AN ABSTRACT OF THE THESIS OF

MINGHUI YANG for the degree of DOCTOR OF PHILOSOPHY in STATISTICS presented on February 8, 1988

Title: A HAZARDOUS-INSPECTION MODEL WITH COSTLY REPAIR

Abstract approved: Redacted for Privacy

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An inspection-repair model is developed that presumes inspection is hazardous to the system being inspected. The form of the optimal inspection-repair policy is determined for different ranges of the model's parameters. The model is formulated as a partially observable Markov decision process with finite numbers of states and actions. The model is analyzed by dynamic programming and mathematical induction. The results obtained are compared to those previously found for a more simple model having no action of repair and no costs and rewards. A computer program and some examples are developed to illustrate the use of the new model. Several possible applications areas for the model are suggested, including industrial systems, military systems, and medical systems.

A HAZARDOUS-INSPECTION MODEL WITH COSTLY REPAIR

by

Minghui Yang

A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

Completed February 8, 1988

Commencement June, 1988

APPROVED:

Redacted for Privacy

Associate Professor of Statistics in charge of major

Redacted for Privacy

Chairman of Department of Statistics

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Dean of Graduate School

Date dissertation is presented: February 8, 1988

Typed by: Minghui Yang

ACKNOWLEDGMENT

I am the first student to come from the People's Republic of China and study at the OSU Department of Statistics.

I wish to thank the Department and Dr. David A. Butler for the support given me to complete my education. I have learned a good deal of new knowledge and experience which will benefit my whole life. I will never forget the quiet, beautiful, small town of Corvallis. I will never forget its golden fall, clear summer and rainy winter. Four years is a very short time for a person, but it is long enough to change one's life and one's point of view of the world. No matter where I am, I will use all I have learned at Oregon State University to serve the entire people.

I should like this message to be taken as a personal expression of heartfelt gratitude to Dr. David A. Butler, who has given me unstinted help and unreserved guidance and who spent a lot of time to correct my dissertation, and to Dr. Steven J. Knapp, who has supported me for almost two years and who has given me a lot of advice and guidance to my life and research. I would also like to thank the other professors and students in the Statistics Department, especially Dr. Jeffrey L. Arthur and Dr. Justus Seely, who have given me much help, and Bill Burns and Maryam Bolouri who often gave help when needed. Also I would like to thank Ron Stillinger and Genevieve Downing for their tremendous help.

This research was partially supported by the office of Naval Research under contract N00014-79-C-0751 (NR-047-208).

Special thanks are due my wife, Xiaoping Dong, and daughter, Linda Yang, for their patience and understanding. Thanks are also due my parents, relatives, and my friends. Without their moral support, the dissertation would never have been completed.

KEY WORDS

Markov decision processes

Dynamic programming

Hazardous inspection

Partially observable process

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A HAZARDOUS-INSPECTION MODEL WITH COSTLY REPAIR

1. INTRODUCTION

This paper extends previous hazardous-inspection models of Butler (1979) and Butler and Chou (1981). Their models did not explicitly consider costs, and did not allow for repair of faulty devices. We develop a new model that incorporates both of these features, then determine the optimal inspection-repair policies for this new model.

We are given a system which operates throughout a finite or infinite number of periods and which in each period is subject to failure. Failure of the system is directly observable. Prior to failure, the system will enter a state in which it is functioning but in an impaired manner. This state can be detected only by performing an inspection. After inspection, if the system is found to be in the impaired state, the operator can take appropriate actions to reduce the failure rate of the system. In this respect inspection is valuable. However, the act of inspecting the system when it is not in the impaired state may itself cause the system to become impaired. In this respect inspection is hazardous.

In real life there are many situations where inspection may pose a hazard. For instance, to test certain types of electronic parts one often uses higher voltages or stronger currents than the parts would normally bear. The terms "system", "functional", "impaired", "failed", "inspection", and "repair" used in this thesis not only apply in industrial or laboratory settings, but can be used in many other areas. We assume that readers follow the extended meaning. For example in medical biopsies, to determine if a patient has cancer, a surgeon often needs to remove a small piece of the affected organ from the patient to do the inspection. If the patient does not have cancer, the surgical procedure may adversely affect his or her health. Another example is the use of X-rays for cancer detection. The X-rays themselves may cause cancer when used to detect if a patient has cancer or not.

The major difference between Butler and Chou's model and our new model is the assumption of repair. In their model, repair is not allowed in any state; if some problems have been detected through inspection, certain actions will be taken to reduce the failure rate of the system. The main concern in their model was not to repair the system, but to prolong its lifetime. To achieve this goal, one must inspect the system to determine the true state. In our new model, we assume that the system is repairable if it has not yet failed. We assume that once the system has failed, it is no longer repairable. For example, if we have a running system, we can change parts to keep it in the fully functional state. In this case repair means to change parts. We assume that inspection is free, repair is costly, and a functioning system earns a reward each period. The main concern in our model is to maximize the expected profit (reward minus cost). Repairing the system will cost money in the current period but may increase the expected profit over the long run. Since a failed system can never be repaired and will never earn any reward, one also needs to know the true state to make the appropriate decision. Inspection is needed; yet it should be used very carefully, since it may cause the system to fail.

Many real-world systems can be repaired when they are functioning but not if they are failed. One example is human beings. The failed state is defined as death, the impaired state as some disease, and the fully functional state as good health. Repair here is by some treatment or operation. Repair is clearly impossible in the failed state. However, a doctor can give the same treatment to both a healthy patient or an ill patient. In other words, repair can be done either in the fully functional or the impaired state.

Another example is the maintenance of electronic equipment. To keep it working very well, people often change some components of the equipment even though no problem has occurred. We can consider such action as repair. If people fail to do this, the failure of some parts may cause the whole equipment to fail. The failure of the whole equipment may be too complicated or too expensive to repair; in which case repair is considered to be impossible.

A number of authors have studied inspection models. Luss (1976) developed a Markovian model in which the holding times in the various states were exponentially distributed. In his model the degree of deterioration could be observed through inspections. The costs incurred included costs of inspections, operating costs, costs of preventive repairs, and the costs of repairing a failed system either at an inspection event or immediately after the occurrence of a malfunction. The objective function in his model was to minimize the total expected cost per unit time. Kao (1973) studied optimal replacement rules when changes of states were semi-Markovian. Luss and Kander (1974) examined models where inspection revealed a malfunction occurrence only and the time it took to make an inspection was non-negligible. Ross (1970) discussed several probability models with optimization applications. In his models inspection and repair were costly. Rosenfield (1976) studied Markovian deterioration with uncertain information. Smallwood and Sondik (1973) studied partially observable Markov processes over a finite horizon and analyzed some optimal control policies. In none of the above papers was inspection assumed to have any effect on the true state of the system. Wattanapanom and Shaw (1976) considered a hazardous-inspection model in which the system had an exponential failure distribution in the absence of inspection. An inspection either caused immediate failure or else increased the failure rate. They derived algorithms for finding inspection policies which minimized the overall operating cost. Butler (1979) formulated a Markov-decision-process model of hazardous-inspection and determined the form of the optimal inspection policy for previously inspected systems. Later Butler and Chou (1981) extended the results to a system which had never been inspected. In Butler and Chou's model there was no action of repair allowed in any state. If the system was partially failed it either remained there or else it went to the failed state. Also, there was no costs or rewards associated with any action in their model. Their objective was to maximize the expected lifetime of the system.

Because repair is often allowable for an operating system in real life, and often there are repair and operating costs incurred during the operating process, we will extend Butler and Chou's model (1981) to a new model in which inspection is still hazardous, but for which repair is allowed when the system is not failed. We assume there exist both repair costs and operating rewards. Our objective is to determine the optimal inspection and repair policy which maximizes the total expected profit of the system during its lifetime.

2. THE MODEL

Consider a system whose operation can be classified into one of three categories: fully functional; functional, but impaired; and failed. The failed state is directly observable, but one can distinguish the impaired state from the (fully) functional state only by performing an inspection. Inspection is instantaneous and perfect, which means the true state is always revealed. Repairing a failed system is either impossible or too costly. But when the system is functioning, whether impaired or fully functional, we can repair it. Repair is also instantaneous and perfect. But after repair the system still may change to the impaired state by the start of the next period.

Even though repair and inspection are both instantaneous, we do not allow both to be done in a single period. Such a restriction can arise because the two operations might be carried out by different personnel who must schedule their work in advance.

The usefulness of any information gained by inspection is always delayed one period. That is, if a system is inspected and is found to be partially failed, the failure rate will not be reduced in the current period, but in the following time period.

We assume that there is no cost for inspection. The main reason for this assumption is that the cost of inspection is usually much less than that of repair in real life. But there is a cost for repair. Also, in any period in which the system is not failed, a reward is earned.

The inspection and repair model will be formulated as a Markov decision process. We let X_n denote the true state of the system at the start of period n (before inspection or repair), and we denote the true state space by Ω, where $\Omega = \{ 1, 2, 3, 4 \}$ and $X_n \in \Omega$. The possible states are:

True State	Description
1	functioning (OK)
2	undetected impairment (UI)
3	detected impairment (DI)
4	failed (F)

Let a_n be the action taken at the start of period n, and $\mathcal{A}$ be the action space, where $\mathcal{A} = \{ 0, 1, 2 \}$ and $a_n \in \mathcal{A}$. A generic element of $\mathcal{A}$ is denoted by a. The possible actions are:

Action	Description
0	do nothing
1	inspect
2	repair

We make the Markov assumption that

$$P\{ X_{n+1} = j \mid X_k, a_k, k = 0, 1, 2, \ldots, n \}$$
$$= P\{ X_{n+1} = j \mid X_n, a_n \} \qquad j \in \Omega, \quad n = 0, 1, 2, \ldots$$

For any $a \in A$, and $i, j \in \Omega$, define

$$Q_{ij}(a) = P\{ X_{n+1} = j \mid X_n = i,\ a_n = a \}.$$

We define the one-step Markov transition matrices Q(0), Q(1), and Q(2) as follows where $Q(a) = [\ Q_{ij}(a)\ ]$:

Q(0)

(do nothing)

$$\begin{array}{c|cccc|} & OK & UI & DI & F \\ OK & 1-\alpha_0 & \alpha_0 & 0 & 0 \\ UI & 0 & 1-\beta & 0 & \beta \\ DI & 0 & 0 & 1-\gamma & \gamma \\ F & 0 & 0 & 0 & 1 \end{array}$$

Q(1)

(inspect)

$$\begin{array}{c|cccc|} & OK & UI & DI & F \\ OK & 1-\alpha_1 & \alpha_1 & 0 & 0 \\ UI & 0 & 0 & 1-\beta & \beta \\ DI & 0 & 0 & 1-\gamma & \gamma \\ F & 0 & 0 & 0 & 1 \end{array}$$

Q(2)

(repair)

$$\begin{array}{c|cccc|} & OK & UI & DI & F \\ OK & 1-\alpha_0 & \alpha_0 & 0 & 0 \\ UI & 1-\alpha_0 & \alpha_0 & 0 & 0 \\ DI & 1-\alpha_0 & \alpha_0 & 0 & 0 \\ F & 0 & 0 & 0 & 1 \end{array}$$

We assume that $0 < \alpha_0, \alpha_1, \beta, \gamma < 1$. We also assume that $\beta > \gamma$, and $\alpha_1 > \alpha_0$.

The assumption that $\beta > \gamma$ means that if partial failure has been detected, the information can be used to reduce the failure rate of the system. The assumption that $\alpha_1 > \alpha_0$ means inspection is more hazardous to a fully functional system than is doing nothing.

The one-step transition matrices Q(0) and Q(1) are the same as in Butler and Chou's model (1981). Since our model is an extension of their model, if we do nothing or inspect the system, the transition probability should be the same as in their model.

The interpretation of Q(0) is straightforward. If the true state is the OK state, at the next period the system will remain in the OK state with probability $1-\alpha_0$, and go to the UI state with probability α_0. If the true state is the UI state, since nothing is done to the system, it remains in the UI state with probability $1-\beta$ at the next period and goes to the failed state with probability β. When the impaired state has been detected, certain actions will be taken to reduce the failure rate. The system will have probability γ (which is less than β) to go to the failed state. Once the system is failed, it will stay there forever.

Q(2) is also easy to interpret. After repair the system is just like a new system and will be put into operation right away since repair is instantaneous. Between then and the start of the next period, it may enter the impaired state.

We need to say a little more about Q(1). Inspection is only available in true states OK and UI (they look no different to an observer). In states DI and F the transition probabilities should be the same as if nothing was done to the system. If the system is in state UI, it will change into DI right after inspection. But there exists one period delay of information, so the failure rate is the same as if the system was in state UI at the start of the period.

Because the true states OK and UI look no different to an observer, the only information available on a system that is running now is how many periods have elapsed since it was put into operation or last inspected. We define an observed state space $\mathcal{S}$ for solving this problem. Since DI and F are directly observable, they are included in $\mathcal{S}$.

Observed State	Description
-1	system failed.
0	detected partial failure.
$j \geq 1$	system not failed, last inspection OK, last inspection j periods ago.
$j^* \geq 1$	system not failed, put into operation as new (or repaired) j periods ago, has not been inspected since that time.

Define $S^* = \{ 1^*, 2^*, 3^*, \ldots \}$, $S = \{ -1, 0, 1, 2, 3, \ldots \}$, and $\mathcal{S} = S \cup S^*$. We denote a generic element of $\mathcal{S}$ by s. If $s = i^*$, then $s \in S^*$. If $s = i$, then $s \in S$. Notice that both i^* and i have the same numerical values, so s+1 either represents i+1 where $i \in S$ or $(i+1)^*$ where $(i+1)^*$ and $i^* \in S^*$. Also, both i and i^* can either represent the state variable or its numerical value, depending upon the context.

The observed state transition probabilities also have the Markov property. Let Z_n denote the observed state at period n and let

$$P_{st}(a) = P\{ Z_{n+1} = t \mid Z_n = s,\ a_n = a \}$$

for all $s, t \in \mathcal{S}$, all $a \in \mathcal{A}$ and $n \geq 0$.

We need to define several probability functions which represent the probability of the true state given the current observed state. For a system which has been inspected some time in the past, we let

$$K_j = P\{ X_{n+j} = 2 \mid X_n = 1,\ a_n = 1,\ a_{n+k} = 0,\ 0 < k < j\} \qquad j \in S.$$

This quantity is the j-step transition probability that the true state is UI at stage n+j, given that at stage n the system was inspected and found to be OK, and the action of doing-nothing was taken for the subsequent j periods. Similarly,

$$L_j = P\{ X_{n+j} = 1 \mid X_n = 1,\ a_n = 1,\ a_{n+k} = 0,\ 0 < k < j\} \qquad j \in S,$$

is the j-step transition probability that the true state is OK at stage n+j, given that at stage n the system was inspected and found to be OK, and the action of doing-nothing was taken for the subsequent j periods. For a system which has never been inspected before, we let

$$K_j^* = P\{ X_j = 2 \mid X_0 = 1,\ a_k = 0,\ \ 0 \le k < j \} \qquad j^* \in S^*.$$

Since a newly repaired system is equivalent to a new one, we also have

$$K_j^* = P\{ X_{n+j} = 2 \mid X_n = x,\ a_n = 2,\ a_{n+k} = 0,\ 0 < k < j \}$$
$$x = 1,\ 2,\ 3;\ j^* \in S^*.$$

Similarly, we can define L_j^* equivalently in either one of two ways:

$$L_j^* = P\{ X_j = 1 \mid X_0 = 1,\ a_k = 0,\ \ 0 \le k < j \} \qquad j^* \in S^*,$$

$$L_j^* = P\{ X_{n+j} = 1 \mid X_n = x,\ a_n = 2,\ a_{n+k} = 0,\ 0 < k < j \}$$
$$x = 1,\ 2,\ 3;\ j^* \in S^*.$$

The quantities K_j^* and L_j^* are the j-step transition probabilities that the true state is UI or OK, respectively, at stage n+j, given that at stage n the system was put into operation or was repaired, and the action of doing-nothing was taken for the subsequent j periods. Finally, let

$$N_j = K_j + L_j,$$
$$N_j^* = K_j^* + L_j^*.$$

Then K_s/N_s is the probability that the true state is 2 (UI), given the observed state is s for all $s \ge 1$, and L_s/N_s is the probability that the true state is 1 (OK), given the observed state is $s \ge 1$.

