PERFORMANCE CONSIDERATIONS OF PARALLEL CONTROL SYSTEMS

by

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LIST OF SYMBOLS

Bca - Armature conductor copper density in ampere per square inch

Bccp - Commutating field conductor copper density in ampere per square inch

D - Denominator of the open-loop transfer function

D' - Armature core diameter in inches

e_1 - Actuating signal 1

e_2 - Actuating signal 2

J_{LM} - Equivalent inertia of load referred to motor shaft in slug-feet squared.

J_{LM1} - Equivalent inertia of load referred to shaft 1 in slug-feet squared

J_{LM2} - Equivalent inertia of load referred to motor shaft 2 in slug-feet squared

J_{M1} - Actual inertia of motor 1 in slug-feet squared

J_{M2} - Actual inertia of motor 2 in slug-feet squared

J_{ML} - Total inertia referred to the load shaft in slug-feet squared

J_{TM} - Total inertia referred to two motor shaft that two motors are required to accelerate in slug-feet squared

J_{TM1} - Total inertia referred to motor shaft 1 in slug-feet squared

J_{TM2} - Total inertia referred to motor shaft 2 in slug-feet squared

K - System gain of the type 1 positional control system in per second ($K = K_0/Ke$)

K_0 - Amplifier gain of the type 1 positional control system in volts per radian

K_1 - Amplifier gain of the parallel control system in volts per radian ($K_1 = n_1 K_{01}$)

K_2 - Amplifier gain2of the parallel control system in volts per radian ($K_2 = n_2K_{02}$)

Ke_1 - Voltage constant of motor 1 in volts per radian per second

Ke_2 - Voltage constant of motor 2 in volts per radian per second

K_J - Motor armature inertia factor

Ks - Armature slot activity = Armature slot copper area/Armature slot area

K_T - Torque constant of motor in pound-feet per ampere

K_{T1} - Torque constant of motor 1 in pound-feet per ampere

K_{T2} - Torque constant of motor 2 in pound-feet per ampere

L - Armature core length (iron only) in inches

N - Numerator of the open-loop transfer function

n_1 - Gear ratio of motor 1 to load

n_2 - Gear ratio of motor 2 to load

pc - Poles of the closed-loop transfer function

Pc - Number of commutating pole

P_M - Number of main pole

Prms - The rms power in horse power

R_1 - Armature resistance 1 including external resistance of brushes, lines, and internal resistance of voltage source in ohms

R_2 - Armature resistance 2 including external resistance of brushes, lines, and internal resistance of voltage source in ohms

T_1 - Transmitted torque 1 in pound-feet

T_2 - Transmitted torque 2 in pound-feet

Td_1 - Torque developed by motor 1 in pound-feet

Td_2 - Torque developed by motor 2 in pound-feet

T_L - Torque on load shaft in pound-feet

T_m - Motor and load time constant in seconds

T_{ML} - Equivalent torque on the load shaft in pound-feet

T_0 - Motor and load time constant without source impedance of the motor power supply in seconds

Trms - The rms torque in pound-feet

zm - Zeros of the closed-looped transfer function

ζ - The damping ratio of system response

θ - Angles between armature winding end turns and armature end punching edges in radians

θ_0 - Angular position of load in radians

θ_1 - Angular position of motor 1 in radians

θ_2 - Angular position of motor 2 in radians

σ_o - The real part of the dominant pole

ω_o - The imaginary part of the dominant pole

ω_n - The undamped natural frequency in per second

ρ_o - The dominant pole

K_{01} - Amplifier gain 1 of the type 1 positional control system in volts per radian

K_{02} - Amplifier gain 2 of the type 2 positional control system in volts per radian

K_C - Commutation compensation in per unit

PERFORMANCE CONSIDERATIONS OF PARALLEL CONTROL SYSTEMS

I. INTRODUCTION

Control systems with two or more forward paths in parallel have been developed and applied to various control situations (4, p. 345-354; 8, p. 230-231; 12, p. 503-509; 15, p. 384-387; 17, p. 10-16). The increase in complexity and variety of control systems has required the investigation of the performance characteristics of systems with two or more similar elements in parallel.

The work on the feedback-feedforward control systems has been presented in many papers. In situations where the noise associated with the input signal is small, R. E. Graham found that it may be possible to reduce the dynamic errors obtained with a given servo systems by the use of forward-acting equalization external to the control loop (6, p. 649-650). The feedforward loops are applied to the design of the operational amplifier, based on the fact that the use of feedforward loops provides a means of adding zeros to the transfer function (7, p. 529-536). A zero or a pole can be produced by the use of a differentiator or an integrator in feedforward loop thereby modifying the original system. The ideal differentiator or integrator cannot be obtained easily, so the R-C lead, lag, and lead-lag networks have been developed for this purpose (3, p. 124-129).

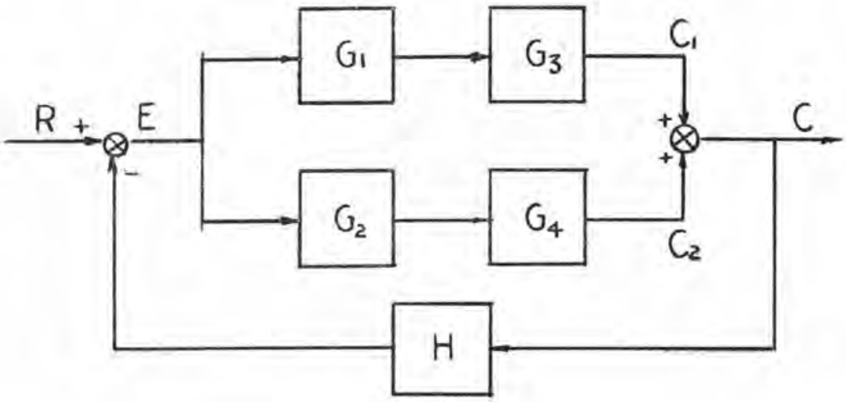
A great deal of work on the feedback-feedforward systems has been carried out by J. R. Moore, in which the design of precision automatic control systems was attempted by a combination of coarse and fine adjustment. The coarse controller provides the major proportion of the output and acts as the open-cycle system, while the vernier acts as the closed cycle system (10, p. 1421-1436; 14, p. 284-286). The equivalent transfer function for the feedback-feedforward control systems is evaluated by the logarithmic frequency characteristic method (LFC) (1, p. 62-67).

As described above, the feedforward path includes unity, integration or differentiation for the purpose of improving the system performance. The high energy is applied to the open-loop and the low energy to the closed-loop, respectively, of the open- and closed-loop cycles system. The parallel control systems to be considered in this thesis, however, consist of two or more control systems in parallel for controlling an output load. In Figure 1, two parallel control systems are illustrated, in which the completely parallel control system is shown in Figure 1(a) and the partially parallel control system in Figure 1(b).

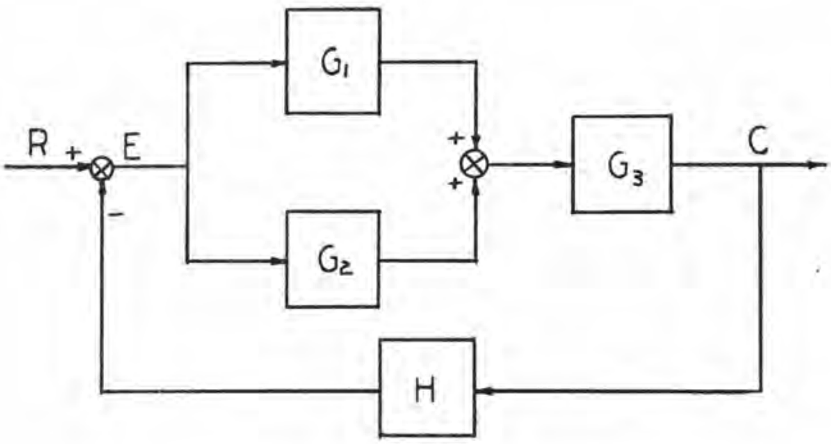
This thesis attempts to study on the performance consideration of parallel control systems consisting of two elements in parallel. Hence each system element is specified as the type 1 positional control system which is most frequently used in servomechanisms. In order to combine these two systems in parallel, spur gears and differential gears are used in the completely parallel control system. The system using spur gears is defined as the completely parallel control system A, and the system using differential gears as the completely parallel control system B, and they are shown in Figures 2 and 3, respectively. In the partially parallel control systems, control elements which produce a manipulated signal are combined in parallel. In cases where the compensating networks are included in control elements, a study on the partially parallel control systems can follow a realization procedure by using compensating networks as done by M. Smith (13, p. 332-336). If only amplifiers are used as control elements, less significance is noted in the system performance of the parallel control system. (The work on the partially parallel open-cycle cloded-cycle system has been carried out by Moore (10, p. 1089-1094).) Consequently, this thesis is concerned with completely parallel control system with realistic values for the time constants of the controlled systems.

Various combinations of two control systems in which each system has four different motor and load time constants are used to investigate the transient response of the parallel control systems. The motor and load time constant is varied by changing the source impedance of motor power supply and the d-c motor design factors (9, p. 1089-1094). The figures of merit of system response for each parallel control system are evaluated by both the approximate equations (11, p. 516-529) and the analog computer.

The Appendices include a table of several combinations of the transfer functions of completely parallel control systems, derivations of the transfer functions of the completely parallel control systems A and B and the motor and load time constant in design terms.



(a)



(b)

Figure 1. BLOCK DIAGRAMS OF THE PARALLEL CONTROL SYSTEMS

(a) THE COMPLETELY PARALLEL CONTROL SYSTEM

(b) THE PARTIALLY PARALLEL CONTROL SYSTEM

II. PARALLEL CONTROL SYSTEMS

1. Completely Parallel Control Systems

The general transfer function of the completely parallel control systems as shown in Figure 1(a) is derived below.

