Supplement to
ANALYSIS OF LONG CYLINDERS OF SANDWICH CONSTRUCTION UNDER UNIFORM EXTERNAL LATERAL PRESSURE
Facings of Moderate and Unequal Thicknesses

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Supplement to

ANALYSIS OF LONG CYLINDERS OF SANDWICH CONSTRUCTION UNDER UNIFORM EXTERNAL LATERAL PRESSURE[1]

Facings of Moderate and Unequal Thicknesses

By

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Summary

A previous analysis of the problem of sandwich cylinders subjected to uniform external lateral loading is extended in order that results may be applied to sandwich cylinders having relatively thick facings of unequal thickness. In the development of the stability criteria, the effect of the stiffness of the individual facings on the stability of the composite cylinder is taken into account. Solutions are obtained from which the stresses and displacements in a stable sandwich cylinder may be determined, and an expression for the determination of the load at which a sandwich cylinder becomes elastically unstable is derived. The sandwich cylinder is assumed to consist of isotropic shell facings and an orthotropic core.

Introduction

This report is a supplement to a previous report that contains a theoretical analysis of sandwich cylinders acted upon by uniform external lateral pressure. Formulas were derived for the stresses in the cylinder, and an expression was obtained from which critical

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[2] Maintained at Madison, Wis., in cooperation with the University of Wisconsin.

pressures may be determined. In that report it was assumed that the facings of the cylinder were thin enough to render membrane theory applicable and that the facings were of equal thickness.

The purpose of this supplementary report is to present solutions for the stresses and for the critical pressures that apply to sandwich cylinders having moderately thick facings of unequal thickness. This requires that the bending moment and the transverse shear in the individual facings be considered in the development of the stability criteria. This extension of the previous work is felt to be of importance in view of the fact that an analysis based on membrane facings may prove inadequate as a design criterion in cases where relatively thick facings are used. It is assumed throughout that buckling takes place at stresses below the elastic limit of the sandwich materials.

The method of analysis used here follows closely the method used in the original report. The same core assumption, namely, that only transverse shear stress and normal stress on planes parallel to the facings are present, is used. It is felt that this assumption applies very well to all practical sandwich constructions because of the relatively low load-carrying capacity in the tangential direction, of the core materials as compared to that of the facing materials. The facings are assumed to be homogeneous and isotropic, and, as indicated previously, they are analyzed on the basis of shell theory rather than membrane theory.

Notation

r, θ	polar coordinates
a	radius to middle surface of outer facing
b	radius to middle surface of inner facing
t_o	thickness of outer facing
t_i	thickness of inner facing
t	thickness of either facing when $t_o = t_i$
E	modulus of elasticity of facings
μ	Poisson's ratio of facings
E_c	modulus of elasticity of core in direction normal to facings
$G_{r\theta}$	modulus of rigidity of core in $r - \theta$ plane

q	intensity of uniform external lateral loading
σ_r, σ_r', σ_{rc}	normal stress in the radial direction acting on the outer facing, on the inner facing, and in the core, respectively
$\bar{\sigma}_r$, $\bar{\sigma}_r'$, $\bar{\sigma}_{rc}$	small normal stress in the radial direction acting on the outer facing, on the inner facing, and in the core, respectively
$\bar{\tau}_{r\theta}$, $\bar{\tau}_{r\theta}'$	small shear stresses acting on outer and inner facings, respectively
$\bar{\tau}_{r\theta c}$	small transverse shear stress in the core
N_θ, N_θ'	direct stress resultants in the tangential direction in the plane of outer and inner facing, respectively
$\bar{N}_\theta$, $\bar{N}_\theta'$	small direct stress resultants in the tangential direction in the plane of outer and inner facings, respectively
$\bar{M}_\theta$, $\bar{M}_\theta'$	small bending moments per unit length of outer and inner facing, respectively
$\bar{Q}_\theta$, $\bar{Q}_\theta'$	small resultant shear forces per unit length of outer and inner facing, respectively
u, u', u_c	radial displacements of outer facing, inner facing, and core respectively
$\bar{u}$, $\bar{u}'$, $\bar{u}_c$	small radial displacements of outer facing, inner facing, and core, respectively
$\bar{v}$, $\bar{v}'$, $\bar{v}_c$	small tangential displacements of outer facing, inner facing, and core, respectively
ϵ_θ, ϵ_θ'	unit tangential strains in outer and inner facing, respectively
$\bar{\epsilon}_\theta$, $\bar{\epsilon}_\theta'$, $\bar{\epsilon}_{\theta c}$	small unit tangential strains in outer facing, inner facing, and core, respectively
$\bar{\chi}_\theta$, $\bar{\chi}_\theta'$	small changes in curvature of outer and inner facing, respectively
n	number of waves in circumference of cylinder
α	one-half the central angle of curved panel
β	$\dfrac{E_c a\,(1 - \mu^2)}{E t_o}$

γ $\quad \dfrac{qa\,(1-\mu^2)}{Et_o}$

δ_n $\quad \dfrac{E_c}{G_{r\theta}} - \dfrac{n^2}{2}$

ϕ_o $\quad \dfrac{t_o^{\,2}}{12a^2}$

ϕ_i $\quad \dfrac{t_i^{\,2}}{12b^2}$

ψ $\quad \dfrac{Et_o\,(1 - b^2/a^2)}{2G_{r\theta}\,b\,(1-\mu^2)}$

A, B, A_n, B_n, C_n, H_n arbitrary constants

Stress Analysis

The sandwich cylinder is assumed to be long enough that the effect of the constraints at the ends is negligible. Under a condition of uniform external loading, each cross section remains circular. The dimensions of a cross section of the cylinder and the positive directions of the polar coordinates, r and θ, are indicated in figure 1. In order to avoid any confusion in regard to signs, the intensity of the external load is assumed to act in the positive r direction; obviously, a negative value of q signifies compressive loading. In the analysis which follows, the assumption is made that the core extends to the middle surfaces of the facings and that the load q is applied to the middle surface of the outer facing. This assumption amounts to neglecting the half-thicknesses of the facings as compared to their radii.

Equilibrium of Core

As previously stated, the assumption is made that, in general, the core transmits only normal stresses in the direction perpendicular to the facings and transverse shear stresses. In the case of uniform external loading, it is noted that the transverse shear stress is zero from considerations of symmetry, and the only stress present in the core is the normal stress in the radial direction, σ_{rc}. Considering the equilibrium of the differential element of the core shown in figure 2, the summation of forces in the radial direction results in the following equation:

$$- \sigma_{rc} \, r \, d\theta + \left(\sigma_{rc} + \frac{d\sigma_{rc}}{dr}\right)\left(r + dr\right) d\theta = 0$$

The differential element is considered to be of unit length in the longitudinal direction. The above equilibrium equation reduces to

$$\frac{d\sigma_{rc}}{dr} + \frac{\sigma_{rc}}{r} = 0 \tag{1}$$

The solution of equation (1) may be written as

$$\sigma_{rc} = E_c \frac{A}{r} \tag{2}$$

where, for convenience, the constant of integration is represented by E_c A. E_c represents the modulus of elasticity of the core in the radial direction, and A is an arbitrary constant. The radial displacement of the core, u_c, is related to the radial stress, σ_{rc}, by the following equation:

$$\sigma_{rc} = E_c \frac{du_c}{dr} \tag{3}$$

From equation (2) it follows that

$$\frac{du_c}{dr} = \frac{A}{r}$$

Integration of the above equation yields

$$u_c = A \log r/a + B \tag{4}$$

The use of log r/a instead of log r merely alters the arbitrary constant of integration.