We also define $A(s)$ to be the set of all available actions given the observed state s, where $s \in S$.

Proposition 1

i) For all $s \in S$ and $a \in A(s)$

$$P_{-1,s}(a) = \begin{cases} 1 & s = -1 \\ 0 & \text{otherwise} \end{cases}$$

ii) For all $s \in S$

$$P_{0,s}(0) = \begin{cases} \gamma & s = -1 \\ 1-\gamma & s = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$P_{0,s}(2) = \begin{cases} 1 & s = 1^* \\ 0 & \text{otherwise} \end{cases}$$

iii) For all $s, t \in S$, such that $s \geq 1$

$$P_{s,t}(0) = \begin{cases} \beta \cdot K_s/N_s & t = -1 \\ (1-\beta) \cdot K_s/N_s + L_s/N_s & t = s+1 \\ 0 & \text{otherwise} \end{cases}$$

$$P_{s,t}(1) = \begin{cases} \beta\ K_s/N_s & t = -1 \\ (1-\beta)\ K_s/N_s & t = 0 \\ L_s/N_s & t = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$P_{s,t}(2) = \begin{cases} 1 & t = 1^* \\ 0 & \text{otherwise} \end{cases}$$

Proof

All the results except those for $P_{s,t}(2)$ have already been established by Butler and Chou (1981). The formula for $P_{s,t}(2)$ follows from the fact that repair is instantaneous and always leads to the OK state. □

Proposition 2

A

i) $L_i = (1-\alpha_1)\cdot(1-\alpha_0)^{i-1}$ for all $i \in S$.

ii) $L_i* = (1-\alpha_0)^i$ for all $i^* \in S^*$, where the numerical value of i^* is i.

B

i) $K_i = \alpha_1\cdot(1-\beta)^{i-1} + \alpha_0\cdot(1-\alpha_1)\cdot[(1-\beta)^{i-1} - (1-\alpha_0)^{i-1}]/(\alpha_0 - \beta)$
$i \in S,\ \alpha_0 \neq \beta.$

ii) $K_i = \alpha_1\cdot(1-\alpha_0)^{i-1} + \alpha_0\cdot(i-1)\cdot(1-\alpha_1)\cdot(1-\alpha_0)^{i-1}$
$i \in S,\ \alpha_0 = \beta.$

iii) $K_i* = \alpha_0\cdot[(1-\beta)^i - (1-\alpha_0)^i]/(\alpha_0 - \beta)$
$i^* \in S^*,\ \alpha_0 \neq \beta.$

iv) $K_i* = i\cdot\alpha_0\cdot(1-\alpha_0)^{i-1}$ $i^* \in S^*,\ \alpha_0 = \beta.$

Proof

See Butler and Chou (1981). □

Our goal is to obtain the maximum expected profit during n periods of operation of a system. If a system has never been inspected before since it was put into operation, or it was repaired some periods ago and we have done nothing since that time, we call it a *new system*. If a system has been inspected before and we have done nothing since that time, we call it an *old system*. We make the following assumptions:

1) If the system runs (the true state is OK, or UI, or DI) we earn a positive reward, R.
2) If the system is failed, it does not earn any reward.
3) If the system is repaired, a positive cost C is incurred.
4) There is a finite n-period horizon. There is no terminal reward or cost. Later this assumption will be relaxed.

Let r(s, a) be the profit earned in stage n, when in observed state s and when action a is taken. Let V(s, n, a) be the optimal total expected profit in an n-period problem, starting in observed state s and taking initial action a. Let $V(s,n) = \max_{a \in A(s)} \{V(s, n, a)\}$ be the optimal total expected profit in an n-period problem, starting in observed state s. In this paper, V(·, n) is called the *profit function*. Also, later we will extend the n-period problem to an infinite-horizon problem, so we need to introduce a discount factor δ, where $0 < \delta < 1$, since both the reward and the cost in the future are less important than that in the present time.

By the principle of optimality of dynamic programming (Hillier and Lieberman, 1980), the following recursive equation relates $V(\cdot, n)$ to $V(\cdot, n-1)$

$$V(s, n) = \max_{a \in A(s)} \{ r(s, a) + \delta \cdot \sum_{t \in S} P_{st}(a) \cdot V(t, n-1) \}.$$

Proposition 3 (Recursive Equations for $V(s, n)$)

i) $V(s, 0) = 0$ for all $s \in S$.

ii) $V(-1, n) = 0$ for all $n \geq 0$.

iii) $V(0, n) = \max\{R + \delta \cdot (1-\gamma) \cdot V(0,n-1),\ R - C + \delta \cdot V(1^*,n-1)\}$.

iv)
$$\begin{aligned} V(s, n) = \max\{ & R + \delta \cdot [(1-\beta) \cdot K_s/N_s + L_s/N_s] \cdot V(s+1, n-1), \\ & R + \delta \cdot [(1-\beta) \cdot K_s/N_s] \cdot V(0, n-1) + \delta \cdot L_s/N_s \cdot V(1, n-1), \\ & R - C + \delta \cdot V(1^*, n-1) \} \end{aligned}$$

where $s \in S - \{-1, 0\}$ and $n \geq 0$.

Proof

(i) and (ii) are the direct results of the previous assumptions. The proofs of (iii) and (iv) are based on the above recursive equations and on Proposition 1. We only prove (iv) here.

When the observed state is $s \geq 1$, if the do nothing action is taken, since the system runs we will earn a reward R. The system will either go to observed state $s+1$ with probability $(1-\beta)\cdot K_s/N_s + L_s/N_s$ and earn future expected profits $\delta\cdot V(s+1, n-1)$, or else go to state -1 with probability $\beta\cdot K_s/N_s$ and earn zero future expected profits. This is the first expression in the right-hand side of $V(s,n)$. If the inspection action is taken, the expected profit in the current period is still R. With probabilities $(1-\beta)\cdot K_s/N_s$, L_s/N_s, and $\beta\cdot K_s/N_s$ the system will go to observed states 0, 1, and -1, and earn future expected profits $\delta\cdot V(0, n-1)$, $\delta\cdot V(1, n-1)$, and 0, respectively. This is the second expression in the right-hand side of the equation for $V(s, n)$. If the repair action is taken, the current expected profit is R - C, and with probability 1 the system will go to state 1^* at the next period, earning future expected profit $\delta\cdot V(1^*, n-1)$. □

3. PRELIMINARY ANALYSIS

Proposition 4

K_i^*/N_i^* is nondecreasing in i^*, and $(1-\beta)\cdot K_i^*/N_i^* + L_i^*/N_i^*$ and L_i^*/N_i^* are nonincreasing in i^*.

Proof

See Butler and Chou (1981). □

Proposition 5

If $\alpha_0 \leq \beta$, then $\lim_{s\to\infty} K_s/N_s = \alpha_0/\beta$.

Proof

See Butler and Chou (1981). □

Lemma 1

$V(s,n)$ is nondecreasing in n for all $s \geq 0$. Also, for all $n \geq 1$ and $s \geq 0$, $R \leq V(s, n) \leq (1-\delta^n)\cdot R/(1-\delta)$.

Proof

The proof is by induction on n. When n = 1,

$$V(0, n) = \max\{ R, R - C \} = R.$$

$$V(s, n) = \max\{ R, R, R - C \} = R.$$

So the lemma holds for n = 1.

When $n = 2$,

$$V(0,\ n) = \max\{\ R + \delta\cdot(1-\gamma)\cdot R,\ R - C + \delta\cdot R\}$$
$$\geq R + \delta\cdot(1-\gamma)\cdot R$$
$$> R = V(0,\ 1).$$

Also, $(1-\delta^2)\cdot R/(1-\delta) = (1+\delta)\cdot R$, $(1+\delta)\cdot R \geq R + \delta\cdot(1-\gamma)\cdot R$ and $(1+\delta)\cdot R \geq R - C + \delta\cdot R$. Thus $V(0,\ 2) \leq (1-\delta^2)\cdot R/(1-\delta)$.

For $s \geq 1$ and $n = 2$,

$$V(s,\ n) = \max\{\ R + \delta\cdot[(1-\beta)\cdot K_s/N_s + L_s/N_s]\cdot R,$$
$$R + \delta\cdot[(1-\beta)\cdot K_s/N_s]\cdot R + \delta\cdot L_s/N_s\cdot R,$$
$$R - C + \delta\cdot R\ \}$$
$$> R = V(s,\ 1).$$

Since $(1-\delta^2)\cdot R/(1-\delta) = (1+\delta)\cdot R$,

$$R + \delta\cdot[(1-\beta)\cdot K_s/N_s + L_s/N_s]\cdot R \leq R + \delta\cdot R = (1+\delta)\cdot R,$$
$$R + \delta\cdot[(1-\beta)\cdot K_s/N_s]\cdot R + \delta\cdot L_s/N_s\cdot R \leq R + \delta\cdot R = (1+\delta)\cdot R,$$
$$R - C + \delta\cdot R \leq (1+\delta)\cdot R,$$

$(1-\delta^2)\cdot R/(1-\delta)$ is greater than each term in the expression for $V(s,\ 2)$. Thus for $n = 1,\ 2$ the lemma holds. Suppose the lemma holds for $n = k$. When $n = k + 1$,

$$V(0,\ k+1) = \max\{\ R + \delta\cdot(1-\gamma)\cdot V(0,\ k),\ R - C + \delta\cdot V(1^*,\ k)\ \}.$$

Also, we have

$$V(0,\ k) = \max\{\ R + \delta\cdot(1-\gamma)\cdot V(0,\ k-1),\ R - C + \delta\cdot V(1^*,\ k-1)\ \}.$$

To show that $V(0,\ n)$ is nondecreasing in n, it is sufficient to show that each term in the maximum operator of the right-hand side is nondecreasing. The reason is by the following argument.

Let X = max{ a, b } and Y = max{ c, d }. We want to show that if $a \geq c$ and $b \geq d$, then $X \geq Y$. Suppose X = a, then by the definition of X, we know that $a \geq b$. Since $a \geq c$ and $b \geq d$, we have $a \geq b \geq d$. Thus it is clear that $a \geq \max\{ c, d \}$, i.e. $X \geq Y$. Using the same argument, we can show that it is also true if X = b. We will use this result later without further explanation.

The difference between the first term of V(0, k+1) and the first term of V(0, k) is

$$[\, R + \delta\cdot(1-\gamma)\cdot V(0,\, k)] - [R + \delta\cdot(1-\gamma)\cdot V(0,\, k-1)]$$
$$= \delta\cdot(1-\gamma)\cdot[\, V(0,\, k) - V(0,\, k-1)] \geq 0 \qquad \text{(by hypothesis).}$$

The difference between the two second terms is

$$[\, R - C + \delta\cdot V(1^*,\, k)] - [\, R - C + \delta\cdot V(1^*,\, k-1)]$$
$$= \delta\cdot[V(1^*,\, k) - V(1^*,\, k-1)] \geq 0.$$

So V(0, n) is nondecreasing in n. Also, by the inductive hypothesis, we know that $V(0,\, k) \leq (1-\delta^k)\cdot R/(1-\delta)$, and $V(1^*,\, k) \leq (1-\delta^k)\cdot R/(1-\delta)$. Thus

$$\begin{aligned} R + \delta\cdot(1-\gamma)\cdot V(0,\, k) &\leq R + \delta\cdot V(0,\, k) \\ &\leq R + \delta\cdot(1-\delta^k)\cdot R/(1-\delta) \\ &= (1-\delta^{k+1})\cdot R/(1-\delta), \end{aligned}$$

and

$$R - C + \delta \cdot V(1^*, k) \le R + \delta \cdot (1-\delta^k) \cdot R/(1-\delta)$$
$$= (1-\delta^{k+1}) \cdot R/(1-\delta).$$

So $V(0, k+1) \le (1-\delta^{k+1}) \cdot R/(1-\delta)$.

For $s \ge 1$, the maximum operator in the recursive equation for $V(s, n)$ contains three terms. To show $V(s, n)$ is nondecreasing in n, it is sufficient to show that each term is nondecreasing in n. If $X = \max\{ a, b, c \}$, then it is equivalent to write X as

$$X = \max\{ \max\{ a, b \}, c \}.$$

Denote $\max\{ a, b \}$ by d, then $X = \max\{ d, c \}$. Using the preceding argument, it is easy to obtain the conclusion needed.

The pairwise differences of the three terms in the maximum operators of $V(s,k+1)$ and $V(s,k)$ are

$$\delta \cdot [(1-\beta) \cdot K_s/N_s + L_s/N_s] \cdot [V(s+1,k+1) - V(s+1,k)],$$
$$\delta \cdot (1-\beta) \cdot K_s/N_s \cdot [V(0,k+1) - V(0, k)],$$

and

$$\delta \cdot [V(1^*, k+1) - V(1^*, k)].$$

By the inductive hypothesis, it is easy to see that all of them are greater than or equal to zero. So $V(s, n)$ is nondecreasing in n for all $s \geq 1$.

Also, by the inductive hypothesis, $V(s,k) \leq (1-\delta^k)\cdot R/(1-\delta)$ for all $s \geq 0$, so

$$R + \delta\cdot[(1-\beta)\cdot K_s/N_s + L_s/N_s]\cdot V(s+1, k)$$
$$\leq R + \delta\cdot V(s+1, k) \leq R + \delta\cdot(1-\delta^k)\cdot R/(1-\delta) = (1-\delta^{k+1})\cdot R/(1-\delta),$$

$$R + \delta\cdot[(1-\beta)\cdot K_s/N_s]\cdot V(0, k) + \delta\cdot L_s/N_s\cdot V(1, k)$$
$$\leq R + \delta\cdot[K_s/N_s + L_s/N_s]\cdot(1-\delta^k)\cdot R/(1-\delta)$$
$$= R + \delta\cdot(1-\delta^k)\cdot R/(1-\delta) = (1-\delta^{k+1})\cdot R/(1-\delta),$$

$$R - C + \delta\cdot V(1^*, n-1) \leq R + \delta\cdot(1-\delta^k)\cdot R/(1-\delta) = (1-\delta^{k+1})\cdot R/(1-\delta).$$

Thus $V(s, k+1) \leq (1-\delta^{k+1})\cdot R/(1-\delta)$ for $s \geq 1$. Since $V(s, n)$ is nondecreasing in n, and when n = 1, $V(s, 1) = R$ for all $s \geq 0$, the result follows. □

By Lemma 1, for $s \geq 0$, the sequence { $V(s, 1)$, $V(s, 2)$, $V(s, 3)$, ... } is nondecreasing in n and each term is bounded by $R/(1-\delta)$, so $\lim_{n\to\infty} V(s, n)$ exists, which means the finite-horizon optimal values converge to the infinite-horizon optimal values as n approaches infinity. Denote this limit by $V(s)$.

