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Title: ZERO-REST-MASS FIELDS FROM THE FORMULA OF A TUBE IN COMPLEX PROJECTIVE THREE-SPACE

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The formula of the volume of the tube in complex projective three-dimensional space is studied.

This formula is a polynomial in the radius of the tube with coefficients (Chern forms) which are topological invariants. We are motivated by these topological invariants to look for a physical significance of the formula of the volume of the tube using twistor theory, where a twistor space can be viewed as a complex projective three-space.

It is shown here that the formulae of tubes around the complex projective one-dimensional space and around the complex projective two-dimensional space in twistor space represent zero-rest-mass fields of spin $\frac{3}{2}$ and spin 2 respectively. Thus they give rise to coupled neutrino (spin $\frac{1}{2}$) fields and a weak gravitational Einstein's field respectively. This is demonstrated by using Penrose correspondence between the complex projective three-space and Minkowski space. Penrose transform is utilized to transfer cohomology data

from the complex projective three-space to the Minkowski space.

ZERO-REST-MASS FIELDS FROM THE FORMULA OF A TUBE IN COMPLEX PROJECTIVE THREE-SPACE

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ZERO-REST-MASS FIELDS FROM THE FORMULA OF

A TUBE IN COMPLEX PROJECTIVE THREE-SPACE

Introduction

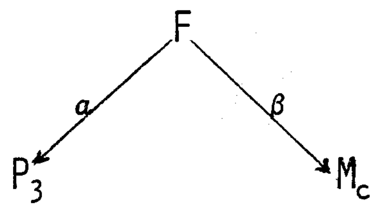
In this paper we consider the formula of the tube around submanifolds in complex projective three space. F. Flaherty has shown [5] that the volume of the tube, in this case, is a polynomial in the radius of the tube with coefficients which are metric invariants. The formula is found by transferring Weyl's formulas of tubes from embedded Riemannian submanifolds in a Euclidean space to complex projective space via the natural projection. A. Gray [7] has also shown that the coefficients in the formula of the volume of the tube around a submanifold P in complex projective space can be completely expressible in terms of the Käehler form and the Chern forms of P, thus revealing the fact that the coefficients are topological invariants.

In this paper we are motivated by this topological invariance character of the coefficients to look for a physical significance of the formula of the tube around submanifolds of the complex projective three space. This complex projective three space proved to be a visualization of twistor space discovered by R. Penrose [13], where physical fields can appropriately be studied. What is done in this paper is to interpret the formula of the volume of the tube around the complex projective one-dimensional projective space and around the complex projective two-dimensional space in twistor space as zero-rest-mass fields of spin $\frac{3}{2}$ and 2. These zero-rest-mass fields correspond to coupled neutrino (spin-half) fields and a weak Einstein's gravitational field respectively. In general, zero-rest-mass fields are differential equations in (spinors) geometrical objects generalizing tensors. The proof we gave to show the above-mentioned interpretation depends heavily on twistor theory, which was first introduced by R. Penrose [13]. The major role it has played has been in setting up a general correspondence which translates certain important physical field equations in space-time into holomorphic structures on a related complex manifold known as twistor space (see Penrose [18], R. O. Wells [24], and N. Woodhouse [27].

This correspondence was first given by Penrose in terms of a contour integral expression which yields general analytic solutions of the field equations, where a holomorphic function--defined on a suitable domain--is inserted into the integrand. The reader may also refer to Penrose-MacCallum [19]. Penrose later realized [17] that these holomorphic functions should be interpreted as Čech cocycles representing elements of cohomology groups.

The geometry of Penrose correspondence was later used by Woodhouse [27] to transfer cohomology on subsets of twistor space ($P_3(\mathbb{C})$) to spinor fields on subsets of the Minkowski space. This process describes zero-rest-mass fields in terms of cohomology classes on $P_3(\mathbb{C})$. The detail of this process can be found in a recent paper written by Eastwood, R. Penrose and R. O. Wells [4].

The following is a brief outline of this paper. In Chapter 1 the basic material from complex manifold theory and Hermitian geometry, used intensively later in explaining Penrose correspondence and deriving the formula of the tube in complex projective space due to F. Flaherty, is discussed. In particular, the Käehler metric and the Study-Fubini metric, Grasmannian manifolds, and some basic formulae of Käehlerian geometry are introduced. Since zero-rest-mass fields will be described in terms of differential equations in spinors, Chapter 2 is devoted to the properties of these spinors and the relationship they bear to Minkowski space. In fact, it is shown that there is a one-to-one correspondence between the two-components spinors and the points of Minkowski space. How one can view spinors as a vector bundle is shown in Chapter 3. This vector bundle has been constructed globally from transition functions. The reader may consult P. Gilky [6] for further details of this construction. In Chapter 4, twistors are introduced and are described geometrically (Penrose [13]) as a null vector in Minkowski space. Two alternative pictures of visualizing twistors, one of which is the complex projective three space picture, are given. In Chapter 5, complex manifold techniques are used to analyze Penrose correspondence between the complex projective three space $P_3(\mathbb{C})$ and Minkowski space $\mathbb{M}$. This correspondence is given via a double fibration of the following type (see Wells [23]).



The geometry of this correspondence is then used to transform data on $\mathbb{P}_3(\mathbb{C})$ to data on $\mathbb{M}$. This is achieved by pulling back cohomology classes from open subsets of $\mathbb{P}_3$ to $\mathbb{F}$ by the mapping μ and then integrating this pullback class over the fibers appropriately, obtaining "vector valued" functions on open subsets of $\mathbb{M}$ which will satisfy zero-rest-mass field equations. Chapter 6 is devoted to the derivation of the formula of the volume of the tube around submanifolds of the complex projective space using the Study-Fubini metric. This chapter relies heavily on the Käehlerian geometry described in Chapter 1.

In Chapter 7, the goal stated at the beginning, i.e., that the formulae of the tube around $\mathbb{P}_1(\mathbb{C})$ and $\mathbb{P}_2(\mathbb{C})$ in twistor space represent zero-rest-mass fields of spin $\frac{3}{2}$ and 2, respectively, is attained. First, the Chern forms, the coefficients in the formulae, are viewed as twistor functions which are the first cohomology classes with coefficients in the hyperplane section bundle. The proof is an induced cohomology argument. Indeed, we pull back forms using the map α and integrate over the fibers of the projection β [look at the figure]. After that, a covariant derivative operator is applied to the integral and proves that the result is zero. As a result, it is shown that the tube formula, in both cases, gives rise to zero-rest-mass fields.

Chapter I. Complex Manifolds and Hermitian Geometry

In this chapter we give the material from complex manifold theory that is needed for formulating the tube formula in complex projective space. This material is also needed for reformulating twistor theory in complex manifold terminology.

We establish the complex structure on a differentiable manifold which is used to define a Hermitian metric and a Käehler metric on the complex manifold. Of particular interest later is the Study-Fubini metric, which is given in section (v). Basic formulae of Käehlerian geometry are derived at the end of the chapter. These will be used in Chapter 6.

Since we will be discussing projective twistor space, where the projective space will be the home of twistors, complex projective space is introduced as a special complex Grassman manifold. The Grassman manifolds are, therefore, defined; and they themselves will be needed later when we eventually discuss the geometry of Penrose correspondance. For details of this chapter, the reader may refer to [11] and [12].

(i) Definition (1):

A complex structure on a real vector space V is a linear endomorphism J of V such that $J^2 = -1$ (identity).

Now a real vector space V with a complex structure J can be turned into a complex vector space by defining multiplication by complex numbers as follows:

$$(a + ib) = aX + bJX \qquad (1)$$

$$X \in V \quad a_1 b \in \mathbb{R}$$

Clearly, the real dimension m of V must be even, and $\frac{m}{2}$ is the complex dimension.

Conversely, given a complex vector space V of complex dimension n, let J be the linear endomorphism of V defined by

$$JX = iX \quad \text{for } X \in V \quad ,$$

and if we consider V as a real vector space of real dimension 2n, then J is a complex structure of V.

Proposition (1): Let J be a complex structure on a 2n-dimensional real vector space V. Then there exist elements $X_1, \ldots, X_n$ of such that $\{X_1, \ldots X_n, JX_1, \ldots JX_n\}$ is a basis for V.

Proof: Turn V into an n-dimensional complex vector space by (1). Let $X_1, \ldots X_n$ be a basis for V as a complex vector space. It is easy to see that $\{X_1, \ldots X_n, JX_1, \ldots JX_n\}$ is a basis for V as a real vector space.

Example: Let $\mathbb{C}^n$ be the complex vector space of n-tiples $Z = (z^1, \ldots z^n)$. If we set $z^k = x^k + iy^k$, $x^k, y^k \in \mathbb{R}$, $k = 1, \ldots n$, then $\mathbb{C}^n$ can be identified with the real vector space $\mathbb{R}^{2n}$ of 2n-tibles of real numbers $(x^1, x^2, \ldots x^n, y^1, \ldots y^n)$.

$$(z^1, \ldots z^n) \longrightarrow (x^1, \ldots x^n, y^1, \ldots y^n)$$

The complex structure of $\mathbb{R}^{2n}$ induced from that of $\mathbb{C}^n$ maps $(x^1, \ldots x^n, y^1, \ldots y^n)$ into $(y^1, \ldots y^n, -x^1, \ldots -x^n)$. This is called the Canonical complex structure of $\mathbb{R}^{2n}$. In terms of the natural basis for $\mathbb{R}^{2n}$, it is given by the matrix

$$J_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \quad I_n \equiv \text{identity matrix.}$$

Let V^* be the dual of the real vector space V. Then the complex structure J on V induces a complex structure on V^*, denoted also by J and defined as follows:

$$\langle Jx, x^* \rangle = \langle x, Jx^* \rangle$$
$$x \in V, \; x^* \in V^*.$$

Now, if V^c is the complexification of V, i.e., $V^c = V \otimes_{\mathbb{R}} \mathbb{C}$, then V is a real subspace of V^c in a natural manner. More generally, the tensor space $T^r_s(V)$ of type (r, s) over V can be considered as a real subspace of the tensor space $T^r_s(V^c)$ in a natural manner. The complex conjugation in V^c is the real linera endomorphism defined by

$$Z = X + iY \longrightarrow \bar{Z} = X - iY \qquad X, Y \in V$$

and the complex conjugation of V^c extends in a natural manner to that of $T^r_s(V^c)$.

Assume now that V is a real 2n-dimensional vector space with a complex structure J. Then J can be uniquely extended to a complex linear endomorphism of V^c, and the extended endomorphism denoted also by J satisfies the equation $J^2 = -I$. The eigenvalues of J are therefore i and -i. We set:

$$V^{1,0} = \{Z \in V^c \; ; \; JZ = iZ\}$$
$$V^{0,1} = \{Z \in V^c \; ; \; JZ = -iZ\}$$

Then the following proposition follows easily.

Proposition (2):

(1) $V^{1,0} = \{x - iJx \ , \ x \in V\}$ and
$V^{0,1} \quad \{x + iJx \ , \ x \in V\}$

(2) $V^c = V^{1,0} + V^{0,1}$

(Complex vector space direct sum.)

(3) The complex conjugation in V^c defines a real linear isomorphism between $V^{1,0}$ and $V_{0,1}$. If we write $V^{*c} = V_{1,0} + V_{0,1}$, then the proof of the following proposition is trivial.

Proposition (3):

$$V_{1,0} = \{x^* \in V^{*c} \ ; \langle x, x^* \rangle = 0 \forall x \in V^{0,1}\}$$

$$V_{0,1} = \{x^* \in V^{*c} \ ; \langle x, x^* \rangle = 0 \forall x \in V^{1,0}\}$$

Let $\wedge V^{*c}$ be the exterior algebra. Denote by $\wedge^{p,q}V^{*c}$ the subspace of $\wedge V^{*c}$ spanned by $\alpha \wedge \beta$ where $\alpha \in \wedge^p V_{1,0}$ and $\beta \in \wedge^q V_{0,1}$. Then the exterior algebra $\wedge V^{*c}$ may be decomposed as follows:

$$\wedge V^{*c} = \sum_{r=0}^{n} \wedge^r V^{*c} \quad \text{with}$$

$$\wedge^r V^{*c} = \sum_{p+q=r} \wedge^{p,q} V^{*c} \quad ,$$

and the complex conjugation in V^{*c} extended to $\wedge V^{*c}$ in a natural manner to give a linear isomorphism between $\wedge^{p,q}V^{*c}$ and $\wedge^{q,p}V^{*c}$.

If $\{e^1, \ldots e^n\}$ is a basis for the complex vector space $V_{1,0}$, then $\{e^{-1}, \ldots e^{-n}\}$ (where $e^{-k} = \overline{e^k}$) is a basis for $V_{0,1}$ and the set of elements:

$$e^{j_1}\wedge\ldots\wedge e^{j_p}\wedge e^{-k_1}\wedge\ldots\wedge e^{-k_q}$$

$$1\leqslant j_1\leqslant\ldots\leqslant j_p\leqslant n$$
$$1\leqslant k_1\leqslant\ldots\leqslant k_q\leqslant n$$

is a basis for $\wedge^{p,q}V^{*c}$ over the field of complex numbers.

Definition (2):

A Hermitian inner product on a real vector space V with a complex structure J is an inner product h such that

$$h(JX,JY) = h(X,Y) \qquad \text{for } X,Y \in V.$$

From this definition, it follows that

$$h(JX,X) = 0 \qquad \text{for every } X \in V.$$

Proposition (4): Let h be a Hermitian inner product in a 2n-dimensional real vector space V with a complex structure J. Then there exist elements $X_1, X_2, \ldots, X_n$ of V such that $\{X_1, \ldots, X_n, JX_1, \ldots, JX_n\}$ is an orthonormal basis for V, with respect to the inner product h.

Proof: We use induction on V. If X_1 is a unit vector, then $\{X_1, JX_1\}$ is orthonormal. Let W be the subspace spanned by X_1 and JX_1 and let $W^\perp$ be the orthogonal complement, so that $V = W + W^\perp$. Then $W^\perp$ is invariant by J. By the inductive hypothesis, $W^\perp$ has an orthonormal basis of the form $\{X_2, \ldots, X_n, JX_2, \ldots, JX_n\}$

Proposition (5): Let h be a Hermitian inner product in a real vector space V with a complex structure J. Then h can be extended uniquely to a complex symmetric bilinear form, denoted also by h, of V^c, and it satisfies the following conditions:

$$(1)\ h(\bar{z},\bar{w}) = \overline{h(z,w)} \qquad \text{for } z,w \in V^c.$$

$$(2)\ h(z,\bar{z}) > 0 \qquad \forall\ z \in V^c \quad z \neq 0.$$

$$(3)\ h(z,\bar{w}) = 0 \qquad \text{for } z \in V^{1,0} \text{ and } w \in V^{0,1}.$$

Conversely, every complex symmetric bilinear form h on V^c satisfying (1), (2), (3) is the natural extension of a Hermitian inner product of V.

Proof: Straightforward.

Now to each Hermitian inner product h on V with respect to a complex structure J, we can associate an element φ of $\Lambda^2 V^*$ as follows:

$$\varphi(X,Y) = h(X,JY) \text{ for } X,Y \in V.$$

φ is skew-symmetric since:

$$\varphi(Y,X) = h(Y,JX) = h(JX,Y) = h(JX,-J^2Y) = h(X,-JY) = -\varphi(X,Y).$$

φ is also invariant by J.

Proposition (6): Let h and φ as above, $\{z_1,\ldots, z_n\}$ a basis for $V^{1,0}$ over $\mathbb{C}$ and $\{\xi^1,\ldots, \xi^n\}$ the dual basis for $V_{1,0}$. Set

$$h_{j\bar{k}} = h(z_j,\bar{z}_k) \qquad \text{for } j,k = 1, \ldots,n.$$

Then:

$$(1)\quad h_{j\bar{k}} = \overline{h}_{k\bar{j}} \qquad \text{for } j,k = 1, \ldots, n.$$

$$(2)\quad \varphi = -2i \sum_{j,k=1}^{n} h_{j\bar{k}}\, \xi^j \wedge \bar{\xi}^k$$

Proof: (1) follows from Proposition (5). To prove (2), let $z,w \in V^c$ and write:

$$z = \sum_{j=1}^{n} (\xi^j(z) z_j + \overline{\xi^j}(z) \bar{z}_j),$$

$$w = \sum_{j=1}^{n} (\xi^j(w) z_j + \overline{\xi^j}(w) \bar{z}_j).$$

A simple calculation then shows:

$$\varphi(z,w) = -i \sum_{j,k}^{n} h_{j\bar{k}} (\xi^j(z) \bar{\xi}^k(w) - \xi^j(w) \bar{\xi}^k(z)) = -2i \sum_{j,k=1}^{n} h_{j\bar{k}} \xi^j \wedge \bar{\xi}^k.$$

Definition (3):

An almost complex structure on a real differentiable manifold M is a tensor field J which is, at every $x \in M$, an endomorphism of $T_x(M)$ such that $J^2 = -1$.

A manifold with a fixed almost complex structure is called an almost complex manifold. To show that every complex manifold carries a natural almost complex structure, we consider the space $\mathbb{C}^n$ of n-tiples of complex numbers $(z^1, z^2, \ldots, z^n)$ with

$$z^j = x^j + iy^j, \; j = 1, \ldots, n.$$

With respect to the coordinate system $(x^1, \ldots, x^n, y^1, \ldots, y^n)$, we define an almost complex structure J on $\mathbb{C}^n$ by

$$J(\partial/\partial x^j) = \partial/\partial y^j, \; J(\partial/\partial y^j) = -\partial/\partial x^j.$$

To define an almost complex structure on a complex manifold M, we transfer the almost complex structure of $\mathbb{C}^n$ to M by means of charts. Since a mapping f of an open subset of $\mathbb{C}^n$ into $\mathbb{C}^m$ preserves the almost complex structure of $\mathbb{C}^n$ and $\mathbb{C}^m$ (i.e., $f_* \circ J = J \circ f_*$) if, and only if, f is holomorphic, we see that an almost complex structure can be thus defined on M independent of the choice of charts.

An almost complex structure J on a manifold M is called a complex structure if M is an underlying differentiable manifold of a complex manifold which induces J in the way just described.

Define the torsion of J to be the tensor field N of type (1, 2) given by

$N(X,Y) = 2\{[JX,JY] - [X,Y] - J[X,JY] - J[JX,Y]\}$ for $X,Y \in \mathfrak{X}(M)$.

Let $x^1,\ldots, x^n$ be a local coordinate system in M. By setting $X = \partial/\partial x^j$, $Y = \partial/\partial x^k$ in the equation defining N, we see that the components N^i_{jk} of N with respect to $x^1,\ldots, x^{2n}$ may be expressed in terms of the components J^i_j of J and its partial derivatives as follows:

$$N^i_{jk} = 2\sum_{h=1}^{2n}(J^h_j\partial_h J^i_k - J^h_k\partial_k J^i_j - J^i_h\partial_j J^h_k + J^i_h\partial_k J^h_j) .$$

An almost complex structure is said to be integrable if it has no torsion.

Theorem (1): An almost complex structure is a complex structure if, and only if, it has no torsion.

Proof: We shall prove that a complex structure has no torsion. The converse will be omitted. Let $z^1,\ldots, z^n$, $z^j = x^j + iy^j$, be a complex local coordinates system in a complex manifold M. An almost complex structure J can be defined on M as before by

$$J(\partial/\partial x^j) = \partial/\partial y^j \ , \ J(\partial/\partial y^j) = -\partial/\partial x^j$$

on $\mathbb{C}^n$ and transferring this to M by means of charts. It is clear then that the components of J with respect to the local coordinate system $x^1, \ldots, x^n, y^1, \ldots, y^n$ are constant functions in the coordinate neighborhood and hence have vanishing partial derivatives. By the expression above for N^i_{jk}, it is clear that the torsion in N is zero.

(ii) The Complex Grassman Manifold:

The complex Grassman manifold $G_{p,q}(\mathbb{C})$ of p-planes in $\mathbb{C}^{p+q}$ is the set of p-dimensional complex subspaces in $\mathbb{C}^{p+q}$ with the structure of a complex manifold defined as follows: Let $z^1, \ldots, z^{p+q}$ be the natural coordinate system in $\mathbb{C}^{p+q}$, each z^j being considered as a complex linear mapping $\mathbb{C}^{p+q} \longrightarrow \mathbb{C}$. For each set $a = \{a_1, \ldots, a_p\}$ of integers such that $1 \leq a_1 < \ldots < a_p \leq p+q$, let U_a be the subset of $G_{p,q}$ consisting of p-dimensional subspaces $S^{p,q}$ such that $Z^{a_1}|S, \ldots, Z^{a_p}|S$ are linearly independent. We shall define a mapping φ_a of U_a into the space $M(q,p;\mathbb{C})$ of $p \cdot q$ complex matrices which may be identified with $\mathbb{C}^{pq}$.

Let $\{a_{p+1}, \ldots, a_{p+q}\}$ be the complement of $\{a_1, \ldots, a_p\}$ in $\{1, \ldots, p+q\}$. Since for each $S \in U_a$, $Z^{a_1}|S, \ldots, Z^{a_p}|S$ form a basis of the dual space of S, we may write

$$Z^{a_{p+q}}|S = \sum_{j=1}^{p} S^k_j (Z^{a_j}|S), \qquad k = 1, \ldots, q .$$

Set $\varphi_a(S) = S^k_j \in M(q,p;\mathbb{C})$. It is easy to see that φ_a maps U_a (1-1) onto $M(q,p;\mathbb{C})$ and that the family of $\binom{p+q}{p}$ charts

(U_a, φ_a) which turns $G_{p,q}$ into a complex manifold of complex dimension pq. The group $GL(p+q, \mathbb{C})$ acting in $\mathbb{C}^{p+q}$ sends every p-dimensional subspace into a p-dimensional subspace and hence can be considered as a transformation group acting on $G_{p,q}(\mathbb{C})$. The action is holomorphic and transitive. If S_o denotes the p-dimensional subspace spanned by the first p-elements of the natural basis of $\mathbb{C}^{p+q}$, then the isotropy subgroup at S_o is given by

$$H = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in GL(p+q, \mathbb{C}) \right\}$$

where 0 denotes the zero matrix with p columns and q rows. Thus $G_{p,q}(\mathbb{C})$ is a quotient space $GL(_{p+q}, \mathbb{C})/H$ and the natural projection: $GL(_{p+q}, \mathbb{C}) \longrightarrow G_{p,q}(\mathbb{C})$ is holomorphic.