Equilibrium of Facings

With reference to the differential elements of the facings shown in figure 3, it is seen that, when the cylinder is acted upon by an external loading of uniform intensity, q, the only forces in the facings are the tensile forces per unit length N_θ and N_θ'. σ_r and σ_r' in figure

3 represent the stresses exerted by the core upon the outer and inner facings, respectively. An equilibrium equation can be obtained for each facing by summing forces in the radial direction on each differential element, considered to be of unit length in the longitudinal direction. The equilibrium equation which pertains to the outer facing is

$$qa\ d\theta - \sigma_r a\ d\theta - N_\theta\ d\theta = 0$$

This equation reduces to

$$N_\theta = a\ (q - \sigma_r)$$

or, since $\sigma_r = (\sigma_{rc})_{r = a}$

$$N_\theta = a \left[q - (\sigma_{rc})_{r = a} \right] \tag{5}$$

In a similar manner, the equilibrium equation of the inner facing is found to be

$$N_\theta' = b\ (\sigma_{rc})_{r = b} \tag{6}$$

From the application of Hooke's law along with the assumption that the tangential stress in the facings is uniformly distributed through their thicknesses, the following relationships are obtained:

$$N_\theta = Et_o\ \epsilon_\theta \tag{7}$$

and

$$N_\theta' = Et_i\ \epsilon_\theta' \tag{8}$$

If the right-hand sides of equations (5) and (7) and the right-hand sides of equations (6) and (8) are equated, the following two equations result:

$$\epsilon_\theta = \frac{a}{Et_o} \left[q - (\sigma_{rc})_{r = a} \right]$$

and

$$\epsilon_\theta' = \frac{b}{Et_i} (\sigma_{rc})_{r = b}$$

Since $\epsilon_\theta = \frac{u}{a}$ and $\epsilon_\theta' = \frac{u'}{b}$, these two equations become

$$u = \frac{a^2}{Et_o}\left[q - (\sigma_{rc})_{r=a}\right] \tag{9}$$

and

$$u' = \frac{b^2}{Et_i}(\sigma_{rc})_{r=b} \tag{10}$$

If continuity of displacements at the interfaces is assumed, then

$$u = (u_c)_{r=a}$$

and

$$u' = (u_c)_{r=b}$$

The above relationships in conjunction with equations (2) and (4) enable equations (9) and (10) to be written as follows:

$$B = \frac{a^2}{Et_o}\left(q - E_c \frac{A}{a}\right) \tag{11}$$

and

$$A \log b/a + B = \frac{E_c b}{Et_i} A \tag{12}$$

Equations (11) and (12) may be solved for A and B with the following results:

$$A = \frac{qa}{E_c} k \tag{13}$$

and

$$B = \frac{qa^2}{Et_o}(1 - k) \tag{14}$$

where

$$k = \frac{1}{1 + \frac{b}{a}\frac{t_o}{t_i} - \frac{Et_o}{E_c a}\log b/a} \tag{15}$$

The substitution of the value of A given by equation (13) into equation (2) yields the following expression for the radial stress in the core:

$$\sigma_{rc} = q \frac{a}{r} k \tag{16}$$

When the above value of σ_{rc} is substituted into equations (5) and (6), the following two equations result:

$$N_\theta = qa \ (1 - k) \tag{17}$$

and

$$N_\theta' = qak \tag{18}$$

The substitution of the values of A and B given by equations (13) and (14) into equation (4) yields the following expression for the core displacement:

$$u_c = \frac{qa^2}{Et_o} [1 - k \ (1 - \frac{Et_o}{E_c a} \log r/a)] \tag{19}$$

Since $u = (u_c)_{r = a}$ and $u' = (u_c)_{r = b}$, then

$$u = \frac{qa^2}{Et_o} (1 - k) \tag{20}$$

and

$$u' = \frac{qab}{Et_i} k \tag{21}$$

Equations (16) - (21) completely describe the stresses and displacements in the sandwich cylinder subjected to uniform external, lateral loading. In each of these equations the value of k is given by equation (15).

Stability Analysis

In discussing the stability of the sandwich cylinder under uniform external lateral pressure, the equilibrium of a slightly deformed element of the cylinder must be considered. It is assumed that the stress situation that exists in this deformed element differs only slightly from the stress situation that existed just before buckling. The stresses before buckling are given by equations (16), (17), and (18). Since the cross section of the deformed cylinder is no longer circular, the small changes in the stresses and the displacements resulting from buckling will be functions of θ as well as r. Following the system of notation used in the original report, a bar is placed over the appropriate symbol to denote the small stresses, strains, and displacements that occur when the cylinder goes from the initial uniformly stressed circular form to the slightly deformed configuration. Again, it is assumed that the core extends to the middle surfaces of the facings and that the load q is applied to the middle surface of the outer facing. This assumption is now somewhat more restrictive than it was in the case of axial symmetry, since it now means that the effect of the interface shear on the bending of the facings and the displacements due to the bending of the individual facings are neglected. However, it is felt that, for cylinders with shell-type facings, the results based on this assumption should be of sufficient accuracy.

Equilibrium of the Core

Since the cylinder is now considered to be slightly deformed, the core is also slightly deformed. A differential element of this core is shown in figure 4. In addition to the radial stress, $q \frac{a}{r} k$, given by equation (16), a small radial stress, $\bar{\sigma}_{rc}$, and a small transverse shear stress, $\bar{\tau}_{r\theta c}$, due to buckling must be taken into account. The differential element is considered to be in equilibrium under the action of the stresses shown. Since the small change in the geometry of the core introduces only small terms of higher order into the equilibrium equations, the differential element in figure 4 is shown in its undeformed state. The analysis of the core is exactly the same as that given in the original report, and for this reason only the basic equations will be repeated here.

The equilibrium equations that apply to the core are obtained by summing forces in the radial and tangential directions in figure 4. These equilibrium equations are:

$$\frac{\partial \bar{\sigma}_{rc}}{\partial r} + \frac{\bar{\sigma}_{rc}}{r} + \frac{1}{r} \frac{\partial \bar{\tau}_{r\theta c}}{\partial \theta} = 0 \tag{22}$$

and

$$\frac{\partial \bar{\tau}_{r\theta c}}{\partial r} + \frac{2\bar{\tau}_{r\theta c}}{r} = 0 \tag{23}$$

The following stress-displacement relations are also applicable:

$$\sigma_{rc} = E_c \frac{\partial \bar{u}_c}{\partial r} \tag{24}$$

and

$$\bar{\tau}_{r\theta c} = G_{r\theta} \left[\frac{1}{r} \frac{\partial \bar{u}_c}{\partial \theta} + \frac{\partial \bar{v}_c}{\partial r} - \frac{\bar{v}_c}{r}\right] \tag{25}$$

It was shown in the original report that the small radial and tangential displacements of the core can be completely determined insofar as their dependence on r is concerned. Then, on the assumption that during buckling the circumference of the cylinder subdivides into n waves, the displacement functions are written as follows:

$$\bar{u}_c = \left[A_n + B_n \frac{a}{r} + C_n \log \frac{r}{a}\right] \cos n\theta \tag{26}$$

and

$$\bar{v}_c = \left[- n A_n + \frac{1}{n} \delta_n B_n \frac{a}{r} - n C_n \left(1 + \log \frac{r}{a}\right) + H_n \frac{r}{a}\right] \sin n\theta \tag{27}$$

where

$$\delta_n = \frac{E_c}{2G_{r\theta}} - \frac{n^2}{2}$$

and

A_n, B_n, C_n, and H_n are arbitrary constants.

After the substitution of the expressions for $\bar{u}_c$ and $\bar{v}_c$ given by equations (26) and (27) into equations (24) and (25), the expressions for the small core stresses become

$$\bar{\sigma}_{rc} = \frac{E_c}{r}\left[- B_n \frac{a}{r} + C_n\right] \cos n\theta \tag{28}$$

and

$$\bar{\tau}_{r\theta c} = - \frac{1}{n} \frac{E_c}{r} B_n \frac{a}{r} \sin n\theta \tag{29}$$

It may be easily verified that equations (28) and (29) satisfy the equilibrium equations, equations (22) and (23).

Equilibrium of Facings

Figure 5 shows the differential elements of the facings of the slightly deformed cylinder. In addition to the forces that exist just before buckling (given by equations (16), (17), and (18)), the small forces and moments that arise during buckling are shown in the figure. In considering the equilibrium of the facing elements, account is taken of the rotation and stretching of the facings which occurs during buckling. As described in the original report, the initial central angle, $d\theta$, becomes

$$\left(1 + \frac{1}{a} \frac{\partial \bar{v}}{\partial \theta} - \frac{1}{a} \frac{\partial^2 \bar{u}}{\partial \theta^2}\right) d\theta \qquad \text{and} \qquad \left(1 + \frac{1}{b} \frac{\partial \bar{v}'}{\partial \theta} - \frac{1}{b} \frac{\partial^2 \bar{u}'}{\partial \theta^2}\right) d\theta$$

for the outer and inner facings, respectively, and the areas of the differential elements of the outer and inner facings become $(1 + \bar{\epsilon}_\theta)$ a $d\theta$ and $(1 + \bar{\epsilon}'_\theta)$ b $d\theta$, respectively. Three equations of equilibrium can be written for each facing; the differential elements are considered to be of unit length in the longitudinal direction.

