successful for the out-of-plane motion equation in yielding an equation with polynomial coefficients.
4. The same solution procedure as was used for the in-plane motion was applicable for the out-of-plane motion. Series solutions were developed using the Frobenius method. Through the use of two separate expansions along with the appropriate series matching conditions, then the out-of-plane motion equation was solved for cables with a depth-to-span ratio of up to 1.09.
5. Based on the theory and equations developed for the normal-mode motion of the cable, then the overall motion of the cable was determined when subjected to an arbitrary oscillatory forcing function. A method and program were developed to evaluate the maximum displacements and tension at any point along a cable when one end of the cable was subjected to a prescribed sinusoidally varying tangential displacement of given frequency and magnitude. Resonance diagrams and corresponding displacement and tension

distributions were constructed using the data obtained from the analysis of forced motion of the cable.

The study and analysis of cable dynamics is a very broad topic and has applications in many different fields. Due to the scope of the subject, several areas in the analysis of cable dynamics were neglected and would lend themselves to further investigation. Areas in which further investigation and analysis would be warranted are as follows:

1. The solutions obtained for both the in-plane and out-of-plane equations of motion theoretically converge for depth-to-span ratios of up to 1.09. It may be desirable to analyze cable systems with depth-to-span ratios of greater than 1.09. There are two possible methods of increasing the solution range to handle depth-to-span ratios of greater than 1.09. First; a third series solution could be written about some point, say $\xi = 4$, and this solution could be matched to the second series solution which was written about the point $\xi = 2$. This technique would involve manipulation of three power series solutions. A second technique would be to write a power series solution about some point, say $\xi = 4$, and match this solution directly to the first series solution written about the point $\xi = 0$. This method would involve operations with only two power series solutions. The drawback of this

second method is that an excessive number of terms in the power series may be necessary to achieve the desired accuracy at the matching point with the first series.

2. The natural frequency ratios were calculated using 80 terms in each of the necessary series expansions. Through a series of trial runs, it was found that there was no significant difference between expansions constructed with 80 terms and 100 terms. These trial runs were for the first four derivatives of the expression for the tangential displacement of the cable at a frequency ratio of 25 and a cable angle of 77°. For frequency ratios of less than 25 or for cable angles, less than 77°, then it would be expected that fewer than 80 terms would be required to provide sufficient accuracy for computations of the natural frequency ratios. Hence it may be desirable to optimize the computer programs so that they could generate data with the necessary accuracy using the fewest number of terms in the series expansions. It would be possible to reduce computational costs in this way.
3. The general equations of motion were derived and included linear damping of the cable from its surrounding medium. The cable solutions given herein neglected the damping of the cable. For mooring lines, it would be expected that viscous damping of the cable would significantly affect the response

of the cable. Hence it would be desirable to include the effects of damping in the cable dynamics solutions. Lamotte [8] did some preliminary work in determining the natural frequency ratios for linearly damped cables with depth-to-span ratios of less than .76. The next logical step would be to extend Lamotte's work to cables with depth-to-span ratios of greater than .76. This could be accomplished by writing another series solution about an appropriate point and then matching the new series solution with the series solution developed by Lamotte. The technique would be similar to the one used in this paper for matching solutions for the undamped cable.

4. Besides linear damping, the derived equations of motion included a term for the extensibility of the cable. However in solving the equations of motion, it was assumed that the cable was inextensible. Equations 2.33 and 2.34 represent three equations in three unknowns U, V, and T. Some preliminary work was done with the three equations to reduce them to a single equation in one unknown U. In its most general form, the single differential equation which was derived from the three given equations was intractable. Only in the case when the cable was assumed to be inextensible does the single differential equation governing the tangential

displacement reduce to something which is manageable. Hence some analysis could be done to include the effects of extensible cables on the normal mode motions. It does not appear to be feasible to approach this problem by trying to solve Equations 2.33 and 2.34 in their most general forms.

5. The cable displacements and tensions were determined for the case where one end was forced to oscillate sinusoidally in the tangential direction at a frequency of ω_f. In order to be able to simulate an arbitrary displacement at one end of the cable, then the cable must also be able to oscillate in the normal directions. Hence it may be desirable to modify Equation 2.76 to include a sinusoidally oscillating normal displacement at position ξ_2.
6. Since the tension in the cable was not affected by the transverse motion of the cable, then the most interesting quantity that could be gleaned from the study of forced motion of the cable in the out-of-plane direction would be the displacement. It may be of interest to determine the displacements in the cable when one end is subjected to a sinusoidally oscillating transverse displacement at frequency ω_f. Analysis similar to that used in Section II.4.3 for the in-plane forced motion of the cable could be used for the out-of-plane motion of the cable.

7. Another area of work that could be expanded upon would be to use superposition techniques to determine the response of the cable to an arbitrary periodic input displacement at one end of the cable.

Recommendations 5 through 7 concern forced motion of the undamped, inextensible cable. Hence work could also be carried out on these three items when the cable is damped and extensible.

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APPENDICES

Data Table A-1. Cable parameter as a function of the sag parameter.

$b/\sqrt{\ell^2 - h^2}$	a/b
0.00	0.00000
0.01	0.06864
0.02	0.07725
0.03	0.08348
0.04	0.08862
0.05	0.09312
0.10	0.11111
0.15	0.12597
0.20	0.13973
0.25	0.15320
0.30	0.16681
0.35	0.18093
0.40	0.19587
0.45	0.21197
0.50	0.22964
0.55	0.24939
0.60	0.27195
0.65	0.29834
0.70	0.33015
0.75	0.37003
0.80	0.42275
0.85	0.49828
0.90	0.62233
0.95	0.89670

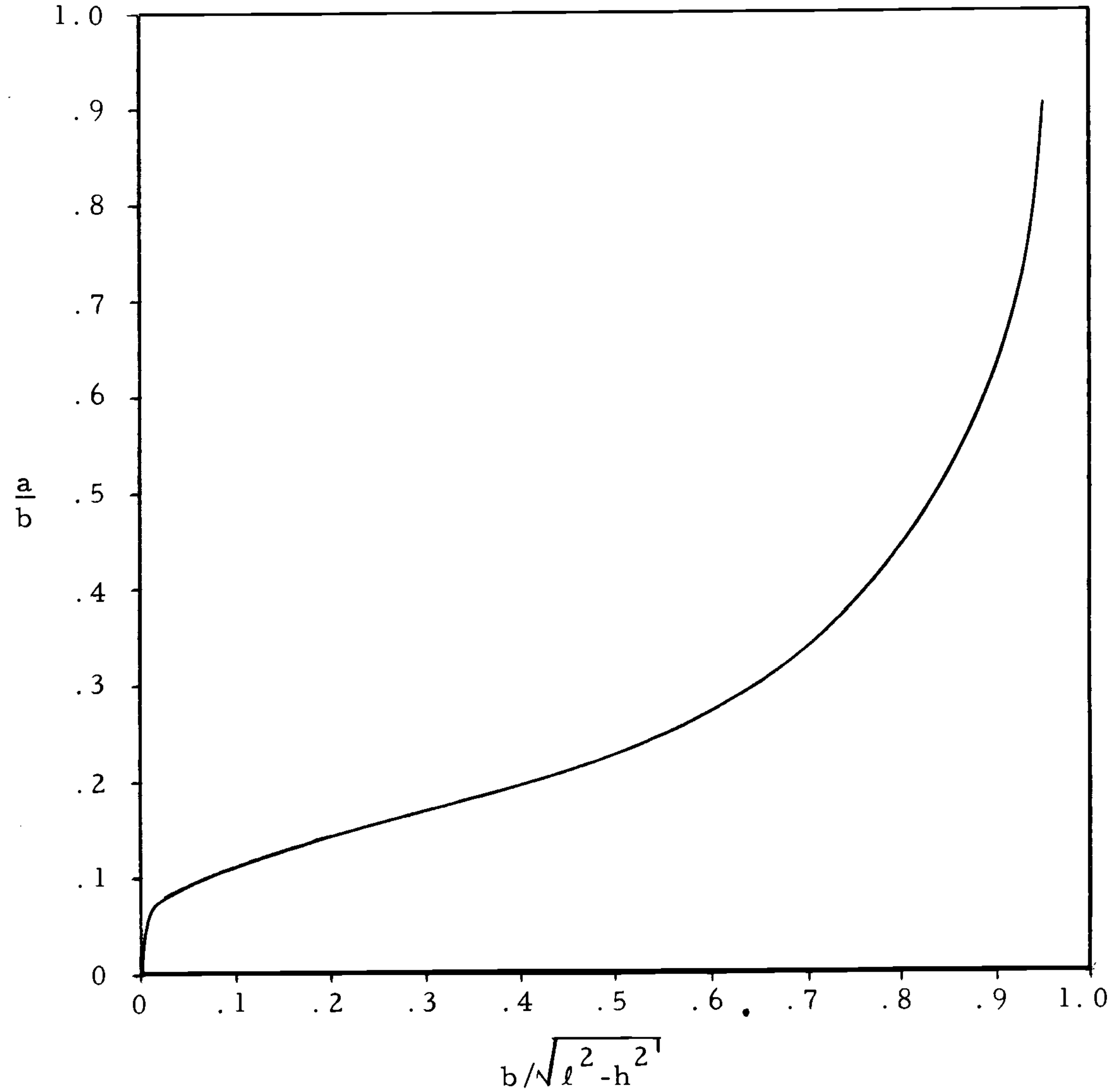


Figure A-2. Cable parameter a/b versus sag parameter $b/\sqrt{\ell^2 - h^2}$.

Appendix A-3. Convergence region program

```
      PROGRAM REGION (INPUT,OUTPUT,TAPE60=INPUT,TAPE61=OUTPUT)
    1 READ (60,6) HOB,X2
      IF (EOF(60)) 5,2
    2 S2 = X2*SQRT(2./X2+1.)
      S2A = SQRT(1.+S2*S2)
      S1 = -1.0
    3 S1A = SQRT(1.+S1*S1)
      WRITE (61,8) S1,S1A
      S1N = S1-(S1A-HOB*ALOG(S1A+S1)-S2A+HOB*ALOG(S2A+S2))/(S1/S1A-HOB/
     $(S1A+S1)*(S1/S1A+1.))
      IF (ABS((S1N-S1)/S1N).LE..0001) GO TO 4
      S1 = S1N
      GO TO 3
    4 X1 = S1N/ABS(S1N)*(SQRT(1.+S1N*S1N)-1.)
      RLOA = S2-S1N
      BOA = ALOG((S2A+S2)/(SQRT(1.+S1N*S1N)+S1N))
      BOSR = 1./SQRT((RLOA/BOA)**2-HOB*HOB)
      WRITE (61,7) X2,X1,HOB,BOA,BOSR
      GO TO 1
    5 STOP
C
    6 FORMAT (2F5.2)
    7 FORMAT (//1X,4HX2 =,F6.3,5X,4HX1 =,F6.3,5X,10HH OVER B =,F5.2,5X,
     $10HB OVER A =,F6.3,5X//,1X,26HB OVER SQRT(L,SQD-H,SQD) =,F6.3)
    8 FORMAT (1X,2(5X,F8.4))
      END
```

Data Table A-4. Cable equation convergence region data.