The open-loop transfer function for the system shown in Figure 1(a) is

$$G(s) = G_1(s)\, G_3(s) + G_2(s)\, G_4(s) \tag{1}$$

Then the closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+H(s)G(s)} = \frac{G_1(s)G_3(s) + G_2(s)G_4(s)}{1+H(s)[G_1(s)G_3(s) + G_2(s)G_4(s)]} \tag{2}$$

Assuming $H(s) = K$ and assuming that each open-loop transfer function is as follows:

$$G_1(s) = \frac{N_1(s)}{D_1(s)}, \quad G_2(s) = \frac{N_2(s)}{D_2(s)} \quad G_3(s) = \frac{N_3(s)}{D_3(s)} \tag{3}$$

and $G_4(s) = \dfrac{N_4(s)}{D_4(s)}$

in which the individual terms on the right are analytic; they contain no poles and no zeros in the right half plane, and so $G_1(s)$, $G_2(s)$, $G_3(s)$ and $G_4(s)$ represent stable open-loop systems. By substituting equation (3) into equation (2), the closed-loop transfer function is obtained

$$\frac{C(s)}{R(s)} = \frac{\dfrac{N_1(s)}{D_1(s)}\dfrac{N_3(s)}{D_3(s)} + \dfrac{N_2(s)}{D_2(s)}\dfrac{N_4(s)}{D_4(s)}}{1 + K\left[\dfrac{N_1(s)}{D_1(s)}\dfrac{N_3(s)}{D_3(s)} + \dfrac{N_2(s)}{D_2(s)}\dfrac{N_4(s)}{D_4(s)}\right]} \tag{4}$$

Hence, so far as stability concerned, the bracket term in the denominator of equation (4) only needs to be considered. In other words, the open-loop transfer function should be inspected for the system stability. It was specified that the individual transfer function is stable at equation (3), therefore the closed-loop transfer function is said to be stable.

The open-loop transfer function is

$$G(s) = \frac{N_1(s)}{D_1(s)}\frac{N_3(s)}{D_3(s)} + \frac{N_2(s)}{D_2(s)}\frac{N_4(s)}{D_4(s)} \tag{5}$$

$$= \frac{N_1(s)N_3(s)D_2(s)D_4(s) + N_2(s)N_4(s)D_1(s)D_3(s)}{D_1(s)D_2(s)D_3(s)D_4(s)} \tag{6}$$

In general, the individual transfer functions in control systems do not contain zero terms except when lead and lead-lag compensators are used. However, equation (6) contains new zero terms in the open-loop transfer function. As shown in Table 5 of Appendix I, these zeros are a function of the system gain and are located near poles. Consequently a more stable system could be achieved by adjusting the system gain to cancel a less stable pole. Since the range of system gain may be restricted to meet the specification of transient characteristics, the above procedure is not feasible. The fact that the zero of the transfer cuntion is a function of system gains prevents to use the root loci for obtaining the resultant system transfer function. The Frequency-response method is applicable in this case and is therefore used in this study.

Table 5 in Appendix I shows various combinations of the transfer functions of completely parallel control systems.

2. Mechanisms Combining Two Control Systems in Parallel

In this thesis two mechanical components, differential gears and spur gears, combining two control systems in parallel, are used in the positional control systems of the completely parallel control systems.

The spur gear and pinion are the most common type of gearing used in control application. When applied to the completely parallel control systems as shown in Figure 2, this gearing obviously is not to be used when it is necessary to add or subtract the controlled position outputs of the two systems. However, it is used to add the torques of the two motor shafts.

Differential gears are used in control systems to add or subtract the positions or velocities of two shafts, mostly as a device to detect an actuating or error signal. In this thesis differential gears are used as a device to add the output positions of two control systems as shown in Figure 3. The output position is equal to the sum of the positions of two shafts. On the other hand, torque on load shat T_L and toques on each motor shaft T_1 and T_2 are all the same (5, p. 315; 334-335).

The relationships between the position, and torque, T, for spur gears and differential gears are as follows: For spur gears,

$$\theta_0 = \theta_1/n_1 = \theta_2/n_2 \tag{7}$$

$$T_L = n_1T_1 + n_2T_2 \tag{8}$$

For differential gears,

$$\theta_0 = \theta_1/n_1 + \theta_2/n_2 \tag{9}$$

$$T_2 = n_1T_1 = n_2T_2 \tag{10}$$

On the basis of the above relationships, the system transfer function for each case is developed and the system performance is analyzed in the following sections.

3. Completely Parallel Control System A

Two type 1 positional control systems are used. Each has the following transfer function:

$$G(s) = \frac{K_o}{Ke\ s\ (\ JZs/K_TKe + FZ/K_TKe + 1)} \tag{11}$$

In many cases of practical importance the armature circuit impedance may be considered to be resistive. The coefficient of mechanical friction, F, is also neglected. This can be done because in most motors the electrical damping is very much larger than the mechanical friction. Typical values of the ratio FR/K_TKe are in the order of 0.01. Therefore, equation (11) becomes

$$G(s) = \frac{K}{s\ (Tm\ s + 1)} \tag{12}$$

where $Tm = J\ R/K_T Ke$ in second.

The motor and load time constant Tm is defined as the time required for motor to speed up 63.2 per cent of its final velocity, when supplied from a constant voltage source. In general, the motor and load time constant Tm in the type 1 positional control system has a value ranging from about 0.01 to 0.1 second in motors of standard design connected to a zero-impedance source (5, p. 209-210).

In completely parallel control system A using spur gear and two pinions, the system operation is based on the relationships given in equations (7) and (8), and the system transfer function is developed by referring to these equations and the electric equation for the d-c motor. Hence, from equation (8)

$$T_L(s) = n_1 T_1(s) + n_2 T_2(s) = J_L\ s^2 \theta_0(s) \tag{13}$$

The developed torques by two motors are shown as follows:

$$Td_1(s) + Td_1(s) = J_{M1} s^2 \theta_1(s) + T_1(s) + J_{M2} s^2 \theta_2(s) + T_2(s) \tag{14}$$

The value of motor inertia for a minimum rms motor torque with the load inertia and gear ratio is fixed is (2, p. 111)

$$J_M = J_L/n^2 \tag{15}$$

According to equations (13) and (14), the torque equation is developed by:

$$n_1 Td_1(s) + n_2 Td_2(s) = n_1 J_{M1} s^2 \theta_1(s) + n_2 J_{M2} s^2 \theta_2(s) + T_L(s) \quad (16)$$

$$= (n_1^2 J_{M1} + n_2^2 J_{M2} + J_L) s^2 \theta_O(s) \quad (17)$$

$$= J_{LM} s^2 \theta_O(s) \quad (18)$$

from which $J_{LM} = n_1^2 J_{M1} + n_2^2 J_{M2} + J_L$ (19)

Based on the transfer function of the type 1 positional control system, equation (12), and above the relationships, the open-loop transfer function of completely parallel control system A is then shown as follows:

$$G(S) = \frac{\theta(S)}{E(S)} = \frac{(n_1 K_1 K_{T1}/R_1) + (n_2 K_2 K_{T2}/R_2)}{S\left[(J_L + J_{M1} n_1^2 + J_{M2} n_2^2)s + (K_{T1} K_{e1} n_1^2/R_1) + (K_{T2} K_{e2} n_2^2/R_2)\right]} \quad (20)$$

or

$$= \frac{(n_1 K_1 K_{T1} R_2 + n_2 K_2 K_{T2} R_1)/(K_{T1} K_{e1} R_2 n_1^2 + K_{T2} K_{e2} R_1 n_2^2)}{S\left[\frac{(J_L + J_{M1} n_1^2 + J_{M2} n_2^2) R_1 R_2}{K_{T1} K_{e1} n_1^2 R_2 + K_{T2} K_{e2} n_2^2 R_1} s + 1\right]} \quad (21)$$

A derivation of equation (20) is made in Appendix II and the block diagram for this sytem is shown in Figure 2(b). The open-loop transfer function is then known to be the type 1 system with one time constant according to equation (21). It should be noted, however, that the inductance term which will make another time constant is neglected here because of the small inductance in the d-c motor armature. In this thesis the basic performance considerations of the parallel control systems are given. The use of compensating network to improve the system performance is not considered.