<u>(iii) The Complex Projective Space</u>:

The n-dimensional complex projective space $P_n(\mathbb{C})$ is the Grassman manifold $G_{n,1}(\mathbb{C})$. Now let $\mathbb{C}^*$ be the multiplicative group of non-zero complex numbers. This group acts freely on $\mathbb{C}^{n+1} - \{o\}$ by $(c,z) \in \mathbb{C}^* x (\mathbb{C}^{n+1} - \{o\}) \longrightarrow cz \in \mathbb{C}^{n+1} - \{o\}$. Let $z^0, z^1, \ldots, z^n$ be the natural coordinate system in $\mathbb{C}^{n+1}$. For each j, j=0,..., n let U_j^* be the set of points of $\mathbb{C}^{n+1} - \{o\}$ where $z^j \neq o$ and let U_j be the image of U_j^* under the natural projection

$$\mathbb{C}^{n+1} - \{o\} \longrightarrow (\mathbb{C}^{n+1} - \{o\})/\mathbb{C}^*.$$

It is easy to see that, considering $z^0/z^j, \ldots, z^{j-1}/z^j, z^{j+1}/z^j, \ldots, z^n/z^j$ as functions defined on U_j, we may identify

$P_n(\mathbb{C})$ with $(\mathbb{C}^{n+1} - \{o\})/\mathbb{C}^*$, whose complex manifold structure is defined by the family of coordinate neighborhoods U_j with local coordinate system $z^0/z^j, \ldots, z^{j-1}/z^j, z^{j+1}/z^j, \ldots z^n/z^j$. called the inhomogeneous coordinate system of $P_n(\mathbb{C})$ in U_j. The coordinate system $z^0, z^1, \ldots, z^n$ of $\mathbb{C}^{n+1}$ is called a homogeneous coordinate system of $P_n(\mathbb{C})$. Homogeneous coordinates of a point of $P_n(\mathbb{C}) \approx (\mathbb{C}^{n+1} - \{o\})/\mathbb{C}$ is, by definition, the coordinates of a point of $\mathbb{C}^{n+1} - \{o\}$ representing it. Thus, homogeneous coordinates are defined up to a non-zero constant factor.

The above may be rephrased more geometrically as follows: $\mathbb{C}^{n+1} - \{o\}$ is a principal fibre bundle over $P_n(\mathbb{C}) = (\mathbb{C}^{n+1} - \{o\})/\mathbb{C}^*$ with group $\mathbb{C}^*$. If we denote by Π the projection $\mathbb{C}^{n+1} - \{o\} \longrightarrow P_n(\mathbb{C})$, then local triviality $\Psi_j : \Pi^{-1}(U_j) \approx U_j \times \mathbb{C}^*$ is given by $\Psi_j(z) = (\Pi(z), z^j) \in U_j \times \mathbb{C}^*$ for $z = (z^0, \ldots, z^n) \in \mathbb{C}^{n+1} - \{o\}$. The transition functions $\Psi_{kj} : U_j \wedge U_k \longrightarrow \mathbb{C}^*$ are given by

$$\Pi_{kj} = z^k/z^j .$$

Let S^{2n+1} be the unit sphere in $\mathbb{C}^{n+1}$ defined by $|z^0|^2 + \ldots + |z^n|^2 = 1$ and S' the multiplicative group of complex numbers of absolute value 1. Then S^{2n+1} is a principal fibre bundle over $P_n(\mathbb{C})$ with group S'; indeed, it is a subbundle of $\mathbb{C}^{n+1} - \{o\}$ in a natural manner. Let Π be the projection of $S^{2n+1} \longrightarrow P_n(\mathbb{C})$. Then local triviality $\Psi_j : \Pi^{-1}(U_j) \approx U_j \times S'$ is given by

$$\Psi_j(z) = (\Pi(z),\ z^j/|z^j|) \in U_j \times S'$$

$$\text{for } Z = (z^0, \dots, z^n) \in U_j \times S^{2n+1}.$$

The transition functions are given by

$$\Psi_{kj} : U^j \wedge U^k \longrightarrow S'$$

$$\Psi_{kj} = |z|^j z^k / |z|^k z^j .$$

Definitions:

(4) A Hermitian metric on an almost complex manifold M is a Reimannian metric g invariant by the almost complex structure J, i.e., $g(JX, JY) = g(X,Y)$ for any vector fields X and Y.

A Hermitian metric thus defines a Hermitian inner product on each tangent space $T_x(M)$ with respect to the complex structure defined by J.

(5) An almost complex manifold (respectively a complex manifold) with a Hermitian metric is called an almost Hermitian manifold (respectively a Hermitian manifold).

Proposition (7): Every almost complex manifold admits a Hermitian metric provided it is paracompact.

Proof: Given an almost complex manifold M, take any Riemannian metric g (which exists provided M is paracompact); we obtain a Hermitian metric h by setting $h(X,Y) = g(X,Y) + g(JX,JY)$ for any vector fields X and Y.

Remark: By proposition (5), every Hermitian metric g on an almost complex manifold M can be extended uniquely to a complex symmetric tensor field of covariant degree 2, also denoted by g, such that:

(1) $g(\bar{z},\bar{w}) = \overline{g(z,w)}$ for z,w complex vector fields.

(2) $g(z,\bar{z}) > 0$ for any $z \neq 0$.

(3) $g(z,\bar{w}) = 0$ for any vector field z of type (1,0) and any vector field w of type (0,1).

Conversely, every complex symmetric tensor field g with properties (1), (2), and (3) is the natural extension of a Hermitian metric on M.

Definition (6):

The fundamental 2-form Φ of an almost Hermitian manifold M with almost complex structure J and metric g is defined by

$$\Phi(X,Y) = g(X,JY) \qquad \forall \text{ vector fields } X \text{ and } Y.$$

Since g is invariant by J, so is Φ, i.e.,

$$\Phi(JX,JY) = \Phi(X,Y).$$

Proposition (8): Let M be an almost Hermitian manifold with almost complex structure J and metric g. Let Φ be the fundamental 2-form, N the torsion of J and ∇ the covariant differentiation of the Riemannian connection defined by g. Then for any vector fields X, Y, and Z on M we have:

$$4g((\nabla_X J)Y,Z) = 6d\Phi(X,JY,JZ) - 6d\Phi(X,Y,Z) + g(N(Y,Z)\ ,\ JX).$$

Proof: Proof is direct calculation (omitted). The following theorem is useful:

Theorem (2): For an almost Hermitian manifold M with almost complex structure J and metric g, the following conditions are equivalent:

(1) The Riemannian connection defined by g is almost complex.

(2) The almost complex structure has no torsion and the fundamental 2-form is closed.

Proof: Assume (1). Since the Riemannian connection is almost complex and has no torsion, J has no torsion. Since both g and J are parallel with respect to the Riemannian connection, so is Φ, as easily seen from the definition of Φ. In particular, Φ is closed.

Assume (2). By proposition (8) we have $\nabla_X J = 0$ for all vector fields X.

Corollary (1): For a Hermitian manifold M, the following conditions are equivalent:

(1) The Riemannian connection defined by the Hermitian metric is almost complex.

(2) The fundamental 2-form Φ is closed.

Definitions:

(7) A Hermitian metric on an almost complex manifold is called a Käehler metric if the fundamental 2-form is closed.

(8) An almost complex manifold (respectively a complex manifold) with a Käehler metric is called an almost Käehler manifold (respectively Käehler manifold).

(9) A 2n-dimensional manifold with a closed 2-form Φ

which is non-degenerate at each point of M is called a symplectic manifold.

(iv) Käehlerian Geometry:

We shall express various tensor field in terms of complex local coordinate systems. In this section, let M be an n-dimensional complex manifold and $z^1, \ldots, z^n$ a complex local coordinate system in M. Greek indices α, β, γ run from 1 to n, while Latin capitals A, B, C,..., mn run from 1 to n and $\bar{1}$ to $\bar{n}$. Set:

$$Z_\alpha = \partial/\partial z^\alpha \text{ , } Z_{\bar{\alpha}} = \partial/\partial \bar{z}^\alpha = Z_\alpha \quad (1)$$

and

$$g_{AB} = g(Z_A, Z_B) \quad (2)$$

Then, by proposition (5):

$$g_{\alpha\beta} = g_{\bar{\alpha}\bar{\beta}} = 0 \quad (3)$$

and $(g_{\alpha\bar{\beta}})$ is an n x n Hermitian matrix.

Write:

$$ds^2 = 2\sum_{\alpha,\beta} g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta \quad (4)$$

for the metric g. By proposition (6), the fundamental 2-form is given by:

$$\Phi = -2i\sum_{\alpha,\beta} g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta. \quad (5)$$

A necessary and sufficient condition for g to be a Käehler metric is given by:

$$\partial g_{\alpha\bar{\beta}}/\partial z^\gamma = \partial g_{\gamma\bar{\beta}}/\partial z^\alpha \quad (6)$$

or

$$\partial g_{\alpha\bar{\beta}}/\partial \bar{z}^\gamma = \partial g_{\alpha\bar{\gamma}}/\partial z^{\bar{\beta}}.$$

Given any affinitive connection with covariant differentiation ∇ on M, we set $$\nabla_{Z_B} Z_C = \sum_A \Gamma^A_{BC} Z_A \quad . \quad (7)$$

Then $$\Gamma^A_{BC} = \Gamma^{\bar{A}}_{\bar{B}\bar{C}} . \quad (8)$$

Since $JZ_\alpha = iZ_\alpha$ and $JZ_{\bar{\alpha}} = -iZ_{\bar{\alpha}}$, it follows that the connection is almost complex if and only if:

$$\Gamma^\alpha_{B\bar{\gamma}} = \Gamma^{\bar{\alpha}}_{B\gamma} = 0. \quad (9)$$

We also see, by direct calculation, that an almost complex connection has no torsion if and only if:

$$\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta} \ , \ \Gamma^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}} = \Gamma^{\bar{\alpha}}_{\bar{\gamma}\bar{\beta}} \quad (10)$$

and other $\Gamma^A_{BC} = 0$.

For a Käehler manifold, the Γ^A_{BC}'s are determined by the metric as follows:

$$\sum_\alpha g_{\alpha\bar{\varepsilon}} \Gamma^\alpha_{\beta\gamma} = \partial g_{\bar{\varepsilon}\beta} / \partial z^\gamma \quad ,$$

$$\sum_\alpha g_{\bar{\alpha}\varepsilon} \Gamma^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}} = \partial g_{\varepsilon\bar{\beta}} / \partial \bar{z}^\gamma \quad . \quad (11)$$

Set $$R(Z_C, Z_D) Z_B = \sum_A K^A_{BCD} Z_A \quad (12)$$

and define $$K_{ABCD} = g(R(Z_C, Z_D) Z_B, Z_A)$$

so that $$K_{ABCD} = \sum_E g_{AE} K^E_{BCD} . \quad (13)$$

From the fact that the connection is almost complex, so that $R(Z_C, Z_D)$ commutes with J, it follows that: $$K^{\alpha}_{\bar{\beta}CD} = K^{\bar{\alpha}}_{\beta CD} = 0. \quad (14)$$

From (3), (13), and (14) we obtain: $$K_{\alpha\beta CD} = K_{\bar{\alpha}\bar{\beta}CD} = 0. \quad (15)$$

We know that: $$K_{ABCD} = -K_{ABDC} = -K_{BACD} = K_{CDAB} \ .$$

This, together with (14) and (15) yields: $$K^{A}_{B\gamma\delta} = K^{A}_{B\bar{\gamma}\bar{\delta}} = 0 \quad (16)$$

$$K_{AB\gamma\delta} = K_{AB\bar{\gamma}\bar{\delta}} = 0.$$

Consequently, only components of the following types can be different from zero:

$$\begin{matrix} K^{\alpha}_{\beta\gamma\bar{\delta}} \ , & K^{\alpha}_{\beta\bar{\gamma}\delta} \ , & K^{\bar{\alpha}}_{\bar{\beta}\gamma\bar{\delta}} \ , & K^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}\delta} \ , \\ K_{\alpha\bar{\beta}\gamma\bar{\delta}} \ , & K_{\alpha\bar{\beta}\bar{\gamma}\delta} \ , & K_{\bar{\alpha}\beta\gamma\bar{\delta}} \ , & K_{\bar{\alpha}\beta\bar{\gamma}\delta} \ . \end{matrix} \quad (17)$$

In the formula $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$

we set $X = Z_{\gamma}$, $Y = Z_{\bar{\delta}}$ and $Z = Z_{\beta}$.

Using (7), (9) and (10), we obtain: $$K^{\alpha}_{\beta\gamma\bar{\delta}} = -\partial\Gamma^{\alpha}_{\beta\gamma}/\partial\bar{z}^{\delta}. \quad (18)$$

From (3), (11), (13) and (18) we obtain:

$$K_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{\partial^2 g_{\alpha\bar{\beta}}}{\partial z^{\gamma}\partial\bar{z}^{\delta}} - \sum_{\tau,\varepsilon} g^{\bar{\varepsilon}\tau} \frac{\partial g_{\alpha\bar{\varepsilon}}}{\partial z^{\gamma}} \frac{\partial g_{\bar{\beta}\tau}}{\partial\bar{z}^{\delta}} \ , \quad (19)$$

where $(g^{\alpha\bar{\beta}})$ is the inverse matrix to $(g_{\alpha\bar{\beta}})$.

Since the components K_{AB} of the Ricci tensor are given by $K_{AB} = \sum_{C} K^{C}_{ACB}$ we obtain

from (14), (16) and (18):

$$K_{\alpha\bar{\beta}} = -\sum_{\gamma} \partial\Gamma^{\gamma}_{\alpha\gamma}/\partial\bar{z}^{\beta} \ , \ K_{\bar{\alpha}\beta} = \overline{K_{\alpha\bar{\beta}}} \ , \ K_{\alpha\beta} = K_{\bar{\alpha}\bar{\beta}} = 0 \ .$$

Now let G be the determinant of the matrix $(g_{\alpha\bar{\beta}})$. Then we have:

$$\frac{\partial G}{\partial z^{\alpha}} = G\sum_{\beta,\gamma} g^{\beta\bar{\gamma}}\frac{\partial g_{\beta\bar{\gamma}}}{\partial z^{\alpha}} \ . \quad (20)$$

Applying (11), we obtain:

$$\sum_{\gamma} \Gamma^{\gamma}_{\alpha\gamma} = \frac{\partial \log G}{\alpha} \ . \quad (21)$$

This, together with (20), gives us:

$$K_{\alpha\bar{\beta}} = -\frac{\partial^{2}\log G}{\partial z^{\alpha}\partial z^{\beta}} \ . \quad (22)$$

In the same way as we obtained the local coordinate expression (5) for Φ, we obtain the following expression for the Ricci form:

$$\rho = -2i\sum_{\alpha,\beta} K_{\alpha\bar{\beta}}dz^{\alpha}\wedge d\bar{z}^{\beta} \ . \quad (23)$$

Using (22), ρ may be written as:

$$\rho = 2i\partial\bar{\partial}\log G \ . \quad (24)$$

Let F be a real-valued function in a coordinate neighborhood of a complex manifold M, and set:

$$g_{\alpha\bar{\beta}} = \frac{\partial^{2}F}{\partial z^{\alpha}\partial\bar{z}^{\beta}} \ . \quad (25)$$

Since F is real, $(g_{\alpha\bar{\beta}})$ is always Hermitian. If it is positive definite, then:

$$2\sum_{\alpha,\beta} g_{\alpha\bar{\beta}} dz^{\alpha} d\bar{z}^{\beta}$$

is a Käehler metric by (6). Every Käehler metric can be locally written as:

$$ds^2 = 2\sum_{\alpha,\beta} g_{\alpha\bar{\beta}} dz^{\alpha} d\bar{z}^{\beta}$$

with $g_{\alpha\bar{\beta}}$ given by (25).

Now let e(x) be a field of unitary frames, defined for x in a neighborhood of M. Its dual coframe field consists of n complex-valued linear differential forms θ_{α} of type (1,0) such that the Hermitian metric can be written as:

$$ds^2 = \sum_{\alpha} \theta_{\alpha} \bar{\theta}_{\alpha} \quad .$$

The connection forms $\theta_{\alpha\bar{\beta}}$ are characterized by the conditions:

$$\theta_{\alpha\beta} + \theta_{\beta\alpha} = 0$$

$$d\theta_{\alpha} = \sum_{\beta} \theta_{\beta} \wedge \theta_{\beta\alpha} \tag{26}$$

and they can be interpreted geometrically as defining the covariant differential:

$$De_{\alpha} = \sum_{\beta} \theta_{\alpha\beta} \otimes e_{\beta} . \tag{27}$$

The curvature forms $\Theta_{\alpha\beta}$ are then defined by:

$$d\theta_{\alpha\beta} = \sum_{\gamma} \theta_{\alpha\gamma} \wedge \theta_{\gamma\beta} + \Theta_{\alpha\beta} \quad (28)$$

and we have: $$\Theta_{\alpha\beta} = -\Theta_{\beta\alpha} = \sum_{\gamma,\delta} R_{\alpha\beta\gamma\delta}\theta_{\gamma} \wedge \bar{\theta}_{\delta} \quad .$$

The skew-Hermitian symmetry of $\Theta_{\alpha\beta}$ expressed by the above equation is equivalent to the symmetry conditions $R_{\alpha\beta\gamma\delta} = \bar{R}_{\beta\alpha\delta\gamma}$.

The Bianchi identities obtained by exterior differentiation of (26) and (28) give further symmetry relations:

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\beta\alpha\delta} = R_{\alpha\delta\gamma\beta} \quad ,$$

and the equation: $$d\Theta_{\alpha\beta} + \sum_{\gamma} \Theta_{\alpha\gamma} \wedge \theta_{\gamma\beta} - \sum_{\gamma} \theta_{\alpha\gamma} \wedge \Theta_{\gamma\beta} = 0 \quad .$$

The quantities $R_{\alpha\beta\gamma\delta}$ define the holomorphic sectional curvature to every tangent vector of M.

Let $$\xi = \sum_{\alpha} \xi_{\alpha} e_{\alpha} \neq 0$$

be a tangent vector at x. Then the holomorphic sectional curvature is defined to be:

$$R(x,\xi) = 2 \sum_{\alpha,\ldots,\delta} R_{\alpha\beta\gamma\delta}\xi_{\alpha}\xi_{\gamma}\xi_{\beta}\xi_{\delta} / (\sum_{\alpha} \xi_{\alpha}\xi_{\alpha})^{2} \quad . \quad (29)$$

M is said to be of constant holomorphic sectional curvature K if $R(x,\xi) = K$ for all (x,ξ). This is expressed by the condition:

$$R_{\alpha\beta\gamma\delta} = \frac{1}{4}(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})K$$

or

$$\Theta_{\alpha\beta} = \frac{1}{4}K(\theta_\beta \wedge \theta_\alpha + \delta_{\alpha\beta}\sum_\gamma \theta_\gamma \wedge \theta_\gamma). \quad (30)$$

All the above are independent of the choice of a frame field. Let

$$e^*_\alpha = \sum_\beta u_{\alpha\beta} e_\beta$$

be a new frame field, where $u_{\alpha\beta}$ are complex-valued C^∞-functions such that $(u_{\alpha\beta})$ is a unitary matrix. Let θ^*_α, $\theta^*_{\alpha\beta}$ be the forms relative to the frame field e^*_α. Then, by definition and (27), we have:

$$\theta^*_\alpha = \sum \bar{u}_{\alpha\beta}\theta_\beta$$

and

$$\theta^*_{\alpha\beta} = \sum_\gamma du_{\alpha\gamma}\bar{u}_{\beta\gamma} + \sum_{\gamma,\delta} u_{\alpha\gamma}\theta_{\gamma\delta}\bar{u}_{\beta\delta} \quad .$$

(v) The Study-Fubini Metric:

Let M be a complex manifold of dimension $n + 1$, whose points are the ordered enniples of complex numbers: $Z = (Z_0, \ldots, Z_n)$. In M we introduce the

Hermitian scalar product: $(w,z) = \sum_A z_A \bar{w}_A$, $w = (w_0, \ldots, w_n)$ (31)

The unitary group U(n + 1) in n + 1 variables is the group of all linear homogeneous transformations on z_A leaving the scalar product (31) invariant. Let M^*_{n+1} be the subset of M_{n+1} obtained by the deletion of the zero vector. Then the complex projective space P_n is the orbit space of M^*_{n+1} under the action of the group $z \longrightarrow \delta z$, $\delta \neq 0$. We have thus the projection $\Pi : M^*_{n+1} \longrightarrow P_n$. To a point $p \in P_n$ a vector $z \in \Pi^{-1}(p)$ is called a homogeneous coordinate vector of p; p is frequently identified with z. Put

$$z_0 = z/(z,z)^{\frac{1}{2}} ,$$

so that $(z_0, z_0) = 1$. Then the Study-Fubini metric is given by:

$$ds^2 = (dz_0, dz_0) - (dz_0, z_0)(z_0, dz_0) \quad . \qquad (32)$$

To study this metric, let z_A be a unitary frame in M_{n+1}, so that

$$(z_A, z_B) = \delta_{AB} \quad .$$

In the space of all unitary frames in M_{n+1}, let ω_{AB} be defined by

$$dz_A = \sum_B \omega_{AB} z_B \quad ,$$

so that we have $\omega_{AB} = -\bar{\omega}_{BA} = (dz_A, z_B)$.

Then ω_{AB} are the Maurer-Cartan forms of

U(n+1) and satisfy the structure equations:

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} \quad . \qquad (33)$$

The metric (32) can then be written as:

$$ds^2 = \sum_\alpha \omega_{0\alpha} \bar{\omega}_{0\alpha}$$

This is of the form $ds^2 = \sum_\alpha \theta_\alpha \bar{\theta}_\alpha$ if we set

$$\theta_\alpha = \omega_{0\alpha} \quad .$$

Equations (26) and (28) will be satisfied,

provided that we choose

$$\theta_{\beta\alpha} = \omega_{\beta\alpha} - \delta_{\beta\alpha} \omega_{00} \quad .$$

These are therefore the connection forms

of the Study-Fubini metric. By (33) we

find the curvature forms of this metric to be:

$$\Theta_{\alpha\beta} = \theta_\beta \wedge \bar{\theta}_\alpha + \delta_{\alpha\beta} \sum_\gamma \theta_\gamma \wedge \bar{\theta}_\gamma \quad .$$

Comparing with (30), we see that the metric (32) has constant holomorphic sectional curvature equal to 4. From definition of the metric, U(n+1) acts on P_n as a group of isometrics.

