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ELIZABETH LINGFOON YIP for the DOCTOR OF PHILOSOPHY
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C. S. Ballantine

This paper studies necessary and sufficient conditions for a matrix to be conjunctive with its adjoint. The problem is solved completely in the usual complex case, in which it is shown that a matrix is conjunctive to its adjoint iff it is conjunctive to a real matrix. The problem is extended to pairs of fields $\mathcal{F}$, $\mathcal{E}$, where $[\mathcal{F} : \mathcal{E}] = 2$ and characteristic $\mathcal{F} \neq 2$. It is shown that if a matrix is conjunctive to a matrix over $\mathcal{E}$, it is then conjunctive to its adjoint. To achieve this result, we first show a matrix over any field $\mathcal{E}$ is congruent over $\mathcal{E}$ to its transpose. We also show that it is sufficient to consider non-singular pencils by proving the uniqueness up to conjunctivity of the non-singular summand of the pencil $\lambda H + \mu K$, where λ and μ are indeterminates over $\mathcal{E}$, $H^* = H$ and $K^* = K$, when $\lambda H + \mu K$ is decomposed (by conjunctivity over $\mathcal{F}$) into a direct sum of its minimum-indices part and a non-singular part.

Matrices Conjunctive with Their Adjoints

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Elizabeth Lingfoon Yip

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Signature redacted for privacy.

Professor of Mathematics
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Signature redacted for privacy.

Chairman of Department of Mathematics

Signature redacted for privacy.

Dean of Graduate School

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Typed by Clover Redfern for Elizabeth Lingfoon Yip

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MATRICES CONJUNCTIVE WITH THEIR ADJOINTS

I. INTRODUCTION AND PRELIMINARIES

1. Introduction

Let $\mathcal{F}$ be a field of characteristic not equal to 2 and $\mathcal{E}$ be a subfield of index 2 in $\mathcal{F}$ (i.e., $\mathcal{F}$ is a vector space of dimension 2 over $\mathcal{E}$). Then there is a unique automorphism $*$ of $\mathcal{F}$ whose fixed field is $\mathcal{E}$:

$$(a+b)^* = a^* + b^*,$$

$$(ab)^* = a^*b^*,$$

and $a^* = a$ iff $a \in \mathcal{E}$. Because $\mathcal{F}$ has dimension 2 over $\mathcal{E}$, we also have that $*$ is involuntary: $(a^*)^* = a$. We call a^* the conjugate of a. If A is a matrix over $\mathcal{F}$, we denote by A^* the transpose of the conjugate of A. Then we have the following properties:

$$(A^*)^* = A,$$

$$(A+B)^* = A^* + B^*,$$

$$(AB)^* = B^*A^*,$$

$$(aA)^* = a^*A^*$$

when $A+B$ and AB are well defined and $a \in \mathcal{F}$. We shall say S is $*$-symmetric iff $S = S^*$, and say S is $*$-congruent to a matrix T iff there exists a non-singular matrix C such that $C^*SC = T$. We say a matrix S is of type 1 iff S is $*$-congruent to S^*; we say S is of type 2 iff S is $*$-congruent to a matrix over $\mathcal{E}$.

We also consider our matrix S as a linear transformation from $\mathcal{V}$, an n-dimensional vector space over $\mathcal{F}$, into $\mathcal{V}^* = \{f: \mathcal{V} \to \mathcal{F} \mid f(x+y) = f(x) + f(y),\ f(\alpha x) = \alpha^* f(x)\}$. We call $\mathcal{V}^*$ the $*$-dual of $\mathcal{V}$. Then S^* is a linear transformation from $(\mathcal{V}^*)^* = \mathcal{V}$ into $\mathcal{V}^*$. Elements of both $\mathcal{V}$ and $\mathcal{V}^*$ are being written as column vectors over $\mathcal{F}$. Suppose $x \in \mathcal{V} = \mathcal{F}^{n \times 1}$, $f \in \mathcal{V}^*$, define $\langle x, f\rangle = x^* f$. Then $\langle x, Sy\rangle = \langle y, S^* x\rangle^*$ for all $x, y \in \mathcal{V}$. In this context, S is $*$-symmetric iff $\langle x, Sy\rangle = \langle y, Sx\rangle^*$ for all $x, y \in \mathcal{V}$. S is $*$-congruent to T iff there exists a one-to-one, onto, linear transformation $C: \mathcal{V} \to \mathcal{V}$ such that $\langle Cx, SCy\rangle = \langle x, Ty\rangle$ for all $x, y \in \mathcal{V}$. If $\{e_1, \ldots, e_n\}$ is a basis for $\mathcal{V}$, then $\langle e_i, Se_j\rangle = e_i^* Se_j$ is the (i,j) entry of the matrix of S with respect to the basis $\{e_1, \ldots, e_n\}$. If $\mathcal{U}$ is a subspace of $\mathcal{V}$, we define $\mathcal{U}^0 = \{x \in \mathcal{V} : x^*\mathcal{U} = 0\}$. Suppose $\mathcal{V} = \bigoplus_{j=1}^{m} \mathcal{V}_j$. Define

$$\mathcal{V}_j^* = (\mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_{j-1} \oplus \mathcal{V}_{j+1} \oplus \cdots \oplus \mathcal{V}_{m-1} \oplus \mathcal{V}_m)^0$$

$$= \mathcal{V}_1^0 \cap \mathcal{V}_2^0 \cap \cdots \cap \mathcal{V}_{j-1}^0 \cap \mathcal{V}_{j+1}^0 \cap \cdots \cap \mathcal{V}_{m-1}^0 \cap \mathcal{V}_m^0 \ ;$$

then $\mathcal{V}_j^*$ acts as a $*$-dual of $\mathcal{V}_j$, i.e., $\mathcal{V}_j^0 \cap \mathcal{V}_j^* = 0$, and $\mathcal{V}^* = \bigoplus_{j=1}^{m} \mathcal{V}_j^*$.

If $n = \sum_{i=1}^{p} k_i$, and we partition the matrix

$$S = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1p} \\ S_{21} & & & \\ \vdots & & & \\ S_{p1} & & \cdots & S_{pp} \end{bmatrix},$$

where S_{ij} is a $k_i \times k_j$ matrix, we write $S = (S_{ij})$. If C is a matrix such that $C = M_1 \oplus \cdots \oplus M_p$, we write $C = \mathrm{diag}(M_1, \ldots, M_p)$, and call C a direct sum of $M_1, \ldots, M_p$.

We have the following facts which will be useful in the work following (we shall not prove those quoted from the literature):

<u>Fact 1.</u> If $S = (S_{ij})$, $C = \mathrm{diag}(I_k, -I_k, I_k, \ldots)$, and $P = (P_{ih}) = C^*SC$, then $P_{ih} = (-1)^{i+h} S_{ih}$.

<u>Fact 2.</u> If $C = \mathrm{diag}(I_k, jI_k, I_k, \ldots)$ where $j^* = -j$, and $P = C^*SC$, then

$$\begin{aligned} P_{ih} &= S_{ih} && \text{if } i \text{ and } h \text{ are odd} \\ &= jS_{ih} && \text{if } i \text{ is odd and } h \text{ is even} \\ &= -jS_{ih} && \text{if } i \text{ is even and } h \text{ is odd} \\ &= -j^2 S_{ih} && \text{if } i \text{ and } h \text{ are even.} \end{aligned}$$

The proofs of Fact 1 and Fact 2 are routine computations.

<u>Fact 3.</u> If $S = A \oplus O$ where A is non-singular and O is a zero matrix, and if S is *-congruent to $B \oplus O$ where B is non-singular and of the same dimension as A, then A is *-congruent to B.

<u>Proof:</u> Suppose C is a non-singular matrix such that $C^*SC = B \oplus O$. Partition $C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$ accordingly. Thus

$$C^*SC = C^*(A \oplus O)C = \begin{bmatrix} C_{11}^*AC_{11} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus $C_{11}^*AC_{11} = B$. Since A and B are both non-singular, C_{11} is also non-singular. Thus A is *-congruent to B.

Suppose H and K are *-symmetric and K is non-singular. Suppose $(K^{-1}H-\beta I)$ is nilpotent for some $\beta \in \mathcal{E}$. Let m be a positive integer such that $(K^{-1}H-\beta I)^m = 0$, but $(K^{-1}H-\beta I)^{m-1} \neq 0$. If we define an inner product $(x, y) = x^*Ky$, then from Mal'cev [7], we can choose a basis for $\mathcal{V}$ so that $K^{-1}H$ is block diagonal with each diagonal block a Jordan canonical block of eigenvalue β, and the Gram matrix of the inner product is conformably block diagonal with each diagonal block of the form

$$\varepsilon \begin{bmatrix} 0 & & & 1 \\ & & 1^{\cdot^{\cdot^{\cdot}}} & \\ & 1 & & \\ 1 & & & 0 \end{bmatrix},$$

where $\varepsilon \in \mathcal{E}$. Thus we have:

<u>Fact 4</u>. Suppose $S = \lambda H + \mu K$, where λ and μ are indeterminants over $\mathcal{E}$ or $\mathcal{F}$, both H and K are $*$-symmetric, K is non-singular, and $(K^{-1}H-\beta I)$ is nilpotent for some $\beta \in \mathcal{E}$. Then there exists for $\mathcal{V}$ a basis such that (the matrix of) S is block diagonal with each block of the form

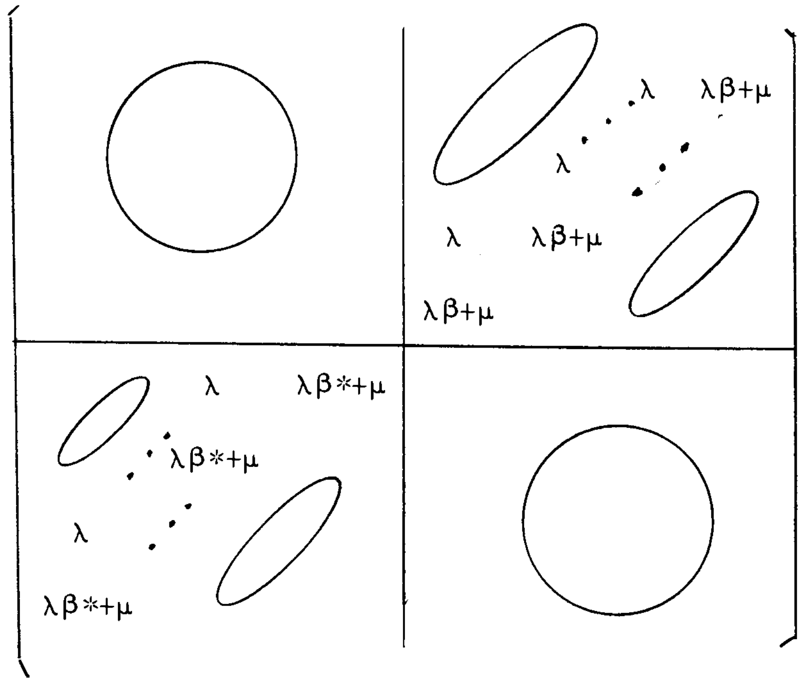
$$\varepsilon \begin{bmatrix} & & & \lambda & \lambda\beta+\mu \\ & & \cdot^{\cdot^{\cdot}} & \lambda\beta+\mu & \\ & \lambda & \cdot^{\cdot^{\cdot}} & & \\ \lambda & \lambda\beta+\mu & & & \\ \lambda\beta+\mu & & & & \end{bmatrix}$$

where $\varepsilon \in \mathcal{E}$.

We define E_n as the $n \times n$ matrix with 1's on the anti-diagonal and zero everywhere else, F_n as the matrix with 1's on the first super-anti-diagonal and zero everywhere else. Also define $N_n = E_n F_n$.

<u>Fact 5</u>. If K and H are $*$-symmetric and $(K^{-1}H-\beta)(K^{-1}H-\beta^*)$ is nilpotent with $\beta \neq \beta^*$, and if λ, μ are

indeterminates over $\mathcal{F}$, then there exists a basis for $\mathcal{V}$ such that $\lambda H + \mu K$ is block diagonal with each block of the form:



<u>Fact 6</u> [12]. If H is symmetric and K is skew-symmetric over $\mathcal{E}$, and λ and μ are indeterminates over $\mathcal{E}$, then the pencil $\lambda H + \mu K$ is congruent over $\mathcal{E}$ to the following form

$$O \oplus \bigoplus_{i=1}^{n} \begin{bmatrix} 0 & L_{\varepsilon_i} \\ \hat{L}_{\varepsilon_i} & 0 \end{bmatrix} \oplus (\lambda H_0 + \mu K_0),$$

where

$$O \bigoplus_{i=1}^{n} \begin{bmatrix} 0 & L_{\varepsilon_i} \\ \hat{L}_{\varepsilon_i} & 0 \end{bmatrix}$$

is called the minimum-indices part of the pencil $\lambda H + \mu K$, and

$$L_{\varepsilon_i} = \underbrace{\begin{bmatrix} \lambda & & & \\ \mu & \lambda & & \\ & \mu & \ddots & \\ & & \ddots & \lambda \\ & & & \mu \end{bmatrix}}_{\varepsilon_i}, \quad L_{\varepsilon_i} = \underbrace{\begin{bmatrix} \lambda & -\mu & & & \\ & \lambda & -\mu & & \\ & & \ddots & \ddots & \\ & & & \lambda & -\mu \\ & & & & \lambda & -\mu \end{bmatrix}}_{\varepsilon_i+1}$$

and $\det(\lambda H_0+\mu K_0)$ is a non-zero polynomial.

<u>Fact 7</u>. Witt's Theorem [9]. If A, B_1, B_2 are *-symmetric non-singular matrices over $\mathcal{F}$, then $A \oplus B_1$, *-congruent to $A \oplus B_2$ implies B_1 is *-congruent to B_2.

2. Uniqueness of the Non-Singular Core of a *-Symmetric Pencil

<u>Theorem I. 2. 1.</u> Let H and K be *-symmetric. Then the pencil $\lambda H + \mu K$, where λ and μ are indeterminates over $\mathcal{E}$, is *-congruent to $L \oplus M$, where

$$L = O \overset{k}{\underset{i=1}{\oplus}} \begin{bmatrix} 0 & L_{\varepsilon_i} \\ L^*_{\varepsilon_i} & 0 \end{bmatrix} .$$

L_{ε_i} is as defined in Fact 6, $L^*_{\varepsilon_i}$ is the conjugate-transpose of L_{ε_i}, and $M = \lambda H_0 + \mu K_0$ is a non-singular pencil. If $\lambda H + \mu K$ is *-congruent also to $L \oplus N$ where N is a non-singular pencil, then M is $*$-congruent to N.

Turnbull proved that a $*$-symmetric pencil can always be reduced to the form $L \oplus M$, where L is the minimum-indices part and M is a non-singular pencil [11]. In this section we shall only prove the uniqueness (up to $*$-congruency) of the non-singular summand M.

Before we proceed with the proof of the theorem, we shall discuss some of the properties of the pencil $L \oplus M$.

Suppose $\mathcal{U}$ is a subspace of $\mathcal{V}$, and S, T are linear transformations from $\mathcal{V}$ to $\mathcal{V}^*$. For our discussion in this section, we need to define, for $\mathcal{U} \subseteq \mathcal{V}$, $(S^{-1}T)\mathcal{U} = \{x \in \mathcal{V} : Sx \in T\mathcal{U}\}$, and, inductively, define $(S^{-1}T)^i\mathcal{U} = \{x \in \mathcal{V} : Sx \in T(S^{-1}T)^{i-1}\mathcal{U}\}$, for $i = 2, 3, \ldots$. $(S^{-1}T)^0\mathcal{U} = \mathcal{U}$.

Also note that if we let $\Lambda = \lambda H + \mu K$ be a pencil of $*$-symmetric matrices, considered as the matrix of a pencil of bilinear forms, then, for each co-ordinate subspace, the matrix of this pencil of forms restricted to this subspace is the corresponding principal submatrix of Λ, and conversely.

<u>Lemma 1.</u> Let $\mathcal{V}^*$ be the $*$-dual of $\mathcal{V}$, let $\mathcal{V}_j$ $(j = 1, 2, \ldots, k)$ be subspaces of $\mathcal{V}$ such that $\mathcal{V} = \bigoplus_{j=1}^{k} \mathcal{V}_j$, and let $\mathcal{V}^* = \bigoplus_{j=1}^{k} \mathcal{V}_j^*$ be the corresponding $*$-dual direct sum defined in Section 1 of this chapter. If S and T are linear

transformations such that $S, T: \mathcal{V} \to \mathcal{V}^*$, $S\mathcal{V}_j \subseteq \mathcal{V}_j^*$, and $T\mathcal{V}_j \subseteq \mathcal{V}_j^*$, then $(S^{-1}T)^i\mathcal{V} = \bigoplus_{j=1}^{k} [\mathcal{V}_j \cap (S^{-1}T)^i\mathcal{V}_j]$.

Proof of Lemma 1. The proof proceeds by induction on i. If $i = 0$, the lemma is obviously true. Suppose it is true for $i \leq h-1$. Consider $i = h$. It is enough to show $(S^{-1}T)^h\mathcal{V} \subseteq \bigoplus_{j=1}^{k} (\mathcal{V}_j \cap (S^{-1}T)^h\mathcal{V}_j)$. Suppose $x \in (S^{-1}T)^h\mathcal{V}$, and $x = \sum_{j=1}^{k} x_j$, with $x_j \in \mathcal{V}_j$. Then $Sx = \sum_{j=1}^{k} Sx_j \in T(S^{-1}T)^{h-1}\mathcal{V}$.

By our induction hypothesis,

$$T(S^{-1}T)^{h-1}\mathcal{V} = T(\bigoplus_{j=1}^{k} \mathcal{V}_j \cap (S^{-1}T)^{h-1}\mathcal{V}_j)$$

$$= \bigoplus_{j=1}^{k} T(\mathcal{V}_j \cap (S^{-1}T)^{h-1}\mathcal{V}_j) .$$

Thus

$$Sx = \sum_{j=1}^{k} Sx_j = \sum_{j=1}^{k} Ty_j ,$$

where $y_j \in (\mathcal{V}_j \cap (S^{-1}T)^{h-1}\mathcal{V}_j)$ and $Ty_j \in \mathcal{V}_j^*$. But $Sx_j \in \mathcal{V}_j^*$. Therefore $Sx_j = Ty_j \in T(S^{-1}T)^{h-1}\mathcal{V}_j$, so $x_j \in \mathcal{V}_j \cap (S^{-1}T)^h\mathcal{V}_j$. Thus $x \in \bigoplus_{j=1}^{k} \mathcal{V}_j \cap (S^{-1}T)^h\mathcal{V}_j$. q.e.d.