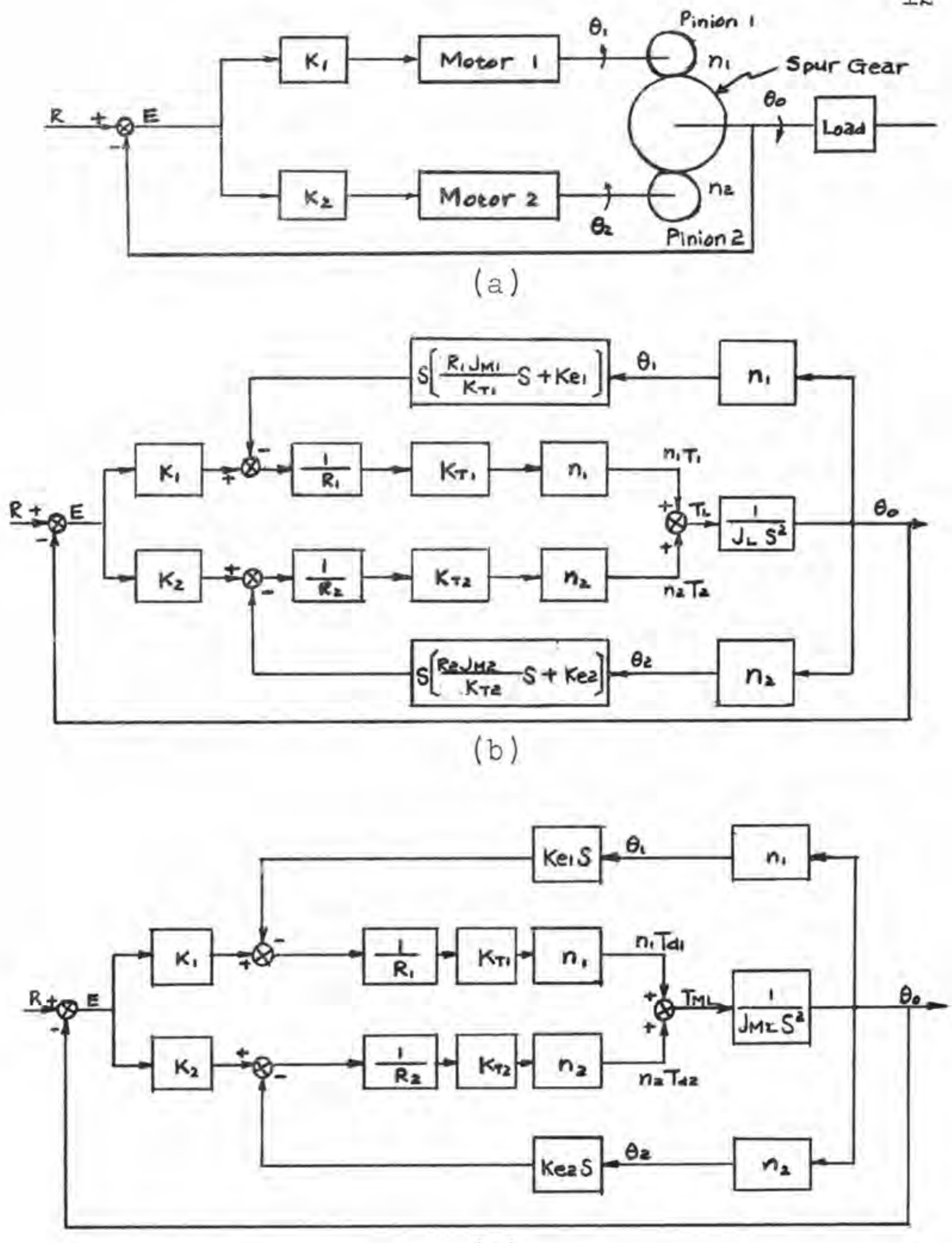


Figure 2. THE COMPLETELY PARALLEL CONTROL SYSTEM A
(a) SCHEMATIC DIAGRAM FOR A POSITIONAL CONTROL SYSTEM
(b) BLOCK DIAGRAM FOR A POSITIONAL CONTROL SYSTEM
(c) BLOCK DIAGRAM FOR A POSITIONAL CONTROL SYSTEM

As a special case of system performance, consider first the either T_{d1} or T_{d2} becomes zero by an open circuit between the amplifier and motor. Accordingly, the load torque T_L is only related to either T_{d2} or T_{d1}. On the other hand, if the input to amplifier 1 is open circuited the motor armature circuit will permit current to flow when a voltage is produced by the rotation of motor 1 by motor 2. The developed torque of motor 1 is on the opposite direction to that of motor 2. This adds to the total load torque on motor 2 and thereby reduces the performance of the system. The transfer function for the later case can be simply obtained by setting $K_2 = 0$ in equation (21).

4. Completely Parallel Control System B

In the same manner as stated in the completely parallel control system A, operation of the completely parallel control system B using differential gears is also described on the basis of equations (9) and (10). The transfer function for this system consisting of two type 1 control systems in parallel is obtained by the following procedures.

From equations (9) and (10),

$$T_L(s) = J_L s^2 \theta_0(s) = J_L\ s^2 \left(\frac{\theta_1}{n_1} + \frac{\theta_2}{n_2}\right)(s) \tag{22}$$

and

$$J_{LM} = J_L/{n_1}^2 = J_L/{n_2}^2 \tag{23}$$

As a result, torque on each motor shaft is,

$$Td_1(s) = J_{M1}\ s^2\theta_1(s) + T_1(s) \tag{24}$$

$$= J_{M1}\ s^2\theta_1(s) + \frac{J_L}{n_1} s^2\theta_0(s) \tag{25}$$

$$Td_2(s) = J_{M2}\ s^2\theta_2(s) + T_2(s) \tag{26}$$

$$= J_{M2}\ s^2\theta_2(s) + \frac{J_L}{n_2} s^2\theta_0(s) \tag{27}$$

Based on the above relationships and the electric equation for d-c motor armature circuit, the open-loop transfer function of the completely parallel control system B is obtained as follows:

$$G(S) = \frac{\theta_o(S)}{E(S)} = \frac{\left[\frac{R_1 J_{M1}}{K_{o1}K_{T1}} + \frac{R_2 J_{M2}}{K_{o2}K_{T2}}\right] s + \left[\frac{K_{e1}}{K_{o1}} + \frac{K_{e2}}{K_{o2}}\right]}{\frac{R_1 R_2}{K_{o1}K_{o2}K_{T1}K_{T2}}\left(J_{TM1}J_{TM2} - \frac{J}{n_1^2 n_2^2}\right) s \left[s^2 + \frac{R_1 J_{TM1} K_{T2} K_{e2} + R_2 J_{TM2} K_{T1} K_{e1}}{R_1 R_2 (J_{TM1} J_{TM2} - J_L^2/n_1^2 n_2^2)} s + \frac{K_{T1} K_{T2} K_{e1} K_{e2}}{R_1 R_2 (J_{TM1} J_{TM2} - J_L^2/n_1^2 n_2^2)}\right]} \tag{28}$$

where $J_{TM1} = J_{M1} + J_L/{n_1}^2$

$J_{TM2} = J_{M2} + J_L/{n_2}^2$

A derivation of the transfer function is given in Appendix III. The schematic diagram and block diagram, based on the above open-loop transfer function, are shown in Figure 3.

As the open-loop transfer function shows, a zero term which is a function of the system gain has appeared in the resultant transfer function. In evaluation of the figures of merit by the approximate equations as

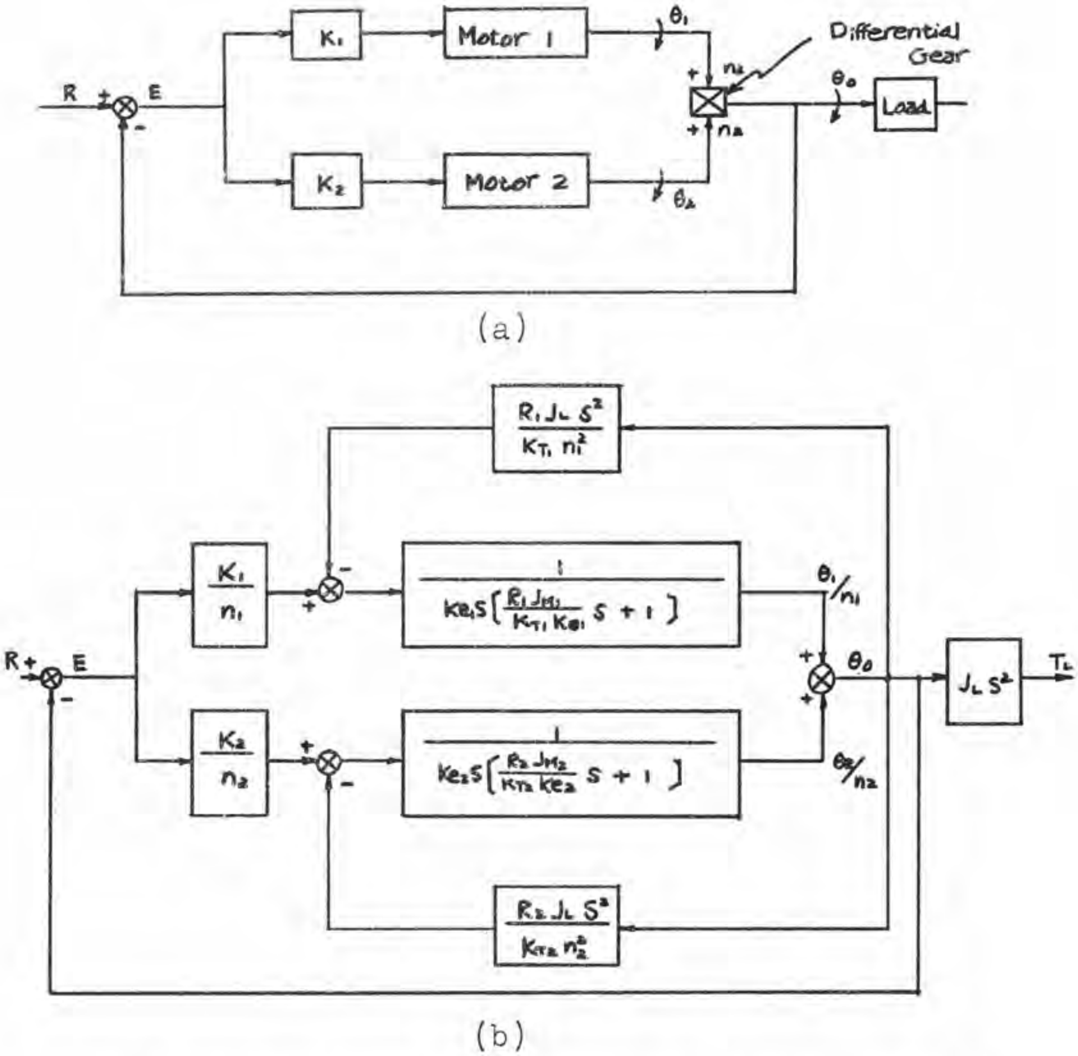


Figure 3. THE COMPLETELY PARALLEL CONTROL SYSTEM B
(a) SCHEMATIC DIAGRAM FOR A POSITIONAL CONTROL SYSTEM
(b) BLOCK DIAGRAM FOR A PSOTIONAL CONTROL SYSTEM

given in the following chapter, it is known that this zero lies near a pole. This makes it possible to neglect this zero for a first order approximation.

It was shown that operation of two motors for this system is governed by equations (9) and (10). Consider a special case in which the motor 1 cannot develop torque, because the armature circuit is open, motor 2 can develop torque, and the load requires a torque to rotate it. Equation (10) states that no torque will be transmitted by the differential gearing. Consequently, load torque also becomes zero, by which no change in the output position of load occurs. For this condition, differential gears act like spur gears, so that the output position of the motor 2 causes the opposite rotation of the output position of motor 1. Consider that the input to amplifier 1 is open and that the armature circuit is normal. Any rotation of motor 1 by the gearing will produce a torque because of the generator action. Therefore some torque can be transmitted to the load and rotation will result. However, the rotation of the motor 1 is in the opposite direction to the motor 2, so that the output position of load is determined by the difference between the output positions of motors 1 and 2.

