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Title: THEORY OF MICROMORPHIC MATERIALS AND APPLICATIONS

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A general theory of micromorphic materials was developed by Eringen for the prediction of continuum behavior of materials with inner structure, such as, granular solids, composite materials, anisotropic and polymeric fluids. Recently Eringen derived balance laws of micromorphic mechanics from a different point of view. His derivation of the master functional balance laws are reviewed and the specific forms of balance laws are obtained.

In the study of viscoelastic behavior of non-Newtonian fluids it is well known that the introduction of generalized deformation-rate tensors in the constitutive equations plays an important role. In the present investigation we have introduced the concept of combined generalized measures in the micromorphic constitutive equations proposed by Eringen. In order to incorporate the viscoelastic effects, generalized measures of not only the first deformation-rate tensor involving velocity gradients, but also those of the second deformation-rate tensor involving acceleration gradients are introduced in the

development of constitutive equations. The use of such generalized measures in the place of ordinary measures in constitutive equations eliminates the need for using higher order deformation-rates as shown by Narasimhan and Sra. In the present investigation we have developed a new constitutive theory for micromorphic materials by introducing in the classical constitutive equations generalized measures of deformation-rate tensors and microdèformation-rate tensors.

Using the constitutive equation based on concepts of generalized measures and basic equations of micromorphic materials of grade one, the problem of micropolar fluid flows in converging and diverging channels are investigated. The behavior of micropolar fluids is investigated in terms of a suitably defined Reynolds number and two other fluid parameters--the viscoelastic parameter and the micro-deformation parameter.

For vanishing viscoelastic parameter but for non-vanishing micro-deformation parameter, exact solutions are obtained in terms of elliptic functions. In this case it is possible to identify the Reynolds number with the Reynolds number defined in terms of the maximum velocity by other workers. Classical solutions for Newtonian fluids are found to appear as a special case of our more general solutions for micropolar fluids. For non-vanishing viscoelastic parameter, exact solutions are again obtained in terms of elliptic functions and elliptic integrals. Numerical and graphical solutions are obtained to study the micropolar fluid flow between converging and diverging channels.

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NOTATION

Symbols	Meaning
a_k	k-th component of acceleration vector
a_r, a_θ, a_z	physical components of acceleration in cylindrical coordinates
b_{ij}	second deformation-rate tensor components
B^*	generalized measure of the second deformation-rate tensor
d_{ij}	first deformation-rate tensor components
D^*	generalized measure of the first deformation-rate tensor
f_k	body force vector
$f^{k\ell_1 \ldots \ell_m}$	body force moment tensor components
g_{ij}	metric tensor components
i_k	micro-inertia
$i_{k\ell}$	micro-inertia moment
I	identity tensor
k, k'	dimension-correcting constants
n, n'	measure indices
p	pressure of fluid
q, q'	irreversibility indices
r, θ, z	cylindrical coordinates
R	Reynolds number
$\bar{t}_{k\ell}$	micro-stress average

Symbols	Meaning
T	stress tensor
$t_{rr}, t_{r\theta}$, etc.	physical components of stress tensor in cylindrical coordinates
$t^{k\ell\ell_1 \ldots \ell_m}$	stress-moment tensor components
$\bar{t}^{k\ell\ell_1 \ldots \ell_m}$	stress-average moment tensor components
$t^{k\ell m}$	first stress moments
t	time
tr	trace
v_k	k-th component of velocity vector
v_r, v_θ, v_z	physical components of velocity vector in cylindrical coordinates
$w_{k\ell}$	spin tensor components
δ_{ij}	Kronecker delta
ρ	density of fluid
α, β	material parameters
μ	coefficient of viscosity
$\dot{\sigma}_{k\ell}$	inertial spin
$\epsilon_{k\ell m}$	alternating tensor
ν	kinematic coefficient of viscosity
ν_k	micro-rotation vector
ν_r, ν_θ, ν_z	physical components of micro-rotation vector

Symbols	Meaning
$\nu_{k\ell}$	gyration tensor components
$\frac{d}{dt}$	material time-derivative
,	partial differentiation
;	covariant differentiation
am(u, k)	amplitude u
cn(u, k)	cosine amplitude u; Jacobian elliptic function
dn(u, k)	delta amplitude u; Jacobian elliptic function
sn(u, k)	sine amplitude u; Jacobian elliptic function
$E(\phi, k)$	Legendre's incomplete elliptic integral of the second kind
E	complete elliptic integral of the second kind
$F(\phi, k)$	incomplete elliptic integral of the first kind
K	complete elliptic integral of the first kind
H(u)	Jacobian Eta function
k	modulus of Jacobian elliptic functions
$k' = \sqrt{1-k^2}$	complementary modulus
$\Theta(u)$	Jacobian Theta function
Z(u)	Jacobian Zeta function
$\Pi(\phi, n^2, k)$	Legendre's incomplete elliptic integral of the third kind

THEORY OF MICROMORPHIC MATERIALS AND APPLICATIONS

1. INTRODUCTION

1.1 History of Oriented Media in General

According to the classical theory of continuum mechanics, a physical body may be regarded as a dense aggregate of continuum particles which do not possess any internal structure. Consequently the classical theory of continuum mechanics admits only the translational displacement field as an independent field.

In some physical bodies, physical anisotropy is observable. For example, the stress required to effect a given extension in an elastic crystal differs with the direction in which the extension is to occur. For the description of deformation of such materials, a mathematical model should admit not only the displacement field but also certain directional field as an independent field. In other words, each continuum particle is endowed with its own directions which suffer deformation during its motion. However, the directions selected at each continuum particle will not be arbitrary.

That the physical bodies should be presented as assemblies not only of points but also of directions associated with the points, in brief, as oriented bodies, was suggested by Duhem (1893). Theories based on this idea were constructed by E. and F. Cosserat (1907).

But during the half century following the publication of their profound work, scant attention has been given to it.

The theory of oriented bodies in general and invariant form was treated by Ericksen and Truesdell (1958). A set of directors is assigned to each particle. Thus a deformation consists in a displacement of the particles as well as independent rotations and stretches of the directors. In the special case when the directors are material elements, the presence of directors adds nothing to the description of materials. In general, the directors are neither material nor invariable. In order to develop theories of rods, shells, and anisotropic solids, Ericksen and Truesdell (1958) further assumed that there are only three linearly independent directors.

In order to describe the behavior of liquid crystals and suspensions of large molecules, Ericksen (1963) has proposed and developed a class of theories of oriented media involving one single director, whose behavior in time is governed by a differential equation. Ericksen's starting point is the fully general equations of balance for momentum, moment of momentum, and energy for every part of a body in its present configuration. He assumed the basic laws of conservation of mass, linear momentum, moment of momentum and energy, respectively, in the form;

$$\frac{d}{dt}\int_V \rho dv = 0,$$

$$\frac{d}{dt}\int_V p_i dv = \int_S t_{ij} da_j + \int_V f_i dv,$$

$$\frac{d}{dt}\int_V m_{ij} dv = \int_S x_{[i}t_{j]k} da_k + \int_V x_{[i}f_{j]} dv, \tag{1.1.1}$$

$$\frac{d}{dt}\int_V E dv = \int_S t_{ij}\dot{x}_i da_j + \int_V f_i \dot{x}_i dv - \int_S q_i da_i,$$

where ρ, p_i, m_{ij}, E, t_{ij}, f_i, q_i and $\frac{d}{dt}$ are density of mass, linear momentum, moment of momentum, energy, stress tensor, body force per unit volume, heat flux and material derivative, and a bracket denotes the exterior product of the respective quantities. Ericksen envisaged a fluid having at each particle a single preferred direction represented by a vector n_i of variable magnitude. The momentum, moment of momentum, and energy of the medium are taken as sums of two parts, one being the classical quantity for an ordinary point-medium and the other the contribution from the director motion;

$$p_i = \rho \dot{x}_i,$$

$$m_{ij} = x_{[i}p_{j]} + \rho n_{[i}\dot{n}_{j]}, \tag{1.1.2}$$

$$E = \rho\epsilon + \frac{1}{2}\rho(\dot{x}_i\dot{x}_i + \dot{n}_i\dot{n}_i),$$

where ϵ is the internal energy density. For isothermal flow of incompressible fluids, Ericksen derived a constitutive equation in the form

$$t_{ij} = -p\delta_{ij} + (\lambda_1 + \lambda_2 d_{km} n_k n_m) n_i n_j + \lambda_3 d_{ij} + \lambda_4 (d_{ik} n_k n_j + d_{jk} n_k n_i) , \quad (1.1.3)$$

where the λ_i depend on n^2. Hence to within an arbitrary isotropic pressure p, the stress at a particle at any time is uniquely determined by n_i and the gradients of the velocity at the same particle and time.

In order to allow for effects of anisotropy in a simple material capable of flow, Noll (1955) proposed the constitutive equation,

$$T + pI = f(L, R, \rho), \quad (1.1.4)$$

where R is the tensor of finite rotation from a fixed reference configuration and L denotes the velocity gradient. The dependence of the constitutive equation on R allows for response differing in direction.

Green (1964) has taken up Noll's theory for the case of a transversely isotropic material. He infers that the stress for such a material is given by an isotropic function of D, deformation-rate tensor, and the tensor product $\underset{\sim}{n} \otimes \underset{\sim}{n}$ where $\underset{\sim}{n} = R\underset{\sim}{e}_R$, $\underset{\sim}{e}_R$ being a certain constant vector. On the assumption that the function is a polynomial, he obtained the constitutive equation

$$T + pI = \alpha_1 M + \alpha_2 D + \alpha_3 D^2 + \alpha_4(MD+DM) + \alpha_5(MD^2+D^2M), \qquad (1.1.5)$$

where $M = \underset{\sim}{n} \otimes \underset{\sim}{n}$ and it has been assumed that the material is incompressible, and where α_i are functions of the following invariants; trD^2, trD^3, trM, $trMD$, $trMD^2$, where tr denotes trace of the respective quantities indicated.

Eringen and Suhubi (1964) have formulated and developed a continuum theory of micro-elastic solids, which takes into account the micromotions and microrotations of the materials. They assumed that the volume element dV centered at a material point X contains many microvolume elements dV' centered at X', $X' = X + \Xi$, where Ξ is the position vector of X' with respect to the center of mass X of dV. After deformation, X' goes into x' so that $x' = x + \xi$, where ξ is the relative position vector of x' with respect to x. Thus

$$x' = x'(X', t) = x(X, t) + \xi(X, \Xi, t), \qquad (1.1.6)$$

and each of the microvolume element dV' possesses a distinct intrinsic motion defined only over its volume element. Eringen and Suhubi assume that fundamental laws of motion for the microelement

dv' in the deformed body having its mass center at $\mathbf{x}'$ are based on those of the theory of classical continuous media. The laws of motion for the volume element dv are then obtained by taking various order statistical moment about the origin. They obtain the basic laws of motion,

$$\frac{\partial \rho}{\partial t} + (\rho v_k)_{,k} = 0,$$

$$\frac{\partial i_{km}}{\partial t} + i_{km,r} v_r - i_{mr} \nu_{rk} - i_{kr} \nu_{rm} = 0,$$

$$t_{k\ell,k} + \rho(f_\ell - \dot{v}_\ell) = 0, \tag{1.1.7}$$

$$t_{m\ell} - s_{m\ell} + \lambda_{k\ell m,k} + \rho(\ell_{\ell m} - \dot{\sigma}_{\ell m}) = 0,$$

$$t_{k\ell} v_{\ell,k} + (s_{k\ell} - t_{k\ell})\nu_{k\ell} + \lambda_{k\ell m} \nu_{m\ell,k} + q_{k,k} + \rho h - \rho\dot{\epsilon} = 0,$$

where ρ, f_ℓ, i_{km}, s_{km}, ℓ_{km}, $\dot{\sigma}_{km}$, ϵ, q_k, h, $t_{k\ell}$, v_k, $\nu_{k\ell}$ and $\lambda_{k\ell m}$ are mass density, body force, micro-inertia moments, micro-stress average, first body moment per unit mass, inertial spin, internal energy density, heat flux, heat source per unit mass, stress tensor, velocity vector, gyration tensor and first stress moments. They have also proposed a set of three constitutive equations, $t_{k\ell}$, $s_{k\ell}$ and $\lambda_{k\ell m}$, which are functions of macro- as well as micro-deformations of the materials.

1.2 Motivation for the Present Investigation

As we have described so far, to obtain properly invariant theories of anisotropic fluids, one must incorporate a vector, a set of vectors, or higher order tensors in theories of non-classical materials. These vectors and tensors represent material properties which originate in the internal structure of continuum particles, for example, in molecular configurations, displacements, and interactions of configurations.

From a continuum point of view, these representations must be valid regardless of the nature of the molecular and submolecular units of which the physical body is composed. Classical theories of materials lack these representations. Thus it is customary in classical fluid dynamical considerations to assume that the stress tensor is symmetric and no account is taken of its dependence on the internal spin or microdisplacements. The argument is based on conservation of angular momentum and makes no allowance for possible internal angular momentum which may be composed of contributions from molecular rotations and from submolecular forms such as electronic and nuclear spin. The coupling that exists between the internal spin and vorticity is not taken into account in classical continuum theory. This coupling leads to an antisymmetric stress tensor which is functionally related to the difference of vorticity and internal spin field.

In order to construct a suitable mathematical model which can describe the macroscopic manifestation of microstructure or internal configuration, one must regard a physical body, not merely as a dense aggregate of particles but rather as a collection of particles each containing sufficient number of smaller particles whose deformations are independent of the motion of the particle to which they belong.

It is, then, necessary to reformulate basic fluid mechanical concepts to account for some structural aspects of fluid flow and reconsider the linear and angular momentum of fluid mechanics. With suitable constitutive equations appropriate to such non-classical bodies, one must obtain generalized equations of motions for the velocity and spin fields.

Eringen (1964) proposed the simplest continuum theory which does not rely on the molecular theories or statistical mechanics. Generalizing his theory, Eringen (1970) formulated a theory of continuum mechanics of micromorphic materials in which the local microscopic structure and intrinsic motions of the particles are important. The foundation of the theory is consistent with the known principles of continuum mechanics and a properly invariant theory of constitutive equations.

This theory has great promise in explaining many new phenomena hitherto unknown or treated partly through statistical-mechanical approach. To name a few, this theory is capable of explaining the phenomenon of surface tension, couple stress, liquid crystal behavior, fluid containing additives, internal spin, and micro-anisotropy in oriented bodies, and it provides a firm foundation for the theory of polar materials. All of the classical theories of solids and fluids can be included in the theory of micromorphic materials.

It is the object of the present investigation to present nonlinear theories of micromorphic materials, develop suitable constitutive equations, and then apply them to investigate the steady flow of micro-polar fluids between two non-parallel planes which either converge or diverge. This class of problems finds application in various industrial and biophysical cases such as the flow of liquid crystals like cholesterol, fluids containing additives and similar other synthetics which are frequently used in industry.

1.3 Organization of the Present Investigation

In Chapter 1, a review of the various existing theories of anisotropic fluids is made. Chapter 2 is devoted to the theory of micromorphic materials and the constitutive equations for micro-morphic fluids. The theory of micropolar fluids, a subclass of micromorphic fluids, is also presented.

In Chapter 3 the theory of micromorphic fluids is applied to the investigation of the steady radial flows of micropolar fluids between two non-parallel planes. The basic equations of micromorphic materials of grade one are formulated.

Chapter 4 and 5 deal with solutions of the problems of micropolar fluids in diverging and converging channels respectively. Solutions for large as well as small Reynolds numbers are discussed.

Chapter 6 contains the summary and conclusions.

2. THEORY OF MICROMORPHIC MATERIALS

2.1 Introduction

In a series of papers, Eringen (1964, 1970) gave a general theory of micromorphic continua for prediction of continuum behavior of materials with inner structure. His theory is intended for the prediction of thermomechanical behavior of granular solids, composite materials, anisotropic and polymeric fluids.

In Section 2, Eringen's new derivation of the balance laws of micromorphic theory is presented. First, the master functional balance laws are presented. Afterwards, the specific forms are obtained. Micropolar media as a special case of micromorphic materials are also discussed. Constitutive equations based on the concept of generalized measures are reviewed.

2.2 Balance Laws of Micromorphic Mechanics

A material point X' having rectangular coordinates X'_K in the undeformed body occupies a spatial point x', referred to the same coordinate system, in the deformed body, at time t. The motion and the inverse motion are respectively defined by one-to-one mapping

$$x' = x'(X, t), \quad X' = X'(x', t). \tag{2.2.1}$$

The axiom of continuity of matter requires that the Jacobian

$$J' = \det\left(\frac{\partial x'_k}{\partial X'_K}\right) \neq 0, \qquad (2.2.2)$$

in a neighborhood of material point X' at time t, except possibly at some singular points, lines and surfaces.

If a definite motion $x = x(X, t)$ is given, any time-dependent scalar, vector or tensor field ψ may be regarded either as a function $\psi(X, t)$ of the particle and the time, or as a function $\psi(x, t)$ of the place and the time.

The material derivative of $\psi(X, t)$, $\dot{\psi}(X, t)$, is related to the spatial derivative by

$$\dot{\psi} = \frac{\partial \psi}{\partial t} + \text{grad}\, \psi \dot{x}, \qquad (2.2.3)$$

where $\text{grad}\, \psi$ denotes the spatial gradient of ψ.

The material time derivative of the integral over material volume V is given by

$$\frac{d}{dt}\int_V \psi dV = \int_V (\dot{\psi} + \psi v^k_{;k}) dV, \qquad (2.2.4)$$

where a semicolon denotes a covariant differentiation. Integral of any tensor field τ defined over the material surface is related to the integral over material volume by

$$\int_S \tau^k da_k = \int_V \tau^k_{;k} dV . \tag{2.2.5}$$

The global balance laws of classical continuum mechanics may be expressed in the form,

$$\frac{d}{dt}\int_V \psi dV = \int_S \tau^k da_k + \int_V g dV, \tag{2.2.6}$$

where ψ, τ^k, and g are some appropriate tensor fields defined over material volume V and material surface S. If the global balance laws (2.2.6) are valid for all parts of the body, we obtain the local balance laws,

$$\frac{\partial \psi}{\partial t} + (\psi v^k)_{;k} = \tau^k_{;k} + g . \tag{2.2.7}$$

Tensorial quantities associated with the particle X' shall be denoted by primed capital letters and those associated with the place x' by primed small letters. For example, $\rho' = \rho(x')$, $\psi' = \psi(x')$, $\phi' = \phi(x')$.

The mathematical model of the continuum theory assumes that any field quantity can be assigned to an infinitesimally small body. Molecular theories of matter, however, have shown that such a continuous measure cannot be assigned to a body smaller than some critical volume.

If we desire to refine continuum theory so that we can explain mechanical phenomena arising out of a discontinuous measure assigned to a micro-volume or micro-element smaller than some critical volume, we must take into account the motions as well as the deformations of these microelements. For this purpose we shall assume that a continuous measure representing any field quantity can be assigned to a body and this field quantity is an average of discontinuous measure assigned to micro-element. This implies that the continuum theory is valid at each particle of media, and all micro-elements behave essentially like classical continuum particles. However, since these micro-elements are embedded in some continuum particles, they give rise to an internal configuration or micro-structure of particles. Thus any change in internal configuration due to displacements of micro-elements or interaction among themselves will certainly affect the behavior of micro-elements as a whole and consequently the behavior of particles in which they are embedded. Since the motions of micro-elements are in no way constrained by the motion of particles, that is, by the displacements of particles or fluid flow, the internal degrees of freedom of micro-elements may play important role in determining macroscopic properties of materials, and in particular the rotation and spin of micro-elements may need to be recognized explicitly in the kinematical description of materials. Therefore, one must define an appropriate averaging process to obtain a continuous measure representing internal structure.