Chapter II. Spinor Theory

In this chapter, a background of spinor theory needed in the later chapters is revealed. Section (i) is an introduction of a two-component spinor. This is used in section (ii) to establish a one-to-one correspondence between the points of Minkowski space and the two-component spinors, being defined as second-rank Hermitian metrics. For further detail, see [3]. In section (iii), we develop some spinor calculus and use it in section (iv) to show the isomorphism between the group $\mathbb{SL}(2,\mathbb{C})$ of unimodular (2 x 2) complex matrices and the two-fold (universal) covering of the connected component of the Lorentz group $\mathbb{O}(1,3)$. The reader might consult [20]. In section (v), a spinor is interpreted (up to sign) as a null flag which physically represents a polarized photon. This interpretation follows that of Penrose [14]. In section (vi), we give spinor equivalents of certain vectors and tensors [see 20]. The spinor equivalents will be useful later in the geometrical interpretation of a twistor as a null line and also when we come to consider zero-rest-mass fields.

(i) Definition of a Two-Component Spinor:

These are geometrical objects Ψ_A which are defined over a two-dimensional complex space $\mathbb{S}$ (the spin space) and obey the transformation law:

$$\Psi'_A = t_A^{\ B}\Psi_B \qquad (A,B = 1,2) \quad (2.1)$$

where the transformation matrix $\|t_A^B\|$ is, in general, complex and non-singular. For the purpose of applications of spinor theory, we restrict ourselves to transformations of spin coordinates with unimodular matrices $\|s_B^A\|$. These are just two unimodulars, which are:

$$s_B^A = t_B^A t^{-\frac{1}{2}} \qquad t = |t_B^A| = \det. t_B^A \quad (2.2)$$

The two values of $\|s_B^A\|$ arise from ambiguity in sign of $t^{-\frac{1}{2}}$.

The effect of restricting ourselves to unimodular transformations (2.2) is achieved by assigning a suitable weight to Ψ_A. In general, a tensor of weight ω is a form or geometrical object transforming according to the law:

$$\Psi^{\alpha\beta\ldots}_{\gamma\delta\ldots} = T^{\alpha}_{\kappa} T^{\beta}_{\lambda} \ldots t^{\mu}_{\gamma} t^{\nu}_{\delta} \ldots t^{\omega} \Psi^{\kappa\lambda}_{\mu\nu} \qquad \text{where } t = |t^{\mu}_{\gamma}| \quad .$$

Thus, if Ψ_A is to be of weight $-\frac{1}{2}$, its transformation law is:

$$\Psi'_A = t_A^B t^{-\frac{1}{2}} \Psi_B = s_A^B \Psi_B \quad (2.3)$$

We call quantities transforming in this way "covariant regular spinors of first rank." Similarly, a geometric object Ψ^A of weight $\frac{1}{2}$ transforming contragrediently to Ψ_A according to:

$$\Psi^{A'} = T_B^A t^{\frac{1}{2}} \Psi^B = S_B^A \Psi^B \quad (2.4)$$

$$\text{where} \qquad T_B^A t_C^B = S_B^A s_C^B = \delta_C^A \quad (2.5)$$

will be called a "contravariant regular spinor of first rank." By analogy with tensor theory, we associate covariant and contravariant components according to the rule:

$$\Psi_A = \epsilon_{AB} \Psi^B \quad (2.6)$$

$$\text{where} \qquad \|\epsilon_{AB}\| = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix} = \|\epsilon^{AB}\| \quad (2.7)$$

These are called the fundamental numerical metric spinors of second rank. The inverse equations to (2.6) are:

$$\Psi^A = \epsilon^{AB}\Psi_B = \epsilon^{BA}\epsilon_{BC}\Psi^C = \delta^A_C\Psi^C \quad .(2.8)$$

The transformations laws of ϵ_{AB} and ϵ^{AB} are:

$$\epsilon_{AB} = t^C_A t^D_B t^{-1}\epsilon_{CD} = s^C_A s^D_B \epsilon_{CD}$$
$$\epsilon^{AB} = T^A_C T^B_D t\epsilon^{CD} = s^A_C s^B_D \epsilon^{CD} \quad , \qquad (2.9)$$

from which it follows that ϵ_{AB} and ϵ^{AB} are of weight -1 and +1 respectively. We can define higher-rank spinors as quantities which transform as products of simple (first-rank) spinors, so that the metric spinors ϵ_{AB} and ϵ^{AB} are of second rank, as already noted. Moreover, the weights ½ and -½ for Ψ_A are consistent with the rules:

$$\Psi_A = \epsilon_{AB}\Psi^B$$
$$\Psi^A = \epsilon^{BA}\Psi_B \quad .$$

It follows that:

$$\Phi^A\Psi_A = -\Phi_A\Psi^A \qquad (2.10)$$

so that every odd-rank spinor has zero absolute value, viz.:

$$\Phi_A\Phi^A = 0$$
$$\Psi_{ABC}\Psi^{ABC} = 0 \qquad (2.11)$$

etc., and in particular: $\Phi^A\Psi_B\chi_B + \Phi_A\Psi_B\chi^A + \Phi_B\Psi^A\chi_A = 0$.

The geometric object which has the components Ψ_1, Ψ_2 is closely analogous to a covariant vector; but the coefficients of the transformations (2.1), being, in general, complex instead of real, allow the more general transformation law:

$$\chi'_A = t_A^B t^{\omega} \bar{t}^{-d} \chi_B \quad (2.12)$$

for a geometric object with components χ_A. The indices ω and d are usually called the weight and anti-weight respectively, and when $\omega = d$ the transformation may be written as:

$$\chi'_A = t_A^B |t|^{2\omega} \chi_B \quad ,$$

where $|t| = (t\bar{t})^{\frac{1}{2}}$ is the absolute value of the determinant; in which case χ_A is said to be of absolute weight 2ω.

Consider the transformation:

$$\chi_{A'}^{B} = \bar{s}_A^C s_D^B \chi_{C'}^{D} \quad (2.13)$$

where primed indices serve as a reminder that a bar (complex conjugate) is to be placed over the corresponding t_A^C or s_A^C in the law of transformation, e.g.,

$$\chi^{A\ldots C'\ldots}_{E\ldots G'\ldots} = T_P^A \ldots \bar{T}_R^C \ldots t_E^T \ldots \bar{t}_G^U \ldots t^{\omega} \bar{t}^{-d} \chi^{P\ldots R'\ldots}_{T\ldots U'\ldots} \quad . \quad (2.14)$$

It is always possible to choose the weights ω and d so that the transformation can be written in terms of the unimodular matrices $\|s_B^A\|$ and $\|S_B^A\|$.

The raising and lowering of primed indices is accomplished with the aid of the spinors $\epsilon^{A'B'}$ and $\epsilon_{A'B'}$ by the rules:

$$\Psi_{A'} = \epsilon_{A'B'} \Psi^{B'}$$
$$\Psi^{A'} = \epsilon^{B'A'} \Psi_{B'} \quad (2.15)$$

where $\|\epsilon_{A'B'}\| = \|\epsilon^{A'B'}\| = \left\| \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \right\|$ and $\Psi_{A'}$ and $\Psi^{A'}$ are of anti-weights -½ and ½ respectively. Moreover, $\epsilon_{A'B'}$ and $\epsilon^{A'B'}$ are of anti-weights -1 and +1, and have the transformation laws:

$$\begin{aligned} \epsilon_{A'B'} &= \bar{s}_{A}^{C}\bar{s}_{B}^{D}\epsilon_{C'D'} \\ \epsilon^{A'B'} &= \bar{s}_{C}^{A}\bar{s}_{D}^{B}\epsilon^{C'D'} \end{aligned} \qquad (2.16)$$

It follows from the above considerations that the complex conjugate of a 2-spinor equation is obtained by priming all unprimed indices.

(ii) The Correspondence Between Minkowski Space $\mathbb{M}$ and Second-Rank Hermitian Matrices

Consider a Minkowski space $\mathbb{M}$ which is $\mathbb{R}^4$-equipped with a flat Lorentz (pseudo)metric of signature -2, i.e., where, in an appropriate coordinate the metric tensor is diagonal with diagonal entries {+1,-1,-1,-1}.

There is quite a remarkable correspondence between second-rank Hermitian matrices and points of $\mathbb{M}$. To establish this correspondence, consider a Hermitian matrix $X = \| X_{A'B} \|$. The condition of Hermiticity is:

$$\bar{X}_{A'B} = X_{B'A} \qquad (2.17)$$

$X_{A'B}$ transforms under general constant spin (i.e., complex binary) transformations as:

$$X'_{A'B} = \bar{t}_{A}^{C}t_{B}^{D}t^{-\frac{1}{2}}\bar{t}^{-\frac{1}{2}}X_{C'D} = \bar{s}_{A}^{C}s_{B}^{D}X_{C'D} \qquad (2.18)$$

and we see that the determinant of this relation yields:

$$| X_{A'B} | = | X_{C'D} | \qquad (2.19)$$

or $X_{1\dot{1}}X_{2\dot{2}} - X_{1\dot{2}}X_{2\dot{1}}$

is invariant, i.e., the determinant is an invariant under general binary transformations.

Alternatively we note that:

$$|X_{A\dot{B}}| = \frac{1}{2}\epsilon^{\dot{C}\dot{A}}\epsilon^{DB}X_{A\dot{B}}X_{C\dot{D}} = \frac{1}{2}X_{A\dot{B}}X^{A\dot{B}} \quad ,$$

from which the above conclusion follows. The essential point now is that every Hermitian matrix of second order can be written in terms of four real parameters p, q, r, s,

as, for example:

$$\| X_{A\dot{B}} \| = \begin{Vmatrix} p + q & r + is \\ r - is & p - q \end{Vmatrix} \quad . \tag{2.20}$$

The quadratic form: $g_{\mu\nu}x^{\mu}x^{\nu} = -(x^1)^2 - (x^2)^2 - (x^3)^2 + (x^4)^2$ can be expressed in the product form:

$$g_{\mu\nu}x^{\mu}x^{\nu} = (p+q)(p-q) - (r+is)(r-is)$$

and in a great variety of ways according to various possible (1-1) identifications of individuals in the sets p, q, r, s and x^1, x^2, x^3, x^4. The correspondence (2.20) is (1-1); the inverse equations are:

$$\begin{aligned} p &= \tfrac{1}{2}(X_{1\dot{1}} + X_{2\dot{2}}) \\ q &= \tfrac{1}{2}(X_{1\dot{1}} - X_{2\dot{2}}) \\ r &= \tfrac{1}{2}(X_{1\dot{2}} + X_{2\dot{1}}) \\ s &= \frac{1}{2i}(X_{1\dot{2}} - X_{2\dot{1}}) \quad . \end{aligned} \tag{2.21}$$

(iii) The σ-Spin Tensors:

We shall use a formalism in which the above identification chosen is:

$p = x^4$, $q = x^3$, $r = x^1$, $s = x^2$

with which (2.20) takes the form:

$$\| X_{A'B} \| = \begin{Vmatrix} X^4 + X^3 & x^1 + ix^2 \\ X^1 - iX^2 & x^4 - x^3 \end{Vmatrix} \qquad (2.22)$$

This may be written in the form:

$$X_{A'B} = \sigma_{\mu A'B} x^{\mu} \qquad (2.23)$$

where the spin-tensors $\sigma_{\mu A'B}$ are given by the matrix equations:

$$\| \sigma_{1A'B} \| = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}$$

$$\| \sigma_{2A'B} \| = \begin{Vmatrix} 0 & i \\ -i & 0 \end{Vmatrix}$$

$$\| \sigma_{3A'B} \| = \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix} \qquad (2.24)$$

$$\| \sigma_{4A'B} \| = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}$$

The inverse equations (2.21) may be written as: $x^{\mu} = \frac{1}{2}\sigma^{\mu A'B} X_{A'B}$

with

$$\| \sigma^{1A'B} \| = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}$$

$$\| \sigma^{2A'B} \| = \begin{Vmatrix} 0 & -i \\ i & 0 \end{Vmatrix}$$

$$\| \sigma^{3A'B} \| = \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix} \qquad (2.25)$$

$$\| \sigma^{4A'B} \| = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} .$$

The σ's obviously satisfy the conditions

$$\sigma^{\mu A'B}\sigma_{\nu A'B} = 2\delta^{\mu}_{\nu}$$
$$\sigma^{\mu A'B}\sigma_{\mu C'D} = 2\delta^{A'}_{C'}\delta^{B}_{D} \,. \qquad (2.26)$$

Multiplying the latter by $\in^{C'F'}\in^{DG}$ and summing, we obtain the useful relation:

$$\sigma^{\mu A'B}\sigma_{\mu}^{F'G} = 2\in^{A'F'}\in^{BG} \qquad (2.27)$$

From equations (2.24) and (2.25) one can verify that:

$$\sigma_{\mu A'B} = \sigma_{\mu B'A} \qquad \text{and} \qquad \bar{\sigma}^{\mu A'B} = \sigma^{\mu B'A} \;; \qquad (2.28)$$

that is, each of the matrices $\|\sigma_{\mu A'B}\|$ and $\|\sigma^{\mu A'B}\|$ is Hermitian.

Another important relation satisfied by the σ-spinors is:

$$\sigma^{\mu A'B}\sigma_{\nu A'C} + \sigma_{\nu}{}^{A'B}\sigma^{\mu}{}_{A'C} = 2\delta^{\mu}_{\nu}\delta^{B}_{C} \qquad (2.29)$$

from which (2.26) obtains by contraction on the indices B,C. This relation may be easily expressed in the equivalent forms:

$$\sigma_{\mu A'B}\sigma_{\nu}^{A'C} + \sigma_{\nu A'B}\sigma_{\mu}^{A'C} = 2\sigma_{\mu\nu}\delta^{C}_{B}$$
or
$$\bar{\sigma}_{\mu B'A}\sigma_{\nu}^{A'C} + \bar{\sigma}_{\nu B'A}\sigma_{\mu}^{A'C} = 2\sigma_{\mu\nu}\delta^{C}_{B} \qquad (2.30)$$

by (2.28). Similarly, from (2.29) and (2.28) we obtain:

$$\bar{\sigma}^{\mu B'}{}_{A}\,\sigma^{\nu A'}{}_{C} + \bar{\sigma}^{\nu B'}{}_{A}\,\sigma^{\mu A'}{}_{C} = -2\sigma^{\mu\nu}\delta^{B}_{C}$$

which can be written in the matrix form:

$$\bar{\sigma}^{\mu}\sigma^{\nu} + \bar{\sigma}^{\nu}\sigma^{\mu} = -2\sigma^{\mu\nu} \qquad (2.31)$$

$$\text{with} \qquad \sigma^{\mu} = \|\sigma^{\mu A'}{}_{B}\|$$

However, we can develop the σ-spin tensors symbolically without the use of any explicit representation. This may be done by defining the σ's by the two conditions:

$$\bar{\sigma}^{\mu}_{A'B} = \sigma^{\mu}_{B'A} \quad (2.32a)$$

$$\sigma_{\mu}^{B'A}\sigma_{\nu B'C} = \sigma_{\mu\nu}\delta^{A}_{C} + \tfrac{1}{2}\sigma^{-\frac{1}{2}}\epsilon_{\mu\nu\alpha\beta}\sigma^{\alpha B'A}\sigma^{\beta}_{B'C} \quad (2.32b)$$

Then all the previous results follow, as well as additional ones of importance. For example, from (2.32b) and its

conjugate: $$\sigma_{\mu}^{A'B}\sigma_{\nu C'B} = \sigma_{\mu\nu}\delta^{A'}_{C'} - \tfrac{1}{2}\sigma^{-\frac{1}{2}}\epsilon_{\mu\nu\alpha\beta}\sigma^{\alpha A'B}\sigma^{\beta}_{C'B} \quad (2.32c)$$

we obtain: $$\sigma_{\mu}^{B'A}\sigma_{\nu B'C} + \sigma_{\nu}^{B'A}\sigma_{\nu B'C} = 2\sigma_{\mu\nu}\delta^{A}_{C} \quad (2.32d)$$

$$\sigma_{\mu}^{B'A}\sigma_{\nu B'C} - \sigma_{\nu}^{B'A}\sigma_{\mu B'C} = \sigma^{-\frac{1}{2}}\epsilon_{\mu\nu\alpha\beta}\sigma^{\alpha B'A}\sigma^{\beta}_{B'C} \;.\quad (2.32e)$$

Similarly, we note from (2.32b) that: $$\epsilon_{\mu\nu\lambda\kappa}\sigma^{\mu}_{B'A}\sigma^{\nu B'C}\sigma^{\lambda}_{D'C}$$

$$= \tfrac{1}{2}\sigma^{-\frac{1}{2}}\epsilon_{\mu\nu\lambda\kappa}\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha B'A}\sigma_{\beta}^{B'C}\sigma^{\lambda}_{D'C}$$

$$= \sigma^{-\frac{1}{2}}\delta^{\alpha\beta}_{\lambda\kappa}\sigma_{\alpha B'A}\sigma_{\beta}^{B'C}\sigma^{\lambda}_{D'C}$$

$$= 2\sigma^{-\frac{1}{2}}(\sigma_{\lambda\kappa}\delta^{C}_{A} - \sigma_{\kappa B'A}\sigma_{\lambda}^{B'C})\sigma^{\lambda}_{D'C}$$

$$= -6\sigma^{-\frac{1}{2}}\sigma_{\kappa D'A} \quad (2.32f)$$

This is needed to establish (2.26b) independently. Using (2.32d), the above also leads to the identity:

$$\sigma^{\mu}_{B'A}\sigma^{\nu B'C}\sigma^{\lambda}_{D'C} = \sigma^{\mu\nu}\sigma^{\lambda}_{D'A} - \sigma^{\mu\lambda}\sigma^{\nu}_{D'A} + \sigma^{\nu\lambda}\sigma^{\mu}_{D'A} - \sigma^{-\frac{1}{2}}\epsilon^{\mu\nu\lambda\kappa}\sigma_{\kappa D'A} \;.$$

Results:

It follows that the light cone is the locus of points of $\mathbb{M}$ which correspond to singular Hermitian matrices. The rows of such a matrix are proportional:

$$\| X_{A^{\prime}B} \| = \| \chi_{A^{\prime}} \Phi_B \| \quad ,$$

so that a vector in $\mathbb{M}$ determined by:

$$x^{\mu} = \tfrac{1}{2}\sigma^{\mu A^{\prime}B} \chi_{A^{\prime}} \Phi_B$$

is a null vector, i.e., using (2.27) and (2.11):

$$x_{\mu} x^{\mu} = \tfrac{1}{4}\sigma_{\mu\nu} \sigma^{\mu A^{\prime}B} \sigma^{\nu C^{\prime}D} \chi_{A^{\prime}} \Phi_B \chi_{C^{\prime}} \Phi_D$$

$$= \tfrac{1}{2}(\chi_{A^{\prime}} \chi^{A^{\prime}})(\Phi_B \Phi^B) = 0 \quad .$$

Similarly, a vector determined by a positive definite Hermitian matrix is time-like, e.g., for:

$$x^{\mu} = \tfrac{1}{2}\sigma^{\mu A^{\prime}B}(\Psi_{A^{\prime}} \Psi_B + \Phi_{A^{\prime}} \Phi_B) \quad ,$$

$$x_{\mu} x^{\mu} = \Phi_A \Psi^A \Phi_{B^{\prime}} \Psi^{B^{\prime}} > 0$$

if Φ_A is not proportional to Ψ_A.

(iv) The isomorphism between Lorentz Transformations in M and the Group of Linear and Antilinear Geometric Transformations in Spin Space:

We have seen in section (ii) that the determinant of $X_{A^{\prime}B}$ is invariant under the binary unimodular transformation

$$X_{A^{\prime}B} = s_{A^{\prime}}^{\;-C^{\prime}} s_B^{\;D} X_{C^{\prime}D}$$

or in matrix form: $$\chi = s^{*}\chi s$$

where $$s = \| s^{A}_{B} \| .$$

Equivalently, under this transformation: $$x_{\mu}x^{\mu} = \tfrac{1}{2}\chi_{A'B}\chi^{A'B}$$

goes into $$x'_{\mu}x^{\mu'} = \tfrac{1}{2}\chi_{A'B}\chi'^{A'B}$$

$$= \tfrac{1}{2}s^{-C}_{A}s^{D}_{B}S^{-A}_{E}S^{B}_{F}\chi_{C'D}\chi^{E'F}$$

$$= \tfrac{1}{2}\sigma_{C'D}\sigma^{\tau E'F}s^{-C}_{A}s^{D}_{B}S^{-A}_{E}S^{B}_{F}x^{\sigma}x_{\tau}$$

$$= L^{\mu}_{\sigma}\ell^{\tau}_{\mu}x^{\sigma}x_{\tau} = x^{\sigma}x_{\sigma}$$

where we have introduced: $$L^{\mu}_{\sigma}\ell^{\tau}_{\mu} = \tfrac{1}{2}\sigma_{\sigma C'D}\sigma^{\tau E'F}s^{-C}_{A}s^{D}_{B}S^{-A}_{E}S^{B}_{F} = \delta^{\tau}_{\sigma} .$$

This relation is satisfied by: $$L^{\mu}_{\nu} = \tfrac{1}{2}\sigma^{\mu A'B}s^{-C}_{A}s^{D}_{B}\sigma_{\nu C'D} \qquad (2.33)$$

or:

$$\begin{Vmatrix} L^1_1 & L^1_2 & L^1_3 & L^1_4 \\ L^2_1 & L^2_2 & L^2_3 & L^2_4 \\ L^3_1 & L^3_2 & L^3_3 & L^3_4 \\ L^4_1 & L^4_2 & L^4_3 & L^4_4 \end{Vmatrix} = \tfrac{1}{2} \begin{Vmatrix} s^1_2 & s^2_2 & s^1_1 & s^2_1 \\ -is^1_2 & -is^2_2 & is^1_1 & is^2_1 \\ s^1_1 & s^2_1 & -s^1_2 & -s^2_2 \\ s^1_1 & s^2_1 & s^1_2 & s^2_2 \end{Vmatrix} \cdot \begin{Vmatrix} s^{-2}_1 & s^{-2}_1 & s^{-2}_1 & s^{-1}_1 \\ s^{-1}_1 & is^{-1}_1 & -s^{-2}_1 & s^{-2}_1 \\ s^{-2}_2 & -is^{-2}_2 & s^{-1}_2 & s^{-1}_2 \\ s^{-1}_2 & is^{-1}_2 & -s^{-2}_2 & s^{-2}_2 \end{Vmatrix} .$$

This is a Lorentz transformation. It is an element of the proper Lorentz group since its determinant is unity and

$$L^4_4 = \tfrac{1}{2}(s_1^{-1}s_1^{1} + s_1^{-2}s_1^{2} + s_2^{-1}s_2^{1} + s_2^{-2}s_2^{2}) \geq 1 \quad .$$

On the other hand, the matrix $s = \|s^A_B\|$ is an element of the binary unimodular (spinor) group of linear transformations; so we obtain the result that the geometric spin transformation, of the type $\Phi_A \longrightarrow \Phi'_A = s_A^{\ B}\Phi_B$, induces the proper Lorentz transformation $x^\mu \longrightarrow x^{\mu'}$ with the matrix given by (2.32). It can also be verified that the correspondence $a \longrightarrow a'$ has the properties: (1) $e \longrightarrow e'$, (2) $a^{-1} \longrightarrow (a')^{-1}$, (3) $ab \longrightarrow a'b'$. However this isomorphism is (2-1), since every proper Lorentz transformation may be represented in the form (2.33); but the two matrices s and -s generate the same Lorentz transformation.