Lemma 2

i) $V(1, n) \geq (1-\beta)\cdot V(0, n)$ for all n.

ii) $V(1) \geq (1-\beta)\cdot V(0)$.

Proof

i)

The proof is by induction on n. When n = 0 or n = 1, the result obviously holds. Suppose it holds for n = k. When n = k+1 we know both V(1, k+1) and V(0, k+1) are no less than R > 0 and have the common term $R - C + \delta\cdot V(1^*, k)$ in Equations (iii) and (iv) of Proposition 3. If this common term is nonpositive, we do not need to consider it, since neither V(1, k+1) nor V(0, k+1) is nonpositive. If $R - C + \delta\cdot V(1^*, k) > 0$, then

$$R - C + \delta\cdot V(1^*, k) > (1-\beta)\cdot[R - C + \delta\cdot V(1^*, k)].$$

By the inductive hypothesis,

$$\begin{aligned} & R + \delta\cdot(1-\beta)\cdot K_1/N_1\cdot V(0, k) + \delta\cdot L_1/N_1\cdot V(1, k) \\ & \geq R + \delta\cdot(1-\beta)\cdot K_1/N_1\cdot V(0, k) + \delta\cdot(1-\beta)\cdot L_1/N_1\cdot V(0, k) \\ & = R + \delta\cdot(1-\beta)\cdot V(0, k) \\ & \geq (1-\beta)\cdot[R + \delta\cdot(1-\gamma)\cdot V(0, k)]. \end{aligned}$$

So

$$\begin{aligned} V(1,k+1) & \geq (1-\beta)\cdot\max\{ R + \delta\cdot(1-\gamma)\cdot V(0, k), R - C + \delta\cdot V(1^*, k) \} \\ & = (1-\beta)\cdot V(0, k+1). \end{aligned}$$

ii)

By taking the limit of (i) as n approaches infinity, the result follows. □

Lemma 3

i) If $\alpha_0 > \alpha_1\beta$, then K_i/N_i is nondecreasing in i, and L_i/N_i and $(1-\beta)\cdot K_i/N_i + L_i/N_i$ are nonincreasing in i.

ii) If $\alpha_0 = \alpha_1\beta$, then K_i/N_i, L_i/N_i and $(1-\beta)\cdot K_i/N_i + L_i/N_i$ are constant in i.

iii) If $\alpha_0 < \alpha_1\beta$, then K_i/N_i is nonincreasing in i, and L_i/N_i and $(1-\beta)\cdot K_i/N_i + L_i/N_i$ are nondecreasing in i.

Proof

See Butler and Chou (1981). □

Lemma 4

i) If $\alpha_0 \geq \alpha_1\beta$, then V(i,n) and V(i) are nonincreasing in $i \geq 1$.

ii) If $\alpha_0 \leq \alpha_1\beta$, then V(i,n) and V(i) are nondecreasing in $i \geq 1$.

Proof

i)

There are three terms in the maximum operator in the recursive equation for V(i, n). We use induction to show that all three terms are nonincreasing in $i \geq 1$ for all n.

When $n = 0$, $V(i, 0) = 0$ for all $i \geq 1$. When $n = 1$, the three terms are R, R, R-C, respectively, so $V(i, 1)$ is nonincreasing in $i \geq 1$. Suppose the lemma holds for $n = k$. When $n = k+1$, the first term is

$$R + \delta \cdot [(1-\beta) \cdot K_i/N_i + L_i/N_i] \cdot V(i+1, k).$$

By Lemma 3, $(1-\beta) \cdot K_i/N_i + L_i/N_i$ is nonincreasing, and by the inductive hypothesis $V(i+1,k)$ is nonincreasing, so $[(1-\beta) \cdot K_i/N_i+L_i/N_i] \cdot V(i+1,k)$ is nonincreasing in $i \geq 1$ for all n.

The second term is

$$\begin{aligned} & R + \delta \cdot [(1-\beta) \cdot K_i/N_i] \cdot V(0, k) + \delta \cdot L_i/N_i \cdot V(1, k) \\ &= R + \delta \cdot (1-\beta) \cdot V(0, k) + \delta \cdot L_i/N_i \cdot [V(1, k) - (1-\beta) \cdot V(0, k)]. \end{aligned}$$

$R + \delta \cdot (1-\beta) \cdot V(0, k)$ is a constant and $V(1, k) - (1-\beta) \cdot V(0, k) \geq 0$. By Lemma 2, L_i/N_i is nonincreasing in $i \geq 1$, so the second term is nonincreasing in $i \geq 1$.

The third term is $R - C + \delta \cdot V(1^*, k)$ which is a constant. Thus, $V(i, n)$ is nonincreasing in $i \geq 1$. Since $V(i) = \lim_{n \to \infty} V(i, n)$, $V(i)$ is nonincreasing in $i \geq 1$.

ii)

Using similar arguments, we can show that when $\alpha_0 \leq \alpha_1\beta$, V(i, n) and V(i) are nondecreasing in i $\geq$ 1. This completes the proof. □

Lemma 5

i) If $\alpha_0 \geq \alpha_1\beta \geq \gamma$, then V(0, n) $\geq$ V(1, n) and V(0) $\geq$ V(1).

ii) If $\gamma \geq \alpha_1\beta$, then V(1, n) $\geq$ V(0, n) and V(1) $\geq$ V(0).

Proof

i)

The proof is by induction on n. For n = 1, the result is obvious. Suppose the lemma holds for n = k. By Lemma 4, V(2, k) $\leq$ V(1, k). Also,

$$(1-\beta)\cdot K_1/N_1 + L_1/N_1 = 1 - \alpha_1\beta \leq 1-\gamma.$$

Thus the first term in the expression for V(1, k+1) given in Proposition 3 is

$$\begin{aligned} & R + \delta\cdot[(1-\beta)\cdot K_1/N_1 + L_1/N_1]\cdot V(2, k) \\ \leq\ & R + \delta\cdot(1-\alpha_1\beta)\cdot V(2, k) \\ \leq\ & R + \delta\cdot(1-\gamma)\cdot V(1, k) \\ \leq\ & R + \delta\cdot(1-\gamma)\cdot V(0, k) \end{aligned}$$

by the inductive hypothesis.

The second term in the expression for V(1,k+1) is

$$R + \delta\cdot[(1-\beta)\cdot K_1/N_1]\cdot V(0, k) + \delta\cdot L_1/N_1\cdot V(1, k)$$
$$\leq R + \delta\cdot[(1-\beta)\cdot K_1/N_1]\cdot V(0, k) + \delta\cdot(1-\beta)\cdot L_1/N_1\cdot V(0, k)$$
$$\leq R + \delta\cdot(1-\beta)\cdot V(0, k)$$
$$\leq R + \delta\cdot(1-\gamma)\cdot V(0, k)$$

since $\beta > \gamma$.

The third term in the expression for V(1, k+1) and the second term for the expression for V(0, k+1) are both $R - C + \delta\cdot V(1^*, k)$. So we have

$$V(1,k+1) \leq \max\{R + \delta\cdot(1-\gamma)\cdot V(0,k),\ R - C + \delta\cdot V(1^*,k)\} = V(0,k+1).$$

By taking limits as n approaches infinity, $V(1) \leq V(0)$.

ii)

By Proposition 3,

$$V(1, n) = \max\{ R + \delta\cdot(1-\alpha_1\beta)\cdot V(2, n-1),$$
$$R + \delta\cdot[\alpha_1\cdot(1-\beta)\cdot V(0,n-1) + (1-\alpha_1)\cdot V(1,n-1)],$$
$$R - C + \delta\cdot V(1^*, n-1) \},$$

$$V(0, n) = \max\{ R + \delta\cdot(1-\gamma)\cdot V(0, n-1),\ R - C + \delta\cdot V(1^*, n-1) \}.$$

Since $V(1, n) \geq R + \delta\cdot[\alpha_1\cdot(1-\beta)\cdot V(0,n-1) + (1-\alpha_1)\cdot V(1,n-1)]$, we only need to show that

$$R + \delta\cdot[\alpha_1\cdot(1-\beta)\cdot V(0,n-1) + (1-\alpha_1)\cdot V(1,n-1)] \geq R + \delta\cdot(1-\gamma)\cdot V(0, n-1).$$

The above inequality is true for n = 1. Suppose it is true for n = k. Then for n = k+1

$$R + \delta\cdot[\alpha_1\cdot(1-\beta)\cdot V(0,k) + (1-\alpha_1)\cdot V(1,k)]$$
$$\geq R + \delta\cdot[\alpha_1\cdot(1-\beta)\cdot V(0,k) + (1-\alpha_1)\cdot V(0, k)]$$
$$= R + \delta\cdot(1-\alpha_1\beta)\cdot V(0,k)$$
$$\geq R + \delta\cdot(1-\gamma)\cdot V(0,k)$$

since $\gamma \geq \alpha_1\beta$.

The assertion regarding $V(1) \geq V(0)$ follows by letting n approach infinity. □

Proposition 6

$V(i^*, n)$ and $V(i^*)$ are nonincreasing in $i^* \in S^*$.

Proof

The proof is by induction on n. For n = 1, $V(i^*, n) = R$ for $i^* \in S^*$ and is therefore nonincreasing. Assume the result holds for n = k. Since the third term in the recursive equation for $V(i^*,k+1)$ is a constant, we only need to consider the first two terms. The first term

$$R + \delta\cdot[(1-\beta)\cdot K_i*/N_i* + L_i*/N_i*]\cdot V((i+1)^*, k)$$

is nonincreasing in i^*, since $(1-\beta)\cdot K_i*/N_i* + L_i*/N_i*$ is nonincreasing in i^* (by Proposition 4) and $V((i+1)^*, k)$ is nonincreasing in i^* (by the inductive hypothesis).

The second term in the recursive equation for $V(i^*, k+1)$ is

$$R + \delta\cdot(1-\beta)\cdot K_i{*}/N_i{*}\cdot V(0, k) + \delta\cdot L_i{*}/N_i{*}\cdot V(1, k)$$
$$= R + \delta\cdot(1-\beta)\cdot V(0, k) + \delta\cdot L_i{*}/N_i{*}\cdot[V(1, k) - (1-\beta)\cdot V(0, k)].$$

It is clear that the above is nonincreasing in i^*, since $R + \delta\cdot(1-\beta)\cdot V(0,k)$ is a constant, $L_i{*}/N_i$ nonincreasing by Proposition 4, and $V(1, k) - (1-\beta)\cdot V(0, k)$ is nonnegative by Lemma 2. The result thus holds for $n = k+1$.

Since $V(i^*)$ is the limit of $V(i^*, n)$ as n approaches infinity, it leads to the conclusion. □

Lemma 6

$K_i{*}/N_i{*} \leq K_i/N_i$ for all i. Furthermore, $L_i{*}/N_i{*} \geq L_i/N_i$, and $(1-\beta)\cdot K_i{*}/N_i{*} + L_i{*}/N_i{*} \geq (1-\beta)\cdot K_i/N_i + L_i/N_i$ for all i.

Proof

See Butler and Chou (1981). □

Proposition 7

$V(i^*, n) \geq V(i, n)$ for all $i \geq 1$, $n \geq 1$, and $V(i^*) \geq V(i)$ for all $i \geq 1$.

Proof

The equations for $V(i^*, n)$ and $V(i, n)$ given in Proposition 3 contain the common term $R - C + \delta \cdot V(1^*, n-1)$. So we only need to compare the other two terms.

When n = 1, the difference between the first terms and the difference between the second terms in the recursive equations for $V(i^*, n)$ and $V(i, n)$ are both equal to zero, so the proposition holds. Suppose the proposition holds for n = k, where k > 1. When n = k+1, the difference of the first terms in the equations for $V(i^*, k+1)$ and $V(i, k+1)$ is

$$\delta \cdot [(1-\beta) \cdot K_i^*/N_i^* + L_i^*/N_i^*] \cdot V((i+1)^*, k)$$
$$-\delta \cdot [(1-\beta) \cdot K_i/N_i + L_i/N_i] \cdot V(i+1, k).$$

By Lemma 6, $(1-\beta) \cdot K_i^*/N_i^* + L_i^*/N_i^* \geq (1-\beta) \cdot K_i/N_i + L_i/N_i$. By the inductive hypothesis, $V((i+1)^*, k) - V(i+1, k) \geq 0$. So we have

$$\delta \cdot [(1-\beta) \cdot K_i^*/N_i^* + L_i^*/N_i^*] \cdot V((i+1)^*, k)$$
$$- \delta \cdot [(1-\beta) \cdot K_i/N_i + L_i/N_i] \cdot V(i+1, k)$$
$$\geq \delta \cdot [(1-\beta) \cdot K_i/N_i + L_i/N_i] \cdot [V((i+1)^*, k) - V(i+1, k)]$$
$$\geq 0.$$

The difference of the second terms in the equations for $V(i^*, k+1)$ and $V(i, k+1)$ is

$$\delta\cdot(1-\beta)\cdot(K_i^*/N_i^* - K_i/N_i)\cdot V(0,\ k) + \delta\cdot(L_i^*/N_i^* - L_i/N_i)\cdot V(1,\ k)$$
$$= \delta\cdot(L_i^*/N_i^* - L_i/N_i)\cdot[V(1,\ k) - (1-\beta)\cdot V(0,\ k)]$$
$$\geq 0,$$

by Lemma 2 and Lemma 6. The result regarding the infinite-horizon case follows by taking limits as n approaches infinity. □

Lemma 7

i) If $\alpha_0\beta \geq \gamma$, then $V(0,\ n) \geq V(1^*,\ n)$ and $V(0) \geq V(1^*)$.

ii) If $\alpha_1\beta \leq \gamma$, or if $\alpha_0\beta \leq \gamma$ and $\alpha_0 \leq \alpha_1\beta$, then $V(1^*,\ n) \geq V(0,\ n)$ and $V(1^*) \geq V(0)$.