III. DETERMINATION OF MOTOR AND LOAD TIME CONSTANT

In control system analysis whenever linear control theory is used, a basic assumption is made in which adequate power and torque, or force, are available in the control system power element to meet the requirements of the desired controlled variable motion. Under this assumption, the selection and design of d-c motors have been carried out by Lebenbaum and Chestnut, respectively (9, p. 1089-1094; 2, p. 89-118).

It is already known on page 10 that the time constant of the motor and load is defined as follows:

$$Tm = R\ J/K_T Ke \quad \text{second} \tag{29}$$

where K_T = torque constant of motor in pound-feet per ampere,

Ke = voltage constant of motor in volts per radian per second,

J = motor and load inertia referred to motor shaft in slug-feet squared,

R = armature resistance including external resistance of brushes, lines, and internal resistance of voltage source in ohms.

The armature circuit inductance and steady-state load and friction torques have been neglected in this derivation.

When this equation is converted to design terms, which is given in Appendix IV, it is found that the time constant is directly proportional to functions of the ratios of armature diameter to armature length, load inertia to motor inertia, and commutating field conductor copper density to armature copper density; and inversely proportional to functions of the pole face flux density, and the ratios of pole pitch to pole arc, armature slot depth to armature diameter, and pole face flux density to armature tooth flux density at the tooth root.

If these deisgn factors and ratios are so given that the time constant becomes a minimum, the following equation is obtained (9, 1089 p):

$$(Tm)min = \left[1.25 \times K_J/Ks(1 + J_L/J_M)\right]\left[1 + \frac{\pi}{P_M \cos\theta}(D'/L) + (1.2Kc)(Bccp/Bca)(P_L/P_M)\right] \text{ seconds at 75 degrees centigrade} \quad (30)$$

From equations (29) and (30), it is known that the time constant of the motor and load is only changed by a variation in the armature circuit resistance and total motor inertia, because the values of the generated voltage and torque constants are fixed by the motor speed and voltage. Both of these factors are functions of the ratios of armature diameter to length. If the motor and load inertias are shown separately, the motor time constant is rewritten as

$$Tm = \frac{1}{K_T Ke} \quad (R\ J_M + R\ J_L) \tag{31}$$

it can be seen that while the first term in the bracketed expression increases with increase in the ratio, as in the case for a motor negligible load inertia, the second term decreases. Therefore, at some point, the motor and load time constant will reach a minimum value as the armature diameter-to-length ratio is increased. It should be noted, however, that an increased ratio may result in larger motors as well as larger generators, so that, cost, size, and weight consideration must be weighed against the motor time constant requirement.

Other factors which limit the minimum value of motor time constant are the maximum possible values of the flux density at the root of the armature tooth, the slot activity factor, the ratio of pole arc to pole pitch, and for larger size machines, the armature slot which can be used. The main reduction of motor and load time constant is then made by increasing the pole face flux density to a value equal to approximately one-half of the flux density at the root of the armature tooth.

So far the possible method of changing the time constant of the motor and load by considering the design factor of motor itself has been considered. However, when the d-c motor is fed from the electronic power source, the total armature circuit resistance can be changed by changing the power amplifier stage of the amplifier. For the fixed values of K_T, Ke, J and Ra, the motor and load time constant can be increased by increasing the output impedance of motor power supply.

In order to determine the various motor and load time constants, the following illustration is considered. It is desired to determine the position control system motor rating and gear ratio for an electric motor to meet the following requirement:

Load inertia = 119,000 pound-feet squared,

friction load = 900 pound-feet,

and shock and unbalance load = none.

The motor selection is based on a typical 115 V 1500 rpm d-c motor with voltage constant Ke = 2.0 volts/rad/sec and torque K_T = 1.4 lb-ft/amps. The ratio of K_T/Ke is a constant and equal to 550/746 = 0.737. Various d-c motors producing the same rms power Prms = 1.38 HP are selected as shown in Table 1. The rms power, Prms, is defined as

$$\text{Prms} = \frac{\text{Trms x Vrms}}{550} \qquad \text{HP} \qquad (32)$$

where $$\text{Trms} = \sqrt{\frac{1}{T} \int_0^T (\text{torque})^2 \, dt} \qquad \text{pound-feet}$$

$$\text{Vrms} = \sqrt{\frac{1}{T} \int_0^T (\text{speed})^2 \, dt} \qquad \text{radians per sec,}$$

in which T is the duration of the period over which the rms power is to be determined. Torque and speed are the instantaneous values of these quantities over the interval from 0 to T. A specific evaluation for these values is given in the reference by Chestnut (2, p. 103-109).

The motor and load inertia referred to the motor shaft, J_{TM}, for spur gears has been shown in the previous section as

$$J_{TM} = J_M + J_L/n^2 \tag{33}$$

Based upon this equation, various inertia for each parallel control system can be evaluated. In case of differential gears, it should be noted that the motor selection will be different, because the fundamental equations are different.

In case of the bevel gear differential,

$$n_1 = n_2 \tag{34}$$

then

$$J_L/{n_1}^2 = J_L/{n_2}^2 \tag{35}$$

From equation (16) on page 10, the following relationship can be shown:

$$J_{TM1} = J_{TM2} \tag{36}$$

In case of the spur gear differential,

$$n_1 \neq n_2 \tag{37}$$

then $J_L/{n_1}^2 \neq J_L/{n_2}^2$ (38)

However, the instantaneous torque applied to the differentials must be the same (assume 1:1:1) according to the operational equation for differential gears. (Hence, the speeds do not need to be the same.) Therefore, the following relationship can also be shown in this case:

$$J_{TM1} \neq J_{TM2} \tag{39}$$

In other words, if all motors to be used in the parallel control system B produce the same horse power, in this illustration, Prms = 1.38 HP, the bevel differentials require to use motors which develop the identical motor torque, while the spur gear differentials can use motors which develop the different motor torques. In transient response study, the spur gear differentials are used to combine motors that have the different motor inertias, by which the various motor combinations are accomplished.

Table 1. TORQUE, SPEED, AND MOTOR AND LOAD TIME CONSTANT FOR LOAD INERTIA 119,000 FT-LB SQUARED AND MOTOR RMS POWER 1.38 HP.

Applied Voltage, Volts	Gear Ratio, n	Maximum Motor Speed, rpm	The rms Torque, Trms, lb-ft	Actual Inertia of Motor, J_M, $lb\text{-}ft^2$	Total Inertia Referred to Motor Shaft, J_{TM}, $slug\text{-}ft^2$	Voltage Constant of Motor, Ke, volt/rads/sec
115	360	1800	6.75	1.4	0.0725	1.50
115	280	1400	9.00	2.5	0.1200	2.00
115	260	1300	9.12	2.6	0.1350	2.02
115	300	1500	7.90	2.0	0.1040	1.76

Torque Constant of Motor, K_T, lb-ft/amps	Motor Armature Resistance, Ra, ohms	Source Resistance of Motor Power Supply, R_O, ohms	Motor and Load Time Constant, Tm, seconds	Motor and Load Time Constant Without Source Resistance To, second
1.10	0.64	0.5	0.05	0.0282
1.47	0.94	1.5	0.10	0.0384
1.49	0.86	4.1	0.25	0.0386
1.29	1.20	8.0	0.40	0.0550

IV. TRANSIENT RESPONSE

The transient response is a main feature to be considered in performance of control systems, and the improvement of response is achieved by the use of various compensators. In parallel control systems without including any compensator, several combinations of two motors which are specified by the different motor and load time constants show some interesting transient response characteristics. In the present section, the transient response of each parallel control system is investigated, in which the gain of the control element is fixed for the purpose of comparing the response of type 1 positional control system and parallel control system for the corresponding motor and load time constant.

In evaluating the figures of merit of system response, the approximate equations derived from a pole-zero configuration of the s-plane are used, in which a pair of poles close to the imaginary axis are selected as the dominant poles for all cases (11, p. 516-529). The transient response is obtained by simulating the system on an analog computer, from which the figures of merit are evaluated.

The following figures of merit are used to judge the transient performance.

1. Mp, peak overshoot (The amount of the first overshoot)
2. Tp, peak time (The time to reach the peak of the first overshoot)
3. Ts, settling time (The time for response to reach and remain within 2 per cent of final value)
4. N, number of oscillations (Up to settling time)

1. The Approximate Equations

For a step input the transform of the system response is of the form

$$C(s) = \frac{K\,A(s)}{B(s)} = \frac{K \prod_{m=1}^{w} (s - Zm)}{s \prod_{c=1}^{v} (s - Pc)} \qquad (40)$$

The term K in the equation is the static loop sensitivity for a unity feedback system. Assume that the system represented by equation (40) has a dominant complex pole $p_o = -\sigma_o + j\omega_o = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$.

Then the time to reach the peak of the first overshoot is;

$$Tp = \frac{1}{\omega_o}\left[\pi/2 - \sum_{\text{all zeros}} \theta + \sum_{\substack{\text{all poles except dominant} \\ \text{pair}}} \phi\right]$$

$$= \frac{1}{\omega_o}\left[\pi/2 - \begin{pmatrix}\text{sum of angles from} \\ \text{zeros to the} \\ \text{dominant pole} \\ -\sigma_o + j\omega_o\end{pmatrix} + \begin{pmatrix}\text{Sum of angles} \\ \text{from the other} \\ \text{poles to the} \\ \text{dominant pole} \\ -\sigma_o + j\omega_o\end{pmatrix}\right] \qquad (41)$$

The amount of the first overshoot is therefore;

$$Mp = \frac{K\,A(0)}{B(0)} + \frac{2\omega_o}{\omega_n^2}\left|K\,\frac{A(p_o)}{B'(p_o)}\right| e^{-\sigma_o Tp} \qquad (42)$$

The settling time is;

$$Ts = 4/\sigma_o = 4/\zeta\omega_n \tag{43}$$

The number of oscillations is;

$$N = 2\,\omega_o/\pi\sigma_o = 2\sqrt{1-\zeta^2}/\pi\zeta \tag{44}$$

The above equations are only for stable system whose transfer functions contain no poles on the imaginary axis and no zeros at the origin, and contain first order pole only.