Our goal here is to introduce a formalism for the description of space-time which makes the spinor structure emerge as more basic, even, than the pseudo-Riemannian structure. The formalism, as described above, is based on the isomorphism between the group $SL(2,\mathbb{C})$ of unimodular (2 x 2) complex matrices, and the twofold (universal) covering of the connected component of the Lorentz group $O(1,3)$.

Alternatively, to express this isomorphism in a direct form, consider the general (2 x 2) Hermitian matrix

$$\begin{pmatrix} X^{00'} & X^{01'} \\ X^{10'} & X^{11'} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + x^1 & x^2 + ix^3 \\ x^2 - ix^3 & x^0 - x^1 \end{pmatrix} . \quad (2.34)$$

If this matrix is multiplied on the left by a unimodular (2 x 2) complex matrix t_B^U and on the right by the conjugate transpose of this matrix, then both the Hermiticity and the value of the determinant of the above matrix will be preserved. Thus we obtain a linear transformation of (x^1, x^2, x^3, x^4) which preserves both the reality of x^a

and the form: $$g_{ab}x^a x^b = (x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2$$

where $$(g_{ab}) = \text{diagonal } (1, -1, -1, -1) .$$

This is, therefore, a Lorentz transformation on x^a:

$$x^a \longrightarrow L^a_b x^b \quad , \qquad (2.35)$$

which is continuous with the identity, since t_B^U is continuous with δ_B^U. Conversely, it follows that any such transformation (2.35) arises from precisely two unimodular matrices t_B^U and $(-t_B^U)$. Thus (2.35) is equivalent to:

$$X^{UU'} \longrightarrow t_B^U t_B^{-U'} X^{BB'} \quad .$$

The Hermitian matrices as such, corresponding to a vector (x^1, x^2, x^3, x^4), will be called a spin-vector.

An object built up of tensor products of the spin space $\mathfrak{S}$, its dual $\dot{\mathfrak{S}}$ and the complex conjugates of these two spaces will be called a spinor field on $\mathbf{M}$.

We will regard each $\bar{X}^{U'}$ as the complex conjugate of the corresponding X^U and write:

$$\overline{X^U} = \bar{X}^{U'} \; ; \; \overline{W^{A'}} = \bar{W}^{A} \quad .$$

Now the system is built up with the proviso that contractions and index substitutions can occur only between index labels of the same type (primed or unprimed), ordering of indices of the same type is necessary, e.g.:

$$\rho^{AA'\alpha}_{B'B} = \rho^{A'A\alpha}_{BB'} = \rho^{AA'\alpha}_{BB'\ldots} \neq \rho^{AA'\alpha}{}_{BB'} \quad .$$

Now consider twice the determinant of (2.34):

$$X^{UU'}X^{BB'}\epsilon_{UB}\epsilon_{U'B'} = X^a X^b g_{ab} \; .$$

This becomes:

$$X^a X^b(\epsilon_{AB}\epsilon_{A'B'} - g_{ab}) = 0$$

$$\Longrightarrow \quad g_{ab} = \epsilon_{AB}\epsilon_{A'B'}, \qquad (2.36)$$

which expresses the fundamental way in which the metric of M is defined by its spinor structure. In particular, the (+, -, -, -) signature is built into the formalism by the identification (2.36).

(v) The Interpretation of a Spin-Vector:

Any spin-vector has a graphic space-time interpretation (up to sign) as a null flag. Physically, this is a polarized photon (Penrose [14]). To see this, let $\omega^A \in S_p$ be the spin-vector at p. The world-tensor associated with ω^A is the world-vector

$$U^a = \omega^A \bar{\omega}^{A'} \quad .$$

Since (2.34) is invariant under $\omega^A \longrightarrow e^{i\theta}\omega^A$, many spin-vectors will correspond to the same null world-vector. To interpret the "phase" of ω^A, let us "square" ω^A instead: $\omega^A\omega^B$. Then, to introduce as many primed indices as unprimed, we multiply by $\epsilon^{A'B'}$. Finally, to get a real-world tensor, we add the complex conjugate:

$$P^{ab} = \omega^A\omega^B\epsilon^{A'B'} + \epsilon^{AB}\bar{\omega}^{A'}\bar{\omega}^{B'} \quad .$$

Now, obviously P^{ab} is skew and simple:

$$P^{ab} = U^aK^b - K^aU^b$$

where $$K^a = \omega^A\bar{K}^{A'} + K^A\bar{\omega}^{A'} \quad ,$$

K^A being any element of $S_{(p)}$ subject to $\omega_A K^A = 1$,

whence $$\omega^A K^B - K^A\omega^B = \epsilon^{AB} \quad .$$

The vector K^a is a real, spacelike of length $\sqrt{2}$ and orthogonal to U^a:

$$\overline{K^a} = K^a \ , \ K^aK_a = -2 \ , \ K^aU_a = 0 \quad .$$

It is defined by P^{ab} up to the addition of a "real" multiple of U^a. The direction K^a therefore defines a half-plane element at P (two-dimensional) which is tangent to the null cone along the null direction of U^a. Thus ω^A defines a kind of null flag in the tangent space at p, whose flag pole is the future-pointing null vector U^a and whose flag plane is this null 2-plane through U^a.

(vi) Spinor Equivalents of Certain Vectors and Tensors:

Proposition: If ξ^a is a null vector, then the associated spinor $\xi^{BX'}$ can be written as a product of two 1-rank spinors, i.e.,

$$\xi^{BX'} = \xi^B \bar{\eta}^{X'} \quad .$$

Moreover, if ξ^a is real null, then:

$$\xi^{BX'} = \xi^B \bar{\xi}^{X'} \quad .$$

Proof:

Write:

$$\xi^a \xi_a = \xi^{BX'} \xi_{BX'} = \epsilon_{BC} \epsilon_{X'Y'} \xi^{BX'} \xi^{CY'}$$

$$= \epsilon_{12} \epsilon_{1'2'} \xi^{11'} \xi^{21'} + \text{three more}$$

$$\text{non-zero terms} = 2 \det(\xi^{AX'}) \quad .$$

Then ξ^a is null if and only if $\xi^{AX'}$, regarded as a (2 x 2) matrix, is of rank 1, i.e.,

$$\xi^a \xi_a = 0 \quad \Longleftrightarrow \quad \xi^{AX'} = \xi^A \bar{\eta}^{X'} \quad .$$

Moreover, if ξ^a is real null,

$$\xi^{AX'} = \bar{\xi}^{AX'} \quad \Longrightarrow \quad \xi^A \bar{\eta}^{X'} = \eta^A \bar{\xi}^{X'}$$

$$\Longrightarrow \quad \eta^A = \lambda \xi^A \text{ for some real } \lambda.$$

Absorbing $|\lambda|^{\frac{1}{2}}$ into ξ^A leaves, for real ξ^a:

$$\xi^a \xi_a = 0 \quad \Longleftrightarrow \quad \xi^{AX'} = \pm \xi^A \bar{\xi}^{X'} \quad \Longleftrightarrow \quad \xi^a \longleftrightarrow \pm \xi^A \bar{\xi}^{X'} \quad .$$

Thus, a real null vector determines a 1-spinor up to a factor of modulus unity. The significance of this factor is that we can define the future-pointing null vectors to be those for which this factor is +1, and the past-pointing null vectors to be those for which this factor is -1. This division of the null half-cones of $\mathcal{M}$ into two classes, "future" and "past," can be made continuously over the whole manifold and, thus, $\mathcal{M}$ is time-orientable.

Now consider a bivector (skew-symmetric 2, 2-index tensor):

$$F_{ab} = F_{[ab]} \quad .$$

The spinor equivalent $F_{AW'BX'}$ must have the same symmetry:

$$F_{AW'BX'} = -F_{BX'AW'}$$

which implies:

$$F_{AW'BX'} = \tfrac{1}{2}(F_{AW'BX'} - F_{BX'AW'})$$

$$= \tfrac{1}{2}(F_{AW'BX'} - F_{BW'AX'} + F_{BW'AX'} - F_{BX'AW'})$$

$$= \tfrac{1}{2}(\epsilon_{AB}F_{HW'X'}{}^{H} + \epsilon_{W'X'}F_{BP'A}{}^{P'}) \quad .$$

Now let:

$$\Phi_{AB} = \tfrac{1}{2}F_{BP'A}{}^{P'} \quad .$$

Then:

$$\Phi_{AB} = -\tfrac{1}{2}F^{P'}{}_{ABP'} = \tfrac{1}{2}F_{AP'B}{}^{P'} = \Phi_{BA}$$

by (2.10) and the rule for moving dummy indices.

Similarly,

$$F_{HW'X'}{}^{H} = F_{HX'W'}{}^{H} \quad .$$

In particular, if F_{ab} is real, then:

$$F_{AW'BX'} = \bar{F}_{AW'BX'} \qquad \text{and} \qquad \tfrac{1}{2}F^{H}{}_{HW'X'} = \Phi_{W'X'} \quad ,$$

so that for any real bivector:

$$F_{ab} \longleftrightarrow \epsilon_{AB}\bar{\Phi}_{W'X'} + \Phi_{AB}\epsilon_{W'X'} \quad ,$$

where $\Phi_{AB} = \tfrac{1}{2}F_{AP'B}{}^{P'}$ is a symmetric 2-spinor.

Thus a real bivector determines a symmetric 2-spinor, and conversely.

Chapter III. The Spin-Bundle

In this chapter we show how one can view spinors as a vector bundle, thus demonstrating the fact that they are, indeed, generalitions of tensors (Wells [23]). We introduce zero-rest-mass fields in section (ii) and give Maxwell's field equations in spinor form as an example. In section (iii), the Dirac approach to spinors as a Clifford algebra is used to construct spin bundles from transition functions. Also given is a necessary and sufficient condition for the existence of the spin structure. The reader may refer to Bott and Atiyah [1] and P. Gilky [6] for details.

(i) Spinors as a Vector Bundle:

Let $\mathcal{M}_n$ be a Riemannian manifold of dimension n. Then the mapping $E \xrightarrow{\pi} \mathcal{M}$, where π is a surjective differentiable map, is called a vector bundle of rank n if:

(a) for each $p \in \mathcal{M}$, $\pi^{-1}(p)$ is a linear vector space of dimension n, the fiber over P;

(b) for each $p \in \mathcal{M}$ there is a neighborhood U of P such that $\pi^{-1}(U) \cong U \times \mathbb{R}^n$.

A section of E over $\mathcal{M}$ is a mapping $S : \mathcal{M} \longrightarrow E$, such that $\pi \circ S = id_p$, i.e., to each point $P \in \mathcal{M}$, S(p) is a point in the fiber $E_p = \pi^{-1}(p)$ over p. In local coordinates, if $\{\eta_a\}$ span E_p and $S^a \in C^\infty(U,R)$, coordinates of the section, then

$$S/U = \sum \eta_a S^a \quad .$$

Now suppose M is a Riemannian manifold with a metric g given by $ds^2 = g_{ab}dx^a dx^b$; then the metric g gives an isomorphism

$$T(M) \cong_g T^*(M) \quad ,$$

because, if $v = v^a \partial/\partial x^a$, $w = w_b dx^b$, then this isomorphism is given by

$$w_b = v^a g_{ab} \quad .$$

In general, a tensor is

$$T = T^{abc\ldots}_{def} \frac{\partial}{\partial x^a} \otimes \frac{\partial}{\partial x^b} \otimes \frac{\partial}{\partial x^c} \otimes \ldots \otimes$$

$$\otimes\, dx^d \otimes dx^e \otimes dx^f \otimes \ldots \quad ,$$

which is a section of

$$T(M) \otimes T(M) \otimes \ldots \otimes T^*(M) \otimes \ldots \otimes T^*(M) \quad ,$$

and $T^{abc\ldots}_{def\ldots}$ are smooth functions on a chart (P,U).

Now spinor calculus is a generalization of tensor calculus. To see this, let $S = (\mathbb{C}^2 \ , \epsilon)$ where ϵ is a skew symmetric non-degenerate complex bilinear form and S is the basic space of spinors, from which higher order spinors are derived (just as the tangent space equipped with inner product induces all higher-order Riemannian tensors). Since ϵ is non-degenerate, we can use it to identify S with the complex linear dual S^* of S. We have

$$S \otimes_R C = S^{1,0} \otimes S^{0,1}$$

$$S^* \otimes_R C = S^*_{1,0} \otimes S^*_{0,1}$$

where $S^{1,0}$and $S^{0,1}$ are the +i and -i eigenspaces of the almost-complex structure J of S (look at chapter 1, section (i)).

We can identify $S^{1,0}$ with $S^*_{1,0}$ by means of ϵ. Note that S is complex-linearly equivalent to $S^{1,0}$ and that S is conjugate-complex-linearly equivalent to $S^{0,1}$.

Now, as defined before (Chapter 2, section (iv)), a spinor is an element of the tensor product of some combination of the above four spaces, e.g.,

$$S^{1,0} \otimes S^{0,1} \otimes S^*_{1,0} \otimes S^*_{0,1} \quad .$$

Choose a basis $\{e_A \, , \, A = 0,1\}$ for $\mathbb{C}^2$ so that

$$\epsilon = [\epsilon_{AB}] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

in these coordinates.

Examples:

(i) A spinor $\xi \in S^{1,0}$ can be written as

$$\xi = \xi^A e_A \quad , \quad \xi^A \in \mathbb{C} \quad A = 0,1 \quad .$$

(ii) $\xi \in S^{1,0} \otimes S^{0,1}$ can be represented as

$$\xi = \xi^{A\bar{B}} e_A \otimes \bar{e}_B$$

where $\bar{e}_B = e_{\bar{B}}$ is a basis for $S^{0,1}$. In complex analysis notation for $\mathbb{C}^2$, we would have

$$\{e_A\} = \{\partial/\partial z^0 \, , \, \partial/\partial z^1\} \quad ,$$

$$\{\bar{e}_A\} = \{\partial/\partial \bar{z}^0 \, , \, \partial/\partial \bar{z}^1\}$$

with the dual basis:

$$\{e^A\} = \{dz^0 \, , \, dz^1\} \quad ,$$

$$\{\overline{e^A}\} = \{\overline{dz^0} \, , \, \overline{dz^1}\} \quad .$$

(iii) A more general spinor might be

$$\xi = \xi^{A\bar{C}}_{B\bar{D}} e_A \otimes e^B \otimes \bar{e}_C \otimes \overline{e^D}$$

To raise or lower indices, we use

$$\xi_B = \xi^A \epsilon_{AB} , \ \xi^A = \epsilon^{AB} \xi_B ;$$

for instance, if

$$\xi^A = \begin{bmatrix} \xi^0 \\ \xi^1 \end{bmatrix} ,$$

then,

$$\xi^0 = \xi_1 , \ \xi^1 = -\xi_0 , \text{ i.e., } \begin{bmatrix} \xi_1 \\ -\xi_0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} = \xi^A .$$

We can consider $M_0 \times F(S)$, where $F(S)$ is the tensor algebra of spinors, and have spinors at points of Minkowski space. Moreover, there is a mapping:

$$T(M_C^A)_0 \cong S^{1,0} \otimes S^{0,1}$$

where M_C^A is affine complexified Minkowski space considered as 2 x 2 complex matrices (look at Chapter 2, section ii), given by sending

$$x^a \longrightarrow \begin{bmatrix} u & \xi \\ \bar{\xi} & v \end{bmatrix} = \begin{bmatrix} p^{00'} & p^{01'} \\ p^{10'} & p^{11'} \end{bmatrix} = x^{AA'} .$$

Here x^a is a coordinate representation of a holomorphic tangent vector at 0 in M_C^A with respect to a frame for which the metric g on Minkowski space M_0 is of the form:

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 .$$

Now Lorentz transformation on x^a induces a mapping by $SL(2,\mathbb{C})$ acting on the 2 x 2 matrix $x^{AA'}$ given by

$$[x^{AA'}] \longrightarrow s[x^{AA'}]s^* \qquad s \quad SL(2,\mathbb{C})$$

(Chapter 2, section (iv)). The action of $SL(2,\mathbb{C})$ on $x^{AA'}$ is just the action of $SL(2,\mathbb{C})$ on $S^{1,0} \otimes S^{0,1}$, induced from the action of $SL(2,\mathbb{C})$ on C^2. Since this action preserves the skew-symmetric form ϵ, it also preserves the full spinor algebra. Thus we have an embedding of tensor algebras:

$$F(M_C^A)_o \longrightarrow M_C^A \times F(S)$$

where $F(M_o)$ is the full tensor algebra of Minkowski space.

This embedding is $SL(2,\mathbb{C})$ equivariant and is induced by the mapping $\quad x^a \longrightarrow x^{AA'} \quad$, taking

$$T(M_C^A) \longrightarrow S^{1,0} \otimes S^{0,1}.$$ We then have the diagram:

$$\begin{array}{ccc} F(M_C^A)_o & \longrightarrow & S^{1,0} \otimes S^{0,1} \\ \uparrow & & \uparrow \\ T(M_0)_o & \longrightarrow & \{x^{AA'} = \overline{x^{A'A}}\} \text{ Hermitian} \end{array}$$

Therefore, we have a means of transforming {tensor algebra objects} into {spinor algebra objects}.

<u>(ii) Zero-Rest-Mass Field Equations</u>:

Using the isomorphism $\quad x^a \longrightarrow x^{AA'} \quad$, we can think of $x^{AA'}$ as coordinates in M_o. Since these are good coordinates, we may consider $\partial/\partial x^{AA'}$, and other combinations of derivatives, as a means of expressing differential equations. We want to reformulate physical fields using the spinor representation. Let $\{x^{AA'}\}$ be the spinor coordinates of $\mathbb{M}_o$, and define

$$\nabla_{AA'} = \partial/\partial x^{AA'} \ , \ \nabla^{AA'} = \partial/\partial x_{AA'} \ .$$

Definition (1):

The zero-rest-mass field equations are the spinor differential equations, satisfying

$$\nabla^{AA'} \varphi_{ABC\ldots D} = 0 \ ,$$

$$\nabla^{AA'} \psi_{A'B'C'\ldots D'} = 0 \ ,$$

where $\varphi_{ABC\ldots D}$, $\psi_{A'B'C'\ldots D'}$ are symmetric spinors with 2s indices, $s = \frac{1}{2}, 1, \frac{3}{2}, \ldots.$

For $s = \frac{1}{2}$: Solutions correspond to neutrinos; this is the Dirac-Weyl equation of an electron for mass zero.

For $s = 1$: Solutions correspond to photons; these are Maxwell's equations.

For $s = 2$: Solutions correspond to "weak gravitational fields"; these are the linearized Einstein equations for a Lorentzian metric h, where $g = \eta + \epsilon h$, g a space-time metric, h a perturbation, and η- the metric for flat space-time.

<u>Example</u>: Maxwell's equations as zero-rest-mass field equations:

Classically they take the form:

$$\frac{\partial B}{\partial t} + \text{Curl } E = 0 \quad , \quad E \equiv \text{electric field}$$

$$\frac{\partial E}{\partial t} - \text{Curl } B = j \quad , \quad B \equiv \text{magnetic field}$$

$$\text{div} B = 0 \quad , \quad j \equiv \text{current}$$

$$\text{div} E = \sigma \quad , \quad \sigma \equiv \text{charge density.}$$

These are invariant with respect to Lorentz transformations. Consider the homogeneous Maxwell's equations, where $j = \sigma = 0$.

Define a Maxwell 2-tensor or 2-form, by defining the skew-symmetric matrix

$$F_{ab} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{bmatrix}$$

and setting

$$F = F_{ab} dx^a \wedge dx^b \quad .$$

The metric $ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$ on M_o induces a Hodge * operator:

$$* : \Lambda^p T^*(M_o) \longrightarrow \Lambda^{4-p} T^*(M_o) \ ,$$

such that, if $\alpha = \alpha_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}$, then

$$(*\alpha)_{j_1,\ldots,j_{4-p}} = \pm \alpha^{i_1 \ldots i_p} \quad ,$$

where $\{i_1, \ldots, i_p, j_1, \quad, j_{4-p}\}$ is an odd or even permutation of {0, 1, 2, 3}, which determines the above sign, and

$$\alpha^{i_1 \ldots i_p} = g^{i_1 k_1} g^{i_2 k_2} \ldots g^{i_p k_p} \alpha_{k_1 \ldots k_p} ,$$

where

$$[g^{ij}] = \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

We find that $**(P\text{-form}) = (-1)^{P+1}(P\text{-form})$. So: $*^2 = -1$ when acting on 2-forms in M_o. Therefore $*$ has eigenvalues $\pm$ i in this case. Considering $\mathbb{C}$-valued 2-forms on M_o, we have

$$\Lambda^2 T^*(M_o) \otimes \mathbb{C} = \Lambda^2_+(M_o) \oplus \Lambda^2_-(M_o)$$

where Λ^2_+ and Λ^2_- denote the +i and -i eigenspaces. So, any 2-form $w \in \Lambda^2(M_o)$ has a decomposition $w = w^+ + w^-$, where

$$w^+ = \tfrac{1}{2}(w - i{*w}) \ , \ w^- = \tfrac{1}{2}(w + i{*w})$$

satisfy $*w^+ = iw$, $*w^- = -iw$. We say that w is self-dual if $w = w^+$ and anti-self-dual if $w = w^-$.

<u>Proposition (1)</u>: (1) Maxwell's homogeneous equations become

$$dF = 0 \quad , \quad d{*F} = 0 \ .$$

(2) If the Maxwell tensor is rewritten as $F = F^+ + F^-$, then Maxwell's equations become $dF^+ = dF^- = 0$. Where $d*$ is the Hodge adjoint to d and $= \pm *d*$, we know from Chapter 2, section (vi), that $F_{ab} \sim F_{AA'BB'} = \epsilon_{AB}\psi_{A'B'} + \epsilon_{A'B'}\varphi_{AB}$.

Proposition (2):

(1) F is real if, and only if, $\psi_{A'B'} = \bar{\varphi}_{AB}$.

(2) F is self-dual (i.e., *F = iF) if, and only if, $F = \epsilon_{AB}\psi_{A'B'}$.