Proof

To show this lemma is true, we only need to establish the finite-horizon case since the infinite-horizon case follows by taking limits as n approaches infinity.

i)

When n = 1, $V(0,\ 1) = V(1^*,\ 1) = R$, so the lemma holds. Assume the lemma holds for n = k. Now let n = k+1. Notice that there are three terms in the recursive equation for $V(1^*, n)$, and two in the recursive equation for $V(0,\ n)$. We first show that the first term in the equation for $V(0,\ n)$ is greater than or equal to the first two terms in $V(1^*,\ n)$. Note that

$$(1-\beta)\cdot K_1^*/N_1^* + L_1^*/N_1^* = 1 - \alpha_0\beta \le 1 - \gamma.$$

Also,

$$V(2^*, k) \le V(1^*, k) \le V(0, k),$$

by Proposition 6 and the inductive hypothesis. Thus

$$R + \delta\cdot[(1-\beta)\cdot K_1^*/N_1^* + L_1^*/N_1^*]\cdot V(2^*,k) \le R + \delta\cdot(1 - \gamma)\cdot V(0, k).$$

Also,

$$R + \delta\cdot(1-\beta)\cdot K_1^*/N_1^*\cdot V(0, k) + \delta\cdot L_1^*/N_1^*\cdot V(1, k)$$
$$\le R + \delta\cdot(1-\beta)\cdot K_1^*/N_1^*\cdot V(0, k) + \delta\cdot L_1^*/N_1^*\cdot V(1^*,k)$$

by Proposition 7. Thus

$$R + \delta\cdot(1-\beta)\cdot K_1^*/N_1^*\cdot V(0, k) + \delta\cdot L_1^*/N_1^*\cdot V(1,k)$$
$$\le R + \delta\cdot[(1-\beta)\cdot K_1^*/N_1^* + L_1^*/N_1^*]\cdot V(0,k)$$
$$= R + \delta\cdot(1-\alpha_0\beta)\cdot V(0, k)$$
$$\le R + \delta\cdot(1-\gamma)\cdot V(0, k)$$

by the inductive hypothesis.

The last terms in the recursive equations for $V(0,k+1)$ and $V(1^*,k+1)$ are the same, namely $R - C + \delta\cdot V(1^*, k)$. The result thus holds for the finite-horizon case.

ii)

Case 1) $\alpha_1\beta \le \gamma$.

In this case $V(1,n) \ge V(0,n)$ by Lemma 5, and $V(1^*,n) \ge V(1,n)$ by Proposition 7. Thus $V(1^*,n) \ge V(0,n)$.

Case 2) $\alpha_0\beta \le \gamma$ and $\alpha_0 \le \alpha_1\beta$.

The proof is by induction on n. When n = 1, $V(0,1) = V(1^*,1) = R$, so the lemma holds. Assume ii) holds for n = k. Now let n = k+1. We know

$$\begin{aligned} V(1^*, k+1) = \max\{ & R + \delta\cdot(1-\alpha_0\beta)\cdot V(2^*, k), \\ & R + \delta\cdot[\alpha_0\cdot(1-\beta)\cdot V(0, k) + (1-\alpha_0)\cdot V(1, k)], \\ & R - C + \delta\cdot V(1^*, k) \}, \end{aligned}$$

$$V(0, k+1) = \max\{ R + \delta\cdot(1-\gamma)\cdot V(0, k),\ R - C + \delta\cdot V(1^*, k) \}.$$

Since $V(1^*, k+1) \ge R + \delta\cdot(1-\alpha_0\beta)\cdot V(2^*, k)$, we only need to show that $R + \delta\cdot(1-\alpha_0\beta)\cdot V(2^*, k) \ge R + \delta\cdot(1-\gamma)\cdot V(0, k)$.

$$\begin{aligned} R + \delta\cdot(1-\alpha_0\beta)\cdot V(2^*, k) & \ge R + \delta\cdot(1-\gamma)\cdot V(2^*, k) \\ & \ge R + \delta\cdot(1-\gamma)\cdot V(2, k) && \text{by Proposition 7,} \\ & \ge R + \delta\cdot(1-\gamma)\cdot V(1, k) && \text{by Lemma 4,} \\ & \ge R + \delta\cdot(1-\gamma)\cdot V(0, k) && \text{by the inductive hypothesis.} \end{aligned}$$

The last terms in the recursive equations for $V(0, k+1)$ and $V(1^*, k+1)$ are the same for both, namely $R - C + \delta\cdot V(1^*, k)$. □

From the above analysis of the profit functions $V(s,n)$ for $s > 0$, we know that a new system (one that has never been inspected since it was put into operation or was repaired) is better that an old system (one that has been inspected since it was put into operation or was repaired). But when we compare a system which is known to be impaired with a system which was never inspected before, we can not say which is better. If a system is known to be impaired, it must have been inspected previously. Because the information obtained by inspection can reduce its failure rate, under some circumstances, a system known to be impaired might be better than a system that has never been inspected and whose true state is therefore unknown. We also can see the importance of information.

The profit function $V(\cdot, n)$ can give some information about which system might make more expected profit, given the observed state. But to answer the question of which action to take, one must check the optimal policy.

4. DETERMINATION OF OPTIMAL INSPECTION-REPAIR POLICIES

Now we proceed to establish the form of the optimal inspection and repair policy for various ranges of the parameters α_0, α_1, β, γ.

For $s \geq 1$, $V(s, n)$ is the maximum of three terms. Denote the three pairwise differences of those terms as $D(s, n)$, $E(s, n)$, and $H(s, n)$. Then

$$D(s, n) = \delta\cdot[(1-\beta)\cdot K_s/N_s + L_s/N_s]\cdot V(s+1, n-1) - \delta\cdot(1-\beta)\cdot K_s/N_s\cdot V(0, n-1) - \delta\cdot L_s/N_s\cdot V(1, n-1)$$

$$E(s, n) = C + \delta\cdot[(1-\beta)\cdot K_s/N_s + L_s/N_s]\cdot V(s+1, n-1) - \delta\cdot V(1^*, n-1),$$

$$H(s, n) = C + \delta\cdot(1-\beta)\cdot K_s/N_s\cdot V(0, n-1) + \delta\cdot L_s/N_s\cdot V(1, n-1) - \delta\cdot V(1^*, n-1),$$

Each term in the right-hand-sides of the above expressions has a limit, so the limits of $D(s, n)$, $E(s, n)$, and $H(s, n)$ exist as n approaches infinity. Define

$$D(s) = \lim_{n\to\infty} D(s, n),\ E(s) = \lim_{n\to\infty} E(s, n),\ H(s) = \lim_{n\to\infty} H(s, n).$$

To determine the optimal policy for the finite-horizon case, we must study the sign changes of $D(s, n)$, $E(s, n)$, and $H(s, n)$ as s varies. We will show that each function has at most one sign change. Thus the same property will hold for $D(s)$, $E(s)$, and $H(s)$, and so the infinite-horizon results will follow directly.

Property 1

$D(s, n) \leq 0$ implies inspecting is better than doing nothing when in state s at stage n. In this case, to determine the optimal action we only need to consider the sign of $H(s, n)$. If $H(s, n) \leq 0$ then repair is the optimal action. If $H(s, n) \geq 0$ then inspection is the optimal action.

Property 2

If $D(s, n) \geq 0$ then doing nothing is better than inspecting when in state s at stage n. In this case, to determine the optimal action we only need to consider the sign of $E(s, n)$. If $E(s, n) \geq 0$ then doing nothing is the optimal action. If $E(s, n) \leq 0$ then repair is the optimal action.

Properties 1 and 2 tell us how to find the optimal action. We look at the sign of $D(s, n)$ first. If it is nonpositive, we only check the sign of $H(s, n)$. If it is nonnegative, we only check the sign of $E(s, n)$. This simplifies the work in finding the optimal policies.

For fixed n, $D(s, n)$, $E(s, n)$, and $H(s, n)$ are functions of s. If we let the X-axis represent s, and let the Y-axis represent the real numbers, then $D(s, n)$, $E(s, n)$, and $H(s, n)$ are three curves in

R^2. From the sign-change properties of the three curves, the form of the optimal policy will be determined.

4.1 Old system, $\alpha_0 \leq \alpha_1\beta$

In this section we will derive the form of the optimal policy for the parameter range $\alpha_0 \leq \alpha_1\beta$ when the observed state s belongs to S - { -1, 0 }, i.e. the system is an old system.

Theorem 1

D(i, n) and D(i) cross zero at most once and from below as i increases from 1.

Proof

To show this we will prove there exists a j, such that D(i,n) $\geq$ 0 if i $\geq$ j, and D(i, n) is nondecreasing if i < j. We let j be infinite if D(i, n) is nondecreasing for all i $\geq$ 1, and j be zero if D(i,n) $\geq$ 0 for all i $\geq$ 1.

Since $\alpha_0 \leq \alpha_1\beta$, by Lemma 3 and Lemma 4, $[(1-\beta)\cdot K_i/N_i + L_i/N_i]$, L_i/N_i, and V(i, n) are all nondecreasing in i. Now γ must fall into one of the following two ranges:

Case 1: $\alpha_1\beta \leq \gamma$.

Case 2: $\gamma \leq \alpha_1\beta$.

We claim that for Case 1 $D(i, n) \geq 0$ for all $i \geq 1$, and for Case 2 there exists a j (j might be infinite) such that if $i \geq j$ then $D(i, n) \geq 0$, and if $i \leq j$ then $D(i, n)$ is nondecreasing in i.

<u>Proof for Case 1</u>: $\alpha_1\beta \leq \gamma$

Since $\alpha_1\beta \leq \gamma$, by Lemma 5 $V(1,n-1) \geq V(0,n-1)$. Also, $\alpha_0 \leq \alpha_1\beta$, so $V(i+1, n-1) \geq V(1, n-1)$ for all $i \geq 1$. Thus

$$\begin{aligned} D(i,n) &= \delta\cdot(1-\beta)\cdot K_i/N_i\cdot[V(i+1,n-1)-V(0,n-1)] \\ &\qquad - \delta\cdot L_i/N_i\cdot[V(i+1,n-1) - V(1,n-1)] \\ &\geq 0 \qquad \text{for all } i \geq 1. \end{aligned}$$

<u>Proof for Case 2</u>: $\gamma \leq \alpha_1\beta$

Since $\alpha_0 \leq \alpha_1$, $\alpha_0\beta \leq \alpha_1\beta$. Now if $\gamma \leq \alpha_0\beta$, then by Proposition 7 and Lemma 7 we have $V(0,n) \geq V(1^*,n) \geq V(1,n)$. But $V(0,n) \geq V(1,n)$ implies $V(0, n) \geq V(i, n)$ for all $i \geq 1$.

If $\alpha_0\beta \leq \gamma$, then by Lemma 7 we know $V(1^*, n) \geq V(0, n)$. Also, $V(i^*, n)$ is nonincreasing in i^* by Proposition 6, $V(i, n)$ is nondecreasing in i by Lemma 4, and $V(i^*,n) \geq V(i,n)$ by Proposition 7. Thus we have

$$V(1^*,n) \geq V(2^*,n) \geq \ldots \geq V(i^*,n) \geq \ldots \geq V(i,n) \geq \ldots \geq V(1,n).$$

Because $V(1^*, n) \geq V(0, n)$, if there exists a finite $i^* > 1$, such that $V(0, n) \geq V(i^*, n)$, then $V(0, n) \geq V(i, n)$ for all i. If such i^* does not exist, but $\lim_{i^* \to \infty} V(i^*, n) \geq V(0, n) \geq \lim_{i \to \infty} V(i, n)$, we also have $V(0, n) \geq V(i, n)$ for all i. The only other possibility is that there is a finite positive integer j such that $V(i, n) \geq V(0, n)$ for all $i \geq j$, and $V(i, n) \leq V(0, n)$ for $i \leq j$. Thus we have only two possible circumstances for $\gamma \leq \alpha_1 \beta$:

(1) $V(0, n) \geq V(i, n)$ for all i.

(2) There exists a finite integer $j \geq 1$, such that $V(i,n) \geq V(0,n)$ for all $i \geq j$ and $V(i, n) \leq V(0, n)$ for $i \leq j$.

If (1) is true, then

$$D(i,n) = \delta \cdot (1-\beta) \cdot K_i/N_i \cdot [V(i+1,n-1) - V(0,n-1)] + \delta \cdot L_i/N_i \cdot [V(i+1,n-1) - V(1,n-1)]$$

is nondecreasing, because the first term on the right-hand side is

$$\begin{aligned} &\delta \cdot (1-\beta) \cdot K_i/N_i \cdot [V(i+1, n-1) - V(0, n-1)] \\ &= -\delta \cdot (1-\beta) \cdot K_i/N_i \cdot [V(0, n-1) - V(i+1, n-1)], \end{aligned}$$

and both K_i/N_i and $[V(0, n-1) - V(i+1, n-1)]$ are nonincreasing and nonnegative. Thus this term is nondecreasing in all i. Also, L_i/N_i and $V(i+1, n-1)$ are nondecreasing in i, and $V(i+1,n-1) - V(1,n-1)$ is nonnegative. Thus the second term $\delta \cdot L_i/N_i \cdot [V(i+1,n-1) - V(1,n-1)]$ is nondecreasing in all i. Hence D(i,n) is nondecreasing in i for $i \geq 1$.

If (2) is true, then for $i \geq j$ we have $V(i+1, n-1) \geq V(0, n-1)$. Also, we still have $V(i+1, n-1) \geq V(1, n-1)$, so

$$\begin{aligned} D(i,n) &= \delta \cdot (1-\beta) \cdot K_i/N_i \cdot [V(i+1,n-1)-V(0,n-1)] \\ &\quad + \delta \cdot L_i/N_i \cdot [V(i+1,n-1)-V(1,n-1)] \\ &\geq 0 \quad \text{for all } i \geq j. \end{aligned}$$

When $1 < i \leq j$, $V(i, n-1) \leq V(0, n-1)$. Thus the first term of the right-side hand of D(i-1, n) is

$$\delta \cdot (1-\beta) \cdot K_{i-1}/N_{i-1} \cdot [V(i, n-1) - V(0, n-1)],$$

and the second term is

$$\delta \cdot L_i/N_i \cdot [V(i,n-1) - V(1,n-1)].$$

By the preceding argument, both of them are nondecreasing. Thus D(i-1, n) is nondecreasing in i where $1 < i \leq j$.

Because in all cases either $D(i,n) \geq 0$, or D(i,n) is nondecreasing, it crosses zero at most once and from below. The same conclusion holds for D(i) by taking limits as n approaches infinity.

□

In our model the states s are discrete. In Theorem 1 D(i, n) crosses zero means there exists a j such that $D(j, n) \leq 0$ and $D(j+1, n) \geq 0$. But we may not be able to find a j such that $D(j, n) = 0$. For convenience, consider the following extension of the various functions of s defined in this paper. For the remainder of the thesis, we will use small italics letters t, u, v, etc (except s) to represent real numbers. The italics letter s we used before still means the generic element of S. Consider a graph of s versus $D(s, n)$. If we let the X-axis represent real numbers, and the Y-axis represent the values of $D(s,n)$, then it is clear that the graph of $D(s, n)$ is discrete since $D(s, n)$ is only defined at integers greater than or equal to 1. Now change $D(s, n)$ from discrete into continuous by connecting its values at s and $s+1$ with a straight line for all $s \geq 1$. Then it is clear that $D(t, n)$ is well defined for any real number $t \geq 1$. For any closed interval $[s, s+1]$, $D(t, n)$ is the same as the original function at the two end points s and $s+1$, and is a piecewise straight line otherwise. We will use the symbol $D(\cdot, n)$ to represent such an extension of $D(s, n)$ where "$\cdot$" is one of the small italics letters. We make the same extension to all other functions of s, such as $E(s, n)$, $H(s, n)$, $V(s, n)$, $V(s)$ etc.

__Definition__ __1__

A crossing point of $D(t, n)$ or $D(t)$ is said to be finite if there exists a u, where $1 \leq u < +\infty$, such that $D(u,n) = 0$, or $D(u) = 0$.