The equation of the system response with unity feedback is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{45}$$

Substituting equation (11) into equation (45), and comparing the characteristic equations of the resultant transfer function and equation (45), the following relationships are derived:

$$\zeta = K_T\ Ke/2\omega_n JR,\ \omega_n = 1/2\zeta Tm,\quad \text{and } K_o = Tm\,\omega_n^2\,Ke \tag{46}$$

in which the damping ratio and the undamped natural frequency can be changed by various values of the motor and load time constant and the amplifier gain K_0. (This has been discussed in the previous section.) Based on these equations, the figures of merit of the type 1 positional control system are calculated and given in Table 2.

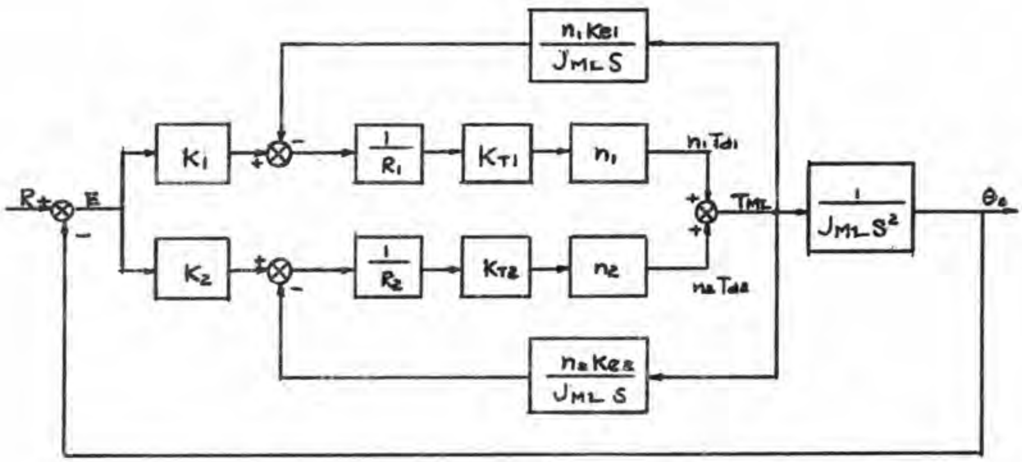
In order to compare the transient responses of the type 1 positional control system and the parallel control systems, the response equations for both the parallel control systems A and B are obtained from equations (20) and (27), respectively, as follows:

$$\frac{\theta_o(S)}{R(S)}=\frac{(n_1 K_1 K_{T1} R_2+n_2 K_2 K_{T2} R_1)/(J_L+J_{M1} n_1^2+J_{M2} n_2^2) R_1 R_2}{S^2+\frac{R_2 K_{T1} K_{e1} n_1^2+R_1 K_{T2} K_{e2} n_2^2}{(J_L+J_{M1} n_1^2+J_{M2} n_2^2) R_1 R_2} s+\frac{n_1 K_1 K_{T1} R_2+n_2 K_2 K_{T2} R_1}{(J_L+J_{M1} n_1^2 \; J_{M2} n_2^2) R_1 R_2}} \tag{47}$$

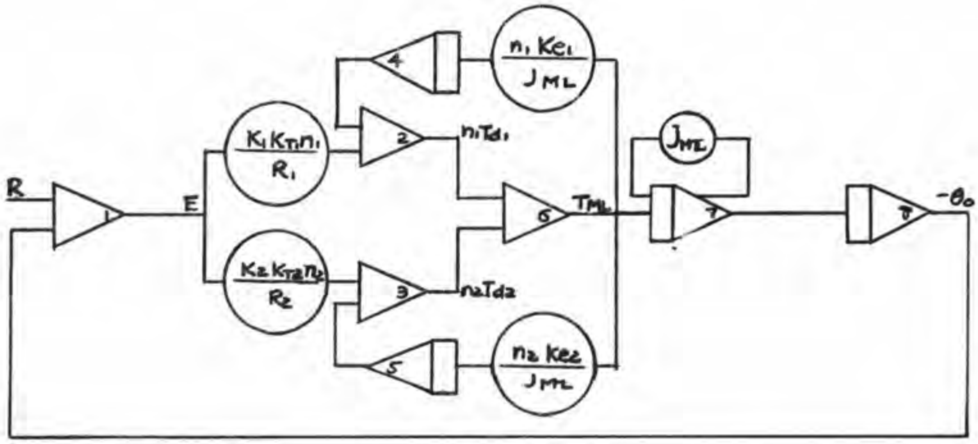
For the completely parallel control system B, from equation (27),

$$\frac{\theta_o(S)}{R(S)}=\frac{\left[\frac{R_1 J_{M1}}{K_{o1} K_{T1}}+\frac{R_2 J_{M2}}{K_{o2} K_{T2}}\right] S+\left[\frac{K_{e1}}{K_{o1}}+\frac{K_{e2}}{K_{o2}}\right]}{\frac{R_1 R_2}{K_{o1} K_{o2} K_{e1} K_{e2}}\left(J_{TM1} J_{TM2}-\frac{J_L^2}{n_1^2 n_2^2}\right)\left\{S^3+\left[\frac{R_1 J_{TM1} K_{T2} K_{e2}+R_2 J_{TM2} K_{T1} K_{e1}}{R_1 R_2\left(J_{TM1} J_{TM2}-\frac{J_L^2}{n_1^2 n_2^2}\right)}\right] S^2+\left[\frac{K_{T1} K_{T2} K_{e1} K_{e2}}{R_1 R_2 (J_{TM1} J_{TM2}-J_L^2/n_1^2 n_2^2)}+\frac{R_1 K_{o2} K_{T2} J_{M1}+R_2 K_{o1} K_{T1} J_{M2}}{R_1 R_2 (J_{TM1} J_{TM2}-J_L^2/n_1^2 n_2^2)}\right] S+\frac{(K_{e1} K_{o2}+K_{e2} K_{o1}) K_{T1} K_{T2}}{R_1 R_2 (J_{TM1} J_{TM2}-J_L^2/n_1^2 n_2^2)}\right\}} \tag{48}$$

The figures of merit of the parallel control systems A and B are evaluated by the approximate equations, by substituting each constant from Table 1 into the above equations, (47) and (48). They are given in Tables 3 and 4.



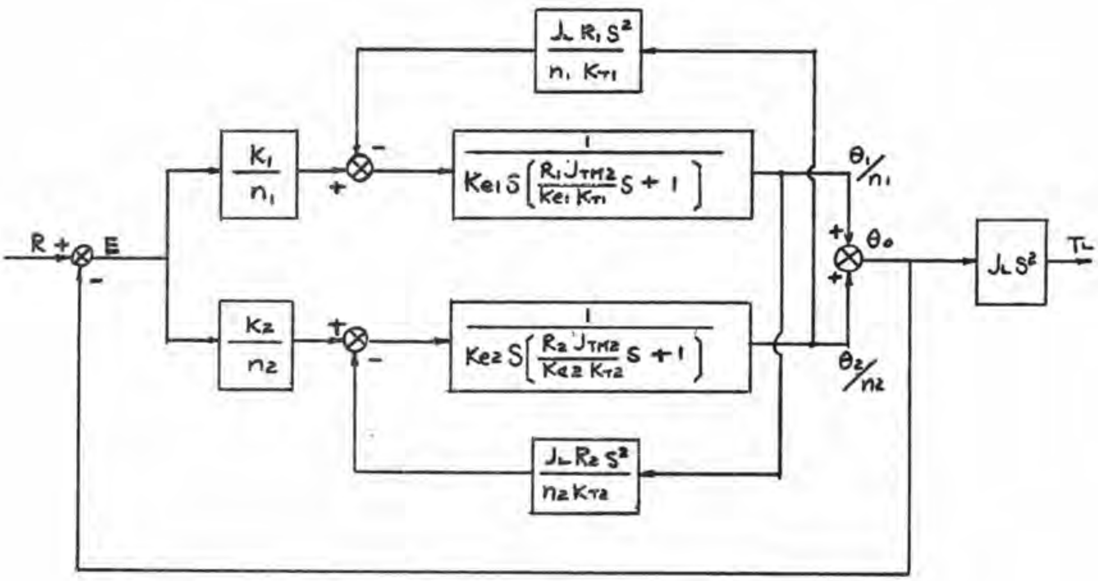
(a)



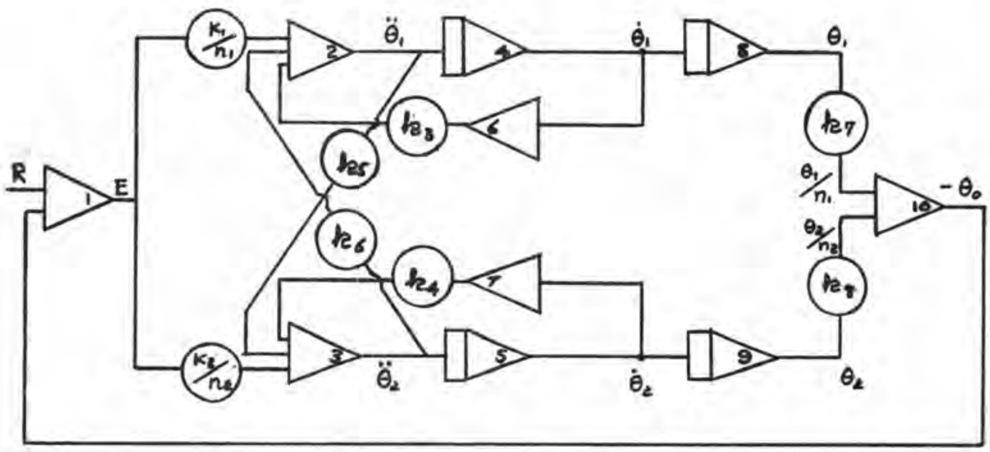
(b)

Figure 4. COMPUTER SIMULATION FOR THE COMPLETELY PARALLEL CONTROL SYSTEM A

(a) BLOCK DIAGRAM FOR THE COMPUTER SIMULATION
(b) COMPUTER SIMULATION



(a)



(b)

Figure 5. COMPUTER SIMULATION FOR THE COMPLETELY PARALLEL CONTROL SYSTEM B.