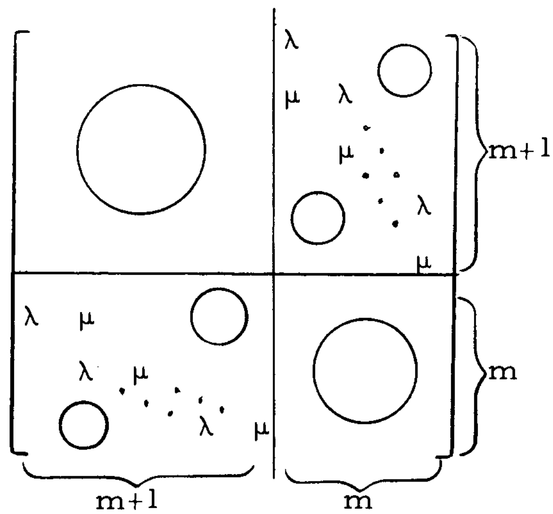
Lemma 2. Let (as usual) characteristic $\mathcal{F} \neq 2$, $H = H^*$ and $K = K^*$ be linear maps of $\mathcal{V}$ into $\mathcal{V}^*$. Let λ, μ be indeterminates over $\mathcal{F}$. Suppose $\lambda H + \mu K$ is $*$-congruent over $\mathcal{F}$ to $L \oplus M_\infty \oplus M_0 \oplus M_1$, where L has no elementary divisors, M_∞ has elementary divisors only of the form μ^q, and M_0 has elementary divisors only of the form λ^q, and M_1 has elementary divisors neither of the form λ^q nor of the form μ^q and $M = M_\infty \oplus M_0 \oplus M_1$ is a non-singular pencil. Then the $*$-congruency type of M over $\mathcal{F}$ is determined as follows:

Let $\mathcal{U} = (K^{-1}H)^n\mathcal{V}$, $\mathcal{W} = (H^{-1}K)^n\mathcal{V}$. (Here $n = \dim \mathcal{V}$ as usual.) Then the pencil Λ of $*$-bilinear forms $x^*(\lambda H + \mu K)y$ with x and y restricted to $\mathcal{U}$ is $*$-congruent over $\mathcal{F}$ to the pencil of $*$-bilinear forms whose matrix is $O \oplus M_0 \oplus M_1$, and Λ restricted to $\mathcal{W}$ is $*$-congruent over $\mathcal{F}$ to the pencil of $*$-bilinear forms whose matrix is $O \oplus M_\infty \oplus M_1$.

<u>Proof of Lemma 2</u>: Without loss of generality, we may assume $\lambda H + \mu K = L \oplus M_\infty \oplus M_0 \oplus M_1$ and

$$L = \bigoplus_{i=1}^{k'} 0 \oplus \left\{ \bigoplus_{i=1}^{k} \begin{bmatrix} 0 & L_{\varepsilon_i} \\ L_{\varepsilon_i}^* & 0 \end{bmatrix} \right\}.$$

Let $\mathcal{V} = \mathcal{R} \oplus \mathcal{V}_\infty \oplus \mathcal{V}_0 \oplus \mathcal{V}_1$. Here $\mathcal{R}$, $\mathcal{V}_\infty$, $\mathcal{V}_0$, $\mathcal{V}_1$ denote the co-ordinate subspaces corresponding to L, M_∞, M_0, M_1, respectively. Denote by $\mathcal{R}_i = \operatorname{span}\{x_1, \ldots, x_{2m+1}\}$, $m \geq 0$, one of the co-ordinate subspaces corresponding to one of the direct summands in L, where x_j denotes the jth co-ordinate vector. Let $\mathcal{R}_i^* = \text{span}\{\hat{x}_1, \ldots, \hat{x}_{2m+1}\}$ be the corresponding subspace of $\mathcal{R}^*$, thus $\mathcal{R}_i^*$ acts as a $*$-dual of $\mathcal{R}_i$. Note that the matrix of $\lambda H + \mu K$ restricted to $\mathcal{R}_i$ is



For simplicity in notation we drop the subscript i in $\mathcal{R}_i$. Thus

$$
\begin{array}{ll}
Hx_1 = \hat{x}_{m+2}, & Kx_1 = 0 \\
Hx_2 = \hat{x}_{m+3}, & Kx_2 = \hat{x}_{m+2} \\
\vdots & Kx_3 = \hat{x}_{m+3} \\
Hx_m = \hat{x}_{2m+1}, & \vdots \\
Hx_{m+1} = 0, & Kx_{m+1} = \hat{x}_{2m+1} \\
Hx_{m+2} = \hat{x}_1, & Kx_{m+2} = \hat{x}_2 \\
Hx_{m+3} = \hat{x}_2, & Kx_{m+3} = \hat{x}_3 \\
\vdots & \vdots \\
Hx_{2m} = \hat{x}_{m-1}, & Kx_{2m} = \hat{x}_m \\
Hx_{2m+1} = \hat{x}_m, & Kx_{2m+1} = \hat{x}_{m+1}
\end{array}
$$

Thus

$$\mathcal{R} \cap (K^{-1}H)\mathcal{R} = \text{span}\{x_1, x_2, \ldots, x_{2m}\}$$
$$\mathcal{R} \cap (K^{-1}H)^2\mathcal{R} = \text{span}\{x_1, x_2, \ldots, x_{2m-1}\}$$
$$\vdots$$
$$\mathcal{R} \cap (K^{-1}H)^{m+1}\mathcal{R} = \text{span}\{x_1, x_2, \ldots, x_{m+1}\}$$
$$= \mathcal{R} \cap (K^{-1}H)^k\mathcal{R} \quad \text{for all} \quad k \geq m+1 .$$

Thus the matrix of Λ restricted to $\mathcal{R} \cap (K^{-1}H)^n\mathcal{R}$ is O. (This O matrix is of order $m+1$ here.)

K is non-singular on $\mathcal{V}_\infty$; hence $K\mathcal{V}_\infty = \mathcal{V}_\infty^*$ and $H\mathcal{V}_\infty \subseteq \mathcal{V}_\infty^*$, so $K^{-1}H\mathcal{V}_\infty \subseteq \mathcal{V}_\infty$ and $K^{-1}H$ is a nilpotent <u>mapping</u> on $\mathcal{V}_\infty$. Thus $(K^{-1}H)^n\mathcal{V}_\infty = 0$.

H is non-singular on $\mathcal{V}_0$. Thus, $H\mathcal{V}_0 = \mathcal{V}_0^*$. $(K^{-1}H)\mathcal{V}_0 \subseteq K^{-1}\mathcal{V}_0^* = \{x: Kx \in \mathcal{V}_0^*\} = \mathcal{V}_0$, since $K(\mathcal{V}_0) \subseteq \mathcal{V}_0^*$. Thus $(K^{-1}H)\mathcal{V}_0 = \mathcal{V}_0$. Suppose $(K^{-1}H)^i\mathcal{V}_0 = \mathcal{V}_0$ for $i \leq k-1$. Then $x \in (K^{-1}H)^k\mathcal{V}_0$ iff $Kx \in H(K^{-1}H)^{k-1}\mathcal{V}_0 = H\mathcal{V}_0 = \mathcal{V}_0$. Thus $\mathcal{V}_0 \cap (K^{-1}H)^n\mathcal{V}_0 = \mathcal{V}_0$.

K and H restricted to $\mathcal{V}_1$ are non-singular. Thus $(K^{-1}H)^n\mathcal{V}_1 = \mathcal{V}_1$.

Therefore $x^*\Lambda y = x^*(\lambda H+\mu K)y$, with x, y restricted to $\mathcal{U} = (K^{-1}H)^n\mathcal{V}$, is *-congruent to a pencil of *-bilinear forms whose matrix is $O \oplus M_0 \oplus M_1$.

Similarly, we can show that $x^*\Lambda y$, with x, y restricted to

$\mathcal{W} = (H^{-1}K)^n \mathcal{V}$ is *-congruent to a pencil of *-bilinear forms whose matrix is $O \oplus M_\infty \oplus M_1$.

Proof of Theorem I.2.1. Write

$$L \oplus M = \lambda H + \mu K = \Lambda,$$

$$L \oplus N = \lambda H_1 + \mu K_1 = \Lambda_1 .$$

Without loss of generality, we may assume $M = M_{\infty 1} \oplus M_{01} \oplus M_{11}$, $N = M_{\infty 2} \oplus M_{02} \oplus M_{12}$, where $M_{\infty 1}$ and $M_{\infty 2}$ have elementary divisors only of the form μ^q, M_{01} and M_{02} have elementary divisors only of the form λ^q, and M_{11} and M_{12} have no elementary divisors of the form μ^q nor of the form λ^q, and recall that M and N are non-singular pencils. If C is a non-singular matrix over $\mathcal{F}$, (we also consider C as a non-singular map of $\mathcal{V}$ into $\mathcal{V}$), such that $C^*\Lambda C = C^*(L \oplus M)C = L \oplus N$, then $C^*HC = H_1$ and $C^*KC = K_1$.

By Lemma 2, $x^*\Lambda y$, and $x^*\Lambda_1 y$, with x, y restricted to $(K^{-1}H)^n \mathcal{V}$ and $(K_1^{-1}H_1)^n \mathcal{V}$, respectively, is equal to $u^*(O \oplus M_{01} \oplus M_{11})_v$ and $u^*(O \oplus M_{02} \oplus M_{12})_v$, respectively, with $u, v \in (K^{-1}H)^n \mathcal{V}$ and $(K_1^{-1}H_1)^n \mathcal{V}$, respectively.

We want to show by induction on i that

$[(C^*KC)^{-1}(C^*HC)]^i \mathcal{V} = C^{-1}(K^{-1}H)^i C \mathcal{V}$.

For $i = 1$:

$$x \in [(C^*KC)^{-1}C^*HC)]\mathcal{V}$$

iff $$C^*KCx \in C^*HC\mathcal{V}$$

iff $$KCx \in HC\mathcal{V}$$

iff $$Cx \in K^{-1}HC\mathcal{V}$$

iff $$x \in C^{-1}(K^{-1}H)C\mathcal{V}.$$

Suppose

$$[(C^*KC)^{-1}(C^*HC)]^{k-1}\mathcal{V} = C^{-1}(K^{-1}H)^{k-1}C\mathcal{V}.$$

Then

$$x \in [(C^*KC)^{-1}(C^*HC)]^{k}\mathcal{V}$$

iff $$(C^*KC)x \in C^*HC[(C^*KC)^{-1}(C^*HC)]^{k-1}\mathcal{V}$$

iff $$KCx \in HC[(C^*KC)^{-1}(C^*HC)]^{k-1}\mathcal{V}$$

(by our induction hypothesis)

$$= HCC^{-1}(K^{-1}H)^{k-1}C\mathcal{V}$$

$$= H(K^{-1}H)^{k-1}C\mathcal{V}$$

iff $$Cx \in K^{-1}H(K^{-1}H)^{k-1}C\mathcal{V}$$

iff $$x \in C^{-1}(K^{-1}H)^{k}C\mathcal{V}.$$

Let $\{x_1, x_2, \ldots, x_m\}$ be a basis for $\mathcal{U} = (K^{-1}H)^n\mathcal{V}$. Then $\{C^{-1}x_1, C^{-1}x_2, \ldots, C^{-1}x_m\}$ is a basis for

$$C^{-1}(K^{-1}H)^n\mathcal{V} = C^{-1}(K^{-1}H)^nC\mathcal{V}$$

$$= [(C^*KC)^{-1}(C^*HC)]^n\mathcal{V}.$$

Therefore

$$(C^{-1}x_j)^*\Lambda_1(C^{-1}x_j) = x_i^*(C^*)^{-1}\Lambda_1 C^{-1}x_j$$

$$= x_i^*(C^*)^{-1}C^*\Lambda CC^{-1}x_j$$

$$= x_i^*\Lambda x_j \ .$$

Thus $x^*\Lambda y$ with x, y restricted to $\mathcal{U}$ is $*$-congruent to $u_1^*(O \oplus M_{02} \oplus M_{12})v_1$ with $u_1, v_1 \in (K_1^{-1}H_1)$. Thus $O \oplus M_{01} \oplus M_{11}$ is $*$-congruent to $O \oplus M_{02} \oplus M_{12}$. Thus by Fact 3, $M_{01} \oplus M_{11}$ is $*$-congruent to $M_{02} \oplus M_{12}$. Let $M_{ij} = \lambda H_{ij} + \mu K_{ij}$ $i = 0, 1$, $j = 1, 2$, where H_{ij} and K_{ij} are $*$-symmetric. Then

$$C_1^*(H_{01} \oplus H_{11})C_1 = H_{02} \oplus H_{12}$$

$$C_1^*(K_{01} \oplus K_{11})C_1 = K_{02} \oplus K_{12} \ .$$

Thus

$$C_1^{-1}(H_{01}^{-1}K_{01} \oplus H_{11}^{-1}K_{11})C_1$$

$$= [C_1^*(H_{01} \oplus H_{11})C_1]^{-1}[C_1^*(K_{01} \oplus K_{11})C_1]$$

$$= H_{02}^{-1}K_{02} \oplus H_{12}^{-1}K_{12} \ .$$

Partition C_1 accordingly: $C_1 = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$. Let $B_{ij} = H_{ij}^{-1}K_{ij}$. Then

$$(B_{01} \oplus B_{11})C_1 = C_1(B_{02} \oplus B_{12})$$

$$\begin{bmatrix} B_{01}C_{11} & B_{01}C_{12} \\ B_{11}C_{21} & B_{11}C_{22} \end{bmatrix} = \begin{bmatrix} C_{11}B_{02} & C_{12}B_{12} \\ C_{21}B_{02} & C_{22}B_{22} \end{bmatrix}$$

$$B_{11}\, C_{21} = C_{21}\, B_{02} .$$

Since B_{11} and B_{02} have no eigenvalues in common, $C_{12} = 0$ [12]. Similarly, $C_{21} = 0$. Thus $C_{11}^{*}M_{01}C_{11} = M_{02}$, $C_{22}^{*}M_{11}C_{22} = M_{12}$. M_{01}, M_{02}, M_{11} and M_{12} are non-singular, thus C_{11} and C_{22} are non-singular. So M_{01} is *-congruent to M_{02} and M_{11} is *-congruent to M_{12}. Likewise, $L \oplus M$ restricted to $(H^{-1}K)^n \mathcal{V}$ is *-congruent to $O \oplus M_{\infty 1} \oplus M_{11}$; $L \oplus N$ restricted to $(H_1^{-1}K_1)^n \mathcal{V}$ is *-congruent to $O \oplus M_{\infty 2} \oplus M_{12}$, where $M_{\infty 1}$ and $M_{\infty 2}$ have elementary divisors of the form μ^q. With an approach similar to that of our consideration on $\mathcal{U}$, we can show that $M_{\infty 1}$ is *-congruent to $M_{\infty 2}$. Since

$$M = M_{01} \oplus M_{\infty 1} \oplus M_{11}$$

$$N = M_{02} \oplus M_{\infty 2} \oplus M_{12} ,$$

we have that M is *-congruent to N. q.e.d.

II. GENERAL CASE

1. Congruency of a Matrix to Its Transpose

If $H = H'$, $K = K'$, H and K are over $\mathcal{E}$, and λ and μ are indeterminates over $\mathcal{E}$, then $\lambda H + \mu K$ is congruent to the following:

$$[II.1.1] \qquad L \oplus M_0 \oplus M_\infty \oplus M_1$$

where L is described in Fact 6 of Chapter I, M_0 has elementary divisors only of the form λ^q, and M_∞ has elementary divisors only of the form μ^q, M_1 has neither elementary divisors of the form λ^q nor μ^q, and $M_0 \oplus M_\infty \oplus M_1$ is non-singular.

In this section, we shall prove:

Theorem II.1.1. Every matrix over $\mathcal{E}$ is congruent over $\mathcal{E}$ to its transpose.

Lemma 3. Let $\mathcal{V}$ be finite dimensional over $\mathcal{E}$ with (as always) characteristic $\mathcal{E} \neq 2$. Let $H: \mathcal{V} \to \mathcal{V}'$ be symmetric, $K: \mathcal{V}' \to \mathcal{V}$ be skew, and $B = KH: \mathcal{V} \to \mathcal{V}$ be non-singular. Let $q(t)$ be irreducible in $\mathcal{E}[t]$ and let $q(t^2)^m$ be the minimum polynomial for B on $\mathcal{V}$. (Here $\mathcal{V}'$ is the (ordinary) dual of $\mathcal{V}$.) Then there is a maximum B-cyclic subspace $\mathcal{U}$ of $\mathcal{V}$ such that

$\mathcal{U} \cap (H\mathcal{U})^0 = 0.$

Proof of Lemma: Let $H_1 = Hq(B^2)^{m-1}$. Then $H_1' = H_1 \neq 0$, thus there exists $e \in \mathcal{V}$ such that $e'H_1e \neq 0$. Let $\mathcal{U}$ be the B-cyclic subspace generated by e. To show that $\mathcal{U}$ is maximum cyclic and that $\mathcal{U} \cap (H\mathcal{U})^0 = 0$, it is sufficient to show that for $f(t) \in \mathcal{E}[t]$, $(f(B)e)'H\mathcal{U} = 0 \Rightarrow q(t^2)^m | f(t)$.