(3) F is anti-self-dual (i.e., *F = -iF) if, and only if, $F = \epsilon_{A'B'}\varphi_{AB}$.

As we have seen, $F_{ab} \sim F_{AA'BB'}$ is split into the sum of two quantities $\epsilon_{AB}\psi_{A'B'}$ where $\psi_{A'B'}$ is a variable symmetric spinor field. The product $\epsilon_{AB}\varphi_{A'B'}$ is self-dual and corresponds to F^+. Similarly, $\epsilon_{A'B'}\varphi_{AB}$ is of the same nature. So, the 6 independent coefficients of $F = F_{ab}$ are replaced by the 6 independent spinor quantities φ_{00}, $\varphi_{10} = \varphi_{01}$, φ_{11} , $\psi_{0'0'}$, $\psi_{0'1'} = \psi_{1'0'}$ and $\psi_{1'1'}$; and the relation between $\{\varphi_{AB} , \psi_{A'B'}\}$, $\{F_{ab}\}$ and $\{E_i , B_i\}$ is a purely linear algebraic one.

Proposition (3): If $F = F_{ab}dx^a \wedge dx^b \sim \epsilon_{AB}\psi_{A'B'} + \epsilon_{A'B'}\varphi_{AB}$, then Maxwell's equations for F take the form

$$\nabla^{AA'}\varphi_{AB} = 0 \quad , \quad \nabla^{AA'}\psi_{A'B'} = 0 .$$

(iii) Construction of a Spin-Bundle:

There is a general approach to spinors where these two-component spinors are just a special case. This approach, which is global in nature, is due to Dirac. To outline this representation, we start with C_m, a real Clifford algebra, on $e_1, \ldots, e_m$ $(m = 2n)$. The group spin $(m) \subset C_m$ is an even products of unit vectors in $\mathbb{R}^n$ which acts on C_m by left multiplication:

$$a \longmapsto \begin{array}{c} w \longmapsto aw \\ \text{a linear transformation} \\ \text{of } C_m \longrightarrow C_m \end{array} .$$

The reader is referred to Bott and Atiyah [1] for further details.

Now replace C_m by $C_m \otimes \mathbb{C}$ and consider the linear transformations:

$T_1, T_2, \ldots, T_n$ of $C_m \otimes \mathbb{C}$ into itself, where T_k = right multiplication by $e_{2k-1}e_{2k}$. Then one shows that

$$T_k^2 = -I \qquad \forall\, k$$

$$T_rT_s = T_sT_r \qquad \forall\, r,s$$

T_k commutes with the action of spin (m) for each k.

Each T_k has its $\pm i$ eigenspace. With $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$.

Define $\{V_\epsilon = w \in C_m \otimes \mathbb{C} \mid T_k w = \epsilon_k w_k\}$, k = 1, 2,..., n to be the associated simultaneous eigenspace. Write

$$C_m \otimes \mathbb{C} = \oplus \sum_{\epsilon} V_\epsilon \qquad 2^n \text{ possible } \epsilon\text{'s} \quad ,$$

where each V_ϵ is invariant under the action of spin (m). Moreover, one can show that all the $V_{\epsilon's}$ have the same dimension, and the representation of spin (m) in any two V's are equivalent. Therefore, each V_ϵ has dimension 2^n [ ].

Now let $P^\pm : C_m \otimes \mathbb{C} \longrightarrow C_m^\pm \otimes \mathbb{C}$ be, respectively, the projection onto the "even" and "odd" components. Since $P^\pm$ commutes with the action of spin (m) and with all the $T_{k's}$, each V_ϵ is sent into itself by $P^\pm$. Putting

$$V_\epsilon^\pm = P^\pm V_\epsilon$$

$$= V_\epsilon \cap (C_m^\pm \otimes \mathbb{C})$$

we obtain $V_\epsilon = V_\epsilon^+ \oplus V_\epsilon^-$ for each ϵ where the subspaces $V^\pm$ are invariant under the action of spin (m) on V_ϵ^+. One can show that the representations of spin (m) on V_ϵ^+ and V_ϵ^- are irreducible and inequivalent [1].

Definition (2):

The half-spin representation $\Delta^\pm$ of spin (m) are the representations by left multiplication in the spaces V_ϵ^+ and V_ϵ^-.

Now let $\mathbb{M}$ be an oriented 2n dimensional Riemannian manifold and $\mathbb{F}$ the oriented orthonormal cotangent frame bundle. The group SO(m) acts on $\mathbb{F}$ from the right by the standard multiplication rule. This action is free, principal and has local sections. Then for each $p \in \mathbb{M}$

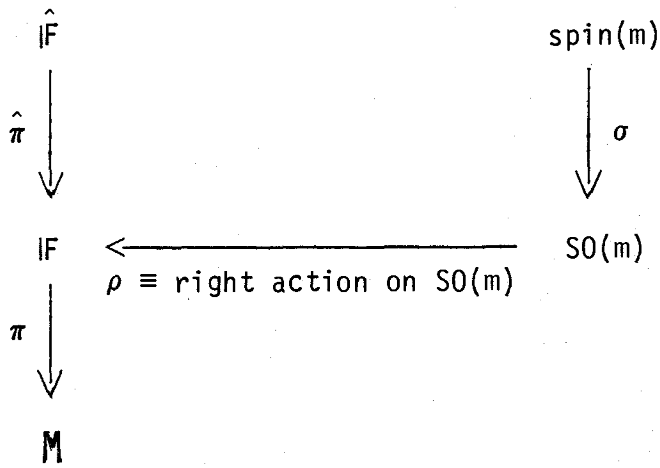
$$\mathbb{F} \times \mathbb{R}^m / SO(m) \cong T_p\mathbb{M}$$

the cotangent bundle of $\mathbb{M}$ at p [11].

Now since the map σ : spin (m) $\longrightarrow$ SO(m) is a double covering map [1], we can define a spin structure $\hat{\mathbb{F}}$ to be a topological space equipped with a 2-to-1 covering map:

$\hat{\pi} : \hat{\mathbb{F}} \longrightarrow \mathbb{F}$ such that for each $p \in \mathbb{M}$ $\hat{\pi}^{-1}(p)$ is the non-trivial covering of $\mathbb{F}_p$. Using simple connectivity of spin (m) and the fact that $\hat{\pi} : \hat{\mathbb{F}} \longrightarrow \mathbb{F}$ is a covering map, one can show the following:

<u>Proposition (4)</u>: There exists a unique right action of spin (m) on $\hat{\mathbb{F}}$ which commutes with σ, $\hat{\pi}$ and the right action of SO(m) on $\mathbb{F}$.



Suppose now that $\{U_\alpha\}$ is a covering of M by coordinates neighborhoods and f_α is a local orthonormal frame over U_α. We can associate a cycle with values in SO(m): For U_α and U_β let:

$$g_{\alpha\beta} : U_\alpha \wedge U_\beta \longrightarrow SO(m) \qquad \text{be defined}$$

by $\qquad f_\alpha g_{\alpha\beta} = f_\beta \qquad$. This is a cocycle because obviously

$$g_{\alpha\beta} g_{\beta\nu} = g_{\alpha\nu} \qquad \text{on } U_\alpha \wedge U_\beta \wedge U_\nu .$$

From the above, it is clear that the existence of $\hat{\mathbb{F}}$ is equivalent to the existence of a lifting of this cocycle, through σ: spin (m) $\longrightarrow$ SO(m) to a cocycle with values in spin(m). We will give a theorem that guarantees the existence of a lifting to a cocycle in spin(m) and, hence, the spin structure. First, let $\{g_{\alpha\beta}\}$ be the transition functions of the vector bundle over M satisfying the cocycle condition: $g_{\alpha\beta} g_{\beta\nu} g_{\nu\alpha} = I$.

Choose a lift $\hat{g}_{\alpha\beta}$ so that:

$\sigma(\hat{g}_{\alpha\beta}) = g_{\alpha\beta}$. Let $E_{\alpha\beta\nu} = \hat{g}_{\alpha\beta}\hat{g}_{\beta\nu}\hat{g}_{\nu\alpha}$.

Then $\sigma(E_{\alpha\beta\nu}) = I$.

So, if there exists a spin structure, then we can choose a lift so that $E_{\alpha\beta\nu} = \pm I$. Using Čech cohomology, one can show that $\delta\in = I$ (Gilkey [6]). Any other E arising from another lift changes by a coboundary. Since E is a 2-Čech chain,

$E \in H^2(M,Z_2)$. Therefore, if there exists a spin structure, then we can choose lifts $\hat{g}_{\alpha\beta}$ satisfying the cocycle condition. This implies that $E_{\alpha\beta\nu} = I$ and, hence, E is trivial in cohomology. The converse of this argument is obviously true, and therefore we have proved the following:

<u>Theorem (3)</u>: $E \in H^2(X_1 Z_2)$ (E is called the 2nd Steffel-Whitney class.)

E is trivial $\Longleftrightarrow$ there exists spin structure.

Chapter IV. Twistor Theory

Twistors were introduced by R. O. Wells in 1966 to describe the geometry of Minkowski space where the ordinary space-time concepts can be translated into twistor terms. The primary geometrical object is not a point in Minkowski space but a null straight line (a twistor) or, more generally, a twisting congruence of null lines. It turns out that twistor algebra has the same type of universality in relation to the conformal group, that the two-component spinor algebra has in relation to the Lorentz group [13]. Thus, twistor theory is applicable to quantum field theory and free fields of zero-rest-mass. It is also used to formulate other fields such as Yang Mill's fields [10].

In section (ii) of this chapter we describe a twistor geometrically as a null line in Minkowski space. In section (v) we give a necessary and sufficient condition for two null lines to intersect in twistor terms. This incidence will be very useful in algebraic manipulation of twistors. Section (iv) is devoted to two alternative pictures of visualizing the twistors, namely null lines in Minkowski space or points of complex projective three space. We also get other geometrical correspondences between the two pictures (Penrose correspondence). For details the reader may see [15] and [16].

(i) Spinors Review:

Since the discussion here depends essentially on the use of spinors, a very brief review of the ideas required and already introduced in Chapter 2, will be given

(a) the translation from world tensors to spinors is achieved using a quantity: $\sigma_j^{JJ'}$ (a hermitian (2x2) matrix for each j)

and its inverse $\sigma^j_{JJ'}$ subject to

$$\sigma_j^{JJ'} \sigma_k^{KK'} \varepsilon_{JK} \varepsilon_{J'K'} = g_{jk}$$

$$\sigma_j^{JJ'} \sigma^i_{JJ'} = \delta^i_j$$

The ε_{JK}, are skew-symmetric Levi-Civita symbols and are used for raising and lowering spinor indices:

e.g., $\xi^A \varepsilon_{AB} = \xi_B$, $\varepsilon^{AB}\xi_B = \xi^A$.

(b) Any tensor (e.g., X_k^{ij}) has a spinor translation which is written using the same base symbol, but with each tensor index replaced by the corresponding pair of spinor indices, e.g.,

$$X_k^{ij} \longleftrightarrow X_{KK'}^{II'JJ'} = X_k^{ij} \sigma_i^{II'} \sigma_j^{JJ'} \sigma^k_{KK'}$$

(c) Under Complex Conjugation, the roles of primed and unprimed indices are interchangeable, so that reality of tensors is expressed as Hermiticity of Spinors.

(d) Since our Minkowski space M will be flat, we can choose $\sigma_j^{JJ'}$ constant and $= 2^{-1/2}$ times the unit matrix and Pauli matrices.

Then

$$(x^1, x^2, x^3, x^4) \longleftrightarrow \begin{bmatrix} x^{11'} & x^{12'} \\ x^{21'} & x^{22'} \end{bmatrix}$$

$$= 2^{-1/2} \begin{bmatrix} x^1 + x^2 & x^3 + i\, x^4 \\ x^3 - i\, x^4 & x^1 - x^2 \end{bmatrix} .$$

(e) The spinor translation of a complex null vector of α^j (i-c $\alpha^j \alpha_j = 0$) has the form $\alpha^j \longleftrightarrow \alpha^{JJ'} = \beta^J \gamma^{J'}$. If α^j is real and future pointing, then we can take $\gamma^{J'}$ to be the complex conjugate of β^J i-e $\alpha^j \longleftrightarrow \beta^J \bar{\beta}^{J'}$.

(ii) <u>Geometric Description of a Twistor</u>:

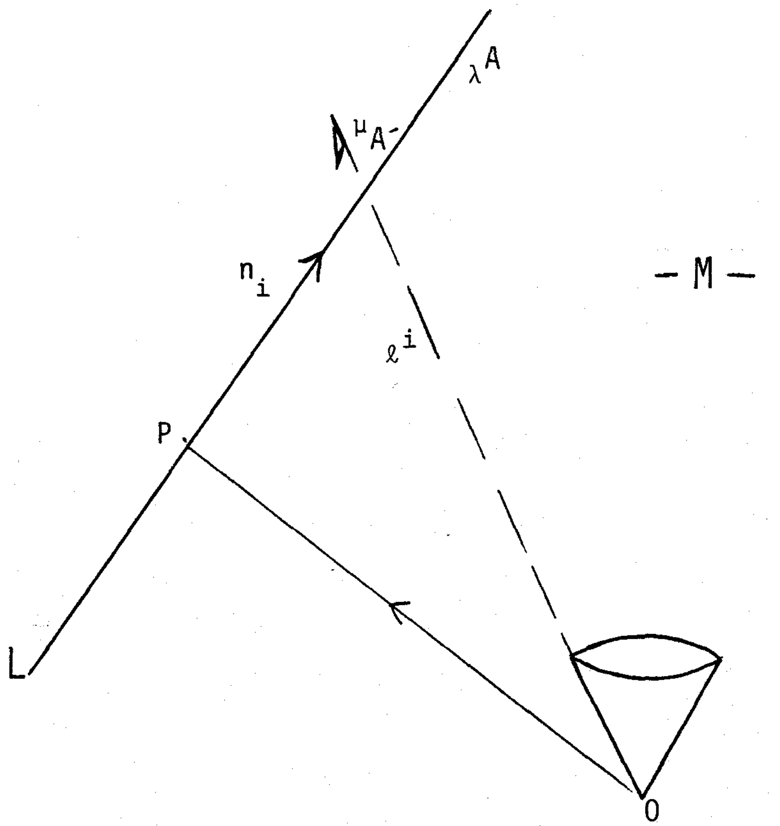
Consider a null straight line L in Minkowski space M. Choose a set (X^i) of Minkowski coordinates with origin 0 and let ℓ^i be the position vector of some point P on L (see figure next page). To assign a set of coordinates for L , we may select a vector n^i along L and the moment

$$m^{ij} = \ell^i n^j - \ell^j n^i \tag{4.1}$$

of the vector n^i (acting at P) about 0 .

Then the ratios of the 10 quantities (n^i, m^{ij}) will uniquely define L . In addition to the requirement here that n^i be null

$$n^i n_i = 0 . \tag{4.2}$$



There are the consistency relations for (1)

$$\varepsilon_{ijkl}\; m^{ij}\; n^k = 0 \tag{4.3}$$

Equation (3) represents just three independent conditions which together with (2) reduce the set of nine ratios in (n^i, m^{ij}) to just <u>five</u> independent real numbers. (This is consistent with the fact that null lines in M form an ∞^5 system, for choosing p as the intersection of L with a fixed space like hyperplane, we have ∞^3 choices for P and ∞^2 for the null direction at P.)

Let us now represent in spinor terms the quantities n^i, m^{ij} which define the null line L. We have

$$n^j \longleftrightarrow n^{JJ} = \lambda^J \bar{\lambda}^{J'}$$

from (1) $m^{jk} \longleftrightarrow \ell^{JJ'}\lambda^{K}\bar{\lambda}^{K'} - \lambda^{J}\bar{\lambda}^{J'}\ell^{KK'}$

$$= i\,\varepsilon^{JK}\,\mu^{J'}\,\bar{\lambda}^{K'} - i\,\mu^{(J}\lambda^{K)}\,\varepsilon^{J'K'}$$

where $$\mu_{A'} = -\,i\,\lambda^{A}\,\ell_{AA'} \qquad (4.4)$$

Thus λ^A and $\mu_{A'}$ together determine ℓ^j and m^{jk} . We may think of λ^A as defining the direction of L and $\mu_{A'}$ as effectively giving us the moment of λ^A (acting at P) about O.

(iii) <u>Some Remarks</u>:

(i) From (4) if λ^A is multiplied by any complex factor, then L is unchanged if $\mu_{A'}$ is multiplied by the same factor.

(ii) (4) implies also that $\mu_{A'}$ is independent of the choice of p on L since if $\ell_{AA'} \longrightarrow \ell_{AA'} + a\lambda_A\bar{\lambda}_{A'}$ then $\mu_{A'}$ is unchanged since $\lambda^A\lambda_A = 0$.

(iii) A particular choice of p which is of interest is the intersection of L with the null cone of 0. From (2.4) and the fact that ℓ^j is real null it follows that

$$\ell_{AA'} = i\,(\lambda^{B}\bar{\mu}_{B})^{-1}\,\bar{\mu}_{A}\,\mu_{A'} \quad .$$

Thus the null direction defined by $\mu_{A'}$ is that of the null line through 0 which meets L (see the figure).

(iv) The exceptional case $\lambda^{B}\bar{\mu}_{B} = 0$ corresponds to L lying in a null hyperplane thrugh 0 . This follows from (4) since n^j would be necessarily orthogonal to any choice of ℓ_j . In this case

λ^A and $\bar{\mu}^A$ are proportional, so that null direction of $\bar{\mu}_A$ is that of L.

The null line L can now be assigned for coordinates, the three complex ratios of four complex quantities

$$L^0 = \lambda^0,\ L^1 = \lambda^1,\ L^2 = \mu_{0'},\ L^3 = \mu_{1'}$$

which we write as $\quad L^\alpha = (\lambda^A, \mu_{A'}) \qquad \alpha = 0,1,2,3 \qquad$ (4.5)

There are six real parameters, so we expect to find one real relation connecting λ^A and $\mu_{A'}$. This is obtained from (4) since the reality of ℓ^i implies $\ell_{AA'}$ is Hermitian, whence

$$(\lambda^A \neq 0) \qquad \mathrm{Re}\ (\lambda^A\ \bar{\mu}_A) = 0\ . \tag{4.6}$$

Condition (6) is also sufficient to ensure the existence of null line L associated with λ^A and $\mu_{A'}$. For if $\lambda^A\ \bar{\mu}$ is purely imaginary then $\ell_{AA'} = (i\ \lambda^B\ \bar{\mu}_B)^{-1}\ \bar{\mu}_A \mu_{A'}$ gives a point through which we choose L with direction λ^A.

(iv) Definitions:

A null twistor of valence $[^1_0]$ is a quantity with components (L^α) given as in (5) with Condition (6).

If $\mathrm{Re}(\lambda^A\ \bar{\mu}_A) > 0$ then L^α is called a right-handed twistor (valence $[^1_0]$) and if $\mathrm{Re}\ (\lambda^A\ \bar{\mu}_A) < 0$ it is called a left-handed twistor.

(v) Incidence of Null Lines in Twistor Terms:

Algebraic rules for manipulation of twistors will have as their basis the idea of incidence between null lines.

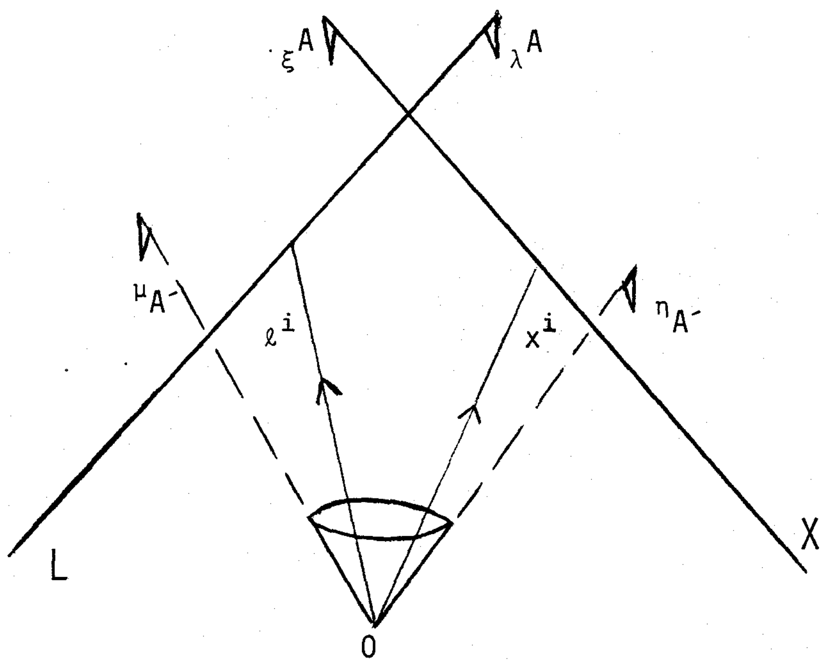
Start with two null lines

$$L^{\alpha} = (\lambda^{A}, \mu_{A'}) \quad , \quad X^{\alpha} = (\xi^{A}, \eta_{A'}) \qquad \alpha = 0,1,2,3$$

we have by (4)

$$\mu_{A'} = -i\ \lambda^{A} \ell_{AA'} \ , \ \eta_{A'} = -i\ \xi^{A}\ x_{AA'} \tag{4.7}$$

where ℓ_i and x_i are the position vectors of the lines L and X respectively. Suppose now that X and L do intersect, then $\ell^i = x^i$ for the coordinate vector of the intersection point.



Incidence of two null lines in Minkowski space.

Then by Equation (7):

$$\xi^A \bar{\mu}_A = i\xi^A \ell_{AA} \bar{\lambda}^{A'} = i\xi^A x_{AA'} \bar{\lambda}^{A'} = -\eta_{A'} \bar{\lambda}^{A'} \qquad (4.8)$$

Define the complex conjugate of a twistor L^α to be $\bar{L}_\alpha$ (valence $[{}^0_1]$) where $(\bar{L}_\alpha) = (\bar{\mu}_A , \bar{\lambda}^{A'})$ (4.9)

or in component form:

$$\bar{L}_0 = \bar{L}^2 , \bar{L}_1 = \bar{L}^3 , \bar{L}_2 = \bar{L}^0 \; \bar{L}_3 = \bar{L}^1$$

Then Equation (4.8) tells us that a necessary condition for L and X to meet is

$$X^\alpha \bar{L}_\alpha = 0 \qquad (4.10)$$

since

$$X^\alpha \bar{L}_\alpha = \xi^A \bar{\mu}_A + \eta_{A'} \bar{\lambda}^{A'} \qquad (4.11)$$

Note that condition (4.6) for L^α to represent a real null line is

$$L^\alpha \bar{L}_\alpha = 0 \qquad (4.12)$$

So L intersects itself!