(a) BLOCK DIAGRAM FOR THE COMPUTER SIMULATION

(b) COMPUTER SIMULATION

Table 2(A). FIGURES OF MERIT OF THE TYPE 1 POSITION CONTROL SYSTEMS EVALUATED BY APPROXIMATE EQUATIONS.

Damping Ratio, ζ	Amount of the First Overshoot, Mp, Per-Unit	Motor & Load Time Constant, Tm, second	Amplifier Gain, K_0, volts/rads.	Undamped Natural Frequency ω_n, $second^{-1}$	The Peak Time, Tp, second	The Settling Time, Ts, second
0.50	1.16	0.05	30.0	20.0	0.182	0.40
0.50	1.16	0.10	20.0	10.0	0.362	0.80
0.50	1.16	0.25	8.1	4.0	0.904	2.00
0.50	1.16	0.40	4.4	2.5	1.440	3.20

Table 2(B). FIGURES OF MERIT OF THE TYPE 1 POSITION CONTROL SYSTEMS EVALUATED BY ANALOG COMPUTER

Damping Ratio, ζ	Amount of the First Overshoot, Mp, Per-Unit	Motor & Load Time Constant, Tm, second	Amplifier Gain, K_0, volts/rads.	The Peak Time, Tp second	The Settling Time, Ts, second
0.50	1.16	0.05	30.0	0.18	0.50
0.50	1.16	0.10	20.0	0.36	0.90
0.50	1.16	0.25	8.1	0.90	2.40
0.50	1.16	0.40	4.4	1.20	3.70

Table 3(A). FIGURES OF MERIT OF THE COMPLETELY PARALLEL CONTROL SYSTEM A EVALUATED BY APPROXIMATE EQUATIONS

Motor & Load Time Constant 1, Tm_1, second	Motor & Load Time Constant 2, Tm_2, second	Peak Time, Tp, second	Settling Time, Ts, second	Amount of the First Overshoot, Mp, Per-unit	Number of Oscillation, N, Per-unit	Damping Ratio, ζ, Per-unit	Undamped Natural Frequency, ω_n, per second
0.05	0.10	0.306	0.439	1.060	0.717	0.662	13.80
0.05	0.25	0.276	0.523	1.120	1.330	0.584	13.10
0.05	0.40	0.292	0.533	1.128	0.970	0.547	12.70
0.10	0.25	0.794	0.908	1.029	0.570	0.741	5.96
0.10	0.40	0.747	1.060	1.059	0.710	0.667	5.67
0.25	0.40	1.037	1.880	1.037	0.605	0.724	2.75

Table 3(B). FIGURES OF MERIT OF THE COMPLETELY PARALLEL CONTROL SYSTEM B EVALUATED BY ANALOG COMPUTER

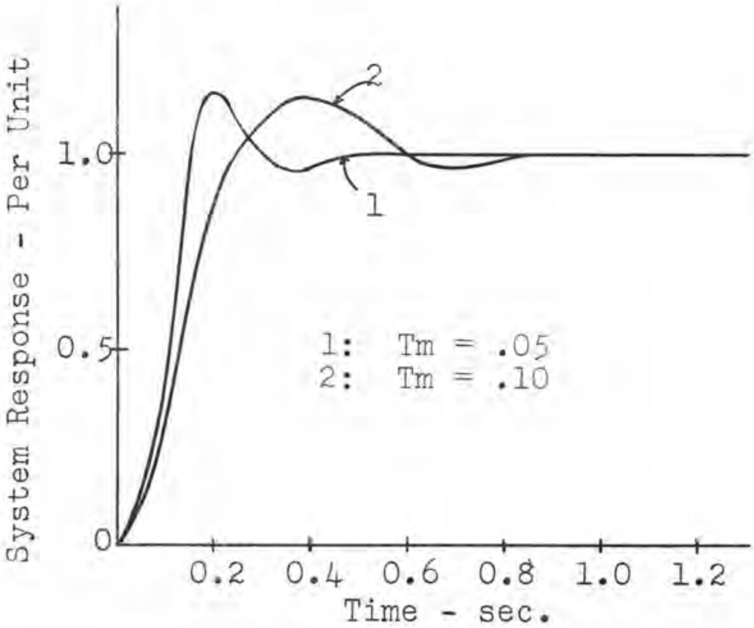
Motor & Load Time Constant 1, Tm_1, second	Motor & Load Time Constant 2, Tm_2, second	Peak Time, Tp, second	Settling Time, Ts, second	Amount of the First Overshoot, Mp, Per-unit	Number of Oscillation, N, Per-unit
0.05	0.10	0.309	0.480	1.10	0.9
0.05	0.25	0.280	0.550	1.18	1.3
0.05	0.40	0.290	0.580	1.19	1.0
0.10	0.25	0.820	1.100	1.05	0.8
0.10	0.40	0.810	1.120	1.10	1.0
0.25	0.40	1.100	1.950	1.08	0.8

Table 4(A). FIGURES OF MERIT OF THE COMPLETELY PARALLEL CONTROL SYSTEM B EVALUATED BY APPROXIMATE EQUATIONS.

Motor & Load Time Constant 1, Tm_1, second	Motor & Load Time Constant 2, Tm_2, second	Peak Time, Tp, second	Settling Time, Ts, second	Amount of the First Overshoot, Mp, Per-Unit	Number of Oscillation, N, Per-Unit	Damping Ratio, ζ, Per-Unit	Undamped Natural Frequency ω_n $second^{-1}$
0.05	0.10	0.444	1.09	1.212	1.210	0.469	7.79
0.05	0.25	0.567	1.05	1.106	0.401	0.835	4.47
0.05	0.40	0.737	1.58	1.210	0.502	0.788	3.22
0.10	0.25	0.512	1.21	1.178	1.240	0.462	7.14
0.10	0.40	1.190	2.29	1.160	0.810	0.619	2.83
0.25	0.40	1.140	3.48	1.253	1.580	0.374	3.08

Table 4(B). FIGURES OF MERIT OF THE COMPLETELY PARALLEL CONTROL SYSTEM B EVALUATED BY ANALOG COMPUTER.

Motor & Load Time Constant 1, Tm_1, second	Motor & Load Time Constant 2, Tm_2, second	Peak Time, Tp, second	Settling Time, Ts, second	Amount of the First Overshoot, Mp, Per-unit	Number of Oscillation, N, Per-unit
0.05	0.10	0.22	0.85	1.28	1.6
0.05	0.25	0.40	0.90	1.12	0.6
0.05	0.40	0.52	1.35	1.14	0.7
0.10	0.25	0.45	1.45	1.16	1.1
0.10	0.40	1.25	2.25	1.12	0.8
0.25	0.40	1.15	4.10	1.32	1.4



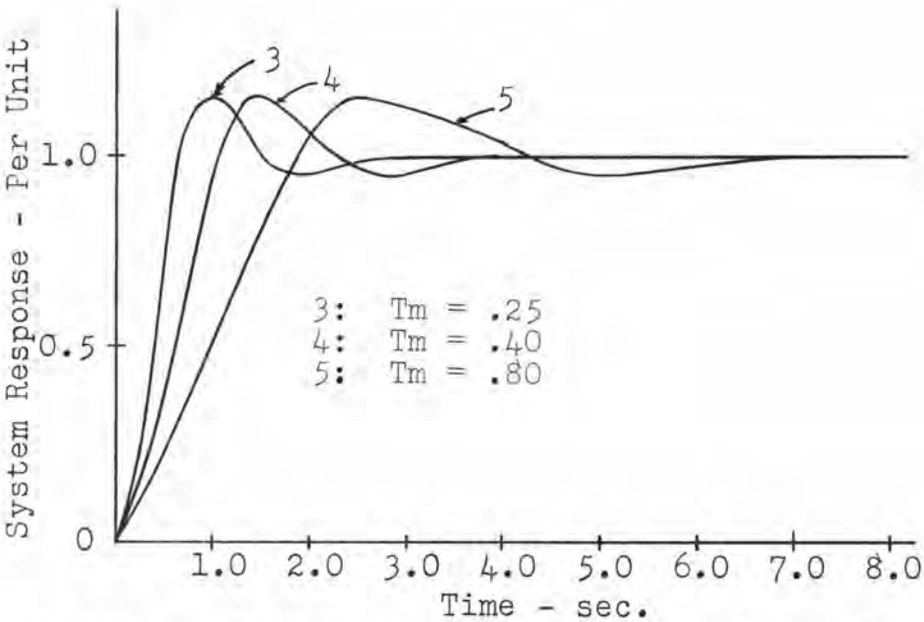


Figure 6. RESPONSE FOR THE TYPE 1 POSITIONAL CONTROL SYSTEM; WHERE DAMPING RATIO =0.5

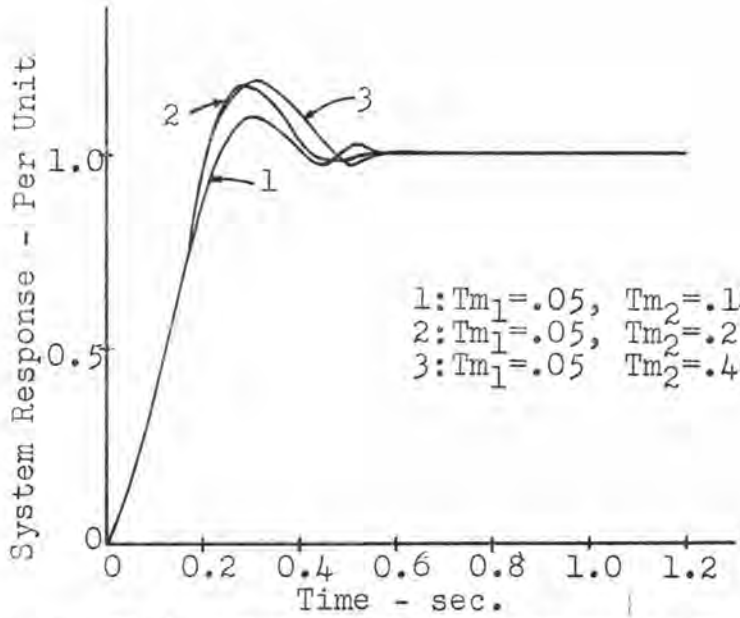


Figure 7(a). RESPONSE FOR VARIOUS COMBINATIONS OF THE COMPLETELY PARALLEL CONTROL SYSTEM A.