Considering first the differential element of the outer facing, the summation of forces in the direction normal to the surface yields

$$q\,(1 + \bar{\epsilon}_\theta)\,a\,d\theta - (qk + \bar{\sigma}_r)(1 + \bar{\epsilon}_\theta)\,a\,d\theta$$

$$- [qa\,(1 - k) + \bar{N}_\theta]\left(1 + \frac{1}{a}\frac{\partial \bar{v}}{\partial \theta} - \frac{\partial^2 \bar{u}}{\partial \theta^2}\right) d\theta + \frac{\partial \bar{Q}_\theta}{\partial \theta}\,d\theta = 0$$

If the relationship, $\bar{\epsilon}_\theta = \frac{\bar{u}}{a} + \frac{1}{a}\frac{\partial \bar{v}}{\partial \theta}$, is used and small quantities of higher order -- that is, products of barred quantities -- are neglected, the above equation becomes

$$\bar{N}_\theta - \frac{\partial \bar{Q}_\theta}{\partial \theta} = -\,a\,\bar{\sigma}_r + qa\,(1 - k)\left(\frac{\bar{u}}{a} + \frac{1}{a}\frac{\partial^2 \bar{u}}{\partial \theta^2}\right) \tag{30}$$

The summation of forces in the tangential direction yields the following equation:

$$-\left[qa\,(1 - k) + \bar{N}_\theta\right] + \left[qa\,(1 - k) + \bar{N}_\theta + \frac{\partial \bar{N}_\theta}{\partial \theta}\,d\theta\right]$$

$$- \bar{\tau}_{r\theta}\,(1 + \bar{\epsilon}_\theta)\,a\,d\theta + \bar{Q}_\theta\left(1 + \frac{1}{a}\frac{\partial \bar{v}}{\partial \theta} - \frac{1}{a}\frac{\partial^2 \bar{u}}{\partial \theta^2}\right) d\theta = 0$$

If small quantities of higher order are again neglected, the above equation may be written as

$$\frac{\partial \bar{N}_\theta}{\partial \theta} + \bar{Q}_\theta = a\,\bar{\tau}_{r\theta} \tag{31}$$

The third equilibrium equation of the outer facing is obtained by equating to zero the summation of moments about point $\underline{0}$, shown in figure 5. Thus

$$- \bar{M}_\theta + \left(\bar{M}_\theta + \frac{\partial M_\theta}{\partial \theta}\,d\theta\right) + \bar{Q}_\theta\,a\,d\theta = 0$$

The above equation reduces to

$$\frac{\partial \bar{M}_\theta}{\partial \theta} + a\,\bar{Q}_\theta = 0 \tag{32}$$

Similarly the three equilibrium equations which pertain to the inner facing are obtained by summing forces in the normal and tangential directions and by summing moments about the point $\underline{O'}$ using the differential element of the inner facing shown in figure 5. These equilibrium equations are:

$$\overline{N}'_\theta - \frac{\partial \overline{Q}_\theta}{\partial \theta} = b\ \overline{\sigma}'_r + qak\left(\frac{\overline{u}'}{b} + \frac{1}{b}\frac{\partial^2 \overline{u}'}{\partial \theta^2}\right) \qquad (33)$$

$$\frac{\partial \overline{N}'_\theta}{\partial \theta} + \overline{Q}'_\theta = -\ b\ \overline{\tau}'_{r\theta} \qquad (34)$$

and

$$\frac{\partial \overline{M}'_\theta}{\partial \theta} + b\ \overline{Q}'_\theta = 0 \qquad (35)$$

If equation (32) is solved for $\overline{Q}_\theta$ and the resulting value is substituted into equations (30) and (31), $\underline{\text{the}}$ three equilibrium equations of the outer facing are reduced to the following two equations:

$$N_\theta + \frac{1}{a}\frac{\partial^2 \overline{M}_\theta}{\partial \theta^2} = -\ a\ \overline{\sigma}_r + q\ (1 - k)\left(\overline{u} + \frac{\partial^2 \overline{u}}{\partial \theta^2}\right) \qquad (36)$$

and

$$\frac{\partial \overline{N}_\theta}{\partial \theta} - \frac{1}{a}\frac{\partial \overline{M}_\theta}{\partial \theta} = a\ \overline{\tau}_{r\theta} \qquad (37)$$

In like manner, if equation (35) is solved for $\overline{Q}'_\theta$ and this value is substituted into equations (33) and (34), the $\underline{\text{three}}$ equilibrium equations of the inner facing are reduced to the following two equations:

$$\overline{N}'_\theta + \frac{1}{b}\frac{\partial^2 \overline{M}'_\theta}{\partial \theta^2} = b\ \overline{\sigma}'_r + q\ \frac{a}{b}\ k\left(\overline{u}' + \frac{\partial^2 \overline{u}'}{\partial \theta^2}\right) \qquad (38)$$

and

$$\frac{\partial \overline{N}'_\theta}{\partial \theta} - \frac{1}{b}\frac{\partial \overline{M}'_\theta}{\partial \theta} = -\ b\ \overline{\tau}'_{r\theta} \qquad (39)$$

From the application of Hooke's law, the following two expressions relating the small tangential forces per unit length, $\underline{\overline{N}_\theta}$ and $\underline{\overline{N}'_\theta}$, to the small tangential strains, $\underline{\overline{\epsilon}_\theta}$ and $\underline{\overline{\epsilon}'_\theta}$, are obtained:

$$\bar{N}_\theta = \frac{Et_o}{1 - \mu^2} \bar{\epsilon}_\theta \quad (40)$$

and

$$\bar{N}'_\theta = \frac{Et_i}{1 - \mu^2} \bar{\epsilon}'_\theta \quad (41)$$

Also, the following two equations relating the bending moments in the facings, $\bar{M}_\theta$ and $\bar{M}'_\theta$, to the changes in curvature in the facings, $\bar{\chi}_\theta$ and $\bar{\chi}'_\theta$, are applicable:[3]

$$\bar{M}_\theta = - \frac{Et_o^3}{12(1 - \mu^2)} \bar{\chi}_\theta \quad (42)$$

and

$$\bar{M}'_\theta = - \frac{Et_i^3}{12(1 - \mu^2)} \bar{\chi}'_\theta \quad (43)$$

Since[4]

$$\bar{\epsilon}_\theta = \frac{\bar{u}}{a} + \frac{1}{a} \frac{\partial \bar{v}}{\partial \theta}$$

$$\bar{\epsilon}'_\theta = \frac{\bar{u}'}{b} + \frac{1}{b} \frac{\partial \bar{v}'}{\partial \theta}$$

$$\bar{\chi}_\theta = \frac{1}{a^2} \left(\frac{\partial \bar{v}}{\partial \theta} - \frac{\partial^2 \bar{u}'}{\partial \theta^2} \right)$$

and

$$\bar{\chi}'_\theta = \frac{1}{b^2} \left(\frac{\partial \bar{v}'}{\partial \theta} - \frac{\partial^2 \bar{u}'}{\partial \theta^2} \right)$$

equations (40) - (43) may be written as follows:

$$\bar{N}_\theta = \frac{Et_o}{1 - \mu^2} \left(\frac{\bar{u}}{a} + \frac{1}{a} \frac{\partial \bar{v}}{\partial \theta} \right) \quad (44)$$

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[3] Timoshenko, S., and Goodier, J. Theory of Elasticity. New York, 1951.

[4] Timoshenko, S. Theory of Plates and Shells, First Edition. New York 1940.

$$\bar{N}'_\theta = \frac{Et_i}{1 - \mu^2}\left(\frac{\bar{u}'}{b} + \frac{1}{b}\frac{\partial \bar{v}'}{\partial \theta}\right) \tag{45}$$

$$\bar{M}_\theta = - \frac{Et_o^3}{12\ (1 - \mu^2)\ a^2}\left(\frac{\partial \bar{v}}{\partial \theta} - \frac{\partial^2 \bar{u}}{\partial \theta^2}\right) \tag{46}$$

and

$$\bar{M}'_\theta = - \frac{Et_i^3}{12\ (1 - \mu^2)\ b^2}\left(\frac{\partial \bar{v}'}{\partial \theta} - \frac{\partial^2 \bar{u}'}{\partial \theta^2}\right) \tag{47}$$

The requirement is now made that there be continuity of displacements at the interfaces; that is,

$$\bar{u} = (\bar{u}_c)_{r = a} \tag{48}$$

$$\bar{u}' = (\bar{u}_c)_{r = b} \tag{49}$$

$$\bar{v} = (\bar{v}_c)_{r = a} \tag{50}$$

and

$$\bar{v}' = (\bar{v}_c)_{r = b} \tag{51}$$

The use of equations (48) - (51) enables equations (44) - (47) to be written as follows:

$$\bar{N}_\theta = \frac{Et_o}{a\ (1 - \mu^2)}\left(\bar{u}_c + \frac{\partial \bar{v}_c}{\partial \theta}\right)_{r = a} \tag{52}$$

$$\bar{N}'_\theta = \frac{Et_i}{b\ (1 - \mu^2)}\left(\bar{u}_c + \frac{\partial \bar{v}_c}{\partial \theta}\right)_{r = b} \tag{53}$$

$$\bar{M}_\theta = - \frac{Et_o^3}{12a^2\ (1 - \mu^2)}\left(\frac{\partial \bar{v}_c}{\partial \theta} - \frac{\partial^2 \bar{u}_c}{\partial \theta^2}\right)_{r = a} \tag{54}$$

and

$$\bar{M}'_\theta = - \frac{Et_i^3}{12b^2\ (1 - \mu^2)}\left(\frac{\partial \bar{v}_c}{\partial \theta} - \frac{\partial^2 \bar{u}_c}{\partial \theta^2}\right)_{r = b} \tag{55}$$