Support Parameter h/b	Right Boundary ξ_2	Minimum Left Boundary ξ_1	Cable Parameter a/b	Minimum Sag Parameter $b/\sqrt{\ell^2-h^2}$
0.00	4.0	-4.000	0.218	0.468
0.25	4.0	-2.916	0.231	0.503
0.50	4.0	-1.977	0.247	0.545
0.75	4.0	-1.208	0.269	0.593
1.00	4.0	-0.634	0.297	0.648
1.50	4.0	-0.057	0.380	0.761
2.00	4.0	0.051	0.506	0.854
2.50	4.0	0.377	0.690	0.918
3.00	4.0	0.907	0.970	0.957
3.50	4.0	1.580	1.445	0.980
4.00	4.0	2.364	2.445	0.993

```
      PROGRAM CABLE (INPUT,CUTPUT,TAPE60=INPUT,TAPE61=OLTPUT)
      DIMENSION R1(4), R2(4), A(200,4), B(200,4), V(4,4), D(4,4), UR(4,4
     $), VR(4,8), ALPHA(4,4), UL(4,4), BETA(4,4), DET(2)
      COMMON N,Y(2)
    1 READ (60,34) HOB,BOSR,N,Y0,XR
      IF (EOF(60)) 31,2
    2 WRITE (61,35) N,Y0,XR
      BOA = 1.0
    3 EX = EXP(BOA)
      EXI = 1./EX
      SIH = (EX-EXI)*.5
      COH = (EX+EXI)*.5
      BOAN = BOA-(BOA/SIH-BOSR)/((1./SIH)*(1.-BOA*COH/SIH))
      IF (ABS((BOA-BOAN)/BOAN)-.001) 5,5,4
    4 BOA = BOAN
      GO TO 3
    5 AOBN = 1./BOAN
      AOLH = AOBN*AOBN*BOSR*BOSR
      HLOA = BOAN*(HOB+SQRT(BOSR*BOSR*HOB*HOB+1.)/BOSR)
      A1 = (SQRT(1.+AOLH)-1.)
      A2 = A1+2.
      S1 = .5*(A1*HLOA-1./(A1*HLOA))
      S2 = .5*(A2*HLOA-1./(A2*HLOA))
      X1 = S1/ABS(S1)*(SQRT(1.+S1*S1)-1.)
      X2 = SQRT(1.+S2*S2)-1.
      WRITE (61,36) HOB,BOSR,AOBN,X1,X2
      YA = Y0
      DO 6 I=1,4
         R2(I) = I-1.
    6 R1(I) = R2(I)*.5
    7 DEL = .0005*ABS(YA)
      Y(1) = YA-DEL/2.
      Y(2) = Y(1)+DEL
      DO 29 L=1,2
         IF (X2.GE..7.AND.X1.LT..7) GO TO 13
         IF (X2.GE..7.AND.X1.GE..7) GO TO 11
         IF (X1.LT.0.0) GO TO 9
C     X2 LESS THAN .7 AND X1 GREATER THAN 0.
         CALL ACOEF (L,A)
         II = 1
         M = 1
         CALL VMAT (II,X1,M,A,R1,V)
         II = 3
         CALL VMAT (II,X2,M,A,R1,V)
         DO 8 I=1,4
            DO 8 J=1,4
    8    D(I,J) = V(I,J)
         GO TO 27
C     X2 LESS THAN .7 AND X1 BETWEEN -.7 AND 0.
    9    CALL ACOEF (L,A)
         II = 3
         M = 1
         CALL VMAT (II,X2,M,A,R1,V)
         DO 10 I=3,4
            DO 10 J=1,4
   10    D(I,J) = V(I,J)
         GO TO 19
C     X2 AND X1 BOTH GREATER THAN .7.
   11    X1R = X1-2.0
         X2R = X2-2.0
         CALL BCOEF (L,B)
         II = 1
         M = 2
         CALL VMAT (II,X1R,M,B,R2,V)
         II = 3
         CALL VMAT (II,X2R,M,B,R2,V)
         DO 12 I=1,4
            DO 12 J=1,4
   12    D(I,J) = V(I,J)
         GO TO 27
C     X2 GREATER THAN .7, THREE POSSIBILITIES EXIST FOR X1.
   13    CALL ACOEF (L,A)
         CALL BCOEF (L,B)
         M = 1
         CALL AMAT (XR,A,R1,M,UR)
         XRM = XR-2.0
         M = 2
         CALL AMAT (XRM,B,R2,M,VR)
         DO 15 I=1,4
            DO 15 J=5,8
               IF (I+4.EQ.J) GO TO 14
               VR(I,J) = 0.0
               GO TO 15
   14          VR(I,J) = 1.0
   15    CONTINUE
         CALL INVERS (VR)
         DO 16 I=1,4
            DO 16 J=1,4
               DO 16 K=1,4
                  IF (K.EQ.1) ALPHA(I,J) = 0.0
   16    ALPHA(I,J) = VR(I,K)*UR(K,J)+ALPHA(I,J)
         X2B = X2-2.0
         II = 3
         M = 2
         CALL VMAT (II,X2B,M,B,R2,V)
         DO 17 I=3,4
            DO 17 J=1,4
               DO 17 K=1,4
                  IF (K.EQ.1) D(I,J) = 0.0
   17    D(I,J) = V(I,K)*ALPHA(K,J)+D(I,J)
         IF (X1.LE.-.7) GO TO 21
         IF (X1.LT.0.0) GO TO 19
C     X1 IS BETWEEN 0.0 AND .7
         II = 1
         M = 1
         CALL VMAT (II,X1,M,A,R1,V)
         DO 18 I=1,2
            DO 18 J=1,4
   18    D(I,J) = V(I,J)
         GO TO 27
C     X1 IS BETWEEN -.7 AND 0.0
   19    X1A = ABS(X1)
         II = 1
         M = 1
         CALL VMAT (II,X1A,M,A,R1,V)
         DO 20 I=1,2
            DO 20 J=1,4
               V(I,J) = (-1)**(I+J)*V(I,J)
   20    D(I,J) = V(I,J)
         GO TO 27
C     X1 LESS THAN OR EQUAL TO -.7
   21    X1B = -X1-2.0
         DO 23 I=1,4
            DO 23 J=1,4
               IF (J.EQ.2.OR.J.EQ.4) GO TO 22
               UL(I,J) = UR(I,J)
               GO TO 23
   22          UL(I,J) = -UR(I,J)
   23    CONTINUE
         DO 24 I=1,4
            DO 24 J=1,4
               DO 24 K=1,4
                  IF (K.EQ.1) BETA(I,J) = 0.0
   24    BETA(I,J) = VR(I,K)*UL(K,J)+BETA(I,J)
         II = 1
         M = 2
         CALL VMAT (II,X1B,M,B,R2,V)
         DO 25 J=1,4
   25    V(2,J) = -V(2,J)
         DO 26 I=1,2
            DO 26 J=1,4
               DO 26 K=1,4
                  IF (K.EQ.1) D(I,J) = 0.0
```