If we replace ψ in (2.2.4) by $\psi\phi$ where ϕ is a tensor-valued function and integrate over the local material volume, we obtain

$$\frac{d}{dt}\int_V \psi'\phi' dV = \int_V (\overline{\psi'\phi'}^{\bullet} + \psi'\phi' v^k_{;k}) dV$$
$$= \int_V \{\psi'\dot{\phi}' + (\dot{\psi}' + \psi' v^k_{;k})\phi'\} dV, \tag{2.2.8}$$

where V denotes the local material volume.

If we replace τ^k in (2.2.5) by $\phi\tau^k$ and integrate over the local material volume, we obtain

$$\int_S \phi'\tau'^k da'_k = \int_V (\phi'\tau'^k)_{;k} dV , \tag{2.2.9}$$

where S denotes the local material surface. In developing the theory of micromorphic materials, we make the fundamental assumption that (2.2.7) is valid at all parts of materials; that is, it is valid at all micro-elements of materials. This means that all micro-elements behave essentially like classical continuum particles. With this assumption, (2.2.8) reduces to the form,

$$\frac{d}{dt}\int_V \psi'\phi' dV = \int_V \{\psi'\dot{\phi}' + (\tau'^k_{;k} + g')\phi'\} dV. \tag{2.2.10}$$

The substraction of (2.2.10) from (2.2.9) yields

$$\int_S \phi'\tau'^k da'_k - \frac{d}{dt}\int_V \psi'\phi' dV$$
$$= \int_V \{(\phi'\tau'^k)_{;k} - \psi'\dot{\phi}' - (\tau'^k_{;k} + g')\phi'\} dV. \qquad (2.2.11)$$

This integral expression over the finite material volume $\underset{\sim}{V}$ and material surface $\underset{\sim}{S}$ can be written in the form

$$\int_{\underset{\sim}{S}} \ll\phi'\tau'^k\gg da_k - \frac{d}{dt}\int_{\underset{\sim}{V}} <\psi'\phi'> dV$$
$$= \int_{\underset{\sim}{V}} \{<\phi'\tau'^k>_{;k} - <\psi'\dot{\phi}'> - <(\tau'^k_{;k} + g')\phi'>\} dV, \qquad (2.2.12)$$

where the local average quantities are defined by

$$\ll\phi'\tau'^k\gg = \frac{1}{S}\int_S \phi'\tau'^k da'_k, \qquad (2.2.13)$$

$$<\phi'\psi'> = \frac{1}{V}\int_V \phi'\psi' dV. \qquad (2.2.14)$$

The integral expression (2.2.12) may be called the global balance laws of micromorphic materials. If the global balance laws of micromorphic materials are valid at all parts of materials, then we obtain the expression for local balance laws of micromorphic materials,

$$\langle\langle \phi'\tau'^k \rangle\rangle_{;k} - \overline{\langle \psi'\phi' \rangle}^{\bullet} - \langle \psi'\phi' \rangle v^k_{;k}$$
$$= \langle \phi'_{;k}\tau'^k \rangle - \langle \psi'\dot{\phi}' \rangle - \langle g'\phi' \rangle \ . \tag{2.2.15}$$

We may expand $\phi' = \phi(x')$ in power series about the place x,

$$\phi(x') = \phi(x) + \sum_M \phi_{;M}(x)\xi^{(M)}, \tag{2.2.16}$$

where

$$\phi_{;M}(x) = \phi_{;\ell_1 \ldots \ell_m}(x) \ ,$$
$$\xi^{(M)} = \xi^{(\ell_1} \ldots \xi^{\ell_m)}/m! \ , \tag{2.2.17}$$

and a parenthesis enclosing indices indicates symmetrization. The substitution of (2.2.16) into (2.2.13) and (2.2.14) yields various average quantities;

$$\langle\langle \phi'\tau'^k \rangle\rangle = \phi\tau^k + \sum_M \phi_{;M}\tau^{kM} \ ,$$

$$\langle\langle \phi'\tau'^k \rangle\rangle_{;k} = \phi_{;k}\tau^k + \phi\tau^k_{;k} + \sum_M (\phi_{;Mk}\tau^{kM} + \phi_{;M}\tau^{kM}_{;k}) \ , \tag{2.2.18}$$

$$<\phi'_{;k}\tau'^{k}> = \sum_{M} \phi_{;kM}\bar{\tau}^{kM},$$

$$<\phi' g'> = \phi g + \sum_{M} \phi_{;M} g^{M} ,$$

(2.2.18 cont.)

$$<\phi'\psi'> = \phi\psi + \sum_{M} \phi_{;M}\psi^{M} ,$$

$$<\dot{\phi}'\psi'> = \phi_{;k}\psi v^{k} + \sum_{M} \phi_{;Mk}\psi^{M} v^{k} + \sum_{M} \phi_{;M}\hat{\psi}^{M} ,$$

where by definitions

$$\tau^{k} = \ll\tau'^{k}\gg , \qquad \tau^{kM} = \ll\tau'^{k}\xi^{(M)}\gg ,$$

$$\bar{\tau}^{kM} = <\tau'^{k}\xi^{(M)}> , \qquad g = <g'> , \tag{2.2.19}$$

$$g^{M} = <g'\xi^{(M)}> , \qquad \psi = <\psi'> ,$$

$$\psi^{M} = <\psi'\xi^{(M)}> , \qquad \hat{\psi}^{kM} = <\psi'\dot{\xi}^{k(M)}> .$$

If we substitute these average quantities into the local balance laws (2.2.15) and equate the coefficients of ϕ and $\phi_{;M}$, then we have

$$\dot{\sigma} = g + \tau^{k}_{;k} , \tag{2.2.20}$$

$$\dot{\sigma}^{M} = g^{M} + \tau^{kM}_{;k} + \tau^{(M)} - \bar{\tau}^{(M)} , \tag{2.2.21}$$

where

$$\overset{\bullet}{\sigma} = \overset{\bullet}{\psi} + \psi v^k_{;k} , \tag{2.2.22}$$

$$\overset{\bullet}{\sigma}{}^M = \overset{\bullet}{\psi}{}^M + \psi^M v^k_{;k} - \hat{\psi}^M . \tag{2.2.23}$$

Thus (2.2.20) and (2.2.21) are the general form of the local balance laws of the micromorphic theory of materials.

If we assume that

$$\overset{\bullet}{\xi}{}^k = \nu^k_\ell \xi^\ell , \tag{2.2.24}$$

then $\hat{\psi}^M$ has the expression

$$\hat{\psi}^{\ell_1 \cdots \ell_m} = \nu^{\ell_1}_k \psi^{k\ell_2 \cdots \ell_m} + \nu^{\ell_2}_k \psi^{\ell_1 k \ell_3 \cdots \ell_m} + \cdots + \nu^{\ell_m}_k \psi^{\ell_1 \cdots \ell_{m-1} k} , \tag{2.2.25}$$

where ν^k_ℓ is gyration tensor. The specific equations for the physically meaningful quantities such as mass density, stress tensor, energy and entropy are all derivable from these equations.

<u>Mass density and micro-inertia tensors</u>. The balance laws for the mass density and micro-inertia tensors are obtained by taking

$$\psi' = \rho', \quad \tau'^k = 0, \quad g' = 0, \tag{2.2.26}$$

in (2.2.19). Thus

$$\psi = \rho = \langle \rho' \rangle ,$$

$$\psi^{\ell_1 \cdots \ell_m} = \rho i^{\ell_1 \cdots \ell_m} , \tag{2.2.27}$$

$$\tau^{k} = 0, \qquad \tau^{k\ell_1\cdots\ell_m} = 0, \tag{2.2.27}$$

$$\bar{\tau}^{k\ell_1\cdots\ell_m} = 0, \quad g^{k\ell_1\cdots\ell_m} = 0 ,$$

where ρ and $i^{\ell_1\cdots\ell_m}$ are the mass density and micro-inertia tensors respectively. The substitution of (2.2.27) into (2.2.22) and (2.2.23) gives

$$\begin{aligned} \dot{\sigma} &= \dot{\rho} + \rho v^{k}_{;k} , \\ \dot{\sigma}^{\ell_1\cdots\ell_m} &= \rho \dot{i}^{\ell_1\cdots\ell_m} - \rho \nu_k^{\ell_1} i^{k\ell_2\cdots\ell_m} - \cdots \\ &\quad - \rho \nu_k^{\ell_m} i^{\ell_1\cdots\ell_{m-1}k} . \end{aligned} \tag{2.2.28}$$

The combination of (2.2.20), (2.2.21) and (2.2.28) yields

$$\begin{aligned} &\dot{\rho} + \rho v^{k}_{;k} = 0 , \\ &\dot{i}^{\ell_1\cdots\ell_m} = \nu_k^{\ell_1} i^{k\ell_2\cdots\ell_m} + \cdots + \nu_k^{\ell_m} i^{\ell_1\cdots\ell_{m-1}k} . \end{aligned} \tag{2.2.29}$$

These equations are the balance laws for the mass density and micro-inertia tensors respectively. The latter is a new balance law which does not appear in classical continuum mechanics.

<u>Momenta.</u> The balance laws of momenta are obtained by taking

$$\psi'^{k} = \rho' v'^{k}, \quad \tau'^{k\ell} = t'^{k\ell}, \quad g'^{k} = \rho' f'^{k}\,, \tag{2.2.30}$$

where v'^{k}, $t'^{k\ell}$ and f'^{k} are the velocity vector, the stress tensor and the body force per unit mass. The substitution of (2.2.30) into (2.2.19) gives

$$\begin{aligned} \psi^{k} &= \langle \rho' v'^{k} \rangle = \rho v^{k} + \rho \nu^{k\ell} i_{\ell}\,, \\ \psi^{k\ell_1 \cdots \ell_m} &= \rho v^{k} i^{\ell_1 \cdots \ell_m} + \rho \nu^{k}_{\ell} i^{\ell\ell_1 \cdots \ell_m}\,, \end{aligned} \tag{2.2.31}$$

$$\begin{aligned} &\tau^{k\ell} = t^{k\ell}\,, && g^{k} = \rho f^{k}\,, \\ &\tau^{k\ell\ell_1 \cdots \ell_m} = t^{k\ell\ell_1 \cdots \ell_m}\,, && \bar{\tau}^{k\ell\ell_1 \cdots \ell_m} = \bar{t}^{k\ell\ell_1 \cdots \ell_m}\,, \\ &g^{\ell\ell_1 \cdots \ell_m} = \rho f^{\ell\ell_1 \cdots \ell_m}\,, \end{aligned} \tag{2.2.32}$$

where $t^{k\ell}$, $t^{k\ell\ell_1 \cdots \ell_m}$, $\bar{t}^{k\ell\ell_1 \cdots \ell_m}$, f^{k} and $f^{\ell\ell_1 \cdots \ell_m}$ are the stress tensor, stress-moment tensor, stress-average moment tensor, body force vector and body force-moment tensor.

The substitution of (2.2.31) and (2.2.32) into (2.2.22) and (2.2.23), together with balance laws for the mass density and micro-inertia tensors (2.2.29), yields

$$\begin{aligned} \dot{\sigma}^{k} &= \rho \dot{v}^{k} + \rho(\dot{\nu}^{k}_{\ell} + \nu^{k}_{r} \nu^{r}_{\ell}) i^{\ell}\,, \\ \dot{\sigma}^{k\ell_1 \cdots \ell_m} &= \rho \dot{v}^{k} i^{\ell_1 \cdots \ell_m} + \rho(\dot{\nu}^{k}_{\ell} + \nu^{k}_{r} \nu^{r}_{\ell}) i^{\ell\ell_1 \cdots \ell_m}\,. \end{aligned} \tag{2.2.33}$$

These equations together with the local balance laws (2.2.20) and (2.2.21) give

$$t^{k\ell}_{;\ell} + \rho(f^k - \dot{v}^k) = \rho(\dot{\nu}^k_\ell + \nu^k_r \nu^r_\ell) i^\ell ,$$

$$t^{\ell k \ell_1 \cdots \ell_m}_{;\ell} + t^{k\ell_1 \cdots \ell_m} - \bar{t}^{k(\ell_1 \cdots \ell_m)} + \rho f^{k\ell_1 \cdots \ell_m}$$
$$= \rho i^{\ell_1 \cdots \ell_m} \dot{v}^k + \rho(\dot{\nu}^k_\ell + \nu^k_r \nu^r_\ell) i^{\ell \ell_1 \cdots \ell_m} . \qquad (2.2.34)$$

2.3 Constitutive Equations of Micromorphic Materials

The constitutive equations characterize the macroscopic properties of micromorphic materials. But material properties are of local nature and may change from particle to particle in a physical body. It is, therefore, necessary to develop concepts which describe the local behavior of micromorphic materials.

In continuum mechanics, each particle of a material may be assigned certain physical quantities such as stress, polarization, chemical constitution, directors, crystallographic directions, and temperature, etc. What constitutes an independent set of such variables will depend on the nature of the material and on the range of physical phenomena encompassed by the particular theory under consideration.

The constitutive equations are subject to invariance requirements of continuum mechanics. Application of the principle of objectivity, which states that the constitutive equations must be form-invariant to arbitrary rigid motions of the frame of reference, leads to more specific form for constitutive equations.

For the micromorphic theory, we need three constitutive equations; the constitutive equation for the stress distribution, the constitutive equation for the micro-stress average distribution and the constitutive equation for the stress-moment distribution.

We will consider now each one of these so as to be able to apply them for our physical problems later. First of all, we will modify Eringen's constitutive theory by introducing the generalized measures of the first and second deformation-rate tensors.

Eringen(1964) proposed the following form of the constitutive equations for simple micro-fluids undergoing small micro-deformation.

$$\begin{aligned} t = {} & \alpha_0 I + \alpha_1 d + \alpha_2 d^2 + \alpha_3 \bar{b} + \alpha_4 \bar{b}^T + \alpha_5 d\bar{b} + \alpha_6 \bar{b}d + \alpha_7 d\bar{b}^T \\ & + \alpha_8 \bar{b}^T d + \alpha_9 d^2 \bar{b} + \alpha_{10} \bar{b}d^2 + \alpha_{11} d^2 \bar{b} + \alpha_{12} \bar{b}^T d^2 \qquad (2.3.1) \\ & + \alpha_{13} d\bar{b}d^2 + \alpha_{14} d\bar{b}^T d^2 , \end{aligned}$$

$$\bar{t} = \beta_0 I + \beta_1 d + \beta_2 d^2 + \beta_3(\bar{b}+\bar{b}^T) + \beta_4(d\bar{b}+\bar{b}^T d)$$

$$+ \beta_5(\bar{b}d+d\bar{b}^T) + \beta_6(d^2\bar{b}+\bar{b}^T d^2) + \beta_7(\bar{b}d^2+d^2\bar{b}^T) \qquad (2.3.2)$$

$$+ \beta_8(d\bar{b}d^2+d^2\bar{b}^T d) + \beta_9(d^2\bar{b}d+d\bar{b}^T d^2),$$

$$t_{k\ell m} = (\gamma_1 a_{mrr} + \gamma_2 a_{rmr} + \gamma_3 a_{rrm})\delta_{k\ell}$$

$$+ (\gamma_4 a_{\ell rr} + \gamma_5 a_{r\ell r} + \gamma_6 a_{rr\ell})\delta_{km}$$

$$+ (\gamma_7 a_{krr} + \gamma_8 a_{rkr} + \gamma_9 a_{rrk})\delta_{\ell m} \qquad (2.3.3)$$

$$+ \gamma_{10} a_{k\ell m} + \gamma_{11} a_{km\ell} + \gamma_{12} a_{\ell km} + \gamma_{13} a_{mk\ell}$$

$$+ \gamma_{14} a_{\ell mk} + \gamma_{15} a_{m\ell k},$$

where $\bar{b}_{k\ell} = w_{k\ell} + \nu_{k\ell}$, $\bar{b}^T$ is the transpose of $\bar{b}$, and α_i, β_i, γ_i are functions of kinematic invariants.

Generalized measures in constitutive equations. Eringen's constitutive equations do not take into account the fact that in non-Newtonian fluids generalized measures of acceleration gradients play an important role in describing various viscoelastic phenomena. In the present investigation, in order to incorporate microviscoelastic effects, we introduce generalized measures of not only the first and second deformation-rate tensors involving velocity and acceleration gradients, but also those of the microdeformation-rate tensor involving gyration tensor. The use of such generalized measures in the place of ordinary measures eliminates the need for using higher order deformation-rates as shown by Narasimhan and Sra (1969). Hence our constitutive theory will involve generalized measures of the deformation-rate tensors combined with those of the microdeformation-rate tensor. We start with the constitutive equations proposed by Narasimhan and Sra (1969)

$$T = -pI + 2\mu D^{*} + 4\eta B^{*} , \qquad (2.3.4)$$

where D^{*} and B^{*} are the first and second generalized deformation-rate tensors. The components of D^{*} and B^{*} are given by

$$d^{*}_{ij} = \frac{k}{m^{q} n^{q}} [\delta_{ij} - (\delta_{ij} - 2md_{ij})^{n/2}]^{q} , \qquad (2.3.5)$$

$$b^{*}_{ij} = \frac{k'}{m'^{q'} n'^{q'}} [\delta_{ij} - (\delta_{ij} - 2m'b_{ij})^{n'/2}]^{q'} , \qquad (2.3.6)$$

where m, m', k and k' are dimension correcting constants; n, n' are measure indices; and q, q' are irreversibility indices of generalized measures. For $n = n' = 2$, $q = q' = 1$, and $k = k' = 1$, these generalized measures reduce to the ordinary deformation-rate tensors d_{ij} and b_{ij} respectively. Here b_{ij} is defined by

$$b_{ij} = \frac{1}{2} (a_{i;j} + a_{j;i} + 2v_{m;i} v^{m}_{;j}) , \qquad (2.3.7)$$

where a_i denotes acceleration vector. Substituting (2.3.5) and (2.3.6) into (2.3.4) and using Cayley-Hamilton theorem, we obtain

$$T = -pI + a_1 D + a_2 D^2 + b_1 B + b_2 B^2, \tag{2.3.8}$$

where a_1, a_2, b_1 and b_2 are known functions of invariants of deformation-rate tensors. For micromorphic fluids undergoing microdeformation, we may modify the constitutive equation (2.3.4) and propose a new constitutive equation for micromorphic fluid,

$$T = -pI + 2\mu D^* + 2\mu_0 \overline{B}^* + 2\mu_1 \overline{B}^{*T} + 4\eta B^*, \tag{2.3.9}$$

where μ_0, μ_1 are the response coefficients and the components of $\overline{B}^*$ are defined by

$$\overline{b}^*_{ij} = \frac{\overline{k}}{\overline{m}^{\overline{q}}\,\overline{n}^{\overline{q}}} [\delta_{ij} - (\delta_{ij} - 2\overline{m}\,\overline{b}_{ij})^{\overline{n}/2}]^{\overline{q}}, \tag{2.3.10}$$

where $\overline{m}$ and $\overline{k}$ are dimension correcting constants, $\overline{n}$ and $\overline{q}$ are measure index and irreversibility index of generalized measures. If we put $n' = \overline{n} = 2$, $q' = \overline{q} = 1$ and $k' = \overline{k} = 1$, then from (2.3.9) we obtain

$$T = -pI + 2\mu D + 2\mu_2 D^2 + 2\mu_0 \overline{B} + 2\mu_1 \overline{B}^T + 4\eta B, \tag{2.3.11}$$

which is the constitutive equation involving the stress distribution for micropolar fluids and which we shall use instead of (2.3.1) in the applications of micromorphic theory.

