Title: MATRICES CONJUNCTIVE WITH THEIR ADJOINTS

This paper studies necessary and sufficient conditions for a matrix to be conjunctive with its adjoint. The problem is solved completely in the usual complex case, in which it is shown that a matrix is conjunctive to its adjoint iff it is conjunctive to a real matrix. The problem is extended to pairs of fields \( \mathcal{F}, \mathcal{E} \), where \( [\mathcal{F}:\mathcal{E}] = 2 \) and characteristic \( \mathcal{F} \neq 2 \). It is shown that if a matrix is conjunctive to a matrix over \( \mathcal{E} \), it is then conjunctive to its adjoint. To achieve this result, we first show a matrix over any field \( \mathcal{E} \) is congruent over \( \mathcal{E} \) to its transpose. We also show that it is sufficient to consider non-singular pencils by proving the uniqueness up to conjunctivity of the non-singular summand of the pencil \( \lambda \mathcal{H} + \mu \mathcal{K} \), where \( \lambda \) and \( \mu \) are indeterminates over \( \mathcal{E} \), \( \mathcal{H}^* = \mathcal{H} \) and \( \mathcal{K}^* = \mathcal{K} \), when \( \lambda \mathcal{H} + \mu \mathcal{K} \) is decomposed (by conjunctivity over \( \mathcal{F} \)) into a direct sum of its minimum-indices part and a non-singular part.
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MATRICES CONJUNCTIVE WITH THEIR ADJOINTS

I. INTRODUCTION AND PRELIMINARIES

1. Introduction

Let \( \mathcal{F} \) be a field of characteristic not equal to 2 and \( \mathcal{E} \) be a subfield of index 2 in \( \mathcal{F} \) (i.e., \( \mathcal{F} \) is a vector space of dimension 2 over \( \mathcal{E} \)). Then there is a unique automorphism \( * \) of \( \mathcal{F} \) whose fixed field is \( \mathcal{E} \):

\[
(a+b)^* = a^* + b^*,
\]

\[
(ab)^* = a^*b^*,
\]

and \( a^* = a \) iff \( a \in \mathcal{E} \). Because \( \mathcal{F} \) has dimension 2 over \( \mathcal{E} \), we also have that \( * \) is involuntary: \( (a^*)^* = a \). We call \( a^* \) the conjugate of \( a \). If \( A \) is a matrix over \( \mathcal{F} \), we denote by \( A^* \) the transpose of the conjugate of \( A \). Then we have the following properties:

\[
(A^*)^* = A,
\]

\[
(A+B)^* = A^* + B^*,
\]

\[
(AB)^* = B^*A^* ,
\]

\[
(aA)^* = a^*A^*
\]

when \( A+B \) and \( AB \) are well defined and \( a \in \mathcal{F} \). We shall say \( S \) is \( * \)-symmetric iff \( S = S^* \), and say \( S \) is \( * \)-congruent to a
matrix $T$ iff there exists a non-singular matrix $C$ such that $C^*SC = T$. We say a matrix $S$ is of type 1 iff $S$ is $*$-congruent to $S^*$; we say $S$ is of type 2 iff $S$ is $*$-congruent to a matrix over $\mathbb{C}$.

We also consider our matrix $S$ as a linear transformation from $\mathcal{V}$, an $n$-dimensional vector space over $\mathbb{F}$, into $\mathcal{V}^*$ defined by $\mathcal{V}^* = \{f: \mathcal{V} \to \mathbb{F} | f(x+y) = f(x) + f(y), f(ax) = a*f(x)\}$. We call $\mathcal{V}^*$ the $*$-dual of $\mathcal{V}$. Then $S^*$ is a linear transformation from $(\mathcal{V}^*)^* = \mathcal{V}^*$ into $\mathcal{V}^*$. Elements of both $\mathcal{V}$ and $\mathcal{V}^*$ are being written as column vectors over $\mathbb{F}$. Suppose $x \in \mathcal{V} = \mathbb{F}^{n \times 1}$, $f \in \mathcal{V}^*$, define $\langle x, f \rangle = x^*f$. Then $\langle x, Sy \rangle = \langle y, S^*x \rangle^*$ for all $x, y \in \mathcal{V}$. In this context, $S$ is $*$-symmetric iff $\langle x, Sy \rangle = \langle y, Sx \rangle^*$ for all $x, y \in \mathcal{V}$. $S$ is $*$-congruent to $T$ iff there exists a one-to-one, onto, linear transformation $C: \mathcal{V} \to \mathcal{V}$ such that $\langleCx, SCy \rangle = \langle x, Ty \rangle$ for all $x, y \in \mathcal{V}$. If $\{e_1, \ldots, e_n\}$ is a basis for $\mathcal{V}$, then $\langle e_i, Se_j \rangle = e_i^*S^*e_j$ is the $(i, j)$ entry of the matrix of $S$ with respect to the basis $\{e_1, \ldots, e_n\}$. If $\mathcal{U}$ is a subspace of $\mathcal{V}$, we define $\mathcal{U}^0 = \{x \in \mathcal{V}: x^*\mathcal{U} = 0\}$. Suppose $\mathcal{V} = \bigoplus_{j=1}^{m} \mathcal{U}_j$. Define $\mathcal{V}^*_j = (\mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_{j-1} \oplus \mathcal{V}_{j+1} \oplus \cdots \oplus \mathcal{V}_{m-1} \oplus \mathcal{V}_m)^0$ $= \mathcal{V}_1^0 \odot \mathcal{V}_2^0 \odot \cdots \odot \mathcal{V}_{j-1}^0 \odot \mathcal{V}_{j+1}^0 \cdots \odot \mathcal{V}_{m-1}^0 \odot \mathcal{V}_m^0$.
then \( \mathcal{U}_j^* \) acts as a \(^*\)-dual of \( \mathcal{U}_j \), i.e., \( \mathcal{U}_j^0 \cap \mathcal{U}_j^* = 0 \), and
\[
\mathcal{U}_j^* = \bigoplus_{j=1}^{m} \mathcal{U}_j^*.
\]

If \( n = \sum_{i=1}^{p} k_i \), and we partition the matrix
\[
S = \begin{pmatrix}
S_{11} & S_{12} & \cdots & S_{1p} \\
S_{21} & \cdots \\
\vdots \\
S_{p1} & \cdots & S_{pp}
\end{pmatrix},
\]
where \( S_{ij} \) is a \( k_i \times k_j \) matrix, we write \( S = (S_{ij}) \). If \( C \) is a matrix such that \( C = M_1 \oplus \cdots \oplus M_p \), we write
\[
C = \text{diag}(M_1, \ldots, M_p),
\]
and call \( C \) a direct sum of \( M_1, \ldots, M_p \).

We have the following facts which will be useful in the work following (we shall not prove those quoted from the literature):

**Fact 1.** If \( S = (S_{ij}), C = \text{diag}(I_{k_1}, -I_{k_1}, I_{k_1}, \ldots), \) and
\[
P = (P_{ih}) = C \ast SC,
\]
then \( P_{ih} = (-1)^{i+h} S_{ih} \).

**Fact 2.** If \( C = \text{diag}(I_{k_1}, jI_{k_1}, I_{k_1}, \ldots) \) where \( j^* = -j \), and \( P = C \ast SC \), then
\[
P_{ih} = S_{ih} \quad \text{if \( i \) and \( h \) are odd}
\]
\[
= jS_{ih} \quad \text{if \( i \) is odd and \( h \) is even}
\]
\[
= -jS_{ih} \quad \text{if \( i \) is even and \( h \) is odd}
\]
\[
= -j^2 S_{ih} \quad \text{if \( i \) and \( h \) are even.}
\]
The proofs of Fact 1 and Fact 2 are routine computations.

**Fact 3.** If \( S = A \oplus O \) where \( A \) is non-singular and \( O \) is a zero matrix, and if \( S \) is \(*\)-congruent to \( B \oplus O \) where \( B \) is non-singular and of the same dimension as \( A \), then \( A \) is \(*\)-congruent to \( B \).

**Proof:** Suppose \( C \) is a non-singular matrix such that \( C^*SC = B \oplus O \). Partition \( C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \) accordingly. Thus

\[
C^*SC = C^*(A \oplus O)C = C_{11}AC_{11} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}.
\]

Thus \( C_{11}AC_{11} = B \). Since \( A \) and \( B \) are both non-singular, \( C_{11} \) is also non-singular. Thus \( A \) is \(*\)-congruent to \( B \).

Suppose \( H \) and \( K \) are \(*\)-symmetric and \( K \) is non-singular. Suppose \( (K^{-1}H-\beta I) \) is nilpotent for some \( \beta \in \mathcal{E} \). Let \( m \) be a positive integer such that \( (K^{-1}H-\beta I)^m = 0 \), but \( (K^{-1}H-\beta I)^{m-1} \neq 0 \). If we define an inner product \( (x, y) = x^*Ky \), then from Mal'cev [7], we can choose a basis for \( \mathcal{V} \) so that \( K^{-1}H \) is block diagonal with each diagonal block a Jordan canonical block of eigenvalue \( \beta \), and the Gram matrix of the inner product is conformably block diagonal with each diagonal block of the form...
We define $E_n$ as the $n \times n$ matrix with 1's on the anti-diagonal and zero everywhere else, $F_n$ as the matrix with 1's on the first super-anti-diagonal and zero everywhere else. Also define $N_n = E_n F_n$.

**Fact 5.** If $K$ and $H$ are $*$-symmetric and

$$(K^{-1}H-\beta)(K^{-1}H-\beta^*)$$

is nilpotent with $\beta \neq \beta^*$, and if $\lambda, \mu$ are
indeterminates over $\mathcal{F}$, then there exists a basis for $\mathcal{V}$ such that $\lambda H + \mu K$ is block diagonal with each block of the form:

$$
\begin{pmatrix}
\lambda & \lambda \beta + \mu \\
\lambda \beta + \mu & \lambda & \lambda \beta + \mu \\
\lambda \beta + \mu & \lambda \beta + \mu & \lambda
\end{pmatrix}
$$

**Fact 6.** If $H$ is symmetric and $K$ is skew-symmetric over $\mathcal{E}$, and $\lambda$ and $\mu$ are indeterminates over $\mathcal{E}$, then the pencil $\lambda H + \mu K$ is congruent over $\mathcal{E}$ to the following form

$$
\bigoplus_{i=1}^{n} \begin{pmatrix}
0 & L_{\varepsilon}^{\wedge}_{i} \\
L_{\varepsilon}^{\wedge}_{i} & 0
\end{pmatrix} \oplus (\lambda H_{0} + \mu K_{0}),
$$

where

$$
\bigoplus_{i=1}^{n} \begin{pmatrix}
0 & L_{\varepsilon}^{\wedge}_{i} \\
L_{\varepsilon}^{\wedge}_{i} & 0
\end{pmatrix}
$$
is called the minimum-indices part of the pencil $\lambda H + \mu K$, and

$$L_{\epsilon_i} = \begin{pmatrix} \lambda & \mu & \epsilon_i \\ \mu & \lambda & \epsilon_i \\ \epsilon_i & \epsilon_i \\ \epsilon_i & \epsilon_i + 1 \end{pmatrix}, \\
L_{\epsilon_i}^* = \begin{pmatrix} \lambda & -\mu \\ \mu & -\mu \\ \epsilon_i & -\mu \\ \epsilon_i + 1 & \lambda \end{pmatrix}$$

and $\det(\lambda H_0 + \mu K_0)$ is a non-zero polynomial.

**Fact 7.** Witt's Theorem [9]. If $A, B_1, B_2$ are $*-\text{symmetric}$ non-singular matrices over $\mathcal{F}$, then $A \oplus B_1, B_2$ is $*-\text{congruent}$ to $A \oplus B_2$ implies $B_1$ is $*-\text{congruent}$ to $B_2$.

2. **Uniqueness of the Non-Singular Core of a $*-\text{Symmetric}$ Pencil**

**Theorem 1.2.1.** Let $H$ and $K$ be $*-\text{symmetric}$. Then the pencil $\lambda H + \mu K$, where $\lambda$ and $\mu$ are indeterminates over $\mathcal{E}$, is $*-\text{congruent}$ to $L \oplus M$, where

$$L = 0 \oplus \bigoplus_{i=1}^{k} \begin{pmatrix} 0 & L_{\epsilon_i} \\ L_{\epsilon_i}^* & 0 \end{pmatrix}.$$ 

$L_{\epsilon_i}$ is as defined in Fact 6, $L_{\epsilon_i}^*$ is the conjugate-transpose of $L_{\epsilon_i}$, and $M = \lambda H_0 + \mu K_0$ is a non-singular pencil. If $\lambda H + \mu K$ is $*-\text{congruent}$ also to $L \oplus N$ where $N$ is a non-singular pencil,
then $M$ is $*$-congruent to $N$.

Turnbull proved that a $*$-symmetric pencil can always be reduced to the form $L \oplus M$, where $L$ is the minimum-indices part and $M$ is a non-singular pencil [11]. In this section we shall only prove the uniqueness (up to $*$-congruency) of the non-singular summand $M$.

Before we proceed with the proof of the theorem, we shall discuss some of the properties of the pencil $L \oplus M$.

Suppose $\mathcal{U}$ is a subspace of $\mathcal{V}$, and $S, T$ are linear transformations from $\mathcal{V}$ to $\mathcal{V}^*$. For our discussion in this section, we need to define, for $\mathcal{U} \subseteq \mathcal{V}$, $(S^{-1}T)\mathcal{U} = \{x \in \mathcal{V} : Sx \in T\mathcal{U}\}$, and, inductively, define $(S^{-1}T)^i\mathcal{U} = \{x \in \mathcal{V} : Sx \in T(S^{-1}T)^{i-1}\mathcal{U}\}$, for $i = 2, 3, \ldots$. $(S^{-1}T)^0\mathcal{U} = \mathcal{U}$.

Also note that if we let $\Lambda = \lambda H + \mu K$ be a pencil of $*$-symmetric matrices, considered as the matrix of a pencil of bilinear forms, then, for each co-ordinate subspace, the matrix of this pencil of forms restricted to this subspace is the corresponding principal submatrix of $\Lambda$, and conversely.

Lemma 1. Let $\mathcal{V}^*$ be the $*$-dual of $\mathcal{V}$, let $\mathcal{V}_j$ ($j = 1, 2, \ldots, k$) be subspaces of $\mathcal{V}$ such that $\mathcal{V} = \bigoplus_{j=1}^{k} \mathcal{V}_j$, and let $\mathcal{V}^* = \bigoplus_{j=1}^{k} \mathcal{V}_j^*$ be the corresponding $*$-dual direct sum defined in Section 1 of this chapter. If $S$ and $T$ are linear
transformations such that $S, T : \mathcal{V} \rightarrow \mathcal{V}^*$, $S \mathcal{V}_j \subseteq \mathcal{V}_j^*$, and $T \mathcal{V}_j \subseteq \mathcal{V}_j^*$, then $(S^{-1}T)^i \mathcal{V} = \bigoplus_{j=1}^k [\mathcal{V}_j \cap (S^{-1}T)^i \mathcal{V}_j].$

**Proof of Lemma 1.** The proof proceeds by induction on $i$.

If $i = 0$, the lemma is obviously true. Suppose it is true for $i \leq h-1$. Consider $i = h$. It is enough to show

$$(S^{-1}T)^h \mathcal{V} \subseteq \bigoplus_{j=1}^k (\mathcal{V}_j \cap (S^{-1}T)^h \mathcal{V}_j).$$

Suppose $x \in (S^{-1}T)^h \mathcal{V}$, and $x = \sum_{j=1}^k x_j$, with $x_j \in \mathcal{V}_j$. Then $Sx = \sum_{j=1}^k Sx_j \in T(S^{-1}T)^h \mathcal{V}$. By our induction hypothesis,

$$T(S^{-1}T)^h \mathcal{V} = T\left(\bigoplus_{j=1}^k (S^{-1}T)^h \mathcal{V}_j\right).$$

Thus

$$Sx = \sum_{j=1}^k Sx_j = \sum_{j=1}^k Ty_j,$$

where $y_j \in (\mathcal{V}_j \cap (S^{-1}T)^h \mathcal{V}_j)$ and $Ty_j \in \mathcal{V}_j^*$. But $Sx_j \in \mathcal{V}_j^*$. Therefore $Sx_j = Ty_j \in T(S^{-1}T)^h \mathcal{V}_j$, so $x_j \in \mathcal{V}_j \cap (S^{-1}T)^h \mathcal{V}_j$. Thus $x \in \bigoplus_{j=1}^k (\mathcal{V}_j \cap (S^{-1}T)^h \mathcal{V}_j)$.