Appendix B-1. In-plane natural frequency program.

```
26     D(I,J) = V(I,K)*BETA(K,J)+D(I,J)
27     DO 28 I=1,4
28     WRITE (61,32) (D(I,J),J=1,4)
       DET(L) = D(1,1)*(D(2,2)*D(3,3)*D(4,4)+D(3,2)*D(4,3)*D(2,4)+D(4,
   $   2)*D(3,4)*D(2,3)-D(2,4)*D(3,3)*D(4,2)-D(3,4)*D(4,3)*D(2,2)-D(4,
   $   4)*D(3,2)*D(2,3))-D(1,2)*(D(2,1)*D(3,3)*D(4,4)+D(3,1)*D(4,3)*D
   $   (2,4)+D(4,1)*D(3,4)*D(2,3)-D(4,1)*D(3,3)*D(2,4)-D(4,3)*D(3,4)*D
   $   (2,1)-D(4,4)*D(2,3)*D(3,1))+D(1,3)*(D(2,1)*D(3,2)*D(4,4)+D(3,1)
   $   *D(4,2)*D(2,4)+D(4,1)*D(3,4)*D(2,2)-D(4,1)*D(3,2)*D(2,4)-D(4,2)
   $   *D(3,4)*D(2,1)-D(4,4)*D(2,2)*D(3,1))
       DET(L) = DET(L)-D(1,4)*(D(2,1)*D(3,2)*D(4,3)+D(3,1)*D(4,2)*D(2,
   $   3)+D(4,1)*D(3,3)*D(2,2)-D(4,1)*D(3,2)*D(2,3)-D(4,2)*D(3,3)*D(2,
   $   1)-D(4,3)*D(2,2)*D(3,1))
       WRITE (61,33) L,DET(L)
29  CONTINUE
    IF (DET(1)*DET(2).LT.0.0) GO TO 30
    DER = (DET(2)-DET(1))/DEL
    YA = Y(2)-DET(2)/DER
    WRITE (61,37) YA
    GO TO 7
30  WRITE (61,38) Y0,DEL,YA
    GO TO 1
31  STOP
C
32  FORMAT (1X,4(E14.7,5X))
33  FORMAT (1X,4HDET(,I1,2H)=,2X,E14.7)
34  FORMAT (2F8.3,I3,F8.6,F8.2)
35  FORMAT (//15H NO. OF TERMS =,I3,5X,25HINITIAL FRECUENCY GUESS =,F6
   $.2,5X,16HMATCHING POINT =,F6.2)
36  FORMAT (1X,10HH OVER B =,F8.3,5X,31HB OVER SQRT. OF L.SQRD-H.SQRD
   $=,F8.3,5X,11H2A OVER B =,F8.3/1X,18HLEFT BOUNDARY X1 =,F8.3,5X,
   $19HRIGHT BOUNDARY X2 =,F8.3)
37  FORMAT (1X,30HINTERMEDIATE FREQUENCY VALUE =,F8.4)
38  FORMAT (1X,25HINITIAL FREQUENCY GUESS =,F6.2,5X,7HDELTA =,F8.5,5X,
   $23HFINAL FREQUENCY VALUE =,F8.4)
    END

    SUBROUTINE ACOEF (L,A)
    DIMENSION A(200,4)
    COMMON N,Y(2)
    ZL = Y(L)*Y(L)
    NUM = N+6
    DO 2 IR=1,4
       RR = IR
       R = (RR-1.0)/2.
       A(1,IR) = 0.0
       A(2,IR) = 0.0
       A(3,IR) = 0.0
       A(4,IR) = 1.0
       A(5,IR) = -A(4,IR)*(4.*R*R+R+1.+ZL)/((R+1.)*(2.*R+1.))
       DO 1 J=6,NUM
          G = J
          K1 = J-1
          K2 = J-2
          K3 = J-3
          K4 = J-4
          F1 = G+R-5.
          F2 = G+R-6.
          F3 = G+R-7.
          F4 = G+R-8.
          F5 = G+R-9.
          F6 = G+R-4.
          F7 = G+R-1.
          F8 = G+R
          A(J,IR) = -A(K1,IR)*(2.*F2*F3*(4.*F6+7.)+(51.+2.*ZL)*F6-3.*
   $      (32.+ZL))/(F6*(4.*F6*F2+3.))-A(K2,IR)*(F2*(5.*F3*F4*F7+(72.+
   $      5.*ZL)*F6-6.*(32.+ZL))-ZL)/(F6*F1*(4.*F6*F2+3.))-A(K3,IR)*F3
   $      *(F4*F5*F8+4.*(6.+ZL)*F6-4.*(21.+ZL))/(F6*F1*(4.*F6*F2+3.))-
   $      A(K4,IR)*ZL*F4/(F6*(4.*F6*F2+3.))
1      CONTINUE
       DO 2 J=4,NUM
          A(J-3,IR) = A(J,IR)
2   CONTINUE
    RETURN
    END

    SUBROUTINE BCOEF (L,B)
    DIMENSION B(200,4)
    COMMON N,Y(2)
    ZL = Y(L)*Y(L)
    NUM = N+8
    DO 2 IR=1,4
       RR = IR
       R = RR-1.
       B(1,IR) = 0.0
       B(2,IR) = 0.0
       B(3,IR) = 0.0
       B(4,IR) = 0.0
       B(5,IR) = 0.0
       B(6,IR) = 1.0
       B(7,IR) = -B(6,IR)*(688.+352.*(R-3.))/(192.*(R+1.))
       B(8,IR) = -B(7,IR)*(688.+352.*(R-2.))/(192.*(R+2.))-B(6,IR)*
   $   (585.+72.*ZL+996.*(R-2.)+252.*(R-2.)*(R-3.))/(192.*(R+2.)*(R+1.
   $   ))
       B(9,IR) = -B(8,IR)*(688.+352.*(R-1.))/(192.*(R+3.))-B(7,IR)*
   $   (585.+72.*ZL+996.*(R-1.)+252.*(R-1.)*(R-2.))/(192.*(R+3.)*(R+2.
   $   ))-B(6,IR)*(102.+99.*ZL+(627.+102.*ZL)*(R-1.)+526.*(R-1.)*(R-2.
   $   )+88.*(R-1.)*(R-2.)*(R-3.))/(192.*(R+3.)*(R+2.)*(R+1.))
       DO 1 J=10,NUM
          G = J
          K1 = J-1
          K2 = J-2
          K3 = J-3
          K4 = J-4
          K5 = J-5
          K6 = J-6
          F1 = G+R-6.
          F2 = G+R-7.
          F3 = G+R-8.
          F4 = G+R-9.
          F5 = G+R-10.
          F6 = G+R-11.
          F7 = G+R-12.
          F8 = G+R-13.
          F9 = G+R-14.
          B(J,IR) = -B(K1,IR)*(688.+352.*F5)/(192.*F1)-B(K2,IR)*(585.+
   $      72.*ZL+996.*F5+252.*F5*F6)/(192.*F1*F2)-B(K3,IR)*(102.+99.*
   $      ZL+(627.+102.*ZL)*F5+526.*F5*F6+88.*F5*F6*F7)/(192.*F1*F2*F3
   $      )-B(K4,IR)*(-ZL+(72.+105.*ZL)*F5+(216.+53.*ZL)*F5*F6+120.*F5
   $      *F6*F7+15.*F5*F6*F7*F8)/(192.*F1*F2*F3*F4)-B(K5,IR)*((12.+36
   $      .*ZL)*F6+(24.+12.*ZL)*F6*F7+10.*F6*F7*F8+F6*F7*F8*F9)/(192.*
   $      F1*F2*F3*F4)-B(K6,IR)*(4.*ZL*F7+ZL*F7*F8)/(192.*F1*F2*F3*F4)
1      CONTINUE
       DO 2 J=6,NUM
          B(J-5,IR) = B(J,IR)
2   CONTINUE
    RETURN
    END

    SUBROUTINE AMAT (X,A,R,M,U)
    DIMENSION A(200,4), R(4), U(4,4)
    COMMON N,Y(2)
    DO 7 J=1,4
       IF (M.EQ.1) GO TO 1
       NA = R(J)
       B = X**NA
       D = B/X
       E = D/X
       F = E/X
       GO TO 2
1      B = X**R(J)
       D = B/X
       E = D/X
       F = E/X
```

```
2     U(1,J) = A(N+1,J)
      DO 3 IA=1,N
         I = N-IA
3     U(1,J) = U(1,J)*X+A(I+1,J)
      AUX1 = U(1,J)
      U(1,J) = AUX1*B
      U(2,J) = A(N+1,J)*N
      DO 4 IA=2,N
         I = N+2-IA
4     U(2,J) = U(2,J)*X+(I-1)*A(I,J)
      AUX2 = U(2,J)
      U(2,J) = AUX2*B+R(J)*AUX1*D
      U(3,J) = A(N+1,J)*N*(N-1)
      DO 5 IA=3,N
         I = N+3-IA
5     U(3,J) = U(3,J)*X+(I-2)*(I-1)*A(I,J)
      AUX3 = U(3,J)
      U(3,J) = AUX3*B+R(J)*(R(J)-1.)*E*AUX1+2.*R(J)*D*AUX2
      U(4,J) = A(N+1,J)*N*(N-1)*(N-2)
      DO 6 IA=4,N
         I = N+4-IA
6     U(4,J) = U(4,J)*X+(I-1)*(I-2)*(I-3)*A(I,J)
      U(4,J) = U(4,J)*B+R(J)*(R(J)-1.)*(R(J)-2.)*F*AUX1+3.*R(J)*(R(J)
 $    -1.)*E*AUX2+3.*R(J)*D*AUX3
7 CONTINUE
  RETURN
  END

   SUBROUTINE VMAT (II,X,M,A,R,V)
   DIMENSION A(200,4), R(4), V(4,4)
   COMMON N,Y(2)
   I1 = II+1
   DO 5 J=1,4
      IF (M.EQ.1) GO TO 1
      NA = R(J)
      C = X**NA
      D = C/X
      GO TO 2
1     C = X**R(J)
      D = C/X
2     V(II,J) = A(N+1,J)
      DO 3 IA=1,N
         I = N-IA
3     V(II,J) = V(II,J)*X+A(I+1,J)
      AUX1 = V(II,J)
      V(II,J) = AUX1*C
      V(I1,J) = A(N+1,J)*N
      DO 4 IA=2,N
         I = N+2-IA
4     V(I1,J) = V(I1,J)*X+(I-1)*A(I,J)
      V(I1,J) = V(I1,J)*C+AUX1*R(J)*D
5 CONTINUE
  RETURN
  END

   SUBROUTINE INVERS (C)
   DIMENSION C(4,8), X(4)
   COMMON N,Y(2)
   DO 3 I=1,4
      D = 0.0
      DO 2 J=1,4
         IF (ABS(C(I,J))-ABS(D)) 2,2,1
1        D = C(I,J)
2     CONTINUE
      DO 3 J=1,8
3 C(I,J) = C(I,J)/D
  DO 12 M=1,4
      KK = 2*4+1-M
      K = 4+2-M
      D = 0.0
      L = 1
      DO 5 I=2,K
         IF (ABS(C(I-1,1))-D) 5,5,4
4        L = I-1
         D = ABS(C(L,1))
5     CONTINUE
      IF (L-1) 6,8,5
6     DO 7 J=1,KK
         D = C(L,J)
         C(L,J) = C(1,J)
         C(1,J) = D
7     CONTINUE
8     CONTINUE
      DO 9 I=1,4
9     X(I) = C(I,1)
      DO 11 J=2,KK
         D = C(1,J)/X(1)
         DO 10 I=2,4
10       C(I-1,J-1) = C(I,J)-X(I)*D
11    C(4,J-1) = D
12 CONTINUE
   RETURN
   END
```