With the aid of equations (48) - (55) and the fact that

$$\overline{\tau}_{r\theta} = (\overline{\tau}_{r\theta c})_{r = a} \tag{56}$$

$$\overline{\tau}_{r\theta}' = (\overline{\tau}_{r\theta c})_{r = b} \tag{57}$$

$$\overline{\sigma}_r = (\overline{\sigma}_{rc})_{r = a} \tag{58}$$

and

$$\overline{\sigma}_r' = (\overline{\sigma}_{rc})_{r = b} \tag{59}$$

the equilibrium equations of the facings, equations (36) - (39), may be expressed entirely in terms of the core displacements and stresses. For example, if the right hand sides of equations (52), (54), (56), (58), and (48) are substituted for $\overline{N}_\theta$, $\overline{M}_\theta$, $\overline{\tau}_{r\theta}$, $\overline{\sigma}_r$, and $\overline{u}$, respectively, in equilibrium equation (36), the result is

$$\frac{Et_o}{a\ (1 - \mu^2)} \left(\overline{u}_c + \frac{\partial \overline{v}_c}{\partial \theta}\right)_{r = a} - \frac{Et_o^3}{12\ (1 - \mu^2)\ a^3} \left(\frac{\partial^3 \overline{v}_c}{\partial \theta^3} - \frac{\partial^4 \overline{u}_c}{\partial \theta^4}\right)_{r = a}$$

$$= - a\ (\overline{\sigma}_{rc})_{r = a} + q\ (1 - k) \left(\overline{u}_c + \frac{\partial^2 \overline{u}_c}{\partial \theta^2}\right)_{r = a} \tag{60}$$

In a similar manner, equations (37) - (39) may be transformed, respectively, to the following three equations:

$$\frac{Et_o}{a\ (1 - \mu^2)} \left(\frac{\partial \overline{u}_c}{\partial \theta} + \frac{\partial^2 \overline{v}_c}{\partial \theta^2}\right)_{r = a} + \frac{Et_o^3}{12\ (1 - \mu^2)\ a^3} \left(\frac{\partial^2 \overline{v}_c}{\partial \theta^2} - \frac{\partial^3 \overline{u}_c}{\partial \theta^3}\right)_{r = a}$$

$$= a\ (\tau_{r\theta c})_{r = a} \tag{61}$$

$$\frac{Et_i}{b\ (1 - \mu^2)} \left(\overline{u}_c + \frac{\partial \overline{v}_c}{\partial \theta}\right)_{r = b} - \frac{Et_i^3}{12\ (1 - \mu^2)\ b^3} \left(\frac{\partial^3 \overline{v}_c}{\partial \theta^3} - \frac{\partial^4 \overline{u}_c}{\partial \theta^4}\right)_{r = b}$$

$$= b\ (\overline{\sigma}_{rc})_{r = b} + q\ \frac{a}{b}\ k \left(\overline{u}_c + \frac{\partial^2 \overline{u}_c}{\partial \theta^2}\right)_{r = b} \tag{62}$$

and

$$\frac{Et_i}{b\,(1-\mu^2)}\left(\frac{\partial \bar{u}_c}{\partial \theta}+\frac{\partial^2 \bar{v}_c}{\partial \theta^2}\right)_{r=b} + \frac{Et_i^3}{12\,(1-\mu^2)\,b^3}\left(\frac{\partial^2 \bar{v}_c}{\partial \theta^2}-\frac{\partial^3 \bar{u}_c}{\partial \theta^3}\right)_{r=b}$$

$$= -\,b\,(\bar{\tau}_{r\theta c})_{r=b} \qquad (63)$$

If the expressions given by equations (26) - (29) for the core displacements and stresses are substituted into equations (60) - (63), four equations containing the parameters $\underline{A_n}$, $\underline{B_n}$, $\underline{C_n}$, and $\underline{H_n}$ are obtained. These four equations are:

$$[(n^2-1)\,\gamma\,(1-k)-(n^2-1)]\,A_n+[(n^2-1)\,\gamma\,(1-k)+\delta_n\,(1+n^2\,\phi_o)$$
$$+(1+n^4\,\phi_o)-\beta\,]\,B_n+[\beta-n^2\,(1+n^2\,\phi_o)]\,C_n$$
$$+[n\,(1+n^2\,\phi_o)]\,H_n=0 \qquad (64)$$

$$(n^2-1)\,A_n+[-\delta_n\,(1+\phi_o)-(1+n^2\,\phi_o)+\frac{\beta}{n^2}]\,B_n$$
$$+[n^2\,(1+\phi_o)]\,C_n+[-n\,(1+\phi_o)]\,H_n=0 \qquad (65)$$

$$[(n^2-1)\,\gamma k\,\frac{t_o}{t_i}-(n^2-1)]\,A_n+[(n^2-1)\,\frac{a}{b}\,\gamma k\,\frac{t_o}{t_i}+\delta_n\,\frac{a}{b}(1+n^2\,\phi_i)$$
$$+\frac{a}{b}\,(1+n^4\,\phi_i)+\beta\,\frac{t_o}{t_i}]\,B_n+[(n^2-1)\,\gamma k\,\frac{t_o}{t_i}\log b/a-\beta\,\frac{b}{a}\,\frac{t_o}{t_i}$$
$$-(n^2-1)\log b/a-n^2\,(1+n^2\,\phi_i)]\,C_n+[n\,b/a\,(1+n^2\,\phi_i)]\,H_n=0$$
$$(66)$$

and

$$(n^2-1)\,A_n+[-\delta_n\,\frac{a}{b}\,(1+\phi_i)-\frac{a}{b}\,(1+n^2\,\phi_i)-\frac{\beta}{n^2}\,\frac{t_o}{t_i}]\,B_n$$
$$+[(n^2-1)\log b/a+n^2\,(1+\phi_i)]\,C_n+[-n\,b/a\,(1+\phi_i)]\,H_n=0$$
$$(67)$$

where, in each of the above equations,

$$\gamma=\frac{qa\,(1-\mu^2)}{Et_o}$$

$$\delta_n = \frac{E_c}{2G_{r\theta}} - \frac{n^2}{2}$$

$$\phi_o = \frac{t_o^{\,2}}{12a^2}$$

$$\phi_i = \frac{t_i^{\,2}}{12b^2}$$

and

$$\beta = \frac{E_c a \, (1 - \mu^2)}{E t_o}$$

Each of the terms in equations (64) - (67) contains one of the parameters $\underline{A_n}$, $\underline{B_n}$, $\underline{C_n}$, and $\underline{H_n}$ that appear in the displacement equations of the cylinder. A buckled form of equilibrium is possible only if equations (64) - (67) yield solutions for these parameters which are different from zero. This requires that the determinant of the coefficients of these parameters must be equal to zero. This determinant may be written as follows:

$$\begin{vmatrix} \gamma (1 - k) & \gamma (1 - k) + (\delta_n + n^2) \phi_o - \frac{\beta}{n^2} & \frac{\beta}{n^2 - 1} - n^2 \phi_o & \phi_o \\ n^2 - 1 & - \delta_n (1 + \phi_o) - (1 + n^2 \phi_o) + \frac{\beta}{n^2} & n^2 (1 + \phi_o) & - (1 + \phi_o) \\ \gamma k \frac{t_o}{t_i} & \gamma k \frac{a}{b} \frac{t_o}{t_i} + (\delta_n + n^2) \phi_i \frac{a}{b} + \frac{\beta}{n^2} \frac{t_o}{t_i} & \gamma k \frac{t_o}{t_i} \log b/a - \frac{\beta}{n^2 - 1} \frac{b}{a} \frac{t_o}{t_i} - n^2 \phi_i & \phi_i \, b/a \\ n^2 - 1 & - \delta_n \frac{a}{b} (1 + \phi_i) - \frac{a}{b} (1 + n^2 \phi_i) - \frac{\beta}{n^2} \frac{t_o}{t_i} & (n^2 - 1) \log b/a + n^2 (1 + \phi_i) & - b/a (1 + \phi_i) \end{vmatrix} = 0 \qquad (68)$$

Since terms containing $\underline{\gamma}$ appear in only two of the four rows, the expansion of the determinant shown above results in a quadratic equation in $\underline{\gamma}$. The two values of $\underline{\gamma}$ that satisfy the quadratic equation are, in general, widely separated negative roots. The root that is the lower in absolute value is proportional to the critical load on the cylinder and is called $\underline{\gamma_{cr}}$. After $\underline{\gamma_{cr}}$ has been determined, the critical load is obtained in accordance with the definition of $\underline{\gamma}$ previously given; that is,

$$q_{cr} = \frac{Et_o}{a(1 - \mu^2)} \gamma_{cr}$$

The physical significance of the fact that the determinant has two eigenvalues will be discussed later.