**Lemma 2.** Let (as usual) characteristic $\mathcal{F} \neq 2$, $H = H^*$ and $K = K^*$ be linear maps of $\mathcal{V}$ into $\mathcal{V}^*$. Let $\lambda, \mu$ be...
indeterminates over \( \mathcal{J} \). Suppose \( \lambda H + \mu K \) is \(*\)-congruent over \( \mathcal{J} \) to \( L \oplus M_\infty \oplus M_0 \oplus M_1 \), where \( L \) has no elementary divisors, \( M_\infty \) has elementary divisors only of the form \( \mu^q \), and \( M_0 \) has elementary divisors only of the form \( \lambda^q \), and \( M_1 \) has elementary divisors neither of the form \( \lambda^q \) nor of the form \( \mu^q \) and \( M = M_\infty \oplus M_0 \oplus M_1 \) is a non-singular pencil. Then the \(*\)-congruency type of \( M \) over \( \mathcal{J} \) is determined as follows:

Let \( \mathcal{U} = (K^{-1}H)^{n}\mathcal{U}, \quad \mathcal{W} = (H^{-1}K)^{n}\mathcal{U}. \) (Here \( n = \dim \mathcal{V} \) as usual.) Then the pencil \( \Lambda \) of \(*\)-bilinear forms \( x^*(\lambda H + \mu K)y \) with \( x \) and \( y \) restricted to \( \mathcal{U} \) is \(*\)-congruent over \( \mathcal{J} \) to the pencil of \(*\)-bilinear forms whose matrix is \( O \oplus M_0 \oplus M_1 \), and \( \Lambda \) restricted to \( \mathcal{W} \) is \(*\)-congruent over \( \mathcal{J} \) to the pencil of \(*\)-bilinear forms whose matrix is \( O \oplus M_\infty \oplus M_1 \).

**Proof of Lemma 2:** Without loss of generality, we may assume \( \lambda H + \mu K = L \oplus M_\infty \oplus M_0 \oplus M_1 \) and

\[
L = \bigoplus_{i=1}^{k'} 0 \oplus \bigoplus_{i=1}^{k} \begin{bmatrix} 0 & L_{\xi_i} \\ L_{\xi_i}^* & 0 \end{bmatrix}
\]

Let \( \mathcal{U} = R \oplus \mathcal{U}_\infty \oplus \mathcal{U}_0 \oplus \mathcal{U}_1 \). Here \( R, \mathcal{U}_\infty, \mathcal{U}_0, \mathcal{U}_1 \) denote the co-ordinate subspaces corresponding to \( L, M_\infty, M_0, M_1 \), respectively. Denote by \( R_i = \text{span} \{x_1, \ldots, x_{2m+1} \} \), \( m \geq 0 \), one of
the co-ordinate subspaces corresponding to one of the direct sum-
mmands in \( L \), where \( x_j \) denotes the \( j \)th co-ordinate vector.

Let \( R_i^* = \text{span}\{\hat{x}_1, \ldots, \hat{x}_{2m+1}\} \) be the corresponding subspace of
\( R^* \), thus \( R_i^* \) acts as a \(*\)-dual of \( R_i \). Note that the matrix
of \( \lambda H + \mu K \) restricted to \( R_i \) is

\[
\begin{bmatrix}
\lambda & & \\
\mu & \lambda & \\
\vdots & & \ddots & \lambda \\
\mu & \cdots & \lambda & \mu
\end{bmatrix}
\]

For simplicity in notation we drop the subscript \( i \) in \( R_i \). Thus

\[
\begin{align*}
Hx_1 &= \hat{x}_{m+2}, & Kx_1 &= 0 \\
Hx_2 &= \hat{x}_{m+3}, & Kx_2 &= \hat{x}_{m+2} \\
& \vdots & & \vdots \\
Hx_m &= \hat{x}_{2m+1}, & Kx_m &= \hat{x}_{m+2} \\
Hx_{m+1} &= 0, & Kx_{m+1} &= \hat{x}_{2m+1} \\
Hx_{m+2} &= \hat{x}_1, & Kx_{m+2} &= \hat{x}_2 \\
Hx_{m+3} &= \hat{x}_1, & Kx_{m+3} &= \hat{x}_3 \\
& \vdots & & \vdots \\
Hx_{2m} &= \hat{x}_{m-1}, & Kx_{2m} &= \hat{x}_m \\
Hx_{2m+1} &= \hat{x}_m, & Kx_{2m+1} &= \hat{x}_{m+1}
\end{align*}
\]
\[ \mathcal{R} \cap (K^{-1}H) \mathcal{R} = \text{span } \{x_1, x_2, \ldots, x_{2m}\} \]
\[ \mathcal{R} \cap (K^{-1}H)^2 \mathcal{R} = \text{span } \{x_1, x_2, \ldots, x_{2m-1}\} \]
\[ \vdots \]
\[ \mathcal{R} \cap (K^{-1}H)^{m+1} \mathcal{R} = \text{span } \{x_1, x_2, \ldots, x_{m+1}\} \]
\[ = \mathcal{R} \cap (K^{-1}H)^k \mathcal{R} \quad \text{for all } k \geq m+1. \]

Thus the matrix of \( \Lambda \) restricted to \( \mathcal{R} \cap (K^{-1}H)^n \mathcal{R} \) is \( 0 \).

(This \( 0 \) matrix is of order \( m+1 \) here.)

\( K \) is non-singular on \( \mathcal{U}_\infty \); hence \( K \mathcal{U}_\infty = \mathcal{U}_\infty^* \) and \( H \mathcal{U}_\infty \subseteq \mathcal{U}_\infty^* \), so \( K^{-1}H \mathcal{U}_\infty \subseteq \mathcal{U}_\infty \) and \( K^{-1}H \) is a nilpotent mapping on \( \mathcal{U}_\infty \). Thus \( (K^{-1}H)^n \mathcal{U}_\infty = 0 \).

\( H \) is non-singular on \( \mathcal{U}_0 \). Thus, \( H \mathcal{U}_0 = \mathcal{U}_0^* \).

\( (K^{-1}H) \mathcal{U}_0 \subseteq K^{-1} \mathcal{U}_0^* = \{x : Kx \in \mathcal{U}_0^*\} = \mathcal{U}_0 \), since \( K(\mathcal{U}_0) \subseteq \mathcal{U}_0^* \).

Thus \( (K^{-1}H) \mathcal{U}_0 = \mathcal{U}_0 \). Suppose \( (K^{-1}H)^i \mathcal{U}_0 = \mathcal{U}_0 \) for \( i \leq k-1 \).

Then \( x \in (K^{-1}H)^k \mathcal{U}_0 \iff Kx \in H(K^{-1}H)^{k-1} \mathcal{U}_0 = H \mathcal{U}_0 = \mathcal{U}_0 \).

Thus \( \mathcal{U}_0 \bigcap (K^{-1}H)^n \mathcal{U}_0 = \mathcal{U}_0 \).

\( K \) and \( H \) restricted to \( \mathcal{U}_1 \) are non-singular. Thus \( (K^{-1}H)^n \mathcal{U}_1 = \mathcal{U}_1 \).

Therefore \( x^* \Lambda y = x^* (\lambda H + \mu K)y \), with \( x, y \) restricted to \( \mathcal{U} = (K^{-1}H)^n \mathcal{U} \), is \( * \)-congruent to a pencil of \( * \)-bilinear forms whose matrix is \( 0 \oplus M_0 \oplus M_1 \).

Similarly, we can show that \( x^* \Lambda y \), with \( x, y \) restricted to
\[ W = (H^{-1}K)^nU \] is \(^*\)-congruent to a pencil of \(^*\)-bilinear forms whose matrix is \( O \oplus M_\infty \oplus M_1 \).

**Proof of Theorem 1.2.1.** Write

\[ L \oplus M = \lambda H + \mu K = \Lambda, \]
\[ L \oplus N = \lambda H_1 + \mu K_1 = \Lambda_1. \]

Without loss of generality, we may assume \( M = M_{\infty 1} \oplus M_{01} \oplus M_{11} \), \( N = M_{\infty 2} \oplus M_{02} \oplus M_{12} \), where \( M_{\infty 1} \) and \( M_{\infty 2} \) have elementary divisors only of the form \( \mu^q \), \( M_{01} \) and \( M_{02} \) have elementary divisors only of the form \( \lambda^q \), and \( M_{11} \) and \( M_{12} \) have no elementary divisors of the form \( \mu^q \) nor of the form \( \lambda^q \), and recall that \( M \) and \( N \) are non-singular pencils. If \( C \) is a non-singular matrix over \( \mathcal{F} \), (we also consider \( C \) as a non-singular map of \( U \) into \( \mathcal{G} \)) such that \( C^*\Lambda C = C^*(L \oplus M)C = L \oplus N \), then \( C^*H_1C = H_1 \) and \( C^*K_1C = K_1 \).

By Lemma 2, \( x^*\Lambda y \), and \( x^*\Lambda_1 y \), with \( x, y \) restricted to \( (K^{-1}H)^nU \) and \( (K^{-1}H_1)^nU \), respectively, is equal to \( u^*(O \oplus M_{01} \oplus M_{11})v \) and \( u^*(O \oplus M_{02} \oplus M_{12})v \), respectively, with \( u, v \in (K^{-1}H)^nU \) and \( (K^{-1}H_1)^nU \), respectively.

We want to show by induction on \( i \) that
\[ [(C^*KC)^{-1}(C^*HC)]^iU = C^{-1}(K^{-1}H)^iC^*U. \]
For $i = 1$:

$$
x \in [(C*KC)^{-1}C*HC)]^n \cup
$$

iff

$$
C*KCx \in C*HC \cup
$$

iff

$$
KCx \in HC \cup
$$

iff

$$
Cx \in K^{-1}HC \cup
$$

iff

$$
x \in C^{-1}(K^{-1}H)C \cup.
$$

Suppose

$$
[(C*KC)^{-1}(C*HC)]^k \cup = C^{-1}(K^{-1}H)^{k-1}C \cup.
$$

Then

$$
x \in [(C*KC)^{-1}(C*HC)]^k \cup
$$

iff

$$
(C*KC)x \in C*HC[(C*KC)^{-1}(C*HC)]^k \cup
$$

iff

$$
KCx \in HC[(C*KC)^{-1}(C*HC)]^k \cup
$$

(by our induction hypothesis)

$$
= HCC^{-1}(K^{-1}H)^{k-1}C \cup
$$

$$
= H(K^{-1}H)^{k-1}C \cup
$$

iff

$$
Cx \in K^{-1}H(K^{-1}H)^{k-1}C \cup
$$

iff

$$
x \in C^{-1}(K^{-1}H)^{k}C \cup.
$$

Let $\{x_1, x_2, \ldots, x_m\}$ be a basis for $\cup = (K^{-1}H)^n \cup$. Then

$$
\{C^{-1}x_1, C^{-1}x_2, \ldots, C^{-1}x_m\} \text{ is a basis for}
$$

$$
C^{-1}(K^{-1}H)^n \cup = C^{-1}(K^{-1}H)^nC \cup
$$

$$
= [(C*KC)^{-1}(C*HC)]^n \cup.
$$
Therefore
\[
\left( C^{-1} x_j \right)^* \Lambda_1 \left( C^{-1} x_j \right) = x_i^* \left( C^* \right)^{-1} \Lambda_1 C^{-1} x_j
\]
\[
= x_i^* \left( C^* \right)^{-1} C^* A C C^{-1} x_j
\]
\[
= x_i^* \Lambda x_j.
\]
Thus \( x^* \Lambda y \) with \( x, y \) restricted to \( \mathcal{U} \) is \(*\)-congruent to
\[
u_i^* (O \oplus M_{02} \oplus M_{12}) v_j \text{ with } u_1, v_1 \in (K_1^{-1} H_1).
\]
Thus \( O \oplus M_{01} \oplus M_{11} \) is \(*\)-congruent to \( O \oplus M_{02} \oplus M_{12} \). Thus by Fact 3, \( M_{01} \oplus M_{11} \) is \(*\)-congruent to \( M_{02} \oplus M_{12} \). Let
\[
M_{ij} = \lambda H_{ij} + \mu K_{ij} \quad i = 0, 1, j = 1, 2, \text{ where } H_{ij} \text{ and } K_{ij} \text{ are}
\]
\(*\)-symmetric. Then
\[
C_1^* (H_{01} \oplus H_{11}) C_1 = H_{02} \oplus H_{12}
\]
\[
C_1^* (K_{01} \oplus K_{11}) C_1 = K_{02} \oplus K_{12}.
\]
Thus
\[
C_1^{-1} \left( H_{01}^{-1} K_{01} \oplus H_{11}^{-1} K_{11} \right) C_1
\]
\[
= [C_1^* (H_{01} \oplus H_{11}) C_1]^{-1} [C_1^* (K_{01} \oplus K_{11}) C_1]
\]
\[
= H_{02}^{-1} K_{02} \oplus H_{12}^{-1} K_{12}.
\]
Partition \( C_1 \) accordingly: \( C_1 = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \). Let
\[
B_{ij} = H_{ij}^{-1} K_{ij}.
\]
Then
\[(B_{01} \oplus B_{11})^{C_{1}} = C_{1}(B_{02} \oplus B_{12})\]

\[
\begin{bmatrix}
B_{01}C_{11} & B_{01}C_{12} \\
B_{11}C_{21} & B_{11}C_{22}
\end{bmatrix}
= \begin{bmatrix}
C_{11}B_{02} & C_{12}B_{12} \\
C_{21}B_{02} & C_{22}B_{22}
\end{bmatrix}
\]

\[B_{11}C_{21} = C_{21}B_{02}.\]

Since \(B_{11}\) and \(B_{02}\) have no eigenvalues in common, \(C_{12} = 0\) [12]. Similarly, \(C_{21} = 0\). Thus \(C_{11}M_{01}C_{11} = M_{02}\).

\(C_{22}M_{11}C_{22} = M_{12}\). \(M_{01}, M_{02}, M_{11}\) and \(M_{12}\) are non-singular, thus \(C_{11}\) and \(C_{22}\) are non-singular. So \(M_{01}\) is *-congruent to \(M_{02}\) and \(M_{11}\) is *-congruent to \(M_{12}\). Likewise, \(L \oplus M\) restricted to \((H^{-1}K)\) is *-congruent to \(O \oplus M_{\infty 1} \oplus M_{11}\);

\(L \oplus N\) restricted to \((H^{-1}K)\) is *-congruent to \(O \oplus M_{\infty 2} \oplus M_{12}\), where \(M_{\infty 1}\) and \(M_{\infty 2}\) have elementary divisors of the form \(\mu^q\). With an approach similar to that of our consideration on \(L\), we can show that \(M_{\infty 1}\) is *-congruent to \(M_{\infty 2}\). Since

\[M = M_{01} \oplus M_{\infty 1} \oplus M_{11}\]

\[N = M_{02} \oplus M_{\infty 2} \oplus M_{12},\]

we have that \(M\) is *-congruent to \(N\). q.e.d.
II. GENERAL CASE

1. Congruency of a Matrix to Its Transpose

If $H = H'$, $K = K'$, $H$ and $K$ are over $\mathcal{E}$, and $\lambda$ and $\mu$ are indeterminates over $\mathcal{E}$, then $\lambda H + \mu K$ is congruent to the following:

$$[\text{II. 1. 1}] \quad L \oplus M_0 \oplus M_\infty \oplus M_1$$

where $L$ is described in Fact 6 of Chapter I, $M_0$ has elementary divisors only of the form $\lambda^q$, and $M_\infty$ has elementary divisors only of the form $\mu^q$, $M_1$ has neither elementary divisors of the form $\lambda^q$ nor $\mu^q$, and $M_0 \oplus M_\infty \oplus M_1$ is non-singular.

In this section, we shall prove:

**Theorem II. 1. 1.** Every matrix over $\mathcal{E}$ is congruent over $\mathcal{E}$ to its transpose.

**Lemma 3.** Let $\mathcal{U}$ be finite dimensional over $\mathcal{E}$ with (as always) characteristic $\mathcal{E} \neq 2$. Let $H: \mathcal{U} \rightarrow \mathcal{U}'$ be symmetric, $K: \mathcal{U}' \rightarrow \mathcal{U}$ be skew, and $B = KH: \mathcal{U} \rightarrow \mathcal{U}$ be non-singular. Let $q(t)$ be irreducible in $\mathcal{E}[t]$ and let $q(t^2)^m$ be the minimum polynomial for $B$ on $\mathcal{U}$. (Here $\mathcal{U}'$ is the (ordinary) dual of $\mathcal{U}$.)

Then there is a maximum $B$-cyclic subspace $\mathcal{U}$ of $\mathcal{U}$ such that
Proof of Lemma: Let \( H_1 = Hq(B^2)^{m-1} \). Then \( H_1' = H_1 \neq 0 \), thus there exists \( e \in U \) such that \( e'H_1e \neq 0 \). Let \( U \) be the B-cyclic subspace generated by \( e \). To show that \( U \) is maximum cyclic and that \( U \cap (HU)^0 = 0 \), it is sufficient to show that for \( f(t) \in \mathcal{E}[t] \), \((f(B)e)'HU = 0 \Rightarrow q(t^2)^m | f(t)\).

Suppose \((f(B)e)'HU = 0\). Write \( f(t) = [q(t^2)]^\ell p(t) \) where \( q(t^2)^\ell p(t) \). Let \( r(t) = \text{g.c.d}(p(t), p(-t)) \). Then \( r(t) = t^k s(t^2) \), for some \( k \geq 0 \), and some \( s(t) \in \mathcal{E}[t] \), such that \( t^k s(t^2) \). We claim that \( \text{g.c.d}(r(t), q(t^2)) = 1 \). Let \( m(t) = \text{g.c.d}(r(t), q(t^2)) \).