Appendix B-2. Out-of-plane natural frequency program

```
      PROGRAM TRANS
      DIMENSION R(2), SR(2), C(200,2), V(2,2), D(2,2), S(200,2), FR(2,2)
     $, GR(2,2), GRI(2,2), ALPHA(2,2), FL(2,2), BETA(2,2), DET(2)
      COMMON N,W(2)
    1 READ (60,24) HOB,BOSR,N,W0,XR
      IF (EOF(60)) 23,2
    2 WRITE (61,25) N,W0,XR
      BOA = 1.0
    3 EX = EXP(BOA)
      EXI = 1./EX
      SIH = (EX-EXI)*.5
      COH = (EX+EXI)*.5
      BOAN = BOA-(BOA/SIH-BOSR)/((1./SIH)*(1.-BOA*COH/SIH))
      IF (ABS((BOA-BOAN)/BOAN)-.001) 5,5,4
    4 BOA = BOAN
      GO TO 3
    5 AOBN = 1./BOAN
      AOLH = AOBN*AOBN*BOSR*BOSR
      HLOA = BOAN*(HOB+SQRT(BOSR*BOSR*HOB*HOB+1.)/BOSR)
      A1 = (SQRT(1.+AOLH)-1.)
      A2 = A1+2.
      S1 = .5*(A1*HLOA-1./(A1*HLOA))
      S2 = .5*(A2*HLOA-1./(A2*HLOA))
      X1 = S1/ABS(S1)*(SQRT(1.+S1*S1)-1.)
      X2 = SQRT(1.+S2*S2)-1.
      WRITE (61,26) HOB,BOSR,AOBN,X1,X2
      WA = W0
      R(1) = 0.0
      R(2) = 0.5
      SR(1) = 0.0
      SR(2) = 1.0
    6 DEL = .0005*ABS(WA)
      W(1) = WA-DEL/2.0
      W(2) = W(1)+DEL
      DO 21 L=1,2
         IF (X2.GE.1.5.AND.X1.LT.1.5) GO TO 12
         IF (X2.GE.1.5.AND.X1.GE.1.5) GO TO 10
         IF (X1.LT.0.0) GO TO 8
C     X2 LESS THAN 1.5 AND X1 GREATER THAN 0.
         CALL CCOEF (L,R,C)
         II = 1
         M = 1
         CALL BOUMAT (II,X1,M,C,R,V)
         II = 2
         CALL BOUMAT (II,X2,M,C,R,V)
         DO 7 I=1,2
            DO 7 J=1,2
    7    D(I,J) = V(I,J)
         GO TO 20
C     X2 LESS THAN 1.5 AND X1 BETWEEN -1.5 AND 0.
    8    CALL CCOEF (L,R,C)
         II = 2
         M = 1
         CALL BOUMAT (II,X2,M,C,R,V)
         DO 9 J=1,2
    9    D(2,J) = V(2,J)
         GO TO 16
C     X2 AND X1 BOTH GREATER THAN 1.5.
   10    X1R = X1-2.0
         X2R = X2-2.0
         CALL SCOEF (L,SR,S)
         II = 1
         M = 2
         CALL BOUMAT (II,X1R,M,S,SR,V)
         II = 2
         CALL BOUMAT (II,X2R,M,S,SR,V)
         DO 11 I=1,2
            DO 11 J=1,2
   11    D(I,J) = V(I,J)
         GO TO 20
C     X2 GREATER THAN 1.5, THREE POSSIBILITIES EXIST FOR X1
   12    CALL CCOEF (L,R,C)
         CALL SCOEF (L,SR,S)
         M = 1
         CALL UMAT (XR,C,R,M,FR)
         XRM = XR-2.0
         M = 2
         CALL UMAT (XRM,S,SR,M,GR)
         CALL INVERS (GR,GRI)
         DO 13 I=1,2
            DO 13 J=1,2
               DO 13 K=1,2
                  IF (K.EQ.1) ALPHA(I,J) = 0.0
   13    ALPHA(I,J) = GRI(I,K)*FR(K,J)+ALPHA(I,J)
         X2R = X2-2.0
         II = 2
         M = 2
         CALL BOUMAT (II,X2R,M,S,SR,V)
         DO 14 J=1,2
            DO 14 K=1,2
               IF (K.EQ.1) D(2,J) = 0.0
   14    D(2,J) = V(2,K)*ALPHA(K,J)+D(2,J)
         IF (X1.LE.-1.5) GO TO 17
         IF (X1.LT.0.0) GO TO 16
C     X1 IS BETWEEN 0.0 AND 1.5.
         II = 1
         M = 1
         CALL BOUMAT (II,X1,M,C,R,V)
         DO 15 J=1,2
   15    D(1,J) = V(1,J)
         GO TO 20
C     X1 IS BETWEEN -1.5 AND 0.
   16    X1A = ABS(X1)
         II = 1
         M = 1
         CALL BOUMAT (II,X1A,M,C,R,V)
         D(1,1) = V(1,1)
         D(1,2) = -V(1,2)
         GO TO 20
C     X1 IS LESS THAN OR EQUAL TO -1.5.
   17    X1B = -X1-2.0
         FL(1,1) = FR(1,1)
         FL(1,2) = -FR(1,2)
         FL(2,1) = FR(2,1)
         FL(2,2) = -FR(2,2)
         DO 18 I=1,2
            DO 18 J=1,2
               DO 18 K=1,2
                  IF (K.EQ.1) BETA(I,J) = 0.0
   18    BETA(I,J) = GRI(I,K)*FL(K,J)+BETA(I,J)
         II = 1
         M = 2
         CALL BOUMAT (II,X1B,M,S,SR,V)
         DO 19 J=1,2
            DO 19 K=1,2
               IF (K.EQ.1) D(1,J) = 0.0
   19    D(1,J) = V(1,K)*BETA(K,J)+D(1,J)
   20    DET(L) = D(1,1)*D(2,2)-D(2,1)*D(1,2)
   21 CONTINUE
      IF (DET(1)*DET(2).LE.0.0) GO TO 22
      DER = (DET(2)-DET(1))/DEL
      WA = W(2)-DET(2)/DER
      WRITE (61,27) WA
      GO TO 6
   22 WRITE (61,28) W0,DEL,WA
      GO TO 1
   23 STOP
C
   24 FORMAT (2F8.3,I3,2F6.2)
   25 FORMAT (//15H NO. OF TERMS =,I3,5X,25HINITIAL FREQUENCY GUESS =,F6
     $.2,5X,16HMATCHING POINT =,F6.2)
   26 FORMAT (1X,10HH OVER B =,F8.3,5X,31HB OVER SQRT. OF L.SQRD-H.SQRD
     $=,F8.3,5X,11H2A OVER B =,F8.3/1X,18HLEFT BOUNDARY X1 =,F8.3,5X,
```

```
     $19HRIGHT BOUNDARY X2 =,F8.3)
 27  FORMAT (1X,30HINTERMEDIATE FREQUENCY VALUE =,F8.4)
 28  FORMAT (1X,25HINITIAL FREQUENCY GUESS =,F6.2,5X,7HDELTA =,F8.5,5X,
     $23HFINAL FREQUENCY VALUE =,F8.4)
     END

      SUBROUTINE CCOEF (L,R,C)
      DIMENSION C(200,2), R(2)
      COMMON N,W(2)
      WS = W(L)*W(L)
      NUM = N+3
      DO 1 IR=1,2
         C(1,IR) = 1.0
         C(2,IR) = -C(1,IR)*(R(IR)**2+WS)/(2.*(1.+R(IR))**2-(1.+R(IR)))
         DO 1 J=3,NUM
            G = J
            J1 = J-1
            J2 = J-2
            F1 = G+R(IR)-1.
            F2 = G+R(IR)-2.
            F3 = G+R(IR)-3.
            DEN = 2.*F1*F2+F1
 1    C(J,IR) = -C(J1,IR)*(F2*F3+F2+WS)/DEN-C(J2,IR)*(+WS)/DEN
      RETURN
      END

      SUBROUTINE SCOEF (L,SR,S)
      DIMENSION S(200,2), SR(2)
      COMMON N,W(2)
      WS = W(L)*W(L)
      NUM = N+5
      DO 1 IR=1,2
         S(1,IR) = 1.0
         S(2,IR) = -S(1,IR)*(6.*(SR(IR)-1.)+3.)/(8.*(1.+SR(IR)))
         S(3,IR) = -S(2,IR)*(3.*(1.+SR(IR)))/(8.*(2.+SR(IR))*(1.+SR(IR))
     $   )-S(1,IR)*3.*WS/(8.*(2.+SR(IR))*(1.+SR(IR)))
         DO 1 J=4,NUM
            G = J
            J1 = J-1
            J2 = J-2
            J3 = J-3
            F1 = G+SR(IR)-1.
            F2 = G+SR(IR)-2.
            F3 = G+SR(IR)-3.
            F4 = G+SR(IR)-4.
            DEN = 8.*F1*F2
 1    S(J,IR) = -S(J1,IR)*(6.*F2*F3+3.*F2)/DEN-S(J2,IR)*(F3*F4+F3+3.*WS)
     $/DEN-S(J3,IR)*WS/DEN
      RETURN
      END

      SUBROUTINE UMAT (X,C,R,M,F)
      DIMENSION C(200,2), R(2), F(2,2)
      COMMON N,W(2)
      DO 5 IR=1,2
         IF (M.EQ.1) GO TO 1
         NA = R(IR)
         B = X**NA
         D = X**(NA-1)
         GO TO 2
 1       B = X**R(IR)
         D = X**(R(IR)-1.)
 2       F(1,IR) = C(N+1,IR)
         DO 3 J=1,N
            JA = N-J
 3       F(1,IR) = F(1,IR)*X+C(JA+1,IR)
         FAUX = F(1,IR)
         F(1,IR) = F(1,IR)*B
         F(2,IR) = C(N+1,IR)*N
         DO 4 J=2,N
            JA = N+2-J
 4       F(2,IR) = F(2,IR)*X+(JA-1)*C(JA,IR)
         F(2,IR) = F(2,IR)*B+R(IR)*FAUX*D
 5    CONTINUE
      RETURN
      END

      SUBROUTINE INVERS (A,B)
      DIMENSION A(2,2), B(2,2)
      COMMON N,W(2)
      DEN = A(1,1)*A(2,2)-A(2,1)*A(1,2)
      B(1,1) = A(2,2)/DEN
      B(1,2) = -A(1,2)/DEN
      B(2,1) = -A(2,1)/DEN
      B(2,2) = A(1,1)/DEN
      RETURN
      END

      SUBROUTINE BOUMAT (II,X,M,C,R,V)
      DIMENSION C(200,2), R(2), V(2,2)
      COMMON N,W(2)
      DO 4 J=1,2
         IF (M.EQ.1) GO TO 1
         NA = R(J)
         D = X**NA
         GO TO 2
 1       D = X**R(J)
 2       V(II,J) = C(N+1,J)
         DO 3 IA=1,N
            I = N-IA
 3       V(II,J) = V(II,J)*X+C(I+1,J)
         V(II,J) = V(II,J)*D
 4    CONTINUE
      RETURN
      END
```

Date Table B-3. In-plane natural frequency ratios.

Support Parameter h/b	Sag Parameter $b/\sqrt{\ell^2-h^2}$	Boundary ξ's		In-Plane Natural Frequency Ratios					
		ξ_1	ξ_2	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
0.00	0.9679	-0.100	0.100	6.67	9.91	13.80	17.11	20.81	24.17
0.00	0.9382	-0.200	0.200	4.52	6.92	9.61	11.99	14.56	16.95
0.00	0.9106	-0.300	0.300	3.55	5.58	7.75	9.70	11.78	13.73
0.00	0.8849	-0.400	0.400	2.98	4.78	6.63	8.33	10.11	11.81
0.00	0.8608	-0.500	0.500	2.59	4.23	5.87	7.40	8.97	10.49
0.00	0.8500	-0.547	0.547	2.44	4.02	5.59	7.05	8.54	9.99
0.00	0.8382	-0.600	0.600	2.30	3.82	5.31	6.70	8.13	9.15
0.00	0.8170	-0.700	0.700	2.09	3.50	4.87	6.17	7.48	8.74
0.00	0.800	-0.785	0.785	1.94	3.28	4.57	5.79	7.02	8.23
0.00	0.700	-1.383	1.383	1.34	2.35	3.30	4.21	5.12	6.02
0.00	0.600	-2.223	2.223	0.98	1.76	2.50	3.21	3.91	4.60
0.00	0.500	-3.468	3.468	0.74	1.33	1.92	2.47	3.02	3.56
0.25	0.98	-0.006	0.180	8.50	12.47	17.38	21.49	26.17	30.35
0.25	0.96	-0.034	0.282	5.78	8.67	12.06	14.99	18.23	21.18
0.25	0.94	-0.073	0.380	4.53	6.94	9.64	12.02	14.61	17.00
0.25	0.92	-0.120	0.477	3.77	5.88	8.17	10.22	12.41	14.46
0.25	0.90	-0.173	0.575	3.25	5.15	7.15	8.97	10.89	12.70
0.25	0.88	-0.232	0.676	2.85	4.60	6.39	8.03	9.75	11.41
0.25	0.86	-0.297	0.779	2.54	4.17	5.80	7.09	8.57	10.34
0.25	0.85	-0.331	0.833	2.41	3.98	5.53	6.98	8.46	9.90
0.25	0.80	-0.525	1.116	1.92	3.25	4.53	5.74	6.96	8.15
0.25	0.70	-1.041	1.798	1.32	2.34	3.28	4.19	5.09	5.98
0.25	0.60	-1.800	2.719	0.97	1.75	2.49	3.19	3.89	4.58
0.25	0.55	-2.314	3.317	0.85	1.52	2.18	2.80	3.42	4.03

Data Table B-3. Continued.