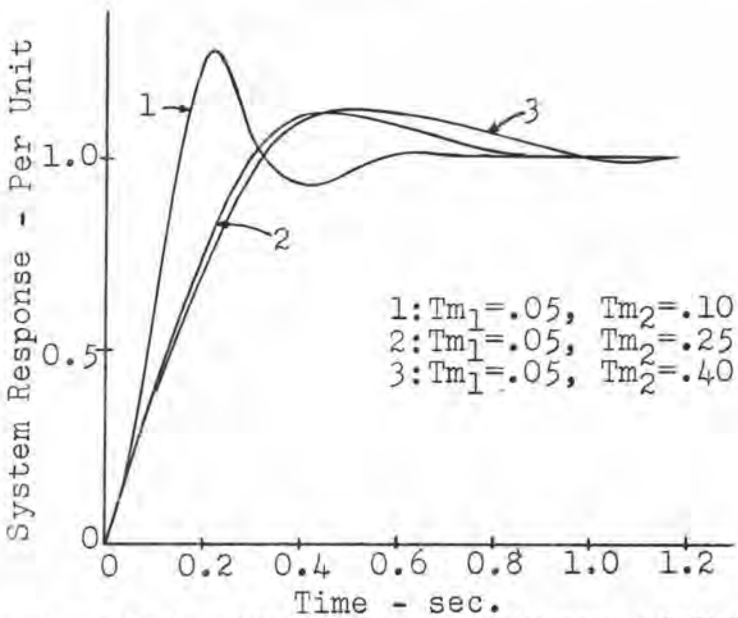


Figure 7(b). RESPONSE FOR VARIOUS COMBINATIONS OF THE COMPLETELY PARALLEL CONTROL SYSTEM B

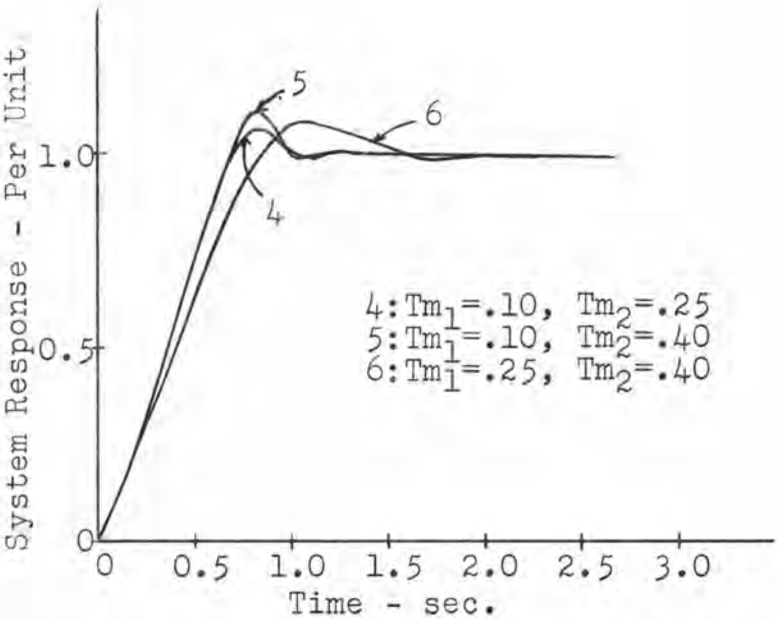


Figure 8(a). RESPONSE FOR VARIOUS COMBINATIONS OF THE COMPLETELY PARALLEL CONTROL SYSTEM A.

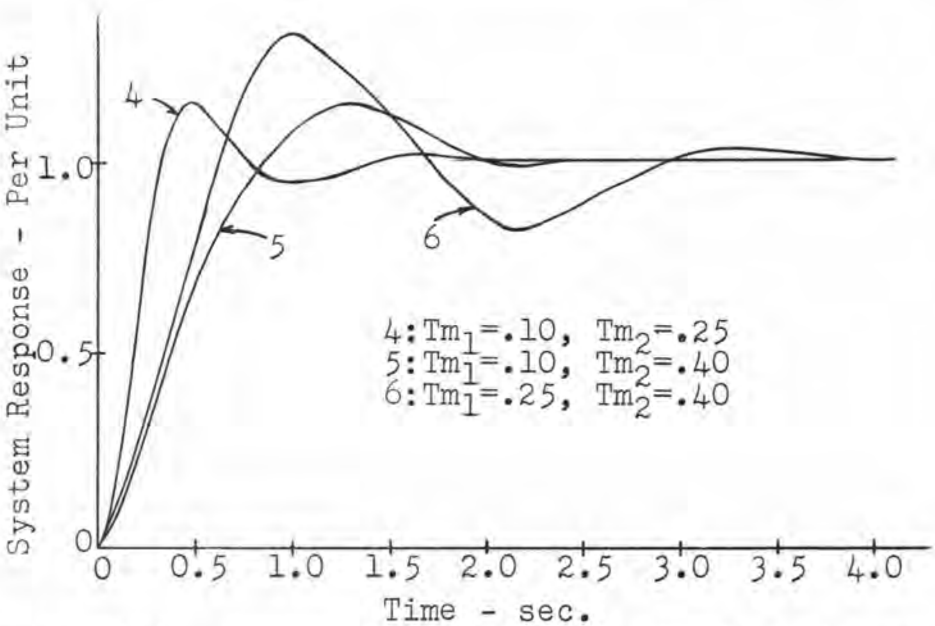


Figure 8(b). RESPONSE FOR VARIOUS COMBINATIONS OF THE COMPLETELY PARALLEL CONTROL SYSTEM B.

2. Simulation of an Analog Computer

A representation of the dynamics of parallel control systems on the analog computer is based on the dynamic equations as shown below:

For the completely parallel control system A,

$$e_1(s) = Td_1(s)R_1/K_{T1} + Ke_1 s\theta_1(s) \tag{49}$$

$$e_2(s) = Td_2(s)R_2/K_{T2} + Ke_2 s\theta_1(s) \tag{50}$$

For the completely parallel control system B,

$$e_1(s) = (R_1 J_{TM1} s^2/K_{T1} + Ke_1 s)\theta_1(s) + (J_L R_1/n_1 n_2 K_{T1}) s^2 \theta_2(s) \tag{51}$$

$$e_2(s) = (R_2 J_{TM2} s^2/K_{T2} + Ke_2 s)\theta_2(s) + (J_L R_2/n_1 n_2 K_{T2}) s^2 \theta_1(s) \tag{52}$$

The use of differentiators in the inner feedback loop shown in Figures 2(b) and 3(b) is avoided in simulating the dynamic equation on the analog computer.

Although the computer representation is usually such as to permit the identification of each component of the system as well as each physical variable characterizing the behavior of the component (12, p. 503-509), the simulation for this thesis was carried out on a modified system in order to eliminate the use of differentiators which are very troublesome. As a result, many system parameters are combined into a single system term.

The analog simulations for both the completely parallel control systems A and B are shown in Figures 4 and 5, respectively.

The system transient response curves were taken from this computer simulation. The figures of merit for system response can be obtained directly from the curves.

3. Transient Responses

Various combinations of two motor and load time constants are used to investigate the transient response of the parallel control systems, whose figures of merit are evaluated by two methods discussed in the previous sections. Both evaluations show a similar result.

The figures of merit of the type 1 positional control system given in Table 2 are referred, so that a comparison can be made with the transient response of the parallel control systems. All the transient responses for different motor and load time constants of type 1 positional control systems are so obtained that the damping ratio is maintained at 0.50 and the undamped natural frequency is changed. The damping ratio of 0.50 is generally a desirable value in system performance. At the same time, the gain corresponding to each system of the parallel control systems remains the same as in the type 1 positional control system with corresponding motor and load time constant. Consequently, the figures of merit for the parallel control systems A and B are evaluated and given in Tables 3 and 4, respectively. The transient response curves are also shown in Figures 6, 7 and 8.

In the completely parallel control system A, which is expressed by a transfer function of the type 1 system with one time constant, the significance of the transient response is mostly due to the smaller motor and load time constant. However, these combinations still show more increase in the settling time than the type 1 positional control system under the same conditions as before. The amount of the first overshoot is improved considerably over the type 1 positional control system.

If there is a slight difference in two motor and load time constants, the peak time is specified by the system with the smaller motor and load time constant, while the settling time is determined by the system with the larger motor and load time constant. The amount of the first overshoot is determined by the difference between the two motor and load time constants. For small differences it is quite similar to the type 1 positional control system, and for larger difference it becomes smaller.

In the completely parallel control system B, the peak time of system response is proportional to the sum of the peak time of each transient response for the type 1 positional control system with the corresponding motor and load time constant. The settling time is also approximately proportional to the settling time of each of the type 1 positional control system, except for a combination with a time constant of 0.4, for which the settling time of the system response is changed. As a whole, the first overshoot is improved, except for the following two sets of combinations, 0.05 and 0.10, and 0.25 and 0.40. A numerical comparison can be made from the data given in Table 2 and 4. However, since there is no exact relation between the transient responses of the type 1 positional control system and the parallel control systems as described above, a general statement cannot be made.

A comparison of the transient response of the completely parallel control systems A and B can be explained as follows; the peak time and settling time are improved in the completely parallel control system A over that of the system B; the amount of the first overshoot is improved in the completely parallel control system A over that of the system B, except for the combinations of 0.05 and 0.10, and 0.05 and 0.40, which are not improved.

It should be noted that it is difficult to determine the relationship between the factors to obtain the motor and load time constant, as studied in the previous section, so that the exceptions in the transient responses are due to the motor selection which is determined by the design view point.

Stability of the parallel control systems is a function of the system gain; therefore, the transient response can be varied by adjusting the system gain within a certain range. However, more desirable response can be achieved by the use of a compensator, designed on the basis of the figures of merit given above.

V. CONCLUSIONS

The method of combining two control systems in parallel determines the characteristics of the parallel control systems, and the application of these methods depends on the load control requirement. If the purpose of the control system is to obtain a higher torque and control a small range of position, the completely parallel control system A, using spur gears, is desirable. On the other hand, if the requirement is a small torque and a large range of position change, the completely parallel control system B, using differential gears, would be preferable.