Since \( q(t^2) \) is even and \( r(t) \) is even or odd, then \( m(t) = t^\ell n(t^2) \) for some \( \ell \geq 0 \) and \( n(t) \in \mathcal{E}[t] \), such that \( t^\ell | q(t^2) \). It follows that \( q(t^2) = t^{2m} h(t^2) \) for some \( m \geq 0 \) and some \( h(t) \in \mathcal{E}[t] \) such that \( t^\ell | h(t^2) \). Thus \( q(t) = t^{2m} h(t) \). But \( q(0) \neq 0 \), since \( B \) is non-singular \( \Rightarrow k = 0 \), so \( m(t) = n(t^2) \). Thus \( n(t^2) | q(t^2) \), which implies \( n(t) | q(t) \), which in turn implies \( n(t) = 1 \) or \( n(t) = q(t) \). If \( n(t) = q(t) \), then \( q(t^2) = n(t^2) = m(t) | r(t) \), so \( q(t^2) | p(t) \), which is a contradiction. \( \Rightarrow n(t) = 1 \). Thus \( m(t) = 1 \). Since \( \text{g.c.d}(r(-t), q(t^2)) = m(-t) = 1 \). Thus \( r(-B) \) is non-singular, so \( r(-B)U = U \). Let \( r(t) = a(t)p(t) + b(t)p(t) \), for some \( a(t), b(t) \in \mathcal{E}[t] \). Now, \( 0 = (f(B)e)'HU \), which implies for all \( i \geq 0 \), that
Since $K = HB$ and $H' = H$, $B'H = (HB)' = K' = -K = -HB$. Inductively, we can show $HB^i = (-B^i)'H$. Thus,

$$[\text{II. 1.2}] = (f(B)e)'HB^i e = [q(B^2)^{\ell} p(B)e]'HB^i e$$

Thus

$$[\text{II. 1.2}] = ((-B)^i e)'H[q(B^2)^{\ell} p(-B)e]$$
$$= [[(-B)^i e)'H[q(B^2)^{\ell} p(-B)e]]'$$
$$= [q(B^2)^{\ell} p(-B)e]'H(-B)^i e .$$

Thus

$$[q(B^2)^{\ell} p(-B)e]'H \mathcal{U} = 0 = [q(B^2)^{\ell} p(B)e]'H \mathcal{U},$$

so

$$0 = [q(B^2)^{\ell} p(-B)e]'H \mathcal{U} \supset [q(B^2)^{\ell} p(-B)e]'H b(-B) \mathcal{U}$$
$$= [q(B^2)^{\ell} p(-B) b(B)e]'H \mathcal{U}$$

and

$$0 = [q(B^2)^{\ell} p(B)e]'H \mathcal{U} \supset [q(B^2)^{\ell} p(B)e]'H a(-B) \mathcal{U}$$
$$= [q(B^2)^{\ell} p(B) a(B)e]'H \mathcal{U}$$

Thus

$$0 = [q(B^2)^{\ell} [p(B) a(B) + p(-B) b(B)]e]'H \mathcal{U}$$
$$= [q(B^2)^{\ell} r(B)e]'H \mathcal{U}$$
$$= [q(B^2)^{\ell} e]'H r(-B) \mathcal{U}$$
$$= [q(B^2)^{\ell} e]'H \mathcal{U} .$$
\[ [q(B^2)^i] \cdot H q(B^2)^i e = e' H q(B^2)^i e = 0 \quad \text{for all} \quad i \geq m. \quad \text{Since} \quad e' H q(B)^m e \neq 0, \quad f \geq m. \quad \text{Thus} \quad q(t^2)^m | f(t). \quad \text{q.e.d.} \]

**Proof of Theorem II.1.1.** Without loss of generality we may consider the pencil [II.1.1] \( L \oplus M_0 \oplus M_\infty \oplus M_1 \) described in the first paragraph of this section; and we replace the indeterminates \( \lambda, \mu \) with 1. We shall show separately \( L, M_0, M_\infty, \) and \( M_1 \) are congruent over \( \mathcal{E} \) to their respective transposes.

Consider \( L. \) For each non-zero block (of order \( 2\varepsilon_i + 1 \)) of \( L, \) multiply on the right and left by \( I_{\varepsilon_i + 1} \oplus E_{\varepsilon_i}. \) Thus in \( L_{\varepsilon_i}, \) the first column becomes the last column, the second column becomes the second to the last column and etc.; in \( L_{\varepsilon_i}, \) the first row becomes the last row, and the second row becomes the second to the last row and etc. Thus, we have for each non-zero block of \( L \)

\[
\begin{bmatrix}
0 & Z_{\varepsilon_i} \\
\hat{Z}_{\varepsilon_i} & 0
\end{bmatrix}
\]

[II.1.3]

where

\[
Z_{\varepsilon_i} = \begin{bmatrix}
\circ & 1 & 1 \\
1 & \circ & 1 \\
1 & 1 & \circ
\end{bmatrix}_{\varepsilon_i}, \quad Z_{\varepsilon_i} = \begin{bmatrix}
\circ & 1 & 1 & -1 \\
1 & -1 & -1 & \circ \\
-1 & \circ & \circ & \circ
\end{bmatrix}_{\varepsilon_i}.
\]

Multiply [II.1.3] on the right and left by \( \text{diag}(1, -1, 1, -1, \ldots, 1); \)
we get the transpose of [11.1.3]. Thus $L$ is congruent to $L'$.

Next consider $M_0$. Let $V_0$ be the co-ordinate subspace corresponding to the direct summand $M_0$ in $S = H + K = L \oplus M_\infty \oplus M_0 \oplus M_1$. $H$ is non-singular and $B = (H^{-1}K)$ is nilpotent on $V_0$. Let $m$ be a positive integer such that $B^m = 0$, and $B^{m-1} \neq 0$. We shall consider the cases $m$ even and $m$ odd separately. We shall prove the following:

(i) if $m$ is odd, there exists a maximum $B$-cyclic subspace $U$ such that $U \cap (HU)^0 = 0$;

(ii) if $m$ is even, there exist two maximum $B$-cyclic subspaces $U$ and $W$ such that

$$0 = U \cap W = U \cap (HW)^0 = W \cap (HU)^0 = (U \oplus W) \cap (H(U \oplus W))^0,$$

$$U \subseteq (HU)^0, \text{ and } W \subseteq (HW)^0.$$

Suppose $m$ is odd. $(HB^{m-1})' = HB^{m-1} \neq 0$. There exists $e \in V$ such that $e'HB^{m-1}e \neq 0$. Let $U$ be the $B$-cyclic subspace generated by $e$. Suppose $(B^l p(B)e)'HU = 0$, where $p(t) \in E[t]$ such that $t_{m} p(t)$. Since $t_{m}$ is the minimum polynomial of $B$ on $V_0$, $p(-B)$ is non-singular and hence $p(-B)U = U$. Thus $0 = (B^l p(B)e)'HU = (B^l e)'H p(-B)U = (B^l e)'HU$. Thus $l \geq m$. We therefore have completed the proof of (i).

The following discussion is not needed in the proofs of (i) and
(ii), but useful for later purposes. We note that $B \mathcal{U} \subseteq \mathcal{U}$ because $\mathcal{U}$ is $B$-cyclic. $B(H \mathcal{U})^0 \subseteq (H \mathcal{U})^0$ because if $x \in (H \mathcal{U})^0$, then $0 = x'H \mathcal{U} \supseteq x'H B \mathcal{U} = -(Bx)'H \mathcal{U}$, which implies $Bx \in (H \mathcal{U})^0$.

Let $e_1 = e + \sum_{i=1}^{m-1} a_i B^i e$; we can choose $a_1, a_2, \ldots, a_{m-1}$ so that $e_1'H B^k e_1 = 0$ for $k \leq m-2$. Let $\mathcal{U}_1$ be the $B$-cyclic subspace generated by $e_1$. $\mathcal{U}_1$ has all the properties proved for $\mathcal{U}$. $B$ restricted to $\mathcal{U}_1$ is the Jordan canonical block of eigenvalue $0$, $H$ restricted to $\mathcal{U}_1$ is a matrix $H_0$ with non-zero elements on the anti-diagonal and zero everywhere else. Thus $K = HB$ restricted to $\mathcal{U}_1$ is a matrix $K_0$ with non-zero entries on the first super-anti-diagonal and zero everywhere else. Multiply $H_0 + K_0$ on the right and left by $\text{diag}(1, -1, \ldots, 1)$. Since $m$ is odd, by Fact 1, we have multiplied the first super-anti-diagonal by $-1$, and have not changed the anti-diagonal. Thus we obtain the transpose of $H_0 + K_0$.

In the next paragraph, we shall prove (ii).

Suppose $m$ is even; since $HB^{m-1} \neq 0$ there exist $e, f \in \mathcal{U}$ such that $e'H B^{m-1} f \neq 0$; we can choose $e, f$ so that $e'H B^{m-1} f = 1$. Let $\mathcal{U}, \mathcal{W}$ be the $B$-cyclic subspaces generated by $e$ and $f$ respectively. First, we want to show $\mathcal{U} \cap \mathcal{W} = 0$.

If $x \in \mathcal{U} \cap \mathcal{W}$, then $x = B^k p(B)e = B^k q(B)f$, for some $p(t), q(t) \in \mathcal{E}[t]$ such that $t^k p(t)$, $t^k q(t)$. Without loss of generality,
assume \( k \geq l \). If \( k \geq m \), then \( x = 0 \). Therefore assume \( k < m \).

Note, since \( t_p(t) \), \( t_q(t) \), \( p(B) \) and \( q(B) \) are non-singular, \( r(B) = p(B)q(B)^{-1} \) is non-singular and we can even take \( r(t) \in \mathcal{E}[t] \) here, and \( r(-B) \) is also non-singular. \( x = B^k p(B)e = B^l q(B)f \) and \( k \geq l \). Therefore

\[
0 = B^l [q(B)f - B^k p(B)e] = q(B) B^l (f - B^k r(B)e) = B^l (f - B^k r(B)e)
\]

since \( q(B) \) is non-singular. \( B^l (f - B^k r(B)e) = 0 \) implies

\[
0 = e'HB^m f - e'HB^m r(B)e .
\]

If \( k \geq l + 1 \), \( e'HB^m r(B)e = e'HB^m f = 0 \); thus

\[
0 = e'HB^m f \text{ which is a contradiction. If } k = l ,
\]

\[
e'HB^m r(B)e = e'HB^m e = e'HB^m - e'HB^m e
\]

because \( m - 1 \) is odd (and hence \( HB^m \) is skew). Thus

\[
e'HB^m e = 0 = e'HB^m f \text{ which is a contradiction. Thus } k \geq m,
\]

which implies \( x = 0 \). Thus \( \mathcal{U} \cap \mathcal{W} = 0 \).

Secondly, we want to show \( \mathcal{U} \cap (\mathcal{H}_\mathcal{W})^0 = 0 \). Suppose \( (B^k p(B)e) \mathcal{H} \mathcal{W} = 0 \) for some \( p(t) \in \mathcal{E}[t] \) such that \( t_p(t) \). Thus \( p(B) \) and \( p(-B) \) are non-singular, so \( p(-B) \mathcal{U} = \mathcal{U} \).
0 = (B^k p(B)e)'H \mathcal{W} = (B^k e')Hp(-B)\mathcal{W} \\
\quad = e'H(-B)^k \mathcal{W} \\
\quad \supset e'H(-B)^{k+i} \mathcal{W} \quad \text{for all } i \geq 0

Thus \( k \geq m \). Thus \( \mathcal{U} \cap (H \mathcal{W})^0 = 0 \). Likewise we can show that \( \mathcal{W} \cap (H \mathcal{U})^0 = 0 \).

Thirdly, we want to show \( (\mathcal{U} \oplus \mathcal{W}) \cap (H(\mathcal{U} \oplus \mathcal{W}))^0 = 0 \).

If \( x \in (\mathcal{U} \oplus \mathcal{W}) \cap (H(\mathcal{U} \oplus \mathcal{W}))^0 \), then \( x = B^k p(B)e + B^l q(B)f \) where \( p(t), q(t) \in \mathcal{E} [t] \) and \( t \not\equiv p(t), q(t) \). Without loss of generality, assume \( l \geq k \). Suppose \( k \leq m-1 \). Then

\[ 0 = [B^k p(B)e + B^l q(B)f]'H(\mathcal{U} \oplus \mathcal{W}) \]
\[ \supset [B^k p(B)e + B^l q(B)f]'H(-B)^{m-1-k}(\mathcal{U} \oplus \mathcal{W}) \]
\[ = [B^{m-1} p(B)e + B^{l+m-1-k} q(B)f]'H(\mathcal{U} \oplus \mathcal{W}) \]
\[ \supset [B^{m-1} p(B)e + B^{l+m-1-k} q(B)f]'H \mathcal{W} \]
\[ = [B^{m-1} p(B)e]'H \mathcal{W} + [B^{l+m-1-k} q(B)f]'H \mathcal{W} \]
\[ \supset p(0)[B^{m-1} e]'Hf + [B^{l+m-1-k} q(B)f]'Hf. \]

If \( l > k \), then \( B^{l+m-1-k} = 0 \). If \( l = k \),

\[ [B^{l+m-1-k} q(B)f]'Hf = [B^{m-1} q(B)f]'Hf = q(0)[B^{m-1} f]'Hf. \]
Since \( m - 1 \) is odd, then \( H^B m - 1 \) is skew. Thus \( f' H^B m - 1 f = 0 \). Both \( p(0) \) and \( (B^m - 1 e)' H f \) are nonzero. Thus we have a contradiction. Therefore \( \ell > m - 1 \). Thus we have shown \( (U \oplus \mathcal{U}) \cap [H(U \oplus \mathcal{U})]^0 = 0 \).

Fourthly, let \( p(t) = \sum_{j \geq 0} a_j t^{2j} \) and let

\[
\hat{f} = f + Bp(B)e
\]

\[
= f + \sum_{j \geq 0} (a_j B^{2j+1})e.
\]

Let \( \mathcal{W} \) be the \( B \)-cyclic subspace generated by \( \hat{f} \). We can choose \( a_0, a_1, \ldots \) such that \( H \mathcal{W} \subseteq \mathcal{W}^0 \).

\[
0 = \hat{f}' H^B m - 2 \hat{f} = (f + \sum_{j \geq 0} a_j B^{2j+1} e)' H^B m - 2 (f + \sum_{j \geq 0} a_j B^{2j+1} e)
\]

\[
= f' H^B m - 2 f - 2a_0 e' H^B m - 1 f.
\]

Thus we can solve for \( a_0 \). If we consider \( \hat{f}' H^B m - 4 \hat{f} = 0 \), we will get an expression in terms of \( f' H^B m - 4 f, a_0, f' H^B m - 3 e, e' H^B m - 2 f \) and \( e' H^B m - 1 f \), and solve for \( a_1 \). Likewise, we can solve for \( a_k \) by considering \( \hat{f}' H^B m - 2(k+1) f \). In like manner, let \( e' = e + Bq(B)f \)

where \( q(t) = \sum_{j \geq 0} \beta_j t^{2j} \). We can pick \( \beta_0, \beta_1, \ldots \) so that \( \mathcal{U} \), the
B-cyclic subspace generated by \( \hat{e} \) satisfies the property \( H \hat{\mathcal{U}} \subseteq \hat{\mathcal{U}}^0 \). \( \hat{\mathcal{U}} \) and \( \hat{\mathcal{W}} \) obviously satisfy the first 3 properties we proved for \( \mathcal{U} \) and \( \mathcal{W} \). Thus we have completed the proof of (ii). Further, let \( \hat{f} = f + \sum_{j=2}^{m} a_j B^{j-1} f \) we can choose \( a_2, a_3, \ldots \) so that \( e^H B \hat{\mathcal{K}} f = 0 \) for \( k \leq m \). Let \( \hat{\mathcal{W}} \) be the cyclic subspace generated by \( \hat{f} \); then the pair \( \hat{\mathcal{U}}, \hat{\mathcal{W}} \) also satisfies the properties we have proved for the pair \( \mathcal{U}, \mathcal{W} \). B restricted to \( \hat{\mathcal{U}} \oplus \hat{\mathcal{W}} \) is the direct sum of 2 \( m \)-dimensional Jordan blocks of eigenvalues 0.

\( H \), restricted to \( \hat{\mathcal{U}} \oplus \hat{\mathcal{W}} \), is

\[
H_0 = \begin{bmatrix} 0 & -S \\ S & 0 \end{bmatrix}, \quad \text{where } S = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ -1 & \cdots & \cdots & 0 \end{bmatrix};
\]

and \( K \), restricted to \( \hat{\mathcal{U}} \oplus \hat{\mathcal{W}} \), is

\[
K_0 = \begin{bmatrix} 0 & T \\ -T & 0 \end{bmatrix}, \quad \text{where } T = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ -1 & \cdots & \cdots & 0 \end{bmatrix}.
\]

Multiply by \( \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \) on the right and left of \( H_0 + K_0 \) we get \( -H_0 - K_0 \). Since \( H_0 \) is of even dimension, if we multiply by \( \text{diag}(1, -1, \ldots, -1) \) on the right and left of \( -H_0 - K_0 \), then by Fact 1, the anti-diagonal has been multiplied by \( -1 \), and the first
super-anti-diagonal is not changed. Thus we get $H_0K_0$.