Support Parameter h/b	Sag Parameter $b/\sqrt{\ell^2 - h^2}$	Boundary ξ's ξ_1	ξ_2	In-Plane Natural Frequency Ratios ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
0.50	0.97	.001	.430	6.56	9.74	13.57	16.82	20.47	26.84
0.50	0.95	-0.005	.562	4.88	7.40	10.30	12.82	15.59	18.13
0.50	0.90	-0.068	.872	3.13	4.98	6.92	8.68	10.55	12.31
0.50	0.85	-0.179	1.183	2.34	3.86	5.37	6.78	8.23	9.63
0.50	0.80	-0.331	1.514	1.86	3.16	4.42	5.60	6.80	7.96
0.50	0.75	-0.525	1.876	1.53	2.67	3.74	4.76	5.78	6.78
0.50	0.70	-0.768	2.282	1.29	2.30	3.22	4.12	5.00	5.88
0.50	0.65	-1.070	2.746	1.11	1.98	2.80	3.59	4.37	5.14
0.50	0.60	-1.446	3.285	0.96	1.73	2.46	3.16	3.84	4.52
1.00	0.98	0.137	0.835	7.31	10.73	14.99	18.53	22.57	28.80
1.00	0.95	0.042	1.157	4.36	6.62	9.24	11.51	14.00	16.10
1.00	0.90	0.000	1.607	2.82	4.49	6.27	7.87	9.57	11.17
1.00	0.85	-0.027	2.034	2.12	3.50	4.91	6.21	7.54	8.83
1.00	0.80	-0.103	2.468	1.70	2.90	4.07	5.17	6.29	7.37
1.00	0.75	-0.224	2.927	1.41	2.47	3.48	4.44	5.40	6.33
1.00	0.70	-0.395	3.424	1.20	2.14	3.02	3.87	4.71	5.54
2.00	0.98	0.638	2.033	5.83	8.55	11.96	14.97	18.08	20.98
2.00	0.95	0.374	2.605	3.49	5.30	7.42	9.24	11.25	13.08
2.00	0.93	0.266	2.925	2.85	4.42	6.19	7.74	9.43	10.98
2.00	0.90	0.153	3.367	2.27	3.62	5.08	6.39	7.78	9.08

Data Table B-3. Continued.

Support Parameter h/b	Sag Parameter $b/\sqrt{\ell^2-h^2}$	Boundary ξ's ξ_1	ξ_2	Lowest In-Plane Natural Frequency Ratio ω_1	ω_2	ω_2	ω_4	ω_5	ω_6
2.50	0.98	0.937	2.681	5.31					
2.50	0.96	0.685	3.170	3.62					
2.50	0.94	0.516	3.582	2.85					
3.00	0.98	1.250	3.343	4.90					
3.00	0.96	0.942	3.924	3.34					
3.50	0.99	1.856	3.576	6.59					

Data Table B-4. Out-of-plane natural frequency ratios.

Support Parameter h/b	Sag Parameter $b/\sqrt{\ell^2-h^2}$	Boundary ξ's		Out-of-Plane Natural Frequency Ratios					
		ξ_1	ξ_2	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
0.00	0.97	-0.093	0.093	3.64	7.23	10.84	14.44	18.04	21.65
0.00	0.95	-0.160	0.160	2.79	5.51	8.25	10.99	13.73	16.48
0.00	0.90	-0.340	0.340	1.91	3.74	5.59	7.44	9.29	11.15
0.00	0.85	-0.547	0.547	1.51	2.92	4.36	5.80	7.24	8.69
0.00	0.80	-0.785	0.785	1.27	2.41	3.60	4.78	5.98	7.16
0.00	0.75	-1.061	1.061	1.09	2.05	3.06	4.07	5.08	6.09
0.00	0.70	-1.383	1.383	0.96	1.78	2.65	3.52	4.40	5.26
0.00	0.60	-2.223	2.223	0.76	1.36	2.03	2.70	3.37	4.04
0.00	0.50	-3.468	3.468	0.62	1.06	1.60	2.11	2.63	3.14
0.25	0.97	-0.018	0.232	3.59	7.13	10.68	14.24	17.80	21.36
0.25	0.95	-0.052	0.331	2.74	5.44	8.14	10.85	13.56	16.27
0.25	0.90	-0.173	0.575	1.89	3.69	5.52	7.35	9.19	11.02
0.25	0.85	-0.331	0.833	1.50	2.89	4.32	5.74	7.17	8.60
0.25	0.80	-0.525	1.116	1.25	2.39	3.57	4.74	5.93	7.11
0.25	0.75	-0.759	1.434	1.08	2.04	3.04	4.04	5.04	6.05
0.25	0.70	-1.041	1.798	0.95	1.77	2.63	3.50	4.36	5.23
0.25	0.65	-1.383	2.221	0.85	1.54	2.30	3.06	3.82	4.57
0.25	0.60	-1.800	2.719	0.76	1.36	2.03	2.69	3.36	4.03
0.25	0.55	-2.314	3.317	0.68	1.20	1.80	2.38	2.96	3.55
0.50	0.97	0.001	0.430	3.45	6.87	10.29	13.71	17.14	20.57
0.50	0.95	-0.005	0.562	2.64	5.24	7.85	10.47	13.08	15.70
0.50	0.90	-0.068	0.872	1.82	3.58	5.35	7.13	8.91	10.69
0.50	0.85	-0.179	1.183	1.45	2.81	4.20	5.59	6.98	8.38

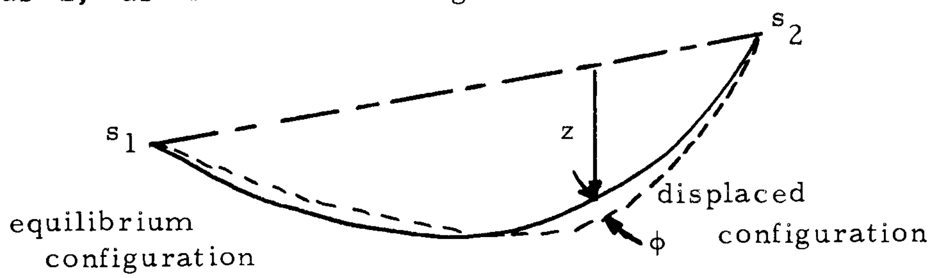
Data Table B-4. Continued.

Support Parameter h/b	Sag Parameter $b/\sqrt{\ell^2-h^2}$	Boundary ξ's ξ_1	ξ_2	Out-of-Plane Natural Frequence Ratios ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
0.50	0.80	-0.331	1.514	1.22	2.34	3.48	4.64	5.79	6.94
0.50	0.75	-0.525	1.876	1.05	2.00	2.97	3.95	4.94	5.92
0.50	0.70	-0.768	2.282	0.93	1.74	2.58	3.43	4.29	5.14
0.50	0.65	-1.070	2.746	0.83	1.53	2.27	3.02	3.77	4.51
0.50	0.60	-1.446	3.285	0.74	1.35	2.01	2.66	3.32	3.97
1.00	0.97	0.094	0.952	3.08	6.14	9.21	12.27	15.34	18.41
1.00	0.95	0.042	1.157	2.37	4.71	7.06	9.41	11.76	14.11
1.00	0.90	0.000	1.607	1.64	3.25	4.86	6.48	8.10	9.71
1.00	0.85	-0.027	2.034	1.31	2.57	3.84	5.12	6.40	7.68
1.00	0.80	-0.103	2.468	1.10	2.16	3.23	4.30	5.37	6.44
1.00	0.75	-0.224	2.927	0.96	1.87	2.79	3.70	4.62	5.53
1.00	0.70	-0.395	3.424	0.86	1.65	2.44	3.23	4.02	4.84
2.00	0.97	0.529	2.244	2.44	4.90	7.37	9.83	12.28	14.74
2.00	0.95	0.374	2.605	1.88	3.78	5.68	7.55	9.45	11.35
2.00	0.93	0.266	2.925	1.59	3.18	4.76	6.33	7.94	9.54
2.00	0.90	0.153	3.367	1.33	2.64	3.92	5.25	6.59	7.93

Appendix B-5. Rayleigh's Method for Estimation of the First Out-of-Plane Natural Frequency

Rayleigh's method uses an energy relationship to estimate the lowest natural frequency of vibration of both lumped-parameter and distributed parameter systems. The crux of the method is to assume an approximate mode shape for the vibration of the system. Using this mode shape, the maximum potential and kinetic energies of the system may then be calculated. The frequency of vibration of the system will appear in the expression for the kinetic energy. When the maximum kinetic energy is equated to the maximum potential energy, then the resulting equation will yield the unknown natural frequency of vibration of the system.

Consider a cable to be suspended between two end points s_1 and s_2. Let the cable be modeled as a rigid wire. As an assumed lowest mode shape, let the rigid wire oscillate sinusoidally through an angle ϕ as measured from the equilibrium position of the cable. The rotation axis is the line connecting the end points s_1 and s_2. Let the distance from the axis of rotation of the cable to any point on the cable be denoted as z, as indicated in the figure below.