The reason we use this definition is for technical convenience, since we would otherwise need to use supremums or infimums. In the remainder of this thesis, if the state variable is denoted by any italics letter other than s, then the corresponding function is the extended one. Notice that for an old system, s may be denoted by i, j; for a new system, s may be denoted by i^*, j^*. In the following we will use this notation without any further explanation.

Fact 1

Any extended function has the same increasing or decreasing properties as the original one. Also, if the extended function crosses zero at most once, so does the original one.

We now study the functions $E(\ell, n)$ and $H(\ell, n)$ for an old system.

Proposition 8

i) Both E(i, n) and H(i, n) are nondecreasing functions of i. Hence the extended functions $E(\ell, n)$ and $H(\ell, n)$ are also nondecreasing in $\ell \geq 1$, and they cross zero at most once and from below.

ii) If there exists $1 \leq u < +\infty$, such that $D(\ell, n)$ crosses zero at u, then $E(\ell, n)$ and $H(\ell, n)$ intersect at u, i.e. $E(u, n) = H(u, n)$. When $1 \leq \ell \leq u$, $H(\ell, n) \geq E(\ell, n)$. When $\ell \geq u$, $E(\ell, n) \geq H(\ell, n)$.

The same conclusion also holds for the infinite-horizon case.

Proof

i)

$$E(i,n) = C + \delta\cdot[(1-\beta)\cdot K_i/N_i + L_i/N_i]\cdot V(i+1,n-1) - \delta\cdot V(1^*,n-1).$$

Note that $C - \delta\cdot V(1^*, n-1)$ is a constant when n is fixed. When $\alpha_0 \le \alpha_1\beta$, both $[(1-\beta)\cdot K_i/N_i + L_i/N_i]$ and $V(i+1,n-1)$ are nondecreasing and nonnegative, so their product $\delta\cdot[(1-\beta)\cdot K_i/N_i + L_i/N_i]\cdot V(i+1, n-1)$ is nondecreasing in i. Hence E(i, n) is nondecreasing in i.

$$\begin{aligned} H(i, n) &= C + \delta\cdot(1-\beta)\cdot K_i/N_i\cdot V(0,n-1) \\ &\qquad + \delta\cdot L_i/N_i\cdot V(1,n-1) - \delta\cdot V(1^*,n-1) \\ &= C - \delta\cdot[V(1^*,n-1)-V(1,n-1)] \\ &\qquad - \delta\cdot K_i/N_i\cdot[V(1,n-1)-(1-\beta)\cdot V(0,n-1)]. \end{aligned}$$

Note that $C - \delta\cdot[V(1^*, n-1) - V(1, n-1)]$ is a constant. Also, K_i/N_i is nonincreasing in i and $[V(1, n-1)-(1-\beta)\cdot V(0, n-1)] \ge 0$, so $-\delta\cdot K_i/N_i\cdot[V(1, n-1) - (1-\beta)\cdot V(0, n-1)]$ is nondecreasing in i. Thus H(i, n) is nondecreasing in i.

So both $E(\ell, n)$ and $H(\ell, n)$ are nondecreasing in $\ell \ge 1$, and they cross zero at most once and from below.

ii) Notice that $D(\ell, n) = E(\ell, n) - H(\ell, n)$, so $D(\ell, n) = 0$ implies that $E(\ell,n) = H(\ell,n)$. If $D(\ell,n) \le 0$, then $E(\ell,n) \le H(\ell,n)$. If $D(\ell, n) \ge 0$, then $E(\ell, n) \ge H(\ell, n)$. It is easy to verify that their limits also have the same property. □

Definition 2

If $D(t, n)$ crosses zero at $1 \leq u < +\infty$, we call the value of either $E(u, n)$ or $H(u, n)$ the *Y-value*. In the infinite-horizon case this definition is also applied.

Definition 3

A point at which either $D(\cdot, n)$, or $E(\cdot, n)$, or $H(\cdot, n)$ changes sign is called a *critical point* of the function. In the infinite-horizon case this definition is also applied.

Since the Y-value might be greater than zero, equal to zero, or less than zero, we have the following results.

Proposition 9

Suppose u is a critical point of $D(t, n)$. If the Y-value is positive, then either $H(t, n)$ crosses zero at w, where $1 \leq w < u$, or else $H(t, n) > 0$ for all $t \leq u$. If the Y-value is negative, then either $E(i,n)$ crosses zero at v, where $u < v < +\infty$, or else $E(t,n) < 0$ for all $t \geq u$. If the Y-value is zero, then $E(t, n)$ and $H(t, n)$ intersect zero at u.

Proof

$H(t, n)$ is nondecreasing in $t \geq 1$, so $H(1, n)$ is the minimum of $H(t, n)$. Now suppose the Y-value is positive. If $H(1, n) > 0$, then $H(t, n) > 0$ for all t, so $H(t, n) > 0$ for all $t \leq u$. If $H(1, n) \leq 0$, since $u > 1$ and the Y-value is positive, $H(\ell, n)$ must cross zero at w, where $1 \leq w < u$.

If the Y-value is negative, then $E(1, n) < 0$. Similarly since $E(\ell, n)$ is nondecreasing in ℓ, $E(1, n)$ is the minimum of $E(\ell, n)$. If $E(\ell, n)$ crosses zero at v and $E(u, n) < 0$, we know $v > u$. If $E(\ell, n)$ does not cross zero at all, we must have $E(\ell, n) < 0$ for all ℓ. The case when the Y-value is zero is trivial. □

Recall that if $E(\ell, n)$ and $H(\ell, n)$ intersect at u, the Y-value is $E(u, n) = H(u, n)$. So if u is finite then the Y-value is finite. We also know that $1 \leq u < +\infty$. We will study the sign of the Y-value and the shape of the three curves, D, E, H since they play important roles in determining the optimal policy. We will do the analysis for $u = 1$ and $u > 1$, and for the Y-value is positive, negative, and zero. But first we will establish an important corollary.

Corollary

Assume that u, v, and w are critical points of $D(\cdot,n)$, $E(\cdot,n)$, and $H(\cdot,n)$ respectively. If $w < u$, then $v > w$. If $v > u$, then $w > v$.

Proof

By the definition of u, v, and w, we have $D(u,n) = 0$, $E(v,n) = 0$, and $H(w, n) = 0$. If $w < u$, then $H(u, n) > H(w, n) = 0$. In this case the Y-value is positive. By Proposition 8 we know if $\ell \leq u$, then $H(\ell, n) \geq E(\ell, n)$. So we have $0 = H(w, n) \geq E(w, n)$ which implies $0 = E(v,n) \geq E(w,n)$. Since $E(\ell, n)$ is nondecreasing in $\ell \geq 1$, $v > w$.

Conversely, if $v > u$, then $0 = E(v, n) \geq E(u, n)$, so the Y-value is negative. By Proposition 8, if $\ell \geq u$ then $E(\ell, n) \geq H(\ell, n)$. Thus we have $E(v, n) > H(v, n)$. We know $E(v, n) = 0$ and $H(w, n) = 0$, so $H(w, n) > H(v, n)$. Since $H(\ell, n)$ is nondecreasing in $\ell \geq 1$, $w > v$.

□

This corollary states that if $u > w$, then $H(\ell, n)$ crosses zero before $E(\ell, n)$ does, and if $v > u$, then $E(\ell, n)$ crosses zero before $H(\ell, n)$ does. We now want to use the above results to determine the form of the optimal policies. We will establish eight separate cases, labeled C1 through C8, then combine their results into a single theorem (Theorem 2).

Case 1 : $u = 1$

In this case, since E(1,n) and H(1,n) are the minimal values of $E(\ell, n)$ and $H(\ell,n)$, we know by Proposition 8 that for all $1 \leq \ell < +\infty$, the curve $E(\ell, n)$ is above the curve $H(\ell, n)$, i.e. $E(\ell, n) \geq H(\ell, n)$ for all $\ell \geq 1$ and $n \geq 0$. In this case the Y-value is equal to E(1,n), and is also equal to H(1, n). To determine the form of the optimal policies we only need to look at $E(\ell, n)$.

i) The Y-value is nonnegative.

In this case we know $E(\ell, n)$ is greater than or equal to the Y-value, hence is nonnegative for all $1 \leq \ell < +\infty$, because $E(\ell,n)$ is nondecreasing and the Y-value is equal to E(1, n). Thus do nothing is the optimal policy for all $\ell \geq 1$. ...... (C1)

ii) The Y-value is negative.

In this case if the critical point of $E(\ell, n)$, v, is finite, then for $1 \leq \ell \leq v$, $E(\ell, n) < 0$. So <u>repair</u> is optimal. If $v \leq \ell < +\infty$ then $E(\ell, n) \geq 0$, so <u>do nothing</u> is optimal. This policy is called a <u>repair-do nothing</u> policy. ...... (C2)

If $v = +\infty$ (which means $E(\ell,n)$ does not cross zero), then $w = +\infty$. In this case $E(\ell, n) < 0$ and $H(\ell, n) < 0$ for all $\ell \geq 1$, so <u>repair</u> is the optimal policy for all $\ell \geq 1$. ...... (C3)

<u>Case 2</u>: $1 < u < +\infty$

i) The Y-value is positive.

Both $E(u, n)$ and $H(u, n)$ are equal to the Y-value, hence are positive. If $1 \leq \ell \leq u$, then $H(\ell, n) \geq E(\ell, n)$ and the form of the optimal policy will be determined by the sign of $H(\ell, n)$. If $u \leq \ell < +\infty$, then $E(\ell, n) \geq H(\ell, n)$ and the form of the optimal policy will be determined by the sign of $E(\ell, n)$.

We know that $H(1,n)$ is the minimal value for $H(\ell,n)$ and $H(\ell,n)$ is nondecreasing in $\ell \geq 1$.

If $H(1, n) \geq 0$, then $H(\ell, n) \geq H(1, n) \geq 0$ for $1 \leq \ell \leq u$, and $E(\ell, n) \geq H(\ell, n) \geq H(1, n) \geq 0$ for $u \leq \ell < +\infty$. The optimal policy is <u>inspection-do nothing</u>. <u>Inspection</u> is optimal if $1 \leq \ell \leq u$; and <u>do nothing</u> is optimal if $u \leq \ell < +\infty$. ...... (C4)

If $H(1, n) < 0$, then $E(1, n) \leq H(1, n) < 0$. Since the Y-value is positive and equal to $E(u,n)$ and $H(u,n)$, and both $E(i,n)$ and $H(i,n)$ are nondecreasing in $i \geq 1$, w, the critical point of $H(i, n)$, and v, the critical point of $E(i, n)$ are both less than u. Using both Proposition 8 and Corollary, we know that if $i \leq u$, $H(i, n) \geq E(i, n)$. So $0 = H(w, n) \geq E(w, n)$ and $H(v, n) \geq E(v, n) = 0$, which implies $w < v < u$. Thus, if $1 \leq i \leq w$, $E(i, n) \leq H(i, n) < 0$, repair is optimal. If $w \leq i \leq u$, $H(i, n) \geq 0$, and $H(i, n) \geq E(i, n)$, inspection is optimal. If $u \leq i < +\infty$, $E(i, n) \geq H(i, n) > 0$, do nothing is optimal. So the form of the optimal policy is repair-inspection-do nothing. ...... (C5)

ii) The Y-value is zero.

In this case w, v, and u are equal. If $1 \leq i \leq u$, by Proposition 8 we have $0 > H(i,n) \geq E(i,n)$, so repair is optimal. If $u \leq i < +\infty$, we have $E(i, n) \geq H(i, n) > 0$, so do-nothing is optimal. The form of the optimal policy is repair-do nothing. ...... (C6)

iii) The Y-value is negative.

It is obvious in this case that $H(1, n) < 0$ and $H(i, n) < 0$ for $1 \leq i \leq u$.

When $u \leq i < +\infty$, $E(i, n) > H(i, n)$. We know that $E(i, n)$ crosses zero at most once and from below. Thus either $E(i, n)$ crosses zero at v such that $u \leq v < +\infty$, or $v = +\infty$ which implies $E(i, n) < 0$ for all $i \geq 1$. If v is finite, then $H(i, n) < 0$ for $1 \leq i \leq u$, so repair is optimal. For $u \leq i \leq v$, $E(i, n) < 0$. So repair is again optimal.

For $v \leq i < +\infty$, $E(i, n) \geq 0$. So <u>do nothing</u> is optimal. The form of the optimal policy is <u>repair-do nothing</u>. ...... (C7)

If v is infinite, i.e. $E(i, n) < 0$ for all i, then the optimal policy is always <u>repair</u> since $H(i,n) < 0$ for $1 \leq i \leq u$ and $E(i,n) < 0$ for $u \leq i < +\infty$. ...... (C8)

We summarize the above results in the following theorem.

Theorem 2

Let the points at which $D(\cdot, n)$, $E(\cdot, n)$ and $H(\cdot, n)$ cross zero be u, v, w respectively. If u is finite ($1 \leq u < +\infty$), then the optimal policy has one of the following forms:

(1) <u>Do nothing</u> for all $i \geq 1$ and $n \geq 1$. This occurs when $u = v = 1$ and the Y-value is zero, or when $u = 1$, $v = 0$ ($v = 0$ means $E(i, n) \geq 0$ for all $i \geq 1$).

(2) <u>Repair only</u> for all $i \geq 1$ and $n \geq 1$. This occurs when the Y-value is negative and $v = +\infty$ ($v = +\infty$ means $E(i, n) < 0$ for all i).

(3) <u>Repair-do nothing policy</u>. This occurs when the Y-value is zero or Y-value is nonpositive and v is finite. <u>Repair</u> is optimal if $1 \leq i \leq v$, and <u>do nothing</u> is optimal if $v \leq i < +\infty$.

(4) <u>Inspection-do nothing policy</u>. This occurs when the Y-value is positive, $u > 1$ and $w = 0$ ($w = 0$ means $H(i, n) > 0$ for all $i \geq 1$). <u>Inspection</u> is optimal if $1 \leq i \leq u$, and <u>do nothing</u> is optimal if $u \leq i < +\infty$.

(5) Repair-inspection-do nothing policy. This occurs when the Y-value is positive and $1 < w < u$. Repair is optimal if $1 \leq i \leq w$, inspection is optimal if $w \leq i \leq u$, and do nothing is optimal if $u \leq i < +\infty$.

This theorem also applies to the infinite-horizon case.

Proof

Since u is finite, either $u = 1$, or $1 < u < +\infty$. Whatever u is, the Y-value is either greater than zero, less than zero, or equal to zero. So there are six possible cases. We already discussed the forms of the optimal policy under those six cases and obtained the results (C1) to (C8).

So (1) is the result of (C1).

(2) is the result of (C3) and (C8).

(3) is the result of (C2), (C6), and (C8).

(4) is the result of (C4).

(5) is the result of (C5). □

Notice that in Theorem 2 we assume that $D(i, n)$ crosses zero at u, where $1 \leq u < +\infty$. But what if $u = 0$ (i.e. $D(i, n) \geq 0$ for all $i \geq 1$) or $u = +\infty$ (i.e. $D(i, n) \leq 0$ for all $i \geq 1$)?

In Theorem 1 we already showed that when $\alpha_0 \leq \alpha_1\beta \leq \gamma$, $D(i, n) \geq 0$. Here $u = 0$. We have the following theorem.