In case $m$ is odd, $U_0 = U \oplus (HU)^0$, $U_0' = HU \oplus U^0$, and $H(HU)^0 \subseteq U^0$, $KU = HBU \subseteq HU$.

Thus we can repeat the above process with $U_0'$ replaced by $(HU)^0$. Likewise, for $m$ even, we can repeat the above process with $U_0$ replaced by $(H(U \oplus W))^0$. Thus, after a finite number of steps, we show that $M_0$ is congruent to its transpose.

The process of showing $M_\infty$ congruent to $M'_\infty$ is similar; in this case, instead of $H^{-1}K$, we consider $K^{-1}H$.

Now we consider $M_1$ (and thus we assume $H$ and $K$ are non-singular here). Suppose $C_2$ is a non-singular matrix such that $C_2M_1C_2 = M_1'$. Without loss of generality, write $M_1 = \bigoplus_{i=1}^{k} N_i$ such that if $H_i = \frac{1}{2}(N_i + N_i')$, $K_i = \frac{1}{2}(N_i - N_i')$, and $f_i(x)$ is the minimum polynomial of $K_i^{-1}H_i$, then $f_i(x)$ is prime to $f_j(x)$ for $i \neq j$.

Note that $K(K^{-1}H)K^{-1} = -HK^{-1} = -(K^{-1}H)'$, so the minimum polynomial of $K^{-1}H$ is even; thus we place another restriction on the polynomials $f_i(x); f_i(x) = [q_i(x^2)]^m$ where $q(x)$ is irreducible (and $x \nmid q(x)$). Without loss of generality, we may assume $M_1 = H_1 + K_1$. Lemma 3 (proved in this section) assures us that we can find a basis for $U$, such that $M_1 = \bigoplus_{i=1}^{p} M_{i1}$, where, if $H_{i1} = \frac{1}{2}(M_{i1} + M_{i1}')$ and $K_{i1} = \frac{1}{2}(M_{i1} - M_{i1}')$ then $K_{i1}^{-1}H_{i1}$ is similar to a companion matrix. Thus it suffices to prove the following:
Lemma 4. If $H$ and $K$ as defined above are non-singular, and $K^{-1}H$ is non-derogatory, then $S = H + K$ is congruent to $S'$. 

Proof. Let $m$ be the degree of the minimum polynomial of $A = K^{-1}H$. Thus, for some $e \in \mathcal{U}$, \( \{e, Ae, A^2e, \ldots, A^{m-1}e\} \) is a basis for $\mathcal{U}$. Write $e_j = A^{j-1}e$. Recall $KA^i$ is skew when $i$ is even, is symmetric when $i$ is odd. Thus

\[
e_j^i Ke_i = (-1)^{j-1}e_k^j KA^{i+j-2}e_1 = 0 \text{ for } i + j \text{ even},
\]

\[
e_j^i He_i = (-1)^{j-1}e_k^j HA^{i+j-2}e_1 = 0 \text{ for } i + j \text{ odd}. \]

Let $C: \mathcal{U} \to \mathcal{U}$ such that $Ce_i = (-1)^{i-1}e_i$. Then

\[
e_j^i C^i HCe_i = (Ce_i)^i H (Ce_i) = (-1)^{i+j}e_j^i He_i = e_j^i He_i
\]

because if $i+j$ is odd, $e_j^i He_i = 0$. Also

\[
e_j^i C^i KCe_i = (Ce_i)^i K (Ce_i) = (-1)^{i+j}e_j^i Ke_i = -e_j^i Ke_i
\]

because if $i+j$ is even, $e_j^i Ke_i = 0$. Thus $C'SC = S'$. q.e.d.

2. General Results

In this section, we shall prove some simple results, some of which lay the ground work for the following section.

Proposition II.2.1. Let $S$ be a non-singular matrix: then:
(i) if $S$ is of type 1, then $\det S^* = \pm \det S$.

(ii) if $S$ is of type 2, then $\det S \in E$.

(iii) if $S$ is of type 2, then $S$ is $\ast$-congruent to $\bar{S}$, where $\bar{S}$ is the conjugate matrix of $S$.

(iv) if $S$ is of type 2, then it is of type 1.

**Proof.** (i) If $C^*SC = S^*$, then $C^*S^*C = S$, so we have $(C^*)^2 S^* (C^2) = S^*$. Therefore $\det(CC^*) = \pm 1$. Thus $\det S^* = \pm \det S$.

(ii) If $C^*SC = T$ is over $E$, then $\det S = (\det T)(\det C^{-1})(\det C^{-1})^*$, which is in $E$.

(iii) If $C^*SC = T$ is over $E$, then $C^*S^*C = T^* = T' = C'S'C$. Therefore $(C')^{-1} C^*S^*C(C')^{-1} = S'$. I.e., $(C')^{-1} C^*S^*C(C')^{-1} = (S')^* = \bar{S}$.

(iv) If $C^*SC = T$ is over $E$, then $T$, from Theorem I.2.1, is congruent over $E$ (hence $\ast$-congruent over $F$) to $T'$, and from (iii) of Proposition II.2.1, $S$ is $\ast$-congruent to $\bar{S}$; then $S$ is $\ast$-congruent to $(S')' = S^*$.

**Proposition II.2.2.** Let $H = \frac{1}{2} (S-S^*)$, and let $K = \frac{1}{2j} (S-S^*)$, where $j^* = -j$, and let $A = S^{-1} S^*$. Then

(i) $A$ is similar to $(A^{-1})^*$.

(ii) If $K$ is non-singular, $B = K^{-1} H$ is similar to $B^*$.

(iii) If $H$ is non-singular, $B_1 = H^{-1} K$ is similar to $B_1^*$. 
(iv) If \( S \) is of type 1 (hence also if \( S \) is of type 2), \( A \) is similar to \( A^* \).

(v) If \( S \) is of type 1, i.e., \( C^*SC = S^* \) where \( C \) is non-singular, and \( K \) is non-singular, then \( C^{-1}K^{-1}HC = -K^{-1}H \).

**Proof.** (i) \( SAS^{-1} = S(S^{-1}S^*)S^{-1} = S^*S^{-1} = (A^*)^{-1} \).

(ii) \( KBK^{-1} = K(K^{-1}H)K^{-1} = HK^{-1} = B^* \).

(iii) \( HB_1H^{-1} = H(H^{-1}K)H^{-1} = KH^{-1} = B_1^* \).

(iv) If \( C \) is non-singular such that \( C^*SC = S^* \), then \( C^{-1}AC = C^{-1}(S^{-1}S^*)C = (C^*SC)^{-1}(C^*S^*C) = (S^*)^{-1}S = A^{-1} \). Since \( A \) is also similar to \( (A^*)^{-1} \), thus \( A \) is similar to \( A^* \).

(v) If \( C \) is non-singular such that \( C^*SC = S^* \), then \( C^*HC = H \) and \( C^*KC = -K \). Thus

\[
C^{-1}(K^{-1}H)C = [C^*KC]^{-1}(C^*HC) = -K^{-1}H. \quad \text{q.e.d.}
\]

**Proposition II. 2. 3.** (i) Suppose \( S \) is non-singular, and \( S = S_0 \oplus S_{\infty} \oplus S_1 \) where \( A_0 = S_{0}^{-1}S_{0}^* \) has eigenvalues only at \( 1 \), \( A_{\infty} = S_{\infty}^{-1}S_{\infty}^* \) has eigenvalues only at \(-1\), and \( A_1 = S_1^{-1}S_1^* \) has eigenvalues neither at \( 1 \) nor at \(-1\). If \( C \) is a non-singular matrix such that \( C^*SC = S^* \), then \( C = C_0 \oplus C_{\infty} \oplus C_1 \), where \( C_0 \) is such that \( C_0^*S_0C_0 = S_0^* \), \( C_{\infty} \) is such that \( C_{\infty}^*S_{\infty}C_{\infty} = S_{\infty}^* \), and \( C_1 \) is such that \( C_1^*S_1C_1 = S_1^* \).
(ii) Suppose \( H = \frac{1}{2} (S+S^*) \) and \( K = \frac{1}{2} (S-S^*) \) and non-singular. Let \( B = K^{-1} H = \bigoplus_{i=1}^{p} B_i \), where the \( B_i \) are such that if \( f_i(t) \) is the minimum polynomial of \( B_i \), then \( f_i(t) = q_i(t^2) \) for some \( q_i(t) \in \mathbb{C}[t] \), and g.c.d \( (f_i(t), f_j(t)) = 1 \) when \( i \neq j \). Suppose

\[
S = \bigoplus_{i=1}^{p} S_i,
\]

where if

\[
H_i = \frac{1}{2} (S_i + S_i^*),
\]

\[
K_i = \frac{1}{2j} (S_i - S_i^*),
\]

then \( B_i = K_i^{-1} H_i \). If \( C \) is a non-singular matrix such that \( C^*SC = S^* \), then \( C = \bigoplus_{i=1}^{p} C_i \) such that \( C_i^*S_iC_i = S_i^* \).

**Proof.** (i) Since

\[
C^{-1}AC = C^{-1}(S^{-1}S^*)C = [C^*SC]^{-1}[C^*SC] = (A^*)^{-1},
\]

\[
AC = C(A^*)^{-1}.
\]

Partition \( C = (C_{ih}) \) conformably to \( A \). \( i = 0, \infty, 1 \) and \( h = 0, \infty, 1 \).

Then \( AC = C(A^*)^{-1} \) implies \( A_i C_{ih} = C_{ih}(A_h^*)^{-1} \). If \( i \neq h \), \( A_i \) and \( (A_h^*)^{-1} \) have no eigenvalues in common and thus \( C_{ih} = 0 \).
We have shown that the off-diagonal blocks of $C$ are $0$. Thus (i) is proved.

(ii) Since

$$C^{-1}BC = (C*KC)^{-1}(C*HC)$$

$$= -B$$

Thus $BC = -CB$. Partition $C$ conformably to $B$. Thus

$B_{i}C_{ij} = -C_{ij}B_{i}$, $i, j = 1, 2, ..., p$. If $i \neq j$, then $B_{i}$ and $-B_{j}$ have no eigenvalues in common, thus $C_{ij} = 0$. We have shown that the off-diagonal blocks of $C$ are $0$, thus we have proved (ii).

The next result is based on Taussky-Zassenhaus' result [10].

**Proposition II.1.4.** If $S$ is non-singular and $S^{-1}S^*$ is non-derogatory, and $C*SC = S*$ for $C$ non-singular, then $C*S$ is $*$-congruent to a symmetric matrix over $\mathcal{F}$.

**Proof:** First assume $A = S^{-1}S^*$ is a companion matrix. $S$ is of type 1 therefore $A$ is similar to $A^*$ hence similar to a matrix over $\mathcal{E}$ [2]; since $A$ is a companion matrix, $A$ is a matrix over $\mathcal{E}$. 
Since
\[ C \times S(A)S^{-1}(C^*)^{-1} = C \times S(S^{-1}S^*)S^{-1}(C^*)^{-1} \]
\[ = C \times (S \times S^{-1})(C^*)^{-1} \]
\[ = (C \times S \times C)(C^*SC)^{-1} \]
\[ = S(S^*)^{-1} \]
\[ = A^* = A' . \]

Since \( A \) is a companion matrix, by the Taussky-Zassenhaus result, \( C \times S \) is symmetric.

Now suppose \( A = S^{-1}S^* \) is not a companion matrix, but \( A \) is non-derogatory. There exists \( D \) such that \( D^{-1}AD = A_0 \), where \( A_0 \) is a companion matrix. Let \( T = D^*SD \). Then

\[ S = (D^*)^{-1}TD^{-1}, \quad S^* = (D^*)^{-1}T^*D^{-1}. \]

\[ C \times SC = C \times (D^*)^{-1}TD^{-1}C \]
\[ = S^* \]
\[ = (D^*)^{-1}T^*D^{-1} \]

Thus \( (D^{-1}CD)^*TD^{-1}CD = T^* \). Note

\[ T^{-1}T^* = D^{-1}S^{-1}(D^*)^{-1}D^*S^*D = D^{-1}AD = A_0 \]

is a companion matrix. From what we have shown for the case \( S^{-1}S^* \) being a companion matrix, \( [D^{-1}CD]^*T \) is a symmetric
matrix over $\mathcal{F}$.

$$[D^{-1}CD]^*T = (D^*)C*(D^*)^{-1}D^*SD$$

$$= D^*C^*SD$$

Thus $C^*S$ is $*$-congruent to a symmetric matrix.

3. Non-Singular $*$-Symmetric Pencils with All Eigenvalues in $\mathcal{F}$

Let $H$ and $K$ be $*$-symmetric. Then Theorem 1.2.1 asserts that the pencil $\lambda H + \mu K$, where $\lambda$ and $\mu$ are indeterminates over $\mathcal{E}$, is $*$-congruent to $L \oplus M$ where $L$ is the minimum-indices part as defined in Theorem 1.2.1 and $M$ is the non-singular core, unique up to $*$-congruency. If $T = H_1 + jK_1$, where $H_1$ and $K_1$ are $*$-symmetric and $j^* = -j$, and $jK_1$ is a matrix over $\mathcal{E}$, then $H_1 - jK_1 = T^* = T^* = H_1' + jK_1'$. Thus proving that the matrix $S = H + jK$ is of type 2 is equivalent to proving that the pencil $\lambda H + \mu K$ is $*$-congruent over $\mathcal{E}$ to $\lambda H_1 + \mu K_1'$, where $H_1' = H_1 = H_1^*$, $-K_1 = K_1 = K_1^*$, i.e., $H_1' = H_1^* \in \mathcal{E}^{\text{type}}$, and $jK_1' = -(jK_1)' \in \mathcal{E}^{\text{type}}$. We say a pencil is of type 2 if $\lambda H + \mu K$ is $*$-congruent to $\lambda H_1 + \mu K_1$ where $H_1$ and $K_1$ are as described above. We say a pencil is of type 1 if $\lambda H + \mu K$ is $*$-congruent to $\lambda H - \mu K$.

First we want to show that:
Proposition II. 3. 1. If \( S = H + jK \) where \( H \) and \( K \) are \(*\)-symmetric and \( j^* = -j \), then \( S \) is of type 2 iff the non-singular core of the pencil \( \lambda H + \mu K \) is of type 2.

Proof: Write \( \lambda H + \mu K = L \oplus M \) where \( L \) is the minimum-indices part and \( M \) is the non-singular core.

("only if") If \( \lambda H + \mu K \) is of type 2, then \( \lambda H + \mu K \) is \(*\)-congruent to \( \lambda H_1 + \mu K_1 \) where \( H_1 = H_1^* = H_1^t, -K_1^t = K_1^t = K_1^* \); thus \( (j^{-1}K_1)^* = -j^{-1}K_1 = j^{-1}K_1 \) is a matrix over \( \mathcal{E} \). Let \( K_0 = j^{-1}K_1 \). \( \lambda H_1 + \mu K_0 \) is congruent over \( \mathcal{E} \) to \( L_0 \oplus M_1 \) where

\[
L_0 = \bigoplus_{i=1}^{k_1} \begin{pmatrix} 0 & L_{\epsilon_i}^t \\ L_{\epsilon_i} & 0 \end{pmatrix},
\]

\( L_{\epsilon_i} \) and \( L_{\epsilon_i}^t \) are as defined in Fact 6 and \( M_1 = \lambda H_1 + \mu K_0 \) is a non-singular pencil and \( H_1, K_0 \) are over \( \mathcal{E} \), and \( H_1^t = H_1 \), \( (K_0)^t = -K_0 \). Thus \( \lambda H_1 + \mu K_1 = \lambda H_1 + j\mu K_0 \) is \(*\)-congruent to \( L_1 \oplus \lambda H_1 + \mu jK_0 \), where

\[
L_1 = \bigoplus_{i=1}^{k_1} \begin{pmatrix} 0 & W_{\epsilon_i}^t \\ W_{\epsilon_i} & 0 \end{pmatrix},
\]

where
Because of the uniqueness of the minimum indices, the same minimum indices, \( L_1 \) is \(*\)-congruent to \( L \) [11].

(Recall in the beginning of the proof, we assume \( \lambda H + \mu K = L \oplus M \), where \( L \) is the minimum-indices part and \( M \) is the non-singular core.) Thus \( \lambda H + \mu K \) is \(*\)-congruent to \( \lambda H_1 + \mu K_1 \) where

\[
(\lambda H_1)^* = H_1^* = H_1, \quad (jK_1)^* = jK_1^* = jK_1;
\]

in turn is

\[
\lambda H_1 + \mu K_1 \quad \text{*-congruent to} \quad L_1 \oplus \lambda H_1 + \mu K_0;
\]

which in turn is \(*\)-congruent to

\[
L \oplus \lambda H_1 + \mu K_0.
\]

Let \( K_1 = jK_0 \). Thus \( \lambda H + \mu K = L \oplus M \) is

\(*\)-congruent to \( L \oplus \lambda H_1 + \mu K_1 \). By the uniqueness of the non-singular core of pencils \( M \) is \(*\)-congruent to \( \lambda H_1 + \lambda K_1 \). We said before \( (\lambda H_1)^* = H_1^* = \lambda H_1 \), \( jK_1 = jK_0 \) is a matrix over \( \mathbb{C} \), and

\[
(jK_1)^* = -jK_1^* = -j(jK_0)^* = j^2 K_0^* = j^2 K_0 = jK_1^*.
\]

Thus \( M \) is of type 2.