Condition (4.10) is also sufficient for L and X to intersect. Suppose X and L are not parallel so λ^A and ξ^A are not proportional then $\xi^A \lambda_A \neq 0$. Construct the complex vector:

$$P_j \longleftrightarrow P_{JJ'} = ({}^i/_{\xi^A \lambda_{A'}}) (\lambda_J \eta_{J'} - \xi_J \mu_{J'})$$

we observe that $\mu_{A'} = i\lambda^A P_{AA'}$, $\eta = -i\ \xi^A P_{AA'}$. Thus, when P_j is real, we can satisfy (4.7) by putting $\ell_j = P_j = x_j$ whence X and L must intersect.

Now P_j is real $\longleftrightarrow$ $P_{JJ'}$ is Hermitian. Since λ^A and ξ^A are not proportional, we can test Hermiticity of $P_{JJ'}$ by taking components with respect to λ^A , ξ^A :

$$\lambda^A \bar{\lambda}^{A'} (P_{AA'} - \bar{P}_{A'A}) = i\ \mu_{A'}\ \bar{\lambda}^{A'} + i\ \lambda^A\ \bar{\mu}_A = i\ L^\alpha \bar{L}_\alpha$$

$$\xi^A \xi^{A'} (P_{AA'} - \bar{P}_{A'A}) = i\ \eta_{A'}\ \xi^{A'} + i\ \xi^A\ \bar{\eta}_A = i\ X^\alpha X_\alpha\ ,$$

$$\xi^A \bar{\lambda}^{A'} (P_{AA'} - \bar{P}_{A'A}) = i\ \eta_{A'}\ \bar{\lambda}^{A'} + i\ \xi^A\ \bar{\mu}_A = i\ X^\alpha \bar{L}_\alpha$$

Therefore if $X^\alpha \bar{L}_\alpha = 0$ then P_j is real.

(vi) <u>Two Alternative Pictures of Visualizing Twistors</u>

The members of the ∞^6 system of twistors introduced in the last section can be given complex projective coordinates

$L^\alpha = (L^0, L^1, L^2, L^3)$. That is, it is only the three complex ratios $L^0 : L^1 : L^2 : L^3$ which are significant. This ∞^6 system we may think of as constituting a three-dimensional complex projective space C. The points of C are just the "complexified" null lines (and null lines) of M . In fact we have two alternative pictures of any given situation. For example, we may think of an object L with projective coordinates (L) either as, say, a "complexified" null line of M (M-picture) or simply as a point in a certain projected three-space C (C-picture). In order that the two pictures be completely equivalent, we need to be able to interpret in C, the condition of reality of a null line in M, of incidence between null lines in M and, finally, of points in M. In effect this requires that the conjugation relation $L^\alpha \longleftrightarrow \overline{L}_\alpha$ should have a meaning with regard to the C picture. We have seen that a twistor L^α (valence $[{}^1_0]$) refers to a point in C, a twistor R_α (valence $[{}^0_1]$) therefore refers to the dual concept of a plane R in C , namely the plane of all points X for which $X^\alpha R_\alpha = 0$. The conjugation relation $L^\alpha \longleftrightarrow \overline{L}_\alpha$ therefore describes a point $\longleftrightarrow$ plane correspondence in C ; which we may refer to as a Hermitian Correlation of signature (++--). The signature here refers to the Hermitian form

$$X^\alpha \overline{X} = X^0 \overline{X}^2 + X^1 \overline{X}^2 + X^2 \overline{X}^0 + X^3 \overline{X}^1 \quad .$$

This Hermitian Correlation is an intrinsic part of the geometric structure of C .

The real null lines in M are the points of the five-real-dimensional subset N (top- $S^3 \times S^2$) of C defined by $X^\alpha \overline{X}_\alpha = 0$. Thus N is a hypyersurface if C is regarded as real 6-dimensional manifold (but not in the sense of the complex structure of C).

We refer to the subset of C for which $X^\alpha \overline{X}_\alpha > 0$ holds as C^+ , and the part for which $X^\alpha \overline{X}_\alpha < 0$ as $\overline{C}$. If L is any point of C , we may regard the plane $\overline{L}$ as the polar plane of L , with respect to N , since plarizing $X^\alpha \overline{X}_\alpha$ with L^α yields $X^\alpha \overline{L}_\alpha = 0$, which is the equation of the plane $\overline{L}$.

The Robinson Congruence associated with L is the intersection (top S^3) of the plane $\overline{L}$ with N . When L lies on N , the plane $\overline{L}$ can be thought of as a complex tangent plane to N at L . This is just the case when L lies on its polar plane.

To represent a point of M in terms of the C-picture, we do this using incidence properties of null lines in M . Any point in M can be uniquely represented by an ∞^2 system of null lines in M , namely the generators of the null cone of P . Let K and L be two null lines in M through P . The generators of the null cone of P are then the null lines common to both $\overline{K}$ and $\overline{L}$ (i.e., the generators must meet both K and L). In the C-picture this is an ∞^2 system of lines on N which lie on the intersection of $\overline{K}$ and $\overline{L}$. This intersection is a complex projective line in C .

Conversely, any line P in C which lies entirely on N , represents some point P in M . To see this consider the C-picture and let the line P lie entirely on N . Let K and L be two points on P . Then we have $K^\alpha \overline{K}_\alpha = 0$, $L^\alpha \overline{L}_\alpha = 0$ and, more generally, $(K^\alpha + \beta L^\alpha)(\overline{K} + \overline{\beta}\, \overline{L}_\alpha) = 0$.

Hence $L^\alpha \overline{K}_\alpha = 0$. So the intersection of L^α and K_α will be the point P . Note that, in the C-picture, that a point L lies on a line P , both L and P lying on N is interpreted in the M-picture as the condition that the null line L passes through the point P .

Thus we get the following correspondence between the M-picture and N:

(i) There is a one-to-one correspondence between the null lines in M and the points in N.

(ii) There is a one-to-one relation between the points in M and the complex lines in N.

(iii) The condition for a point to lie on a null line in M is interpreted in N, as the condition for the corresponding line to pass through the corresponding point.

Chapter V. Complex Manifold Techniques, Penrose Correspondence and Penrose Transform

In Chapter 4 we introduced a twistor as a null line or a congruence of null lines in Minkowski space M. We also gave a one-to-one correspondence between lines in Minkowski space and points of the projective complex three space. In this chapter we will consider the basic geometric properties of the above correspondence--Penrose correspondence--using complex manifolds techniques. The geometry of Penrose correspondence will then be used to describe the Penrose transform which transfers cohomology on subsets of P_3 to spinor fields on subsets of M_c.

(i) Penrose Correspondence:

Let M be compactified Minkowski space. $M = M_0 \cup$ (light cone at ∞) where M_0 is the flat Minkowski space (Penrose [13]). Let M_c be the complexification of M which is of the complex dimension. We had, in Chapter 3, the Penrose correspondence between M and N $P(T) \equiv$ projective twistor space; namely,

$$\{\text{Complex lines in N}\} \quad \overset{|\text{---}|}{\longleftrightarrow} \quad \{\text{Points in M}\}$$

$$\{\text{Points in N}\} \quad \overset{|\text{---}|}{\longleftrightarrow} \quad \{\text{Null lines in M}\}$$

The above correspondence can be extended to M_c:

$$\begin{array}{ccc} \{\text{Complex lines in P(T)}\} & \longleftrightarrow & M_c \\ \uparrow & & \uparrow \\ \{\text{Complex lines in N}\} & \longleftrightarrow & M \end{array}$$

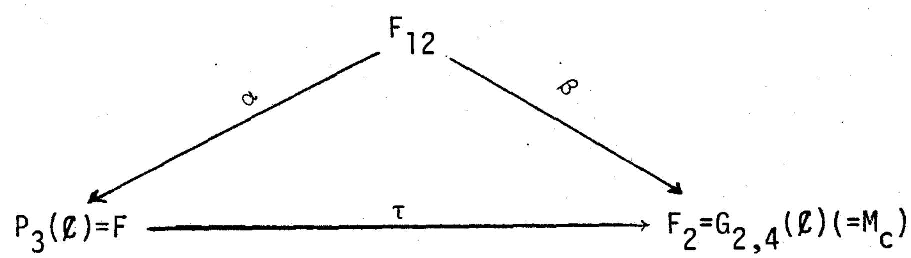
This correspondence is used to transfer problems in mathematical physics in Minkowski space into problems of several complex variables on subsets of $P_3(\mathbb{C})$. Now M_c turns out to be the Grassmanian manifold of two-dimnal complex planes in $\mathbb{C}^4, G_{24}$ which is clearly equivalent to the set of all complex lines in $P_3(\mathbb{C})$ [23].

According to R. O. Wells [23] the above remarkable correspondence of Penrose between M_c and subsets of $P_3(\mathbb{C})$ can be described in terms of a basic double fibration. Briefly, the idea is this: consider $\mathbb{C}^4$. Define the flag manifold $F_{d_1 . . . d_r}$ where $o < d_1 < \ldots < d_r < 4$ as follows:

$$F_{d_1 \ldots d_r} = \{(L_1, L_2, \ldots L_r): L_1 \subset L_2 . . . \subset L_r \subset \ldots \mathbb{C}^4$$

is a nested sequence of subspaces of $\mathbb{C}^4$ with dimension $_cL_j = d_j \quad j = 1, \ldots, r\}$

If $r = 1$, $d_1 = 1$ then $F_1 = P_3$, and if $r = 1$, $d_1 = 2$ then $F_2 = G_{2,4}(\mathbb{C})$. We then consider with respect to $\mathbb{C}^4$ the three flag manifolds F_{12}, F_1 and F_2 and get the following diagram:



We want this diagram to describe Penrose correspondence. So define the analogue of Penrose correspondence τ (which is again called Penrose correspondence) between F_1 and F_2 by $\tau(P) = \beta\,\alpha^{-1}(p)$ and $\tau^{-1}(p) = \alpha\,\beta^{-1}(p)$. One can then prove the following

Proposition: (1) $\tau(p)$ is a two-complex-dimnal projective plane $\cong P_2(C)$ embedded in F_2 .

(2) $\tau^{-1}(p)$ is a one-complex dimensional projective line embedded in F_1 .

<u>Proof</u>: (1) By definition:

$$\alpha^{-1}(p) = \{\text{flags } (L_1^o, L_2) : L_1^o \subset L_2, L_1^o \text{ fixed, } L_2 \text{ variable}\} ,$$

therefore $\beta\,\alpha^{-1}(p) = \{L_2 \subset \mathbb{C}^4 : L_2 \supset L_1^o \text{ -fixed}\}$

i.e., $\beta\,\alpha^{-1}(p)$ is the set of all two-dimensional subspaces of $\mathbb{C}^4$ which contain a fixed one-dimensional subspace L_1^o . This is simply an embedding of $P_2(C)$ in F_2 . Since, if we fix one vector e_1 , and let e_2 vary in a three-dimensional subspace $e_1^{\perp}$ perpendicular to e_1 with respect to some metric on $\mathbb{C}^4$, then the span of $\{e_1, e_2\}$ will span all subspaces $L_2 \supset L_1^o$. But the set of all such e_2's span the set of all complex lines in $e_1^{\perp}$, and hence is isomorphic to $P_2(\mathbb{C})$.

(2) By the same reasoning as (1):

$$\alpha\,\beta^{-1}(p) = \{L_1 \subset \mathbb{C}^4 : L_1 \subset L_2^o , L_2^o \text{ -fixed}\}$$

But L_2^o is two-complex dimensional, and hence $\alpha\,\beta^{-1}(p) \cong P_1(\mathbb{C})$.

(ii) Twistor Structure:

Let Φ be a non-degenerate Hermitian bilinear form on $\mathbb{C}^4$ of signature 0 , i.e., {+, +, -, -, } .

In an appropriate coordinate system the matrix for Φ can be represented as

$$\Phi = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix} .$$

As a quadratic form we can write

$$\Phi(Z) = Z^0\overline{Z^2} + Z^1\overline{Z^3} + Z^2\overline{Z^0} + Z^3\overline{Z^1} .$$

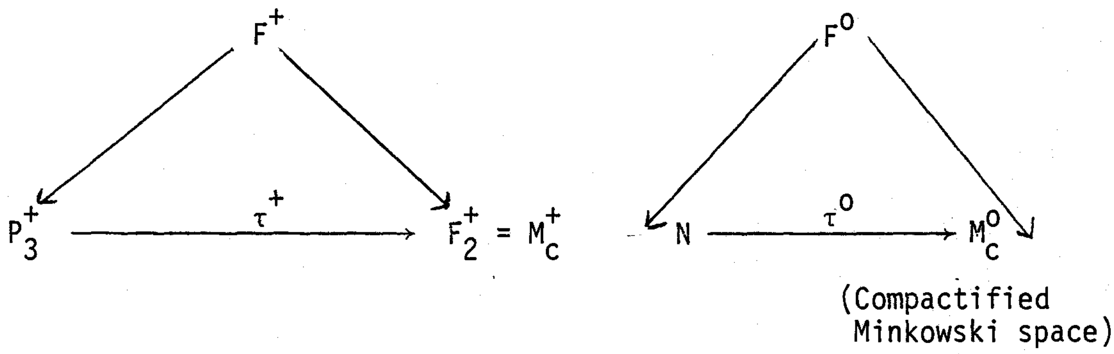
We can define the space of twistors T to be $T = \{\mathbb{C}^4$ with the Hermition form $\Phi\}$. This space is a representation space for SU(2,2) which is a 4-1 covering of conformal group acting on M_c [15].

Now let $T^+ = \{Z \varepsilon T : \Phi(Z) > 0\}$ Positive twistors

$T^0 = \{Z \varepsilon T : \Phi(Z) = 0\}$ Null twistors

$T^- = \{Z \varepsilon T : \Phi(Z) < 0\}$ Negative twistors

In the projectivized twistor space $P(T) = P_3(\mathbb{C})$ we have the corresponding portions P_3^+, P_3^0 and P_3^- . Let $N = P_3^0$. So as we have seen in Chapter 4 N is a real five-dimensional hypersurface in $P(T)$ which divides $P(T)$ into two complex-analytically equivalent parts P_3^+ and P_3^-. We then got the induced correspondence



The basic geometric properties of the above spaces are given by the following proposition (see R. O. Wells [23]).

<u>Proposition</u>: (1) P_3^+ contains a four-complex dimensional family of projective complex lines parameterized by M_c^+ .

(2) M_c^+ is biholomorphically equivalent to the domain of a 2x2 complex matrices whose Hermitian imaginary part is positive definite.

(3) Let M_o be the Hermitian 2x2 matrices, then M_o is a boundary component of M_c^+ .

(4) M is a compact four-dimensional real-analytic submanifold of M_c which is diffeomorphic to $S^1 x S^3$.

(5) N is a compact five-dimensional real-analytic hypersurface in P_3 diffeomorphic to $S^2 x S^3$.

(6) F^+ is biholomorphic to $P_1 x M_c^+$.

Now considering the correspondence τ^+ and τ^o , one can prove the following

<u>Proposition</u>: (1) $\tau^+(p)$ is the intersection of an affine complex two-plane with M_c^+ .

(2) $\tau^{o}(p)$ is a circle S' embedded in M .

(3) $(\tau^{+})^{-1}(p)$ and $(\tau^{o})^{-1}(p)$ are complex projective lines embedded in P_3^+ and N respectively.

Suppose we have twistor homogeneous coordinates $Z^{\alpha} = (Z^0, Z^1, Z^2, Z^3)$ [Chapter 4]. In these coordinates $\Phi(Z^{\alpha}) = Z^0\overline{Z}^2 + Z^1\overline{Z}^3 + Z^3\overline{Z}^1$. We define the dual variables with respect to the Hermitian from Φ by

$$\overline{Z}_0 = \overline{Z}^2,\ \overline{Z}_1 = \overline{Z}^3,\ \overline{Z}_2 = \overline{Z}^0,\ \overline{Z}_3 = \overline{Z}^1$$

and thus $\Phi(Z^{\alpha}) = Z^{\alpha}\overline{Z}_{\alpha}$.

Again, as in Chapter 4, with $\omega^A = (\omega^0, \omega^1) = (Z^0, Z^1)$

$$\Pi_{A'} = (\Pi_{0'}, \Pi_{1'}) = (Z^2, Z^3)$$

$Z^{\alpha} = (Z^0, Z^1, Z^2, Z^3)$ becomes the pair of spinors $(\omega^A, \Pi_{A'})$ which corresponds to $\mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2$. The dual coordinates become:

$$\overline{Z}_{\alpha} = (\overline{\Pi}_A, \overline{\omega}^{A'}) \quad \text{and}$$

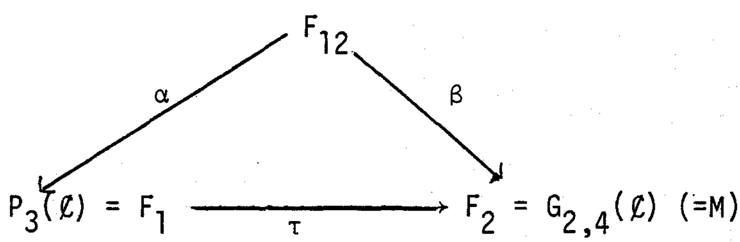
$$\Phi(Z^{\alpha}) = \omega^A\overline{\Pi}_A + \Pi_{A'}\overline{\omega}^{A'}$$

If a particle of zero-test-mass moves along a light ray then $\Pi^A\overline{\Pi}^{A'}$ corresponds to momentum and $\omega^A\overline{\omega}^{A'}$ corresponds to angular momentum. The form $\Phi(Z^{\alpha})$ corresponds to the spin (twist) of the particle. The path of motion of the particle is given by $\omega([\tau^A, \Pi_{A'}])$. The detail of this interpretation can be found in Penrose-MacCallum [19].

(iii) Penrose Transform:

We will see in this section how holomorphic massless fields of positive helicity on M^+ could be described in terms of cohomology classes on $P_3^+(\mathbb{C})$. This result is due to R. Penrose [17], and is extended to negative helicity and to weak (hyperfunction) solutions on the real compactified Minkowski space M by R. O. Wells [25].

We start with Penrose correspondence given by the diagram:



In fact the geometry of the above diagram is used extensively to construct certain fundamental sheaves on P_3 , F and M_c [21].

In general, let X be a complex manifold and let O_X be the sheaf of local holomorphic functions on X . Suppose $V \longrightarrow X$ in a holomorphic vector bundle. Let $O_X(V)$ be the sheaf of local holomorphic sections of V . Consider a holomorphic mapping $X \xrightarrow{f} Y$ of two complex manifolds. If $G = O_Y(V)$ is a locally free sheaf on Y , then we define the pullback sheaf f^*G to be $O_X(f^*(V))$, the sheaf of holomorphic sections of the pullback bundle.

If K is any sheaf of abelian groups on X , then we define a sequence of sheaves $\{f_*^n K\}$ on Y called the direct image sheaves under f . The nth direct image sheaf of K under f , denoted by $f_*^n K$ is the sheaf generated by the presheaf

$$U \longrightarrow H^n(f^{-1}(U), K) \qquad \text{for } U \text{ open in } Y$$

The stalk of $f^n_* K$ at $p \varepsilon Y$ is the direct limit

$$(f^n_* K)_p = \lim_{p \varepsilon U} H^n(f^{-1}(U), K)$$

This is essentially the cohomology along the fiber $f^{-1}(p)$.

With this preparation Penrose transform can be briefly described: For $n \varepsilon Z$, let $H^{-n-2} \longrightarrow P_3(\mathbb{C})$ be the hyperplane section bundle of P_3 raised to the power $-n-2$, i.e., the local sections of H^{-n-2} are the homogeneous functions in the homogeneous coordinates of P_3 of degree $-n-2$. For $n \varepsilon Z$ define:

$$S_{P_3}(n) = O_{P_3}(H^{-n-2}) \qquad \text{the sheaf of holomorphic sections of } H^{-n-2}$$

$$S_F(n) = \alpha^* O_{P_3}(H^{-n-2})$$

$$S_M(n) = \beta_* \alpha^* O_{P_3}(H^{-n-2})$$

These are the basic fundamental sheaves on P_3, F and M_c
So the sheaf $S_F(n)$ and $S_M(n)$ are the natural pullback and push-forward of the sheaf $S_{P_3}(n)$. It can be shown that if $n \geq 0$

$$S_M(n) = O_M(V_n) \quad \text{where } \operatorname{rank}_{\mathbb{C}} V_n = \operatorname{rank} \odot^n(\mathbb{C}^2)$$

$\odot^n$ denotes the nth symmetric tensor product.

Moreover, sections of $S_M(n)$ can be identified with holomorphic spinors of primed type on open subsets of M , i.e., of the form

$A'B'\ldots D'$ [4]. To transform cohomology on subsets of P_3 to spinor fields on subsets of M_C we need differential operators defined by the geometry of the Penrose correspondence. Let $T(F)$ be the tangent bundle to F and let $T_\alpha(F) \subset T(F)$ be the subbundle of vectors tangent to the fibres of the fibration α. This induces a cononical surjection of the dual bundles

$$T^*(F) \xrightarrow{\Pi_\alpha} T^*_\alpha(F) \quad .$$

and a differential form $d_\alpha = \Pi_\alpha \circ d$. This acts on a differential form of any degree r on F, i.e.,

$$E^r(F) \xrightarrow{d} E^r(F, T^*(F))$$

exterior differentiation of scalar r-forms to $T^*(F)$-valued r forms.

Thus we obtain by composition with Π_α a differential operator

$$E^r(F) \xrightarrow{d_\alpha} E^r(F, T^*_\alpha(F))$$

d_α corresponds to "Differentiation along the fibers of α". Now use $T^*_\alpha = T^*_\alpha(F)$ to introduce two new sheaves on F and M, namely

$$S^\alpha_F(n) = O_F(\alpha^* H^{-n-2} \times T^*_\alpha)$$

$$S^\alpha_M(n) = \beta^1_* O_F(\alpha^* H^{-n-2} \times T^*_\alpha)$$

differential form valued sections

which is the extension of the basic sheaves on F and M by the cotangent bundle along the fibers of α. Let us represent the sheaf cohomology in terms of differential forms via the Dolbeault isomorphism (cohomology classes being represented by $\bar{\partial}$-closed (0,1) forms as usual).