The constitutive equation for micro-stress average $\bar{t}_{k\ell}$ is

$$\bar{T} = -pI + 2\mu D + 2\mu_2 D^2 + 2\mu_3(\bar{B} + \bar{B}^T) + 4\eta B\,, \tag{2.3.12}$$

where μ_3 denotes the additional coefficient in the case of micro-stress average. The constitutive equation for the stress-moment tensor shall be retained as in (2.3.3) in our investigation.

2.4 Micropolar Materials as a Special Case of Micromorphic Materials

A micromorphic fluid is called micropolar if for all motions

$$\nu_{k\ell} = -\nu_{\ell k}, \qquad t_{k\ell m} = -t_{km\ell}\,. \tag{2.4.1}$$

Thus micropolar fluids exhibit only microrotational effects and can support surface and body couples. The material points contained in a small volume element, in addition to its usual rigid motion, can undergo only the rotational motions about its center of mass and no micro-stretch of particles is allowed. Physically, micropolar materials may represent materials that are made up of dumbbell molecules. Thus, for example, certain fibrous solids and liquid crystals can be represented with this model.

The condition (2.4.1) implies

$$a_{k\ell m} = -a_{\ell km} , \tag{2.4.2}$$

where $a_{k\ell m} = \nu_{k\ell;m}$ as defined before. For small micro-deformations, the constitutive equation (2.3.3) together with (2.4.1) and (2.4.2) yields

$$\begin{aligned} \gamma_1 - \gamma_2 + \gamma_4 - \gamma_5 &= 0, \\ \gamma_7 - \gamma_8 &= 0, \\ \gamma_{10} - \gamma_{12} + \gamma_{11} - \gamma_{13} &= 0, \end{aligned} \tag{2.4.3}$$

and hence we have from (2.3.3)

$$\begin{aligned} t^{k\ell m} &= (\gamma_1-\gamma_2)(g^{k\ell}a_r^{\ mr} - g^{km}a_r^{\ \ell r}) + (\gamma_{10}-\gamma_{12})(a^{k\ell m} - a^{km\ell}) \\ &+ (\gamma_{14}-\gamma_{15})a^{\ell mk} . \end{aligned} \tag{2.4.4}$$

3. APPLICATIONS OF MICROMORPHIC THEORY TO FLOWS IN CONVERGING AND DIVERGING CHANNELS

3.1 Introduction

Basic equations of micromorphic theory of grade one are given in Section 2. In Section 3, the problems of micropolar fluid flows in converging and diverging channels are formulated, and the basic equations to be solved are derived for incompressible fluid flows. Reynolds numbers and fluid parameters which describe the characteristics of micropolar fluid flows are introduced. Constitutive equations are expressed explicitly as functions of velocity and microrotation fields.

3.2 Basic Equations of Micromorphic Theory of Grade One

The balance laws of the micromorphic theory of grade k are obtained by taking k energy equations, $k+1$ momentum balance equations and $k+2$ mass balance equations. These equations for the micromorphic theory of grade one are given below:

Conservation of mass.

$$\dot{\rho} + \rho v^{k}_{;k} = 0. \qquad (3.2.1)$$

Conservation of microinertia.

$$\dot{i}^{k} - \nu^{k}_{\ m} i^{m} = 0 , \tag{3.2.2}$$

$$\dot{i}^{k\ell} - \nu^{k}_{\ m} i^{m\ell} - \nu^{\ell}_{\ m} i^{km} = 0 . \tag{3.2.3}$$

Balance of momentum.

$$t^{k\ell}_{\ ;k} + \rho(f^{\ell} - a^{\ell}) - \rho i^{k}(\dot{\nu}^{\ell}_{\ k} + \nu^{\ell}_{\ m}\nu^{m}_{\ k}) = 0 . \tag{3.2.4}$$

Balance of moments of momentum.

$$t^{k\ell m}_{\ ;k} + t^{m\ell} - \bar{t}^{(m\ell)} + \rho(f^{\ell m} - i^{m} a^{\ell}) = \rho i^{km}(\dot{\nu}^{\ell}_{\ k} + \nu^{\ell}_{\ n}\nu^{n}_{\ k}) . \tag{3.2.5}$$

Balance of energy.

$$t^{k\ell} v_{\ell ;k} + t^{k\ell m}\nu_{\ell m;k} + (\bar{t}^{(k\ell)} - t^{k\ell})\nu_{\ell k} + q^{k}_{\ ;k} + s^{k}_{\ ;k} + \rho h - \rho\dot{\epsilon} = 0, \tag{3.2.6}$$

where ϵ, q^{k}, s^{k} and h are respectively the internal energy density per unit mass, the heat flux vector, the surface energy vector and the energy source per unit volume.

3.3 Converging and Diverging Flows

We shall consider the steady radial motion of an incompressible micropolar fluid in the region between two non-parallel planes in the absence of external forces. The flow is two-dimensional and may be regarded as occurring in any representative plane perpendicular to the given planes

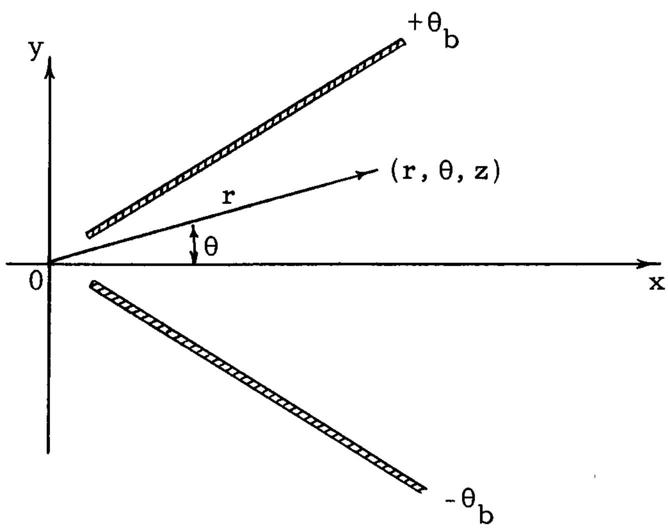

Figure 3.1. Divergent or convergent channels.

In this plane, the coordinates are r, θ, z where r is the distance from 0, the intersection of planes, θ is measured from the x-axis, and z is the coordinate perpendicular to the representative plane. In the following investigation we assume that the walls do not actually reach the point $r = 0$, but fall short of it, so that the singularities which occur there are avoided.

The physical components of velocity in the directions of r, θ, z are v_r, v_θ, v_z respectively. With the assumption $v_\theta = v_z = 0$, the equation of continuity becomes

$$\frac{\partial v}{\partial r} + \frac{v}{r} = 0, \qquad (3.3.1)$$

where $v = v_r(r, \theta)$. This continuity equation can only be satisfied if

$$v = \frac{2\nu G}{r}, \qquad (3.3.2)$$

where G is a function of θ only and ν denotes kinematic viscosity.

<u>Stress tensor.</u> Since the gyration tensor $\nu_{k\ell}$ is antisymmetric for micropolar fluids, it may be represented by a vector ν^m given by $\nu_{k\ell} = \epsilon_{k\ell m}\nu^m$, where $\epsilon_{k\ell m}$ is the alternating tensor. The physical components of the axial vector ν^m are denoted by ν_r, ν_θ and ν_z.

The physical components of stress tensor are

$$t_{rr} = -p + 2\mu\frac{\partial v}{\partial r} + 2\mu_2\left\{\frac{1}{4r^2}\left(\frac{\partial v}{\partial \theta}\right)^2 + \frac{v^2}{r^2}\right\} + \frac{16\eta v^2}{r^2},$$

$$(3.3.3)$$

$$t_{r\theta} = \frac{\mu}{r}\frac{\partial v}{\partial \theta} + 2(\mu_0 - \mu_1)\left\{\frac{1}{2r}\frac{\partial v}{\partial \theta} + \nu_z\right\} - \frac{8\eta v}{r^2}\frac{\partial v}{\partial \theta},$$

$$t_{rz} = -2(\mu_0 - \mu_1)\nu_\theta \ ,$$

$$t_{\theta r} = \frac{\mu}{r}\frac{\partial v}{\partial \theta} - 2(\mu_0 - \mu_1)\left\{\frac{1}{2r}\frac{\partial v}{\partial \theta} + \nu_z\right\} - \frac{8\eta v}{r^2}\frac{\partial v}{\partial \theta} \ ,$$

$$t_{\theta\theta} = -p + \frac{2\mu v}{r} + 2\mu_2\left\{\frac{1}{4r^2}\left(\frac{\partial v}{\partial \theta}\right)^2 + \frac{v^2}{r^2}\right\} + \frac{4\eta}{r^2}\left(\frac{\partial v}{\partial \theta}\right)^2 \ , \qquad (3.3.3)$$

$$t_{\theta z} = 2(\mu_0 - \mu_1)\nu_r \ , \qquad t_{zr} = 2(\mu_0 - \mu_1)\nu_\theta \ ,$$

$$t_{z\theta} = -2(\mu_0 - \mu_1)\nu_r , \qquad t_{zz} = -p.$$

<u>Stress-moment tensor</u>. The non-vanishing components of the stress-moment tensor $t^{k\ell m} = -t^{km\ell}$ are

$$t^{123} = \frac{(\gamma_{10} - \gamma_{12})}{r}\left\{\frac{\partial \nu_z}{\partial z} + \frac{1}{r}\left(\frac{\partial \nu_\theta}{\partial \theta} + \nu_r\right)\right\} + \frac{(\gamma_{14} - \gamma_{15})}{r}\frac{\partial \nu_r}{\partial r} \ ,$$

$$t^{213} = \frac{(\gamma_{12} - \gamma_{10})}{r}\left\{\frac{\partial \nu_z}{\partial z} + \frac{\partial \nu_r}{\partial r}\right\} + \frac{(\gamma_{15} - \gamma_{14})}{r^2}\left\{\frac{\partial \nu_\theta}{\partial \theta} + \nu_r\right\} ,$$

$$t^{312} = \frac{(\gamma_{10} - \gamma_{12})}{r}\left\{\frac{\partial \nu_r}{\partial r} + \frac{1}{r}\left(\frac{\partial \nu_\theta}{\partial \theta} + \nu_r\right)\right\} + \frac{(\gamma_{14} - \gamma_{15})}{r}\frac{\partial \nu_z}{\partial z} \ ,$$

$$(3.3.4)$$

$$t^{112} = \frac{(\gamma_1 - \gamma_2)}{r}\frac{\partial \nu_r}{\partial z} - \frac{\gamma_{16}}{r}\frac{\partial \nu_z}{\partial r} ,$$

$$t^{113} = \frac{(\gamma_2 - \gamma_1)}{r}\left\{\frac{\partial \nu_r}{\partial \theta} - \nu_\theta\right\} + \frac{\gamma_{16}}{r}\frac{\partial \nu_\theta}{\partial r} \ ,$$

$$t^{212} = \frac{(\gamma_1 - \gamma_2)}{r^2}\frac{\partial \nu_\theta}{\partial z} - \frac{\gamma_{16}}{r^3}\frac{\partial \nu_z}{\partial \theta} \ ,$$

$$t^{223} = \frac{(\gamma_1 - \gamma_2)}{r^2} \frac{\partial \nu_\theta}{\partial r} - \frac{\gamma_{16}}{r^3} \{ \frac{\partial \nu_r}{\partial \theta} - \nu_\theta \} ,$$

$$t^{313} = \frac{(\gamma_2 - \gamma_1)}{r} \frac{\partial \nu_z}{\partial \theta} + \gamma_{16} \frac{\partial \nu_\theta}{\partial z} , \tag{3.3.4}$$

$$t^{323} = \frac{(\gamma_1 - \gamma_2)}{r} \frac{\partial \nu_z}{\partial r} - \frac{\gamma_{16}}{r} \frac{\partial \nu_r}{\partial z} ,$$

where $\gamma_{16} = \gamma_1 + \gamma_{10} + \gamma_{15} - \gamma_2 - \gamma_{12} - \gamma_{14}$ and $\gamma_1, \ldots, \gamma_{15}$, are the coefficients which appear in the constitutive equation (2.3.3) of micropolar fluids.

Equations of balance of momentum. Assuming that the motion of gyration tensor $\nu^{k\ell}$ to be steady, we obtain three equations of balance of momentum from (3.2.4);

$$\begin{aligned} &- \frac{\partial p}{\partial r} + \frac{\mu}{r^2} \frac{\partial^2 v}{\partial \theta^2} - \frac{2\mu_2}{r^3} \{ (\frac{\partial v}{\partial \theta})^2 + 4v^2 \} \\ &- 2(\mu_0 - \mu_1) \{ \frac{1}{2r^2} \frac{\partial^2 v}{\partial \theta^2} + \frac{1}{r} \frac{\partial \nu_z}{\partial \theta} - \frac{\partial \nu_\theta}{\partial z} \} \\ &- \frac{8\eta}{r^3} \{ 12v^2 + 2v \frac{\partial^2 v}{\partial \theta^2} + 3(\frac{\partial v}{\partial \theta})^2 \} \\ &= \rho \{ -\frac{v^2}{r} - i_r(\nu_\theta^2 + \nu_z^2) + i_\theta(v \frac{\partial \nu_z}{\partial r} + \nu_r \nu_\theta) + i_z(\nu_r \nu_z - v \frac{\partial \nu_\theta}{\partial r}) \} , \end{aligned} \tag{3.3.5}$$

$$-\frac{1}{r}\frac{\partial p}{\partial\theta} + \frac{2\mu}{r^2}\frac{\partial v}{\partial\theta} + \frac{\mu_2}{r^3}\frac{\partial v}{\partial\theta}\{\frac{\partial^2 v}{\partial\theta^2} + 4v\}$$

$$-2(\mu_0-\mu_1)\{\frac{1}{r^2}\frac{\partial v}{\partial\theta} - \frac{\partial\nu_z}{\partial r} + \frac{\partial\nu_r}{\partial z}\} + \frac{8\eta}{r^3}\frac{\partial v}{\partial\theta}\{\frac{\partial^2 v}{\partial\theta^2} + 2v\} \qquad (3.3.6)$$

$$= \rho\{i_r(\nu_r\nu_\theta - v\frac{\partial\nu_z}{\partial r}) - i_\theta(\nu_r^2+\nu_z^2) + i_z(\nu_\theta\nu_z + v\frac{\partial\nu_r}{\partial r})\},$$

$$-\frac{\partial p}{\partial z} - 2(\mu_0-\mu_1)\{\frac{\partial\nu_\theta}{\partial r} - \frac{1}{r}\frac{\partial\nu_r}{\partial\theta} + \frac{\nu_\theta}{r}\}$$

$$= \rho\{i_r(\nu_r\nu_z + v\frac{\partial\nu_\theta}{\partial r}) + i_\theta(\nu_\theta\nu_z - v\frac{\partial\nu_r}{\partial r}) - i_z(\nu_r^2+\nu_\theta^2)\}, \qquad (3.3.7)$$

where i_r, i_θ, i_z are the physical components of microinertia $i^{.k}$.

Equations of balance of moments of momentum. For microisotropic fluids, the microinertia moment tensor $i^{.k\ell}$ is given by $i^{k\ell} = ig^{k\ell}$, where i is a constant and $g^{k\ell}$ denotes the metric tensor. It can be shown that for steady motions of microinertia moment tensor $i^{.k\ell}$ and gyration tensor $\nu^{k\ell}$, the microinertia vector i^k and microrotation vector ν^k must vanish for microisotropic fluids and hence equations of balance of moments of momentum must also be satisfied.

Equations of motion. Since $i_r = i_\theta = i_z = 0$, $\nu_r = \nu_\theta = \nu_z = 0$ for microisotropic fluids, the equations of balance of momentum (3.3.5), (3.3.6) and (3.3.7) reduce to the forms.

$$-\frac{\partial p}{\partial r}+\frac{\mu}{r^2}\frac{\partial^2 v}{\partial\theta^2}-\frac{2\mu_2}{r^3}\{(\frac{\partial v}{\partial\theta})^2+4v^2\}-\frac{(\mu_0-\mu_1)}{r^2}\frac{\partial^2 v}{\partial\theta^2}$$

$$-\frac{8\eta}{r^3}\{12v^2+2v\frac{\partial^2 v}{\partial\theta^2}+3(\frac{\partial v}{\partial\theta})^2\}=-\frac{\rho v^2}{r}, \quad (3.3.8)$$

$$-\frac{\partial p}{\partial\theta}+\frac{2\mu}{r}\frac{\partial v}{\partial\theta}+\frac{\mu_2}{r^2}\frac{\partial v}{\partial\theta}\{\frac{\partial^2 v}{\partial\theta^2}+4v\}$$

$$-\frac{2(\mu_0-\mu_1)}{r}\frac{\partial v}{\partial\theta}+\frac{8\eta}{r^2}\frac{\partial v}{\partial\theta}\{\frac{\partial^2 v}{\partial\theta^2}+2v\}=0, \quad (3.3.9)$$

$$-\frac{\partial p}{\partial z}=0. \quad (3.3.10)$$

The last equation implies that the pressure is independent of the z-coordinate. The substitution of $v = 2\nu G(\theta)/r$ in (3.3.8) and (3.3.9) gives

$$\frac{\partial p}{\partial r}=\frac{2\nu}{r^3}\{2\rho\nu G^2+(\mu-\mu_0+\mu_1)G''\}$$

$$-\frac{8\nu^2}{r^5}\{\mu_2(G'^2+4G^2)+4\eta(12G^2+2GG''+3G'^2)\}, \quad (3.3.11)$$

$$\frac{\partial p}{\partial\theta}=\frac{4\nu}{r^2}\{(\mu-\mu_0+\mu_1)G'\}$$

$$+\frac{4\nu^2}{r^4}\{\mu_2(G'G''+4GG')+8\eta(G'G''+2GG')\}, \quad (3.3.12)$$

where a prime denotes differentiation with respect to θ.

Differentiating (3.3.11) and (3.3.12) with respect to θ and r respectively and equating the resulting two equations, we obtain, with $\beta = 1 + (\mu_1 - \mu_0)/\mu$,

$$\mu r^2(4\beta G' + 4GG' + \beta G''') = 32\nu\eta(8GG' + 2G'G'' + GG''') . \qquad (3.3.13)$$

Since the fluid parameter β involves the response coefficients of microdeformation-rate tensors, it is referred to as microdeformation parameter in the rest of our thesis.