The kinetic energy of an element of the cable is given by:

$$d(KE) = d(\frac{1}{2} m\dot{w}^2)$$

where $\dot{w}$ is the speed of the cable element and may be expressed as:

$$\dot{w} = z\dot{\phi}$$

The total kinetic energy of the cable is thus given by:

$$KE = \int d(\frac{1}{2} m\dot{w}^2) \approx \frac{1}{2} \rho\dot{\phi}^2 \int_{s_1}^{s_2} z^2 ds$$

The potential energy of an element of the cable is given by:

$$d(PE) = d(mgd)$$

where d is the vertical distance between a point on the cable in its equilibrium configuration and the same point when the cable is rotated through an angle ϕ, and may be expressed as:

$$d = z(1 - \cos \phi)$$

The total potential energy of the cable is thus given by:

$$PE = \int d(mgd) = \rho g \int_{s_1}^{s_2} z(1 - \cos \phi) ds$$

For small angles, the potential energy may be approximated as:

$$PE = \frac{\rho g \phi^2}{2} \int_{s_1}^{s_2} z ds$$

If it is assumed that the cable oscillates sinusoidally, then ϕ may be assumed to have a solution of the form:

$$\phi = \phi_0 \sin \beta t$$

Using the assumed solution for ϕ, the kinetic and potential energies of the cable are given as:

$$KE = \frac{1}{2} \rho \beta^2 \phi_0^2 \cos^2 \beta t \int_{s_1}^{s_2} z^2 ds$$

and

$$PE = \frac{1}{2} \rho g \phi_0^2 \sin^2 \beta t \int_{s_1}^{s_2} z ds$$

The maximum kinetic and potential energies are:

$$KE_{max} = \frac{1}{2} \rho \beta^2 \phi_0^2 \int_{s_1}^{s_2} z^2 ds$$

and

$$PE_{max} = \frac{1}{2} \rho g \phi_0^2 \int_{s_1}^{s_2} z ds$$

Equating the maximum kinetic and potential energies yields:

$$\frac{1}{2}\beta^2\rho\phi_0^2\int_{s_1}^{s_2} z^2 ds = \frac{1}{2}\rho g\phi_0^2\int_{s_1}^{s_2} z ds$$

or solving for β^2:

$$\beta^2 = g\,\frac{\int_{s_1}^{s_2} z ds}{\int_{s_1}^{s_2} z^2 ds}$$

The frequency $\beta(t)$ may be expressed in nondimensional terms through use of:

$$\beta(t) = \omega(\theta)\sqrt{g/a}$$

where a is the catenary parameter. Hence the lowest nondimensional natural frequency for out-of-plane motion of the cable is given by:

$$\omega^2(\theta) = a\,\frac{\int_{s_1}^{s_2} z ds}{\int_{s_1}^{s_2} z^2 ds}$$

Using the cable support parameters b and h, the cable length ℓ, and the catenary geometry equations presented in Section III. 1 of this paper, the integrals in the above equation may be evaluated. With these integrals, then the lowest nondimensional natural frequency for out-of-plane motion may be calculated for any cable geometry.

The following table gives the natural frequencies for various cable geometries.