Then the differential operator d_α extends to differential forms with coefficients in $\alpha^* H^{-n-2}$ since the transition functions for $\alpha^* H^{-n-2}$ can be taken to be constant along the fibers of α and thus would be annihilated by d_α . Therefore we get a mapping

$$H^1(F_+, S_F(n)) \xrightarrow{d_\alpha} H^1(F_+, S_F^\alpha(n))$$

we can define a vector bundle $V_n \longrightarrow M_{+c}$ by defining the fibers of V_n to be

$$V_{n,x} = H^1(\beta^{-1}(x), S_F(n)) .$$

One can then use the theory of direct image sheaves to show that V_n defined in this manner is indeed a holomorphic vector bundle.

There is a natural mapping:

$$I : H^1(F_+, S_F(n)) \longrightarrow H^0(M_+, S_M(n))$$

obtained by restricing a $\bar{\partial}$-closed differential form representing a cohomology class in $H^1(F_+, S_F(n))$ to a differential form on the submanifold $\beta^{-1}(x)$ ($\cong P_1(\mathbb{C})$) which defines a cohomology class in $H^1(\beta^{-1}(x), S_F(n))$, i.e., a point in the vector space V_n . Similarly there is a mapping

$$I_\alpha : H^1(F_+, S_F^\alpha(n)) \longrightarrow H^0(M_+, S_M^\alpha(w)) \quad ;$$

using all the above data we get the following diagram:

$$
\begin{array}{ccccc}
 & & H^1(F_+, S_F(n)) & \xrightarrow{d_\alpha} & H^1(F_+, S_F^\alpha(n)) \\
 & \nearrow^{\alpha^*} & \cong \big\downarrow I & & \cong \big\downarrow I_\alpha \\
H^1(P_+, S_P(n)) & & & & \\
 & \searrow_{p} & & & \\
 & & H^0(M_+, S_M(n)) & \xrightarrow{\nabla_\alpha} & H^0(M_+, S_M^\alpha(n))
\end{array}
$$

Theorem: Suppose $n > 0$, then

(1) I and I_α are isomorphisms and ∇_α is well defined.

(2) $V_{n,x} \cong \otimes^n_{sym} C^2$

(3) The induced differential operator ∇_α is the zero-set-mass operator of spin $s = n/2$.

(4) Ker $d_\alpha = \text{Im}\alpha^*$.

As a corollary of this theorem we get

Corollary (Penrose transform)

If $S > 0$, then

$$H^1(P_+, S_p(n)) \xrightarrow[\cong]{p} \{\text{Ker } \nabla_\alpha : H^0(M_+, S_M(n)) \longrightarrow H^0(M, S_M^\alpha(n))\}$$

$\cong$ {self dual holomorphic solutions of the zero-rest-mass equation of spin s on M_{C+}}

All the detail of the proof of the above theorem is found in the elegant paper [4] by Eastwood, Penrose and Wells. These people study the pullback of the local data in P_3 involving inverse image sheaves and the relative deRham sequence. Then they solve the problem of integration over the fibers for the mapping β . This involves direct image sheaves and a fundamental spectral sequence.

Part (1) of the theorem follows from the Leray spectral sequence for direct image sheaves and appropriate standard cohomology vanishing theorems in several complex variables along either the fibers of β or on M_c . Part (2) is a computation using the theory of compact complex manifolds. Part (3) follows from an appropriate choice of basis for the vector spaces involved. The last part is much deeper and involves solving $\bar{\partial}\mu = f$ problems locally along the fibers of α , i.e., the inhomogeneous Cauchy-Riemann equation for differential forms.

Chapter VI. The Volume of a Tube in Complex Projective Space

In this chapter, the formula of a tube in complex projective space $P_n(\mathbb{C})$, as given by Flaherty, is derived [5]. The Study-Fubini metric, already described in Chapter 1, is used. The geometry of complex projective space, as already discussed in Chapter 1, is also used here. First, we obtain the volume element for the tube by computing the Käehler form for $P_n(\mathbb{C})$ in the Study-Fubini metric. Then the average value of monomials of certain form on the unit sphere in $\mathbb{C}^k$ is computed and used to define scalar invariants S_γ on the submanifold $P \subset P_n(\mathbb{C})$. This S_γ is found to be the Hermitian curvature tensor. Thus, the formula of the volume of the tube is obtained as a polynomial in the radius of the tube with coefficients depending only on the curvature of P. At the end of the chapter, Chern forms are introduced and used to express the formula only in terms of those forms and the Käehler form. The reader may also refer to F. Flaherty [5] and A. Gray [7].

Let us first review the geometry of complex projective space discussed in Chapter 1. If $\mathbb{C}^{n+1}$ is the space of (n + 1)-tiples of complex numbers and $e_0, \ldots, e_n$ a frame field on $\mathbb{C}^{n+1}$; then:

$$de_A = \Sigma e_B \omega_A^B \quad (0 \leqslant B \leqslant n) \ , \ d\omega_B^A = \Sigma \omega_C^A \wedge \omega_B^C \ .$$

Let $P_n(\mathbb{C})$ be the complex projective space and $\mathbb{C}^{n+1} - \{0\} \xrightarrow{\pi} P_n(\mathbb{C})$ the canonical fibering. Then,

$$d\pi = \Sigma e_i \omega_o^i + \Sigma \bar{e}_i \bar{\omega}_o^i \quad (1 \leqslant i \leqslant n) \ .$$

Suppose $e_0,\ldots, e_n$ unitary. Then the Hermitian structure on $P_n(\mathbb{C})$ is given by:

$$ds^2 = 2\Sigma\omega_0^i\overline{\omega}_0^i \qquad (1 \leqslant i \leqslant n)$$

and in homogeneous coordinates:

$$ds^2 = (2/(z,z)^2)((z,z)(dz,dz) - (z,dz)(dz,z)) \quad .$$

The Käehlerian connection on $P_n(\mathbb{C})$ is given by:

$$\pi_j^i = \omega_j^i - \delta_j^i\omega_o^o \qquad (1 \leqslant i \quad \leqslant n) \ ,$$

and the curvature form Ω_j^i is given by:

$$\Omega_j^i = \theta^i \wedge \bar{\theta}^j + \delta_j^i\Sigma_k\theta^k \wedge \bar{\theta}^k$$

for $\theta^1, \theta^2,\ldots,\theta^n$ local coframe field.

Also we have: $$d\theta^i + \Sigma\pi_j^i \wedge \theta^j = 0 \qquad (1 \leqslant j \leqslant n) \quad .$$

Let P be a holomorphic submanifold of $P_n(\mathbb{C})$ of dimension m; then we may choose the θ's so $\theta^1, \theta^2,\ldots, \theta^m$ is a unitary coframe field on P. Moreover, (π_β^α), $1 \leqslant \alpha, \beta \leqslant m$ is the connection form matrix of the Käehlerian connection on P.

Since $\theta^r = 0$ on P for $r = m + 1,\ldots, n$, it follows from the above equation that:

$$\Sigma_\alpha\pi_r^\alpha \wedge \theta^\alpha = 0 \qquad (1 \leqslant \alpha \leqslant m) \quad .$$

As a result: $$\pi_\alpha^r = \Sigma_\beta\bar{S}_{\alpha\beta}^r\theta^\beta \qquad (1 \leqslant \beta \leqslant m)$$

with $$S_{\alpha\beta}^r = S_{\beta\alpha}^r \quad .$$

A tube of radius σ around a submanifold is defined to be the image of the normal σ-disc bundle under the exponential map.

Now let P be a submanifold of $P_n(\mathbb{C})$. A typical point normal to P will be of the form $\exp_w(t,\bar{t})$ where $(t,\bar{t})$ is a complex normal vector. Consider the canonical fibering:

$$\mathbb{C}^{n+1} - \{0\} \xrightarrow{\pi} P_n(\mathbb{C}) \quad .$$

Since this is a Hermitian submersion, we get the following diagram which commutes

$$\begin{array}{ccc} T_p(\mathbb{C}^{n+1}) & \xrightarrow{\pi_*} & T_w P_n(\mathbb{C}) \\ \Big\downarrow \exp_p & & \Big\downarrow \exp_w \\ \mathbb{C}^{n+1} - \{0\} & \xrightarrow[\pi]{} & P_n \mathbb{C} \end{array}$$

where $\pi(p) = w$.

Let $e_1, e_2, \ldots, e_n$ be a unitary frame field on $\mathbb{C}^{n+1}$ adapted to $P_n(\mathbb{C})$, i.e., e_0 is tangent to the fiber π. Thus, $\pi_* e_1, \ldots, \pi_* e_m$ are tangent to P (m = dimension of P) and $\pi e_0 = w$, where w is a local submanifold map of P. It follows from the above diagram that

$$\pi(z) = \exp_w(t, \bar{t})$$

where

$$z = e_0 t_0 + \Sigma e_r t_r / \sqrt{2} + \Sigma \bar{e}_r \bar{t}_r / \sqrt{2} \quad .$$

Since the frame is adapted, the derivative of z is

$$dz = e_0 dt_0 + t_0 de_0 + d(\Sigma e_r t_r / \sqrt{2} + \Sigma \bar{e}_r \bar{t}_r / \sqrt{2})$$

$$= e_0 (dt_0 + t_0 \omega^0_0) + t_0 (\bar{e}_0 \bar{\omega}^0_0 + \Sigma e_\alpha \theta^\alpha + \Sigma \bar{e}_\alpha \bar{\theta}^\alpha) + d(\Sigma e_r t_r / \sqrt{2} + \Sigma \bar{e}_r \bar{t}_r / \sqrt{2}) \quad .$$

Now we may assume that π^r_s and $\bar{\pi}^r_s$ vanish locally. Then it follows that

$$dz = e_0(dt_0 + t_0\omega^0_0) + t_0(\bar{e}_0\bar{\omega}^0_0 + \Sigma e_\alpha\theta^\alpha + \Sigma\bar{e}_\alpha\bar{\theta}^\alpha)$$

$$+\Sigma_{\alpha,r} e_\alpha\pi^\alpha_r t_r/\sqrt{2} + \Sigma_{\alpha,r}\bar{e}_\alpha\bar{\pi}^\alpha_r\bar{t}_r/\sqrt{2} + \Sigma_r e_r dt_r/\sqrt{2} + \Sigma_r\bar{e}_r d\bar{t}_r/\sqrt{2} \quad .$$

The Käehler form φ for $P_n(\mathbb{C})$ in the Study-Fubini metric is given by:

$$\varphi = (\sqrt{-1}/(z,z)^2)((z,z)(dz,dz) - (dz,z) \wedge (z,dz)) \quad .$$

We want to compute φ on the tube, so we normalize the t's (homogeneous coordinates) and substitute to get:

$$\varphi = (\sqrt{-1})t_0^2(\Sigma_\alpha(\theta^\alpha + \Sigma_r\pi^\alpha_r t_r/t_0\sqrt{2}) \wedge (\bar{\theta}^\alpha + \Sigma_r\bar{\pi}^\alpha_r\bar{t}_r/t_0\sqrt{2})$$

$$+(\frac{1}{2t_0^2})(\Sigma_r dt_r \wedge d\bar{t}_r - (\Sigma_r\bar{t}_r dt_r)(\Sigma_r t_r d\bar{t}_r))) \quad .$$

Hence the volume element of the tube is:

$$\varphi^n/n! = (\sqrt{-1})^n t_0^{2n}\det(\delta_{\alpha\beta} - \tfrac{1}{2}\Sigma_\gamma(\Sigma_r t_r S^r_{\alpha\gamma})(\Sigma_r\bar{t}_r\bar{S}_{\gamma\beta}))$$

$$\wedge_\alpha \theta^\alpha \wedge \bar{\theta}^\alpha \wedge_r \tfrac{1}{2}dt_r \wedge d\bar{t}_r \quad .$$

Then we have the following:

<u>Theorem (6.1)</u>: If P is a holomorphic submanifold of $P^n(\mathbb{C})$, D is a compact subdomain of P with smooth boundary. Then the volume of the tube $\nu_\sigma(D)$ is given by:

$$\int_D (\frac{1}{(1+t\bar{t})^{n+1}}) \wedge_\alpha(\sqrt{-1})\,\theta^\alpha \wedge \bar{\theta}^\alpha$$

$$\cdot \int_D \det(\delta_{\alpha\beta} - \tfrac{1}{2}(\Sigma_\gamma(\Sigma_r t_r S^r_{\alpha\gamma})(\Sigma_r\bar{t}_r\bar{S}^r_{\gamma\beta}))) \wedge_r \tfrac{1}{2}(\sqrt{-1})dt_r \wedge d\bar{t}_r \quad .$$

Proof: Consider the substitution $\tau_r = t_r/t_0$ in the fiber coordinates and use the fact that

$$t_0^2(1 + t\bar{t}) = 1 \qquad \text{where} \qquad t\bar{t} = \sum_r t_r \bar{t}_r \quad .$$

Let $k = n - m$ be the co-dimension of P, and let the indices r, s,... vary in the co-dimension range that is $1 \leqslant r, s, \ldots, \leqslant k$. Set:

$$\varphi(t,\bar{t}) = \varphi(t_1, \ldots, t_k, \bar{t}_1, \ldots, \bar{t}_k)$$

and denote the average of $\varphi(t,\bar{t})$ on the unit sphere $\Sigma\xi_r\bar{\xi}_r = 1$ by $<\varphi(t,\bar{t})>$.

Now, since the average value of a monomial

$$t_1^{\xi_1} \ldots t_k^{\xi_k} \bar{t}_k^{\xi_1'} \ldots \bar{t}_k^{\xi_k'}$$

plus its conjugate vanishes on S^{2k-1} unless $\xi_r = \xi_r'$ for all r, one need only to compute the average of monomials of the form $t^{\xi_1} \ldots t^{\xi_k} \bar{t}^{\xi_1} \ldots \bar{t}^{\xi_k}$.

Let $t_r = \rho\eta_r$ where $\Sigma\eta_r\eta_r = 1$ and let $t^\xi = t_1^{\xi_1} \ldots t_k^{\xi_k}$, $|\xi| = \xi_1 + \xi_2 + \ldots + \xi_k$ and $\xi! = \xi_1! \ldots \xi_k!$, the usual multi-index notation for $\xi = (\xi_1, \ldots, \xi_k)$.

Consider
$$\int_{C^k} \bar{e}^{\rho^2} |t_1|^{2\xi_1} \ldots |t_k|^{2\xi_k} \bigwedge_r \tfrac{1}{2}(\sqrt{-1})dt_r \wedge d\bar{t}_r \quad . \tag{6.1}$$

Using the polar coordinates introduced above and Fubini's theorem, the above integral may be written as

$$\int_0^\infty \bar{e}^{\rho^2} \rho^{2|\xi|+2k-1} d\rho \int_{S^{2k-1}} \eta^\xi \bar{\eta}^\xi dv_k$$

where dv_k is the volume element of S^{2k-1} in $\mathbb{C}^k$.

Applying Fubini's theorem directly to (6.1) yields:

$$\prod_{1\leqslant r\leqslant k} \int \bar{e}^{t_r \bar{t}_r} (t_r \bar{t}_r)^{\xi_r} \tfrac{1}{2}(\sqrt{-1})dt_r \wedge d\bar{t}_r \ .$$

Using polar coordinates in the $(t_r,\bar{t}_r)$ plane, this

integral is equal to $$\prod_r \int \bar{e}^{\rho^2} \rho^{2\xi_r+1} d\rho \wedge dv_1 \ .$$

From the definition of gamma function, this last

integral may be written as $$\prod_r (\tfrac{1}{2})(2\pi)\Gamma(\xi_r + 1) \ .$$

In summary, we have $$\int_0^\infty \bar{e}^{\rho^2} \rho^{2|\xi|+2k-1} d\rho \int_{S^{2k-1}} \eta^\xi \bar{\eta}^\xi dv_k$$

$$= \pi^k \prod_{1\leqslant r\leqslant k} \xi_r ! \qquad = \pi^k \xi ! \ .$$

Solving for the integral over S^{2k-1}, one finds that

$$\tfrac{1}{2} \int_{S^{2k-1}} \eta^\xi \bar{\eta}^\xi dv_k = \frac{\pi^k \xi !}{2\int_0^\infty \bar{e}^{\rho^2} \rho^{2|\xi|+2k-1} d\rho} = \frac{\pi^k \xi !}{\Gamma(|\xi| + k)}$$

If $\xi_1 = 0 = \ldots = \xi_k$ then $c_k = \pi^k / \tfrac{1}{2}\Gamma(k)$.

For the average value of $t_r \bar{t}_r$ on S^{2k-1}, we obtain:

$$<t^\xi \bar{t}^\xi> = \frac{\int \eta^\xi \bar{\eta}^\xi dv_k}{\int_{S^{2k-1}} dv_k} = \frac{\pi^k \xi !}{\Gamma(|\xi|+k)} \cdot \frac{\Gamma(k)}{\pi^k} = \frac{\xi !}{k(k+1)\ldots(k+|\xi|-1)} \ . \qquad (6.2)$$

Now we are ready to prove the following theorem

from F. Flaherty.

Theorem (6.2) (Flaherty): If P is a holomorphic submanifold of $P_n(\mathbb{C})$ and D is a compact subdomain of P with smooth boundary, then the volume of the tube $\nu_\sigma(D)$, where σ is the radius of the tube around D, is given by the formula:

$$\mathrm{vol}(\nu_\sigma(D)) = c_k \sum_{0\leq\gamma\leq m} K_\gamma J_\gamma(a) \quad ,$$

where K_γ are given by integrals depending only on the curvature of P, and

$$J_\gamma(a) = \int_0^a (\sin b)^{2\gamma+2k-1}(\cos b)^{2m-2\gamma+1} db \quad ,$$

where $\quad \tan a = \sigma \quad .$

Proof: Consider $\quad \det(\delta_{\alpha\beta}+\tfrac{1}{2}\sum_\gamma(-\sum_\gamma t_r S^r_{\alpha\gamma})(\sum_r \bar{t}_r \bar{S}^r_{\gamma\beta})) \quad .$

This determinant may be expanded in the form $1 + f_2 + \ldots + f_{2m}$

where $\quad f_{2\gamma}(t,\bar{t}) = \Sigma_{\xi+\xi'=2\gamma} A_{\xi,\xi'} t^\xi \bar{t}^{\xi'} \quad .$

Define the scalar invariants S_γ on P by:

$$S_\gamma = k(k+1)\ldots(k+\gamma-1) \langle f_{2\gamma}(t,\bar{t})\rangle \quad .$$

Then it is easy to see that $\quad S_\gamma = \Sigma_{\xi+\xi'=2\gamma} \xi!\, A_{\xi,\xi'} \quad$, by (6.2).

Hence: $$\int_{t\bar{t}\leq\sigma^2} \frac{f_{2\gamma}(t,\bar{t})}{(1+t\bar{t})^{n+1}} \wedge_r \frac{\sqrt{-1}}{2} dt_r \wedge d\bar{t}_r = \frac{c_k S_\gamma}{k(k+1)\ldots(k+\gamma-1)} \cdot \int_0^a \frac{\rho^{2\gamma+2k-1}}{(1+\rho^2)^{n+1}} d\rho \quad .$$

Substitute $\tan b = \rho$ and the integral becomes

$$\frac{c_k S_\gamma}{k(k+1)\ldots(k+\ -1)}\int_0^a (\sin b)^{2\gamma+2k-1}(\cos b)^{2m-2\gamma+1}db$$

where $\sigma = \tan a$.

Let
$$J_\gamma(a) = \frac{\int_0^a (\sin b)^{2\gamma+2k-1}(\cos b)^{2m-2\gamma+1}db}{k(k+1)\ldots(k+\gamma-1)} .$$

Therefore the volume of the tube is:

$$\mathrm{vol}(\nu_\sigma(D)) = c_k \sum_{0\leq\gamma\leq m} K_\gamma J_\gamma(a)$$

where $K_\gamma = \int_D S_\gamma \wedge_\alpha (\sqrt{-1})\ \theta^\alpha \wedge \bar{\theta}^\alpha$.

Let us now determine the nature of the scalar functions S_γ. First, S_γ is a polynomial in $S^r_{\alpha\beta}$, $\bar{S}^r_{\alpha\beta}$ with the property that S_γ is invariant under the unitary groups U(m) , U(k) in the sense that if $S^r_{\alpha\beta}$ is transformed into $\Sigma_p u_{rp} S^p_{\alpha\beta}$ where $u_{rp} \in U(m)$, or if $S^r_{\gamma\beta}$ is transformed into $\Sigma_\gamma u_{\alpha\gamma} S^r_{\gamma\beta}$ where $u_{\alpha\gamma} \in U(m)$, S_γ does not change. In fact, $f_{2\gamma}$ is a sum of principal minors of order γ from the matrix

$$\Sigma_\epsilon(\Sigma_r t_r S^r_{\alpha\epsilon})(\Sigma_r \bar{t}_r \bar{S}^r_{\epsilon\beta})$$

and, thus, S_γ is a sum of averages of principal minors from the same matrix. By Weyl's theory of vector invariants [26], the S_γ are polynomials in the Hermitian forms $\Sigma_r S^r_{\alpha\epsilon} S^r_{\beta\gamma}$.

Thus,
$$\langle \det(-\tfrac{1}{2}\Sigma_\epsilon(\Sigma_r t_r S^r_{\alpha\epsilon})(\Sigma_r \bar{t}_r \bar{S}^r_{\epsilon\beta}))\rangle$$

is a constant multiple of

$$\sum \delta\begin{pmatrix}\beta_1 \cdots \beta_\gamma \\ \alpha_1 \cdots \alpha_\gamma\end{pmatrix}\begin{pmatrix}\mu_1 \cdots \mu_\gamma \\ \lambda_1 \cdots \lambda_\gamma\end{pmatrix} S_{\alpha_1\lambda_1\beta_1\mu_1} \cdots S_{\alpha_\gamma\lambda_\gamma\beta_\gamma\mu_\gamma}$$

where $S_{\alpha\lambda\beta\mu} = \Sigma_r S^r_{\alpha\mu}\bar{S}^r_{\lambda\beta}$ and $\delta\begin{pmatrix}\beta_1 \cdots \beta_\gamma \\ \alpha_1 \cdots \alpha_\gamma\end{pmatrix}$ is the generalized Kronecker symbol : equal to the sign of permutation if $\beta_1, \ldots, \beta_\gamma$ is a rearrangement of $\alpha_1, \ldots, \alpha_\gamma$ or, otherwise, equal to zero. The summation extends over all γ-tiples selected from 1, 2,..., m.

Thus the S_γ are local representations of globally defined scalar invariants on P since $S_{\alpha\lambda\beta\mu}$ is easily seen to be the Hermitian curvature tensor.