Integrating (3.3.13) and simplifying the resulting equation, we obtain

$$\frac{\mu(8\beta G + 4G^2 + 2\beta G'' + a)}{32\nu\eta(8G^2 + G'^2 + 2GG'' - b)} = \frac{1}{r^2} , \qquad (3.3.14)$$

where a and b are integration constants. The right-hand side of this equation is a function of r only whereas the left-hand side is a function of θ only. Setting (3.3.14) to be a constant and integrating the resulting equation, we obtain

$$4\beta G^2 + \frac{4}{3}G^3 + \beta G'^2 + aG = \alpha(\frac{8}{3}G^3 + GG'^2 - bG) + c, \quad (3.3.15)$$

where $\alpha = \text{const.} \times 32\eta/\rho$ and c is an integration constant. Since the fluid parameter α involves the response coefficient containing acceleration gradients, it is referred to as viscoelastic parameter in the rest of the thesis.

We now define the Reynolds number R as $R = |Q|/\nu$, where Q denotes the volumetric flow of fluid per unit radial distance per unit time. If we put $G = \frac{1}{2}RF(\theta)$ in (3.3.15), we obtain

$$(2\beta - \alpha RF)F'^2 = \frac{2}{3}\{(2\alpha-1)RF^3 - 6\beta F^2 - 3(a+b\alpha)R^{-1}F + 6cR^{-2}\}, \quad (3.3.16)$$

which may be rewritten in the form

$$d\theta = \pm\sqrt{3}\left\{\frac{\beta}{R} - \frac{\alpha F}{2}\right\}\frac{dF}{\sqrt{Q(F)}}, \quad (3.3.17)$$

where

$$\begin{aligned} Q(F) &= a_0F^4 + a_1F^3 + a_2F^2 + a_3F + a_4, \\ a_0 &= \alpha(1-2\alpha), \quad a_1 = 2\beta(5\alpha-1)R^{-1}, \\ a_2 &= 3\{\alpha(a+b\alpha) - 4\beta^2\}R^{-2}, \\ a_3 &= -6\{\beta(a+b\alpha) + c\alpha\}R^{-3}, \\ a_4 &= 12c\beta R^{-4}. \end{aligned} \quad (3.3.18)$$

<u>Pressure field.</u> In order to determine the pressure, we integrate (3.3.9) and obtain

$$p = p_1(r) + 4\mu\nu r^{-2}G + 2\nu^2 r^{-4}\{4(\mu_2+4\eta)G^2 + (\mu_2+8\eta)G'^2\}, \quad (3.3.19)$$

where $p_1(r)$ is a function of r only.

If we differentiate (3.3.19) with respect to r and equate the resulting equation with (3.3.8), we obtain

$$\frac{r^3}{\rho}\frac{\partial p_1}{\partial r} = \nu^2(8\beta G+4G^2+2\beta G'') - \frac{32\eta\nu^2}{\rho r^2}\{8G^2+G'^2+2GG''\}. \qquad (3.3.20)$$

From Equations (3.3.15) and (3.3.20), we obtain the expression for $p_1(r)$,

$$p_1 = \frac{a\mu\nu}{2r^2} + \frac{8b\nu^2\eta}{r^4} + \text{constant}. \qquad (3.3.21)$$

The substitution of (3.3.21) into (3.3.19) gives

$$p = -\frac{\mu\nu}{2r^2}(a+4\beta RF)$$

$$+\frac{\nu^2R^2}{r^4}\{\frac{\mu_2}{2}(4F^2+F'^2)+4\eta(2F^2+F'^2+\frac{2b}{R^2})\}. \qquad (3.3.22)$$

4. SOLUTION OF THE PROBLEM OF MICROPOLAR FLUID FLOW IN DIVERGING CHANNELS

4.1 Introduction

It is necessary to distinguish two different types of flow that are described by the equations of motions. The first type, considered in this chapter, is the flow in diverging channels in which it is assumed that a line source with a positive efflux velocity is present at the origin of the diverging channel walls.

It is again necessary to distinguish two solutions that are obtained according as the viscoelastic parameter α vanishes or remains non-zero. The exact solutions for the velocity field for the steady flow of an incompressible micropolar fluid in diverging channels are obtained and the velocity profiles are determined with the aid of Jacobian elliptic functions. The numerical solution for vanishing viscoelastic parameter α are given for diverging channels with the total opening of ten degrees and compared with the classical viscous flow solution.

Solutions for small as well as large Reynolds numbers and for unitary elliptic modularity are discussed. The coupling of the Reynolds number and the opening of the diverging channels is also discussed.

4.2 Method of Solution for Diverging Channel Flows for Non-Vanishing Microdeformation Parameter but for Vanishing Viscoelastic Parameter

When $\alpha = 0$, the leading coefficient a_0 of polynomial $Q(F)$ defined by (3.3.18) vanishes and the differential equation (3.3.16) reduces to

$$3\beta F'^2 = -2RF^3 - 12\beta F^2 - 6aR^{-1}F + 12cR^{-2} . \quad (4.2.1)$$

Putting $\theta = \theta_b$ in this equation, we obtain the constant c,

$$c = \frac{1}{4}\beta R^2 F_b'^2 , \quad (4.2.2)$$

where $F_b' = F'(\theta_b)$. If we differentiate (4.2.1) with respect to θ and set $\theta = \theta_b$, then we obtain the constant a,

$$a = -\beta R F_b'' , \quad (4.2.3)$$

where $F_b'' = F''(\theta_b)$.

The substitution of (4.2.2) and (4.2.3) into (4.2.1) gives

$$3\beta F'^2 = 2RQ_0(F) , \quad (4.2.4)$$

which may be expressed in differential form

$$d\theta = \pm\left(\frac{3\beta}{2R}\right)^{1/2} \frac{dF}{\sqrt{Q_0(F)}} , \quad (4.2.5)$$

where $Q_0(F)$ is a cubic polynomial in F defined by

$$Q_0(F) = -F^3 + 3\beta R^{-1}\{-2F^2 + F_b''F + \frac{1}{2} F_b'^2\} . \tag{4.2.6}$$

In order to obtain physically meaningful solutions of the problem, we assume that $Q_0(F)$ is positive for possible values of F. This assumption implies that the constant c must be positive, and hence the parameter β must also be positive.

If α_1, α_2, and α_3 are the three roots of $Q_0(F) = 0$, then the following relations hold among the roots;

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= -6\beta R^{-1} , \\ \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 &= -3\beta R^{-1} F_b'' , \\ \alpha_1\alpha_2\alpha_3 &= \frac{3}{2} \beta R^{-1} F_b'^2 . \end{aligned} \tag{4.2.7}$$

There are two cases to be considered separately according as only one root is real or all three roots are real.

<u>Case I: $Q_0(F) = 0$ has only one real root.</u> If α_1 is the real root, then α_2 and α_3 are conjugate to each other; i.e.,

$$\begin{aligned} \alpha_2 &= m + in , \\ \alpha_3 &= m - in , \end{aligned} \tag{4.2.8}$$

where m and n are reals. The product of these roots is

$$\alpha_2\alpha_3 = m^2 + n^2 > 0 \; . \tag{4.2.9}$$

Since $\alpha_1\alpha_2\alpha_3 > 0$ to obtain physically meaningful solutions, one must have $\alpha_1 > 0$. Now $Q_0(F)$ can be written as

$$\begin{aligned} Q_0(F) &= (\alpha_1 - F)(F - \alpha_2)(F - \alpha_3) \\ &= (\alpha_1 - F)\{(F - m)^2 + n^2\} , \end{aligned} \tag{4.2.10}$$

and $Q_0(F) > 0$ implies $F < \alpha_1$. Thus the function F is bounded above by α_1. Since F must vanish at the boundaries, the possible range of the function F is, therefore,

$$0 \leq F \leq \alpha_1 \; , \tag{4.2.11}$$

and hence the flow is divergent.

Integrating (4.2.5) with respect to θ and taking negative sign for divergent flow, one has

$$\left(\frac{2R}{3\beta}\right)^{1/2} \int_0^\theta d\theta = - \int_{\alpha_1}^{F} \frac{dF}{\sqrt{Q_0(F)}} \; , \tag{4.2.12}$$

which can be expressed in terms of Jacobi elliptic function. A familiar transformation of elliptic integrals gives

$$\int_F^{a_1} \frac{dF}{\sqrt{Q_0(F)}} = \frac{1}{\sqrt{H}} \, cn^{-1}\left(\frac{H-a_1+F}{H+a_1-F}, k\right) , \tag{4.2.13}$$

where the modulus k and H are defined by

$$k^2 = \frac{H+a_1-m}{2H} , \quad H^2 = (a_1-m)^2 + n^2 . \tag{4.2.14}$$

The combination of (4.2.12) and (4.2.13) yields

$$\theta\left(\frac{2R}{3\beta}\right)^{1/2} = \frac{1}{\sqrt{H}} \, cn^{-1}\left(\frac{H-a_1+F}{H+a_1-F}, k\right) , \tag{4.2.15}$$

and an inversion gives

$$\frac{H-a_1+F}{H+a_1-F} = cn(\bar{m}\theta, k) , \tag{4.2.16}$$

where $\bar{m} = \left(\frac{2RH}{3\beta}\right)^{1/2}$. The desired solution is

$$F = a_1 - \frac{H\{1-cn(\bar{m}\theta, k)\}}{1+cn(\bar{m}\theta, k)} . \tag{4.2.17}$$

If we set $\theta = 0$ in this equation, then since $cn(0, k) = 1$ we get $a_1 = F(0) = F_0$ which gives the maximum velocity in the channel.

Now the boundary condition gives

$$H = \frac{F_0\{1+cn(\bar{m}\theta_b, k)\}}{1-cn(\bar{m}\theta_b, k)} . \tag{4.2.18}$$

The substitution of (4.2.18) into (4.2.17) gives

$$\frac{F}{F_0} = 1 - \frac{\{1+cn(\bar{m}\theta_b,k)\}\{1-cn(\bar{m}\theta,k)\}}{\{1-cn(\bar{m}\theta_b,k)\}\{1+cn(\bar{m}\theta,k)\}} \ . \tag{4.2.19}$$

The first relation in (4.2.7) together with the condition $\alpha_2 + \alpha_3 = 2m$ gives

$$m = -\frac{1}{2}(F_0+6\beta R^{-1}) \ . \tag{4.2.20}$$

The substitution of (4.2.20) into (4.2.14) yields

$$H = \frac{3(2+R^*)F_0}{2(2k^2-1)R^*} \ , \tag{4.2.21}$$

where

$$R^* = \beta^{-1}RF_0 \ . \tag{4.2.22}$$

With the expression for H given by (4.2.21) $\bar{m}$ can be written in the form

$$\bar{m} = \frac{2+R^*}{2k^2-1} \ .$$

Equating (4.2.18) and (4.2.21), we obtain

$$\frac{3(2+R^*)}{2(2k^2-1)R^*} = \frac{1+cn(\bar{m}\theta_b,k)}{1-cn(\bar{m}\theta_b,k)} \ . \tag{4.2.23}$$

Now (4.2.19) may be written in the form

$$\frac{F}{F_0} = 1 - \frac{3(2+R^*)\{1-\mathrm{cn}(\bar{m}\theta, k)\}}{2(2k^2-1)R^*\{1+\mathrm{cn}(\bar{m}\theta, k)\}} . \qquad (4.2.24)$$

From (4.2.23) we obtain

$$\mathrm{cn}\{\frac{(2+R^*)\theta_b}{2k^2-1}, k\} = \frac{6+R^*(5-4k^2)}{6+R^*(1+4k^2)} , \qquad (4.2.25)$$

from which k can be determined. Thus if k is known, then H is obtained from (4.2.21). Subsequently, the velocity profile may be determined from (4.2.24).

Case II: All the three roots of $Q_0(F) = 0$ are reals. When all the three roots are reals, we may assume that $\alpha_3 \leq \alpha_2 \leq \alpha_1$. Then it follows from the first of the relations (4.2.7) that one of the roots must be negative and in this case α_3 must be negative. Since $Q_0(F)$ is positive, $F \leq \alpha_3$ or $\alpha_2 \leq F \leq \alpha_1$. If $F \leq \alpha_3$, the boundary condition that $F = 0$ on the boundary cannot be satisfied. Hence we reject this possibility. So the only possible case is $\alpha_2 \leq F \leq \alpha_1$.

Again, since $\alpha_1 \alpha_2 \alpha_3 > 0$, and $\alpha_3 < 0$ we must have $\alpha_1 > 0$ and $\alpha_2 < 0$. Thus for divergent flow, we have $0 \leq F \leq \alpha_1$.

Now the integral of (4.2.5) gives

$$\left(\frac{2R}{3\beta}\right)^{1/2} \int_0^{\theta} d\theta = - \int_{a_1}^{F} \frac{dF}{\sqrt{Q_0(F)}} . \tag{4.2.26}$$

If we introduce two constants k and m defined by

$$k^2 = \frac{a_1 - a_2}{a_1 - a_3} , \quad m^2 = \frac{(a_1 - a_3)R}{6\beta} , \tag{4.2.27}$$

then

$$a_1 - a_2 = \frac{6\beta m^2 k^2}{R} , \tag{4.2.28}$$

$$a_2 - a_3 = \frac{6\beta m^2 (1-k^2)}{R} . \tag{4.2.29}$$

The first relation in (4.2.7) yields

$$a_1 = \frac{2\beta}{R} \{m^2(1+k^2) - 1\} , \tag{4.2.30}$$

and the substitution of (4.2.30) into (4.2.29) and (4.2.28) gives

$$a_2 = \frac{2\beta}{R} \{m^2(1-2k^2) - 1\} , \tag{4.2.31}$$

$$a_3 = \frac{2\beta}{R} \{m^2(k^2-2) - 1\} . \tag{4.2.32}$$

A transformation of elliptic integral (4.2.26) gives

$$\int_F^{a_1} \frac{dF}{\sqrt{Q_0(F)}} = \frac{2}{\sqrt{a_1 - a_3}} \, \mathrm{sn}^{-1} \left\{ \left(\frac{a_1 - F}{a_1 - a_2} \right)^{1/2}, k \right\} , \tag{4.2.33}$$

where k is the modulus. The combination of (4.2.26) and (4.2.33) yields

$$\left\{\frac{R(\alpha_1-\alpha_3)}{6\beta}\right\}^{1/2}\theta = sn^{-1}\left\{\left(\frac{\alpha_1-F}{\alpha_1-\alpha_2}\right)^{1/2}, k\right\} , \qquad (4.2.34)$$

and an inversion gives

$$\frac{\alpha_1-F}{\alpha_1-\alpha_2} = sn^2(m\theta, k) . \qquad (4.2.35)$$

Hence, the desired solution takes the form

$$F = \alpha_1 - (\alpha_1-\alpha_2)sn^2(m\theta, k) . \qquad (4.2.36)$$

If we put $\theta = 0$ in this equation, then since $sn(0,k) = 0$ we get $\alpha_1 = F(0) = F_0$. The boundary condition $F(\theta_b) = 0$ together with (4.2.28) yields

$$F_0 = \frac{6\beta m^2 k^2}{R} sn^2(m\theta_b, k) , \qquad (4.2.37)$$

and hence from (4.2.36) we have

$$\frac{F}{F_0} = 1 - \frac{sn^2(m\theta, k)}{sn^2(m\theta_b, k)} . \qquad (4.2.38)$$

If we equate (4.2.30) and (4.2.37), then the result is

$$sn^2(m\theta_b, k) = \frac{m^2(1+k^2)-1}{3m^2k^2} . \qquad (4.2.39)$$

The substituion of m^2 obtained from (4.2.30) into the transcendental equation (4.2.39) gives

$$\mathrm{sn}^2\{\theta_b(\frac{2+R^*}{2+2k^2})^{1/2}, k\} = \frac{1}{3}\{\frac{R^*(1+k^2)}{k^2(2+R^*)}\} , \quad (4.2.40)$$

where $R^* = \beta^{-1}RF_0$ and k can be determined by this equation. If k is known, then m is determined from (4.2.30) and subsequently the velocity profile may be determined from (4.2.38) for given θ_b, β and R.

Now the pressure field is obtained from (3.3.22). We have

$$p = \frac{\mu\nu\beta R}{2r^2}\{4F-F_b''\} + \frac{\nu^2\mu_2R^2}{2r^4}\{4F^2+F'^2\} . \quad (4.2.41)$$

The substitution of (4.2.4) into (4.2.41) gives

$$p = \frac{\mu\nu\beta R}{2r^2}\{4F-F_b''\} + \frac{2\mu_2\nu^2R^2}{r^4}\{F^2 - \frac{R}{6\beta}Q_0(F)\} . \quad (4.2.42)$$

The pressure along the central line dividing the two planes, denoted by p_0, is

$$p_0 = \frac{\mu\nu\beta R}{2r^2}\{4F_0-F_b''\} + \frac{2\mu_2\nu^2R^2F_0^2}{r^4} . \quad (4.2.43)$$

The substraction of (4.2.42) from (4.2.43) gives the pressure difference

$$p_0 - p = \frac{2\mu\nu\beta R F_0}{r^2}\{1 - \frac{F}{F_0}\} + \frac{2\mu_2\nu^2 R^2 F_0^2}{r^4}\{1 - \frac{F^2}{F_0^2} + \frac{R Q_0(F)}{6\beta F_0^2}\}. \tag{4.2.44}$$

The pressure difference between the center line and the boundary planes is

$$p_0 - p_b = \frac{2\mu\nu\beta R F_0}{r^2} + \frac{2\mu_2\nu^2 R^2 F_0^2}{r^4}\{1 + \frac{F_b'^2}{4F_0^2}\}, \tag{4.2.45}$$

which shows that the higher the Reynolds number, the greater is the pressure difference.

We next discuss the micromorphic behavior of the fluid flowing in divergent channel on the basis of the solutions obtained for the velocity and pressure fields.

We have drawn in Figure 4.1 velocity profiles for $\alpha = 0$ and $\beta = 1$. In numerical examples, a total angle of ten degrees has been chosen for the divergent as well as the convergent flows. The modified Reynolds number R_0, $R_0 = RF_0$, corresponding to unitary elliptic modularity is 684. One also finds that the modified Reynolds number corresponding to a profile with zero slope at the channels is 1342. For the modified Reynolds number greater than 1342, the phenomenon of back-flow would occur as shown in Figure 4.1 with $R_0 = 5,000$.