Suppose $(f(B)e)'H\mathcal{U} = 0$. Write $f(t) = [q(t^2)]^{\ell} p(t)$ where $q(t^2) \nmid p(t)$. Let $r(t) = \text{g.c.d}\ (p(t), p(-t))$. Then $r(t) = t^k s(t^2)$, for some $k \geq 0$, and some $s(t) \in \mathcal{E}[t]$, such that $t \nmid s(t^2)$. We claim that $\text{g.c.d}\ (r(t), q(t^2)) = 1$. Let $m(t) = \text{g.c.d}\ (r(t), q(t^2))$. Since $q(t^2)$ is even and $r(t)$ is even or odd, then $m(t) = t^{\ell} n(t^2)$ for some $\ell \geq 0$ and $n(t) \in \mathcal{E}[t]$, such that $t \nmid n(t^2)$. Thus $t^{\ell} | q(t^2)$. It follows that $q(t^2) = t^{2m} h(t^2)$ for some $m \geq 0$ and some $h(t) \in \mathcal{E}[t]$ such that $t \nmid h(t^2)$. Thus $q(t) = t^m h(t)$. But $q(0) \neq 0$, since B is non-singular $\therefore$ $k = 0$, so $m(t) = n(t^2)$. Thus $n(t^2) | q(t^2)$, which implies $n(t) | q(t)$, which in turn implies $n(t) = 1$ or $n(t) = q(t)$. If $n(t) = q(t)$, then $q(t^2) = n(t^2) = m(t) | r(t)$, so $q(t^2) | p(t)$, which is a contradiction. $\therefore$ $n(t) = 1$. Thus $m(t) = 1$. Since $\text{g.c.d}\ (r(-t), q(t^2)) = m(-t) = 1$. Thus $r(-B)$ is non-singular, so $r(-B)\mathcal{U} = \mathcal{U}$. Let $r(t) = a(t)p(t) + b(t)p(t)$, for some $a(t), b(t) \in \mathcal{E}[t]$. Now, $0 = (f(B)e)'H\mathcal{U}$, which implies for all $i \geq 0$, that

$$[II.1.2] \qquad 0 = (f(B)e)'HB^i e = [q(B^2)^\ell p(B)e]'HB^i e$$

Since $K = HB$ and $H' = H$, $B'H = (HB)' = K' = -K = -HB$. Inductively, we can show $HB^i = (-B^i)'H$. Thus,

$$\begin{aligned}[II.1.2] &= ((-B)^i e)'H[q(B^2)^\ell p(-B)e] \\ &= [((-B)^i e)'H[q(B^2)^\ell p(-B)e]]' \\ &= [q(B^2)^\ell p(-B)e]'H(-B)^i e\ .\end{aligned}$$

Thus

$$[q(B^2)^\ell p(-B)e]'H\,\mathcal{U} = 0 = [q(B^2)^\ell p(B)e]'H\,\mathcal{U},$$

so

$$\begin{aligned}0 = [q(B^2)^\ell p(-B)e]'H\,\mathcal{U} &\supseteq [q(B^2)^\ell p(-B)e]'Hb(-B)\mathcal{U} \\ &= [q(B^2)^\ell p(-B)b(B)e]'H\mathcal{U}\end{aligned}$$

and

$$\begin{aligned}0 = [q(B^2)^\ell p(B)e]'H\,\mathcal{U} &\supseteq [q(B^2)^\ell p(B)e]'Ha(-B)\mathcal{U} \\ &= [q(B^2)^\ell p(B)a(B)e]'H\,\mathcal{U}\end{aligned}$$

Thus

$$\begin{aligned}0 &= [q(B^2)^\ell [p(B)a(B)+p(-B)b(B)]e]'H\mathcal{U} \\ &= [q(B^2)^\ell r(B)e]'H\mathcal{U} \\ &= [q(B^2)^\ell e]'Hr(-B)\mathcal{U} \\ &= [q(B^2)^\ell e]'H\mathcal{U}.\end{aligned}$$

$\therefore\ [q(B^2)^{\ell} e]' H\, q(B^2)^{i} e = e' H q(B^2)^{\ell + i} e = 0$ for all $i \geq m$. Since $e' H q(B)^{m-1} e \neq 0$, $\ell \geq m$. Thus $q(t^2)^m \mid f(t)$. q.e.d.

Proof of Theorem II.1.1. Without loss of generality we may consider the pencil [II.1.1] $L \oplus M_0 \oplus M_\infty \oplus M_1$ described in the first paragraph of this section; and we replace the indeterminates λ, μ with 1. We shall show separately L, M_0, M_∞, and M_1 are congruent over $\mathcal{E}$ to their respective transposes.

Consider L. For each non-zero block (of order $2\varepsilon_i + 1$) of L, multiply on the right and left by $I_{\varepsilon_i + 1} \oplus E_{\varepsilon_i}$. Thus in L_{ε_i}, the first column becomes the last column, the second column becomes the second to the last column and etc.; in $\hat{L}_{\varepsilon_i}$, the first row becomes the last row, and the second row becomes the second to the last row and etc. Thus, we have for each non zero block of L

$$\text{[II.1.3]} \qquad \begin{bmatrix} 0 & Z_{\varepsilon_i} \\ \hat{Z}_{\varepsilon_i} & 0 \end{bmatrix},$$

where

$$Z_{\varepsilon_i} = \underbrace{\begin{bmatrix} \bigcirc & & & 1 \\ & & 1 & 1 \\ & \cdot^{\cdot^{\cdot}} & 1 & \\ 1 & \cdot^{\cdot^{\cdot}} & & \\ 1 & & \bigcirc & \end{bmatrix}}_{\varepsilon_i}, \qquad \hat{Z}_{\varepsilon_i} = \left.\begin{bmatrix} \bigcirc & & 1 & -1 \\ & 1 & -1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ \cdot^{\cdot^{\cdot}} & & \bigcirc & \\ 1 & -1 & & \end{bmatrix}\right\} \varepsilon_i .$$

Multiply [II.1.3] on the right and left by $\operatorname{diag}(1, -1, 1, -1, \ldots, 1)$;

we get the transpose of [II. 1.3]. Thus L is congruent to L'.

Next consider M_0. Let $\mathcal{V}_0$ be the co-ordinate subspace corresponding to the direct summand M_0 in $S = H + K = L \oplus M_\infty \oplus M_0 \oplus M_1$. H is non-singular and $B = (H^{-1}K)$ is nilpotent on $\mathcal{V}_0$. Let m be a positive integer such that $B^m = 0$, and $B^{m-1} \neq 0$. We shall consider the cases m even and m odd separately. We shall prove the following:

(i) if m is odd, there exists a maximum B-cyclic subspace $\mathcal{U}$ such that $\mathcal{U} \cap (H\mathcal{U})^0 = 0$;

(ii) if m is even, there exist two maximum B-cyclic subspaces $\mathcal{U}$ and $\mathcal{W}$ such that

$$0 = \mathcal{U} \cap \mathcal{W} = \mathcal{U} \cap (H\mathcal{W})^0 = \mathcal{W} \cap (H\mathcal{U})^0 = (\mathcal{U} \oplus \mathcal{W}) \cap (H(\mathcal{U} \oplus \mathcal{W}))^0,$$

$$\mathcal{U} \subseteq (H\mathcal{U})^0, \quad \text{and} \quad \mathcal{W} \subseteq (H\mathcal{W})^0.$$

Suppose m is odd. $(HB^{m-1})' = HB^{m-1} \neq 0$. There exists $e \in \mathcal{V}$ such that $e'HB^{m-1}e \neq 0$. Let $\mathcal{U}$ be the B-cyclic subspace generated by e. Suppose $(B^\ell p(B)e)'H\mathcal{U} = 0$, where $p(t) \in \mathcal{E}[t]$ such that $t \nmid p(t)$. Since t^m is the minimum polynomial of B on $\mathcal{V}_0$, p(-B) is non-singular and hence $p(-B)\mathcal{U} = \mathcal{U}$. Thus $0 = (B^\ell p(B)e)'H\mathcal{U} = (B^\ell e)'Hp(-B)\mathcal{U} = (B^\ell e)'H\mathcal{U}$. Thus $\ell \geq m$. We therefore have completed the proof of (i).

The following discussion is not needed in the proofs of (i) and (ii), but useful for later purposes. We note that $B\mathcal{U} \subseteq \mathcal{U}$ because $\mathcal{U}$ is B-cyclic. $B(H\mathcal{U})^0 \subseteq (H\mathcal{U})^0$ because if $x \in (H\mathcal{U})^0$, then $0 = x'H\mathcal{U} \supseteq x'HB\mathcal{U} = -(Bx)'H\mathcal{U}$, which implies $Bx \in (H\mathcal{U})^0$. Let $e_1 = e + \sum_{i=1}^{m-1} a_i B^i e$; we can choose $a_1, a_2, \ldots, a_{m-1}$ so that $e_1' H B^k e_1 = 0$ for $k \leq m-2$. Let $\mathcal{U}_1$ be the B-cyclic subspace generated by e_1. $\mathcal{U}_1$ has all the properties proved for $\mathcal{U}$. B restricted to $\mathcal{U}_1$ is the Jordan canonical block of eigenvalue = 0, H restricted to $\mathcal{U}_1$ is a matrix H_0 with non-zero elements on the anti-diagonal and zero everywhere else. Thus $K = HB$ restricted to $\mathcal{U}_1$ is a matrix K_0 with non-zero entries on the first super-anti-diagonal and zero everywhere else. Multiply $H_0 + K_0$ on the right and left by $\mathrm{diag}(1, -1, \ldots, 1)$. Since m is odd, by Fact 1, we have multiplied the first super-anti-diagonal by -1, and have not changed the anti-diagonal. Thus we obtain the transpose of $H_0 + K_0$.

In the next paragraph, we shall prove (ii).

Suppose m is even; since $HB^{m-1} \neq 0$ there exist $e, f \in \mathcal{V}$ such that $e'HB^{m-1}f \neq 0$; we can choose e, f so that $e'HB^{m-1}f = 1$. Let $\mathcal{U}$, $\mathcal{W}$ be the B-cyclic subspaces generated by e and f respectively. First, we want to show $\mathcal{U} \cap \mathcal{W} = 0$. If $x \in \mathcal{U} \cap \mathcal{W}$, then $x = B^k p(B)e = B^\ell q(B)f$, for some $p(t), q(t) \in \mathcal{E}[t]$ such that $t \nmid p(t)$, $t \nmid q(t)$. Without loss of generality, assume $k \geq \ell$. If $\ell \geq m$, then $x = 0$. Therefore assume $\ell < m$. Note, since $t \nmid p(t)$, $t \nmid q(t)$, $p(B)$ and $q(B)$ are non-singular, $r(B) = p(B)q(B)^{-1}$ is non-singular and we can even take $r(t) \in \mathcal{E}[t]$ here, and $r(-B)$ is also non-singular. $x = B^k p(B)e = B^\ell q(B)f$ and $k \geq \ell$. Therefore

$$0 = B^\ell[q(B)f - B^{k-\ell}p(B)e]$$

$$= q(B)B^\ell(f - B^{k-\ell}r(B)e)$$

$$0 = B^\ell(f - B^{k-\ell}r(B)e)$$

since $q(B)$ is non-singular. $B^\ell(f-B^{k-\ell}r(B)e) = 0$ implies

$$0 = e'HB^{m-1}(f - B^{k-\ell}r(B)e)$$

$$= e'HB^{m-1}f - e'HB^{m-1+k-\ell}r(B)e .$$

If $k \geq \ell+1$, $e'HB^{m-1+k-\ell}r(B)e = e'HB^m B^{k-\ell-1}r(B)e = 0$; thus $0 = e'HB^{m-1}f$ which is a contradiction. If $k = \ell$, $e'HB^{m-1}r(B)e = e'HB^{m-1}r(0)e = e'HB^{m-1}f$, $e'HB^{m-1}e = -e'HB^{m-1}e$ because $m-1$ is odd (and hence HB^{m-1} is skew). Thus $e'HB^{m-1}e = 0 = e'HB^{m-1}f$ which is a contradiction. Thus $\ell \geq m$, which impliex $x = 0$. Thus $\mathcal{U} \cap \mathcal{W} = 0$.

Secondly, we want to show $\mathcal{U} \cap (H\mathcal{W})^0 = 0$. Suppose $(B^k p(B)e)'H\mathcal{W} = 0$ for some $p(t) \in \mathcal{E}[t]$ such that $t \nmid p(t)$. Thus $p(B)$ and $p(-B)$ are non-singular, so $p(-B)\mathcal{U} = \mathcal{U}$.

$$0 = (B^k p(B)e)'H\mathcal{W} = (B^k e')Hp(-B)\mathcal{W}$$

$$= e'H(-B)^k \mathcal{W}$$

$$\supseteq e'H(-B)^{k+i}\mathcal{W} \quad \text{for all} \quad i \geq 0$$

Thus $k \geq m$. Thus $\mathcal{U} \cap (H\mathcal{W})^0 = 0$. Likewise we can show that $\mathcal{W} \cap (H\mathcal{U})^0 = 0$.

Thirdly, we want to show $(\mathcal{U} \oplus \mathcal{W}) \cap (H(\mathcal{U} \oplus \mathcal{W}))^0 = 0$. If $x \in (\mathcal{U} \oplus \mathcal{W}) \cap (H(\mathcal{U} \oplus \mathcal{W}))^0$, then $x = B^k p(B)e + B^\ell q(B)f$ where $p(t), q(t) \in \mathcal{E}[t]$ and $t \nmid p(t), q(t)$. Without loss of generality, assume $\ell \geq k$. Suppose $k \leq m-1$. Then

$$0 = [B^k p(B)e + B^\ell q(B)f]'H(\mathcal{U} \oplus \mathcal{W})$$

$$\supseteq [B^k p(B)e + B^\ell q(B)f]'H(-B)^{m-1-k}(\mathcal{U} \oplus \mathcal{W})$$

$$= [B^{m-1} p(B)e + B^{\ell+m-1-k} q(B)f]'H(\mathcal{U} \oplus \mathcal{W})$$

$$\supseteq [B^{m-1} p(B)e + B^{\ell+m-1-k} q(B)f]'H\mathcal{W}$$

$$= [B^{m-1} p(B)e]'H\mathcal{W} + [B^{\ell+m-1-k} q(B)f]'H\mathcal{W}$$

$$\supseteq p(0)[B^{m-1}e]'Hf + [B^{\ell+m-1-k} q(B)f]'Hf.$$

If $\ell > k$, then $B^{\ell+m-1-k} = 0$. If $\ell = k$,

$$[B^{\ell+m-1-k} q(B)f]'Hf = [B^{m-1} q(B)f]'Hf$$

$$= q(0)[B^{m-1}f]'Hf.$$

$[B^{m-1}f]'Hf = (-1)^{m-1}f'HB^{m-1}f$. Since $m-1$ is odd, then HB^{m-1} is skew. Thus $f'HB^{m-1}f = 0$. Both $p(0)$ and $(B^{m-1}e)'Hf$ are nonzero. Thus we have a contradiction. Therefore $\ell > m-1$. Thus we have shown $(\mathcal{U} \oplus \mathcal{W}) \cap [H(\mathcal{U} \oplus \mathcal{W})]^0 = 0$.

Fourthly, let $p(t) = \sum_{j \geq 0} a_j t^{2j}$ and let

$$\hat{f} = f + Bp(B)e$$

$$= f + \sum_{j \geq 0} (a_j B^{2j+1})e \,.$$

Let $\hat{\mathcal{W}}$ be the B-cyclic subspace generated by $\hat{f}$. We can choose $a_0, a_1, \ldots$ such that $H\hat{\mathcal{W}} \subseteq \hat{\mathcal{W}}^0$.

$$0 = \hat{f}'HB^{m-2}\hat{f} = (f + \sum_{j \geq 0} a_j B^{2j+1} e)'HB^{m-2}(f + \sum_{j \geq 0} a_j B^{2j+1} e)$$

$$= f'HB^{m-2}f - 2a_0 e'HB^{m-1}f.$$

Thus we can solve for a_0. If we consider $\hat{f}'HB^{m-4}\hat{f} = 0$, we will get an expression in terms of $f'HB^{m-4}f$, a_0, $f'HB^{m-3}e$, $e'HB^{m-2}f$ and $e'HB^{m-1}f$, and solve for a_1. Likewise, we can solve for a_k by considering $\hat{f}'HB^{m-2(k+1)}\hat{f}$. In like manner, let $\hat{e} = e + Bq(B)f$ where $q(t) = \sum_{j \geq 0} \beta_j t^{2j}$. We can pick $\beta_0, \beta_1, \ldots$ so that $\hat{\mathcal{U}}$, the

B-cyclic subspace generated by $\hat{e}$ satisfies the property $H\hat{\mathcal{U}} \subseteq \hat{\mathcal{U}}^0$. $\hat{\mathcal{U}}$ and $\hat{\mathcal{W}}$ obviously satisfy the first 3 properties we proved for $\mathcal{U}$ and $\mathcal{W}$. Thus we have completed the proof of (ii). Further, let $\hat{\hat{f}} = \hat{f} + \sum_{j=2}^{m} a_j B^{j-1} f$ we can choose $a_2, a_3, \ldots$ so that $\hat{e}' H B^k \hat{\hat{f}} = 0$ for $k \leq m$. Let $\hat{\hat{\mathcal{W}}}$ be the cyclic subspace generated by $\hat{\hat{f}}$; then the pair $\hat{\mathcal{U}}$, $\hat{\hat{\mathcal{W}}}$ also satisfies the properties we have proved for the pair $\hat{\mathcal{U}}$, $\hat{\mathcal{W}}$. B restricted to $\hat{\mathcal{U}} \oplus \hat{\hat{\mathcal{W}}}$ is the direct sum of 2 m-dimensional Jordan blocks of eigenvalues 0.

H, restricted to $\hat{\mathcal{U}} \oplus \hat{\hat{\mathcal{W}}}$, is

$$H_0 = \begin{bmatrix} 0 & -S \\ S & 0 \end{bmatrix}, \quad \text{where} \quad S = \begin{bmatrix} & & & & -1 \\ & & & 1 & \\ & & \cdot^{\cdot^{\cdot}} & & \\ & -1 & & & \\ 1 & & & & \end{bmatrix};$$

and K, restricted to $\hat{\mathcal{U}} \oplus \hat{\hat{\mathcal{W}}}$, K_0, is

$$K_0 = \begin{bmatrix} 0 & T \\ -T & 0 \end{bmatrix}, \quad \text{where} \quad T = \begin{bmatrix} & & & -1 & 0 \\ & & 1 & \cdot^{\cdot^{\cdot}} & \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & \\ -1 & 0 & & & \\ 0 & & & & \end{bmatrix}$$

Multiply by $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ on the right and left of $H_0 + K_0$ we get $-H_0 - K_0$. Since H_0 is of even dimension, if we multiply by $\mathrm{diag}(1, -1, \ldots, -1)$ on the right and left of $-H_0 - K_0$, then by Fact 1, the anti-diagonal has been multiplied by -1, and the first super-anti-diagonal is not changed. Thus we get H_0-K_0.