We can determine the constant in question by specializing the formula to a simple geometric figure of co-dimension 1 and with $S_{\alpha\beta} = \delta_{\alpha\beta}$. As a result

$$<\det(-\tfrac{1}{2}\Sigma_\epsilon(\Sigma t_r S^r_{\alpha\epsilon})(\bar{t}_r \bar{S}^r_{\epsilon\beta}))> = \frac{1}{2^\gamma \gamma!\, k(k+1)\ldots(k+\gamma-1)}$$

$$\times \sum \delta(\ )\delta(\ ) S_{\alpha_1\lambda_1\beta_1\mu_1} \cdots S_{\alpha_\gamma\lambda_\gamma\beta_\gamma\mu_\gamma} .$$

Using this last expression, we can rewrite the formula of the tube in $P_n(\mathbb{C})$ as:

$$\mathrm{vol}(\nu_\sigma(D)) = c_k \sum_{0 \leqslant \gamma \leqslant m} \frac{1}{\gamma!} W_\gamma J_\gamma(a) \quad , \quad (6.3)$$

where $W_\gamma = \underline{\mathrm{cte}}. \int_P \Sigma\delta(\ )\delta(\ ) S_{\alpha_1\lambda_1\beta_1\mu_1} \cdots S_{\alpha_\gamma\lambda_\gamma\beta_\gamma\mu_\gamma} \bigwedge_\alpha (\theta^\alpha \wedge \bar{\theta}^\alpha)$.

Now set $\det.(\lambda I + \frac{i}{2\pi}\Omega) = \sum_{k=0}^{n} \lambda^{n-k} c_k(\Omega)$. This defines the basic Chern forms $c_k(\Omega)$, which are given explicitly by:

$$c_k(\Omega) = \frac{1}{k!}\left(\frac{\sqrt{-1}}{2\pi}\right)^k \sum \delta\begin{pmatrix}\alpha_1 \cdots \alpha_k \\ \beta_1 \cdots \beta_k\end{pmatrix} \Omega^{\alpha_1}_{\beta_1} \wedge \ldots \Omega^{\alpha_k}_{\beta_k} .$$

Using $\Omega^{\alpha}_{\beta} = \sum_{\gamma,\delta} S_{\alpha\bar{\beta}\gamma\bar{\delta}}\, \theta^{\alpha} \wedge \bar{\theta}^{\alpha}$ and $\varphi = \sqrt{-1}\sum_{\alpha} \theta^{\alpha} \wedge \bar{\theta}^{\alpha}$

we deduce that
$$P_k \bigwedge_{\alpha} \sqrt{-1}\,(\theta^{\alpha} \wedge \bar{\theta}^{\alpha}) = c_k \wedge \varphi^{n-k} . \quad (6.4)$$

where $P_k = \underline{\text{cte}}. \sum \delta(\ )\delta(\ ) S_{\alpha_1\lambda_1\beta_1\mu_1} \cdots S_{\alpha_k\lambda_k\beta_k\mu_k}$.

Using (6.4) in the formula of the tube (6.3), we get the formula expressed in terms of the Chern forms and Käehler form:

$$\mathrm{vol}(\nu_p(\sigma)) = \sum_{d=0}^{m} R_d(\lambda,\varphi,c_1,\ldots c_d)(\sin\sqrt{\lambda}\sigma)^{2(n-d)} , \quad (6.5)$$

where $R_d = \underline{\text{cte}}.\sum_{a=0}^{m-d} c_a \wedge \varphi^{m-a}$, $\varphi \equiv$ Kaehler form ,

and $c_1,\ldots c_m$ are the Chern forms. The reader may also refer to A. Gray [7].

Chapter VII. Zero-Rest-Mass Fields from the Formula of a Tube in Complex Projective Three-Space

In this chapter we show that the formulae of the volumes of the tubes around $P_1(\mathbb{C})$ and $P_2(\mathbb{C})$ in twistor space represent zero-rest-mass fields of spin $\frac{3}{2}$ and 2, respectively. Thus they give rise to coupled neutrino (spin $\frac{1}{2}$) fields and a weak Einstein's gravitational field respectively.

First, we show that the first Chern forms can be viewed as classes of the first cohomology with coefficients in the hyperplane section bundle of $P_n(\mathbb{C})$. Thus they can be viewed as twistor functions [17]. The second part of this chapter is devoted to the goal of this paper, as stated above. We use the Penrose correspondence to integrate along the fibers of the projection β mentioned in Chapter (5), and then we use spinor calculus to apply the covariant derivative operator to the integral. The Penrose transform is also used in rewriting Chern classes of the hyperplane section bundle H as spinors in twistor space.

We need the following theorem:

Theorem (7.1): The first Chern classes of the hyperplane section bundle of $P_n(\mathbb{C})$ can be viewed as elements of the first cohomology group with coefficients in the bundle.

Proof: Let $\mathcal{O}$ be the sheaf of holomorphic functions and $\mathcal{O}^*$ the sheaf of non-vanishing holomorphic functions on P_n. Consider the sequence of sheaves

$$0 \longrightarrow Z \xrightarrow{\ i\ } \mathcal{O} \xrightarrow{\ exp.\ } \mathcal{O}^* \longrightarrow 0$$

where i is the inclusion map and exp : $\mathcal{O} \longrightarrow \mathcal{O}^*$ is defined by:

$$\exp_U(f)(z) = \exp\,(2\pi i f(z)) \quad .$$

We claim that this sequence is exact. To show this for some small simply-connected neighborhood U of $x \in P_n$, and for some representative $g \in \mathcal{O}^*(U)$ of germs g_x at x , choose

$f_x = \left(\left(\frac{1}{2\pi i}\right)\log g\right)_x$ for some branch of the logarithm function. We have: $\exp_x(f_x) = g_x$.

Suppose that $\exp_x(f_x)$ = the identity element in the abelian group $\mathcal{O}^*$ under multiplication.

This then implies that: $\exp 2\pi i f(z) = 1 \qquad z \in U$ for any $f \in \mathcal{O}(U)$ which is a representative of the germ f_x on a neighborhood U of x. Therefore, f is a constant integer on U, so that $\mathrm{Ker}(\exp_x) = Z$. Hence the sequence is exact.

Now consider the induced cohomology sequence:

$$H^1(P_n,\mathcal{O}) \longrightarrow H^1(P_n,\mathcal{O}^*) \longrightarrow H^2(P_n,Z) \longrightarrow H^2(P_n,\mathcal{O})$$

$H^1(P_n,\mathcal{O}) \cong H^{0,1}(P_n)=0$ (Dolbeault cohomology)

$H^2(P_n,Z) \cong Z$

$H^2(P_n,\mathcal{O}) \cong H^{0,2}(P_n)=0$ (Dolbeault cohomology)

Therefore we obtain: $0 \longrightarrow H^1(P_n\mathbb{C}, \mathcal{O}^*) \longrightarrow Z \longrightarrow 0$.

Now it is a well-known fact in twistor theory [17] that a twistor function can be viewed as a co-cycle defining an element of the above cohomology group $H^1(P_3, \mathcal{O}^*)$. Thus we can view the first Chern classes being defined on the hyperplane section bundle as effectively giving twistor functions.

We have seen in Chapter 6 that the formula of the volume of the tube around a submanifold P of dimension m in P_3 is given by:

$$\mathrm{vol}(\nu_P(\sigma)) = \sum_{d=0}^{m} c_d(\lambda, F, \gamma_1, \ldots \gamma_d)(\sin\sqrt{\lambda}\sigma)^{2(n-d)}$$

where $$c_d(\lambda, F, \gamma_1, \ldots \gamma_d) = \underline{\mathrm{cte}}.\sum_{a=0}^{m-d} \gamma_a \wedge F^{m-a}$$

(cte. notation follows that of P. Griffiths in [8]),

$F \equiv$ Käehler form,

$\gamma_1, \gamma_2, \ldots, \gamma_m$ are the Chern forms,

and $$\gamma_a \wedge F^{m-a} = \int_P \gamma_a \wedge F^{m-a} .$$

Suppose that $P \equiv P_1(\mathbb{C})$ is a submanifold in twistor space of dimension 1. Then:

$$\mathrm{vol}(\nu_{P_1}(\sigma)) = \sum_{d=0}^{1} (\sin\sqrt{\lambda}\sigma)^{2(n-d)} c_d .$$

Now fix σ and write:

$$\mathrm{vol}(\nu_{P_1}(\sigma)) = \sum_{d=0}^{1} \underline{\mathrm{cte}}.c_d = a_0 c_0 + a_1 c_1 ,$$

where cte. here depends on σ .

$$\text{But} \qquad c_0 = \underline{\text{cte}}.\sum_{a=0}^{1}\int \gamma_a \wedge F^{1-a} = \underline{\text{cte}}.\int_{P_1} \gamma_0 \wedge F^1 + \gamma_1 \wedge F^0$$

$$\text{and} \qquad c_1 = \underline{\text{cte}}.\sum_{a=0}^{0}\int \gamma_a \wedge F^{1-a} = \underline{\text{cte}}.\int_{P_1} \gamma_0 \wedge F^1$$

$$\therefore \qquad \text{vol}(\nu_{P_1}(\sigma)) = \int b_0\gamma_0 \wedge F + b_1\gamma_1 \wedge F^0 = \int_{P_1} b_0 F + b_1\gamma_1 \quad . \qquad (7.1)$$

Since $\gamma_0 = 1$

b_0 , b_1 are constants depending on σ .

Similar calculations lead to the volume of the tube around $P_2 \subset T$. It is easy to verify that, in this case:

$$\text{vol}(\nu_{P_2}(\sigma)) = \int_{P_2} d_0\gamma_0 \wedge F^2 + d_1\gamma_1 \wedge F^1 + d_2\gamma_2 \wedge F^0 \qquad (7.2)$$

where d_0 , d_1 , d_2 are constants depending on σ .

Consider first equation (7.1):

$$\text{vol}(\nu_{P_1}) = \int_{P_1} b_0 F + b_1\gamma_1 \quad .$$

Following the elements of the Penrose transform in Chapter 5 closely, one needs to integrate over the bundle. So we have to pull back forms.

Using the half-twistor space:

$$T^+ = \{Z^\alpha : Z^\alpha \bar{Z}_\alpha > 0\} \quad ,$$

we have the induced Penrose correspondence (Chapter 5):

$$\begin{array}{ccc} & F^+ & \\ \alpha \swarrow & & \searrow \beta \\ P_3^+ & \xrightarrow{\ \tau^+\ } & M_c^+ \end{array} \quad .$$

In the coordinates $\quad \alpha : F^+ \longrightarrow T^+$

and $\quad \beta : F^+ \longrightarrow M_c^+$

are given by $\quad (Z^{AA'}, \lambda^B) \longrightarrow Z^\alpha = (iZ^{AB'}\lambda_{B'}, \lambda_{A'})$

and $\quad (Z^{AA'}, \lambda^B) \longrightarrow Z^{AA'}$, respectively.

We also know, from Chapter 5, that F^+ is biholomorphic to $P_1 \times M_c^+$, which is the basic geometric property of the above spaces.

Now in the two-plane $(\omega^A, \lambda_{A'})$ in T, we have $\omega^A = iZ^{AA'}\lambda_{A'}$ where $Z^{AA'}$ parameterizes this two-plane under the above Penrose correspondence. We then obtain the following diagram:

$$(\lambda_{A'}, Z^{AA'}) \longmapsto \{(\omega^A, \lambda_{A'}) : \omega^A = iZ^{AA'}\lambda_{A'}\}$$

$$\begin{array}{ccc}
\Cap & & \\
\mathbb{C}^2 \times M_c & \xrightarrow{\alpha_0} & T \\
\downarrow & & \downarrow \\
F^+ = P_1 \times M_c^+ & \xrightarrow{\alpha} & P_3^+ \\
\text{Projection } \beta \downarrow & & \\
M_c^+ & &
\end{array}$$

The above diagram explains how one may use Penrose transform to transfer cohomology classes to zero-rest-mass fields.

Consider again the first equation (7.1):

$$\mathrm{vol}(\nu_{P_1}) = \int_{P_1} F + \gamma_1 = \int_{P_1} F + \int_{P_1} \gamma_1 \quad . \quad (7.1')$$

The pullback $[a^*(F)]\,(\lambda_{A'}\ ,\ z^{AA'})$ (look at the diagram) is a (1 , 1) form on $P_1 \times M_c^+$. Therefore, integrating over the fibers of the projection β, we obtain for the first integral in the above equation:

$$\int_{P_1} F = \iint_{P_1 \times \{z^{AA'}\}} [a^*F]\,(\lambda_{A'}\ ,\ z^{AA'})$$

$$= \iint_{P_1 \times \{z^{AA'}\}} \theta^{\alpha}(z)d\bar{z}_{\alpha} \wedge \lambda_{E'}d\lambda^{E'} \quad . \quad (7.3)$$

where

$$\theta^{\alpha}(z)d\bar{z}_{\alpha} \equiv (0,1)\ \text{form} \quad ,$$

$$\lambda_{E'}d\lambda^{E'} = W_{\alpha\beta}z^{\alpha}dz^{\beta} \equiv (1,0)\ \text{form} \quad ,$$

$$W_{\alpha\beta} = \begin{bmatrix} 0 & 0 \\ 0 & \varepsilon^{A'B'} \end{bmatrix} \quad 4 \times 4 \ \text{matrix} \quad .$$

Let $I = \theta^{\alpha}(z)d\bar{z}_{\alpha} \wedge \lambda_{E'}d\lambda^{E'}$ be the integrand in (7.3). Differentiating I , we obtain:

$$\frac{\partial}{\partial z^{AA'}} I = \lambda_{E'}d\lambda^{E'} \wedge \frac{\partial}{\partial z^{AA'}}(\theta_{M'}d(i\bar{z}^{MM'}\bar{\lambda}_M) + \theta^{M}d\bar{\lambda}_M)$$

[where we use $\theta^{\alpha}(z)d\bar{z}_{\alpha} = \theta_{M'}d\bar{\omega}^{M'} + \theta^{M}d\bar{\lambda}_M$], and the above

$$= \lambda_{E'} d\lambda^{E'} \left\{ \frac{\partial \theta_{M'}}{\partial \omega^A} \frac{\partial \omega^A}{\partial z^{AA'}} d\bar{\omega}^{M'} + \frac{\partial \theta^M}{\partial \omega^A} \frac{\partial \omega^A}{\partial z^{AA'}} d\bar{\lambda}_M \right\}$$

where $\bar{\omega}^{M'} = iz^{M'M}\bar{\lambda}_M$.

Therefore:

$$\frac{\partial}{\partial Z^{AA'}} I(\lambda_{A'}, z^{AA'}) = i\lambda_{A'}\lambda_{E'} d\lambda^{E'} \left\{ \frac{\partial \theta_{M'}}{\partial \omega^A} d\bar{\omega}^{M'} + \frac{\partial \theta^M}{\partial \omega^A} d\bar{\lambda}_M \right\}$$

The last expression is symmetric in A' , E' .

This implies that:

$$\varepsilon^{A'E'} \frac{\partial}{\partial z^{AA'}} I_{E'} = 0 \quad , \qquad (7.4)$$

since $\varepsilon^{A'E'}$ is skew symmetric.

$$\left[\frac{\partial}{\partial z^{AE'}} I_{E'} = \varepsilon^{A'E'} \frac{\partial}{\partial z^{AA'}} I_{E'} = -\varepsilon^{E'A'} \frac{\partial}{\partial z^{A'E}} I_{A'} = - \frac{\partial}{\partial z^{AE'}} I_{E'} \right.$$

$$\left. \Longrightarrow \varepsilon^{A'E'} \frac{\partial}{\partial z^{AA'}} I_{E'} = \frac{\partial}{\partial z^{AE'}} I_{E'} = 0 \right] .$$

We also know that $\nabla^{AA'} = \varepsilon^{EA}\varepsilon^{E'A'}\nabla_{EE'}$

where $\varepsilon^{EA}\varepsilon^{E'A'}$ raises the indices.

Therefore

$$\partial/\partial z^{AA'} I_{A'} = \nabla^{AA'} I_{A'} = \varepsilon^{EA}\varepsilon^{E'A'}\nabla_{EE'} I_{A'}$$

$$= \varepsilon^{EA}\varepsilon^{A'E'}\nabla_{EA'} I_{E'} \qquad \text{(switching } E' \text{ and } A'\text{)}$$

$$= \varepsilon^{A'E'}\nabla_{AA'} I_{E'} \underset{\text{by (7.4)}}{=} 0 .$$

Hence: $$\partial/\partial z^{AA'} \iint_{P_1 \times \{z^{AA'}\}} \theta^{\alpha}(z) d\bar{z}_{\alpha} \wedge \lambda_{E'} d\lambda^{E'}$$

$$= \partial/\partial z^{AA'} \iint_{P_1 \times \{z^{AA'}\}} I = \iint_{P_1 \times \{z^{AA'}\}} \partial/\partial z^{AA'} I = 0 .$$

Consider now the second integral in equation (7.1´) $\int_{P_1} \gamma_1$.

Since γ_1 is the first Chern class, then, by theorem (7.1), $\gamma_1 \in H^1(P_3, \mathcal{H}^1)$. Therefore, by the Penrose transform, γ_1 can be transformed to a spinor $\phi_{A'B'C'}$ (see Chapter 5). This spinor field is a zero-rest-mass field and, therefore, $\partial/\partial z^{AA'} \phi_{A'B'C'} = 0$. Then, differentiating the above integral we obtain:

$$\partial/\partial z^{AA'} \iint_{P_1 \times \{z^{AA'}\}} \phi_{A'B'C'} = \iint_{P_1 \times \{z^{AA'}\}} \partial/\partial z^{AA'} \phi_{A'B'C'} = 0 .$$

Thus, the formula for the volume of the tube around P_1 represents a zero-rest-mass field of rank 3 and, therefore, it gives a physical field of spin $\frac{3}{2}$.

Now consider the case where $P \equiv P_2(\mathbb{C}) \subset T$. The formula of the tube, in this case, is given by equation (7.2) to be:

$$\mathrm{vol}(\nu_{P_2}) = \int_{P_2} (\gamma_0 \wedge F^2 + \gamma_1 \wedge F^1 + \gamma_2 \wedge F^0)$$

$$= \int_{P_2} F^2 + \int_{P_2} \gamma_1 \wedge F^1 + \int_{P_2} \gamma_2$$

$$= \iint_{P_1 \times \{z^{AA'}\}} a^*(F^2) + \iint_{P_1 \times \{z^{AA'}\}} a^*(\gamma_1 \wedge F^1) + \iint_{P_1 \times \{z^{AA'}\}} a^*(\gamma_2) \quad .$$

Let us first consider the second integral:

$$I = \iint_{P_1 \times \{z^{AA'}\}} a^*(\gamma_1 \wedge F^1) \qquad \text{with } \gamma_1 \in H^1(P_3 , \mathcal{H}^2).$$

It follows from the Penrose transform that we can associate with γ_1 a spinor $\phi_{A'B'C'D'}$ of rank 4.

Writing $\phi_{A'B'C'D'} = \lambda_{A'} \ldots \lambda_{D'}$ we obtain:

$$I_2 = \psi_{A'B'C'D'} = \iint_{P_1 \times \{z^{AA'}\}} \lambda_{A'} \ldots \lambda_{D'} \wedge \theta^{\alpha}(z) d\bar{z}_{\alpha} \wedge \lambda^{E'} d\lambda_{E'}$$

where, as before, integration is along the fibers of β .

Differentiating I_2 we have:

$$\frac{\partial}{\partial z^{AA'}} I_2(z^{AA'}) = \iint_{P_1 \times \{z^{AA'}\}} \lambda_{A'} \ldots \lambda_{D'} \frac{\partial}{\partial z^{AA'}} \theta^{\alpha}(z) d\bar{z}_{\alpha} \wedge \lambda^{E'} d\lambda_{E'} \quad .$$

Let $\lambda_{A'} \ldots \lambda_{D'} \frac{\partial}{\partial z^{AA'}} \theta^{\alpha}(z) d\bar{z}_{\alpha} \wedge \lambda^{E'} d\lambda_{E'} = J$.

Then $$J = \lambda_{B'}\ldots\lambda\, \lambda^{E'} d\lambda_{E'} \wedge \frac{\partial}{\partial z^{AA'}}(\theta_{M'} d(i\bar{z}^{M'}\bar{\lambda}_M) + \theta^M d\bar{\lambda}_M)$$

$$= \lambda_{B'}\ldots\lambda\, \lambda^{E'} d\lambda_{E'} \left\{\frac{\partial\theta_{M'}}{\partial\omega^A}\frac{\partial\omega^A}{\partial z^{AA'}} d\bar{\omega}^{M'} + \frac{\partial\theta^M}{\partial\omega^A}\frac{\partial\omega^A}{\partial z^{AA'}} d\bar{\lambda}_M\right\}$$

$$= i\lambda_{A'}\ldots\lambda\, \lambda^{E'} d\lambda_{E'} \left\{\frac{\partial\theta_{M'}}{\partial\omega^A} d\bar{\omega}^{M'} + \frac{\partial\theta^M}{\partial\omega^A} d\bar{\lambda}_M\right\} .$$

Therefore J is symmetric in A', B', C', D' and so is $\frac{\partial}{\partial z^{AA'}} \psi_{A'B'C'D'}$. This implies that

$$\varepsilon^{A'B'} \frac{\partial}{\partial z^{AA'}} \psi_{B'\ldots D'} = 0 . \quad (7.5)$$

Again using the identity $\nabla^{AA'} = \varepsilon^{BA}\varepsilon^{B'A'}\nabla_{BB'}$,

We get: $$\frac{\partial}{\partial z^{AA'}} \psi_{A'\ldots D'} = \nabla^{AA'}\psi_{A'\ldots D'} = \varepsilon^{BA}\varepsilon^{B'A'}\nabla_{BB'}\psi_{A'\ldots D'}$$

$$= \varepsilon^{BA}\varepsilon^{A'B'}\nabla_{BA'}\psi_{B'\ldots D'} = \varepsilon^{A'B'}\frac{\partial}{\partial z^{AA'}}\psi_{B'\ldots D'} = 0 .$$

(by (7.5))

Thus $$\frac{\partial}{\partial z^{AA'}} I_2(z^{AA'}) = 0 .$$

For the other two integrals $\iint_{P_1\times\{z^{AA'}\}} F^2$, and $\iint_{P_1\times\{z^{AA'}\}} \gamma_2$, we know that a 2-form on P_1 vanishes and, since we are integrating these along the fiber, these two integrals vanish. Thus, the formula of the volume of the tube around the submanifold $P_2 \subset T$ represents a zero-rest-mass field of rank 4, or spin s = 2. This corresponds to "weak gravitational fields," which are linearized Einstein equations. So viewing $P_1(\mathbb{C})$ as a curve, we get coupled-neutrino particles from the formula of the tube around this curve; and viewing $P_2(\mathbb{C})$ as a surface, we get weak gravitaional fields from the formula of the tube around this surface.

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