	A	B	C
β	1.0	1.0	1.0
k^2	1.0	0.5	0.8
$R_0 = RF_0$	684	1,342	5,000

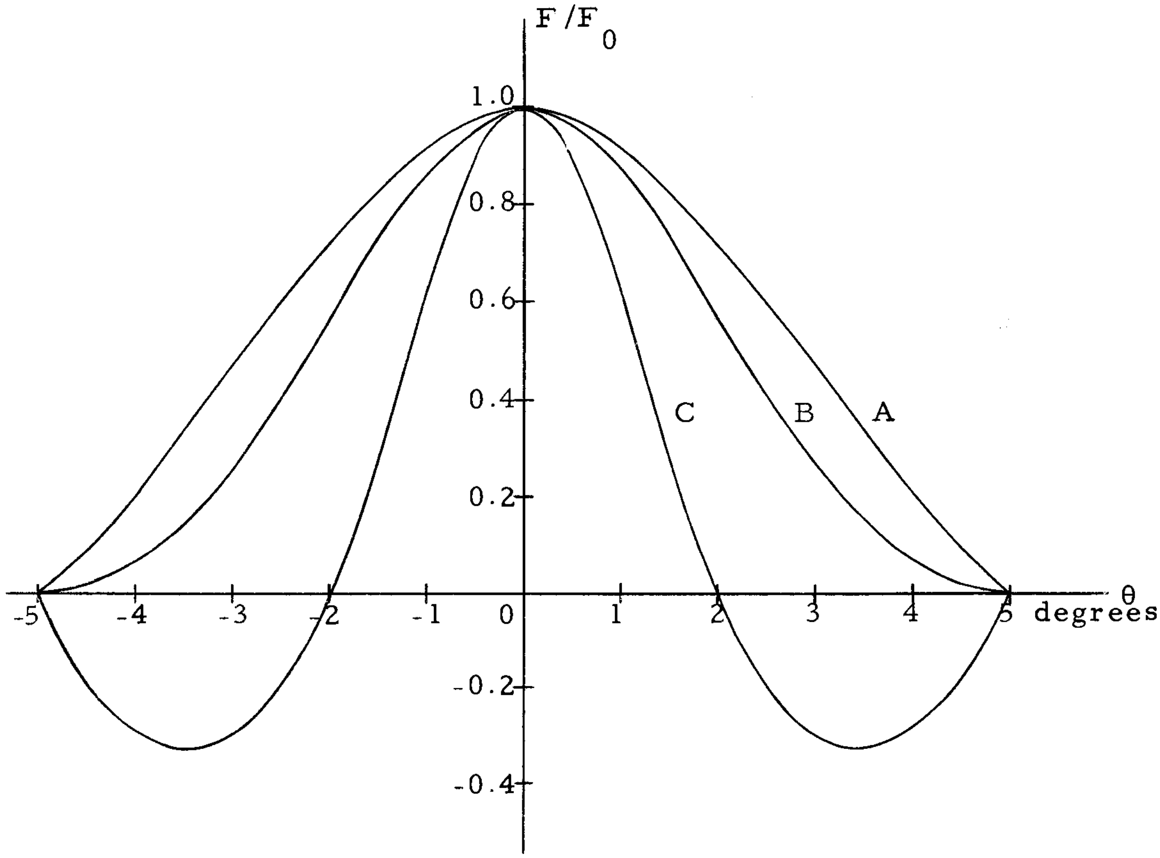

Figure 4.1. Velocity profiles for diverging flow with $\alpha = 0$, $\beta = 1$, variable k^2.

	A	B	C
β	1.0	0.6	0.5
k^2	1.0	1.0	1.0
$R_0 = RF_0$	684	684	684

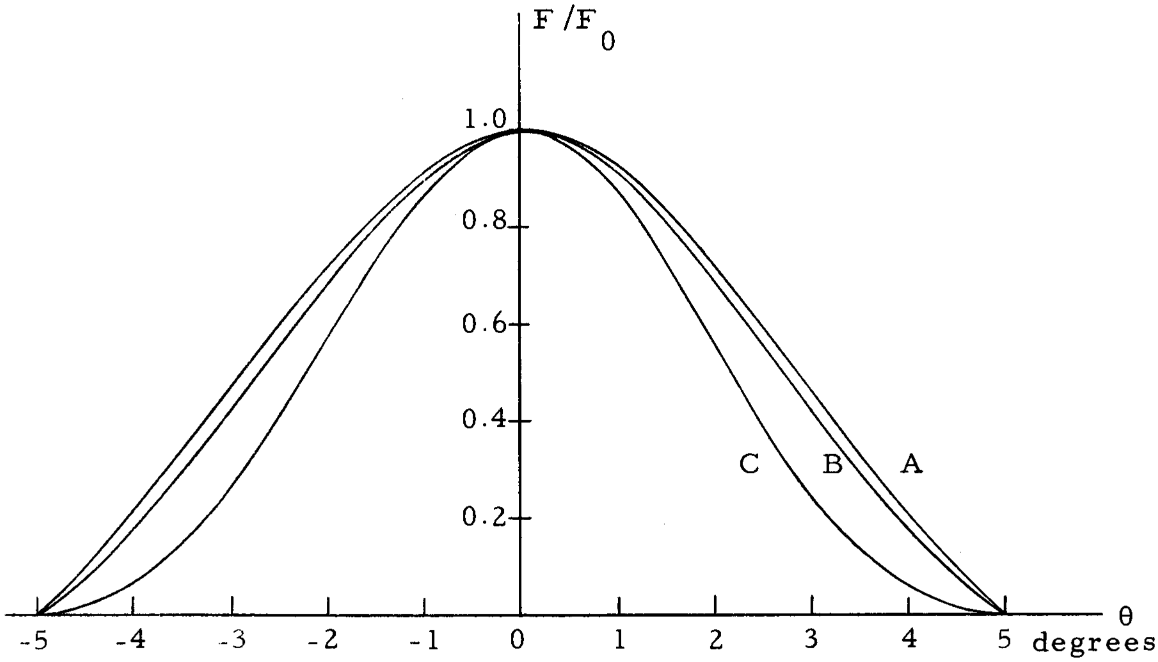

Figure 4.2. Velocity profiles for diverging flow for variable β and $k^2=1$.

Thus if we identify the modified Reynolds number R_0 with the Reynolds number defined in terms of maximum velocity by Millsaps and Pohlhausen (1953), then the velocity profiles shown in Figure 4.1 agree entirely with the velocity profiles given by them. This makes it possible to identify the two Reynolds numbers defined differently. Moreover, the velocity profiles of Newtonian fluids arise as a special case of our more general micropolar fluids if we set $\alpha = 0$ and $\beta = 1$.

Figure 4.2 shows the velocity profiles for three different values of the parameter β with the modified Reynolds number $R_0 = 684$. As β decreases continuously from unity to 0.5, the magnitude of the velocity also decreases continuously and for β less than 0.5 back-flow would occur. The phenomena of back-flow are clearly shown in Figure 4.3 with the modified Reynolds number $R_0 = 1342$.

	A	B	C	D
β	1.0	0.52	0.37	0.22
k^2	1.0	0.6	0.7	0.8
$R_0 = RF_0$	1,342	1,342	1,342	1,342

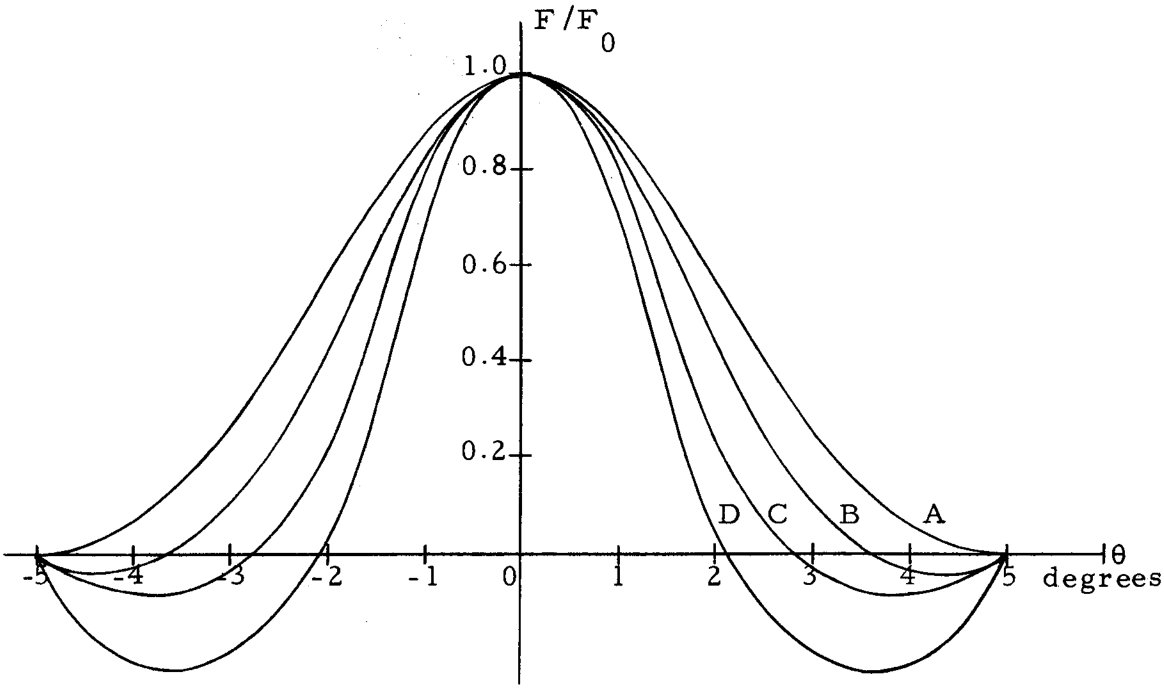

Figure 4.3. Velocity profiles for diverging flow for variables β and k^2.

All three figures show that each of the velocity profiles is more sharply curved in the middle of the channel than the Poiseuille parabola valid for parallel flows. In the case of divergent flows, all of the profiles have inflexion points suggesting flow instability. At large modified Reynolds numbers, the velocity vanishes at four points across the profiles. Thus channels may be placed at any of these points obtaining flows with two symmetrical back-flow regions.

4.3 Solution for Large Reynolds Numbers for Diverging Channel Flows with Vanishing Viscoelastic Parameter

As the modified Reynolds number of the diverging flow becomes larger than the modified Reynolds number corresponding to zero slope at the channels with $\alpha = 0$ and $\beta = 1$, a region of back-flow appear along each channel. As the modified Reynolds number increases further, it is likely that positive jets, in addition to the back-flow regions, will appear along each channel. This process may continue with an increasing number of back-flow regions and positive jets appearing in a symmetrical distribution about the main flow. However, it is to be noted that, since between each forward jet and each back-flow region there is a point with zero velocity, at this point a channel could be placed leading to asymmetrical velocity profiles with the possible number of asymmetrical configurations increasing with increasing modified Reynolds number. When β is less than unity, back-flow occurs at lower modified Reynolds number.

Moreover, at sufficiently large modified Reynolds number it is found that the transcendental equation (4.2.40) has multiple roots. Each member of the set of solutions corresponding to a definite modified Reynolds number leads to a possible symmetrical profile with back-flow regions and positive jets. As the modified Reynolds numbers increase, so do the number of solutions. The effect of increasing modified Reynolds number is to exclude, progressively, more and more of the simpler types of flow.

4.4 Solution for Small Reynolds Numbers and for Unitary Elliptic Modularity for Diverging Channel Flows with Vanishing Viscoelastic Parameter

In case of flow with small Reynolds numbers in a channel of small divergence, the elliptic function of (4.2.34) reduces to its argument,

$$\mathrm{sn}(m\theta, k) \approx m\theta \ , \tag{4.4.1}$$

and the velocity profile is given by

$$\frac{F}{F_0} = 1 - \frac{\theta^2}{\theta_b^2} \ , \tag{4.4.2}$$

which is the familiar parabolic velocity distribution for plane Poiseuille flow. From (4.2.37) we have

$$F_0 \approx \frac{6\beta m^4 k^2 \theta_b^2}{R} . \qquad (4.4.3)$$

If $k = 1$, it follows from (4.2.30) that

$$m^2 = \frac{1}{4}(2+R^*) . \qquad (4.4.4)$$

Using the following identity relation

$$\mathrm{sn}(m\theta, 1) = \tanh m\theta , \qquad (4.4.5)$$

we obtain from (4.2.37)

$$F_0 = \frac{3\beta(2+R^*)}{2R} \tan^2 h\{\theta_b(\frac{2+R^*}{4})^{1/2}\} . \qquad (4.4.6)$$

The velocity profiles are given by

$$\frac{F}{F_0} = 1 - \frac{\tanh^2\{\theta(\frac{2+R^*}{4})^{1/2}\}}{\tanh^2\{\theta_b(\frac{2+R^*}{4})^{1/2}\}} \qquad (4.4.7)$$

and are shown in Figure 4.2 for different values of β, which shows the influence of the micromorphic structure of the medium as well as the effect of geometry of the channel on the velocity field.

4.5 Relation Between the Reynolds Numbers and the Divergence of Channel

If $\alpha = 0$, then from (4.2.12) we have

$$(\frac{2R}{3\beta})^{1/2}\theta_b = \int_0^{a_1} \frac{dF}{\sqrt{(a_1-F)(F-a_2)(F-a_3)}} . \tag{4.5.1}$$

When a_1 is real and a_2, a_3 are imaginary roots of $Q_0(F) = 0$, the product of the two linear factors

$$(F-a_2)(F-a_3) = F^2 - (a_2+a_3)F + a_2a_3 . \tag{4.5.2}$$

The use of the first relation in (4.2.7) in (4.5.2) gives

$$(F-a_2)(F-a_3) = F^2 + (a_1+\frac{6\beta}{R})F + a_2a_3 . \tag{4.5.3}$$

Since $a_2a_3 > 0$, we obtain the following inequality

$$(a_1+\frac{6\beta}{R})F \leq (F-a_2)(F-a_3) . \tag{4.5.4}$$

When a_1, a_2, a_3 are real roots, $a_2a_3 > 0$ since $a_2 < 0$ and $a_3 < 0$. Therefore, the inequality (4.5.4) holds in both cases. Using this inequality in (4.5.1), we get

$$(\frac{2R}{3\beta})^{1/2}\theta_b \leq \int_0^{a_1} \frac{dF}{\sqrt{(a_1-F)(a_1+\frac{6\beta}{R})F}}$$

$$= \frac{1}{\sqrt{(a_1+\frac{6\beta}{R})}} \int_0^{a_1} \frac{dF}{\sqrt{(a_1-F)F}} . \tag{4.5.5}$$

If we put $F = x\alpha_1$, then we have

$$\int_0^{\alpha_1} \frac{dF}{\sqrt{(\alpha_1 - F)F}} = \int_0^1 \frac{dx}{\sqrt{(1-x)x}} = \pi . \tag{4.5.6}$$

This identity relation together with (4.5.5) yields

$$\left(\frac{2R}{3\beta}\right)^{1/2} \theta_b \leq \frac{\pi}{\left(\alpha_1 + \frac{6\beta}{R}\right)^{1/2}} , \tag{4.5.7}$$

or

$$\left(1 + \frac{R\alpha_1}{6\beta}\right)^{1/2} \theta_b \leq \frac{\pi}{2} , \tag{4.5.8}$$

which shows that R and θ_b are coupled and hence cannot be chosen arbitrarily in the case of divergent flows.

If we multiply (4.2.5) by F and integrate the resulting differential equation, then we obtain

$$\left(\frac{2R}{3\beta}\right)^{1/2} \int_0^{\theta_b} F d\theta = - \int_{\alpha_1}^{0} \frac{F dF}{\sqrt{(\alpha_1 - F)(F - \alpha_2)(F - \alpha_3)}} . \tag{4.5.9}$$

If the volume flow per unit time per unit radial distance is such that

$$\int_0^{\theta_b} F d\theta = \frac{1}{2} , \tag{4.5.10}$$

then we get from (4.5.9)

$$\left(\frac{R}{6\beta}\right)^{1/2} \leq \alpha_1 \int_0^{\alpha_1} \frac{dF}{\sqrt{(\alpha_1 - F)(F-\alpha_2)(F-\alpha_3)}}$$

$$= \alpha_1 \theta_b \left(\frac{2R}{3\beta}\right)^{1/2} \tag{4.5.11}$$

$$\leq \frac{\alpha_1 \pi}{\left(\alpha_1 + \frac{6\beta}{R}\right)^{1/2}} ,$$

and hence the following inequality

$$\left(\frac{R}{6\beta}\right)^{1/2} \leq \alpha_1 \theta_b \left(\frac{2R}{3\beta}\right)^{1/2} \leq \frac{\alpha_1 \pi}{\left(\alpha_1 + \frac{6\beta}{R}\right)^{1/2}} , \tag{4.5.12}$$

$$1 \leq 2\alpha_1 \theta_b \leq \frac{\alpha_1 \pi}{\left(\frac{\alpha_1 R}{6\beta} + 1\right)^{1/2}} . \tag{4.5.13}$$

Since $\alpha_1 = F(0) = F_0$ for divergent flow, we have from (4.5.13)

$$\frac{1}{2F_0} \leq \theta_b \leq \frac{\pi}{2\left(1 + \frac{F_0 R}{6\beta}\right)^{1/2}} . \tag{4.5.14}$$

Thus for given $F_0 R/\beta$, θ_b must be chosen so that this inequality holds and in particular

$$\frac{1}{2F_0} \leq \theta_b \leq \frac{\pi}{2} . \tag{4.5.15}$$

From (4.5.14), we obtain the following inequality

$$\frac{2}{\pi}\left(1+\frac{F_0 R}{6\beta}\right)^{1/2} \leq \frac{1}{\theta_b} \leq 2F_0 \; , \qquad (4.5.16)$$

and after some simplification, we get

$$R \leq \frac{6\beta}{F_0}\left(\frac{\pi^2}{4\theta_b^2} - 1\right) \leq \frac{6\beta}{F_0}\left(F_0^2\pi^2 - 1\right) \; , \qquad (4.5.17)$$

or

$$R \leq \frac{6\beta}{F_0\theta_b^2}\left(\frac{\pi^2}{4} - \theta_b^2\right) \leq \frac{6\beta}{F_0}\left(F_0^2\pi^2 - 1\right) \; . \qquad (4.5.18)$$

Thus when θ_b is given, R cannot be chosen arbitrarily and is bound by the inequality (4.5.18).

4.6 Method of Solution for Diverging Flow with Non-Vanishing Viscoelastic Parameter

When the viscoelastic parameter a is different from zero, the leading coefficient of the quartic polynomial $Q(F)$ does not vanish. Differentiating the differential equation (3.3.16), we obtain

$$\left(\beta - \frac{aR}{2}F\right)F'' - \frac{aR}{4}F'^2 = (2a-1)RF^2 - 4\beta F - \frac{a+ba}{R} \; . \qquad (4.6.1)$$

If we put $\theta = \theta_b$ in this equation, then since $F(\theta_b) = 0$ by the boundary condition, we get

$$a + ba = \frac{aR^2}{4}F'^2(\theta_b) - \beta R F''(\theta_b) \; . \qquad (4.6.2)$$

The substitution of (4.2.3) into this equation yields

$$b = \frac{R^2}{4} F'^2(\theta_b) = \frac{c}{\beta} . \tag{4.6.3}$$

Thus three constants a, b, and c are determined if $F(\theta)$ is known. In case of divergent flow, we take negative sign on the right hand side of (3.3.17)

$$d\theta = -\sqrt{3}\,\left(\frac{\beta}{R} - \frac{\alpha F}{2}\right)\frac{dF}{\sqrt{Q(F)}} , \tag{4.6.4}$$

where

$$\begin{aligned} Q(F) &= a_0F^4 + a_1F^3 + a_2F^2 + a_3F + a_4 , \\ a_0 &= \alpha(1-2\alpha) , \\ a_1 &= 2\beta(5\alpha-1)R^{-1} , \\ a_2 &= \frac{3}{4}\alpha^2F_b'^2 - 3\alpha\beta R^{-1}F_b'' - 12\beta^2R^{-2} , \\ a_3 &= -3\alpha\beta R^{-1}F_b'^2 + 6\beta^2R^{-2}F_b'' , \\ a_4 &= 3\beta^2R^{-2}F_b'^2 . \end{aligned} \tag{4.6.5}$$

We may write the quartic $Q(F)$ in the form

$$Q(F) = a_0(F-\alpha_1)(F-\alpha_2)(F-\alpha_3)(F-\alpha_4) , \tag{4.6.6}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ denote the roots of $Q(F) = 0$. These roots satisfy the following relations among themselves:

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \frac{2\beta(1-5\alpha)}{R\alpha(1-2\alpha)},$$

$$\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4 \tag{4.6.7}$$

$$= \frac{3R\alpha(R\alpha F_b'^2 - 4\beta F_b'') - 48\beta}{4R^2\alpha(1-2\alpha)},$$

$$\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4$$

$$= \frac{3\beta(R\alpha F_b'^2 - 2\beta F_b'')}{R^2\alpha(1-2\alpha)}, \tag{4.6.8}$$

$$\alpha_1\alpha_2\alpha_3\alpha_4 = \frac{3\beta^2 F_b'^2}{R^2\alpha(1-2\alpha)}.$$

The second relation of (4.6.8) shows that

$$\alpha_1\alpha_2\alpha_3\alpha_4 > 0, \quad \text{or} \quad \alpha_1\alpha_2\alpha_3\alpha_4 < 0 \tag{4.6.9}$$

according as $a_0 = \alpha(1-2\alpha) > 0$ or $a_0 < 0$.