("if") We only need to prove the minimum-indices part \( L \) of
the pencil is of type 2. Therefore without loss of generality, assume

\[
\Lambda = \lambda H + \mu K = \begin{bmatrix}
0 & L_{\epsilon_i} \\
L_{\epsilon_i}^* & 0
\end{bmatrix},
\]

where \( L_{\epsilon_i} \) is defined in Fact 6 of Section 1 of Chapter I. Multiply on the right and left of \( \Lambda \) by \( \mathbb{I} \oplus E_{\epsilon_i} \). By a calculation similar to a calculation appearing in the proof of Theorem II.1.1, we can see that \( \Lambda \) becomes

\[
[\text{III. 1. 1}]
\]

\[
\Lambda_2 = \begin{bmatrix}
0 & Y_{\epsilon_i} \\
Y_{\epsilon_i}^* & 0
\end{bmatrix},
\]

where

\[
Y_{\epsilon_i} = \begin{bmatrix}
& & & \lambda \\
& \lambda & & \\
& & \mu & \\
\epsilon_i & & & 
\end{bmatrix}
\]

Multiply on the right of [III. 1. 1] by \( C_1 = \text{diag}(1, j, \ldots, j, 1) \) and the left by \( C_1^* \). Since [III. 1. 1] is of odd order, by Fact 2 of Section 1, Chapter I, we have multiplied the entries on the anti-diagonal
alternately by \( j \) and \( -j \) and the entries on the anti-diagonal alternately by 1 and \(-j^2\). Thus \( C_1^* \wedge_2 C_1 = \lambda H_2 + \mu K_2 \) is such that \( H_2 \) is a matrix over \( E \) because its non-zero entries are \( \epsilon \{ 1, -j^2 \} \) and \( jK_2 \) is a matrix over \( E \) because \( K_2 \)'s non-zero entries are \( \epsilon \{ j, -j \} \). q.e.d.

In this section, we shall consider \( S = H + jK \) where \( H, K \) are \(*\)-symmetric and \( j^* = -j \), such that the pencil \( \lambda H + \mu K \) is non-singular, and such that all the eigenvalues of the pencil are in \( \mathcal{F} \). (Note that \( S^{-1}S^* = (H+jK)^{-1}(H-jK) \). If \( H \) is non-singular, then

\[
(H(I+jH^{-1}K))^{-1}H(I-jH^{-1}K) = (I+jH^{-1}K)^{-1}(I-jH^{-1}K).
\]

Therefore \( S^{-1}S^* \) having eigenvalue at 1 is equivalent to \( H^{-1}K \) being nilpotent. Likewise, \( S^{-1}S^* \) having eigenvalues at -1 is equivalent to \( K^{-1}H \) being nilpotent.)

Suppose \( S = \bigoplus_{i=1}^{p} S_i \) and \( H_i = \frac{1}{2} (S_i + S_i^*) \) and \( K_i = \frac{1}{2j} (S_i - S_i^*) \) and \( H_1^{-1}K_1 \) is nilpotent, \( K_2^{-1}H_2 \) is nilpotent, and for \( i \geq 3 \), \( H_i \) and \( K_i \) are non-singular, and \( K_i^{-1}H_i \) is similar to \( -K_i^{-1}H_i \) and has eigenvalues only at \( \beta_i, \beta_i^*, -\beta_i \) and \( -\beta_i^* \), and if \( f_i(t) \) is the minimum polynomial of \( K_i^{-1}H_i \) for \( i \geq 3 \), then g.c.d.

\[
(f_i(t), f_j(t)) = 1 \text{ for } i \neq j.
\]

Then Proposition II.2.3 asserts that \( S \) is of type 1 iff each of the \( S_i \) is of type 1. Therefore, without loss
of generality, we consider the following cases separately:

(i) $K^{-1}H$ has eigenvalue only at $\beta$ and $-\beta$ where $\beta^* = -\beta$, and $K^{-1}H$ is similar to $-K^{-1}H$.

(ii) $K^{-1}H$ has eigenvalues only at $\beta$, $\beta^*$, $-\beta$ and $-\beta^*$, $\beta^* \neq \pm \beta$, and $K^{-1}H$ is similar to $-K^{-1}H$.

(iii) $K^{-1}H$ has eigenvalues only at $\beta$, $-\beta$ where $\beta \in \mathcal{E}$, and $K^{-1}H$ is similar to $-K^{-1}H$.

(iv) $H^{-1}K$ is nilpotent.

(v) $K^{-1}H$ is nilpotent.

Theorem II.3.2. If $K^{-1}H$ is similar to $-K^{-1}H$, and $K^{-1}H$ has eigenvalues only at $\beta$, $\beta^*$, $-\beta$, $-\beta^*$ where $\beta \notin \mathcal{E}$, then $S$ is of type 2.

Proof: First consider $\beta^* = -\beta$. By Fact 5, in the introductory chapter, $S$ is $*$-congruent to a block diagonal matrix whose diagonal blocks are of the form:
Icrote:

-13+j

1

become either

1

or

1

-1j

(R+j)* = j =

(-13+j)* = R - j =

1

[(\pi.3.1)]

\[
\begin{pmatrix}
1 & \beta+j \\
-\beta+j & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & \beta+j \\
-\beta+j & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 1 \\
-\beta+j & -\beta+j
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & \beta+j \\
-\beta+j & 1
\end{pmatrix}
\]

Note: \((\beta+j)^* = -\beta - j = -(\beta+j)\)

\((-\beta+j)^* = \beta - j = -(-\beta+j)\)

Therefore \(j(\beta+j)\) and \(j(-\beta+j)\) are elements in \(E\). We shall
show that each block is \(*\)-congruent to a matrix over \(E\). Therefore
without loss of generality, we may assume \(S = [\pi.3.1]\). Multiply
by \(C = \text{diag}(1, j, 1, j, \ldots)\) to the right and by \(C^*\) on the left of \(S\);
then by Fact 2, the entries on the first super-anti-diagonal of \(S\)
become either 1 or \(-j^2\) or (in case of the middle entry) 0, and
the entries on the anti-diagonal are multiplied alternately by \(j\) and
\(-j\). Thus \(S\) is of type 2.

Next consider \(\beta^* \neq \pm \beta\). By Fact 4, \(S\) is \(*\)-congruent to a
matrix with diagonal blocks of the form:
\[\begin{array}{cccc}
1 & \beta+j & & \\
\ldots & \beta+j & & \\
\beta+j & & & \\
1 & \beta^*+j & & \\
\ldots & \beta^*+j & & \\
\beta^*+j & & & \\
\end{array}\]

\[\begin{array}{cccc}
1 & & & \\
1 & -\beta+j & & \\
\ldots & -\beta+j & & \\
1 & -\beta^*+j & & \\
\ldots & -\beta^*+j & & \\
-\beta^*+j & & & \\
\end{array}\]

\[= M \otimes N,\]
where $M$ is the upper left $2m \times 2m$ block and $N$ is the lower right $2m \times 2m$ block. Let $C = \text{diag}(1, j, 1, \ldots, j) \oplus \text{diag}(1, j, 1, \ldots, j)$.

Consider $C^*(M \oplus N)C = P \oplus Q$. $p_{ij} = q_{ij} = 0$ for $i+j \neq 2m+1$ and $\neq 2m$, and also $p_{m,m} = q_{m,m} = 0$ and

\[
P_{i, 2m-i} = q_{i, 2m-i} = (-1)^2 \quad \text{when} \quad i \quad \text{is even}
\]
\[
1 \quad \text{when} \quad i \quad \text{is odd}.
\]

\[
P_{i, 2m+1-i} = (-1)^{i-1}j(\beta^*+j) = (-1)^{i-1}[j\beta^*+2j^2] \quad i \leq m
\]
\[
= (-1)^{i-1}j(\beta+j) = (-1)^{i-1}[j\beta+j^2] \quad i > m.
\]

\[
q_{i, 2m+1-i} = (-1)^{i-1}(-j\beta+j^2) \quad i \leq m
\]
\[
= (-1)^{i-1}(-j\beta^*+j^2) \quad i > m.
\]

\[Q = \overline{P}.\] Consider

\[
\begin{bmatrix}
I & I
\end{bmatrix}
\begin{bmatrix}
P & 0 \\
0 & \overline{P}
\end{bmatrix}
\begin{bmatrix}
I & P
\end{bmatrix}
= \begin{bmatrix}
I & I
\end{bmatrix}
\begin{bmatrix}
P & P^2 \\
P^*P & \overline{P}
\end{bmatrix}
\begin{bmatrix}
P^*P & \overline{P}
\end{bmatrix}
\begin{bmatrix}
P & P^2 \\
P^*P & \overline{P}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
P+\overline{P} & P^2+\overline{P}^2 \\
P^*P+P^*\overline{P} & P^*P^2+P^*\overline{P}^2
\end{bmatrix},
\]

which is over $\mathcal{E}$. Note that
\[ \det \begin{bmatrix} I & P \\ I & \bar{P} \end{bmatrix} = \det (\bar{P} - P) \]
\[ = \pm j^{2m} (\beta + \beta^*)^{2m} \neq 0, \]

because \( \pm j(\beta + \beta^*) \neq 0 \). Therefore \( M \oplus N \) is of type 2. Thus \( S \) is of type 2.

From this point let \( B_i \) and \( D_i \) denote non-singular diagonal matrices over \( \mathbb{C} \).

**Theorem II.3.3.** (i) If \( H^{-1}K \) is nilpotent, then there is a basis for \( U \) such that \( S = \bigoplus \limits_{i=1}^{h} T_i^0 \) where \( T_i^0 = (E_{n_i} + jF_i) \otimes B_i \), and \( n_i \neq n_j \) for \( i \neq j \).

(ii) If \( K^{-1}H \) is nilpotent, then there is a basis for \( U \) such that \( S = \bigoplus \limits_{i=1}^{h} T_i^\infty \) where \( T_i^\infty = (jE_{n_i} + F_i) \otimes B_i \), and \( n_i \neq n_j \) for \( i \neq j \).

(iii) If \( K^{-1}H \) is similar to \( -K^{-1}H \) and has eigenvalues only at \( \beta \), and \(-\beta\), \( \beta \neq 0 \), \( \beta \in \mathbb{C} \), then there exists a basis for \( U \) such that \( S = \bigoplus \limits_{i=1}^{h} T_i^\beta \), where \( T_i^\beta = F_{n_i} \otimes (B_i \oplus D_i) + E_{n_i} \otimes [(\beta + j)B_i \oplus (-\beta + j)D_i] \), and \( n_i \neq n_j \) for \( i \neq j \).

(Pictures of blocks of \( T_i^z \), \( z = 0, \infty, \beta \) where \( \beta \neq 0 \), \( \beta \in \mathbb{C} \) are in Appendix I).

**Proof:** (i) By Fact 4, \( S \) is \(*\)-congruent to a matrix in diagonal blocks with each block of the form:

\[ W_{\epsilon, n}^0 = \epsilon (E_{n} + jF_n) \; \epsilon \neq n, \; \epsilon \in \mathbb{C}. \]
Thus

\[ S = \bigoplus_{j=1}^{k} \bigoplus_{i=1}^{n_j} \varepsilon_{i}^{j}, n_j \]

where \( n_j \neq n_i \) if \( i \neq j \). Suppose \( S = \bigoplus_{i=1}^{n_1} \varepsilon_{i}^{j}, n_1 \). Write \( n = n_1 \), \( B = H^{-1}K \), and \( k = k_1 \). Then

\{e_1, Be_1, \ldots, B^{n-1}e_1, e_2, Be_2, \ldots, B^{n-1}e_2, \ldots, e_k, \ldots, B^{n-1}e_k\} \]

is a basis for \( \mathcal{U} \) for some \( e_1, \ldots, e_k \in \mathcal{U} \). Rearrange the basis elements in the following manner:

\{e_1, e_2, \ldots, e_k, Be_1, Be_2, \ldots, Be_k, \ldots, B^{n-1}e_1, \ldots, B^{n-1}e_k\} \]

With respect to this new basis \( S \) becomes \( T_1^0 = (E_{n_1}^{j} + jF_{n_1}) \otimes B_1 \)

where \( B_1 = \text{diag}(\varepsilon_1, \ldots, \varepsilon_k) \). Repeat the above process for

\( j = 2, 3, \ldots, h \). Thus if \( S = \bigoplus_{j=1}^{h} \bigoplus_{i=1}^{n_j} \varepsilon_{i}^{j}, n_j \), \( S \) is \( \ast \)-congruent to \( T_1^0 \).

If \( K^{-1}H \) is nilpotent, by Fact 4 \( S \) is \( \ast \)-congruent to a matrix in diagonal blocks with each block of the form:

\[ W_{\varepsilon, n}^\infty = \varepsilon(jE_{n} + F_{n}) \quad \varepsilon \neq 0, \quad \varepsilon \in \mathcal{E}. \]

Going through a similar process as in the case \( H^{-1}K \) being nilpotent, we can show \( S \) is \( \ast \)-congruent to \( \bigoplus_{j=1}^{h} T_j^0 \).

(ii) When \( K^{-1}H \) has eigenvalues only at \( \pm \beta \neq 0, \quad \beta \neq 0 \), by Fact 4, we have \( S \) \( \ast \)-congruent to a direct sum of blocks with these blocks paired off in the form \( U_{\varepsilon, n}^\beta \otimes U_{\varepsilon', n}^\beta \), where
\[
U_{\epsilon, n} = \epsilon(F_n + (\beta + j)E_n), \quad \epsilon \neq 0, \quad \epsilon \in \mathcal{E}.
\]
\[
V_{\delta, n} = \delta(F_n + (-\beta + j)E_n), \quad \delta \neq 0, \quad \delta \in \mathcal{E}.
\]

Without loss of generality, assume

\[
\text{[II.3.2]} \quad S = \bigoplus_{i=1}^{h} \bigoplus_{j=1}^{k_i} U_{\epsilon_i, n_i} \oplus \bigoplus_{i=1}^{k} V_{\delta_i, n_i}.
\]

First consider

\[
\text{[II.3.3]} \quad S = \bigoplus_{i=1}^{k_1} \left( U_{\epsilon_i, n_i} \right)^{\beta} \oplus \bigoplus_{i=1}^{k} \left( V_{\delta_i, n_i} \right)^{\beta}.
\]

Write \( n = n_1, \quad k_1 = k \). Let \( A_0 = (K^{-1}H^{-3}I), \quad A_1 = (K^{-1}H+\beta I) \). A basis of \( \mathcal{U} \) for which \( S \) is of the form \([\text{II.3.3}]\) is of the form:

\[
\{ e_1, A_0 e_1, \ldots, A_0^{n-1} e_1, e_2, \ldots, A_0 e_2, \ldots, A_1 e_1, d_1, A_1 d_1, \ldots, A_1^{n-1} d_1, \ldots, A_1 d_k, A_1^{n-1} d_k \}. \]

Rearrange this basis in the following manner

\[
\{ e_1, e_2, \ldots, e_k, d_1, d_2, \ldots, d_k, A_0 e_1, \ldots, A_0 e_k, A_1 d_1, \ldots, A_1 d_k, A_0 e_1, \ldots, A_0 e_k, A_1 d_1, \ldots, A_1 d_k \}. \]

Then in this new basis \( S \) is of the form

\[
T_{1}^{\beta} = F_n \otimes (B_1 \otimes D_1) + E_n \otimes [(\beta + j)B_1 \otimes (-\beta + j)D_1] \quad \text{where}
\]
\[
B_1 = \text{diag}(\epsilon_1, \epsilon_2, \ldots, \epsilon_k), \quad D_1 = \text{dial}(\delta_1, \ldots, \delta_k) \quad \epsilon_i, \delta_i \in \mathcal{E}. \]

Thus if \( S \) is of \([\text{II.3.2}]\) \( S \) is *-congruent to \( \bigoplus_{j=1}^{k} T_{j}^{\beta} \). q.e.d.

In the following theorem we let \( T_{1}^{z} = T_{1}^{z}, \) where \( z = \infty, 0, \beta \).
and where $\beta \neq 0$, $\beta \in \mathbb{C}$; we also write $n = n_1$, $k = k_1$, $B = B_1$, $E = E_n$, and $F = F_n$.

**Theorem II.3.4.** (i) (a) For $n$ odd, $T^0$ is of type 2 and hence of type 1. For $n$ even, (b) $T^0$ is of type 1 iff $B$ is $*$-congruent to $-B$ (i.e., iff $jB$ is of type 1), and (c) $T^0$ is of type 2 iff $jB$ is of type 2.

(ii) For $n$ even, $T^\infty$ is of type 2 and hence of type 1. For $n$ odd, $T^\infty$ is of type 1 iff $B$ is $*$-congruent to $-B$ (i.e., iff $jB$ is of type 1), and $T^\infty$ is of type 2 iff $jB$ is of type 2.