Support Parameter h/b	Sag Parameter $b/\sqrt{\ell^2-h^2}$	Out-of-Plane Natural Frequency ω; by Rayleigh	Out-of-Plane Natural Frequency ω; Exact
0	0.95	2.80	2.79
0	0.90	1.93	1.91
0	0.80	1.28	1.27
0	0.70	0.97	0.96
0	0.60	0.77	0.76
0	0.50	0.62	0.62

```
      PROGRAM FORCE (INPUT,OUTPUT,TAPE60=INPUT,TAPE61=OUTPUT)
      DIMENSION R1(4), R2(4), A(200,4), B(200,4), V(4,4), D(4,4), UR(4,4
     1), VR(4,8), ALPHA(4,4), UL(4,4), BETA(4,4), DA(4,8), C(4), FD(4),
     *UP(4,4), U(4), AP(4), PF(4)
      COMMON N,Y(2)
    1 READ (60,51) HOB,BOSR,N,Y(1),XR
      IF (EOF(60)) 47,2
    2 READ (60,50) BD(1),BD(2),BD(3),BD(4)
      TMAX = 0.0
      BOA = 1.0
    3 EX = EXP(BOA)
      EXI = 1./EX
      SIH = (EX-EXI)*.5
      COH = (EX+EXI)*.5
      BOAN = BOA-(BOA/SIH-BOSR)/((1./SIH)*(1.-BOA*COH/SIH))
      IF (ABS((BOA-BOAN)/BOAN)-.001) 5,5,4
    4 BOA = BOAN
      GO TO 3
    5 AOBN = 1./BOAN
      AOLH = AOBN*AOBN*BOSR*BOSR
      HLOA = BOAN*(HOB+SQRT(BOSR*BOSR*HOB*HOB+1.)/BOSR)
      A1 = (SQRT(1.+AOLH)-1.)
      A2 = A1+2.
      S1 = .5*(A1*HLOA-1./(A1*HLOA))
      S2 = .5*(A2*HLOA-1./(A2*HLOA))
      X1 = S1/ABS(S1)*(SQRT(1.+S1*S1)-1.)
      X2 = SQRT(1.+S2*S2)-1.
      IF (X2-X1.LT.5.) GO TO 6
      NP = 29
      GO TO 9
    6 IF (X2-X1.LT.4.) GO TO 7
      NP = 19
      GO TO 9
    7 IF (X2-X1.LT.3.) GO TO 8
      NP = 15
      GO TO 9
    8 NP = 11
    9 CONTINUE
      ANP = NP
      NP = NP+1
      WRITE (61,48) N,Y(1),XR,NP
      WRITE (61,49) HOB,BOSR,AOBN,X1,X2
      DO 10 I=1,4
         R2(I) = I-1.
   10 R1(I) = R2(I)*.5
      L = 1
      IF (X2.GE..7) GO TO 14
      IF (X1.LT.0.0) GO TO 12
C     X2 LESS THAN .7 AND X1 GREATER THAN 0.
      CALL ACOEF (L,A)
      II = 1
      M = 1
      CALL VMAT (II,X1,M,A,R1,V)
      II = 3
      CALL VMAT (II,X2,M,A,R1,V)
      DO 11 I=1,4
         DO 11 J=1,4
   11 D(I,J) = V(I,J)
      GO TO 30
C     X2 LESS THAN .7 AND X1 BETWEEN -.7 AND 0.
   12 CALL ACOEF (L,A)
      II = 3
      M = 1
      CALL VMAT (II,X2,M,A,R1,V)
      DO 13 I=3,4
         DO 13 J=1,4
   13 D(I,J) = V(I,J)
      GO TO 20
C     X2 GREATER THAN .7, FOUR POSSIBILITIES EXIST FOR X1.
   14 CALL ACOEF (L,A)
      CALL BCOEF (L,B)
      M = 1
      CALL AMAT (XR,A,R1,M,UR)
      XRM = XR-2.0
      M = 2
      CALL AMAT (XRM,B,R2,M,VR)
      DO 16 I=1,4
         DO 16 J=5,8
            IF (I+4.EQ.J) GO TO 15
            VR(I,J) = 0.0
            GO TO 16
   15       VR(I,J) = 1.0
   16 CONTINUE
      CALL INVERS (VR)
      DO 17 I=1,4
         DO 17 J=1,4
            DO 17 K=1,4
               IF (K.EQ.1) ALPHA(I,J) = 0.0
   17 ALPHA(I,J) = VR(I,K)*UR(K,J)+ALPHA(I,J)
      X2B = X2-2.0
      II = 3
      M = 2
      CALL VMAT (II,X2B,M,B,R2,V)
      DO 18 I=3,4
         DO 18 J=1,4
            DO 18 K=1,4
               IF (K.EQ.1) D(I,J) = 0.0
   18 D(I,J) = V(I,K)*ALPHA(K,J)+D(I,J)
      IF (X1.GE..7) GO TO 28
      IF (X1.LE.-.7) GO TO 22
      IF (X1.LT.0.0) GO TO 20
C     X1 IS BETWEEN 0.0 AND .7
      II = 1
      M = 1
      CALL VMAT (II,X1,M,A,R1,V)
      DO 19 I=1,2
         DO 19 J=1,4
   19 D(I,J) = V(I,J)
      GO TO 30
C     X1 IS BETWEEN -.7 AND 0.0
   20 X1A = ABS(X1)
      II = 1
      M = 1
      CALL VMAT (II,X1A,M,A,R1,V)
      DO 21 I=1,2
         DO 21 J=1,4
            V(I,J) = (-1)**(I+J)*V(I,J)
   21 D(I,J) = V(I,J)
      GO TO 30
C     X1 LESS THAN OR EQUAL TO -.7
   22 X1B = -X1-2.0
      DO 24 I=1,4
         DO 24 J=1,4
            IF (J.EQ.2.OR.J.EQ.4) GO TO 23
            UL(I,J) = UR(I,J)
            GO TO 24
   23       UL(I,J) = -UR(I,J)
   24 CONTINUE
      DO 25 I=1,4
         DO 25 J=1,4
            DO 25 K=1,4
               IF (K.EQ.1) BETA(I,J) = 0.0
   25 BETA(I,J) = VR(I,K)*UL(K,J)+BETA(I,J)
      II = 1
      M = 2
      CALL VMAT (II,X1B,M,B,R2,V)
      DO 26 J=1,4
   26 V(2,J) = -V(2,J)
      DO 27 I=1,2
         DO 27 J=1,4
            DO 27 K=1,4
               IF (K.EQ.1) D(I,J) = 0.0
   27 D(I,J) = V(I,K)*BETA(K,J)+D(I,J)
```

```
      GO TO 30
C     X1 IS GREATER THAN .7.
   28 X1B = X1-2.0
      II = 1
      M = 2
      CALL VMAT (II,X1B,M,B,R2,V)
      DO 29 I=1,2
         DO 29 J=1,4
            DO 29 K=1,4
               IF (K.EQ.1) D(I,J) = 0.0
   29 D(I,J) = V(I,K)*ALPHA(K,J)+D(I,J)
   30 CONTINUE
      DO 31 I=1,4
         DO 31 J=1,4
   31 DA(I,J) = D(I,J)
      DO 33 I=1,4
         DO 33 J=5,8
            IF (I+4.EQ.J) GO TO 32
            DA(I,J) = 0.0
            GO TO 33
   32       DA(I,J) = 1.0
   33 CONTINUE
      CALL INVERS (DA)
      DO 34 I=1,4
         DO 34 K=1,4
            IF (K.EQ.1) C(I) = 0.0
   34 C(I) = DA(I,K)*BD(K)+C(I)
      DO 46 IP=1,NP
         XP = X1+(X2-X1)/ANP*(IP-1.0)
         IF (ABS(XP).GE..7) GO TO 38
C     XP LIES BETWEEN -.7 AND .7.
         XPA = ABS(XP)
         M = 1
         CALL AMAT (XPA,A,R1,M,UP)
         IF (XP.GT.0.0) GO TO 36
         DO 35 I=1,4
            DO 35 J=1,4
   35    UP(I,J) = UP(I,J)*(-1)**(I+J)
   36    CONTINUE
         DO 37 I=1,4
            DO 37 K=1,4
               IF (K.EQ.1) U(I) = 0.0
   37    U(I) = UP(I,K)*C(K)+U(I)
         GO TO 44
   38    IF (ABS(XP).NE.XP) GO TO 41
C     XP IS GREATER THAN OR EQUAL TO .7.
         DO 39 I=1,4
            DO 39 K=1,4
               IF (K.EQ.1) AP(I) = 0.0
   39    AP(I) = ALPHA(I,K)*C(K)+AP(I)
         M = 2
         XPA = XP-2.0
         CALL AMAT (XPA,B,R2,M,UP)
         DO 40 I=1,4
            DO 40 K=1,4
               IF (K.EQ.1) U(I) = 0.0
   40    U(I) = UP(I,K)*AP(K)+U(I)
         GO TO 44
C     XP IS LESS THAN OR EQUAL TO -.7.
   41    CONTINUE
         DO 42 I=1,4
            DO 42 K=1,4
               IF (K.EQ.1) BP(I) = 0.0
   42    BP(I) = BETA(I,K)*C(K)+BP(I)
         M = 2
         XPB = -XP-2.0
         CALL AMAT (XPB,B,R2,M,UP)
         DO 43 I=1,4
            DO 43 K=1,4
               IF (K.EQ.1) U(I) = 0.0
   43    U(I) = UP(I,K)*BP(K)+U(I)
   44    CONTINUE
         XA = ABS(XP)
         XA1 = 2./XA+1.0
         SXA = SQRT(XA1)
         DISTAN = U(1)
         DISNOR = (XA+XP*XP)*SXA*U(2)
         SLOPE = U(1)/(1.+XA)**2+XP/XA*(1.+4.*XA+2.*XP*XP)/(1.+XA)*U(2)+
     $   XP*XP*SXA*U(3)
         TENSN = XP*SXA/(1.+XA)*U(1)-((3.+4.*XP*XP*XA1)*XA*SXA+Y(1)*Y(1)
     $   *XA*SXA*(1.+XA)**3+2.*XA*SXA)*U(2)-(5.*XP**3*SXA*XA1*(1.+XA)+3.
     $   *XP*SXA*(1.+XA))*U(3)-(XA**3*SXA*XA1*(1.+XA)*(1.+XA))*U(4)
         WRITE (61,52) XP
         WRITE (61,53) DISTAN,DISNOR,SLOPE,TENSN
         IF (ABS(TMAX).GT.ABS(TENSN)) GO TO 45
         TMAX = TENSN
   45    CONTINUE
   46 CONTINUE
      WRITE (61,54) TMAX
      GO TO 1
   47 STOP
C
   48 FORMAT (//15H NO. OF TERMS =,I3,5X,25HFORCING FREQUENCY VALUE =,F6
     $.2,5X,16HMATCHING POINT =,F6.2,5X,27HNO. OF POINTS ALONG CABLE =,
     $I3)
   49 FORMAT (1X,10HH OVER B =,F8.3,5X,31HB OVER SQRT. OF L,SQRO-H,SQRO
     $=,F8.3,5X,11H2A OVER B =,F8.3/1X,18HLEFT BOUNDARY X1 =,F8.3,5X,
     $19HRIGHT BOUNDARY X2 =,F8.3)
   50 FORMAT (4F6.2)
   51 FORMAT (2F8.3,I3,2F7.3)
   52 FORMAT (/1X,4HXP =,F7.3)
   53 FORMAT (1X,25HTANGENTIAL DISPLACEMENT =,F8.4,5X,21HNORMAL DISPLACE
     $MENT =,F8.4/1X,16HSLOPE OF CABLE =,F8.3,5X,18HTENSION IN CABLE =,
     $F8.4)
   54 FORMAT (/1X,34HMAXIMUM DYNAMIC TENSION IN CABLE =,F8.4)
      END
      SUBROUTINE ACOEF (L,A)
      DIMENSION A(200,4)
      COMMON N,Y(2)
      ZL = Y(L)*Y(L)
      NUM = N+6
      DO 2 IR=1,4
         RR = IR
         R = (RR-1.0)/2.
         A(1,IR) = 0.0
         A(2,IR) = 0.0
         A(3,IR) = 0.0
         A(4,IR) = 1.0
         A(5,IR) = -A(4,IR)*(4.*R*R+R+1.+ZL)/((R+1.)*(2.*R+1.))
         DO 1 J=6,NUM
            G = J
            K1 = J-1
            K2 = J-2
            K3 = J-3
            K4 = J-4
            F1 = G+R-5.
            F2 = G+R-6.
            F3 = G+R-7.
            F4 = G+R-3.
            F5 = G+R-3.
            F6 = G+R-4.
            F7 = G+R-1.
            F8 = G+R
            A(J,IR) = -A(K1,IR)*(2.*F2*F3*(4.*F6+7.)+(51.+2.*ZL)*F6-3.*
     *      (32.+ZL))/(F6*(4.*F6*F2+3.))-A(K2,IR)*(F2*(5.*F3*F4*F7+(72.+
     *      5.*ZL)*F6-6.*(32.+ZL))-ZL)/(F6*F1*(4.*F6*F2+3.))-A(K3,IR)*F3
     *      *(F4*F5*F8+4.*(6.+ZL)*F6-4.*(21.+ZL))/(F6*F1*(4.*F6*F2+3.))-
     *      A(K4,IR)*ZL*F4/(F6*(4.*F6*F2+3.))
    1    CONTINUE
         DO 2 J=4,NUM
            A(J-3,IR) = A(J,IR)
    2 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE BCOEF (L,B)
      DIMENSION B(200,4)
      COMMON N,Y(2)
      ZL = Y(L)*Y(L)
      NUM = N+8
      DO 2 IR=1,4
         RR = IR
         R = RR-1.
         B(1,IR) = 0.0
         B(2,IR) = 0.0
         B(3,IR) = 0.0
         B(4,IR) = 0.0
         B(5,IR) = 0.0
         B(6,IR) = 1.0
         B(7,IR) = -B(6,IR)*(688.+352.*(R-3.))/(192.*(R+1.))
         B(8,IR) = -B(7,IR)*(688.+352.*(R-2.))/(192.*(R+2.))-B(6,IR)*
     $   (585.+72.*ZL+996.*(R-2.)+252.*(R-2.)*(R-3.))/(192.*(R+2.)*(R+1.
     $   ))
         B(9,IR) = -B(8,IR)*(688.+352.*(R-1.))/(192.*(R+3.))-B(7,IR)*
     $   (585.+72.*ZL+996.*(R-1.)+252.*(R-1.)*(R-2.))/(192.*(R+3.)*(R+2.
     $   ))-B(6,IR)*(102.+99.*ZL+(627.+102.*ZL)*(R-1.)+526.*(R-1.)*(R-2.
     $   )+88.*(R-1.)*(R-2.)*(R-3.))/(192.*(R+3.)*(R+2.)*(R+1.))
         DO 1 J=10,NUM
            G = J
            K1 = J-1
            K2 = J-2
            K3 = J-3
            K4 = J-4
            K5 = J-5
            K6 = J-6
            F1 = G+R-6.
            F2 = G+R-7.
            F3 = G+R-8.
            F4 = G+R-9.
            F5 = G+R-10.
            F6 = G+R-11.
            F7 = G+R-12.
            F8 = G+R-13.
            F9 = G+R-14.
            B(J,IR) = -B(K1,IR)*(688.+352.*F5)/(192.*F1)-B(K2,IR)*(585.+
     $      72.*ZL+996.*F5+252.*F5*F6)/(192.*F1*F2)-B(K3,IR)*(102.+99.*
     $      ZL+(627.+102.*ZL)*F5+526.*F5*F6+88.*F5*F6*F7)/(192.*F1*F2*F3
     $      )-B(K4,IR)*(-ZL+(72.+105.*ZL)*F5+(216.+53.*ZL)*F5*F6+120.*F5
     $      *F6*F7+15.*F5*F6*F7*F8)/(192.*F1*F2*F3*F4)-B(K5,IR)*((12.+36
     $      .*ZL)*F6+(24.+12.*ZL)*F5*F7+10.*F6*F7*F8+F6*F7*F8*F9)/(192.*
     $      F1*F2*F3*F4)-B(K6,IR)*(4.*ZL*F7+ZL*F7*F8)/(192.*F1*F2*F3*F4)
1        CONTINUE
         DO 2 J=6,NUM
            B(J-5,IR) = B(J,IR)
2     CONTINUE
      RETURN
      END

      SUBROUTINE AMAT (X,A,R,M,U)
      DIMENSION A(200,4), R(4), U(4,4)
      COMMON N,Y(2)
      DO 7 J=1,4
         IF (M.EQ.1) GO TO 1
         NA = R(J)
         B = X**NA
         D = B/X
         E = D/X
         F = E/X
         GO TO 2
1        B = X**R(J)
         D = B/X
         E = D/X
         F = E/X
2        U(1,J) = A(N+1,J)
         DO 3 IA=1,N
            I = N-IA
3        U(1,J) = U(1,J)*X+A(I+1,J)
         AUX1 = U(1,J)
         U(1,J) = AUX1*B
         U(2,J) = A(N+1,J)*N
         DO 4 IA=2,N
            I = N+2-IA
4        U(2,J) = U(2,J)*X+(I-1)*A(I,J)
         AUX2 = U(2,J)
         U(2,J) = AUX2*B+R(J)*AUX1*D
         U(3,J) = A(N+1,J)*N*(N-1)
         DO 5 IA=3,N
            I = N+3-IA
5        U(3,J) = U(3,J)*X+(I-2)*(I-1)*A(I,J)
         AUX3 = U(3,J)
         U(3,J) = AUX3*B+R(J)*(R(J)-1.)*E*AUX1+2.*R(J)*D*AUX2
         U(4,J) = A(N+1,J)*N*(N-1)*(N-2)
         DO 6 IA=4,N
            I = N+4-IA
6        U(4,J) = U(4,J)*X+(I-1)*(I-2)*(I-3)*A(I,J)
         U(4,J) = U(4,J)*B+R(J)*(R(J)-1.)*(R(J)-2.)*F*AUX1+3.*R(J)*(R(J)
     $   -1.)*E*AUX2+3.*R(J)*D*AUX3
7     CONTINUE
      RETURN
      END

      SUBROUTINE VMAT (II,X,M,A,R,V)
      DIMENSION A(200,4), R(4), V(4,4)
      COMMON N,Y(2)
      I1 = II+1
      DO 5 J=1,4
         IF (M.EQ.1) GO TO 1
         NA = R(J)
         C = X**NA
         D = C/X
         GO TO 2
1        C = X**R(J)
         D = C/X
2        V(II,J) = A(N+1,J)
         DO 3 IA=1,N
            I = N-IA
3        V(II,J) = V(II,J)*X+A(I+1,J)
         AUX1 = V(II,J)
         V(II,J) = AUX1*C
         V(I1,J) = A(N+1,J)*N
         DO 4 IA=2,N
            I = N+2-IA
4        V(I1,J) = V(I1,J)*X+(I-1)*A(I,J)
         V(I1,J) = V(I1,J)*C+AUX1*R(J)*D
5     CONTINUE
      RETURN
      END

      SUBROUTINE INVERS (C)
      DIMENSION C(4,8), X(4)
      COMMON N,Y(2)
      DO 3 I=1,4
         D = 0.0
         DO 2 J=1,4
            IF (ABS(C(I,J))-ABS(D)) 2,2,1
1           D = C(I,J)
2        CONTINUE
         DO 3 J=1,8
3     C(I,J) = C(I,J)/D
      DO 12 M=1,4
         KK = 2*4+1-M
         K = 4+2-M
         D = 0.0
         L = 1
         DO 5 I=2,K
            IF (ABS(C(I-1,1))-D) 5,5,4
4           L = I-1
            D = ABS(C(L,1))
```

```
 5      CONTINUE
        IF (L-1) 6,8,6
 6      DO 7 J=1,KK
           D = C(L,J)
           C(L,J) = C(1,J)
           C(1,J) = D
 7      CONTINUE
 8      CONTINUE
        DO 9 I=1,4
 9      X(I) = C(I,1)
        DO 11 J=2,KK
           D = C(1,J)/X(1)
           DO 10 I=2,4
10         C(I-1,J-1) = C(I,J)-X(I)*D
11      C(4,J-1) = D
12  CONTINUE
    RETURN
    END
```

Data Table C-2. In-plane forced motion results.