In case m is odd, $\mathcal{V}_0 = \mathcal{U} \oplus (H\mathcal{U})^0$, $\mathcal{V}_0' = H\mathcal{U} \oplus \mathcal{U}^0$, and $H(H\mathcal{U})^0 \subseteq \mathcal{U}^0$, $K\mathcal{U} = HB\mathcal{U} \subseteq H\mathcal{U}$, $K(H\mathcal{U})^0 = HB(H\mathcal{U})^0 \subseteq H(H\mathcal{U})^0 \subseteq \mathcal{U}^0$; thus we can repeat the above process with $\mathcal{V}_0$ replaced by $(H\mathcal{U})^0$. Likewise, for m even, we can repeat the above process with $\mathcal{V}_0$ replaced by $(H(\mathcal{U} \oplus \mathcal{W}))^0$. Thus, after a finite number of steps, we show that M_0 is congruent to its transpose.

The process of showing M_∞ congruent to M_∞' is similar; in this case, instead of $H^{-1}K$, we consider $K^{-1}H$.

Now we consider M_1 (and thus we assume H and K are non-singular here). Suppose C_2 is a non-singular matrix such that $C_2'M_1C_2 = M_1'$. Without loss of generality, write $M_1 = \bigoplus_{i=1}^{k} N_i$ such that if $H_i = \frac{1}{2}(N_i+N_i')$, $K_i = \frac{1}{2}(N_i-N_i')$, and $f_i(x)$ is the minimum polynomial of $K_i^{-1}H_i$, then $f_i(x)$ is prime to $f_j(x)$ for $i \neq j$. Note that $K(K^{-1}H)K^{-1} = -HK^{-1} = -(K^{-1}H)'$, so the minimum polynomial of $K^{-1}H$ is even; thus we place another restriction on the polynomials $f_i(x)$; $f_i(x) = [q_i(x^2)]^m$ where $q(x)$ is irreducible (and $x \nmid q(x)$). Without loss of generality, we may assume $M_1 = H_1 + K_1$. Lemma 3 (proved in this section) assures us that we can find a basis for $\mathcal{V}$, such that $M_1 = \bigoplus_{i=1}^{p} M_{i1}$, where, if $H_{i1} = \frac{1}{2}(M_{i1}+M_{i1}')$ and $K_{i1} = \frac{1}{2}(M_{i1}-M_{i1}')$ then $K_{i1}^{-1}H_{i1}$ is similar to a companion matrix. Thus it suffices to prove the following:

Lemma 4. If H and K as defined above are non-singular, and $K^{-1}H$ is non-derogatory, then $S = H + K$ is congruent to S'.

Proof. Let m be the degree of the minimum polynomial of $A = K^{-1}H$. Thus, for some $e \in \mathcal{V}$, $\{e, Ae, A^2e, \ldots, A^{m-1}e\}$ is a basis for $\mathcal{V}$. Write $e_j = A^{j-1}e$. Recall KA^i is skew when i is even, is symmetric when i is odd. Thus

$e_j'Ke_i = (-1)^j e_1 KA^{i+j-2} e_1 = 0$ for $i + j$ even,

$e_j'He_i = (-1)^j e_1 HA^{i+j-2} e_1 = 0$ for $i + j$ odd. Let $C: \mathcal{V} \to \mathcal{V}$ such that $Ce_i = (-1)^{i-1} e_i$. Then

$$e_j'C'HCe_i = (Ce_j)'H(Ce_i) = (-1)^{i+j} e_j'He_i = e_j'He_i$$

because if $i+j$ is odd, $e_j'He_i = 0$. Also

$$e_j'C'KCe_i = (Ce_j)'K(Ce_i) = (-1)^{i+j} e_j'Ke_i = -e_j'Ke_i$$

because if $i+j$ is even, $e_j'Ke_i = 0$. Thus $C'SC = S'$. q.e.d.

2. General Results

In this section, we shall prove some simple results, some of which lay the ground work for the following section.

Proposition II.2.1. Let S be a non-singular matrix; then:

(i) if S is of type 1, then $\det S^* = \pm \det S$.

(ii) if S is of type 2, then $\det S \in \mathcal{E}$.

(iii) if S is of type 2, then S is $*$-congruent to $\bar{S}$, where $\bar{S}$ is the conjugate matrix of S.

(iv) if S is of type 2, then it is of type 1.

Proof. (i) If $C^*SC = S^*$, then $C^*S^*C = S$, so we have $(C^*)^2 S^* (C^2) = S^*$. Therefore $\det(CC^*) = \pm 1$. Thus $\det S^* = \pm \det S$.

(ii) If $C^*SC = T$ is over $\mathcal{E}$, then $\det S = (\det T)(\det C^{-1})(\det C^{-1})^*$, which is in $\mathcal{E}$.

(iii) If $C^*SC = T$ is over $\mathcal{E}$, then $C^*S^*C = T^* = T' = C'S'\bar{C}$. Therefore $(C')^{-1}C^*S^*C(\bar{C})^{-1} = S'$. I.e., $(C')^{-1}C^*SC(\bar{C})^{-1} = (S')^* = \bar{S}$.

(iv) If $C^*SC = T$ is over $\mathcal{E}$, then T, from Theorem I.2.1, is congruent over $\mathcal{E}$ (hence $*$-congruent over $\mathcal{F}$) to T', and from (iii) of Proposition II.2.1, S is $*$-congruent to $\bar{S}$; then S is $*$-congruent to $(\bar{S})' = S^*$.

Proposition II.2.2. Let $H = \frac{1}{2}(S-S^*)$, and let $K = \frac{1}{2j}(S-S^*)$, where $j^* = -j$, and let $A = S^{-1}S^*$. Then

(i) A is similar to $(A^{-1})^*$.

(ii) If K is non-singular, $B = K^{-1}H$ is similar to B^*.

(iii) If H is non-singular, $B_1 = H^{-1}K$ is similar to B_1^*.

(iv) If S is of type 1 (hence also if S is of type 2), A is similar to A^*.

(v) If S is of type 1, i.e., $C^*SC = S^*$ where C is non-singular, and K is non-singular, then $C^{-1}K^{-1}HC = -K^{-1}H$.

Proof. (i) $SAS^{-1} = S(S^{-1}S^*)S^{-1} = S^*S^{-1} = (A^*)^{-1}$.

(ii) $KBK^{-1} = K(K^{-1}H)K^{-1} = HK^{-1} = B^*$.

(iii) $HB_1H^{-1} = H(H^{-1}K)H^{-1} = KH^{-1} = B_1^*$.

(iv) If C is non-singular such that $C^*SC = S^*$, then $C^{-1}AC = C^{-1}(S^{-1}S^*)C = (C^*SC)^{-1}(C^*S^*C) = (S^*)^{-1}S = A^{-1}$. Since A is also similar to $(A^*)^{-1}$, thus A is similar to A^*.

(v) If C is non-singular such that $C^*SC = S^*$, then $C^*HC = H$ and $C^*KC = -K$. Thus

$$\begin{aligned} C^{-1}(K^{-1}H)C &= [C^*KC]^{-1}(C^*HC) \\ &= -K^{-1}H . \end{aligned}$$

q.e.d.

Proposition II.2.3. (i) Suppose S is non-singular, and $S = S_0 \oplus S_\infty \oplus S_1$ where $A_0 = S_0^{-1}S_0^*$ has eigenvalues only at 1, $A_\infty = S_\infty^{-1}S_\infty^*$ has eigenvalues only at -1, and $A_1 = S_1^{-1}S_1^*$ has eigenvalues neither at 1 nor at -1. If C is a non-singular matrix such that $C^*SC = S^*$, then $C = C_0 \oplus C_\infty \oplus C_1$, where C_0 is such that $C_0^*S_0C_0 = S_0^*$, C_∞ is such that $C_\infty^*S_\infty C_\infty = S_\infty^*$, and C_1 is such that $C_1^*S_1C_1 = S_1^*$.

(ii) Suppose $H = \frac{1}{2}(S+S^*)$ and $K = \frac{1}{2}(S-S^*)$ and non-singular. Let $B = K^{-1}H = \bigoplus_{i=1}^{p} B_i$, where the B_i are such that if $f_i(t)$ is the minimum polynomial of B_i, then $f_i(t) = q_i(t^2)$ for some $q_i(t) \in \mathcal{E}[t]$, and g.c.d $(f_i(t), f_j(t)) = 1$ when $i \neq j$. Suppose

$$S = \bigoplus_{i=1}^{p} S_i ,$$

where if

$$H_i = \frac{1}{2}(S_i + S_i^*) ,$$

$$K_i = \frac{1}{2j}(S_i - S_i^*) ,$$

then $B_i = K_i^{-1}H_i$. If C is a non-singular matrix such that $C^*SC = S^*$, then $C = \bigoplus_{i=1}^{p} C_i$ such that $C_i^* S_i C_i = S_i^*$.

<u>Proof</u>. (i) Since

$$\begin{aligned} C^{-1}AC &= C^{-1}(S^{-1}S^*)C \\ &= [C^*SC]^{-1}[C^*SC] \\ &= (A^*)^{-1} \\ AC &= C(A^*)^{-1} . \end{aligned}$$

Partition $C = (C_{ih})$ conformably to A. $i = 0, \infty, 1$ and $h = 0, \infty, 1$.

Then $AC = C(A^*)^{-1}$ implies $A_i C_{ih} = C_{ih}(A_h^*)^{-1}$. If $i \neq h$, A_i and $(A_h^*)^{-1}$ have no eigenvalues in common and thus $C_{ih} = 0$.

We have shown that the off-diagonal blocks of C are 0. Thus (i) is proved.

(ii) Since

$$C^{-1}BC = (C^*KC)^{-1}(C^*HC)$$
$$= -B$$

Thus $BC = -CB$. Partition C conformably to B. Thus $B_iC_{ij} = -C_{ij}B_j$, $i, j = 1, 2, \ldots, p$. If $i \neq j$, then B_i and $-B_j$ have no eigenvalues in common, thus $C_{ij} = 0$. We have shown that the off-diagonal blocks of C are 0, thus we have proved (ii).

The next result is based on Taussky-Zassenhaus' result [10].

<u>Proposition II.1.4.</u> If S is non-singular and $S^{-1}S^*$ is non-derogatory, and $C^*SC = S^*$ for C non-singular, then C^*S is $*$-congruent to a symmetric matrix over $\mathcal{F}$.

<u>Proof</u>: First assume $A = S^{-1}S^*$ is a companion matrix. S is of type 1 therefore A is similar to A^* hence similar to a matrix over $\mathcal{E}$ [2]; since A is a companion matrix, A is a matrix over $\mathcal{E}$.

Since

$$C^*S(A)S^{-1}(C^*)^{-1} = C^*S(S^{-1}S^*)S^{-1}(C^*)^{-1}$$

$$= C^*(S^*S^{-1})(C^*)^{-1}$$

$$= (C^*S^*C)(C^*SC)^{-1}$$

$$= S(S^*)^{-1}$$

$$= A^* = A' .$$

Since A is a companion matrix, by the Taussky-Zassenhaus result, C^*S is symmetric.

Now suppose $A = S^{-1}S^*$ is not a companion matrix, but A is non-derogatory. There exists D such that $D^{-1}AD = A_0$, where A_0 is a companion matrix. Let $T = D^*SD$. Then

$$S = (D^*)^{-1}TD^{-1}, \quad S^* = (D^*)^{-1}T^*D^{-1} .$$

$$C^*SC = C^*(D^*)^{-1}TD^{-1}C$$

$$= S^*$$

$$= (D^*)^{-1}T^*D^{-1}$$

Thus $(D^{-1}CD)^*TD^{-1}CD = T^*$. Note

$$T^{-1}T^* = D^{-1}S^{-1}(D^*)^{-1}D^*S^*D = D^{-1}AD = A_0$$

is a companion matrix. From what we have shown for the case $S^{-1}S^*$ being a companion matrix, $[D^{-1}CD]^*T$ is a symmetric matrix over $\mathcal{F}$.

$$[D^{-1}CD]*T = (D*)C*(D*)^{-1}D*SD$$

$$= D*C*SD$$

Thus $C*S$ is *-congruent to a symmetric matrix.

3. Non-Singular *-Symmetric Pencils with All Eigenvalues in $\mathcal{F}$

Let H and K be *-symmetric. Then Theorem I. 2. 1 asserts that the pencil $\lambda H + \mu K$, where λ and μ are indeterminates over $\mathcal{E}$, is *-congruent to $L \oplus M$ where L is the minimum-indices part as defined in Theorem I. 2. 1 and M is the non-singular core, unique up to *-congruency. If $T = H_1 + jK_1$, where H_1 and K_1 are *-symmetric and $j^* = -j$, and jK_1 is a matrix over $\mathcal{E}$, then $H_1 - jK_1 = T^* = T' = H_1' + jK_1'$. Thus proving that the matrix $S = H + jK$ is of type 2 is equivalent to proving that the pencil $\lambda H + \mu K$ is *-congruent over $\mathcal{E}$ to $\lambda H_1 + \mu K_1$ where $H_1' = H_1 = H_1^*$, $-K_1 = K_1' = K_1^*$, i. e., $H_1' = H_1^* \in \mathcal{E}^{n \times n}$, and $jK_1 = -(jK_1)' \in \mathcal{E}^{n \times n}$. We say a pencil is of type 2 if $\lambda H + \mu K$ is *-congruent to $\lambda H_1 + \mu K_1$ where H_1 and K_1 are as described above. We say a pencil is of type 1 if $\lambda H + \mu K$ is *-congruent to $\lambda H - \mu K$.

First we want to show that:

Proposition II.3.1. If $S = H + jK$ where H and K are $*$-symmetric and $j^* = -j$, then S is of type 2 iff the non-singular core of the pencil $\lambda H + \mu K$ is of type 2.

Proof: Write $\lambda H + \mu K = L \oplus M$ where L is the minimum-indices part and M is the non-singular core.

("only if") If $\lambda H + \mu K$ is of type 2, then $\lambda H + \mu K$ is $*$-congruent to $\lambda H_1 + \mu K_1$ where $H_1 = H_1' = H_1^*$, $-K_1' = K_1 = K_1^*$; thus $(j^{-1}K_1)^* = -j^{-1}K_1' = j^{-1}K_1$ is a matrix over $\mathcal{E}$. Let $K_0 = j^{-1}K_1$. $\lambda H_1 + \mu K_0$ is congruent over $\mathcal{E}$ to $L_0 \oplus M_1$ where

$$L_0 = O \bigoplus_{i=1}^{k_1} \begin{bmatrix} 0 & L_{\varepsilon_i} \\ \hat{L}_{\varepsilon_i} & 0 \end{bmatrix},$$

L_{ε_i} and $\hat{L}_{\varepsilon_i}$ are as defined in Fact 6 and $M_1 = \lambda \hat{H}_1 + \mu \hat{K}_0$ is a non-singular pencil and $\hat{H}_1$, $\hat{K}_0$ are over $\mathcal{E}$, and $\hat{H}_1' = \hat{H}_1$, $(\hat{K}_0)' = -\hat{K}_0$. Thus $\lambda H_1 + \mu K_1 = \lambda H_1 + j\mu K_0$ is $*$-congruent to $L_1 \oplus \lambda \hat{H}_1 + \mu j \hat{K}_0$, where

$$L_1 = O \bigoplus_{i=1}^{k_1} \begin{bmatrix} 0 & W_{\varepsilon_i} \\ W_{\varepsilon_i}^* & 0 \end{bmatrix}$$

where

$$W_{\varepsilon_i} = \underbrace{\begin{bmatrix} \lambda & & & \\ \mu j & \lambda & & \bigcirc \\ & \mu j & \ddots & \\ & & \ddots & \lambda \\ \bigcirc & & & \mu j \end{bmatrix}}_{\varepsilon_i}$$

Because of the uniqueness of the minimum indices, $\begin{bmatrix} 0 & W_{\varepsilon_i} \\ W_{\varepsilon_i}^* & 0 \end{bmatrix}$ is *-congruent to $\begin{bmatrix} 0 & L_{\varepsilon_i} \\ L_{\varepsilon_i}^* & 0 \end{bmatrix}$. Since $\lambda H_1 + \mu K_1$ and $\lambda H + \mu K$ have the same minimum indices, L_1 is *-congruent to L [11]. (Recall in the beginning of the proof, we assume $\lambda H + \mu K = L \oplus M$, where L is the minimum-indices part and M is the non-singular core.) Thus $\lambda H + \mu K$ is *-congruent to $\lambda H_1 + \mu K_1$ where $H_1^* = H_1' = H_1$, $(jK_1)^* = -jK_1' = jK_1$; $\lambda H_1 + \mu K_1$ in turn is *-congruent to $L_1 \oplus \lambda \hat{H}_1 + \mu j \hat{K}_0$, which in turn is *-congruent to $L \oplus \lambda \hat{H}_1 + \mu j \hat{K}_0$. Let $\hat{K}_1 = j\hat{K}_0$. Thus $\lambda H + \mu K = L \oplus M$ is *-congruent to $L \oplus \lambda \hat{H}_1 + \mu \hat{K}_1$. By the uniqueness of the non-singular core of pencils M is *-congruent to $\lambda \hat{H}_1 + \mu \hat{K}_1$. We said before $(\hat{H}_1)^* = \hat{H}_1' = \hat{H}_1$. $j\hat{K}_1 = j^2 \hat{K}_0$ is a matrix over $\mathcal{E}$, and $(j\hat{K}_1)^* = -j\hat{K}_1^* = -j(j\hat{K}_0)^* \doteq j^2 \hat{K}_0' = -j^2 \hat{K}_0 = -j\hat{K}_1$. Thus M is of type 2.