We shall assume that a_0 is positive in our problem. Then there are two cases to consider separately.

Case I: All the four roots of $Q(F) = 0$ are reals. If α_1, α_2, α_3, α_4 are reals, we may assume without loss of generality that $\alpha_4 < \alpha_3 < \alpha_3 < \alpha_1$. Since $a_0 > 0$, the second relation of (4.6.8) implies that either all the four roots are positive or negative, or two roots are positive and two roots are negative. In case of divergent flow, however, F is positive and hence we have

$$0 \le F \le a_4 \qquad (4.6.10)$$

if all the four roots are positive, and

$$0 \le F \le a_2 \qquad (4.6.11)$$

if two roots are positive and two roots are negative.

When $0 \le F \le a_4$. Integrating (4.6.4), we obtain

$$\frac{2R}{\sqrt{3}} \int_0^{\theta} d\theta = -2\beta \int_{a_4}^{F} \frac{dF}{\sqrt{Q(F)}} + aR \int_{a_4}^{F} \frac{FdF}{\sqrt{Q(F)}} \; . \qquad (4.6.12)$$

The reduction of the first integral of (4.6.12) to a Jacobian elliptic function gives

$$\int_F^{a_4} \frac{dF}{\sqrt{Q_1(F)}} = \gamma \, sn^{-1}\left[\left\{\frac{(a_1-a_4)(a_4-F)}{(a_1-a_3)(a_3-F)}\right\}^{1/2}, k\right], \qquad (4.6.13)$$

where

$$\gamma = \frac{2}{\sqrt{(a_1-a_3)(a_2-a_4)}} ,$$

$$k^2 = \frac{(a_2-a_3)(a_1-a_4)}{(a_1-a_3)(a_2-a_4)} , \qquad (4.6.14)$$

and $Q(F) = a_0 Q_1(F)$.

The second integral of (4.6.12) is given by

$$\int_F^{a_4} \frac{F dF}{\sqrt{Q_1(F)}} = \frac{\gamma a_4}{n^2} \{(n^2 - n_0^2)\,\Pi(u, n^2) + n_0^2 u\}, \qquad (4.6.15)$$

where

$$u = \mathrm{sn}^{-1}\left[\left\{\frac{(a_1 - a_4)(a_4 - F)}{(a_1 - a_3)(a_3 - F)}\right\}^{1/2}, k\right],$$

$$n^2 = \frac{a_1 - a_4}{a_1 - a_3} > 1, \qquad (4.6.16)$$

$$n_0^2 = \frac{a_3(a_1 - a_4)}{a_4(a_1 - a_3)},$$

and the incomplete elliptic integral of the third kind is defined by

$$\Pi(u, n^2) = \int_0^u \frac{du}{1 - n^2 \mathrm{sn}^2 u}, \qquad (4.6.17)$$

which may be put in the form

$$\Pi(u, n^2) = \frac{n\{\Omega_4 - u Z(a, k)\}}{\sqrt{(n^2 - 1)(n^2 - k^2)}}, \qquad (4.6.18)$$

where

$$a = \sin^{-1}(\frac{1}{n}) ,$$

$$KZ(a, k) = KE(a, k) - EF(a, k) , \tag{4.6.19}$$

$$\Omega_4 = \frac{1}{2} \ln \frac{H[F(a, k)+u]}{H(F(a, k)-u]} ,$$

and K, E, F(a, k), E(a, k), Z(a, k) and H are complete elliptic integral of the first kind, complete elliptic integral of the second kind, the normal elliptic integral of the first kind, the normal elliptic integral of the second kind, Jacobian Zeta function and Jacobian Eta function respectively.

The substitution of (4.6.13) and (4.6.15) into (4.6.12) now yields

$$m\theta = (2\beta-\alpha R\alpha_3)u - \alpha R(\alpha_4-\alpha_3)\Pi(u, n^2), \tag{4.6.20}$$

where

$$m = \frac{2R}{\gamma} \sqrt{\frac{a_0}{3}} . \tag{4.6.21}$$

When $\alpha_4 \approx \alpha_3$, the second term of (4.6.20) may be neglected and the resulting equation is

$$\mathrm{sn}^2(\bar{m}\theta, k) = \frac{(\alpha_1-\alpha_4)(\alpha_4-F)}{(\alpha_1-\alpha_3)(\alpha_3-F)} , \tag{4.6.22}$$

where

$$\bar{m} = \frac{m}{2\beta-\alpha R\alpha_3} . \tag{4.6.23}$$

Solving (4.6.22), we get

$$F = \frac{a_3 \mathrm{sn}^2(\bar{m}\theta, k) - a_4 n^2}{\mathrm{sn}^2(\bar{m}\theta, k) - n^2} . \tag{4.6.24}$$

If we put $\theta = 0$, then we obtain

$$a_4 = F(0) = F_0 . \tag{4.6.25}$$

If we put $\theta = \theta_b$, then the boundary condition yields

$$a_4 n^2 = a_3 \mathrm{sn}^2(\bar{m}\theta_b, k) . \tag{4.6.26}$$

The substitution of (4.6.25) and (4.6.26) into (4.6.24) gives

$$\frac{F}{F_0} = \frac{n^2}{n^2 - \mathrm{sn}^2 \bar{m}\theta} \left\{ 1 - \frac{\mathrm{sn}^2 \bar{m}\theta}{\mathrm{sn}^2 \bar{m}\theta_b} \right\} . \tag{4.6.27}$$

When n is large, this equation reduces to

$$\frac{F}{F_0} = 1 - \frac{\mathrm{sn}^2 \bar{m}\theta}{\mathrm{sn}^2 \bar{m}\theta_b} , \tag{4.6.28}$$

which we may identify with (4.2.38). Then n can be regarded as a new parameter representing a.

<u>When $0 \leq F \leq a_2$</u>. Integrating (4.6.4), we obtain

$$\frac{2R}{\sqrt{3}} \int_0^\theta d\theta = -2\beta \int_{a_2}^F \frac{dF}{\sqrt{Q(F)}} + aR \int_{a_2}^F \frac{F dF}{\sqrt{Q(F)}} . \tag{4.6.29}$$

The reduction of the first integral of (4.6.29) to a Jacobian elliptic function gives

$$\int_F^{\alpha_2} \frac{dF}{\sqrt{Q_1(F)}} = \gamma\, sn^{-1}\left[\left\{\frac{(\alpha_1-\alpha_3)(\alpha_2-F)}{(\alpha_2-\alpha_3)(\alpha_1-F)}\right\}^{1/2}, k\right], \quad (4.6.30)$$

where γ and k^2 are given by (4.6.14). The second integral is given by

$$\int_F^{\alpha_2} \frac{F dF}{\sqrt{Q_1(F)}} = \frac{\alpha_2\gamma}{n^2}\{n_1^2 u+(n^2-n_0^2)\Pi(u, n^2)\}, \quad (4.6.31)$$

where

$$n^2 = \frac{\alpha_2-\alpha_3}{\alpha_1-\alpha_3} < k^2,$$

$$n_0^2 = \frac{\alpha_1(\alpha_2-\alpha_3)}{\alpha_2(\alpha_1-\alpha_3)}, \quad (4.6.32)$$

$$u = sn^{-1}\left[\left\{\frac{(\alpha_1-\alpha_3)(\alpha_2-F)}{(\alpha_2-\alpha_3)(\alpha_1-F)}\right\}^{1/2}, k\right].$$

The elliptic integral of the third kind appearing in (4.6.31) may be put in the form

$$\Pi(u, n^2) = u + \frac{n\{uZ(a, k)-\Omega_3\}}{\sqrt{(1-n^2)(k^2-n^2)}}, \quad (4.6.33)$$

where

$$a = \sin^{-1}\left(\frac{n}{k}\right),$$

$$\Omega_3 = \frac{1}{2} \ln \frac{\Theta[F(a,k)+u]}{\Theta[F(a,k)-u]}, \tag{4.6.34}$$

and Θ denotes Jacobi's Theta function. The substitution of (4.6.30) and (4.6.31) into (4.6.29) yields

$$m\theta = (2\beta - \alpha R\alpha_1)u - \alpha R(\alpha_2 - \alpha_1)\Pi(u, n^2), \tag{4.6.35}$$

where m is defined by (4.6.21).

When $\alpha_2 \approx \alpha_1$, the second term of (4.6.35) may be neglected and the resulting equation is

$$\mathrm{sn}^2(\bar{m}\theta, k) = \frac{(\alpha_1 - \alpha_3)(\alpha_2 - F)}{(\alpha_2 - \alpha_3)(\alpha_1 - F)}, \tag{4.6.36}$$

where

$$\bar{m} = \frac{m}{2\beta - \alpha R\alpha_1}. \tag{4.6.37}$$

Solving this equation for F, we have

$$F = \frac{\alpha_2 - \alpha_1 n^2 \mathrm{sn}^2 \bar{m}\theta}{1 - n^2 \mathrm{sn}^2 \bar{m}\theta}. \tag{4.6.38}$$

If we put $\theta = 0$, then $\alpha_2 = F(0) = F_0$. The boundary condition gives

$$F_0 = \alpha_1 n^2 \mathrm{sn}^2 \bar{m}\theta_b, \tag{4.6.39}$$

and the substitution of (4.6.39) into (4.6.38) yields

$$\frac{F}{F_0} = \frac{1}{1-n^2 \operatorname{sn}^2 m\bar{\theta}} \left\{ 1 - \frac{\operatorname{sn}^2 m\bar{\theta}}{\operatorname{sn}^2 m\bar{\theta}_b} \right\} . \qquad (4.6.40)$$

When $n = 0$, this equation reduces to

$$\frac{F}{F_0} = 1 - \frac{\operatorname{sn}^2 m\bar{\theta}}{\operatorname{sn}^2 m\bar{\theta}_b} , \qquad (4.6.41)$$

which may be identified with (4.2.38). Thus n may be regarded as a new parameter representing α.

Figure 4.4 shows velocity profiles for different values of n^2. When n^2 is different from zero, the magnitude of velocity increases slightly but it is noteworthy that in the region of back-flow the increase of velocity is very drastic.

<u>Case II: When only two roots are real.</u> In this case we may assume that α_1 and α_2 are the two real roots and $\alpha_2 < \alpha_1$. Two imaginary roots α_3 and α_4 are conjugate to each other. Since $\alpha_3\alpha_4 = \alpha_3\bar{\alpha}_3 > 0$ and $\alpha_1\alpha_2\alpha_3\alpha_4 > 0$, we must have $\alpha_1\alpha_2 > 0$. Hence either both α_1 and α_2 are positive or negative. When both α_1 and α_2 are positive, the flow is divergent and the values of F are such that

$$0 \leq F \leq \alpha_2 . \qquad (4.6.42)$$

	A	B	C	D
β	1.0	1.0	1.0	1.0
k^2	0.5	0.5	0.8	0.8
n^2	0.5	0	0	0.8
$R^* = \frac{RF_0}{\beta}$	1,342	1,342	5,000	5,000

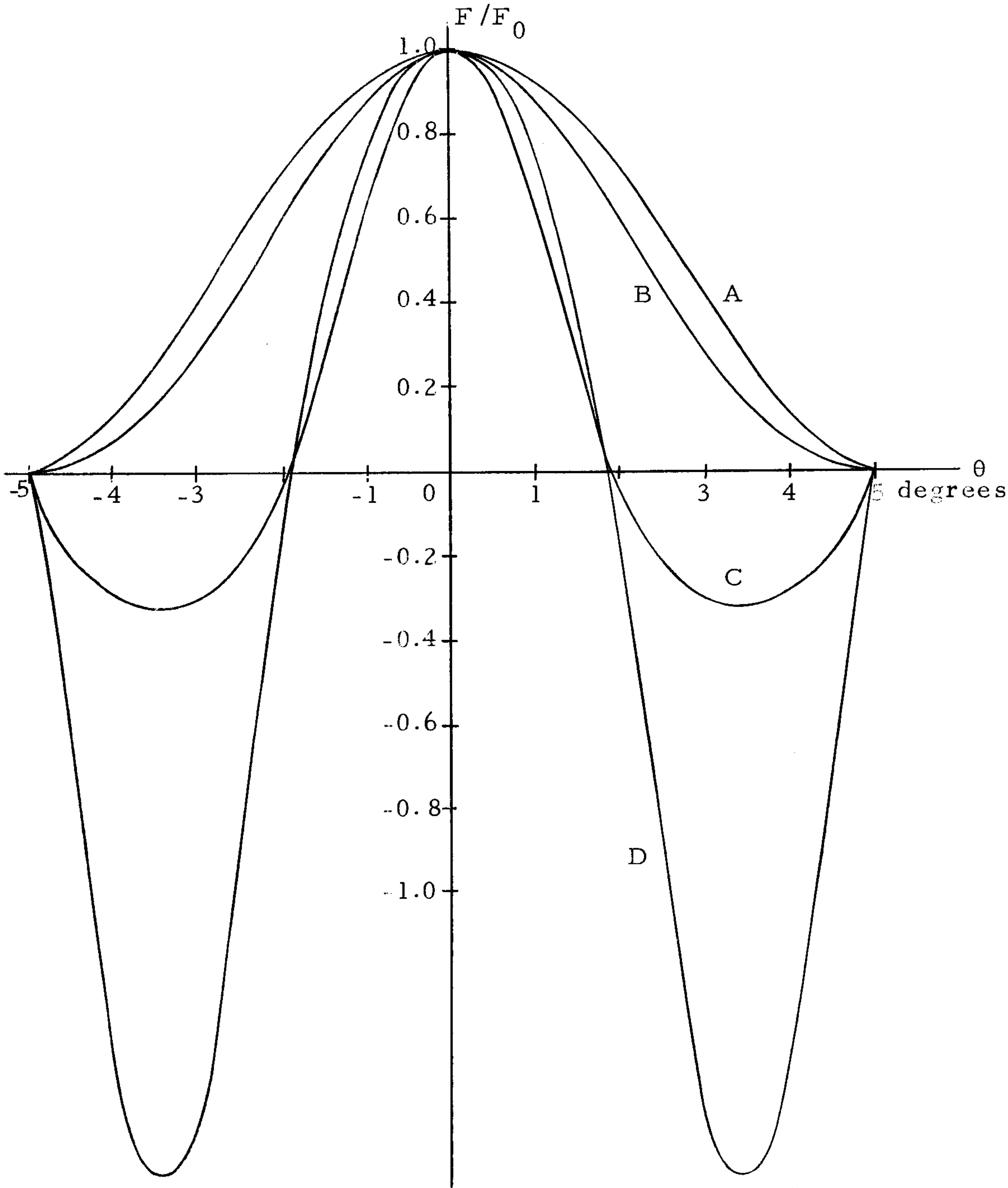

Figure 4.4. Velocity profiles for diverging flow for variables n^2 and k^2.

When both α_1 and α_2 are negative, the flow is convergent and F is such that

$$\alpha_1 \leq F \leq 0 . \qquad (4.6.43)$$

Since the method of solutions is similar for divergent and convergent flows, the solutions will be discussed in Chapter 5.

5. SOLUTIONS OF THE PROBLEM OF MICROPOLAR FLUID FLOW IN CONVERGING CHANNELS

5.1 Introduction

The second type of flow considered in this chapter is the flow in converging channels in which it is assumed that a line sink with a negative influx velocity is located at the intersection of the converging channel walls.

It is necessary to distinguish two solutions which are obtained according as the viscoelastic parameter $\alpha = 0$ or $\alpha \neq 0$. The exact solutions for the velocity field for the steady flow are obtained and the velocity profiles are determined with the aid of elliptic functions. The numerical solutions for $\alpha = 0$ are given for converging channels with the total opening of ten degrees and compared with solutions for diverging channels as well as solutions for classical viscous flow problems.

Solutions for small and large Reynolds numbers are also discussed.

5.2 Method of Solution for Converging Channel Flows for Non-Vanishing Microdeformation Parameter but for Vanishing Viscoelastic Parameter

When $\alpha = 0$, the differential equation to be solved is (4.2.4). In case of convergent flow, the differential expression takes the form

$$d\theta = +\left(\frac{3\beta}{2R}\right)^{1/2} \frac{dF}{\sqrt{Q_0(F)}} , \tag{5.2.1}$$

where $Q_0(F)$ is defined by (4.2.6). Also we have $Q_0(F) > 0$, $c > 0$, and $\beta > 0$.

When $Q_0(F) = 0$ has only one real root, the flow is divergent. When $Q_0(F) = 0$ has three real roots, we have seen that $\alpha_1 > 0$, $\alpha_2 < 0$, $\alpha_3 < 0$, and $\alpha_2 \leq F \leq \alpha_1$. Hence for convergent flow we have $\alpha_2 \leq F \leq 0$.