Forcing Frequency; ω_f	Absolute Value of Maximum Dynamic Tension; $\lvert T_{max} \rvert$
$h/b = 0.00$; $b/\sqrt{\ell^2 - h^2} = 0.90$; $\xi_1 = -0.340$; $\xi_2 = 0.340$; $U(\xi_1) = V(\xi_1) = V(\xi_2) = 0.0$; $U(\xi_2) = 0.01$; Natural frequency ratios: $\omega_1 = 3.29$; $\omega_2 = 5.22$	
0.00	0.028
0.50	0.025
1.00	0.018
1.50	0.012
2.00	0.041
2.50	0.080
3.00	0.138
3.27	0.554
3.31	0.457
3.50	0.139
4.00	0.228
4.50	0.322
5.00	0.543
5.19	2.098
5.23	3.874
5.50	0.248
6.00	0.443
$h/b = 0.0$; $b/\sqrt{\ell^2 - h^2} = 0.70$; $\xi_1 = -1.383$; $\xi_2 = 1.383$; $U(\xi_1) = V(\xi_1) = V(\xi_2) = 0.0$; $U(\xi_2) = 0.01$; Natural frequency ratios: $\omega_1 = 1.34$; $\omega_2 = 2.35$	
0.00	0.169
0.50	0.187
1.00	0.312
1.32	5.218
1.36	3.422
1.50	0.505
1.75	0.263
2.00	0.330
2.25	1.124
2.33	5.164
2.37	7.533

Data Table C-2. Continued.

Forcing Frequency; ω_f	Absolute Value of Maximum Dynamic Tension; $\lvert T_{max} \rvert$
2.50	0.968
2.75	0.498
3.00	0.519

$h/b = 0.50$; $b/\sqrt{\ell^2 - h^2} = 0.90$; $\xi_1 = -0.068$; $\xi_2 = 0.872$;

$U(\xi_1) = V(\xi_1) = V(\xi_2) = 0.0$; $U(\xi_2) = 0.01$;

Natural frequence ratios: $\omega_1 = 3.13$; $\omega_2 = 4.98$

0.00	0.034
0.50	0.032
1.00	0.024
1.50	0.016
2.00	0.053
2.50	0.101
3.00	0.160
3.13	0.178
3.132	0.181
3.133	7.440
3.14	0.178
3.50	0.233
4.00	0.324
4.50	0.460
4.96	3.054
5.00	2.311
5.50	0.473
6.00	0.635

$h/b = 0.50$; $b/\sqrt{\ell^2 - h^2} = 0.70$; $\xi_1 = -0.768$; $\xi_2 = 2.282$;

$U(\xi_1) = V(\xi_1) = V(\xi_2) = 0.0$; $U(\xi_2) = 0.01$;

Natural frequency ratios: $\omega_1 = 1.29$; $\omega_2 = 2.30$

0.00	0.122
0.50	0.128
1.00	0.185
1.27	1.516
1.31	1.884
1.50	0.120

Data Table C-2. Continued.

Forcing Frequency; ω_f	Absolute Value of Maximum Dynamic Tension; $\lvert T_{max} \rvert$
1.75	0.089
2.00	0.144
2.28	6.869
2.32	2.144
2.50	0.406
2.75	0.269
3.00	0.317

Data Table C-3. In-plane forced motion results.

Position on Cable ξ_p	Tangential Displacement $U(\xi_p)$	Normal Displacement $V(\xi_p)$	Dynamic Tension $T(\xi_p)$
$h/b = 0.0$; $b/\sqrt{\ell^2-h^2} = 0.90$; $\xi_1 = -0.340$; $\xi_2 = 0.340$; $U(\xi_1) = V(\xi_1) = V(\xi_2) = 0.0$; $U(\xi_2) = 0.01$; $\omega_f = 2.0$; Natural frequency ratios: $\omega_1 = 3.29$; $\omega_2 = 5.22$			
-0.340	0.000	0.000	-0.010
-0.279	0.000	0.002	-0.009
-0.217	0.000	0.004	-0.008
-0.155	0.001	0.006	-0.006
-0.093	0.001	0.008	-0.004
-0.031	0.003	0.010	-0.003
0.031	0.008	0.009	-0.013
0.093	0.009	0.006	-0.021
0.155	0.010	0.004	-0.027
0.217	0.010	0.002	-0.032
0.279	0.010	0.001	-0.037
0.340	0.010	0.000	-0.041
$h/b = 0.00$; $b/\sqrt{\ell^2-h^2} = 0.70$; $\xi_1 = -1.383$; $\xi_2 = 1.383$; $U(\xi_1 = V(\xi_1) = V(\xi_2) = 0.0$; $U(\xi_2) = 0.01$; $\omega_f = 1.32$; Natural frequence ratios: $\omega_1 = 1.34$; $\omega_2 = 2.35$			
-1.383	0.000	0.000	4.867
-1.132	0.001	-0.025	5.218
-0.880	0.004	-0.051	5.132
-0.629	0.010	0.075	0.117
-0.377	0.023	0.092	0.121
-0.126	0.048	0.082	0.095
0.126	0.055	-0.071	-0.110
0.377	0.032	-0.086	-0.145
0.629	0.020	-0.071	-0.147
0.880	0.013	-0.049	-0.142
1.132	0.011	-0.024	-0.135
1.383	0.010	0.000	-0.129

Data Table C-3. Continued.

Position on Cable ξ_p	Tangential Displacement $U(\xi_p)$	Normal Displacement $V(\xi_p)$	Dynamic Tension $T(\xi_p)$
$h/b = 0.00$; $b/\sqrt{\ell^2 - h^2} = 0.70$; $\xi_1 = -1.383$; $\xi_2 = 1.383$; $U(\xi_1) = V(\xi_1) = V(\xi_2) = 0.0$; $U(\xi_2) = 0.01$; $\omega_f = 1.36$; Natural frequency rations: $\omega_1 = 1.34$; $\omega_2 = 2.35$			
-1.383	-0.000	0.000	-3.223
-1.132	-0.001	0.016	-3.422
-0.880	-0.002	0.033	-3.313
-0.629	-0.006	-0.049	-0.098
-0.377	-0.015	-0.058	-0.099
-0.126	-0.030	-0.048	-0.078
0.126	-0.024	0.059	0.061
0.377	-0.006	0.065	0.074
0.629	0.003	0.052	0.066
0.880	0.008	0.035	0.054
1.132	0.009	0.017	0.040
1.383	0.010	0.000	0.028

Appendix D. Outline of the Derivation and Solution of the Equations of Normal Mode Motion of a Suspended Cable

A synopsis of the derivation and solution for the normal mode motion of the cable is provided to assist the reader in following the steps carried out in this paper.

The cable is considered to be a perfectly flexible, elastic cable suspended in a viscous medium. Hamilton's principle along with the associated Euler-Lagrange equations are used to derive the equations of motion. Hamilton's equation is stated as:

$$\int_{t_1}^{t_2} [\delta L + \delta W_f + \delta W_s] \, dt = 0$$

where L is the Lagrangian (total kinetic energy minus the total potential energy) for the system, δW_f is the increment of virtual work performed on the cable due to linear drag forces from the surrounding medium, and δW_s is the increment of virtual work performed by internal forces on the cable as it is strained between two states. Equations 2.19 and 2.20 are the resulting equations governing the motion and tension in the cable. The equilibrium configuration of the cable is then derived from the general equations of motion of the cable.

For the undamped, inextensible case, the equations of motion are then nondimensionalized and linearized for small variations about the equilibrium state. The result is given by Equations 2.29 and 2.30.

Normal mode solutions are assumed for these equations, leading to Equations 2. 35 and 2. 38, which govern the tangential and out-of-plane displacements. These equations contain the natural frequency ratios ω and nondimensional position along the cable σ. The coefficients of the derivatives of the displacement terms are irrational; hence these equations are not solvable by power series solutions.

The change of independent variable

$$\sigma = \frac{\xi}{|\xi|} \sqrt{(1 + |\xi|)^2 - 1}$$

transforms equations 2. 35 and 2. 38 to equations with polynomial coefficients. For positive values of ξ, equations 2. 42 and 2. 44 govern the tangential and out-of-plane displacements.

A power series solution, written about the singular point $\xi = 0$, of the form

$$y(\xi) = \sum_{k=0}^{\infty} a_k \, \xi^{k+r}$$

will satisfy Equation 2. 42 for the tangential displacement and converge for $0 < \xi < 1$. Since Equation 2. 42 is a fourth order equation, there will be four independent power series and four integration constants which make up the entire solution for the tangential displacement. A second power series solution, written about the ordinary point $\xi = 2$, of the form

$$z(\xi-2) = \sum_{k=0}^{\infty} b_k\,(\xi-2)^{k+r}$$

will also satisfy Equation 2.42 and converge for $0 < \xi < 4$. Again there will be four integration constants associated with the complete solution for the tangential displacement. By matching the first four derivatives of the tangential displacement at some point where both series solutions converge, it is possible to express the integration constants for the second power series solution in terms of the integration constants for the first power series solution. Considerations of cable symmetry and matching at the origin will yield series solutions for the tangential displacement of the cable which are valid for the region $-4 < \xi < 4$ which contain just four constants of integration.

At the attachment points of the cable ξ_1, and ξ_2, the tangential and normal components of the displacement are zero. Since the normal displacement is related to the first derivative of the tangential displacement, the boundary conditions may be expressed as

$$U(\xi_1) = \left.\frac{dU}{d\xi}\right|_{\xi_1} = U(\xi_2) = \left.\frac{dU}{d\xi}\right|_{\xi_2} = 0$$

Equation 2.74 is the matrix equation representation of the boundary conditions. The values of ω which render zero values for the determinant of the square matrix in Equation 2.74 are the nondimensional natural frequency ratios for in-plane oscillation.

The solution of Equation 2.44 for the out-of-plane natural frequency ratios is carried out analgously to the method presented for solution of the in-plane natural frequency ratios.