Analysis of Results

The results of this report are contained in equations (16) through (21), the equations for determining the stresses and displacements in the stable sandwich cylinder, and in equation (68), the determinant from which the critical load on a sandwich cylinder may be obtained. These results apply to long sandwich cylinders that have thin shell facings and are subjected to uniform external, lateral loading.

The use of equations (16) through (21) is self-explanatory; if the dimensions and material properties of a given sandwich cylinder are known, the stresses and displacements in terms of the external load, q, may be easily computed. In the case of sandwich cylinders having facings of equal thickness, equations (16) - (21) reduce to equations (18) - (23) of the original report. It is of interest to compare equations (16) - (21) with results obtained by Reissner.[5] If Reissner's equations are expressed in the notation used here, the following equations for the cylinder stresses result:

$$N_\theta = qa \left[\frac{1 + \frac{Et}{E_c a}(1 - b/a)}{2 + \frac{Et}{E_c a}(1 - b/a)} \right]$$

$$N_\theta' = qa \left[\frac{1}{2 + \frac{Et}{E_c a}(1 - b/a)} \right]$$

and

$$\sigma_{rc} \text{ (at middle surface of core)} = q\left(\frac{2}{1 + b/a}\right)\left[\frac{1}{2 + \frac{Et}{E_c a}(1 - b/a)}\right]$$

Equations (16) - (18), for facings of equal thickness, may be written as follows:

$$N_\theta = qa \left[\frac{\frac{b}{a} - \frac{Et}{E_c a}\log b/a}{1 + b/a - \frac{Et}{E_c a}\log b/a} \right]$$

$$N_\theta' = qa \left[\frac{1}{1 + b/a - \frac{Et}{E_c a}\log b/a} \right]$$

and

$$\sigma_{rc} \text{ (at middle surface of core)} = q\left(\frac{2}{1 + b/a}\right)\left[\frac{1}{1 + b/a - \frac{Et}{E_c a}\log b/a}\right]$$

Since the first term in the series expansion of log b/a is - (1 - b/a), it may be noted that, for cylinders with b/a ratios close to 1, the equations of Reissner yield results that are very nearly the same as those given by equations (16), (17), and (18). Reissner's results are based on the assumption that $a - b << a$ and hence become increasingly less accurate as the cylinder thickness is increased.

[5] Reissner, Eric, Small Bending and Stretching of Sandwich-type Shells. National Advisory Committee on Aeronautics, Tech. Note 1832. 1949.

For the determination of critical loads on long-sandwich cylinders, equation (68) with $n = 2$ is used. The case $n = 1$ represents rigid body translation of the cylinder, and values of $n > 2$ result in higher critical loads than that obtained with $n = 2$. With $n = 2$, equation (68) becomes

$$\begin{vmatrix} \gamma (1 - k) & \gamma (1 - k) + (\delta_2 + 4)\, \phi_o - \frac{\beta}{4} & \frac{\beta}{3} - 4\phi_o & \phi_o \\ 3 & - \delta_2 (1 + \phi_o) - (1 + 4\phi_o) + \frac{\beta}{4} & 4 (1 + \phi_o) & - (1 + \phi_o) \\ \gamma k \frac{t_o}{t_i} & \gamma k \frac{a}{b} \frac{t_o}{t_i} + (\delta_2 + 4)\, \phi_i \frac{a}{b} + \frac{\beta}{4} \frac{t_o}{t_i} & \gamma k \frac{t_o}{t_i} \log b/a - \frac{\beta}{3} \frac{b}{a} \frac{t_o}{t_i} - 4\phi_i & \phi_i \frac{b}{a} \\ 3 & - \delta_2 \frac{a}{b} (1 + \phi_i) - \frac{a}{b} (1 + 4\phi_i) - \frac{\beta}{4} \frac{t_o}{t_i} & 3 \log b/a + 4 (1 + \phi_i) & - \frac{b}{a} (1 + \phi_i) \end{vmatrix} = 0 \qquad (69)$$

Since a literal expansion of the above determinant results in very little simplification, equation (69) is left in this form. Numerical solutions can be obtained quite readily for specific cases. If, in equation (69), ϕ_o and ϕ_i are set equal to zero and $t_o = t_i$, the determinant reduces to that obtained on the assumption of membrane facings presented in the original report.

The two eigenvalues of the determinant correspond to the two configurations shown in figure 6. Obviously, the critical load that corresponds to the configuration shown in figure 6 (b) is considerably higher than that which corresponds to 6 (a) and is of no practical interest in the long sandwich cylinder problem.

A simpler expression for the determination of critical loads is obtained if the modulus of elasticity of the core in the radial direction is assumed to be infinite. Under this assumption, equation (68) becomes

$$\gamma_{cr} = (n^2 - 1)\left(\frac{1 + \frac{b}{a}\frac{t_o}{t_i}}{1 + \frac{b^2}{a^2}\frac{t_o}{t_i}}\right)\left\{\frac{\left(1 - \frac{b}{a}\right)^2 + \left(\phi_o \frac{t_o}{t_i} + \phi_i \frac{b}{a}\right)\left(\frac{t_i}{t_o} + \frac{b}{a}\right) + n^2 \psi \left[\phi_o (1 + \phi_i) \frac{b}{a} + \phi_i (1 + \phi_o) \frac{t_i}{t_o}\right]}{(1 + \phi_o) \frac{t_o}{t_i} + (1 + \phi_i) \frac{b}{a} + n^2 \psi (1 + \phi_o)(1 + \phi_i)}\right\} \qquad (70)$$

where

$$\psi = \frac{Et_o \left(1 - \frac{b^2}{a^2}\right)}{2G_{r\theta}\, b\, (1 - \mu^2)}$$

Equation (70), with n = 2, yields values of γ_{cr} within 3 percent of the values obtained from equation (69) for usual sandwich constructions.

For cylinders having very thin facings (membrane facings) of equal thickness, ϕ_o and ϕ_i are assumed to be zero, and equation (70) reduces to

$$\gamma_{cr} = -\,(n^2 - 1)\,\frac{\left(1 - \frac{b}{a}\right)^2}{\left(1 + \frac{b^2}{a^2}\right)\left[1 + \frac{n^2\, Et\,\left(1 - \frac{b}{a}\right)}{2G_{r\theta}\, b\, (1 - \mu^2)}\right]} \tag{71}$$

Equation (71), for n = 2, becomes

$$\gamma_{cr} = -\,\frac{3\left(1 - \frac{b}{a}\right)^2}{\left(1 + \frac{b^2}{a^2}\right)\left[1 + \frac{2Et\,\left(1 - \frac{b}{a}\right)}{G_{r\theta}\, b\, (1 - \mu^2)}\right]} \tag{72}$$

The value of q_{cr} is then determined from the definition previously given,

$$q_{cr} = \frac{Et}{a\,(1 - \mu^2)}\,\gamma_{cr}$$

Equation (72) may be used as a good approximation to the expression obtained in the original report (equation 64) for the critical load on cylinders with membrane facings.

It is of interest to examine the results given by equation (68) for certain limiting cases other than that of membrane facings. Some of these results are given in table 1. In each of these limiting cases, ϕ_o and ϕ_i are neglected as compared to 1.

Equation (68) may be used for determining the approximate value of the critical load on long sandwich panels in the form of a portion of a cylinder hinged along the edges $\theta = 0$ and $\theta = 2\alpha$ as shown in figure (7). If, in equation (68), $\underline{\pi/\alpha}$ is substituted for $\underline{n}$, the smaller absolute value of $\underline{q}$ obtained from equation (68) represents the critical load on a panel whose dimensions and properties are known. This solution applies to the unsymmetrical type of buckling shown in figure (7). If, as in the case of relatively flat panels, symmetrical buckling with no inflection point between the supports occurs, equation (68) is not applicable.