Theorem 3

If $\alpha_0 \leq \alpha_1\beta \leq \gamma$, then inspection is never better than do nothing or than repair. Furthermore for the finite-horizon case the optimal policy has one of the following forms:

1) When $C \geq \delta\cdot\alpha_1\beta\cdot R\cdot[1-(\delta-\delta\cdot\alpha_1\beta)^{n-1}]/(1-\delta+\delta\cdot\alpha_1\beta)$, do nothing is optimal.

2) When $C \leq \delta\cdot\alpha_1\beta\cdot R\cdot[1-(\delta-\delta\cdot\alpha_1\beta)^{n-1}]/(1-\delta+\delta\cdot\alpha_1\beta)$, there exists an integer $L > 1$ (L might be infinite) such that if $1 \leq i \leq L$, then repair is optimal. If $i > L$, do nothing is optimal. This policy is called repair-do nothing policy.

For the infinite-horizon case, the above conclusions still hold, except in (1) the condition is $C \geq \delta\cdot\alpha_1\beta\cdot R/(1-\delta+\delta\cdot\alpha_1\beta)$, and in (2) the condition is $C \leq \delta\cdot\alpha_1\beta\cdot R/(1-\delta+\delta\cdot\alpha_1\beta)$.

Proof

Finite-horizon:

1)

By the proof of Theorem 1, $\alpha_0 \leq \alpha_1\beta \leq \gamma$ implies $D(\ell, n) \geq 0$ for all $\ell \geq 1$. So by Property 2, doing nothing is at least as good as inspection. To find the form of the optimal policy we only need to determine the sign of $E(\ell,n)$. Since $E(\ell,n)$ is nondecreasing in $\ell \geq 1$, $E(1, n)$ is the minimum of $E(\ell, n)$. If $E(1, n) \geq 0$ then $E(\ell, n) \geq 0$ for all $\ell \geq 1$. If $E(1, n) < 0$, then there exists a positive integer L (L might be infinite) such that for $1 \leq \ell \leq L$, $E(\ell, n) < 0$. In this case repair is optimal. If $\ell > L$, then $E(\ell, n) \geq 0$, and do nothing is optimal. Our task is to determine when $E(1, n) < 0$, and when $E(1, n) \geq 0$.

$$\begin{aligned} E(1, n) &= \delta\cdot[(1-\beta)\cdot K_1/N_1 + L_1/N_1]\cdot V(2, n-1) + C - \delta\cdot V(1^*, n-1) \\ &= \delta\cdot(1-\alpha_1\beta)\cdot V(2, n-1) + C - \delta\cdot V(1^*, n-1). \end{aligned}$$

Since $V(1^*,n-1) \geq V(2^*,n-1)$ by Proposition 6 and $V(2, n-1) \geq V(1, n-1)$ by Lemma 4, we have

$$\delta\cdot(1-\alpha_1\beta)\cdot V(1,n-1) + C - \delta\cdot V(1^*,n-1) \leq E(1,n) \leq \delta\cdot(1-\alpha_1\beta)\cdot V(2,n-1) + C - \delta\cdot V(2^*, n-1).$$

Then the left-hand side (LHS) is greater than or equal to zero implies $E(1, n) \geq 0$, and the right-hand side (RHS) less than or equal to zero implies $E(1, n) \leq 0$.

If LHS ≥ 0, then

$$C \geq \delta\cdot[\alpha_1\beta\cdot V(1, n-1) + V(1^*, n-1) - V(1, n-1)].$$

By Proposition 7, $V(1^*, n-1) - V(1, n-1) \geq 0$, so $C \geq \delta\cdot\alpha_1\beta\cdot V(1, n-1)$. Now we use induction to show

$$V(1, n-1) \geq R\cdot[1-(\delta-\delta\cdot\alpha_1\beta)^{n-1}]/(1-\delta+\delta\cdot\alpha_1\beta).$$

When n = 1 or n = 2, the result obviously holds. Suppose it holds for n = k. When n = k+1, by the recursive equation for V(1,k), we know

$$V(1, k) \geq R + \delta\cdot[(1-\beta)\cdot K_1/N_1 + L_1/N_1]\cdot V(2, k-1).$$

Since $(1-\beta)\cdot K_1/N_1 + L_1/N_1 = 1 - \alpha_1\beta$, and $V(\ell, n)$ is nondecreasing in $\ell \geq 1$, we have

$$V(1, k) \geq R + \delta\cdot(1-\alpha_1\beta)\cdot V(1, k-1).$$

By the inductive hypothesis,

$$V(1, k-1) \geq R\cdot[1-(\delta-\delta\cdot\alpha_1\beta)^{k-1}]/(1-\delta+\delta\cdot\alpha_1\beta),$$

so

$$V(1, k) \geq R + \delta\cdot(1-\alpha_1\beta)\cdot R\cdot[1-(\delta-\delta\cdot\alpha_1\beta)^{k-1}]/(1-\delta+\delta\cdot\alpha_1\beta)$$
$$= R\cdot[1-(\delta-\delta\cdot\alpha_1\beta)^{k}]/(1-\delta+\delta\cdot\alpha_1\beta).$$

Thus the result $V(1, n-1) \geq R\cdot[1-(\delta-\delta\cdot\alpha_1\beta)^{n-1}]/(1-\delta+\delta\cdot\alpha_1\beta)$ follows. So

$$C \geq \delta\cdot\alpha_1\beta\cdot R\cdot[1-\ (\delta-\delta\cdot\alpha_1\beta)^{n-1}]/(1-\delta+\delta\cdot\alpha_1\beta).$$

This implies that if LHS ≥ 0, we must have

$$C \geq \delta\cdot\alpha_1\beta\cdot R\cdot[1-\ (\delta-\delta\cdot\alpha_1\beta)^{n-1}]/(1-\delta+\delta\cdot\alpha_1\beta).$$

2)

We know that if $C \leq \delta\cdot\alpha_1\beta\cdot R\cdot[1 - (\delta-\delta\cdot\alpha_1\beta)^{n-1}]/(1-\delta+\delta\cdot\alpha_1\beta)$, then LHS ≤ 0, but we cannot conclude that $E(1, n) \leq 0$. If we can show that $E(1, n) \leq 0$ in this case, we are done.

To show this, notice that

$$V(1,n-1) \geq R\cdot[1-(\delta-\delta\cdot\alpha_1\beta)^{n-1}]/(1-\delta+\delta\cdot\alpha_1\beta)$$

and

$$V(2,n-1) \geq V(1,n-1).$$

So if

$$C \leq \delta\cdot\alpha_1\beta\cdot R\cdot[1-(\delta-\delta\cdot\alpha_1\beta)^{n-1}]/(1-\delta+\delta\cdot\alpha_1\beta),$$

then

$$C \leq \delta\cdot\alpha_1\beta\cdot V(2,n-1) \leq \delta\cdot[V(2^*,n-1) - V(2,n-1) + \alpha_1\beta\cdot V(2,n-1)].$$

This is equivalent to $\delta\cdot(1-\alpha_1\beta)\cdot V(2, n-1) + C - \delta\cdot V(2^*, n-1) \leq 0$, which means RHS ≤ 0. Hence $E(1, n) \leq 0$.

<u>Infinite horizon</u>:

Since $0 < \delta < 1$, when n approaches infinity, $(\delta-\delta\cdot\alpha_1\beta)^{n-1}$ approaches zero. This completes the proof. □

In the proof of Theorem 1, we showed when $\alpha_0\beta \leq \gamma$, $D(\ell, n)$ is either nonnegative or nondecreasing in $\ell \geq 1$. By Property 2, if $D(\ell, n) \geq 0$ for all ℓ, then doing nothing is at least as good as inspection. By Property 1, if $D(\ell, n) \leq 0$, but $H(\ell, n) \leq 0$, repair is at least as good as inspection. In general, despite the sign of $D(\ell,n)$, when is repair at least as good as inspection? We will answer this question next.

Theorem 4

i) For a finite-horizon problem,
if $C \leq \delta\cdot\alpha_0\cdot R\cdot[1-(\delta-\delta\cdot\gamma)^{n-1}]/(1-\delta+\delta\cdot\gamma)$ and $\alpha_0\beta \leq \gamma$, then repair is at least as good as inspection.

ii) If $C \leq \delta\cdot\alpha_0\cdot R/(1-\delta+\delta\cdot\gamma)$ and $\alpha_0\beta \leq \gamma$, the same conclusion also holds for an infinite-horizon problem.

Proof

i) Finite horizon

It is sufficient to show that $H(i, n) \leq 0$ for all i. Since $\alpha_0 \leq \alpha_1\beta \leq \beta$, $\alpha_0 \leq \beta$. By Proposition 5 we have $\lim_{s\to\infty} K_s/N_s = \alpha_0/\beta$. Also, when $s \geq 1$, K_s/N_s is nonincreasing in s, so $\alpha_0/\beta \leq K_i/N_i$ for all i. By the recursive equation for V(0, k), we know

$$V(0, k) \geq R + \delta\cdot(1-\gamma)\cdot V(0, k-1).$$

Using induction, it is easy to obtain that

$$V(0, n-1) \geq R\cdot[1-(\delta-\delta\cdot\gamma)^{n-1}]/(1-\delta+\delta\cdot\gamma).$$

Thus,

$$C \leq \delta\cdot\alpha_0\cdot R\cdot[\ 1-(\delta-\delta\cdot\gamma)^{n-1}]/(1-\delta+\delta\cdot\gamma)$$
$$\leq \delta\cdot\beta\cdot\alpha_0/\beta\cdot V(0,\ n-1)$$
$$\leq \delta\cdot\beta\cdot K_i/N_i\cdot V(0,\ n-1) \qquad \text{for all i.}$$

Thus $C - \delta\cdot\beta\cdot K_i/N_i\cdot V(0,\ n-1) \leq 0$. Since $\alpha_0\beta \leq \gamma$, $V(1^*,n) \geq V(0,n)$ by Lemma 7. Also, by Proposition 7, $V(1^*,\ n) \geq V(i,\ n) \geq V(1,\ n)$. So $\delta\cdot V(1^*,\ n) - \delta\cdot K_i/N_i\cdot V(0,\ n-1) - \delta\cdot L_i/N_i\cdot V(1,\ n-1) \geq 0$ for all i. Thus,

$$\delta\cdot V(1^*,\ n) - \delta\cdot K_i/N_i\cdot V(0,\ n-1) - \delta\cdot L_i/N_i\cdot V(1,\ n-1)$$
$$\geq C - \delta\cdot\beta\cdot V(0,\ n-1)\cdot K_i/N_i$$

for all i. This is equivalent to

$$C + \delta\cdot(1-\beta)\cdot K_i/N_i\cdot V(0,n-1) + \delta\cdot L_i/N_i\cdot V(1,n-1) - \delta\cdot V(1^*,n-1) \leq 0$$

for all i. Thus $H(i,\ n) \leq 0$ for all i.

ii) Infinite horizon

As n approaches infinity, $(\delta-\delta\cdot\gamma)^{n-1}$ approaches zero. The same result follows. □

If n is large and $\delta = 1$ (no discount), then the conditions of Theorem 3 reduce to $\alpha_0 \leq \alpha_1\beta \leq \gamma$, and $C \geq R$. In such a case, α_0 is relatively small and repair cost is bigger than the operating reward.

So it is wise to do nothing. Also, when n is large, the conditions of Theorem 4 reduce to $\alpha_0 \leq \alpha_1\beta \leq \gamma$, and $C \leq \alpha_0 \cdot R/\gamma < R$. Since γ is relatively large in this case, the information obtained by inspection would not be so attractive. In both cases, either doing nothing or repair is an optimal action.

For an old system when $\alpha_0 \leq \alpha_1\beta$, repair-inspection-do nothing is the optimal policy. This policy may seem a little strange, but it is optimal for systems that are not dependable after inspection. In an old system, the system has been inspected some number of periods ago. Since inspection is hazardous, even if the system looks fine to an observer, one still may want to repair it shortly after an inspection, because it is likely the system is in the impaired state after inspection. If one inspects the system and the system is found to be impaired, since the usefulness of the information for reducing the failure rate is delayed one time period, inspection might not be worthwhile. If the system is not impaired, the hazardous inspection itself might cause the system to be impaired. So the best decision in this case might be to repair the system directly. If the system survives a reasonable number of time periods after an inspection, one may suspect that the inspection did no harm. Since $\alpha_0 \leq \alpha_1\beta$, we must have $\alpha_0 \leq \beta$. The failure rate β is relatively high. The information obtained by inspection can help people to reduce the failure rate, so inspection in this case is a better choice than repair. If the system survives a large number of time periods after an inspection, one may be quite sure that the inspection did no harm since otherwise it would probably have failed long ago. In this circumstance, doing nothing is the best choice.

4.2 New system, $\alpha_0 \leq \alpha_1\beta$

For a new system, when $\alpha_0 \leq \alpha_1\beta$ we have the following results.

Theorem 5

$D(i^*, n)$ crosses zero at most once and from above.

Proof

Rewrite $D(i^*, n)$ as

$$D(i^*, n) = \delta\cdot L_i*/N_i*\cdot[V((i+1)^*, n-1) - V(1, n-1)] + \delta\cdot(1-\beta)\cdot K_i*/N_i*\cdot[V((i+1)^*, n-1) - V(0, n-1)].$$

$V((i+1)^*,n-1) \geq V(i+1,n-1) \geq V(1,n-1)$ by Proposition 7 and Lemma 4. Also, L_i*/N_i* and $V((i+1)^*,n-1)$ are nonincreasing by Proposition 4 and Proposition 6. So the first term on the right-side hand of the equation for $D(i^*, n)$ is nonincreasing and nonnegative.

In the range of i^* such that $V((i+1)^*, n-1) \geq V(0, n-1)$, we have $(1-\beta)\cdot K_i*/N_i*\cdot[V((i+1)^*, n-1) - V(0, n-1)] \geq 0$. Thus $D(i^*, n)$ is nonnegative. In the range of i^*, such that $V(0,n-1) \geq V((i+1)^*,n-1)$, we have

$$\delta\cdot(1-\beta)\cdot K_i^*/N_i^*\cdot[V((i+1)^*,\ n-1) - V(0,\ n-1)]$$
$$= -\delta\cdot(1-\beta)\cdot K_i^*/N_i^*\cdot[V(0,\ n-1) - V((i+1)^*,\ n-1)].$$

Since K_i^*/N_i^* and $V(0, n-1) - V((i+1)^*, n-1)$ are both nondecreasing and nonnegative, the second term on the right-side hand is nonincreasing. Thus $D(i^*,n)$ is nonincreasing.

Since $D(i^*, n)$ is either greater than or equal to zero, or nonincreasing for all $i^* \geq 1$, so is its limit $D(i^*)$. The result follows. □

Proposition 10

If $s \in S^*$ and $n \geq 1$, both $E(s, n)$ and $H(s, n)$ are nonincreasing in s. Hence they cross zero at most once and from above. The same conclusion holds for $s \in S - \{ -1, 0 \}$ if $\alpha_0 \geq \alpha_1\beta$. This proposition can also be extended to the infinite-horizon case.

Notice that this proposition applies either to a new system or else to an old system if $\alpha_0 \geq \alpha_1\beta$.