("if") We only need to prove the minimum-indices part L of the pencil is of type 2. Therefore without loss of generality, assume

$$\Lambda = \lambda H + \mu K = \begin{bmatrix} 0 & L_{\varepsilon_i} \\ L^*_{\varepsilon_i} & 0 \end{bmatrix},$$

where L_{ε_i} is defined in Fact 6 of Section 1 of Chapter I. Multiply on the right and left of Λ by $I \oplus E_{\varepsilon_i}$. By a calculation similar to a calculation appearing in the proof of Theorem II. 1. 1, we can see that Λ becomes

$$[\text{III. 1. 1}] \qquad \Lambda_2 = \begin{bmatrix} 0 & Y_{\varepsilon_i} \\ Y^*_{\varepsilon_i} & 0 \end{bmatrix},$$

where

$$Y_{\varepsilon_i} = \underbrace{\begin{bmatrix} 0 & & & \lambda \\ & & \lambda & \mu \\ & \iddots & \mu & \\ \lambda & \iddots & & 0 \\ \mu & & & \end{bmatrix}}_{\varepsilon_i}$$

Multiply on the right of [III. 1. 1] by $C_1 = \text{diag}(1, j, \ldots, j, 1)$ and the left by C_1^*. Since [III. 1. 1] is of odd order, by Fact 2 of Section 1, Chapter I, we have multiplied the entries on the anti-diagonal alternately by j and $-j$ and the entries on the anti-diagonal alternately by 1 and $-j^2$. Thus $C_1^* \Lambda_2 C_1 = \lambda H_2 + \mu K_2$ is such that H_2 is a matrix over $\mathcal{E}$ because its non-zero entries are $\in \{1, -j^2\}$ and jK_2 is a matrix over $\mathcal{E}$ because K_2's non-zero entries are $\in \{j, -j\}$. q.e.d.

In this section, we shall consider $S = H + jK$ where H, K are $*$-symmetric and $j^* = -j$, such that the pencil $\lambda H + \mu K$ is non-singular, and such that all the eigenvalues of the pencil are in $\mathcal{F}$. (Note that $S^{-1}S^* = (H+jK)^{-1}(H-jK)$. If H is non-singular, then

$$[H(I+jH^{-1}K)]^{-1}H(I-jH^{-1}K) = (I+jH^{-1}K)^{-1}(I-jH^{-1}K) .$$

Therefore $S^{-1}S^*$ having eigenvalue at 1 is equivalent to $H^{-1}K$ being nilpotent. Likewise, $S^{-1}S^*$ having eigenvalues at -1 is equivalent to $K^{-1}H$ being nilpotent.)

Suppose $S = \bigoplus_{i=1}^{p} S_i$ and $H_i = \frac{1}{2}(S_i + S_i^*)$ and $K_i = \frac{1}{2j}(S_i - S_i^*)$ and $H_1^{-1}K_1$ is nilpotent, $K_2^{-1}H_2$ is nilpotent, and for $i \geq 3$, H_i and K_i are non-singular, and $K_i^{-1}H_i$ is similar to $-K_i^{-1}H_i$ and has eigenvalues only at β_i, β_i^*, $-\beta_i$ and $-\beta_i^*$, and if $f_i(t)$ is the minimum polynomial of $K_i^{-1}H_i$, for $i \geq 3$, then g.c.d. $(f_i(t), f_j(t)) = 1$ for $i \neq j$. Then Proposition II.2.3 asserts that S is of type 1 iff each of the S_i is of type 1. Therefore, without loss of generality, we consider the following cases separately:

(i) $K^{-1}H$ has eigenvalue only at β and $-\beta$ where $\beta^* = -\beta$, and $K^{-1}H$ is similar to $-K^{-1}H$.

(ii) $K^{-1}H$ has eigenvalues only at β, β^*, $-\beta$ and $-\beta^*$, $\beta^* \neq \pm\beta$, and $K^{-1}H$ is similar to $-K^{-1}H$.

(iii) $K^{-1}H$ has eigenvalues only at β, $-\beta$ where $\beta \in \mathcal{E}$, and $K^{-1}H$ is similar to $-K^{-1}H$.

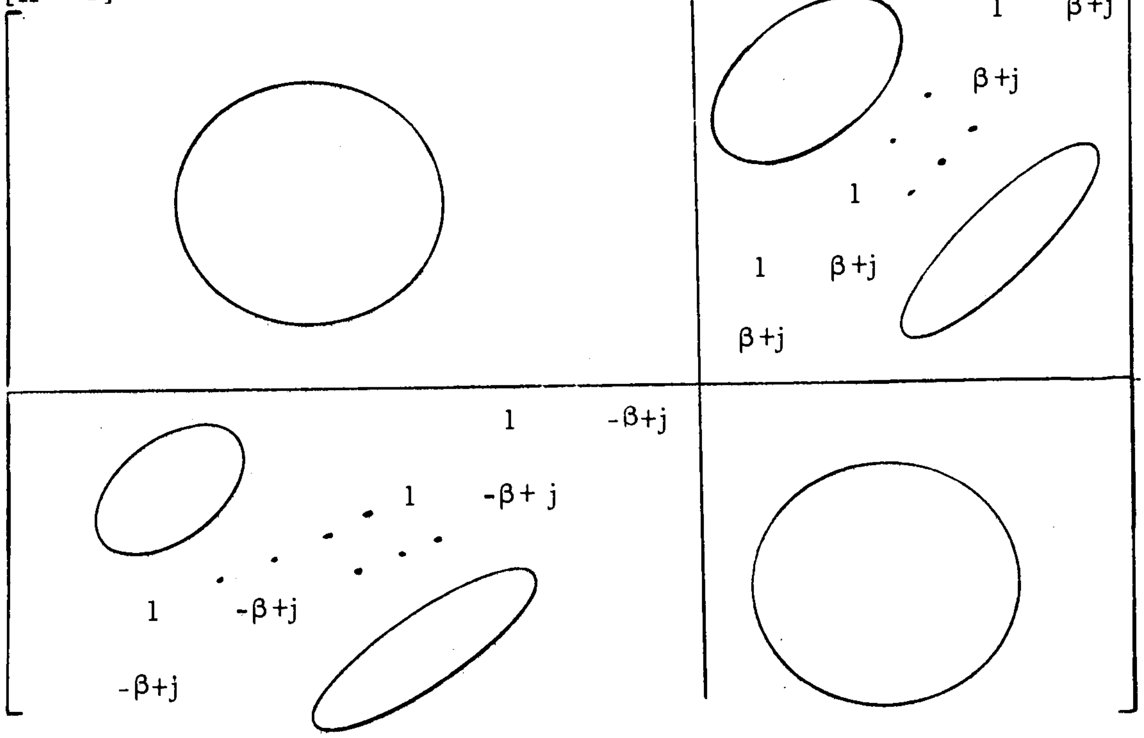
(iv) $H^{-1}K$ is nilpotent,

(v) $K^{-1}H$ is nilpotent.

Theorem II.3.2. If $K^{-1}H$ is similar to $-K^{-1}H$, and $K^{-1}H$ has eigenvalues only at β, β^*, $-\beta$, $-\beta^*$ where $\beta \notin \mathcal{E}$, then S is of type 2.

Proof: First consider $\beta^* = -\beta$. By Fact 5, in the introductory chapter, S is $*$-congruent to a block diagonal matrix whose diagonal blocks are of the form:

[II.3.1]

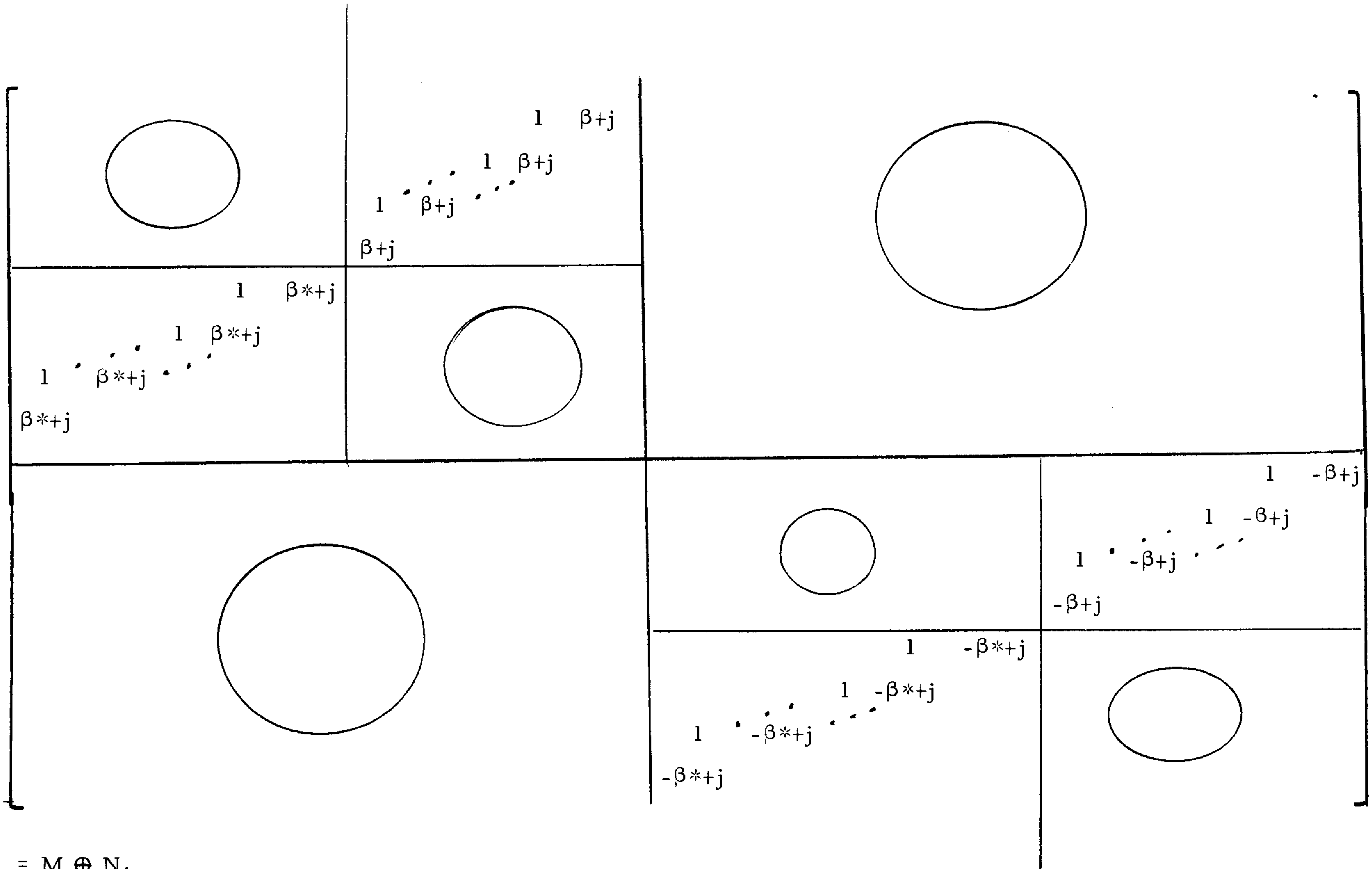


Note: $(\beta+j)^* = -\beta - j = -(\beta+j)$

$(-\beta+j)^* = \beta - j = -(-\beta+j)$

Therefore $j(\beta+j)$ and $j(-\beta+j)$ are elements in $\mathcal{E}$. We shall show that each block is $*$-congruent to a matrix over $\mathcal{E}$. Therefore without loss of generality, we may assume $S = [\text{II}.3.1]$. Multiply by $C = \text{diag}(1, j, 1, j, \ldots)$ to the right and by C^* on the left of S; then by Fact 2, the entries on the first super-anti-diagonal of S become either 1 or $-j^2$ or (in case of the middle entry) 0, and the entries on the anti-diagonal are multiplied alternately by j and $-j$. Thus S is of type 2.

Next consider $\beta^* \neq \pm\beta$. By Fact 4, S is $*$-congruent to a matrix with diagonal blocks of the form:



$= M \oplus N,$

where M is the upper left $2m \times 2m$ block and N is the lower right $2m \times 2m$ block. Let $C = \text{diag}(\underbrace{1, j, 1, \ldots, j}_{2m}) \oplus \text{diag}(\underbrace{1, j, 1, \ldots, j}_{2m})$

Consider $C^*(M \oplus N)C = P \oplus Q$. $p_{ij} = q_{ij} = 0$ for $i+j \neq 2m+1$ and $\neq 2m$, and also $p_{m,m} = q_{m,m} = 0$ and

$$p_{i,\,2m-i} = q_{i,\,2m-i} = -j^2 \quad \text{when } i \text{ is even}$$
$$= 1 \quad \text{when } i \text{ is odd.}$$

$$p_{i,\,2m+1-i} = (-1)^{i-1} j(\beta^*+j) = (-1)^{i-1}[j\beta^*+2j^2] \quad i \leq m$$
$$= (-1)^{i-1} j(\beta+j) = (-1)^{i-1}[j\beta+j^2] \quad i > m.$$

$$q_{i,\,2m+1-i} = (-1)^{i-1}(-j\beta+j^2) \quad i \leq m$$
$$= (-1)^{i-1}(-j\beta^*+j^2) \quad i > m.$$

$\therefore$ $Q = \overline{P}$. Consider

$$\begin{bmatrix} I & I \\ P^* & P' \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & \overline{P} \end{bmatrix} \begin{bmatrix} I & P \\ I & \overline{P} \end{bmatrix} = \begin{bmatrix} I & I \\ P^* & P' \end{bmatrix} \begin{bmatrix} P & P^2 \\ \overline{P} & \overline{P}^2 \end{bmatrix}$$
$$= \begin{bmatrix} P+\overline{P} & P^2+\overline{P}^2 \\ P^*P+P'\overline{P} & P^*P^2+P'\overline{P}^2 \end{bmatrix},$$

which is over $\mathcal{E}$. Note that

$$\det \begin{bmatrix} I & P \\ I & \bar{P} \end{bmatrix} = \det(\bar{P} - P)$$

$$= \pm j^{2m}(\beta+\beta^*)^{2m} \neq 0 ,$$

because $\pm j(\beta\pm\beta^*) \neq 0$. Therefore $M \oplus N$ is of type 2. Thus S is of type 2.

From this point let B_i and D_i denote non-singular diagonal matrices over $\mathcal{E}$.

<u>Theorem II.3.3.</u> (i) If $H^{-1}K$ is nilpotent, then there is a basis for $\mathcal{V}$ such that $S = \bigoplus_{i=1}^{h} T_i^0$ where $T_i^0 = (E_{n_i} + jF_{n_i}) \otimes B_i$, and $n_i \neq n_j$ for $i \neq j$.

(ii) If $K^{-1}H$ is nilpotent, then there is a basis for $\mathcal{V}$ such that $S = \bigoplus_{i=1}^{h} T_i^{\infty}$ where $T_i^{\infty} = (jE_{n_i} + F_{n_i}) \otimes B_i$ $n_i \neq n_j$ for $i \neq j$.

(iii) If $K^{-1}H$ is similar to $-K^{-1}H$ and has eigenvalues only at β, and $-\beta$, $\beta \neq 0$, $\beta \in \mathcal{E}$, then there exists a basis for $\mathcal{V}$ such that $S = \bigoplus_{i=1}^{h} T_i^{\beta}$, where

$T_i^{\beta} = F_{n_i} \otimes (B_i \oplus D_i) + E_{n_i} \otimes [(\beta+j)B_i \oplus (-\beta+j)D_i]$ $n_i \neq n_j$, for $i \neq j$.

(Pictures of blocks of T_1^z, $z = 0, \infty, \beta$ where $\beta \neq 0$, $\beta \in \mathcal{E}$ are in Appendix I).

<u>Proof:</u> (i) By Fact 4, S is $*$-congruent to a matrix in diagonal blocks with each block of the form:

$$W^0_{\varepsilon,n} = \varepsilon(E_n + jF_n) \quad \varepsilon \neq n, \quad \varepsilon \in \mathcal{E}.$$

Thus

$$S = \bigoplus_{j=1}^{h} \left(\bigoplus_{i=1}^{k_j} W^0_{\varepsilon_i, n_j} \right)$$

where $n_j \neq n_i$ if $i \neq j$. Suppose $S = \bigoplus_{i=1}^{k_1} W^0_{\varepsilon_i, n_1}$. Write $n = n_1$, $B = H^{-1}K$, and $k = k_1$. Then

$\{e_1, Be_1, \ldots, B^{n-1}e_1, e_2, Be_2, \ldots, B^{n-1}e_2, \ldots, e_k, \ldots, B^{n-1}e_k\}$ is a basis for $\mathcal{V}$ for some $e_1, \ldots, e_k \in \mathcal{V}$. Rearrange the basis elements in the following manner:

$\{e_1, e_2, \ldots, e_k, Be_1, Be_2, \ldots, Be_k, \ldots, B^{n-1}e_1, \ldots, B^{n-1}e_k\}$. With respect to this new basis S becomes $T^0_1 = (E_{n_1} + jF_{n_1}) \otimes B_1$ where $B_1 = \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_k)$. Repeat the above process for $j = 2, 3, \ldots, h$. Thus if $S = \bigoplus_{j=1}^{h} \bigoplus_{i=1}^{k_i} W^0_{\varepsilon_i, n_j}$, S is *-congruent to $\bigoplus_{j=1}^{h} T^0_{n_j}$.

If $K^{-1}H$ is nilpotent, by Fact 4 S is *-congruent to a matrix in diagonal blocks with each block of the form:

$$W^{\infty}_{\varepsilon, n} = \varepsilon(jE_n + F_n) \qquad \varepsilon \neq 0, \quad \varepsilon \in \mathcal{E}.$$

Going through a similar process as in the case $H^{-1}K$ being nilpotent, we can show S is *-congruent to $\bigoplus_{j=1}^{h} T^{\infty}_j$.