**Proof:** Suppose $n$ is odd. Let $C = \text{diag}(1, j, 1, \ldots, j_1)$.

Recall $T^0 = (E+jF) \otimes B$. Then

\[
(C \otimes I_k)^* T^0 (C \otimes I_k) = (C \otimes I_k)^* ((E+jF) \otimes B) (C \otimes I_k) = C^* (E+jF) C \otimes B
\]

By Fact 2, the entries on the anti-diagonal of $C^* (E+jF) C$ are alternately $-j^2$ or 1, those on the first super anti-diagonal are alternately $\pm j^2 = \pm j^2$, the rest of the entries of $C^* (E+jF) C^*$ are zeroes. Thus $T^0$ is of type 2, and hence is of type 1.

(b) Suppose $n$ is even, and suppose $B$ is $*$-congruent to $-B$. Suppose $P$ is non-singular such that $P^*BP = -B$. Let $C = \text{diag}(1, -1, 1, \ldots)$; then
\[(C \otimes P)^* [(E+jF) \otimes B](C \otimes P) = C^*(E+jF)C \otimes P^*BP\]
\[= (-E+jF) \otimes -B\]
\[= (E-jF) \otimes B.\]

Conversely suppose \(C_1\) is non-singular such that \(C_1^*T_0C_1 = (T_0)^*\).

Let \(H = \frac{1}{2}(T_0^0 + (T_0)^*), \) and \(K = \frac{1}{2j}(T_0^0 - (T_0)^*). \) Then
\[C_1^*H(H^{-1}K)^{n-1}C_1 = -H(H^{-1}K)^{n-1}.\]
\[H(H^{-1}K)^{n-1} = [E \otimes B][E F \otimes I_k]^{n-1} = B \oplus O.\]

This computation will be used again in the proof of part (c). Thus
\[C_1^*(B \oplus O)C_1 = -B \oplus O.\] Thus by Fact 3, \(B\) is \(*\)-congruent to \(-B\).

(c) Now suppose \(n\) is even and \(jB\) is of type 2, i.e., there exists a non-singular \(C_0\) such that \(jC_0^*B C_0\) is over \(E\). Let \(C_2 = \text{diag}(1, j, \ldots, j)\). Then
\[(C_2 \otimes C_0)^* T_0^0 (C_2 \otimes C_0) = [C_2^* \otimes C_0^*][(E+jF) \otimes B](C_2 \otimes C_0)\]
\[= C_2^*(E+jF)C_2 \otimes C_0^*BC_0.\]

The anti-diagonal entries of \(C_2^*(E+jF)C_2\) are alternately \(+j\) and \(-j\); the first super-anti-diagonal entries are alternately \(-j^3\) and \(j\). Thus \(j^{-1}C_2^*(E+jF)C_2\) is over \(E\). Thus (if we multiply the first factor by \(j^{-1}\) and the second factor by \(j\) in the tensor product) we get that \([II.3.4] = j^{-1}C_2^*(E+jF)C_2 \otimes jC_0^*BC_0\) is over \(E\). Thus
\(T^0\) is of type 2.

Conversely suppose \(C_3^*T^0C_3 = W\) is over \(\mathcal{C}\). Let
\[H_1 = \frac{1}{2}(W+W^*), \quad K_1 = \frac{1}{2j}(W-W^*)\] then \(W = H_1 + jK_1\).
\[
W^* = H_1 - jK_1 = H_1' + jK_1'.
\]
Thus \(K_1' = -K_1\). Thus
\[(jK_1)^* = -jK_1 = jK_1',\]
so \(jK_1\) is over \(\mathcal{C}\). Likewise \(H_1 = H_1^* = H_1'\) is over \(\mathcal{C}\). Since \(C_3\) is non-singular, there exists a permutation matrix \(R\) such that \(C_3R\) has its first \(k \times k\) principal sub-matrix, \(D\), non-singular. Thus
\[
jR^*C_3^*(H^{-1}K^{n-1})C_3R = jR^*H_1^{-1}K_1^{-1}R.
\]
Since \(R\) is a permutation matrix, \(jR^*H_1^{-1}K_1^{n-1}R\) is a matrix over \(\mathcal{C}\). Recall \(H^{-1}K^{n-1} = B \oplus O\) and the first principal sub-matrix \(D\) of \(C_3R\) is non-singular. Hence \(jD^*BD\) is the upper left block of \(j(C_3R)^*(B \oplus O)C_3R = j(C_3R)^*(H^{-1}K^{n-1})C_3R\) which is over \(\mathcal{C}\). Thus \(jB\) is of type 2.

(ii) The proof of (ii) is similar to that of (i). q.e.d.

For the following discussion, write \(T^\beta\) as \(T\). Recall that \(\beta \neq 0\), and \(\beta \in \mathcal{C}\), \(n_2 = n\). (Refer to picture in Appendix I.)

**Theorem II. 3. 5.** The following are equivalent.

(1) \(T\) is of type 1.

(2) \(T\) is of type 2.
(3) \( B \) is \( \ast \)-congruent to \((-1)^nD\).

**Lemma 5.** If \( c \in \mathcal{C} \) such that \( c^* \neq \pm c \), \( C = cI \), \( \varepsilon \in \mathcal{C} \),

\[
M = \begin{pmatrix} C & C^* \\ C^* & C \end{pmatrix},
\]

then

\[
M^* \begin{pmatrix} A & 0 \\ 0 & \varepsilon A^* \end{pmatrix} M = \begin{pmatrix} cc^*(A+\varepsilon A^*) & (c^*)^2A+\varepsilon c^2A \\ c^2A+\varepsilon(c^*)^2A^* & cc^*(A+\varepsilon A^*) \end{pmatrix}
\]

The proof of Lemma 5 is routine computation.

**Lemma 6.** Suppose \( C, B, D \) are non-singular. If

\[
C^*(B \oplus (-1)^nD)C = -(B \oplus (-1)^nD)
\]

and

\[
C^*(B \oplus (-1)^{n-1}D)C = B \oplus (-1)^{n-1}D
\]

then \( B \) is \( \ast \)-congruent to \((-1)^{n-1}D\).

**Proof of Lemma 6.** Partition \( C \) conformably:

\[
C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}; \quad \text{then}
\]

\[
C^* \begin{pmatrix} B & 0 \\ 0 & \varepsilon D \end{pmatrix} C = \begin{pmatrix} c_{11}BC_{11} + \varepsilon c_{12}DC_{21} & c_{11}BC_{12} + \varepsilon c_{12}DC_{22} \\ c_{12}BC_{11} + \varepsilon c_{22}DC_{21} & c_{12}BC_{12} + \varepsilon c_{22}DC_{22} \end{pmatrix}
\]

where \( \varepsilon = \pm 1 \).

Referring to the first condition, we have

\[
c_{11}BC_{11} + (-1)^n c_{21}DC_{21} = -B.
\]
Referring to the second condition we have
\[ C_{11}^* B_{11} + (-1)^{n-1} C_{21}^* D_{21} = B. \]

If we add the above two equations together, we get
\[ 2C_{11}^* B_{11} = 0. \]
Thus \((-1)^{n-1} C_{21}^* D_{21} = B.\) Thus \(B\) is \(^*\)-congruent to \((-1)^{n-1} D.\)

**Lemma 7.** If \(T\) is as in Theorem II.3.5, and \(H = \frac{1}{2}(T+T^*),\)
\[ K = \frac{1}{2j}(T-T^*), \]
then
\[
K[(K^{-1}H)^2 - \beta^2 I]^{n-1} = (2\beta)^{n-1}(B \oplus (-1)^{n-1} D) \oplus O \\
H[(K^{-1}H)^2 - \beta^2 I]^{n-1} = (2\beta)^{n-1}(B \oplus (-1)^{n} D) \oplus O
\]

**Proof of Lemma 7.** Recall
\[ T = F \otimes (B \oplus D) + E \otimes [(\beta+j)B \oplus (-\beta+j)D] \text{ and } \beta \in \mathbb{C}. \](Refer to picture.)
\[ H = F \otimes (B \oplus D) + E \otimes (\beta B \oplus -\beta D) \]
\[ K = E \otimes (B \oplus D), \]
so
\[ K^{-1} = E \otimes (B^{-1} \oplus D^{-1}). \]

Let \(k = \text{dim } B.\)
\[ K^{-1}H = E \otimes I_{2k} + I_n \otimes (\beta I_k \oplus -\beta I_k). \]
\[ N = EF \text{ is the } n \times n \text{ matrix with } 1's \text{ on the first sub-diagonal} \]
and zero everywhere else.

\[
K^{-1}H - \beta I = N \otimes I_{2k} + I_n \otimes (O_k \oplus -2\beta I_k),
\]

\[
K^{-1}H + \beta I = N \otimes I_{2k} + I_n \otimes (2\beta I_k \oplus O_k),
\]

so

\[
(K^{-1}H - \beta I)(K^{-1}H + \beta I)
\]

\[
= N^2 \otimes I_{2k} + N \otimes (2\beta I_k \oplus -2\beta I_k)
\]

\[
= (N \otimes I_{2k})(N \otimes I_{2k} + I \otimes (2\beta I_k \oplus -2\beta I_k)).
\]

Thus

\[
[(K^{-1}H)^2 - \beta^2 I]^{n-1} = (N^{n-1} \otimes I_{2k})(N \otimes I_{2k} + I \otimes (2\beta I_k \oplus -2\beta I_k))^{n-1}.
\]

\[
M = (N \otimes I_{2k} + I \otimes (2\beta I_k \oplus -2\beta I_k))^{n-1}
\]

is a \(2kn \times 2kn\) block lower triangular matrix with the diagonal blocks

\[
= (2\beta)^{n-1}I_k \oplus (-2\beta)^{n-1}I_k.
\]

Thus

\[
K[(K^{-1}H)^2 - \beta^2 I]^{n-1} = [E \otimes (B \oplus D)][N^{n-1} \otimes I_{2k}]M
\]

\[
= ((2\beta)^{n-1}B \oplus (-2\beta)^{n-1}D) \oplus O,
\]

\[
= (2\beta)^{n-1}(B \oplus (-1)^{n-1}D) \oplus O.
\]

Similarly,

\[
H[K^{-1}H]^2 - \beta^2 I]^{n-1} = (2\beta)^{n-1}B \oplus (2\beta)^{n-1}(-\beta)D \oplus O
\]