Proof

$$E(s,\ n) = C + \delta\cdot[(1-\beta)\cdot K_s/N_s+L_s/N_s\ ]\cdot V(s+1,n-1) - \delta\cdot V(1^*,n-1).$$

If $s \in S - \{-1, 0\}$ and $\alpha_0 \geq \alpha_1\beta$, or else if $s \in S^*$, then both $V(s+1,n-1)$ and $(1-\beta)\cdot K_s/N_s + L_s/N_s$ are nonincreasing in s. So, $\delta\cdot[(1-\beta)\cdot K_s/N_s+L_s/N_s\ ]\cdot V(s+1,n-1)$ is nonincreasing in s. Also,

$C - \delta \cdot V(1^*, n-1)$ is a constant. Hence $E(s, n)$ is nonincreasing in s.

$$H(s, n) = C + \delta \cdot (1-\beta) \cdot K_s/N_s \cdot V(0,n-1) + \delta \cdot (1-K_s/N_s) \cdot V(1,n-1) - \delta \cdot V(1^*,n-1)$$
$$= C + \delta \cdot [V(1,n-1) - V(1^*, n-1)] - \delta \cdot K_s/N_s \cdot [V(1,n-1) - (1-\beta) \cdot V(0,n-1)].$$

As we know, if $s \in S - \{-1, 0\}$ and $\alpha_0 \geq \alpha_1 \beta$, or else if $s \in S^*$, K_s/N_s is nondecreasing in s, and $V(1, n-1) - (1-\beta) \cdot V(0, n-1) \geq 0$, so $K_s/N_s \cdot [V(1, n-1) - (1-\beta) \cdot V(0, n-1)]$ is nondecreasing in s. Also, $C + \delta \cdot [V(1, n-1) - V(1^*, n-1)]$ is a constant. Hence $H(s, n)$ is nonincreasing in s. □

In the proof of the next theorem, we add a superscript "*" to each state variable, since now we are dealing with a new system.

Theorem 6

Let the points at which $D(t^*, n)$, $E(t^*, n)$ and $H(t^*, n)$ cross zero be u^*, v^*, w^* respectively. If u^* is finite $(1 \leq u^* < +\infty)$, then the optimal policy has one of the following forms:

(1) Repair only for all $t^* \geq 1$ and $n \geq 1$. This occurs when the Y-value is negative, or the Y-value is zero and $u^* = w^* = 1$.

(2) Inspection only for all $t^* \geq 1$. This occurs when the Y-value is positive, $w^* = +\infty$ ($w^* = +\infty$ means $H(t^*, n) \geq 0$ for all $t^* \geq 1$), and $u^* = 1$.

(3) <u>Do nothing-inspection policy</u>. This occurs when the Y-value is positive, $u^* > 1$ and $w^* = +\infty$. <u>Do nothing</u> is optimal if $1 \leq t^* \leq u^*$. <u>Inspection</u> is optimal if $u^* \leq t^* < +\infty$.

(4) <u>Inspection-repair policy</u>. This occurs when the Y-value is positive and $u^* = 1$, $w^* < +\infty$. <u>Inspection</u> is optimal if $1 \leq t^* \leq w^*$. <u>Repair</u> is optimal if $w^* \leq t^* < +\infty$.

(5) <u>Do nothing-repair policy</u>. This occurs when the Y-value is negative and $1 < v^* \leq u^*$, or Y-value is zero. <u>Do nothing</u> is optimal if $1 \leq t^* \leq v^*$. <u>Repair</u> is optimal if $v^* \leq t^* < +\infty$.

(6) <u>Do nothing-inspection-repair policy</u>. This occurs when the Y-value is positive and $1 < u^* < w^*$. <u>Do nothing</u> is optimal if $1 \leq t^* \leq u^*$. <u>Inspection</u> is optimal if $u^* \leq t^* \leq w^*$, and <u>repair</u> is optimal if $w^* \leq t^* < +\infty$.

The same conclusions hold for the infinite-horizon case.

Proof

Since u^* is finite, either $u^* = 1$, or $u^* > 1$. First we assume that $u^* = 1$. We know $u^* = 1$ is the critical point of $D(t^*, n)$ and $D(t^*, n)$ crosses zero at most once and from above by Theorem 5, then for all $t^* \geq 1$, $D(t^*, n) \leq 0$.

i) The Y-value is positive.

In this case $H(1^*, n)$ is the Y-value and is the maximal value of $H(t^*, n)$ because $t^* \geq 1$, and $H(t^*, n)$ is nonincreasing by Proposition 10. By Property 1, to determine the form of the optimal policy, we only need to check the sign of $H(t^*, n)$ by Proposition 1. Since $H(w^*, n) = 0$, $H(1^*, n) > 0$, and $H(t^*, n)$ is nonincreasing, $w^* > 1$. If $1 \leq t^* \leq w^*$, then $H(t^*, n) \geq 0$. So <u>inspection</u> is the optimal action. If $t^* > w^*$, then $H(t^*, n) \leq 0$. Thus <u>repair</u> is the optimal action. We know w^* is either finite or infinite. If w^* is finite then the form of the optimal policy is <u>inspection-repair</u>. If w^* is infinite, then <u>inspection</u> is the optimal policy.

ii) The Y-value is nonpositive.

In this case $H(t^*, n) \leq H(1^*, n) \leq 0$ for all t^*, since $H(1^*, n)$ is the Y-value and $H(t^*, n)$ is nonincreasing in t^*. So <u>inspection</u> is the optimal policy.

Now we assume $u^* > 1$. Then $D(t^*, n) \geq 0$ if $1 \leq t^* \leq u^*$, and $D(t^*,n) \leq 0$ if $t^* \geq u^*$. By Property 1 and Property 2, to determine the form of the optimal policy we need to check the sign of $E(t^*, n)$ for $D(t^*, n) \geq 0$, and the sign of $H(t^*, n)$ for $D(t^*, n) \leq 0$.

i) The Y-value is positive.

$E(u^*, n)$ is the Y-value, also $E(t^*, n)$ is nonincreasing in t^*. For the range of $1 \leq t^* \leq u^*$, $E(t^*,n) \geq E(u^*,n) > 0$ and $D(t^*,n) \geq 0$, so <u>do nothing</u> is the optimal action. If $t^* \geq u^*$, $D(t^*, n) \leq 0$; also $H(t^*, n) \leq H(u^*, n)$, and $H(t^*, n)$ is nonincreasing in t^*, so $w^* > u^*$ where w^* is the critical point of $H(t^*, n)$ and $H(w^*, n) = 0$. So

$H(t^*,n) \geq 0$ for $u^* \leq t^* \leq w^*$, <u>inspection</u> is the optimal action. If $t^* \geq w^*$, then $H(t^*, n) \leq 0$, <u>repair</u> is the optimal action. If $w^* = +\infty$, the form of the optimal policy is <u>do nothing-inspection</u>. If $w^* < +\infty$, the optimal policy is <u>do nothing-inspection-repair.</u>

ii) The Y-value is zero.

In this case $v^* = w^* = u^*$, $E(u^*,n) = H(u^*,n) = 0$. If $1 \leq t^* \leq u^*$, then $E(t^*, n) \geq E(u^*, n) \geq 0$, and <u>do nothing</u> is the optimal action. If $t^* \geq u^*$, $H(t^*, n) \leq H(u^*, n) \leq 0$, and <u>repair</u> is the optimal action. Thus the optimal policy is <u>do nothing-repair</u>.

iii) The Y-value is negative.

Since $E(u^*, n) = H(u^*, n) < 0 = E(w^*, n)$ in this case, $w^* \leq u^*$. If $1 \leq t^* \leq w^*$, then $E(t^*, n) \geq 0$ and <u>do nothing</u> is optimal by Property 2. If $w^* \leq t^* \leq u^*$, then $E(t^*, n) \leq 0$ and <u>repair</u> is the optimal action. If $w^* = 0$, which means $E(t^*, n) \leq 0$ for $w^* \leq u^*$, then <u>repair</u> is optimal for $w^* \leq u^*$.

Now for $t^* \geq u^*$, $H(t^*, n) \leq H(u^*, n) < 0$, so <u>repair</u> is the optimal action in this range. Thus the form of the optimal policy either is <u>do nothing-repair</u> which occurs when $1 \leq w^* \leq u^*$, or <u>repair</u> only for all $t^* \geq 1$ which occurs when $E(t^*, n) \leq 0$ for all $t^* \geq 1$.

Summarizing all the above arguments, we have the results stated in the theorem. By taking limits as n approaches infinity, the same results hold for the infinite-horizon case. □

The optimal policy for a new system makes sense intuitively. When a system is new, one would not do anything to it. After it operates for a while, the system might suffer some problems, so it may be wise to inspect the system. The longer it runs, the worse it likely is. In this case to avoid breakdown of the system, the best action is to repair it directly.

4.3 Old system, $\alpha_0 \geq \alpha_1 \beta$

In the preceding sections, we established the form of the optimal policy for either a new system or an old system when $\alpha_0 \leq \alpha_1 \beta$. But in the real world we often have $\alpha_0 \geq \alpha_1 \beta$. What is the form of the optimal policy in this case? We give a partial answer here. In this section we assume $\alpha_0 \geq \alpha_1 \beta$ and $s \in S - \{ -1, 0 \}$. We state our result for $\alpha_1 \beta \geq \gamma$ first.

Proposition 11

If $\alpha_1 \beta \geq \gamma$, then D(i, n) and D(i) are nonpositive for all i $\geq$ 1.

Proof

By Lemma 4 and Lemma 5, $\alpha_0 \geq \alpha_1 \beta$ and $\alpha_1 \beta \geq \gamma$ implies V(0, n) $\geq$ V(1, n) and V(1, n) $\geq$ V(i, n) for all i $\geq$ 1. Thus

$$D(i,n) = \delta \cdot L_i/N_i \cdot [V(i+1,n-1)-V(1,n-1)] + \delta \cdot (1-\beta) \cdot K_i/N_i \cdot [V(i+1,n-1)-V(0,n-1)].$$

Since $V(i+1, n-1) - V(1, n-1) \leq 0$ and $V(i+1, n-1) - V(0, n-1) \leq 0$, $D(i, n) \leq 0$ for all $i \geq 1$. Since $D(i) = \lim_{n\to\infty} D(i, n)$, $D(i) \leq 0$. □

Proposition 12

The optimal policy when $\alpha_1\beta \geq \gamma$ is inspection-repair, which means there exists a $u \geq 1$ such that when $i \leq u$, inspection is optimal; when $i > u$, repair is optimal.

Proof

Since $D(i, n) \leq 0$, by Property 2 we only need to check the sign changes of $H(i, n)$. $H(i, n)$ is nonincreasing by Proposition 10. Hence there exists a u, such that $H(i,n)$ crosses zero at u. If $i \leq u$, $H(i, n) \leq 0$, inspection is optimal. If $i > u$, $H(i, n) \geq 0$, repair is optimal. Notice that if $u < 1$, then inspection is always optimal. If $u = +\infty$, repair is always optimal. □

For an old system, if $\alpha_0 \geq \alpha_1\beta \geq \gamma$, the large value of α_0 implies that with no intervention the system is likely to go to the impaired state, and the small value of γ implies that information obtained by inspection is of great help in reducing the failure rate of the system. So inspection and repair are the optimal actions in this case. The system here is one that needs intensive care.

We have shown the single sign-change property of D(i, n) and D(i) for $\alpha_1\beta \geq \gamma$; the only uncertain case is when $\alpha_0 \geq \alpha_1\beta$ and $\gamma \geq \alpha_1\beta$. Butler and Chou (1981) showed that in their model D(i, n) was monotonic, so it crossed zero at most once. Is the same conclusion still true in this new model? Although all of the numerical examples described below show D(i, n) changing sign only once, we are as yet unable to prove it.

5. NUMERICAL EXAMPLES

In this chapter, we will develop some numerical examples to illustrate our results. We used Microsoft FORTRAN 77, PC Version 3.31 to write a computer program that computes:

1) the observed probabilities K_s/N_s and L_s/N_s,
2) the profit functions $V(s, n)$,
3) the form of the optimal policy.

The program works as follows:

1) Input the repair cost C, the reward for a running system R, the discount factor δ, the one-step transition probabilities α_0, α_1, β, and γ, and the number of stages n.
2) Use Proposition 2 to calculate the observed probabilities.
3) Use Proposition 3 to calculate the profit functions.
4) Calculate $D(\cdot, n)$, $E(\cdot, n)$, and $H(\cdot, n)$ to find the sign changes of each function.

The computer program does not directly use any theoretical result (except Propositions 2 and 3) to obtain the form of the optimal policy.

Notice that in the above calculations, we must calculate all observed probabilities as well as the profit functions. In most circumstances, the calculation does not take too long. When the stage n = 20, observed state j = 20, it only takes seven seconds for an IBM PC-AT with a numeric coprocessor (80287) to compute the results. When

$n = 63$, $j = 37$, the time becomes twenty seconds. If $n = 120$, $j = 80$, the time is one minute and forty seconds. If $n = 150$, $j = 100$, the time is three minutes and thirty seconds. A period in real life often represents a day, a week, a month, or even a year. Generally, the total number of periods is less than two hundred. So the above computation times are satisfactory for most real situations.

Here, we give several numerical examples. Each example shows the input data and the output data produced by the computer program. The output consists of two parts: the numerical results and the form of the optimal policy. Then we analyze the output data to see how the computer output agrees with our theoretical study.

Example 1

Input data:

Where, $\alpha_0 = 0.1$, $\alpha_1 = 0.35$, $\beta = 0.40$, $\gamma = 0.2$, $\delta = 0.92$, $n = 31$, $1 \leq i \leq 37$, $C = 34.4$, $R = 52.3$. We can see that the input data satisfy $C \leq \delta\cdot\alpha_1\beta\cdot R\cdot[1-(\delta-\delta\cdot\alpha_1\beta)^{n-1}]/(1-\delta+\delta\cdot\alpha_1\beta)$ and $\alpha_0 < \alpha_1\beta < \gamma$.