(ii) When $K^{-1}H$ has eigenvalues only at $\pm\beta \neq 0$, $\beta \in \mathcal{E}$, by Fact 4, we have S *-congruent to a direct sum of blocks with these blocks paired off in the form $U^{\beta}_{\varepsilon, n} \oplus U^{\beta}_{\varepsilon, n}$, where

$$U^{\beta}_{\varepsilon,n} = \varepsilon(F_n+(\beta+j)E_n), \quad \varepsilon \neq 0, \quad \varepsilon \in \mathcal{E},$$

$$V^{\beta}_{\delta,n} = \delta(F_n+(-\beta+j)E_n), \quad \delta \neq 0, \quad \delta \in \mathcal{E}.$$

Without loss of generality, assume

[II.3.2] $$S = \bigoplus_{j=1}^{h} [(\bigoplus_{i=1}^{k_i} U^{\beta}_{\varepsilon_i,n_j} \oplus (\bigoplus_{i=1}^{k} V^{\beta}_{\delta_i,n_j})] .$$

First consider

[II.3.3] $$S = \bigoplus_{i=1}^{k_1}\left(U^{\beta}_{\varepsilon_i,n_1}\right) \oplus \bigoplus_{i=1}^{k_1}\left(V^{\beta}_{\delta_i,n_1}\right).$$

Write $n = n_1$, $k_1 = k$. Let $A_0 = (K^{-1}H-\beta I)$, $A_1 = (K^{-1}H+\beta I)$. A basis of $\mathcal{V}$ for which S is of the form [II.3.3] is of the form: $\{e_1, A_0e_1, \ldots, A_0^{n-1}e_1, e_2, \ldots, A_0^{n-1}e_2, \ldots, A_1^{n-1}e_k, d_1, A_1d_1, \ldots, A_1^{n-1}d_1, \ldots, A_1d_k, A_1^2d_k, \ldots, A_1^{n-1}d_k\}$. Rearrange this basis in the following manner $\{e_1, e_2, \ldots, e_k, d_1, d_2, \ldots, d_k, A_0e_1, \ldots, A_0e_k, A_1d_1, \ldots, A_1d_k, \ldots, A_0^{n-1}e_1, \ldots, A_0^{n-1}e_k, A_1^{n-1}d_1, \ldots, A_1^{n-1}d_k\}$. Then in this new basis S is of the form

$T_1^{\beta} = F_{n_1} \otimes (B_1 \oplus D_1) + E_{n_1} \otimes [(\beta+j)B_1 \oplus (-\beta+j)D_1]$ where $B_1 = \text{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$, $D_1 = \text{dial}(\delta_1, \ldots, \delta_k)$ $\varepsilon_i, d_i \in \mathcal{E}$. Thus if S is of [II.3.2] S is $*$-congruent to $\bigoplus_{j=1}^{k} T_j^{\beta}$. q.e.d.

In the following theorem we let $T^z = T_1^z$, where $z = \infty, 0, \beta$

and where $\beta \neq 0$, $\beta \in \mathcal{E}$; we also write $n = n_1$, $k = k_1$, $B = B_1$, $E = E_n$, and $F = F_n$.

<u>Theorem II.3.4</u>. (i) (a) For n odd, T^0 is of type 2 and hence of type 1. For n even, (b) T^0 is of type 1 iff B is $*$-congruent to $-B$ (i.e., iff jB is of type 1), and (c) T^0 is of type 2 iff jB is of type 2.

(ii) For n even, T^∞ is of type 2 and hence of type 1. For n odd, T^∞ is of type 1 iff B is $*$-congruent to $-B$ (i.e., iff jB is of type 1), and T^∞ is of type 2 iff jB is of type 2.

<u>Proof</u>: Suppose n is odd. Let $C = \text{diag}(\underbrace{1, j, 1, \ldots, j, 1}_{n})$. Recall $T^0 = (E+jF) \otimes B$. Then

$$
\begin{aligned}
(C \otimes I_k)^* T^0 (C \otimes I_k) &= (C \otimes I_k)^*((E+jF) \otimes B)(C \otimes I_k) \\
&= C^*(E+jF)C \otimes B
\end{aligned}
$$

By Fact 2, the entries on the anti-diagonal of $C^*(E+jF)C$ are alternately $-j^2$ or 1, those on the first super anti-diagonal are alternately $\pm j(j) = \pm j^2$, the rest of the entries of $C^*(E+jF)C^*$ are zeroes. Thus T^0 is of type 2, and hence is of type 1.

(b) Suppose n is even, and suppose B is $*$-congruent to $-B$. Suppose P is non-singular such that $P^*BP = -B$. Let $C = \text{diag}(1, -1, 1 \ldots)$; then

$$(C \otimes P)^*[(E+jF) \otimes B](C \otimes P) = C^*(E+jF)C \otimes P^*BP$$
$$= (-E+jF) \otimes -B$$
$$= (E-jF) \otimes B.$$

Conversely suppose C_1 is non-singular such that $C_1^* T^0 C_1 = (T^0)^*$. Let $H = \frac{1}{2}(T^0+(T^0)^*)$, and $K = \frac{1}{2j}(T^0-(T^0)^*)$. Then

$$C_1^* H(H^{-1}K)^{n-1} C_1 = -H(H^{-1}K)^{n-1}.$$

$$H(H^{-1}K)^{n-1} = [E \otimes B][E F \otimes I_k]^{n-1} = B \oplus O.$$

This computation will be used again in the proof of part (c). Thus $C_1^*(B \oplus O)C_1 = -B \oplus O$. Thus by Fact 3, B is $*$-congruent to $-B$.

(c) Now suppose n is even and jB is of type 2, i.e., there exists a non-singular C_0 such that $jC_0^* BC_0$ is over $\mathcal{E}$. Let $C_2 = \text{diag}(1, j, \ldots, j)$. Then

[II.3.4]
$$(C_2 \otimes C_0)^* T^0 (C_2 \otimes C_0) = [C_2^* \otimes C_0^*][(E+jF) \otimes B](C_2 \otimes C_0)$$
$$= C_2^*(E+jF)C_2 \otimes C_0^* BC_0$$

The anti-diagonal entries of $C_2^*(E+jF)C_2$ are alternately $+j$ and $-j$ the first super-anti-diagonal entries are alternately $-j^3$ and j. Thus $j^{-1}C_2^*(E+jF)C_2$ is over $\mathcal{E}$. Thus (if we multiply the first factor by j^{-1} and the second factor by j in the tensor product) we get that $[II.3.4] = j^{-1}C_2^*(E+jF)C_2 \otimes jC_0^* BC_0$ is over $\mathcal{E}$. Thus

T^0 is of type 2.

Conversely suppose $C_3^* T^0 C_3 = W$ is over $\mathcal{E}$. Let $H_1 = \frac{1}{2}(W+W^*)$, $K_1 = \frac{1}{2j}(W-W^*)$ then $W = H_1 + jK_1$, $W^* = H_1 - jK_1 = H'_1 + jK'_1$. Thus $K'_1 = -K_1$. Thus $(jK_1)^* = -jK_1 = jK'_1$, so jK_1 is over $\mathcal{E}$. Likewise $H_1 = H_1^* = H'_1$ is over $\mathcal{E}$. Since C_3 is non-singular, there exists a permutation matrix R such that C_3R has its first $k \times k$ principal submatrix, D, non-singular. Thus

$$jR^*C_3^*(H(H^{-1}K)^{n-1})C_3R = jR^*H_1(H_1^{-1}K_1)^{n-1}R .$$

Since R is a permutation matrix, $jR^*H_1(H_1^{-1}K)^{n_1-1}R$ is a matrix over $\mathcal{E}$. Recall $H(H^{-1}K)^{n-1} = B \oplus O$ and the first principal submatrix D of C_3R is non-singular. Hence jD^*BD is the upper left block of $j(C_3R)^*(B\oplus O)C_3R = j(C_3R^*(H(H^{-1}K)^{n-1})C_3R$ which is over $\mathcal{E}$. Thus jB is of type 2.

(ii) The proof of (ii) is similar to that of (i). q.e.d.

For the following discussion, write T_1^β as T. Recall that $\beta \neq 0$, and $\beta \in \mathcal{E}$, $n_1 = n$. (Refer to picture in Appendix I.)

<u>Theorem II.3.5</u>. The following are equivalent.

(1) T is of type 1.

(2) T is of type 2.

(3) B is $*$-congruent to $(-1)^n D$.

Lemma 5. If $c \in \mathcal{E}$ such that $c^* \neq \pm c$, $C = cI$, $\varepsilon \in \mathcal{E}$, $M = \begin{pmatrix} C & C^* \\ C^* & C \end{pmatrix}$, then

$$M^* \begin{bmatrix} A & 0 \\ 0 & \varepsilon A^* \end{bmatrix} M = \begin{bmatrix} cc^*(A+\varepsilon A^*) & (c^*)^2 A+\varepsilon c^2 A \\ c^2 A+\varepsilon(c^*)^2 A^* & cc^*(A+\varepsilon A^*) \end{bmatrix}$$

The proof of Lemma 5 is routine computation.

Lemma 6. Suppose C, B, D are non-singular. If $C^*(B \oplus (-1)^n D)C = -(B \oplus (-1)^n D)$ and $C^*(B \oplus (-1)^{n-1} D)C = B \oplus (-1)^{n-1} D$ then B is $*$-congruent to $(-1)^{n-1} D$.

Proof of Lemma 6. Partition C conformably: $C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$; then

$$C^* \begin{bmatrix} B & 0 \\ 0 & \varepsilon D \end{bmatrix} C = \begin{bmatrix} C_{11}^* BC_{11} + \varepsilon C_{21}^* DC_{21} & C_{11}^* BC_{12} + \varepsilon C_{12}^* DC_{22} \\ C_{12}^* BC_{11} + \varepsilon C_{22}^* DC_{21} & C_{12}^* BC_{12} + \varepsilon C_{22}^* DC_{22} \end{bmatrix}$$

where $\varepsilon = \pm 1$.

Referring to the first condition, we have

$$C_{11}^* BC_{11} + (-1)^n C_{21}^* DC_{21} = -B.$$

Referring to the second condition we have

$$C_{11}^{*}BC_{11} + (-1)^{n-1}C_{21}^{*}DC_{21} = B.$$

If we add the above two equations together, we get $2C_{11}^{*}BC_{11} = 0$. Thus $(-1)^{n-1}C_{21}^{*}DC_{21} = B$. Thus B is $*$-congruent to $(-1)^{n-1}D$.

Lemma 7. If T is as in Theorem II.3.5, and $H = \frac{1}{2}(T+T^{*})$, $K = \frac{1}{2j}(T-T^{*})$, then

$$K[(K^{-1}H)^2-\beta^2 I]^{n-1} = (2\beta)^{n-1}(B \oplus (-1)^{n-1}D) \oplus O$$

$$H[(K^{-1}H)^2-\beta^2 I]^{n-1} = (2\beta)^{n-1}\beta(B \oplus (-1)^{n}D) \oplus O$$

Proof of Lemma 7. Recall

$T = F \otimes (B \oplus D) + E \otimes [(\beta+j)B \oplus (-\beta+j)D]$ and $\beta \in \mathcal{E}$. (Refer to picture.)

$$H = F \otimes (B \oplus D) + E \otimes (\beta B \oplus -\beta D)$$

$$K = E \otimes (B \oplus D),$$

$\therefore$

$$K^{-1} = E \otimes (B^{-1} \oplus D^{-1}).$$

Let $k = \dim B$.

$$K^{-1}H = E\,F \otimes I_{2k} + I_n \otimes (\beta I_k \oplus -\beta I_k).$$

$N = E\,F$ is the $n \times n$ matrix with 1's on the first sub-diagonal and zero everywhere else.

$$\therefore \qquad K^{-1}H - \beta I = N \otimes I_{2k} + I_n \otimes (O_k \oplus -2\beta I_k) ,$$

$$K^{-1}H + \beta I = N \otimes I_{2k} + I_n \otimes (2\beta I_k \oplus O_k) ,$$

so

$$(K^{-1}H-\beta I)(K^{-1}H+\beta I)$$

$$= N^2 \otimes I_{2k} + N \otimes (2\beta I_k \oplus -2\beta I_k)$$

$$= (N \otimes I_{2k})(N \oplus I_{2k} + I \otimes (2\beta I_k \oplus -2\beta I_k)) .$$

Thus

$$[(K^{-1}H)^2-\beta^2 I]^{n-1} = (N^{n-1} \otimes I_{2k})(N \otimes I_{2k} + I \otimes (2\beta I_k \oplus -2\beta I_k))^{n-1} .$$

$$M = (N \otimes I_{2k} + I \otimes (2\beta I_k \oplus -2\beta I_k))^{n-1}$$

is a 2kn x 2kn block lower triangular matrix with the diagonal blocks $= (2\beta)^{n-1} I_k \oplus (-2\beta)^{n-1} I_k$. Thus

$$K[(K^{-1}H)^2-\beta^2 I]^{n-1} = [E \otimes (B \oplus D)][N^{n-1} \otimes I_{2k}]M$$

$$= ((2\beta)^{n-1}B \oplus (-2\beta)^{n-1}D) \oplus O ,$$

$$= (2\beta)^{n-1}(B \oplus (-1)^{n-1}D) \oplus O .$$

Similarly,

$$H[K^{-1}H)^2-\beta^2]^{n-1} = (2\beta)^{n-1}\beta B \oplus (2\beta)^{n-1}(-\beta)D \oplus O$$

$$= (2\beta)^{n-1}\beta(B \oplus (-1)^n D) \oplus O . \qquad \text{q.e.d.}$$

<u>Proof of Theorem II.3.5</u>. We shall prove (3) => (2) => (1) => (3).

(3) => (2). Suppose Q is non-singular $k \times k$ such that $Q^*DQ = (-1)^n B$. Let $C = I_n \otimes (I_k \oplus Q)$. Let $P = C^*TC$. Then

$$P = [I_n \otimes (I_k \oplus Q)]^*[F \otimes (B \oplus D) + E \otimes [(\beta+j)B \oplus (-\beta+j)D]][I_n \otimes (I_k \oplus Q)]$$

$$= F \otimes (B \oplus Q^*DQ) + E \otimes [(\beta+j)B \oplus (-\beta+j)Q^*DQ]$$

$$= F \otimes (B \oplus (-1)^n B) + E \otimes [(\beta+j)B \oplus (-\beta+j)(-1)^n B] .$$

Let $c \in \mathcal{F}$ such that $c^* \neq \pm c$, and

$$M = \begin{bmatrix} cI_k & c^*I_k \\ c^*I_k & cI_k \end{bmatrix}$$

By Lemma 5

$$R_0 = M^*(B \oplus (-1)^n B)M = \begin{bmatrix} cc^*(B+(-1)^n B) & (c^*)^2 B+(-1)^n c^2 B \\ c^2 B+(-1)^n (c^*)^2 B & cc^*(B+(-1)^n B \end{bmatrix} .$$

Let $\mu = \beta+j$, $\mu^* = \beta-j$. Thus $-\beta+j = -\mu^*$. Then again by Lemma 5,

$$R_1 = M^*(\mu B \oplus (-1)^{n-1} \mu^* B)M$$

$$= \begin{bmatrix} cc^*(\mu B+(-1)^{n-1} \mu^* B) & (c^*)^2 \mu B+(-1)^{n-1} c^2 \mu^* B \\ c^2 \mu B+(-1)^{n-1} (c^*)^2 \mu^* B & cc^*(B+(-1)^{n-1} B) \end{bmatrix}$$

Note

$$(I_n \otimes M^*)P(I_n \otimes M)$$
$$= (I_n \otimes M^*)(F \otimes (B \oplus (-1)^n B) + E \otimes (\mu B + (-1)^{n-1}\mu^* B)(I_n \otimes M)$$
$$= F \otimes (M^*(B \oplus (-1)^n B)M) + E \otimes M^*(\mu B + (-1)^{n-1}\mu^* B)M$$
$$= F \otimes R_0 + E \otimes R_1 .$$

Note that if n is even, R_0 and jR_1 are matrices over $\mathcal{E}$, if n is odd, jR_0 and R_1 are matrices over $\mathcal{E}$. Thus let $U = \text{diag}(1, j, 1, j, \ldots)$. Then

$$(U \oplus I_{2k})^*(F \otimes R_0 + E \otimes R_1)(U \oplus I_{2k}) = U^*FU \otimes R_0 + U^*EU \otimes R_1,$$

by Fact 2, is a matrix over $\mathcal{E}$. This T is of type 2.

Proposition II.2.1 (iv) asserts that (2) => (1).

(1) => (3). Suppose $C^*TC = T^*$. Recall $H = \frac{1}{2}(T+T^*)$, $K = \frac{1}{2j}(T-T^*)$, $C^*K[(K^{-1}H)^2-\beta^2 I]^{n-1}C = -K[(K^{-1}H)^2-\beta^2]^{n-1}$, $C^*H[(K^{-1}H)^2-\beta^2 I]^{n-1}C = H[(K^{-1}H)^2-\beta^2]^{n-1}$. Thus

$$C^*(H+jK)[(K^{-1}H)^2-\beta^2 I]^{n-1}C = (H-jK)[(K^{-1}H)^2-\beta^2]^{n-1} .$$

Lemma 7 asserts that

$$H[(K^{-1}H)^2-\beta^2 I]^{n-1} = (2\beta)^{n-1}\beta(B \oplus (-1)^n D) \oplus O$$
$$K[(K^{-1}H)^2-\beta^2 I]^{n-1} = (2\beta)^{n-1}(B \oplus (-1)^{n-1} D) \oplus O.$$

$$C^*(H+jK)[(K^{-1}H)^2-\beta^2 I]^{n-1}C = C^*[(2\beta)^{n-1}\beta(B \oplus (-1)^n D + j(2\beta)^{n-1}(B \oplus (-1)^{n-1}D) \oplus O]C$$

$$= [(2\beta)^{n-1}\beta(B \oplus (-1)^n D) - j(2\beta)^{n-1}(B \oplus (-1)^{n-1}D] \oplus O$$

By Fact 3, there exists C_{11} non-singular, such that

$$C_{11}^*(B \oplus (-1)^n D)C_{11} = B \oplus (-1)^n D$$

$$C_{11}^*(B \oplus (-1)^{n-1}D)C_{11} = B \oplus (-1)^{n-1}D$$

Thus by Lemma 6, B is $*$-congruent to $(-1)^n D$. q.e.d.