The integration of (5.2.1) yields

$$\left(\frac{2R}{3\beta}\right)^{1/2}\theta = \int_{\alpha_2}^{F} \frac{dF}{\sqrt{Q_0(F)}} . \tag{5.2.2}$$

The reduction of this integral to a Jacobian elliptic function yields

$$\int_{\alpha_2}^{F} \frac{dF}{\sqrt{Q_0(F)}} = \frac{2}{\sqrt{\alpha_1 - \alpha_3}} \, dn^{-1}\left\{\left(\frac{\alpha_2 - \alpha_3}{F - \alpha_3}\right)^{1/2}, k\right\}, \tag{5.2.3}$$

and an inversion gives

$$\frac{\alpha_2 - \alpha_3}{F - \alpha_3} = dn^2\left[\theta\left\{\frac{R(\alpha_1 - \alpha_3)}{6\beta}\right\}^{1/2}, k\right], \tag{5.2.4}$$

where the modulus k is given by (4.2.27). The desired solution is obtained from (5.2.4)

$$F = \alpha_3 + \frac{\alpha_2 - \alpha_3}{dn^2(m\theta, k)}, \tag{5.2.5}$$

where m is given by (4.2.27), that is,

$$m^2 = \frac{R(\alpha_1 - \alpha_3)}{6\beta}. \tag{5.2.6}$$

If we put $\theta = 0$ in the solution, then since $dn(0,k) = 1$ we get $\alpha_2 = F(0) = F_0$. The boundary condition $F(\theta_b) = 0$ gives

$$\alpha_3 = \frac{F_0}{1 - dn^2(m\theta_b, k)}. \tag{5.2.7}$$

The substitution of α_3 into (5.2.5) gives

$$\frac{F}{F_0} = \frac{1}{1 - dn^2(m\theta_b, k)} \left\{ 1 - \frac{dn^2(m\theta_b, k)}{dn^2(m\theta, k)} \right\}. \tag{5.2.8}$$

Since $1 - dn^2(m\theta_b, k) = k^2 sn^2(m\theta_b, k)$, we have from (5.2.7)

$$k^2 sn^2(m\theta_b, k) = \frac{F_0}{\alpha_3}, \tag{5.2.9}$$

and

$$sn^2(m\theta_b, k) = \frac{R^*}{2k^2\{m^2(k^2-2)-1\}}. \tag{5.2.10}$$

From (4.2.27), we have

$$m^2 = \frac{2+R^*}{2(1-2k^2)} . \qquad (5.2.11)$$

The substitution of m^2 into (5.2.10) yields

$$sn^2\left\{\theta_b\left(\frac{2+R^*}{2-4k^2}\right)^{1/2}, k\right\} = \frac{R^*(1-2k^2)}{k^2\{R^*(k^2-2)+6(k^2-1)\}} . \qquad (5.2.12)$$

If the value of k determined by solving this equation is inserted into (5.2.11) and if this value of m and the value of k are substituted into (5.2.8), the velocity profiles for prescribed θ_0, β and R may be calculated.

Figure 5.1 shows the velocity profiles for three modified Reynolds numbers with the microdeformation parameter $\beta = 1$. The velocity profiles which at small modified Reynolds numbers and for small values of θ_b are approximately parabolic become flatter in the middle of the channels and drop to zero in layers near the walls which become narrower as the modified Reynolds numbers increase. None of the profiles has any inflexion points. In Figure 5.1 the curve A is a rough approximation, nevertheless it shows the typical micromorphic behavior for convergent channel flows.

Figure 5.2 shows velocity profiles for three different values of β with fixed modified Reynolds number $R_0 = 685$. As β decreases from unity, the magnitude of the velocity for convergent flow increases while it decreases in the case of divergent flow which is a contrasting behavior. In convergent flow, the velocity profile would always be of the symmetrical pure inflow type.

	A	B	C
k^2	1.0	0.9	0.7
β	1.0	1.0	1.0
$R^* = \frac{RF_0}{\beta}$	41,600	2,995	1,196

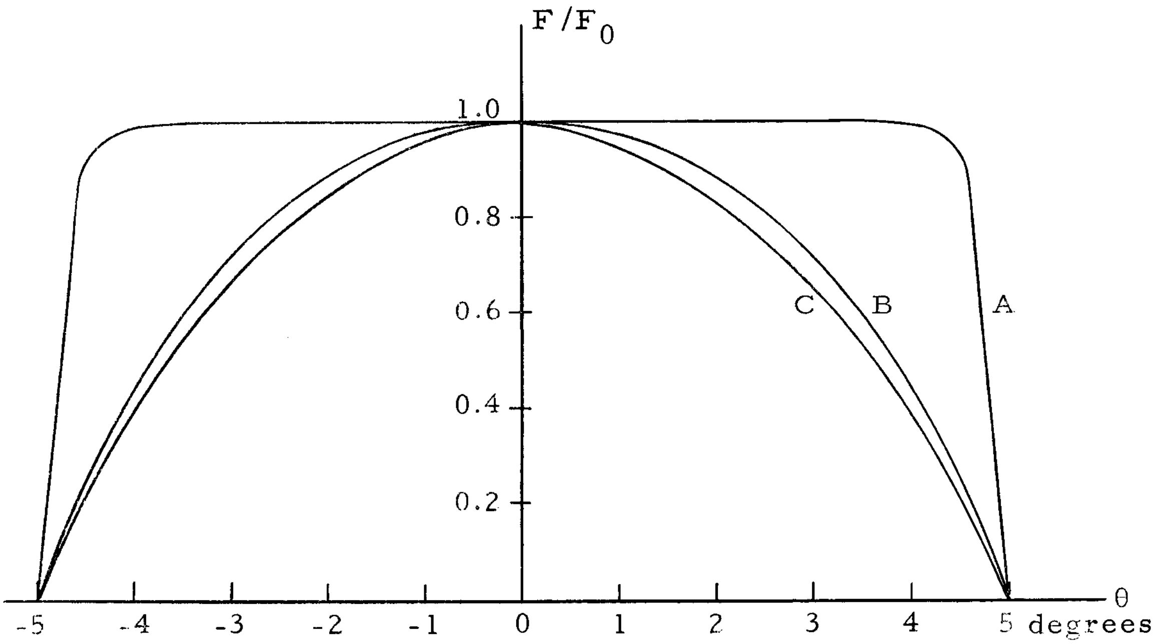

Figure 5.1. Velocity profile for converging flow for $\beta = 1$ and variable k^2.

	A	B	C
β	0.23	0.36	1.0
k^2	0.9	0.8	0.7
$R_0 = RF_0$	685	685	685

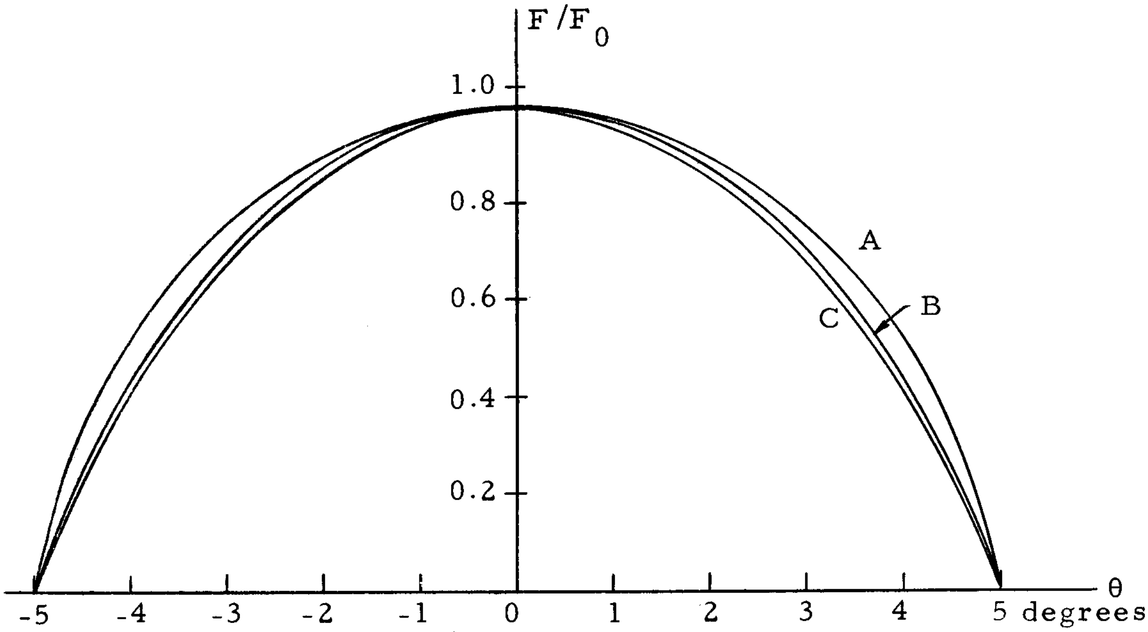

Figure 5.2. Velocity profiles for convering flow for variables β and k^2.

With increasing modified Reynolds number in convergent flow, and with small value of θ_b, the velocity profile exhibits all the well known characteristics of boundary layers near the planes, and an approximately constant velocity across the rest of the channel.

The transcendental equation (5.2.12) that determines the elliptic modularity of the converging flows has multiple roots. The largest value of k leads to a symmetrical profile, while each of the remaining values leads to a symmetrical profile with jets and back-flow regions. This infinitude of possible profiles was first noted by Rosenhead (1940). It should also be noted that asymmetrical profiles may be constructed for converging flows in exactly the same manner as in the case of diverging flows.

5.3 Solution for Large Reynolds Numbers for Converging Channel Flows with Vanishing Viscoelastic Parameter

If we put

$$f = \frac{F}{\alpha_2}, \quad w_1 = -\frac{\alpha_1}{\alpha_2}, \quad w_3 = \frac{\alpha_3}{\alpha_2}, \qquad (5.3.1)$$

in Equation (5.2.2), then we obtain

$$\left(-\frac{2R\alpha_2}{3\beta}\right)^{1/2}\theta = -\int_1^f \frac{df}{[(w_1+f)(1-f)(w_3-f)]^{1/2}}. \qquad (5.3.2)$$

In order that the Reynolds number may be large, w_3 must be close to unity, that is, $\alpha_2 \approx \alpha_3$. If this is the case, the first relation of (4.2.7) shows $w_1 \approx 2$. Now Equation (5.3.2) becomes

$$\left(-\frac{2R\alpha_2}{3\beta}\right)^{1/2}\theta = \int_f^1 \frac{df}{(1-f)\sqrt{2+f}}, \qquad (5.3.3)$$

from which we obtain

$$\left(-\frac{2R\alpha_2}{3\beta}\right)^{1/2}\theta_b = \int_0^1 \frac{df}{(1-f)\sqrt{2+f}} . \tag{5.3.4}$$

Subtraction of (5.3.3) from (5.3.4) yields

$$\left(-\frac{2R\alpha_2}{3\beta}\right)^{1/2}(\theta_b-\theta) = \int_0^f \frac{df}{(1-f)\sqrt{2+f}} . \tag{5.3.5}$$

If we put $2+f = 3\tanh^2\phi$, then (5.3.5) becomes

$$\left(-\frac{2R\alpha_2}{3\beta}\right)^{1/2}(\theta_b-\theta) = \frac{2}{\sqrt{3}}(\phi-1.146) , \tag{5.3.6}$$

and the velocity field is given by

$$\frac{F}{F_0} = 3\tanh^2\left\{\left(-\frac{RF_0}{2\beta}\right)^{1/2}(\theta_b-\theta)+1.146\right\} - 2 , \tag{5.3.7}$$

with $\tanh^{-1}(\frac{2}{3})^{1/2} = 1.146$. Since $\tanh x$ is close to unity when $x \approx 2.5$, it follows that for large R, $F \approx F_0$ except in a narrow layer near each plane, of thickness proportional to $(\frac{\beta}{RF_0})^{1/2}$.

5.4 Solution for Small Reynolds Numbers for Converging Channel Flow with Vanishing Viscoelastic Parameter

For small values of $R^* = RF_0/\beta$ and θ_b, the series development for $dn(m\theta_b, k)$ shows that

$$\mathrm{dn}(m\theta_b, k) \approx 1 - \frac{1}{2} k^2 m^2 \theta_b^2 ,$$

$$\mathrm{dn}^2(m\theta_b, k) \approx 1 - k^2 m^2 \theta_b^2 , \qquad (5.4.1)$$

$$\frac{1}{\mathrm{dn}^2(m\theta, k)} \approx 1 + k^2 m^2 \theta^2 .$$

The substitution of the approximations (5.4.1) into (5.2.7) gives

$$a_3 \approx \frac{F_0}{k^2 m^2 \theta_b^2} , \qquad (5.4.2)$$

and further substitution of (5.4.1) into (5.2.8) gives

$$\frac{F}{F_0} = 1 - \frac{\theta^2}{\theta_b^2} , \qquad (5.4.3)$$

which is the familiar parabolic velocity distribution of plane Poiseuille flow.

As in the case of divergent flow the coupling between the Reynolds number and the channel opening can be discussed. It can be shown that similar coupling behavior occurs in the case of convergent flow also.

5.5 Method of Solution for Converging Channel Flows for Non-Vanishing Viscoelastic Parameter

In the case of convergent flow, the differential equation (3.3.17) becomes

$$d\theta = \sqrt{3}\left(\frac{\beta}{R} - \frac{\alpha F}{2}\right)\frac{dF}{\sqrt{Q(F)}} . \tag{5.5.1}$$

There are two cases to consider separately.

Case I: All the four roots of $Q(F) = 0$ are reals. When all the four roots are negative, F must be chosen such that $\alpha_1 \leq F \leq 0$ and when two roots are negative, F must be chosen such that $\alpha_3 \leq F \leq 0$.

When $\alpha_1 \leq F \leq 0$. From (5.5.1) we have

$$\frac{2R}{\sqrt{3}} \int_0^{\theta} d\theta = 2\beta \int_{\alpha_1}^{F} \frac{dF}{\sqrt{Q(F)}} - \alpha R \int_{\alpha_1}^{F} \frac{FdF}{\sqrt{Q(F)}} . \tag{5.5.2}$$

The first integral on the right-hand side of Equation (5.5.2) is given by

$$\int_{\alpha_1}^{F} \frac{dF}{\sqrt{Q_1(F)}} = \gamma \, sn^{-1}\left[\left\{\frac{(\alpha_2-\alpha_4)(F-\alpha_1)}{(\alpha_1-\alpha_4)(F-\alpha_2)}\right\}^{1/2}, k\right] , \tag{5.5.3}$$

where γ and k are given by (4.6.14). The second integral of (5.5.2) is

$$\int_{\alpha_1}^{F} \frac{FdF}{\sqrt{Q_1(F)}} = \frac{\alpha_1 \gamma}{n^2}\{(n^2-n_0^2)\Pi(u, n^2)+n_0^2 u\} , \tag{5.5.4}$$

where

$$u = \mathrm{sn}^{-1}\left[\left\{\frac{(a_2-a_4)(F-a_1)}{(a_1-a_4)(F-a_2)}\right\}^{1/2}, k\right],$$

$$n^2 = \frac{a_1-a_4}{a_2-a_4} > 1, \tag{5.5.5}$$

$$n_0^2 = \frac{a_2(a_1-a_4)}{a_1(a_2-a_4)}.$$

The elliptic integral of the third kind appearing in (5.5.4) may be put in the form,

$$\Pi(u, n^2) = \frac{n\{\Omega_4 - uZ(a,k)\}}{\sqrt{(n^2-1)(n^2-k^2)}}, \tag{5.5.6}$$

where

$$a = \sin^{-1}\left(\frac{1}{n}\right),$$

$$\Omega_4 = \frac{1}{2}\ln\frac{H[F(a,k)+u]}{H[F(a,k)-u]}. \tag{5.5.7}$$

The substitution of (5.5.3) and (5.5.4) into (5.5.2) gives

$$m\theta = (2\beta - aRa_2)u - aR(a_1-a_2)\Pi(u, n^2), \tag{5.5.8}$$

where m is given by (4.6.21).

When $a_1 \approx a_2$, the second term of (5.5.8) may be neglected and the resulting equation is

$$\frac{(\alpha_2-\alpha_4)(F-\alpha_1)}{(\alpha_1-\alpha_4)(F-\alpha_2)} = sn^2(\bar{m}\theta, k) , \tag{5.5.9}$$

where

$$\bar{m} = \frac{m}{2\beta-\alpha R\alpha_2} . \tag{5.5.10}$$

Solving (5.5.9) for F, we obtain

$$F = \frac{\alpha_1-\alpha_2 n^2 k^{-2}\{1-dn^2(\bar{m}\theta, k)\}}{1-n^2 k^{-2}\{1-dn^2(\bar{m}\theta, k)\}} . \tag{5.5.11}$$

If we put $\theta = 0$, then $\alpha_1 = F(0) = F_0$. The boundary condition yields

$$\frac{\alpha_2 n^2}{k^2} = \frac{F_0}{1-dn^2(\bar{m}\theta_b, k)} . \tag{5.5.12}$$

The substitution of (5.5.12) into (5.5.11) gives

$$\frac{F}{F_0} = \frac{1}{1-dn^2\bar{m}\theta_b}\left\{\frac{dn^2\bar{m}\theta-dn^2\bar{m}\theta_b}{1-n^2k^{-2}+n^2k^{-2}dn^2\bar{m}\theta}\right\} . \tag{5.5.13}$$

If $n^2 = k^2$, this equation reduces to

$$\frac{F}{F_0} = \frac{1}{1-dn^2\bar{m}\theta_b}\left\{1-\frac{dn^2\bar{m}\theta_b}{dn^2\bar{m}\theta}\right\} , \tag{5.5.14}$$

which is similar to (5.2.8). Thus we may identify (5.5.14) with

(5.2.8) and regard n^2 as a new parameter representing α.

<u>When $\alpha_3 \leq F \leq 0$.</u> The integration of (5.5.1) gives

$$\frac{2R}{\sqrt{3}} \int_0^\theta d\theta = 2\beta \int_{\alpha_3}^F \frac{dF}{\sqrt{Q(F)}} - \alpha R \int_{\alpha_3}^F \frac{FdF}{\sqrt{Q(F)}} . \qquad (5.5.15)$$

The first integral of (5.5.15) is given by

$$\int_{\alpha_3}^F \frac{dF}{\sqrt{Q_1(F)}} = \gamma\, sn^{-1}\left[\left\{\frac{(\alpha_2-\alpha_4)(F-\alpha_3)}{(\alpha_2-\alpha_3)(F-\alpha_4)}\right\}^{1/2}, k\right], \qquad (5.5.16)$$

where γ and k are given by (4.6.14).