Numerical Output:

i	D(i,n)	D(i*,n)	E(i,n)	E(i*,n)	H(i,n)	H(i*,n)
1	.0000	2.3511	-22.5719	12.3720	-22.5719	-22.5719
2	.0000	.0000	-18.6305	2.6875	-18.6305	-18.6305
3	.0000	.0000	-15.7926	-1.7125	-15.7926	-15.7926
4	.0000	.0000	-13.7976	-4.4554	-13.7976	-13.7976
5	.0000	.0000	-12.4186	-6.2062	-12.4186	-12.4186
6	.0000	.0000	-11.4764	-7.3406	-11.4764	-11.4764
7	.0000	.0000	-10.8378	-8.0829	-10.8378	-10.8378
8	.0000	.0000	-10.4073	-8.5716	-10.4073	-10.4073
9	.0000	.0000	-10.1182	-8.8948	-10.1182	-10.1182
10	.0000	.0000	-9.9245	-9.1090	-9.9245	-9.9245
11	.0000	.0000	-9.7950	-9.2513	-9.7950	-9.7950
12	.0000	.0000	-9.7084	-9.3460	-9.7084	-9.7084
13	.0000	.0000	-9.6506	-9.4090	-9.6506	-9.6506
14	.0000	.0000	-9.6120	-9.4510	-9.6120	-9.6120
15	.0000	.0000	-9.5863	-9.4789	-9.5863	-9.5863
16	.0000	.0000	-9.5691	-9.4976	-9.5691	-9.5691
17	.0000	.0000	-9.5577	-9.5100	-9.5577	-9.5577
18	.0000	.0000	-9.5500	-9.5182	-9.5500	-9.5500
19	.0000	.0000	-9.5450	-9.5237	-9.5450	-9.5450
20	.0000	.0000	-9.5416	-9.5274	-9.5416	-9.5416
21	.0000	.0000	-9.5393	-9.5299	-9.5393	-9.5393
22	.0000	.0000	-9.5378	-9.5315	-9.5378	-9.5378
23	.0000	.0000	-9.5368	-9.5326	-9.5368	-9.5368
24	.0000	.0000	-9.5361	-9.5333	-9.5361	-9.5361
25	.0000	.0000	-9.5357	-9.5338	-9.5357	-9.5357
26	.0000	.0000	-9.5354	-9.5341	-9.5354	-9.5354
27	.0000	.0000	-9.5352	-9.5344	-9.5352	-9.5352
28	.0000	.0000	-9.5350	-9.5345	-9.5350	-9.5350
29	.0000	.0000	-9.5350	-9.5346	-9.5350	-9.5350
30	.0000	.0000	-9.5349	-9.5347	-9.5349	-9.5349
31	.0000	.0000	-9.5349	-9.5347	-9.5349	-9.5349
32	.0000	.0000	-9.5349	-9.5347	-9.5349	-9.5349
33	.0000	.0000	-9.5348	-9.5348	-9.5348	-9.5348
34	.0000	.0000	-9.5348	-9.5348	-9.5348	-9.5348
35	.0000	.0000	-9.5348	-9.5348	-9.5348	-9.5348
36	.0000	.0000	-9.5348	-9.5348	-9.5348	-9.5348
37	.0000	.0000	-9.5348	-9.5348	-9.5348	-9.5348

Policy Analysis Output:

The Lemma 2 (i.e $V(1, n) \geq (1-\beta)\cdot V(0, n)$) does hold.

For an old system:

The optimal action is: always repair.

D(i,n) is always a constant.

For a new system:

Between observed state 1 and 2, the optimal action changes from do nothing to do nothing or inspection.

Between observed state 2 and 3, the optimal action changes from do nothing or inspection to repair.

After state 2, $D(i^*,n)$ changes from decreasing to a constant.

□

From the numerical output, we can see

(1) D(i, n) does not cross zero. $D(i^*, n)$ crosses zero once and from above, in agreement with Theorem 1 and Theorem 5.

(2) Both E(i, n) and H(i, n) are nondecreasing in i, in agreement with Proposition 8.

(3) Both $E(i^*, n)$ and $H(i^*, n)$ are nonincreasing in i^*, in agreement with Proposition 10.

(4) For $k \leq n$, all of V(0, k), V(1, k), and $V(s+1, k)$ are equal to $R - C + \delta\cdot V(1^*, k-1)$, hence $D(i, n) = 0$ for all $i \geq 1$.

(5) Since $D(i, n) = 0$ and $H(i, n) > 0$ for all i, by Property 1, the optimal policy for an old system is always repair.

(6) Since $D(i^*, n) \geq 0$ for all i^*, and $E(i^*, n)$ changes sign between 2 and 3, by Property 2, the optimal policy for a new system is do nothing-repair.

It is very clear that the computer output supports the above conclusions about the optimal policy. Hence it agrees with our theoretical study. There might exist a tie between some optimal actions, and the computer output reveals when such a tie might occur.

When the number of stages is changed from 31 to 63, and the other parameters are unchanged, all conclusions are exactly the same as the above, which further illustrates the correctness of our theoretical study.

Example 2

Input Data:

$\alpha_0 = 0.1$, $\alpha_1 = 0.35$, $\beta = 0.4$, $\gamma = 0.2$, $n = 31$, and $1 \leq i \leq 37$, as in Example 1. We change δ, C, and R to 0.6, 30.4, and 20.3. Now $\alpha_0 < \alpha_1\beta < \gamma$, $C > R$, and $C > \delta\cdot\alpha_1\beta\cdot R\cdot[1-(\delta-\delta\cdot\alpha_1\beta)^{n-1}]/(1-\delta+\delta\cdot\alpha_1\beta)$. We would like to see what has changed in the optimal policy.

Numerical Output:

i	D(i,n)	D(i*,n)	E(i,n)	E(i*,n)	H(i,n)	H(i*,n)
1	.6912	1.9706	28.6079	32.7451	27.9166	27.9166
2	.8459	1.6827	29.1082	31.8142	28.2622	28.2622
3	.9573	1.5100	29.4684	31.2557	28.5111	28.5111
4	1.0357	1.4024	29.7217	30.9075	28.6860	28.6860
5	1.0898	1.3336	29.8967	30.6853	28.8069	28.8069
6	1.1268	1.2891	30.0163	30.5413	28.8895	28.8895
7	1.1518	1.2600	30.0974	30.4471	28.9455	28.9455
8	1.1687	1.2408	30.1520	30.3850	28.9833	28.9833
9	1.1801	1.2281	30.1887	30.3440	29.0086	29.0086
10	1.1877	1.2197	30.2133	30.3168	29.0256	29.0256
11	1.1928	1.2141	30.2297	30.2988	29.0370	29.0370
12	1.1962	1.2104	30.2407	30.2867	29.0446	29.0446
13	1.1984	1.2079	30.2481	30.2787	29.0496	29.0496
14	1.1999	1.2063	30.2530	30.2734	29.0530	29.0530
15	1.2010	1.2052	30.2562	30.2699	29.0553	29.0553
16	1.2016	1.2044	30.2584	30.2675	29.0568	29.0568
17	1.2021	1.2040	30.2599	30.2659	29.0578	29.0578
18	1.2024	1.2036	30.2608	30.2649	29.0585	29.0585
19	1.2026	1.2034	30.2615	30.2642	29.0589	29.0589
20	1.2027	1.2033	30.2619	30.2637	29.0592	29.0592
21	1.2028	1.2032	30.2622	30.2634	29.0594	29.0594
22	1.2029	1.2031	30.2624	30.2632	29.0595	29.0595
23	1.2029	1.2031	30.2625	30.2631	29.0596	29.0596
24	1.2029	1.2030	30.2626	30.2630	29.0597	29.0597
25	1.2029	1.2030	30.2627	30.2629	29.0597	29.0597
26	1.2030	1.2030	30.2627	30.2629	29.0597	29.0597
27	1.2030	1.2030	30.2627	30.2629	29.0597	29.0597
28	1.2030	1.2030	30.2627	30.2629	29.0597	29.0597
29	1.2030	1.2030	30.2627	30.2629	29.0597	29.0597
30	1.2030	1.2030	30.2627	30.2629	29.0597	29.0597
31	1.2030	1.2030	30.2627	30.2629	29.0597	29.0597
32	1.2030	1.2030	30.2627	30.2629	29.0597	29.0597
33	1.2030	1.2030	30.2627	30.2629	29.0597	29.0597
34	1.2030	1.2030	30.2627	30.2629	29.0597	29.0597
35	1.2030	1.2030	30.2627	30.2629	29.0597	29.0597
36	1.2030	1.2030	30.2627	30.2629	29.0597	29.0597
37	1.2030	1.2030	30.2627	30.2629	29.0597	29.0597

Policy Analysis Output:

The Lemma 2 (i.e $V(1, n) \geq (1-\beta)\cdot V(0, n)$) does hold.

For an old system:

The optimal action is always do nothing.

For a new system:

The optimal action is always do nothing. □

The output shows that $D(i, n) \geq 0$, and $E(i, n) \geq 0$. Also,we know that $C > \delta\cdot\alpha_1\beta\cdot R\cdot[1-(\delta-\delta\cdot\alpha_1\beta)^{n-1}]/(1-\delta+\delta\cdot\alpha_1\beta)$, so do nothing is the optimal policy. This result agrees with Property 2 and Theorem 3.

Example 3

Input Data:

Where, $\alpha_0 = 0.2$, $\alpha_1 = 0.35$, $\beta = 0.40$, $\gamma = 0.12$, $\delta = 0.95$, $n = 22$, $1 \leq i \leq 34$, $C = 54.3$, $R = 62.56$. Here $C < R$ and $\alpha_0 > \alpha_1\beta > \gamma$.

Numerical Output:

i	D(i,n)	D(i*,n)	E(i,n)	E(i*,n)	H(i,n)	H(i*,n)
1	-.3732	-.2132	-7.9926	13.1388	-7.6194	-7.6194
2	-.4215	-.3245	-14.3811	-1.5613	-13.9596	-13.9596
3	-.4536	-.3906	-18.6166	-10.2940	-18.1630	-18.1630
4	-.4756	-.4329	-21.5267	-15.8861	-21.0511	-21.0511
5	-.4911	-.4614	-23.5756	-19.6411	-23.0845	-23.0845
6	-.5022	-.4810	-25.0431	-22.2434	-24.5408	-24.5408
7	-.5103	-.4950	-26.1070	-24.0865	-25.5967	-25.5967
8	-.5162	-.5050	-26.8852	-25.4123	-26.3690	-26.3690
9	-.5205	-.5123	-27.4581	-26.3764	-26.9376	-26.9376
10	-.5237	-.5177	-27.8819	-27.0831	-27.3582	-27.3582
11	-.5261	-.5216	-28.1965	-27.6043	-27.6704	-27.6704
12	-.5279	-.5245	-28.4306	-27.9903	-27.9028	-27.9028
13	-.5292	-.5267	-28.6052	-28.2772	-28.0760	-28.0760
14	-.5302	-.5283	-28.7356	-28.4907	-28.2054	-28.2054
15	-.5309	-.5295	-28.8331	-28.6501	-28.3022	-28.3022
16	-.5315	-.5304	-28.9060	-28.7691	-28.3745	-28.3745
17	-.5319	-.5311	-28.9606	-28.8582	-28.4287	-28.4287
18	-.5322	-.5316	-29.0015	-28.9248	-28.4693	-28.4693
19	-.5324	-.5320	-29.0321	-28.9747	-28.4997	-28.4997
20	-.5326	-.5323	-29.0551	-29.0121	-28.5225	-28.5225
21	-.5327	-.5325	-29.0723	-29.0400	-28.5396	-28.5396
22	-.5328	-.5326	-29.0852	-29.0610	-28.5524	-28.5524
23	-.5329	-.5328	-29.0949	-29.0768	-28.5620	-28.5620
24	-.5330	-.5329	-29.1022	-29.0885	-28.5692	-28.5692
26	-.5330	-.5330	-29.1117	-29.1040	-28.5786	-28.5786
27	-.5331	-.5330	-29.1147	-29.1090	-28.5817	-28.5817
28	-.5331	-.5330	-29.1170	-29.1127	-28.5840	-28.5840
29	-.5331	-.5331	-29.1187	-29.1155	-28.5857	-28.5857
29	-.5331	-.5331	-29.1187	-29.1155	-28.5857	-28.5857
30	-.5331	-.5331	-29.1200	-29.1176	-28.5869	-28.5869
31	-.5331	-.5331	-29.1210	-29.1192	-28.5879	-28.5879
32	-.5331	-.5331	-29.1217	-29.1204	-28.5886	-28.5886
33	-.5331	-.5331	-29.1223	-29.1212	-28.5892	-28.5892
34	-.5331	-.5331	-29.1227	-29.1219	-28.5896	-28.5896

Policy Analysis Output:

The Lemma 2 (i.e $V(1, n) \geq (1-\beta)\cdot V(0, n)$) does hold.

For an old system:

The optimal policy is always repair.

$D(i,n)$ is always decreasing.

For a new system:

Between observed state 1 and 2, the optimal action changes from inspection to repair.

$D(i^*,n)$ is always decreasing. □

The output agrees with both Proposition 10 and Proposition 11 for an old system when $\alpha_0 > \alpha_1\beta > \gamma$.

6. CONCLUSIONS AND FURTHER RESEARCH DIRECTIONS

In the preceding sections, the form of the optimal policies has been established for various ranges of parameters. When the parameters satisfy certain conditions, the form of the optimal policy in our model is predictable. Even though we don't know the exact values of the critical points, we know that the optimal policy is of a three-interval form. There are three actions in our model. In general, there exist at most two critical points l and u such that if the time after the system was put into operation or after last inspection is less than or equal to l, one action is optimal. If the above time is greater than l but less than u, another action becomes optimal. If that time is greater than or equal to u, the third action is optimal. For some values of C, R, γ and δ, we might have only one critical point or even none. We consider these special cases of the three-interval policy.

For the same parameters, the form of the optimal policy for an old system and a new system generally is not the same. The reason for this is that the old system was inspected before and inspection is hazardous.

In our model, the form of the three-interval optimal policy is mainly determined by α_0, α_1, β, γ, not by C, R, and δ. In other words, the structure of the system itself determines the form of the optimal policy; the cost parameters affect the critical points, but not the form of the optimal policy.

When the repair cost C is very high and the reward R is relatively small, the action of repair is no longer available. So Butler and Chou's previous model (1981) may be considered a special case of our model.

Now I would like to point out some possible changes and extensions to the current model. Even a small change to the model may cause a great change in both the results and conclusions. We wish to find a model that is close to real-life and also easy to work with. From the preceding sections, we can see that such a model is very hard to establish and analyze. Any improvement in establishing such a model may lead to progress in obtaining satisfactory results. Some possible changes to the current model are as follows.

1) We might assume that inspection is imperfect, which means that with some positive probability the inspection says the system is OK when in reality it is impaired. Thus if the system is in UI state and inspection is taken, there exists a positive probability $1-r$ that the system stays in UI state. Then we would have one of the following two transition probability matrices:

Case 1: the information gained by inspection is delayed one-period

Q(1)

(inspect)

$$\begin{array}{c} \\ OK \\ UI \\ DI \\ F \end{array} \begin{array}{c} \begin{array}{cccc} OK & UI & DI & F \end{array} \\ \left[\begin{array}{cccc} 1-\alpha_1 & \alpha_1 & 0 & 0 \\ 0 & 1-\tau & \tau(1-\beta) & \tau\beta \\ 0 & 0 & 1-\gamma & \gamma \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

Case 2: the information gained by inspection is immediately available

Q(1)

(inspect)

$$\begin{array}{c} \\ OK \\ UI \\ DI \\ F \end{array} \begin{array}{c} \begin{array}{cccc} OK & UI & DI & F \end{array} \\ \left[\begin{array}{cccc} 1-\alpha_1 & \alpha_1 & 0 & 0 \\ 0 & 1-\tau & \tau(1-\gamma) & \tau\gamma \\ 0 & 0 & 1-\gamma & \gamma \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

The difficult task for this model is to analyze K_s/N_s and L_s/N_s. If they still have increasing or decreasing properties in some ranges of the model's parameters, then this model might have better results than the current model.

2) We assume that repair can be done when the system is failed. This would be the hardest and also the most realistic model to analyze. Now $V(-1, n) \neq 0$ for all $n \geq 1$. This change makes it very hard to determine the optimal policy. We have obtained some partial results for this model, but there are still many uncertainties.

3) We may change some assumptions about the cost and reward. The simplest assumption about repair cost is that it is the same regardless of the state. Alternatively we may assume the repair cost is higher if the system has failed. Both assumptions are reasonable. If repair is done by replacing some parts of the system, then the repair cost is generally the same, whether or not the system is failed. If repair is done by rebuilding defective parts, then the second assumption is more reasonable.

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