<u>Theorem II.3.6</u>. (i) If $S_z = \bigoplus_{i=1}^{h} T_i^z$, with $n_i \neq n_j$ for $i \neq j$, where $z = \infty$ or 0, then S is of type 1 iff each T_i^z is of type 1.

(ii) Let $H = \frac{1}{2}(S_z+S_z^*)$, $K = \frac{1}{2j}(S_z-S_z^*)$. Then if $z = 0$, S_z is of type 1 iff $H(H^{-1}K)^{2m+1}$ is $*$-congruent to $-H(H^{-1}K)^{2m+1}$ for $m = 0, 1, \ldots$; if $z = \infty$, S_z is of type 1 iff $K(K^{-1}H)^{2m}$ is $*$-congruent to $-K(K^{-1}H)^{2m}$.

<u>Proof</u>: The "if" parts of both (i) and (ii) are trivial. To prove the "only if" part of (i) for $z = 0$ it is enough to prove T_i^0 is of type 1 for n_i even (see Theorem II.3.4(i)). Assume $n_i > n_h$ for

$i < h$. Write S_0 as S. If n_1 is odd then T_1^0 is of type 1. If n_1 is even, then S is of type 1 implies $H(H^{-1}K)^{n_1-1}$ is $*$-congruent to $-H(H^{-1}K)^{n_1-1}$. Also $H(H^{-1}K)^{n_1-1} = B_1 \oplus O$, so by Fact 3 B_1 is $*$-congruent to $-B_1$. Hence by Theorem II.3.4(i) T_1^0 is of type 1 also if n_1 is even. Let $H_i = \frac{1}{2}(T_i^0+(T_i^0)*)$, $K_i = \frac{1}{2j}(T_i^0-(T_i^0)*)$; then

$$H(H^{-1}K)^{n_2-1} = H_1(H_1^{-1}K_1)^{n_2-1} \oplus H_2(H_2^{-1}K_2)^{n_2-1} \oplus O.$$

Also

$$H_2(H_2^{-1}K_2)^{n_2-1} = B_2 \oplus O,$$

and (if n_2 is even then) $H(H^{-1}K)^{n_2-1}$ is $*$-congruent to $-H(H^{-1}K)^{n_2-1}$. Thus by Fact 3 $H_1(H_1^{-1}K_1)^{n_2-1} \oplus B_2$ is $*$congruent to $-H_1(H_1^{-1}K_1)^{n_2-1} \oplus -B_2$. Since $H_1(H_1^{-1}K_1)^{n_2-1}$ is $*$-congruent to $-H_1(H_1^{-1}K_1)^{n_2-1}$, then B_2 is $*$-congruent $-B_2$. Thus (if n_2 is even and hence in any case) T_2 is of type 1. If we repeat the same process by considering in like manner $H(H^{-1}K)^{n_i-1}$ for $i = 3,4,\ldots,h$, then we can show each T_i^0 is of type 1. Thus we have shown for $i = 1,\ldots,h$, that T_i^0 is of type 1 if S_0 is of type 1. Thus we have proved (ii) for $z = 0$.

The proof for $z = \infty$ for both (i) and (ii) are similar to the above discussion. In this case we would only replace "even" by "odd" (and vice versa) and T_i^0 by T_i^∞ and $H(H^{-1}K)^{n_i-1}$ by

$K(K^{-1}H)^{n_i-1}$.

<u>Theorem II.3.7.</u> If $S = \bigoplus_{i=1}^{h} T_i^{\beta}$, $\infty \neq \beta \neq 0$, $\beta \in \mathcal{E}$, then the following are equivalent.

(1) S is of type 1;

(2) S is of type 2;

(3) T_1^{β} is of type 1 for each i;

(4) T_1^{β} is of type 2 for each i;

<u>Proof</u>: Theorem II.3.5 asserts (3) <=> (4) and Proposition II.2.1 asserts (2) => (1). Thus the following implications are obvious:

$$\begin{array}{ccc} (2) & \Rightarrow & (1) \\ \Uparrow & & \Uparrow \\ 4 & \Leftrightarrow & 3 \end{array}$$

Thus to complete the proof we have only to show (1) => (3). Let $H = \frac{1}{2}(S+S^*)$, $K = \frac{1}{2j}(S-S^*)$. $H_i = \frac{1}{2}(T_i^{\beta}+(T_i^{\beta})^*)$, $K_i = \frac{1}{2j}(T_i^{\beta}-(T_i^{\beta})^*)$. Suppose $C^*SC = S^*$ where C is non-singular. Let $P_i = K[(K^{-1}H)^2-\beta^2 I]^{n_i-1}$, $Q_i = H[(K^{-1}H)^2-\beta^2 I]^{n_i-1}$; then $C^*P_iC = -P_i$, and $C^*Q_iC = Q_i$. Without loss of generality, assume $n_i > n_h$ for $i < h$. From Lemma 7 $P_1 = (2\beta)^{n_1-1}(B_1 \oplus (-1)^{n_1-1} D_1) \oplus O$, and $Q_1 = (2\beta)^{n_1-1}\beta(B_1 \oplus (-1)^{n_1} D_1) \oplus O$. Apply Fact 3 to $Q_1 + jP_1$ and $Q_1 - jP_1 = C^*(Q_1+jP_1)C$: there exists a non-singular C_1 such that

$$C_1^*(B_1 \oplus (-1)^{n_1-1} D_1)C_1 = -(B_1 \oplus (-1)^{n_1-1} D_1)$$

and

$$C_1^*(B_1 \oplus (-1)^{n_1} D_1)C_1 = B_1 \oplus (-1)^{n_1} D_1 .$$

Then by Lemma 6, B_1 is *-congruent to $(-1)^n D_1$. Thus T_1^β is of type 1 (see Theorem II. 3. 5). Now,

$$P_2 = K_1[(K_1^{-1}H_1)^2 - \beta^2 I]^{n_2-1} \oplus (2\beta)^{n_2-1}(B_2 \oplus (-1)^{n_2-1} D_2) \oplus O,$$

$$Q_2 = H_1[(K_1^{-1}H_1)^2 - \beta^2 I]^{n_2-1} \oplus (2\beta)^{n_2-1}\beta(B_2 \oplus (-1)^{n_2} D_2) \oplus O,$$

and

$$C^*(\beta P_2 + Q_2)C = -\beta P_2 + Q_2.$$

Applying Fact 3 to this last *-congruency, we get that

$$\beta K_1[(K_1^{-1}H_1)^2 - \beta^2 I]^{n_2-1} + H_1[(K_1^{-1}H_1)^2 - \beta^2 I]^{n_2-1} \oplus (2\beta)^{n_2}(B_2 \oplus O)$$

is *-congruent to

$$-\beta K_1[(K_1^{-1}H_1)^2 - \beta^2 I]^{n_2-1} + H_1[(K_1^{-1}H_1)^2 - \beta^2 I]^{n_2-1} \oplus (2\beta)^{n_2}(O \oplus (-1)^{n_2} D_2).$$

Also, since T_1^β is of type 1, we have that

$$\beta K_1[(K_1^{-1}H_1)^2 - \beta^2 I]^{n_2-1} + H_1[(K_1^{-1}H_1)^2 - \beta^2 I]^{n_2-1}$$

is *-congruent to

$$-\beta K_1[(K_1^{-1}H_1)^2-\beta^2 I]^{n_2-1} + H_1[(K_1^{-1}H_1)^2-\beta^2 I]^{n_2-1} .$$

Thus by Witt's Theorem, $B_2 \oplus O$ is *-congruent to $O \oplus (-1)^{n_2} D_2$, so B_2 is *-congruent to $(-1)^{n_2} D_2$. Hence T_2^{β} is of type 1. If we repeat the above process by considering P_i and Q_i for $i = 3, 4, \ldots, h$, then we can show T_i^{β} is of type 1 for each i.

III. THE USUAL COMPLEX CASE

1. General Results

In this section, we take $\mathcal{F}$ to be the complex field and $\mathcal{E}$ to be the real field and we consider the problem from the view point of a criterion Dina Ng proved in her thesis [3].

Proposition III. 1. 1. If $h(y)$ is a real polynomial of simple real roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ and

$$\mathcal{P} = \{[h^{(1)}(y)]^{j_1}, [h^{(2)}(y)]^{j_2}, \ldots, [h^{(n)}(y)]^{j_n} \mid j_i = 0, 1, 2\}$$

and $n = \deg h(y)$, then there exist $g_1, g_2, \ldots, g_n \in \mathcal{P}$ such that the $n \times n$ matrix $P(i, j) = (\operatorname{sgn} g_j(\lambda_i))$ is non-singular.

Proof: Refer to Dina Ng's thesis [3].

Proposition III. 1. 2. Suppose S is an $n \times n$ matrix and $K = \frac{1}{2i}(S - S^*)$, $H = \frac{1}{2}(S + S^*)$. Then

(i) if $H^{-1}K$ is nilpotent, we have that S is of type 1 iff $H(H^{-1}K)^{2m+1}$ is of signature 0 for $m = 0, 1, 2, \ldots$;

(ii) if H and K are as in (i) and $K^{-1}H$ is nilpotent, we have that S is of type 1 iff $K(K^{-1}H)^{2m}$ has signature 0 for $m = 0, 1, 2, \ldots$.

Proof: (i) From Theorem II.3.6(ii), S is of type 1 iff $H(H^{-1}K)^{2m+1}$ is *-congruent to $-H(H^{-1}K)^{2m+1}$ (for $m = 0,1,2,\ldots$). In the usual complex case, this is equivalent to saying that $H(H^{-1}K)^{2m+1}$ has signature 0. q.e.d.

(ii) The proof follows in a similar manner as that of (i).

Theorem III.1.3. Suppose S is an $n \times n$ matrix, $K = \frac{1}{2i}(S-S^*)$, $H = \frac{1}{2}(S+S^*)$, H and K are non-singular, and $K^{-1}H$ is similar to $-K^{-1}H$. If $p(x)$ is the characteristic polynomial of $K^{-1}H$ and $f(x) = \dfrac{p(x)}{\text{g.c.d.}(p(x), p'(x))}$, then $f(x) = h(x^2)$ for some real polynomial $h(y)$. Let

$$\mathcal{P} = \{[h^{(1)}(y)]^{j_1}[h^{(2)}(y)]^{j_2}\ldots[h^{(r)}(y)]^{j_r} \mid j_i = 0, 1, 2\},$$

where $r = \deg h(y)$. S is of type 1 iff $K(f(K^{-1}H))^m \; g((K^{-1}H)^2)$ has signature 0 for $m = 0, 1, 2, \ldots, n-1$ (where n is the order of S) and all $g \in \mathcal{P}$.

Proof: ("only if") If S is of type 1 then $\operatorname{sig} Kq(K^{-1}H) = 0$ for all even polynomials $q(x) \in \mathcal{E}[x]$.

("if") Suppose $K[f(K^{-1}H)^m g((K^{-1}H)^2)]$ has signature zero for $m = 0, 1, 2, \ldots, n-1$ and all $g \in \mathcal{P}$. Without loss of generality, we can assume $S = S_1 \oplus S_2$, where if $H_j = \frac{1}{2}(S_j+S_j^*)$ and $K_j = \frac{1}{2i}(S_j-S_j^*)$, $j = 1, 2$, such that all eigenvalues of $K_2^{-1}H_2$ are non-real, and all eigenvalues of $K_1^{-1}H_1$ are real. We have shown in Chapter II that S_2 is of type 1. Therefore without loss of generality, assume $S = S_1$. There exists a basis for $\mathcal{V}$ such that S of the form

$$S = \bigoplus_{k=1}^{p} \bigoplus_{\ell=1}^{r} T_k^{\beta_\ell}$$

where

$$T_k^{\beta_\ell} = F_{n_k} \otimes (B_{\ell k} \oplus D_{\ell k}) + E_{n_k} \otimes [(\beta_\ell + i)B_{\ell k} \oplus (-\beta_\ell + i)D_{\ell k}] ,$$

and $B_{\ell k}$ and $D_{\ell k}$ are real diagonal matrices, with entries ± 1. Assume $n_i > n_h$ if $i < h$. Note

$$f(x) = \prod_{\ell=1}^{r} (x^2 - \beta_\ell^2), \quad \text{and} \quad h(y) = \prod_{m=1}^{r} (y - \beta_m^2)$$

From our hypothesis,

[III. 1. 1]
$$0 = \operatorname{sig}(Kf(K^{-1}H)^{n_1 - 1} g[(K^{-1}H)^2])$$

$$= \operatorname{sig}[K \prod_{m=1}^{r} [(K^{-1}H)^2 - \beta_m^2 I]^{n_1 - 1} g[(K^{-1}H)^2]$$

Let

$$K_\ell = \frac{1}{2i} (T_1^{\beta_\ell} - (T_1^{\beta_\ell})^*) ,$$

$$H_\ell = \frac{1}{2} (T_1^{\beta_\ell} + (T_1^{\beta_\ell})^*) .$$

[III. 1. 1] equals to

$$\operatorname{sig} \bigoplus_{\ell=1}^{r} K_\ell \Big[\prod_{m=1}^{r} ((K_\ell^{-1} H_\ell)^2 - \beta_m^2 I) \Big]^{n_1 - 1} g((K_\ell^{-1} H_\ell)^2)$$

$$= \operatorname{sig} \bigoplus_{\ell=1}^{r} K_\ell \Big[\prod_{m \neq \ell} ((K_\ell^{-1} H_\ell)^2 - \beta_m^2 I)^{n_1 - 1} \Big] g((K_\ell^{-1} H_\ell)^2)$$

$$\times ((K_\ell^{-1} H_\ell)^2 - \beta_m^2 I)^{n_1 - 1}$$

Note $(K_\ell^{-1} H_\ell)^2 = (\beta_\ell^2)^2 I + M$ where M is strictly lower triangular.

Thus

$$g((K_\ell^{-1} H_\ell)^2) = g(\beta_\ell^2) I + M_1,$$

$$\prod_{m \neq \ell} ((K_\ell^{-1} H_\ell)^2 - \beta_m^2 I) = h'(\beta_\ell^2) I + M_2$$

(because $\prod_{m \neq \ell} (\beta_\ell^2 - \beta_m^2) = h'(\beta_\ell^2)$), where M_1 and M_2 are strictly lower triangular.

$$(K_\ell^{-1} H_\ell)^2 - \beta_\ell^2 I = N_{n_1} \otimes I_{2k}$$

$$M_j (N_{n_1} \otimes I_{2k})^{n_1 - 1} = 0 \quad \text{for} \quad j = 1, 2.$$

Thus

$$[\text{III. 1. 1}] = \operatorname{sig} \bigoplus_{\ell=1}^{r} K_\ell [h'(\beta_\ell^2) I]^{n_1 - 1} g(\beta_\ell^2) (N_{n_1} \otimes I_{2k})^{n_1 - 1}$$

Recall $K_\ell = E_{n_1} \otimes (B_{\ell 1} \oplus (-1)^{n_1 - 1} D_{\ell 1})$.

$$[III.1.1] = \operatorname{sig} \bigoplus_{\ell=1}^{r} h'(\beta_\ell^2)^{n_1-1} g(\beta_\ell^2)[E_{n_1} N_{n_1}^{n_1-1} \otimes (B_{\ell 1} \oplus (-1)^{n_1-1} D_{\ell 1})]$$

$$= \sum_{\ell=1}^{r} \operatorname{sig}[h'(\beta_\ell^2)^{n_1-1} g(\beta_\ell^2)(B_{\ell 1} \oplus (-1)^{n_1-1} D_{\ell 1})]$$

$$= \sum_{\ell=1}^{r} \operatorname{sgn}[h'(\beta_\ell^2)^{n_1-1} g(\beta_\ell^2)] \operatorname{sig}(B_{\ell 1} \oplus (-1)^{n_1-1} D_{\ell 1})$$

Choose g (successively) $= g_1, g_2, \ldots, g_r$ such that $P(j,\ell) = (\operatorname{sgn} g_j(\beta_\ell^2))$ is a non-singular $r \times r$ matrix. (See Proposition III.1.1.) Then $Q(j,\ell) = P(j,\ell)(\operatorname{sgn}(h'(\beta_\ell^2))^{n_1-1})$ is also a non-singular $r \times r$ matrix. Thus $\operatorname{sig}(B_{\ell 1} \oplus (-1)^{n_1-1} D_{\ell 1}) = 0$ for each ℓ. Thus

$$\operatorname{sig} B_{\ell 1} = -\operatorname{sig}(-1)^{n_1-1} D_{\ell 1}$$

$$= \operatorname{sig}(-1)^{n_1} D_{\ell 1}$$

Thus $B_{\ell 1}$ is $*$-congruent to $(-1)^{n_1} D_{\ell 1}$ for each ℓ. Thus $T_1^{\beta_\ell}$ is of type 1 for each ℓ.

Write

$$S_0 = \bigoplus_{\ell=1}^{r} T_1^{\beta_\ell}, \quad \text{and let} \quad S_1 = \bigoplus_{j=2}^{p} \left(\bigoplus_{\ell=1}^{r} T_j^{\beta_\ell} \right),$$

$$H_0 = \frac{1}{2}(S_0+S_0^*), \qquad K_0 = \frac{1}{2i}(S_0-S_0^*)$$

$$H_1 = \frac{1}{2}(S_1+S_1^*), \qquad K_1 = \frac{1}{2i}(S_1-S_1^*) .$$

Then

$$0 = \text{sig } K(f(K^{-1}H)^{n_2-1} g((K^{-1}H)^2))$$

$$= \text{sig } K_0(f(K_0^{-1}H_0)^{n_2-1} g((K_0^{-1}H_0)^2))$$

$$+ \text{sig } K_1(f(K_1^{-1}H_1)^{n_2-1} g((K_1^{-1}H_1)^2)) .$$

Because S_0 is of type 1 and $f(x)^{n_2-1} g(x^2)$ is an even polynomial, repeating the same process as above, by replacing K with K_1, H with H_1, n_1 with n_2, we can show $B_{\ell 2}$ is *-congruent to $(-1)^{n_2} D_{\ell 2}$, for each ℓ; thus $T_2^{\beta_\ell}$ is of type 1 for each ℓ. The proof of the "if" part is completed by similarly considering the fact that

$$0 = \text{sig } K(f(K^{-1}H)^{n_k-1} g[(K^{-1}H)^2])$$

for $k = 3, \ldots, p$.