Conclusions

The purpose of this report was to extend the previous work done in connection with sandwich cylinders subjected to uniform lateral loading by taking into account the effect of the stiffnesses of the individual facings of the cylinder and also by making the results applicable to sandwich cylinders having facings of unequal thickness. The results indicate that for the majority of sandwich cylinders the analysis based on membrane facings is sufficiently accurate. However, for cylinders having relatively thick facings, say in the neighborhood of one-fourth of the core thickness, the facing stiffnesses have an appreciable effect on the critical load on the composite cylinder. Since the magnitude of the effect of the facing stiffnesses is dependent not only on the dimensions of the cylinder but also on the mechanical properties of the core and facing materials, equation (68) of this report should be used for computing the critical load if indications are that this effect may be important. In all cases of sandwich cylinders having facings of unequal thickness, the equations of this report should be used for computing stresses prior to buckling and for the determination of critical loads.

Table 1.--Results obtained with equation 68 for certain limiting cases

E_c	$G_{r\theta}$	t_o	t_i	q_{cr}
E_c	$G_{r\theta}$	o	t_i	$-\dfrac{Et_i^3}{4(1-\mu^2)b^3}\left(\dfrac{b}{a}\right)$
E_c	$G_{r\theta}$	t_o	o	$-\dfrac{Et_o^3}{4(1-\mu^2)a^3}$
0	$G_{r\theta}$	t_o	t_i	$-\dfrac{Et_o^3}{4(1-\mu^2)a^3}$
∞	0	t_o	t_i	$-3\dfrac{Et_o}{a(1-\mu^2)}\left(\dfrac{1+\dfrac{b}{a}\dfrac{t_o}{t_i}}{1+\dfrac{b^2}{a^2}\dfrac{t_o}{t_i}}\right)\left(\phi_o\dfrac{b}{a}+\phi_i\dfrac{t_i}{t_o}\right)$
∞	∞	t	t	$-3\dfrac{Et}{a(1-\mu^2)}\left[\dfrac{\left(1-\dfrac{b}{a}\right)^2+\dfrac{t^2}{12ab}\left(1+\dfrac{b}{a}\right)^2}{1+\dfrac{b^2}{a^2}}\right]$

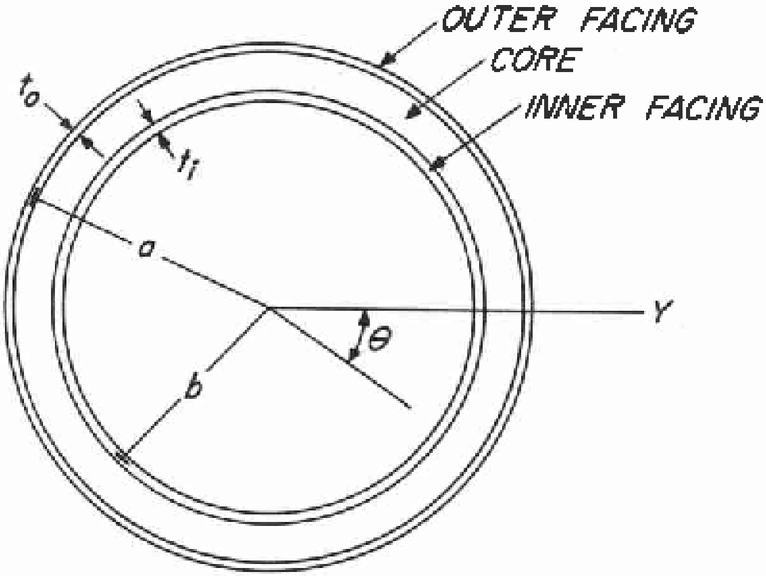

Figure 1.--Cross section of sandwich cylinder.

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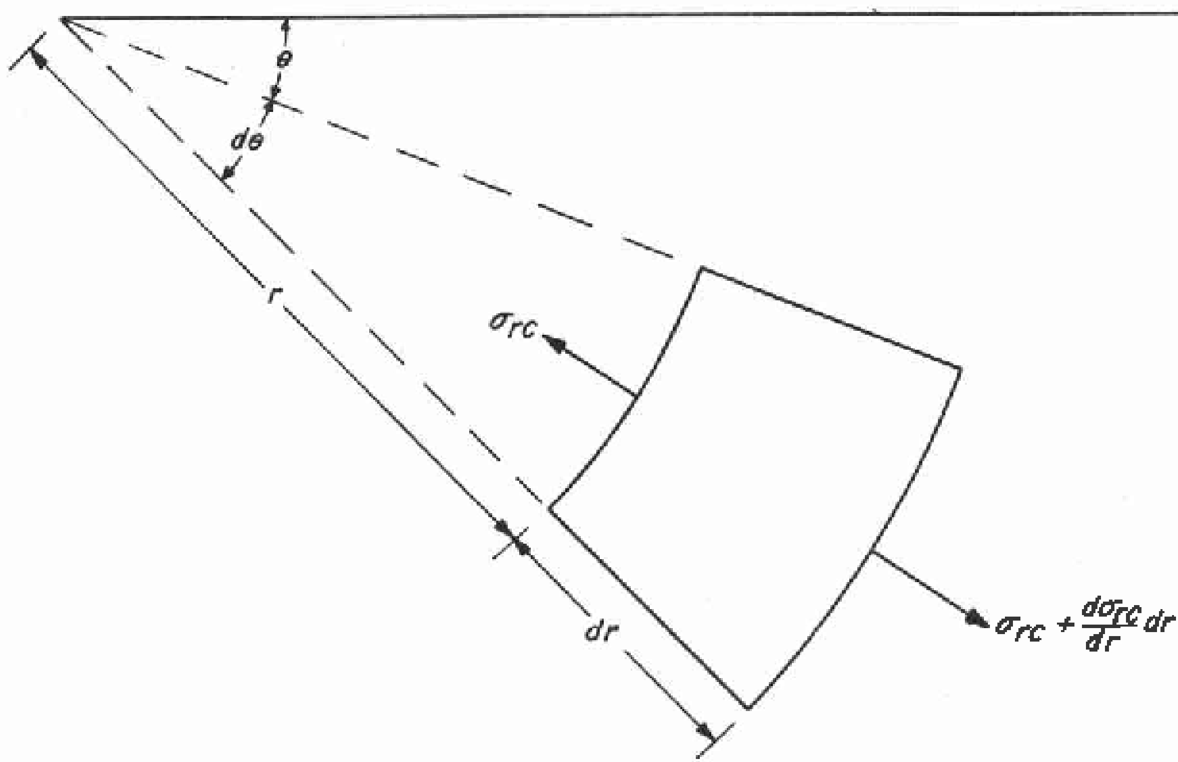

Figure 2.--Differential element of core of uniformly stressed cylinder.

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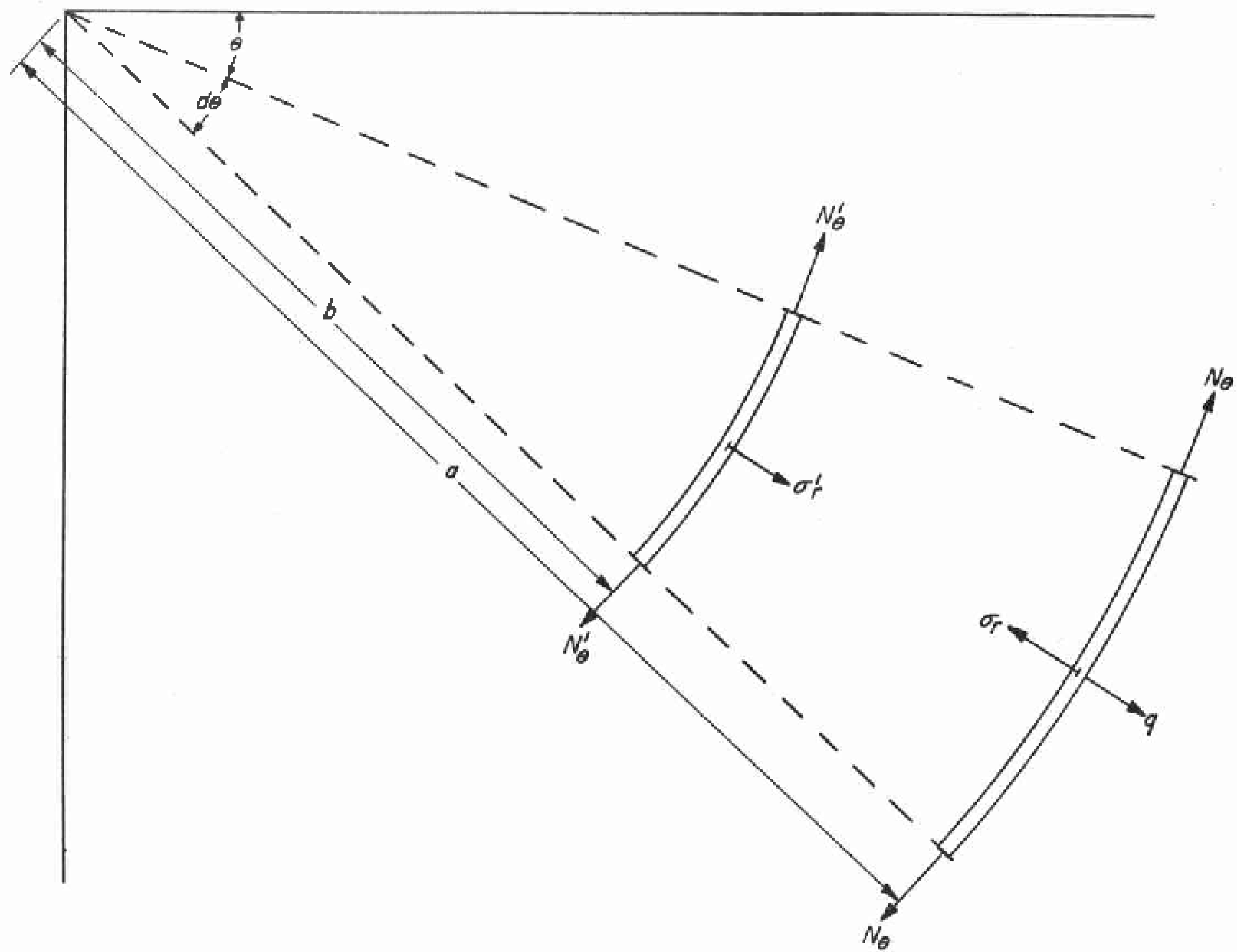