The second integral of (5.5.15) is given by

$$\int_{\alpha_3}^F \frac{FdF}{\sqrt{Q_1(F)}} = \frac{\alpha_3\gamma}{n^2}\{(n^2-n_0^2)\Pi(u, n^2)+n_0^2 u\} , \qquad (5.5.17)$$

where

$$u = sn^{-1}\left[\left\{\frac{(\alpha_2-\alpha_4)(F-\alpha_3)}{(\alpha_2-\alpha_3)(F-\alpha_4)}\right\}^{1/2}, k\right],$$

$$n^2 = \frac{\alpha_2-\alpha_3}{\alpha_2-\alpha_4} < k^2 , \qquad (5.5.18)$$

$$n_0^2 = \frac{\alpha_4(\alpha_2-\alpha_3)}{\alpha_3(\alpha_2-\alpha_4)} .$$

The elliptic integral of the third kind occurring in (5.5.17) may be written in the form

$$\Pi(u, n^2) = u + \frac{n\{uZ(a, k) - \Omega_3\}}{\{(1-n^2)(k^2-n^2)\}^{1/2}} , \qquad (5.5.19)$$

where

$$a = \sin^{-1}(\frac{n}{k}) ,$$

$$\Omega_3 = \frac{1}{2} \ln \frac{\Theta\{F(a, k)+u\}}{\Theta\{F(a, k)-u\}} .$$

The substitution of (5.5.16) and (5.5.17) into (5.5.15) yields

$$m\theta = (2\beta - \alpha R\alpha_4)u - \alpha R(\alpha_3 - \alpha_4)\Pi(u, n^2) . \qquad (5.5.20)$$

When $\alpha_3 \approx \alpha_4$, the second term may be neglected. Subsequently we have

$$m\theta = (2\beta - \alpha R\alpha_4)\mathrm{sn}^{-1}\left\{\frac{(\alpha_2-\alpha_4)(F-\alpha_3)}{(\alpha_2-\alpha_3)(F-\alpha_4)}\right\}^{1/2} , \qquad (5.5.21)$$

from which we may find

$$F = \frac{(\alpha_4 n^2 - \alpha_3 k^2) - \alpha_4 n^2 \mathrm{dn}^2 \bar{m}\theta}{(n^2-k^2) - n^2 \mathrm{dn}^2 \bar{m}\theta} , \qquad (5.5.22)$$

where

$$\bar{m} = \frac{m}{2\beta - \alpha R\alpha_4} .$$

If we put $\theta = \theta_b$, then we obtain

$$\alpha_4 n^2 - \alpha_3 k^2 = \alpha_4 n^2 dn^2 \bar{m}\theta_b \ . \tag{5.5.23}$$

The substitution of this expression into (5.5.22) gives

$$F = \frac{\alpha_4 n^2 dn^2 \bar{m}\theta_b}{n^2 - k^2 - n^2 dn^2 \bar{m}\theta} \left\{ 1 - \frac{dn^2 \bar{m}\theta}{dn^2 \bar{m}\theta_b} \right\} . \tag{5.5.24}$$

If we put $\theta = 0$ in this equation, then we get

$$F_0 = F(0) = \frac{\alpha_4 n^2}{k^2} \{1 - dn^2 \bar{m}\theta_b\} \ . \tag{5.5.25}$$

The combination of (5.5.24) and (5.5.25) yields

$$\frac{F}{F_0} = \frac{1}{1 - dn^2 \bar{m}\theta_b} \left\{ \frac{dn^2 \bar{m}\theta - dn^2 \bar{m}\theta_b}{1 - n^2 k^{-2}(1 - dn^2 \bar{m}\theta)} \right\} . \tag{5.5.26}$$

If $n^2 = k^2$, this equation reduces to the form

$$\frac{F}{F_0} = \frac{1}{1 - dn^2 \bar{m}\theta_b} \left\{ 1 - \frac{dn^2 \bar{m}\theta_b}{dn^2 \bar{m}\theta} \right\} , \tag{5.5.27}$$

which is similar to Equation (5.2.8). Thus it may be possible to identify (5.5.27) with the convergent flow for $\alpha = 0$, and to regard n^2 as a new parameter representing α.

Case II: When only two roots are real. When two roots are real and the other roots are imaginary, two real roots α_1 and α_2

are negative for convergent flow and hence F must be chosen such that

$$\alpha_1 \leq F \leq 0 . \qquad (5.5.28)$$

The integration of (5.5.1) gives

$$\frac{2R}{\sqrt{3}} \int_0^\theta d\theta = 2\beta \int_{\alpha_1}^F \frac{dF}{\sqrt{Q(F)}} - \alpha R \int_{\alpha_1}^F \frac{FdF}{\sqrt{Q(F)}} . \qquad (5.5.29)$$

The reduction of the first integral of (5.5.29) to a Jacobian elliptic function is given by

$$\int_{\alpha_1}^F \frac{dF}{\sqrt{Q_1(F)}} = \frac{1}{\sqrt{AB}} \operatorname{cn}^{-1}\left\{ \frac{(A-B)F+\alpha_1 B-\alpha_2 A}{(A+B)F-\alpha_1 B-\alpha_2 A}, k \right\}, \qquad (5.5.30)$$

where

$$A^2 = (\alpha_1-\alpha_3)(\alpha_1-\bar{\alpha}_3) ,$$

$$B^2 = (\alpha_2-\alpha_3)(\alpha_2-\bar{\alpha}_3) , \qquad (5.5.31)$$

$$k^2 = \frac{(A+B)^2-(\alpha_1-\alpha_2)^2}{4AB} .$$

The second integral of (5.5.29) is given by

$$\int_{\alpha_1}^F \frac{FdF}{\sqrt{Q_1(F)}} = \frac{\alpha_1 B-\alpha_2 A}{A+B} - \frac{n_0 u}{\sqrt{AB}} - \frac{(n-n_0)}{\sqrt{AB}\,(1-n_2^2)} \{\Pi(u, n_2^2)-nf\} , \qquad (5.5.32)$$

where

$$u = cn^{-1}\left[\frac{(A-B)F+\alpha_1 B-\alpha_2 A}{(A+B)F-\alpha_1 B-\alpha_2 A}, k\right],$$

$$n = \frac{B+A}{B-A},$$

$$n_0 = \frac{\alpha_1 B+\alpha_2 A}{\alpha_1 B-\alpha_2 A}, \tag{5.5.33}$$

$$n_2 = \frac{n^2}{n^2-1},$$

and the function f is given by

$$f = \frac{snu}{dnu}, \quad \text{if} \quad n_2 = k^2, \tag{5.5.34}$$

$$f = \left(\frac{1-n^2}{k_1^2}\right)^{1/2} \tan^{-1}\left\{\left(\frac{k_1^2}{1-n^2}\right)^{1/2} \frac{snu}{dnu}\right\}, \tag{5.5.35}$$

if $n_2 < k^2$,

$$f = \frac{1}{2}\left(\frac{n^2-1}{k_1^2}\right)^{1/2} \ln\left\{\frac{k_1 dnu+\sqrt{n^2-1}\, snu}{k_1 dnu-\sqrt{n^2-1}\, snu}\right\}, \tag{5.5.36}$$

if $n_2 > k^2$, and k_1 denotes $k_1 = \sqrt{k^2+k'^2 n^2}$.

The substitution of (5.5.30) and (5.5.32) into (5.5.29) yields

$$\left(\frac{a_0}{3}\right)^{1/2} 2R\theta = \frac{\alpha R(\alpha_2 A-\alpha_1 B)}{A+B} + \frac{(2\beta+\alpha R n_0)u}{\sqrt{AB}}$$

$$+ \frac{\alpha R(n-n_0)}{\sqrt{AB}\,(1-n^2)} \{\Pi(u, n^2)-nf\}. \tag{5.5.37}$$

If $\alpha_1 \approx \alpha_2$, then $n \approx n_0$ and the resulting equation is after neglecting the first and third terms,

$$\left(\frac{a_0}{3}\right)^{1/2} 2R\theta = \frac{(2\beta+\alpha R n_0)u}{\sqrt{AB}} . \qquad (5.5.38)$$

Solving this equation for F, we get

$$F = \frac{(\alpha_2 A-\alpha_1 B)-(\alpha_1 B+\alpha_2 A)\mathrm{cn}(2m\theta)}{(A-B)-(A+B)\mathrm{cn}(2m\theta)} , \qquad (5.5.39)$$

where

$$m = \frac{R}{2\beta+\alpha R n_0} \left(\frac{a_0}{3}\right)^{1/2} . \qquad (5.5.40)$$

If we put $\theta = 0$, then $\alpha_1 = F(0) = F_0$. The boundary condition gives

$$\alpha_2 = \frac{\alpha_1 B\{1+\mathrm{cn}(2m\theta_b)\}}{A\{1-\mathrm{cn}(2m\theta_b)\}} . \qquad (5.5.41)$$

The substitution of (5.5.41) into (5.5.39) yields

$$\frac{F}{F_0} = \frac{1-\left\{\frac{1+\mathrm{cn}(2m\theta_b)}{1-\mathrm{cn}(2m\theta_b)}\right\}\left\{\frac{1-\mathrm{cn}(2m\theta)}{1+\mathrm{cn}(2m\theta)}\right\}}{1-\frac{A}{B}\left\{\frac{1-\mathrm{cn}(2m\theta)}{1+\mathrm{cn}(2m\theta)}\right\}} . \qquad (5.5.42)$$

Since

$$\frac{1-\mathrm{cn}2x}{1+\mathrm{cn}2x} = \mathrm{tn}^2 x\, \mathrm{dn}^2 x , \qquad (5.5.43)$$

the Equation (5.5.42) may be written in the form

$$\frac{F}{F_0} = \frac{1 - \dfrac{tn^2 m\theta dn^2 m\theta}{tn^2 m\theta_b dn^2 m\theta_b}}{1 - \dfrac{A}{B} tn^2 m\theta dn^2 m\theta}, \tag{5.5.44}$$

where $tnm\theta = (snm\theta)/(cnm\theta)$. In the case $\alpha_1 \approx \alpha_2$, $A \approx B$ and (5.5.44) becomes

$$\frac{F}{F_0} = \frac{1 - \dfrac{tn^2 m\theta dn^2 m\theta}{tn^2 m\theta_b dn^2 m\theta_b}}{1 - tn^2 m\theta dn^2 m\theta}. \tag{5.5.45}$$

For the purpose of illustration of the micromorphic behavior in the convergent flow, we have shown the velocity profiles for case I where the function $Q(F)$ has all its roots real. The curves are found to depend on both the parameters k^2 and n^2.

	A	B	C	D
k^2	0.9	0.9	0.9	0.9
n^2	0.9	0.7	0.5	0.1
$\bar{m}$	15.8	15.8	15.8	15.8

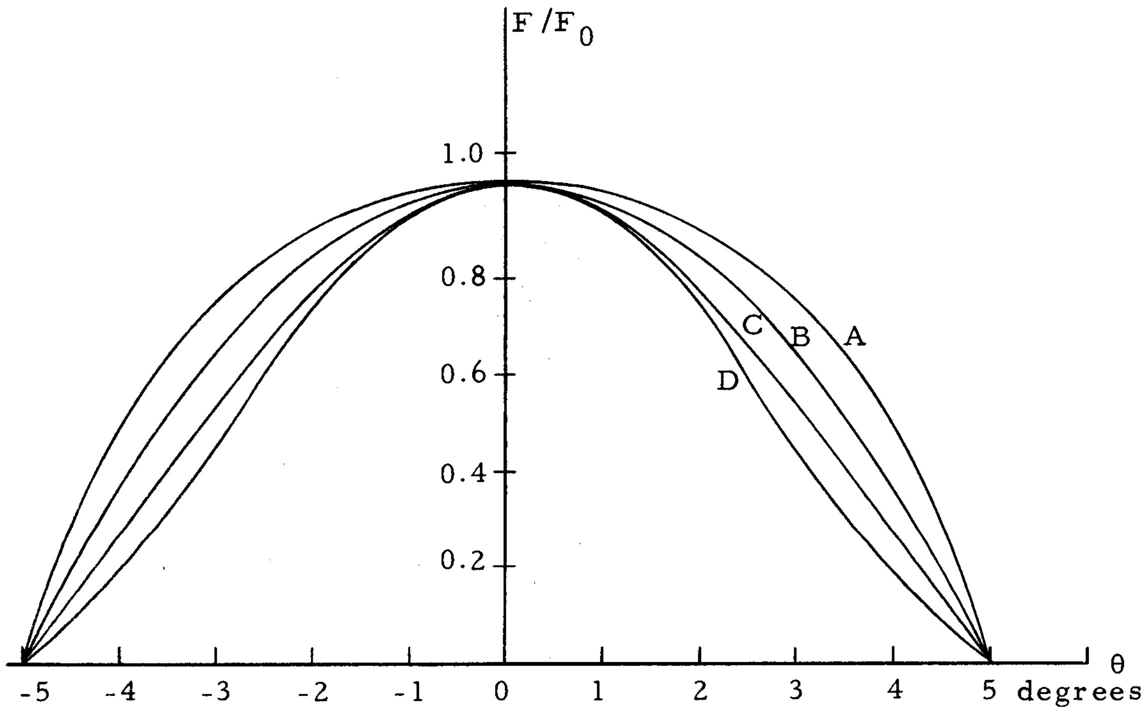

Figure 5.3. Velocity profiles for convergent flow for variables k^2 and n^2.

6. SUMMARY AND CONCLUSIONS

6.1 General Discussion of the Results

In classical continuum mechanics, the molecular or microscopic nature of a material medium is disregarded and the macroscopic properties of the material are determined without taking into account its internal structure. Thus according to the classical theory of continuum mechanics, a physical body is the aggregate of structureless particles to which kinematical or dynamical constitutive variables can be assigned at any time. These variables are supposed to be continuous functions of the spatial coordinates of the particle and time.

The classical theory, however, fails to explain the behavior of oriented media whose properties cannot be explained without introducing material directors assigned at each particle. The classical theory also fails to describe the phenomenon of surface tension, internal spin, couple stress, the behavior of liquid crystals and many other interesting wellknown phenomena exhibited in all experiments.

The fundamental difficulty with the classical theory is owing to the dual nature of materials, namely hydrodynamic and configurational. The configurational aspect requires further investigations. It is, therefore, necessary to introduce the concept of structured particles having mass as the absolute scalar measure, and energy, linear momentum, internal angular momentum, and moment of inertia as measures with densities relative to mass.

The molecular theory of liquids attempts to describe the properties of liquids in terms of the radial distribution function of nearest neighbors around a given reference molecule, and in terms of the distribution of molecular velocities. The number of configurations which a molecule may assume is very large. Consideration of each of them individually would be a task of scope beyond all possibility of realization. In a liquid, the molecular arrangement is so mobile that it can be thought of only in statistical terms. This brings the necessity of adopting a statistical point of view. An appropriate average over the total population of configuration for an assembly of molecules has to be deduced and the need for a concise measure of configuration becomes apparent.

Such a measure has been accomplished by Eringen and Suhubi (1964) who have proposed and developed the simplest continuum theory taking into account local effects without using the molecular point of view and the method of statistical mechanics. Eringen (1970) further generalized his theory and introduced the master balance laws of micromorphic materials.

The theory of micromorphic materials is not to question the real existence of atoms or subatomic particles but to describe phenomena where difference among such particles, as well as the details of their behavior, are unimportant. A micromorphic theory requires more information about a given material, but a full structural specification is unnecessary. A full specification of molecular structure will obviously carry with it extreme mathematical complexity, irrelevant to mechanical questions regarding finite body.

In the present investigation we have developed a new constitutive theory for micromorphic materials by introducing in the classical constitutive equations generalized measures of both micro- and macro-deformation rates, and it is expected that this theory would be better able to describe the dynamic behavior of oriented media than the classical ones. The resulting constitutive theory is then used to investigate the steady radial motion of an incompressible micropolar fluid between two non-parallel planes. It is found that the entire analysis of the micromorphic material behavior can ultimately be expressed in terms of two parameters, one depicting the microdeformation and the other the viscoelastic nature. For vanishing viscoelastic parameter but for non-vanishing microdeformation parameter, we find that exact solutions are possible in terms of elliptic functions for both diverging and converging flows. For diverging flow, the velocity distribution, which at small modified Reynolds numbers and for small opening of the channels is approximately parabolic, alters drastically with increasing modified

Reynolds numbers, the flux becoming more and more concentrated in the middle of the channel, until a critical modified Reynolds number is reached. If the modified Reynolds number is further increased, a purely divergent flow becomes impossible and a back-flow is found to appear near the walls. If the modified Reynolds number is still further increased, the number of possible regions of back-flow and the number of possible solutions also increase. It is possible to identify the modified Reynolds number with the Reynolds number defined in terms of the maximum velocity. If this identification is made, the classical solutions appear as a special case when $\alpha = 0$ and $\beta = 1$.

For convergent flow, when $\alpha = 0$, the velocity distribution, which at small modified Reynolds number and for small opening of the channels is approximately parabolic, becomes flatter in the middle of the channel and drops to zero in layers near the walls with the layers becoming narrower as the modified Reynolds number increases. As parameter β decreases from unity, the magnitude of velocity increases in convergent flow while it decreases in case of divergent flow.

When the viscoelastic parameter does not vanish, it is found that exact solutions are obtained again in terms of elliptic functions and elliptic integrals which involve a quartic polynomial. The figures drawn clearly show the velocity profiles both for convergent and divergent flows in the micromorphic case as well as the classical Newtonian case. The dependence of the velocity profiles on the parameters α and β for different ranges of Reynolds numbers and for different channel openings is evident from the figures. The contrasting behavior of the oriented media from that of the classical viscous fluid thus becomes clear from the figures.

6.2. Guidelines for Further Research

In the present investigation, we have considered the micromorphic theory of grade one only. The balance laws and entropy inequality of the micromorphic theory of grade n can similarly be obtained by taking n energy equations, n entropy inequalities, $n+1$ momentum balance equations, and $n+2$ mass balance equations. These balance laws must be supplemented with a set of constitutive equations which are yet to be developed.

In the problem of the flows in converging and diverging channels, many important and interesting conjectures can be made and many interesting questions can be raised on the bearing of the existence of jets and back-flow regions on the problem of hydrodynamic stability and on the possibility of studying turbulence by examining in greater detail the multiplicities of possible solutions of the equations of motion. It is well known that divergent flow is in practice highly unstable, and it is therefore likely that the flow might in some cases be found to be unstable for small disturbances. Exact solutions for micropolar, incompressible, steady, radial flow in a straight channel, are good approximations to the same problem in certain symmetric, divergent or convergent channels with small wall-curvature. The present research can be extended to non-radial flow in a curved channel.

Also this work can further be extended to determine the thermal distributions for the steady laminar flow of a micropolar fluid between non-parallel plane walls held at a constant temperature which is of considerable interest both theoretically and practically.

Recent experiment with fluids containing minute amount of additives indicate that the skin friction near a rigid body is considerably lower than that in the case of the same fluids without additives. The classical theory is incapable of predicting this new physical phenomenon. It may be possible that this kind of phenomenon can be explained properly with the theory of micromorphic mechanics.

The presence of electromagnetic fields influencing the micromorphic behavior adds a further dimension to the study of micromorphic mechanics. Thus our investigation can be extended to the case of electrically conducting micromorphic fluid flows in the presence of magnetic field.

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APPENDIX

A. 1. The Three Canonical Forms of the Elliptic Integrals

The normal elliptic integrals of the first, second and third kind are defined by

$$F(\phi, k) = \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2 \sin^2\theta}} = \int_0^{y} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} , \qquad (A. 1)$$

$$E(\phi, k) = \int_0^{\phi} \sqrt{1-k^2\sin^2\theta}\; d\theta = \int_0^{y} \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\, dt , \qquad (A. 2)$$

$$\Pi(\phi, n^2, k) = \int_0^{\phi} \frac{d\theta}{(1-n^2\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}$$

$$= \int_0^{y} \frac{}{(1-n^2t^2)\sqrt{(1-t^2)1-k^2t^2)}} , \qquad (A. 3)$$

where $y = \sin\phi$.

The complete elliptic integrals are defined by

$$K = K(k) = F(\tfrac{\pi}{2}, k) ,$$

$$E = E(k) = E(\tfrac{\pi}{2}, k) , \qquad (A. 4)$$

$$\Pi(n^2, k) = \Pi(\tfrac{\pi}{2}, n^2, k).$$

A. 2. Jacobian Elliptic Functions

The inverse function of (A. 1) may be defined by

$$y = \sin\phi = \mathrm{sn}(u, k), \qquad \phi = \mathrm{am}(u, k) \qquad \text{(a. 5)}$$

where $u = F(\phi, k)$. Two other functions are then defined by

$$\mathrm{cn}(u, k) = \sqrt{1-y^2} = \cos\phi$$

$$\mathrm{dn}(u, k) = \sqrt{1-k^2y^2} = \sqrt{1-k^2\sin^2\phi} \quad . \qquad \text{(A. 6)}$$