Proposition III. 1. 4. In the usual complex case, S is of type 1 iff S is of type 2.

Proof: It is only necessary to prove that S is of type 1 implies S is of type 2.

Suppose S is of type 1. Here all roots of $K^{-1}H$ are in $\mathcal{F}$, so, in view of Chapter II, where we have shown that when K and H are non-singular, type 1 is equivalent to type 2, it is only necessary to consider the cases $K^{-1}H$ nilpotent and $H^{-1}K$ nilpotent. Without loss of generality, assume $S = T_1^z$ where $z = 0$ or ∞. In view of Theorem II.3.4, it follows that showing S of type 1 implies S of type 2 is equivalent to showing that, if a diagonal matrix B with entries ± 1 is of signature 0, then iB is of type 2. Suppose B is a diagonal matrix with entries ± 1 and of signature 0, then without loss of generality, write $B = I \oplus -I$. Let $c \in \mathcal{C}$ such that $c^* \neq \pm c$. Let $M = \begin{pmatrix} cI & c^*I \\ c^*I & cI \end{pmatrix}$; then by Lemma 5 of Chapter II

$$iM^*BM = i \begin{bmatrix} 0 & [(c^*)^2 - (c^2)]I \\ [c^2 - (c^*)^2]I & 0 \end{bmatrix}$$

Thus iB is of type 2. q.e.d.

2. Geometrical Approach to 2 x 2 Usual Complex Case

In the 2×2 case, using geometric ideas from Ballantine's work on Positive Definite Matrices [1], we can characterize the problem geometrically.

Define $\Gamma(S) = \{X^*SX \mid x \in \mathcal{C}^{2 \times 1}\}$. We say S is bidefinite iff $\Gamma(S)$ is a line; S is contradefinite iff $\Gamma(S)$ is the whole complex plane.

If S is non-singular; let $\operatorname{sgn}\det S = e^{2i\beta} = \frac{\det S}{|\det S|}$. If $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, define $\gamma \geq 0$ such that

$$1 - \gamma^2 = \frac{a\bar{d} + d\bar{a} - b\bar{b} - c\bar{c}}{2|\det S|}$$

If S is non-singular, S is $*$-congruent to one of the following forms:

$$S_1 = e^{i\beta}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \qquad S_2 = e^{i\beta}\begin{bmatrix} 1 & 0 \\ 2\gamma & 1 \end{bmatrix}$$

S_1 is bidefinite; S_2 is contradefinite if $\gamma > 1$.

If S is singular and non-zero, S is $*$-congruent to one of the following forms

$$S_3 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}, \qquad S_4 = \begin{bmatrix} e^{i\beta} & 0 \\ 0 & 0 \end{bmatrix},$$

where here

$$e^{i\beta} = \frac{\operatorname{trace} S}{|\operatorname{trace} S|} = \operatorname{sgn}\operatorname{tr} S$$

<u>Proposition III. 2. 1.</u> The following are equivalent:

(i) S is type 2,

(ii) S is type 1,

(iii) if S is non-singular then S has real determinant, and either $\det S$ is positive or S is bidefinite or contradefinite;

if S is singular, then either S is contradefinite or trace S is real.

<u>Proof</u>: In view of Proposition III. 1.4, it is enough to prove (ii) => (iii) and conversely. We consider the non-singular cases first.

(ii) => (iii) Suppose S is of type 1, then these exists C such that $C^*SC = S^*$, then $\det(CC^*)\det S = \det S^*$. Also $C^*S^*C = S$ thus $\det(CC^*)\det S^* = \det S$. Thus

$$\frac{\det S^*}{\det S} = \frac{1}{\det(CC^*)} = \frac{\det CC^*}{1}$$

$$\therefore \qquad \det CC^* = \pm 1 \ .$$

Since $\det CC^* = (\det C)(\det C)^* > 0$, $\det CC^* = 1$. Thus $\det S = \det S^*$. Suppose $\det S > 0$; then $e^{i\beta} = \sqrt{\dfrac{\det S}{|\det S|}} = \pm 1$. Without loss of generality, assume $e^{i\beta} = 1$. If S is $*$-congruent to $S_2 = \begin{pmatrix} 1 & 0 \\ 2\gamma & 1 \end{pmatrix}$, S is of type 2, hence is of type 1. Now suppose $\det S < 0$, $e^{i\beta} = \pm i$. Without loss of generality, assume $e^{i\beta} = i$. If S is $*$-congruent to $S_2 = i\begin{pmatrix} 1 & 0 \\ 2\gamma & 1 \end{pmatrix}$, S is of type 1 if iS is $*$-congruent to $iS^* = -(iS)^*$. Let

$$L = iS + (iS)^* = iS - iS^* = \begin{bmatrix} -2 & -2\gamma \\ -2\gamma & -2 \end{bmatrix} .$$

S is of type 1 implies L is $*$-congruent to $iS^* - iS = -(iS)^* - iS = -L$, which in turn implies $\det L < 0$; and $\det L = 4 - 4\gamma^2 = 4(1-\gamma^2) < 0$ iff $1 < \gamma^2$ iff $\gamma > 1$ iff S is contradefinite. If S is $*$-congruent to S_1 then in all cases, S is bidefinite.

(iii) => (ii) Suppose $\det S \neq 0$ and is real. If $\det S > 0$, $e^{i\beta} = \sqrt{\frac{\det S}{|\det S|}} = \pm 1$. Without loss of generality, assume $e^{i\beta} = 1$. If $\det S < 0$, then $e^{i\beta} = \pm i$. Without loss of generality, assume $e^{i\beta} = +i$. First suppose S is bidefinite, if $\det S > 0$; S is $*$-congruent to $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Without loss of generality, let $S = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Let $C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then $C^*SC = S^*$. Thus S is of type 1. If $\det S < 0$, then S is $*$-congruent to $i\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Thus S is of type 2 hence, is of type 1. Suppose S is contradefinite. If $\det S > 0$, S is $*$-congruent to $\begin{pmatrix} 1 & 0 \\ 2\gamma & 1 \end{pmatrix}$. Thus S is of type 2, hence is of type 1. If $\det S < 0$, S is $*$-congruent to $i\begin{pmatrix} 1 & 0 \\ 2\gamma & 1 \end{pmatrix}$. By Lemma 4 of [1], $\begin{pmatrix} 1 & 0 \\ 2\gamma & 1 \end{pmatrix}$ is $*$-congruent to $\begin{pmatrix} e^{i\alpha} & 0 \\ i\sin\alpha & e^{-i\alpha} \end{pmatrix}$ provided $\gamma > |\sin\alpha|$. Choose $\alpha = \pi/2$.
(Note: $\gamma > 1 = |\sin\frac{\pi}{2}|$.) Thus S is of type 2, hence is of type 1.

q.e.d.

Now suppose S is singular. If S is $*$-congruent to S_3, which is contradefinite and real then S is of type 2, hence is of type 1. Therefore we assume $S = S_4$. Here S is $*$-congruent to

S^* iff there exists $c \in \mathcal{C}$ such that $c^*ce^{i\beta} = e^{-i\beta}$ iff $e^{2i\beta} = cc^*$ iff $e^{i\beta} = \pm\sqrt{cc^*} \in \mathbb{R}$ iff S is over $\mathbb{R}$. q.e.d.

The above interpretation cannot be applied to higher dimensional matrices. Consider

$$T_1 = \text{diag}(i, -i, i, i)$$

$$T_2 = \begin{bmatrix} 1 & 0 \\ 2\gamma & 1 \end{bmatrix} \oplus \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \qquad \gamma > 1$$

$$T_3 = \begin{bmatrix} i & 1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} i & 1 \\ 1 & 0 \end{bmatrix}$$

$\Gamma(T_1)$ is the y-axis. Thus T_1 is bidefinite. If $S = \begin{pmatrix} 1 & 0 \\ 2\gamma & 1 \end{pmatrix}$, then $\Gamma(S) \subsetneq \Gamma(T_2)$. $\Gamma(S)$ is the complex plane. Thus $\Gamma(T_2)$ is contra-definite. Det $T_3 = 1$. But T_j, $j = 1, 2, 3$ are not of type 1, because $K_j = \frac{-i}{2}(T_j - T_j^*)$ are not of signature zero.

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APPENDICES

APPENDIX I

Pictures of Matrices

$$T_1^0 = \begin{bmatrix} & & & & jB & B \\ & & & jB & B & \\ & & \iddots & \iddots & & \\ & jB & \iddots & & & \\ jB & B & & & & \\ B & & & & & \end{bmatrix}$$

n_1 blocks

$$T_1^\infty = \begin{bmatrix} & & & & B & jB \\ & & & B & jB & \\ & & \iddots & \iddots & & \\ & B & \iddots & & & \\ B & jB & & & & \\ jB & & & & & \end{bmatrix}$$

n_1 blocks

$$T_1^{\beta} = \begin{bmatrix} & & & \begin{matrix} B & 0 \\ 0 & D \end{matrix} & \begin{matrix} \mu B & 0 \\ 0 & -\mu^* D \end{matrix} \\ & & \begin{matrix} B & 0 \\ 0 & D \end{matrix} & \begin{matrix} \mu B & 0 \\ 0 & -\mu^* D \end{matrix} & \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & \\ \begin{matrix} B & 0 \\ 0 & D \end{matrix} & \begin{matrix} \mu B & 0 \\ 0 & -\mu^* D \end{matrix} & & & \\ \begin{matrix} \mu B & 0 \\ 0 & -\mu^* D \end{matrix} & & & & \end{bmatrix}$$

$\beta \in \mathcal{E}$. n_1 blocks

Let $\mu = \beta+j$, $\mu^* = \beta-j$, $\mu^* = -\beta+j$.

APPENDIX II

Miscellaneous Results

Here we observe that any non-singular 2 x 2 type 1 matrix has determinant over $\mathcal{E}$. We also make observations on fields with -1 as a norm.

Proposition IV. 1. 1. If S is a 2 x 2 non-singular matrix of type 1, then det S is over $\mathcal{E}$.

Lemma 8. If S is a 2 x 2 non-singular matrix and S is of type 1, then S is *-congruent to $\begin{bmatrix} s & s^* \\ -s+as & s \end{bmatrix}$ for some $s \in \mathcal{F}$ and some $a \in \mathcal{E}$.

Proof of Lemma 8. It has been proved earlier that $A = S^{-1}S^*$ is *-congruent to A^* and A^{-1}. Therefore the characteristic polynomial of A is $x^2 - ax + 1$, i.e., $A = \begin{bmatrix} 0 & -1 \\ 1 & a \end{bmatrix}$.

Suppose

$$S^* = SA = \begin{bmatrix} s_{12} & -s_{11}+as_{12} \\ s_{22} & -s_{21}+as_{22} \end{bmatrix} = \begin{bmatrix} s_{11}^* & s_{21}^* \\ s_{12}^* & s_{22}^* \end{bmatrix}$$

Therefore $s_{11}^* = s_{12}$, $s_{22}^* = s_{12}$, so $s_{11} = s_{22}$ and $s_{21}^* = -s_{11} + as_{12}$. Let $s = s_{11}$. We have $S = \begin{pmatrix} s & s^* \\ -s^*+as & a \end{pmatrix}$.

If $T^{-1}T^*$ is similar to $A = \begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix}$, i.e., if there exists a non-singular D such that $D^{-1}(T^{-1}T^*)D = A$, then $S = D^*TD$ has the canonical form we have just shown. Thus it is enough to consider A being a companion matrix.

Proof of Proposition IV. 1. 1. From Lemma 8, when S is 2×2 non-singular type -1 matrix S is $*$-congruent to $S_0 = \begin{pmatrix} s & s^* \\ -s^*+as & a \end{pmatrix}$. $\det S_0 = s^2 + (s^*)^2 + ass^* \in \mathcal{E}$. If C is a non-singular matrix, such that $C^*SC = S_0$, then $(\det C)(\det C^*)(\det S) = \det S_0$, so $\det S = (\det S_0)[(\det C)(\det C^*)]^{-1}$. Therefore $\det S \in \mathcal{E}$.

Proposition IV. 1. 2. Suppose S is an $n \times n$ matrix over $\mathcal{F}$, $H = \frac{1}{2}(S+S^*)$, $K = \frac{1}{2j}(S-S^*)$. If -1 is a norm, and one of the following holds:

(1) $H^{-1}K$ is nilpotent,

(s) $K^{-1}H$ is nilpotent,

then S is of type 1.

Proof: From Theorem II. 3. 4, it is enough to show every diagonal matrix over $\mathcal{E}$ is $*$-congruent to its negative. Since -1 is a norm, $-1 = \alpha\alpha^*$, for some $\alpha \in \mathcal{F}$; if B is any matrix over $\mathcal{E}$, $(\alpha I)^*B(\alpha I) = -B$. q. e. d.

Proposition IV. 1. 3. Here we assume $\mathcal{E}$ and $\mathcal{F}$ to be finite fields, and S, H, and K are as defined in Proposition IV. 1. 2. If the following holds:

(i) $H^{-1}K$ and $K^{-1}H$ are not nilpotent.

(ii) $K^{-1}H$ is similar to $-K^{-1}H$ and all eigenvalues of $K^{-1}H$ are in $\mathcal{E}$, then S is of type 2, hence is of type 1.

<u>Proof</u>: In view of Theorems II. 3. 2, II. 3. 3, II. 3. 4, II. 3. 5, and II. 3. 7, it is enough to show that every diagonal matrix over $\mathcal{E}$ is *-congruent to any diagonal matrix of the same order over $\mathcal{E}$. Let $B = \text{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$, $\varepsilon_i \in \mathcal{E}$. Since $\mathcal{E}$ and $\mathcal{F}$ are finite fields, every element in $\mathcal{E}$ is a norm. Let $\alpha_i \in \mathcal{F}$ such that $\alpha_i \alpha_i^* = \varepsilon_i$. Let $C = \text{diag}(\alpha_1^{-1}, \alpha_2^{-1}, \ldots, \alpha_k^{-1})$ then $C^*BC = I_k$. Thus every diagonal matrix over $\mathcal{E}$ is *-congruent to the identity matrix. q. e. d.

<u>Proposition IV. 1. 4</u>. Here $\mathcal{E}$, $\mathcal{F}$, S, H and K are as defined in Proposition IV. 1. 3. Suppose one of the following holds

(i) $H^{-1}K$ is nilpotent,

(ii) $K^{-1}H$ is nilpotent.

In view of Theorems II. 3. 3 and II. 3. 6, we may, without loss of generality, assume $S = T^z$, $z = \infty, 0$. (Refer to pictures in Appendix I.) Then the following holds:

(i) if $z = 0$, (recall $S = T^0 = (E_n + jF_n) \otimes B$, where B is a diagonal matrix over $\mathcal{E}$) then

(a) when n is odd, S is of type 2,

(b) when n is even, S is of type 2 iff $\dim B$ is even;

(ii) if $z = \infty$ (recall $S = T_n^{\infty} = (F_n + jE_n) \otimes B$, where B is a diagonal matrix over $\mathcal{E}$), then

(a) when n is even, S is of type 2

(b) when n is odd, S is of type 2 iff $\dim B$ is even.

<u>Proof</u>: (i) (a) Refer to Theorem II. 3. 4.

(b) In view of Theorem II. 3. 4, it is enough to show that if B is a diagonal matrix over $\mathcal{E}$, then jB is of type 2 iff $\dim B$ is even. Suppose $\dim B$ is even. In the proof of Proposition IV. 1. 3, we have shown that every diagonal matrix over $\mathcal{E}$ is *-congruent to any other diagonal matrix over $\mathcal{E}$ of the same order and rank. If $\dim B = 2m$, $m \geq 1$, without loss of generality, assume $B = (1, -1, 1, -1, \ldots, 1, -1)$; let $C = \mathrm{diag}(\underbrace{C_0, C_0, \ldots, C_0}_{m})$, then $C^*BC = \mathrm{diag}(D_0, D_0, \ldots, D_0)$ where

$$D_0 = \begin{bmatrix} 0 & (c^*)^2 - c^2 \\ c^2 - (c^*)^2 & 0 \end{bmatrix} .$$

Thus $jC^*BC = C^*jBC$ is over $\mathcal{E}$. Thus jB is of type 2. Conversely, suppose $\dim B$ is odd and jB is of type 2. I.e., there exists a non-singular matrix C such that $C^*jBC = D$, and D is over $\mathcal{E}$. Thus

$$(\det C^*) \det(jB)(\det C) = j^n(\det B)(\det C)^*(\det C)$$
$$= \det W$$

which is over $\mathcal{E}$. But n is odd, so j^n is not over $\mathcal{E}$. Thus we have a contradiction; so jB is not of type 2.

(ii) The proof is similar to that of (i). q.e.d.

<u>Corollary 1.</u> If $\mathcal{E}$ and $\mathcal{F}$ are finite fields, S, H, K are as Proposition IV.1.4, and if one of the following holds:

(i) $H^{-1}K$ is nilpotent

(ii) $K^{-1}H$ is similar to $-K^{-1}H$ and has all its eigenvalues over $\mathcal{F}$, then S is of type 1.

<u>Proof</u>: Refer to Propositions IV.1.2 and IV.1.4.

<u>Corollary 2.</u> If $\mathcal{E}$ and $\mathcal{F}$ are finite fields, S is a 2×2 matrix over $\mathcal{E}$ such that if $H = \frac{1}{2}(S+S^*)$, $K = \frac{1}{2j}(S=S^*)$, the pencil $\lambda H + \mu K$ is non-singular. K is non-singular, $K^{-1}H$ is similar to $-K^{-1}H$, then S is of type 2.

<u>Proof</u>: Since the characteristic polynomial of $K^{-1}H$ would be of the form $x^2 - c$, where $c \in \mathcal{E}$. Since $\mathcal{E}$ is finite, $\mathcal{F}$ is

the only quadratic field over $\mathcal{E}$, thus it contains all the roots of all second degree polynomials. So we can apply Propositions IV. 1. 3 and IV. 1. 4.