Figure 3.--Differential elements of inner and outer facings of uniformly stressed cylinder.

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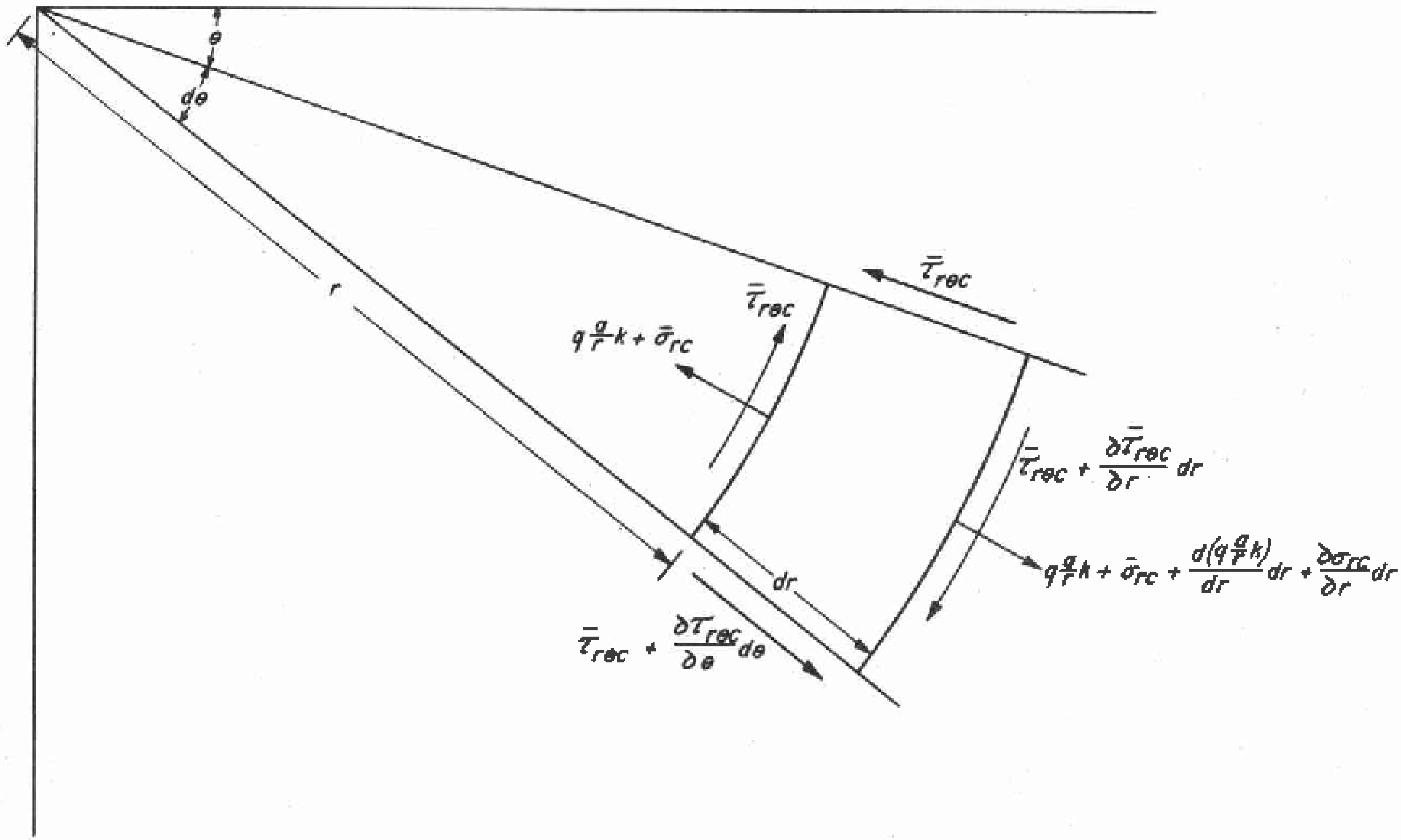

Figure 4.--Differential element of core of slightly deformed cylinder, neglecting changes in geometry of core.

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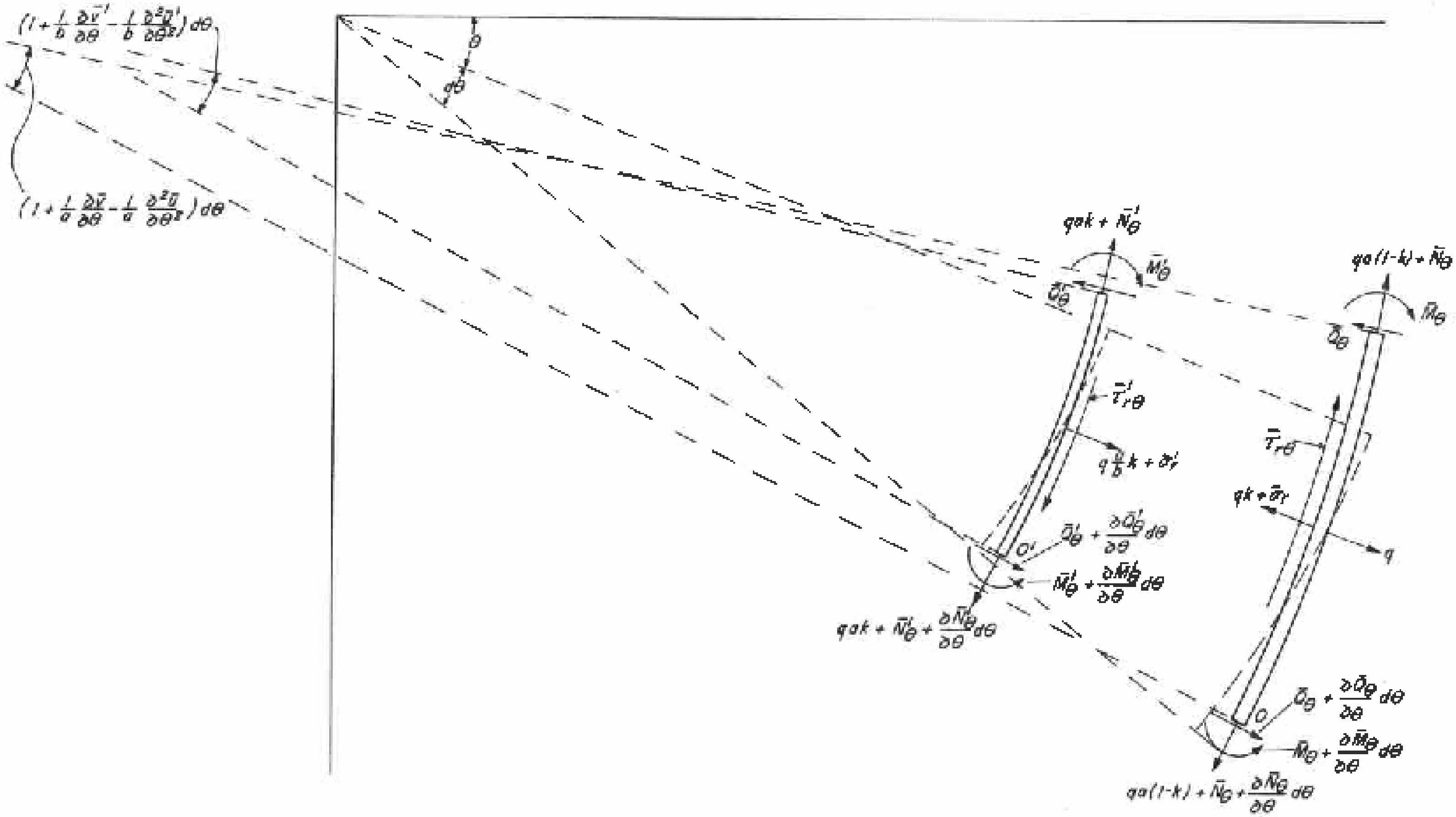

Figure 5.--Differential elements of outer and inner facing of slightly deformed cylinder.