\[
= (2\beta)^{n-1}\beta(B \oplus (-1)^{n}D) \oplus O.
\]

```
Proof of Theorem 11.3.5. We shall prove (3) => (2) => (1) => (3).

(3) => (2). Suppose $Q$ is non-singular $k \times k$ such that $Q^*DQ = (-1)^n B$. Let $C = I_n \otimes (I_k \oplus Q)$. Let $P = C*TC$. Then

$$P = [I_n \otimes (I_k \oplus Q)]*[F \otimes (B \oplus D) + E \otimes [(\beta+j)B \oplus (-\beta+j)D]] [I_n \otimes (I_k \oplus Q)]$$

$$= F \otimes (B \oplus Q*DQ) + E \otimes [(\beta+j)B \oplus (-\beta+j)Q*DQ]$$

$$= F \otimes (B \oplus (-1)^n B) + E \otimes [(\beta+j)B \oplus (-\beta+j)(-1)^n B].$$

Let $c \in \mathcal{J}$ such that $c^* \neq \pm c$, and

$$M = \begin{bmatrix} cI_k & c*I_k \\ c*I_k & cI_k \end{bmatrix}$$

By Lemma 5

$$R_0 = M*(B \oplus (-1)^n B)M = \begin{bmatrix} cc*(B+(-1)^n B) & (c*)^2 B + (-1)^n \ c^2 B \\ c^2 B + (-1)^n (c*)^2 B & cc*(B+(-1)^n B) \end{bmatrix}.$$ 

Let $\mu = \beta+j$, $\mu^* = \beta-j$. Thus $-\beta+j = -\mu^*$. Then again by Lemma 5,

$$R_1 = M*(\mu B \oplus (-1)^{n-1} \mu^* B)M$$

$$= \begin{bmatrix} cc*(\mu B+(-1)^{n-1} \mu^* B) & (c*)^2 \mu B + (-1)^{n-1} \ c^2 \mu^* B \\ c^2 \mu B + (-1)^{n-1} (c*)^2 \mu^* B & cc*(B+(-1)^{n-1} B) \end{bmatrix}.$$ 

Note
\[(I_n \otimes M^*)P(I_n \otimes M)\]
\[= (I_n \otimes M^*)(F \otimes (B \oplus (-1)^n B) + E \otimes (\mu B + (-1)^{n-1} \mu B))(I_n \otimes M)\]
\[= F \otimes (M^*(B \oplus (-1)^n B))M + E \otimes M^*(\mu B + (-1)^{n-1} \mu B)M\]
\[= F \otimes R_0 + E \otimes R_1.\]

Note that if \(n\) is even, \(R_0\) and \(jR_1\) are matrices over \(\mathcal{E}\), if \(n\) is odd, \(jR_0\) and \(R_1\) are matrices over \(\mathcal{E}\). Thus let

\[U = \text{diag}(1, j, 1, j, \ldots).\]

Then

\[(U \oplus I_{2k})(F \otimes R_0 + E \otimes R_1)(U \oplus I_{2k}) = U^*FU \otimes R_0 + U^*EU \otimes R_1,\]

by Fact 2, is a matrix over \(\mathcal{E}\). This \(T\) is of type 2.

Proposition 2.1 (iv) asserts that \((2) \Rightarrow (1)\).

\((1) \Rightarrow (3).\) Suppose \(C^*TC = T^*.\) Recall \(H = \frac{1}{2} (T + T^*),\)
\[K = \frac{1}{2j} (T - T^*),\]
\[C^*K[(K^{-1}H)^2 - \beta^2 I]^{-1}C = -K[(K^{-1}H)^2 - \beta^2 I]^{-1},\]
\[C^*H[(K^{-1}H)^2 - \beta^2 I]^{-1}C = H[(K^{-1}H)^2 - \beta^2 I]^{-1}.\]

Thus
\[C^*(H + jK)[(K^{-1}H)^2 - \beta^2 I]^{-1}C = (H - jK)[(K^{-1}H)^2 - \beta^2 I]^{-1}.\]

Lemma 7 asserts that
\[H[(K^{-1}H)^2 - \beta^2 I]^{-1} = (2\beta)^{-1}\beta(B \oplus (-1)^n D) \oplus O\]
\[K[(K^{-1}H)^2 - \beta^2 I]^{-1} = (2\beta)^{-1}(B \oplus (-1)^{n-1} D) \oplus O.\]
\[ C^*(H+jK)((K^{-1}H)^2-B^2I)^{n-1}C = C^*((2\beta)^{n-1}\beta(B \oplus (-1)^nD) \oplus O)C \]
\[ + j(2\beta)^{n-1}(B \oplus (-1)^nD) \oplus O]C \]
\[ = [(2\beta)^{n-1}\beta(B \oplus (-1)^nD) \]
\[ - j(2\beta)^{n-1}(B \oplus (-1)^nD) \oplus O \]

By Fact 3, there exists \( C_{11} \) non-singular, such that
\[ C_{11}^*(B \oplus (-1)^nD)C_{11} = B \oplus (-1)^nD \]
\[ C_{11}^*(B \oplus (-1)^{n-1}D)C_{11} = B \oplus (-1)^{n-1}D \]

Thus by Lemma 6, \( B \) is \( \ast \)-congruent to \((-1)^nD\). q.e.d.

**Theorem II.3.6.** (i) If \( S = \bigoplus_{i=1}^{h} T_i^z \), with \( n_i \neq n_j \) for \( i \neq j \), where \( z = \infty \) or 0, then \( S \) is of type 1 iff each \( T_i^z \) is of type 1.

(ii) Let \( H = \frac{1}{2}(S_z + S_z^*) \), \( K = \frac{1}{2j}(S_z - S_z^*) \). Then if \( z = 0 \), \( S_z \) is of type 1 iff \( H(H^{-1}K)^{2m+1} \) is \( \ast \)-congruent to \(-H(H^{-1}K)^{2m+1} \) for \( m = 0, 1, \ldots \); if \( z = \infty \), \( S_z \) is of type 1 iff \( K(K^{-1}H)^{2m} \) is \( \ast \)-congruent to \(-K(K^{-1}H)^{2m} \).

**Proof:** The "if" parts of both (i) and (ii) are trivial. To prove the "only if" part of (i) for \( z = 0 \) it is enough to prove \( T_i^0 \) is of type 1 for \( n_i \) even (see Theorem II.3.4(i)). Assume \( n_i > n_h \) for
i < h. Write $S_0$ as $S$. If $n_1$ is odd then $T_1^0$ is of type 1.

If $n_1$ is even, then $S$ is of type 1 implies $H(H^{-1}K)^{n_1-1}$ is *-congruent to $-H(H^{-1}K)^{n_1-1}$. Also $H(H^{-1}K)^{n_1-1} = B_1 \oplus O$, so by Fact 3 $B_1$ is *-congruent to $-B_1$. Hence by Theorem II.3.4(i) $T_1^0$ is of type 1 also if $n_1$ is even. Let $H_i = \frac{1}{2} (T_i^0 (T_i^0)^*)$, $K_i = \frac{1}{2} j (T_i^0 (T_i^0)^*)$; then

$$H(H^{-1}K)^{n_2-1} = H_1(H_1^{-1}K_1)^{n_2-1} \oplus H_2(H_2^{-1}K_2)^{n_2-1} \oplus O.$$ 

Also

$$H_2(H_2^{-1}K_2)^{n_2-1} = B_2 \oplus O,$$

and (if $n_2$ is even then) $H(H^{-1}K)^{n_2-1}$ is *-congruent to $-H(H^{-1}K)^{n_2-1}$. Thus by Fact 3 $H_1(H_1^{-1}K_1)^{n_2-1} \oplus B_2$ is *-congruent to $-H_1(H_1^{-1}K_1)^{n_2-1} \oplus -B_2$. Since $H_1(H_1^{-1}K_1)^{n_2-1}$ is *-congruent to $-H_1(H_1^{-1}K_1)^{n_2-1}$, then $B_2$ is *-congruent $-B_2$.

Thus (if $n_2$ is even and hence in any case) $T_2^0$ is of type 1. If we repeat the same process by considering in like manner

$$H(H^{-1}K)^{n_i-1} \quad \text{for} \quad i = 3, 4, \ldots, h, \quad \text{then we can show each} \quad T_i^0 \quad \text{is of type 1. Thus we have shown for} \quad i = 1, \ldots, h, \quad \text{that} \quad T_i^0 \quad \text{is of type 1 if} \quad S_0 \quad \text{is of type 1. Thus we have proved (ii) for} \quad z = 0.$$

The proof for $z = \infty$ for both (i) and (ii) are similar to the above discussion. In this case we would only replace "even" by "odd" (and vice versa) and $T_i^0$ by $T_i^\infty$ and $H(H^{-1}K)^{n_i-1}$ by
Theorem II.3.7. If \( S = \bigoplus_{i=1}^{h} T_{i}^{\beta}, \infty \neq \beta \neq 0, \beta \in \mathcal{E} \), then the following are equivalent:

1. \( S \) is of type 1;
2. \( S \) is of type 2;
3. \( T_{1}^{\beta} \) is of type 1 for each \( i \);
4. \( T_{1}^{\beta} \) is of type 2 for each \( i \);

Proof: Theorem II.3.5 asserts (3) \(\iff\) (4) and Proposition II.2.1 asserts (2) \(\implies\) (1). Thus the following implications are obvious:

\[
(2) \implies (1) \\
(4) \implies (3)
\]

Thus to complete the proof we have only to show \((1) \implies (3)\).

Let \( H = \frac{1}{2} (S+S^*) \), \( K = \frac{1}{2} (S-S^*) \), \( H_{i} = \frac{1}{2} (T_{i}^{\beta}+(T_{1}^{\beta})^*) \), \( K_{i} = \frac{1}{2} (T_{i}^{\beta}-(T_{1}^{\beta})^*) \). Suppose \( C^*SC = S^* \) where \( C \) is non-singular.

Let \( P_{i} = K[ (K^{-1}H)_{1}^{2} - \beta_{1}^{2} I_{1}^{n_{i-1}} ] \), \( Q_{i} = H[ (K^{-1}H)_{1}^{2} - \beta_{1}^{2} I_{1}^{n_{i-1}} ] \); then

\( C^*P_{i}C = -P_{i} \), and \( C^*Q_{i}C = Q_{i} \). Without loss of generality, assume \( n_{i} > n_{h} \) for \( i < h \). From Lemma 7 \( P_{1} = (2\beta) (B_{1} \oplus (-1) D_{1}) \oplus Q_{1} \), and \( Q_{1} = (2\beta) (B_{1} \oplus (1) D_{1}) \oplus Q_{1} \). Apply Fact 3 to \( Q_{1} + jP_{1} \) and \( Q_{1} - jP_{1} = C^*(Q_{1}+jP_{1})C \); there exists a non-singular \( C_{1} \) such that
and
\[
C_1^*(B_1 \oplus (-1)^{n_1-1} D_1)C_1 = -(B_1 \oplus (-1)^{n_1-1} D_1).
\]

Then by Lemma 6, \( B_1 \) is \(*\)-congruent to \((-1)^{n_1} D_1\). Thus \( T_1^\beta \) is of type 1 (see Theorem II.3.5). Now,

\[
P_2 = K_1[(K_1^{-1} H_1)^2 - \beta^2 I] \oplus (2\beta)^{n_2-1} (B_2 \oplus (-1)^{n_2-1} D_2) \oplus O,
\]

\[
Q_2 = H_1[(K_1^{-1} H_1)^2 - \beta^2 I] \oplus (2\beta)^{n_2-1} \beta(B_2 \oplus (-1)^{n_2-1} D_2) \oplus O,
\]

and

\[
C*(\beta P_2 + Q_2)C = -\beta P_2 + Q_2.
\]

Applying Fact 3 to this last \(*\)-congruency, we get that

\[
\beta K_1[(K_1^{-1} H_1)^2 - \beta^2 I]^{n_2-1} + H_1[(K_1^{-1} H_1)^2 - \beta^2 I]^{n_2-1} \oplus (2\beta)^{n_2} (B_2 \oplus O)
\]

is \(*\)-congruent to

\[
-\beta K_1[(K_1^{-1} H_1)^2 - \beta^2 I]^{n_2-1} + H_1[(K_1^{-1} H_1)^2 - \beta^2 I]^{n_2-1} \oplus (2\beta)^{n_2} (O \oplus (-1)^{n_2} D_2).
\]

Also, since \( T_1^\beta \) is of type 1, we have that

\[
\beta K_1[(K_1^{-1} H_1)^2 - \beta^2 I]^{n_2-1} + H_1[(K_1^{-1} H_1)^2 - \beta^2 I]^{n_2-1}
\]
is $\ast$-congruent to

$$-\beta K_1 [(K_1^{-1}H_1)^2 - \beta^2 I]^{n_2 - 1} + H_1 [(K_1^{-1}H_1)^2 - \beta^2 I]^{n_2 - 1}. $$

Thus by Witt's Theorem, $B_2 \oplus \mathbb{O}$ is $\ast$-congruent to $\mathbb{O} \oplus (-1)^{n_2} D_2$, so $B_2$ is $\ast$-congruent to $(-1)^{n_2} D_2$. Hence $T_2^\beta$ is of type 1.

If we repeat the above process by considering $P_i$ and $Q_i$ for $i = 3, 4, \ldots, h$, then we can show $T_i^\beta$ is of type 1 for each $i$. 
III. THE USUAL COMPLEX CASE

1. General Results

In this section, we take \( \mathbb{F} \) to be the complex field and \( \mathbb{E} \) to be the real field and we consider the problem from the viewpoint of a criterion Dina Ng proved in her thesis [3].

**Proposition III.1.1.** If \( h(y) \) is a real polynomial of simple real roots \( \lambda_1, \lambda_2, \ldots, \lambda_n \) and

\[
\bigcirc = \{ [h^{(1)}(y)]^1, [h^{(2)}(y)]^2, \ldots, [h^{(n)}(y)]^n \mid j_i = 0, 1, 2 \}
\]

and \( n = \deg h(y) \), then there exist \( g_1, g_2, \ldots, g_n \in \bigcirc \) such that the \( n \times n \) matrix \( P(i,j) = (\text{sgn} \ g_j(\lambda_i)) \) is non-singular.

**Proof:** Refer to Dina Ng's thesis [3].

**Proposition III.1.2.** Suppose \( S \) is an \( n \times n \) matrix and \( K = \frac{1}{2} (S-S^*) \), \( H = \frac{1}{2} (S+S^*) \). Then

(i) if \( H^{-1}K \) is nilpotent, we have that \( S \) is of type 1 iff \( H(H^{-1}K)^{2m+1} \) is of signature 0 for \( m = 0, 1, 2, \ldots \);

(ii) if \( H \) and \( K \) are as in (i) and \( K^{-1}H \) is nilpotent, we have that \( S \) is of type 1 iff \( K(K^{-1}H)^{2m} \) has signature 0 for \( m = 0, 1, 2, \ldots \).
Proof: (i) From Theorem II.3.6(ii), $S$ is of type 1 iff
\[ H(H^{-1}K)^{2m+1} \text{ is } \ast\text{-congruent to } -H(H^{-1}K)^{2m+1} \] (for $m = 0, 1, 2, ...$).

In the usual complex case, this is equivalent to saying that
\[ H(H^{-1}K)^{2m+1} \text{ has signature } 0. \text{ q.e.d.} \]

(ii) The proof follows in a similar manner as that of (i).

Theorem III.1.3. Suppose $S$ is an $n \times n$ matrix,

\[ K = \frac{1}{2i}(S-S^*) \quad H = \frac{1}{2}(S+S^*) \quad H \text{ and } K \text{ are non-singular, and} \]

$K^{-1}H$ is similar to $-K^{-1}H$. If $p(x)$ is the characteristic polynomial of $K^{-1}H$ and $f(x) = \frac{p(x)}{g. c. d.(p(x), p'(x))}$, then $f(x) = h(x^2)$ for some real polynomial $h(y)$. Let

\[ \mathcal{O} = \{[h(1)(y)]^j_1 h(2)(y)]^j_2 \ldots [h(r)(y)]^j_r \mid j_i = 0, 1, 2 \}, \]

where $r = \deg h(y)$. $S$ is of type 1 iff $K(f(K^{-1}H))^m g((K^{-1}H)^2)$ has signature 0 for $m = 0, 1, 2, ..., n-1$ (where $n$ is the order of $S$) and all $g \in \mathcal{O}$.

Proof: ("only if") If $S$ is of type 1 then $\text{sig } Kq(K^{-1}H) = 0$ for all even polynomials $q(x) \in \mathcal{E}[x]$.

("if") Suppose $K[f(K^{-1}H)^m g((K^{-1}H)^2)]$ has signature zero for $m = 0, 1, 2, ..., n-1$ and all $g \in \mathcal{O}$. Without loss of generality, we can assume $S = S_1 \oplus S_2$, where if $H_j = \frac{1}{2} (S_j + S_j^*)$ and

\[ K_j = \frac{1}{2i} (S_j - S_j^*) \quad j = 1, 2, \text{ such that all eigenvalues of } K_j^{-1}H_j \text{ are} \]
non-real, and all eigenvalues of \( K_1^{-1}H_1 \) are real. We have shown in Chapter II that \( S_2 \) is of type 1. Therefore without loss of generality, assume \( S = S_1 \). There exists a basis for \( V \) such that \( S \) of the form

\[
S = \bigoplus_{k=1}^{p} \bigoplus_{\ell=1}^{r} \beta_{k\ell}
\]

where

\[
T_{k\ell} = F_{n_k} \bigoplus (B_{k\ell} \oplus D_{k\ell}) + E_{n_k} \bigoplus [(\beta_{k\ell} + i)B_{k\ell} \oplus (-\beta_{k\ell} + i)D_{k\ell}],
\]

and \( B_{k\ell} \) and \( D_{k\ell} \) are real diagonal matrices, with entries \( \pm 1 \).

Assume \( n_i > n_h \) if \( i < h \). Note

\[
f(x) = \prod_{\ell=1}^{r} (x^2 - \beta_{\ell}^2), \quad \text{and} \quad h(y) = \prod_{m=1}^{r} (y - \beta_m^2)
\]

From our hypothesis,

\([III. 1.1]\)

\[
0 = \text{sig}(Kf(K_1^{-1}H) - \text{g}((K_1^{-1}H)^2))
\]

\[
= \text{sig}[K \prod_{m=1}^{r} [(K_1^{-1}H)^2 - \beta_m^2]] \text{g}((K_1^{-1}H)^2)
\]

Let

\[
K_{\ell} = \frac{1}{2i} (T_{1\ell} - (T_{1\ell})^*),
\]

\[
H_{\ell} = \frac{1}{2} (T_{1\ell} + (T_{1\ell})^*).
\]
Note \((K_{i}^{-1}H_{i})^2 = (\beta_{i}^2)^2I + M\) where \(M\) is strictly lower triangular.

Thus

\[
g((K_{i}^{-1}H_{i})^2) = g(\beta_{i}^2)I + M_1,
\]

\[
\prod_{m \neq i} ((K_{i}^{-1}H_{i})^2 - \beta_{m}^2 \mathbf{I}) = h'(\beta_{i}^2)I + M_2
\]

(because \(\prod_{m \neq i} (\beta_{i}^2 - \beta_{m}^2) = h'(\beta_{i}^2)\)), where \(M_1\) and \(M_2\) are strictly lower triangular.

\[
(K_{i}^{-1}H_{i})^2 - \beta_{i}^2 \mathbf{I} = N_{n_1} \otimes I_{2k}
\]

\[
M_j(N_{n_1} \otimes I_{2k})^{n_1-1} = 0 \quad \text{for} \quad j = 1, 2.
\]

Thus

\[
[\text{III. 1. 1}] = \text{sig} \bigoplus_{\ell=1}^{r} K_{\ell}[h'(\beta_{\ell}^2)I] n_1^{-1} g(\beta_{\ell}^2) (N_{n_1} \otimes I_{2k})^{n_1-1}
\]
Recall \( K_{\ell} = E_{n_1} \otimes (B_{\ell 1} \oplus (-1)^{n_1-1} D_{\ell 1}) \).

\[
[\text{III. 1. 1}] = \sum_{\ell=1}^{r} \text{sgn}[h'_{\ell}(\beta^2_{\ell})^n_{n_1-1} g(\beta^2_{\ell})(B_{\ell 1} \oplus (-1)^{n_1-1} D_{\ell 1})] \]

Choose \( g \) (successively) = \( g_1, g_2, \ldots, g_r \) such that

\[ P(j, \ell) = (\text{sgn } g_j(\beta^2_{\ell})) \text{ is a non-singular } r \times r \text{ matrix. (See Proposition III. 1. 1.) Then } Q(j, \ell) = P(j, \ell)(\text{sgn}(h'_{\ell}(\beta^2_{\ell}))^n_{n_1-1}) \text{ is also a non-singular } r \times r \text{ matrix. Thus } \text{sgn}(B_{\ell 1} \oplus (-1)^{n_1-1} D_{\ell 1}) = 0 \text{ for each } \ell. \]

Thus

\[ \text{sgn } B_{\ell 1} = -\text{sgn}(-1)^{n_1-1} D_{\ell 1} \]

\[ = \text{sgn}(-1)^{n_1-1} D_{\ell 1} \]

Thus \( B_{\ell 1} \) is \(*\)-congruent to \((-1)^{n_1} D_{\ell 1}\) for each \( \ell \). Thus \( T_{\ell 1}^{\beta_{\ell}} \) is of type 1 for each \( \ell \).

Write

\[ S_0 = \bigoplus_{\ell=1}^{r} T_{\ell 1}, \text{ and let } S_1 = \bigoplus_{j=2}^{p} \bigoplus_{\ell=1}^{r} T_{j\ell}, \]
\[ H_0 = \frac{1}{2} (S_0 + S_0^*), \quad K_0 = \frac{1}{2i} (S_0 - S_0^*) \]

\[ H_1 = \frac{1}{2} (S_1 + S_1^*), \quad K_1 = \frac{1}{2i} (S_1 - S_1^*) \]

Then

\[ 0 = \text{sig } K(f(K^{-1}H)^{n_2 - 1} g((K^{-1}H)^2)) \]

\[ = \text{sig } K_0(f(K_0^{-1}H_0)^{n_0 - 1} g((K_0^{-1}H_0)^2)) \]

\[ + \text{sig } K_1(f(K_1^{-1}H_1)^{n_2 - 1} g((K_1^{-1}H_1)^2)). \]

Because \( S_0 \) is of type 1 and \( f(x) \quad g(x^2) \) is an even polynomial, repeating the same process as above, by replacing \( K \) with \( K_1 \), \( H \) with \( H_1 \), \( n \) with \( n_2 \), we can show \( B_{l2} \) is \(*\)-congruent to \((-1)^{n_2} D_{l2}\), for each \( l \); thus \( T_{e_l} \) is of type 1 for each \( l \). The proof of the "if" part is completed by similarly considering the fact that

\[ 0 = \text{sig } K(f(K^{-1}H)^{n_k - 1} g([K^{-1}H]^2)) \]

for \( k = 3, \ldots, p \).

**Proposition III. 1.4.** In the usual complex case, \( S \) is of type 1 iff \( S \) is of type 2.

**Proof:** It is only necessary to prove that \( S \) is of type 1 implies \( S \) is of type 2.
Suppose $S$ is of type 1. Here all roots of $K^{-1}H$ are in $\mathcal{F}$, so, in view of Chapter II, where we have shown that when $K$ and $H$ are non-singular, type 1 is equivalent to type 2, it is only necessary to consider the cases $K^{-1}H$ nilpotent and $H^{-1}K$ nilpotent. Without loss of generality, assume $S = T_1^z$ where $z = 0$ or $\infty$. In view of Theorem II.3.4, it follows that showing $S$ of type 1 implies $S$ of type 2 is equivalent to showing that, if a diagonal matrix $B$ with entries $\pm 1$ is of signature 0, then $iB$ is of type 2. Suppose $B$ is a diagonal matrix with entries $\pm 1$ and of signature 0, then without loss of generality, write $B = I \oplus -I$.

Let $c \in \mathbb{C}$ such that $c^* \neq \pm c$. Let $M = (\begin{array}{cc} c & c^* \\ c^* & -c \end{array})$; then by Lemma 5 of Chapter II

$$iM^*BM = i \begin{bmatrix} 0 & [(c^*)^2 - (-c^2)]I \\ [c^2 - (c^*)^2]I & 0 \end{bmatrix}$$

Thus $iB$ is of type 2. q.e.d.

2. Geometrical Approach to 2 x 2 Usual Complex Case

In the $2 \times 2$ case, using geometric ideas from Ballantine's work on Positive Definite Matrices [1], we can characterize the problem geometrically.

Define $\Gamma(S) = \{X^*SX \mid X \in \mathbb{C}^{2 \times 1}\}$. We say $S$ is bidefinite iff $\Gamma(S)$ is a line; $S$ is contradefinite iff $\Gamma(S)$ is the whole
complex plane.

If $S$ is non-singular, let $\text{sgn det } S = e^{2i\beta} = \frac{\det S}{|\det S|}$. If $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, define $\gamma \geq 0$ such that

$$1 - \gamma^2 = \frac{ad - d\bar{a} - b\bar{b} - cc}{2|\det S|}$$

If $S$ is non-singular, $S$ is $*$-congruent to one of the following forms:

$$S_1 = e^{i\beta} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad S_2 = e^{i\beta} \begin{bmatrix} 1 & 0 \\ 2\gamma & 1 \end{bmatrix}$$

$S_1$ is bidefinite; $S_2$ is contradefinite if $\gamma > 1$.

If $S$ is singular and non-zero, $S$ is $*$-congruent to one of the following forms

$$S_3 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}, \quad S_4 = \begin{bmatrix} i\beta & 0 \\ 0 & 0 \end{bmatrix},$$

where here

$$e^{i\beta} = \frac{\text{trace } S}{|\text{trace } S|} = \text{sgn } \text{tr } S$$

**Proposition III.2.1.** The following are equivalent:

(i) $S$ is type 2,

(ii) $S$ is type 1,
(iii) if $S$ is non-singular then $S$ has real determinant, and either $\det S$ is positive or $S$ is bidefinite or contradefinite;

if $S$ is singular, then either $S$ is contradefinite or trace $S$ is real.

**Proof:** In view of Proposition III.1.4, it is enough to prove (ii) $\Rightarrow$ (iii) and conversely. We consider the non-singular cases first.

(ii) $\Rightarrow$ (iii) Suppose $S$ is of type 1, then there exists $C$ such that $C^*SC = S^*$, then $\det(CC^*)\det S = \det S^*$. Also $C^*S^*C = S$ thus $\det(CC^*)\det S^* = \det S$. Thus

$$\frac{\det S^*}{\det S} = \frac{1}{\det(CC^*)} = \frac{\det CC^*}{1}$$

.$$\therefore \det CC^* = \pm 1.$$ Since $\det CC^* = (\det C)(\det C)^* > 0$, $\det CC^* = 1$. Thus $\det S = \det S^*$. Suppose $\det S > 0$; then $e^{i\beta} = \sqrt{\frac{\det S}{|\det S|}} = \pm 1$. Without loss of generality, assume $e^{i\beta} = 1$. If $S$ is $\ast$-congruent to $S_2 = \begin{pmatrix} 1 & 0 \\ 2\gamma & 1 \end{pmatrix}$, $S$ is of type 2, hence is of type 1. Now suppose $\det S < 0$, $e^{i\beta} = \mp i$. Without loss of generality, assume $e^{i\beta} = i$. If $S$ is $\ast$-congruent to $S_2 = i\begin{pmatrix} 1 & 0 \\ 2\gamma & 1 \end{pmatrix}$, $S$ is of type 1 if $iS$ is $\ast$-congruent to $iS^* = -(iS)^*$. Let

$$L = iS + (iS)^* = iS - iS^* = \begin{pmatrix} -2 & -2\gamma \\ -2\gamma & -2 \end{pmatrix}.$$
$S$ is of type 1 implies $L$ is $*$-congruent to

$iS^* - iS = -(iS)^* - iS = -L$, which in turn implies $\det L < 0$; and

$\det L = 4 - 4\gamma^2 = 4(1 - \gamma^2) < 0$ iff $1 < \gamma^2$ iff $\gamma > 1$ iff $S$ is contradefinite. If $S$ is $*$-congruent to $S_1$ then in all cases, $S$ is bidefinite.

(iii) $\Rightarrow$ (ii) Suppose $\det S \neq 0$ and is real. If $\det S > 0$, $e^{i\beta} = \sqrt{\frac{\det S}{|\det S|}} = \pm 1$. Without loss of generality, assume $e^{i\beta} = 1$.

If $\det S < 0$, then $e^{i\beta} = \pm i$. Without loss of generality, assume $e^{i\beta} = +i$. First suppose $S$ is bidefinite, if $\det S > 0; S$ is

$*$-congruent to $(i 0)$ without loss of generality, let $S = (i 0)$.

Let $C = (0 \ 1 \ 1 \ 0)$, then $C^*SC = S^*$. Thus $S$ is of type 1. If $\det S < 0$, then $S$ is $*$-congruent to $i(0 \ -1) = (0 \ -1)$. Thus $S$ is of type 2 hence, is of type 1. Suppose $S$ is contradefinite. If $\det S > 0$, $S$ is $*$-congruent to $(1 \ 0)$. Thus $S$ is of type 2, hence is of type 1. If $\det S < 0$, $S$ is $*$-congruent to $i(1 \ 0)$. By Lemma 4 of [1], $(1 \ 0)$ is $*$-congruent to $e^{i\alpha}$ provided $\gamma > |\sin \alpha|$. Choose $\alpha = \pi/2$.

(Note: $\gamma > 1 = |\sin \frac{\pi}{2}|$.) Thus $S$ is of type 2, hence is of type 1.

$q.e.d.$

Now suppose $S$ is singular. If $S$ is $*$-congruent to $S_3$, which is contradefinite and real then $S$ is of type 2, hence is of type 1. Therefore we assume $S = S_4$. Here $S$ is $*$-congruent to
S* iff there exists \( c \in \mathbb{C} \) such that \( c^* c e^{i\beta} = e^{-i\beta} \) iff
\[ e^{2i\beta} = cc^* \] iff \( e^{i\beta} = \pm \sqrt{cc^*} \in \mathbb{R} \) iff \( S \) is over \( \mathbb{R} \). q.e.d.

The above interpretation cannot be applied to higher dimensional matrices. Consider

\[ T_1 = \text{diag}(i, -i, i, i) \]
\[ T_2 = \begin{bmatrix} 1 & 0 \\ 2\gamma & 1 \end{bmatrix} \oplus \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, \quad \gamma > 1 \]
\[ T_3 = \begin{bmatrix} i & 1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} i & 1 \\ 1 & 0 \end{bmatrix} \]

\( \Gamma(T_1) \) is the y-axis. Thus \( T_1 \) is bidefinite. If \( S = (1 0) \), then \( \Gamma(S) \subseteq \Gamma(T_2) \). \( \Gamma(S) \) is the complex plane. Thus \( \Gamma(T_2) \) is contra-definite. \( \text{Det} \ T_3 = 1 \). But \( T_j, \ j = 1, 2, 3 \) are not of type 1, because
\[ K_j = \frac{-i}{2} (T_j - T_j^*) \] are not of signature zero.
BIBLIOGRAPHY


APPENDICES
APPENDIX I

Pictures of Matrices

\[ T_1^0 = \begin{bmatrix} jB & B \\ jB & B \\ \vdots \\ jB \\ B \end{bmatrix} \]

\[ T_1^\infty = \begin{bmatrix} B & jB \\ B & jB \\ \vdots \\ B \\ jB \end{bmatrix} \]

n_1 blocks
Let $\mu = \beta + j$, $\mu^* = \beta - j$, $\mu^* = -\beta + j$. 

\[ T^\beta_{11} = \]

\[ \begin{bmatrix}
    B & 0 & \mu B & 0 \\
    0 & D & 0 & -\mu^* D \\
    B & 0 & \mu B & 0 \\
    0 & D & 0 & -\mu^* D
\end{bmatrix} \]

\[ \beta \epsilon \mathcal{E}_F \]

\[ n_1 \text{ blocks} \]
Here we observe that any non-singular $2 \times 2$ type 1 matrix has determinant over $E$. We also make observations on fields with $-1$ as a norm.

**Proposition IV. 1. 1.** If $S$ is a $2 \times 2$ non-singular matrix of type 1, then $\det S$ is over $E$.

**Lemma 8.** If $S$ is a $2 \times 2$ non-singular matrix and $S$ is of type 1, then $S$ is $\ast$-congruent to $\begin{bmatrix} s & s^* \\ -s^* as & s \end{bmatrix}$ for some $s \in F$ and some $a \in E$.

**Proof of Lemma 8.** It has been proved earlier that $A = S^{-1}S^\ast$ is $\ast$-congruent to $A^\ast$ and $A^{-1}$. Therefore the characteristic polynomial of $A$ is $x^2 - ax + 1$, i.e., $A = \begin{bmatrix} 0 & -1 \\ 1 & a \end{bmatrix}$.

Suppose

$$S^\ast = SA = \begin{bmatrix} s_{12} & -s_{11} + as_{12} \\ s_{22} & -s_{21} + as_{22} \end{bmatrix} = \begin{bmatrix} s_{11} & s_{21}^* \\ s_{12}^* & s_{22} \end{bmatrix}$$

Therefore $s_{11}^* = s_{12}$, $s_{21}^* = s_{12}$, so $s_{11} = s_{22}$ and $s_{21} = -s_{11} + as_{12}$. Let $s = s_{11}$. We have $S = \begin{bmatrix} s & s^* \\ -s^* + as & s \end{bmatrix}$.

If $T^{-1}T^\ast$ is similar to $A = \begin{bmatrix} 0 & -1 \\ 1 & a \end{bmatrix}$, i.e.,
if there exists a non-singular \( D \) such that \( D^{-1}(T^{-1}T^*)D = A \), then \( S = D^*TD \) has the canonical form we have just shown. Thus it is enough to consider \( A \) being a companion matrix.

**Proof of Proposition IV.1.1.** From Lemma 8, when \( S \) is a 2 x 2 non-singular type \(-1\) matrix \( S \) is \(*\)-congruent to
\[
S_0 = \begin{pmatrix} s & s^* \\ -s^*a & a \end{pmatrix}, \quad \det S_0 = s^2 + (s^*)^2 + ass^* \in \mathbb{C}.
\]
If \( C \) is a non-singular matrix, such that \( C^*SC = S_0 \), then
\[
(det C)(det C^*)(det S) = det S_0, \quad so \quad det S = (det S_0)[(det C)(det C^*)]^{-1}.
\]
Therefore \( det S \in \mathbb{C} \).

**Proposition IV.1.2.** Suppose \( S \) is an \( n \times n \) matrix over \( \mathcal{F} \), \( H = \frac{1}{2}(S+S^*) \), \( K = \frac{1}{2j}(S-S^*) \). If \(-1\) is a norm, and one of the following holds:

1. \( H^{-1}K \) is nilpotent,
2. \( K^{-1}H \) is nilpotent,

then \( S \) is of type \( 1 \).

**Proof:** From Theorem II.3.4, it is enough to show every diagonal matrix over \( \mathbb{C} \) is \(*\)-congruent to its negative. Since \(-1\) is a norm, \(-1 = a(a^*) \) for some \( a \in \mathcal{F} \); if \( B \) is any matrix over \( \mathbb{C} \), \( (aI)^*B(aI) = -B \). q.e.d.

**Proposition IV.1.3.** Here we assume \( \mathbb{C} \) and \( \mathcal{F} \) to be
finite fields, and $S$, $H$, and $K$ are as defined in Proposition IV.1.2. If the following holds:

(i) $H^{-1}K$ and $K^{-1}H$ are not nilpotent.

(ii) $K^{-1}H$ is similar to $-K^{-1}H$ and all eigenvalues of $K^{-1}H$ are in $\mathcal{E}$, then $S$ is of type 2, hence is of type 1.

**Proof:** In view of Theorems II.3.2, II.3.3, II.3.4, II.3.5, and II.3.7, it is enough to show that every diagonal matrix over $\mathcal{E}$ is $\ast$-congruent to any diagonal matrix of the same order over $\mathcal{E}$.

Let $B = \text{diag}(\epsilon_1, \epsilon_2, \ldots, \epsilon_k)$, $\epsilon_i \in \mathcal{E}$. Since $\mathcal{E}$ and $\mathcal{F}$ are finite fields, every element in $\mathcal{E}$ is a norm. Let $a_i \in \mathcal{F}$ such that $a_i a_i^{-1} = \epsilon_i$. Let $C = \text{diag}(a_1^{-1}, a_2^{-1}, \ldots, a_k^{-1})$ then $C \ast BC = I_k$.

Thus every diagonal matrix over $\mathcal{E}$ is $\ast$-congruent to the identity matrix. q.e.d.

**Proposition IV.1.4.** Here $\mathcal{E}$, $\mathcal{F}$, $S$, $H$ and $K$ are as defined in Proposition IV.1.3. Suppose one of the following holds

(i) $H^{-1}K$ is nilpotent,

(ii) $K^{-1}H$ is nilpotent.

In view of Theorems II.3.3 and II.3.6, we may, without loss of generality, assume $S = T^z$, $z = \infty, 0$. (Refer to pictures in Appendix I.) Then the following holds:

(i) if $z = 0$, (recall $S = T^0 = (E_n + jF_n) \otimes B$, where $B$ is a diagonal matrix over $\mathcal{E}$) then
(a) when \( n \) is odd, \( S \) is of type 2,

(b) when \( n \) is even, \( S \) is of type 2 iff \( \dim B \) is even;

(ii) if \( z = \infty \) (recall \( S = T_n^\infty = (F_n + jE_n) \otimes B \), where \( B \) is a diagonal matrix over \( \mathcal{E} \), then

(a) when \( n \) is even, \( S \) is of type 2

(b) when \( n \) is odd, \( S \) is of type 2 iff \( \dim B \) is even.

Proof: (i) (a) Refer to Theorem II.3.4.

(b) In view of Theorem II.3.4, it is enough to show that if \( B \) is a diagonal matrix over \( \mathcal{E} \), then \( jB \) is of type 2 iff \( \dim B \) is even. Suppose \( \dim B \) is even. In the proof of Proposition IV.1.3, we have shown that every diagonal matrix over \( \mathcal{E} \) is *-congruent to any other diagonal matrix over \( \mathcal{E} \) of the same order and rank. If \( \dim B = 2m \), \( m \geq 1 \), without loss of generality, assume \( B = (1, -1, 1, -1, \ldots, 1, -1) \); let \( C = \text{diag}(C_0, C_0, \ldots, C_0) \), then \( C*BC = \text{diag}(D_0, D_0, \ldots, D_0) \) where

\[
D_0 = \begin{bmatrix}
0 & (c*)^2 - c^2 \\
(c^2 - (c*)^2) & 0
\end{bmatrix}
\]

Thus \( jC*BC = C*jBC \) is over \( \mathcal{E} \). Thus \( jB \) is of type 2. Conversely, suppose \( \dim B \) is odd and \( jB \) is of type 2. I.e., there
exists a non-singular matrix \( C \) such that \( C^* j B C = D \), and \( D \) is over \( \mathcal{E} \). Thus

\[
(det C^*) det(jB)(det C) = j^n(det B)(det C)^*(det C)
\]

\[= det \ W \]

which is over \( \mathcal{E} \). But \( n \) is odd, so \( j^n \) is not over \( \mathcal{E} \). Thus we have a contradiction; so \( jB \) is not of type 2.

(ii) The proof is similar to that of (i). q.e.d.

**Corollary 1.** If \( \mathcal{E} \) and \( \mathcal{F} \) are finite fields, \( S, H, K \) are as Proposition IV. 1. 4, and if one of the following holds:

(i) \( H^{-1}K \) is nilpotent

(ii) \( K^{-1}H \) is similar to \( -K^{-1}H \) and has all its eigenvalues over \( \mathcal{F} \), then \( S \) is of type 1.

**Proof:** Refer to Propositions IV. 1. 2 and IV. 1. 4.

**Corollary 2.** If \( \mathcal{E} \) and \( \mathcal{F} \) are finite fields, \( S \) is a 2 x 2 matrix over \( \mathcal{E} \) such that if \( H = \frac{1}{2}(S + S^*), \ K = \frac{1}{2j}(S - S^*) \), the pencil \( \lambda H + \mu K \) is non-singular, \( K \) is non-singular, \( K^{-1}H \) is similar to \( -K^{-1}H \), then \( S \) is of type 2.

**Proof:** Since the characteristic polynomial of \( K^{-1}H \) would be of the form \( x^2 - c \), where \( c \in \mathcal{E} \). Since \( \mathcal{E} \) is finite, \( \mathcal{F} \) is
the only quadratic field over \( \mathbb{Q} \), thus it contains all the roots of all second degree polynomials. So we can apply Propositions IV.1.3 and IV.1.4.