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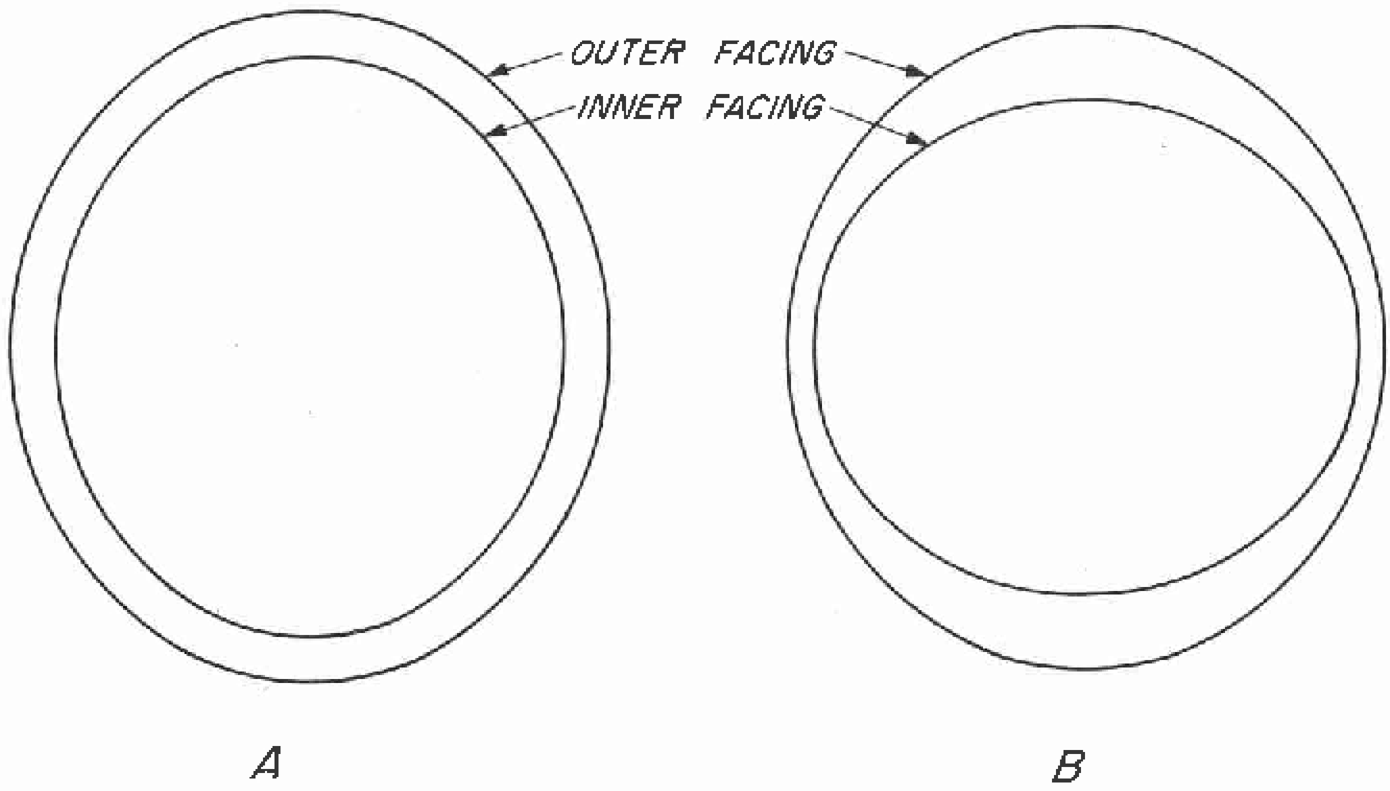

Figure 6.--Configurations corresponding to the two eigenvalues of the stability determinant.

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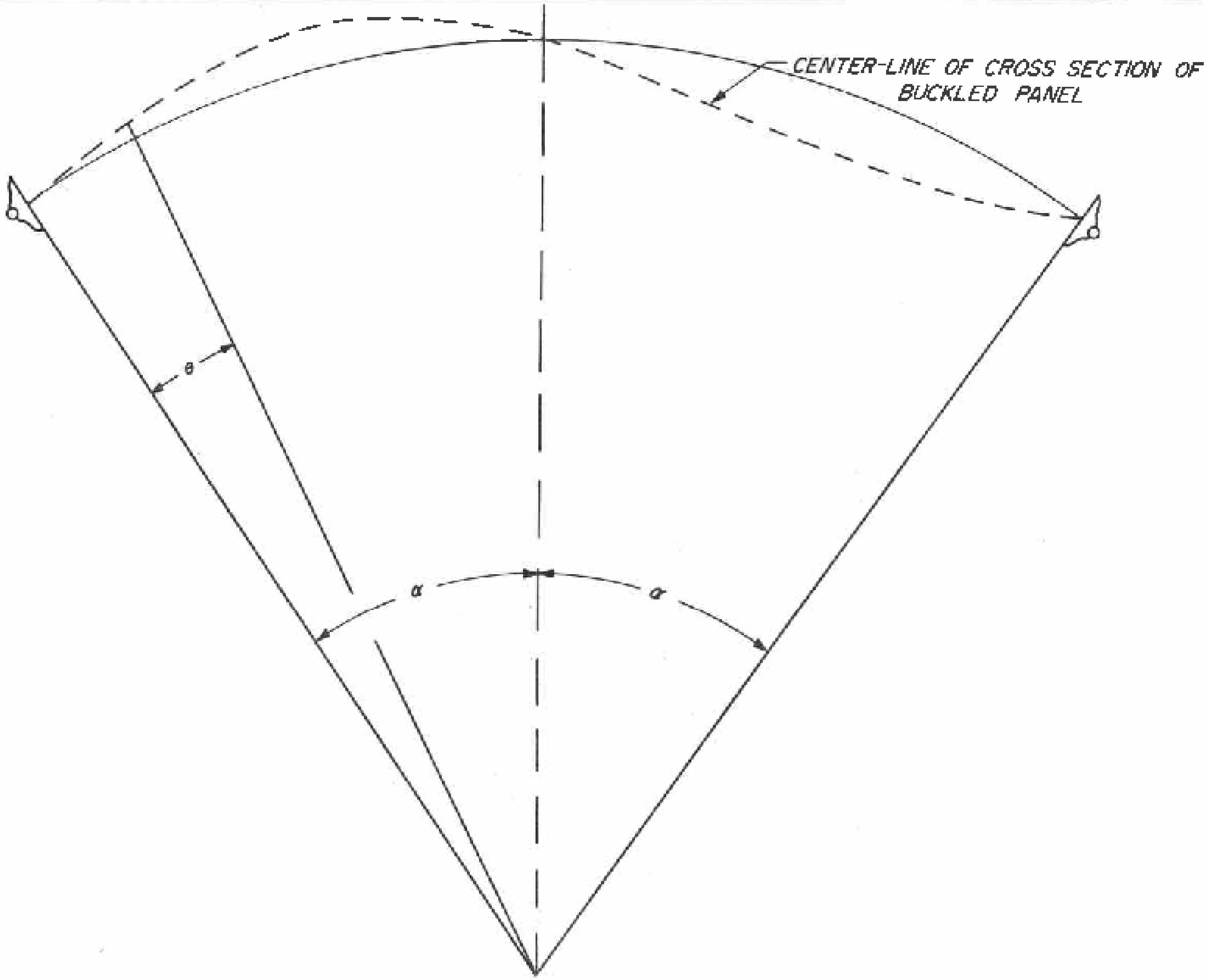

Figure 7.--Cross section of long panel.

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