The open-loop transfer function of the completely parallel control system A is a type 1 system with one time constant. For the completely parallel control system B, the open-loop transfer function belongs to the type 1 system with one zero and two poles. However, since the zero lies near one of the poles, their separate effects might be cancel, so that the transfer function of this system could be thought as being of the type 1 system with one time constant. Stability of both parallel control systems A and B is a function of the system gain.

A combination of two controlled systems in parallel is largely dependent on the design requirement of the transient performance for the over-all system. As far as the response time is concerned, the completely parallel control system A is more advantageous than the system B, while the amount of the first overshoot is taken into consideration, both system A and B give improvement. In general within a system, the combination of a controlled system having a smaller motor and load time constant is desirable. However, it is difficult to obtain the smaller motor and load time constant in the d-c motor design considerations.

The source impedance of motor power supply gives a certain range of variation in the motor and load time constant, in which the source impedance is approximately proportional to the motor and load time constant. One is reminded, however, that the design must be done from the economical view point so that a careful consideration should be given to changing the motor and load time constant. Based on the figures of merit of the parallel control system response obtained from the above procedure, a compensator is normally used to improvement of system performance.

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APPENDICES

APPENDIX I

THE OPEN-LOOP TRANSFER FUNCTIONS OF THE COMPLETELY PARALLEL CONTROL SYSTEMS

The general form of the open-loop transfer function is given by:

$$G(s) = K \; S^{\gamma} \frac{\prod_{j=1}^{m} (S - z_j)}{\prod_{k=1}^{n} (S - p_k)} \tag{53}$$

where K = the static loop sensitivity

γ = a negative integer including zero

Two transfer functions which are specified by various types of system with one time constant term are chosen from equation (53) and combined. The resultant transfer functions are given in Table 5.

Table 5. VARIOUS COMBINATIONS OF TRANSFER FUNCTIONS OF COMPLETELY PARALLEL CONTROL SYSTEMS.

Types of Combinations	Resultant Transfer Functions	Remarks
TYPE 0 & TYPE 0	$\frac{(k_1+k_2)(s+z_1)}{(s+p_1)\ (s+p_2)}$	$z_1 = \frac{k_1p_2+k_2p_1}{k_1 + k_2}$ k_1: gain of type 0 system k_2: gain of type 0 system
TYPE 0 & TYPE 1	$\frac{k_1(s+z_1)(s+z_2)}{s\ (s+p_1)(s+p_2)}$	k_1: gain of type 0 system k_2: gain of type 1 system *(1)

Table 5. (continued)

Types of Combinations	Resultant Transfer Functions	Remarks
TYPE 0 & TYPE 2	$\frac{k_1(s+z_1)(s+z_2)(s+z_3)}{s^2(s+p_1)(s+p_2)}$	k_1: gain of type 0 system k_2: gain of type 2 system *(2)
TYPE 1 & TYPE 1	$\frac{(k_1+k_2)(s+z_1)}{s(s+p_1)(s+p_1)}$ $\quad z_1 = \frac{k_1p_2+k_2p_1}{k_1+k_2}$	k_1: gain of type 1 system k_2: gain of type 1 system
TYPE 1 & TYPE 2	$\frac{k_1(s+z_1)(s+z_2)}{s^2(s+p_1)(s+p_2)}$	k_1: gain of type 1 system k_2: gain of type 2 system *(3)
TYPE 2 & TYPE 2	$\frac{(k_1+k_2)(s+z_1)}{s^2(s+p_1)(s+p_2)}$	$z_1 = \frac{k_1p_2+k_2p_1}{k_1+k_2}$ k_1: gain of type 2 system k_2: gain of type 2 system

WHERE

*(1) $$z_1 = \tfrac{1}{2}\left\{-\left(p_2 + \frac{k_2}{k_1}\right) + \left[\left(p_2 + \frac{k_2}{k_1}\right)^2 - 4\,\frac{k_2}{k_1}\,p_1\right]^{\frac{1}{2}}\right\}$$
$$z_2 = \tfrac{1}{2}\left\{-\left(p_2 + \frac{k_2}{k_1}\right) - \left[\left(p_2 + \frac{k_2}{k_1}\right)^2 - 4\,\frac{k_2}{k_1}\,p_1\right]^{\frac{1}{2}}\right\}$$

*(2) $$(s + z_1)(s + z_2)(s + z_3) = \left(s^3 + p_2 s^2 + \frac{k_2}{k_1}s + \frac{k_2}{k_1}\,p_1\right)$$

*(3) $$z_1 = \tfrac{1}{2}\left\{-\left(p_2 + \frac{k_2}{k_1}\right) + \left[\left(p_2 + \frac{k_2}{k_1}\right)^2 - 4\,\frac{k_2}{k_1}\,p_1\right]^{\frac{1}{2}}\right\}$$
$$z_2 = \tfrac{1}{2}\left\{-\left(p_2 + \frac{k_2}{k_1}\right) + \left[\left(p_2 + \frac{k_2}{k_1}\right)^2 - 4\,\frac{k_2}{k_1}\,p_1\right]^{\frac{1}{2}}\right\}$$

APPENDIX II

DERIVATION OF THE OPEN-LOOP TRANSFER FUNCTION OF THE COMPLETELY PARALLEL CONTROL SYSTEM A

The operational equations of spur gears that combine two systems in parallel are given by equations (7) and (8),

$$\theta_o = \theta_1/n_1 = \theta_2/n_2 \tag{7}$$

$$T_L = n_1T_1 + n_2T_2 \tag{8}$$

In derivation of the electric equations and torque equation for the d-c motor, the inductance term in the d-c motor armature circuit and the mechanical friction of the load are neglected because of the small value.

The electric equations for the d-c motors, 1 and 2, are given by,

$$e_1(s) = E(s)K_1 = Ia_1(s)\ R_1 + Ke_1 s\theta_1(s) \tag{54}$$

$$e_2(s) = E(s)K_2 = Ia_2(s)\ R_2 + Ke_2 s\theta_2(s) \tag{55}$$

The developed torques by two motors are shown as follows:

$$Td_1(s) = J_{M1}\ s^2\theta_1(s) + T_1(s) \tag{56}$$

$$Td_1(s) = J_{M2}\ s^2\theta_2(s) + T_2(s) \tag{57}$$

or

$$Ia_1(s)K_{T1} = J_{M1}\ s^2\theta_1(s) + T_1(s) \tag{58}$$

$$Ia_2(s)K_{T2} = J_{M2}\ s^2\theta_2(s) + T_2(s) \tag{59}$$

in which

$$Td_1(s) = Ia_1(s)K_{T1} \tag{60}$$

$$Td_2(s) = Ia_2(s)K_{T2} \tag{61}$$

Substituting equations (58) and (59) into equations (54) and (55), respectively, we obtain the following equations for the transmitted torques of two motors:

$$T_1(s) = \frac{K_{T1}}{R_1}\left[K_1E(s) - \left(\frac{R_1J_{M1}s^2}{K_{T1}} + Ke_1s\right)\theta_1(s)\right] \tag{62}$$

$$T_2(s) = \frac{K_{T2}}{R_2}\left[K_2E(s) - \left(\frac{R_2J_{M2}s^2}{K_{T2}} + Ke_2s\right)\theta_2(s)\right] \tag{63}$$

or

$$n_1T_1(s) = \frac{n_1^2K_{T1}}{R_1}\left[K_{01}E(s) - \left(\frac{R_1J_{M1}s^2}{K_{T1}} + Ke_1s\right)\theta_0(s)\right] \tag{64}$$

$$n_2T_2(s) = \frac{n_2^2K_{T2}}{R_2}\left[K_{02}E(s) - \left(\frac{R_2J_{M2}s^2}{K_{T2}} + Ke_2s\right)\theta_0(s)\right] \tag{65}$$

where $K_1 = n_1K_{01}$

$K_2 = n_2K_{02}$

Substituting equations (64) and (65) into equation (8), equation (8) then becomes

$$J_L\ s^2\theta_0(s) = \frac{n_1^2K_{T1}K_{01}}{R_1}E(s) - \left(\frac{R_1J_{M1}s^2}{K_{T1}} + Ke_1s\right)\frac{n_1^2K_{T1}}{R_1}\theta_0(s) +$$

$$\frac{n_2^2K_{T2}K_{02}}{R_2}E(s) - \left(\frac{R_2J_{M2}s^2}{K_{T2}} + Ke_2s\right)\frac{n_2^2K_{T2}}{R_2}\theta_0(s) \tag{66}$$

And

$$E(S)\left(\frac{n_1 K_{T1} K_1}{R_1} + \frac{n_2 K_{T2} K_2}{R_2}\right) - \theta_o(S)\left[J_L s^2 + \left(\frac{R_1 J_{M1}}{K_{T1}} s^2 + K_{e1} S\right)\frac{n_1^2 K_{T1}}{R_1} + \left(\frac{R_2 J_{M2}}{K_{T2}} s^2 + K_{e2} S\right)\frac{n_2^2 K_{T2}}{R_2}\right]$$

$$= \theta_o(S) S\left[(J_L + J_{M1} n_1^2 + J_{M2} n_2^2) S + \left(\frac{n_1 K_1 K_{T1}}{R_1} + \frac{n_2 K_2 K_{T2}}{R_2}\right)\right] \quad (67)$$

Consequently, the open-loop transfer function of the parallel control system A can be shown as follows:

$$G(S) = \frac{\theta_o(S)}{E(S)}$$

$$= \frac{(n_1 K_1 K_{T1} R_2 + n_2 K_2 K_{T2} R_1) / (n_1^2 K_{T1} K_{e1} R_2 + n_2^2 K_{T2} K_{e2} R_1)}{S\left[\frac{(J_L + J_{M1} n_1^2 + J_{M2} n_2^2) R_1 R_2}{R_2 n_1^2 K_{T1} K_{e1} + R_1 n_2^2 K_{T2} K_{e2}} S + 1\right]} \quad (68)$$

The block diagram as shown in Figure 2(b) was based on equations (64) and (65).

The open-loop transfer function can also be derived by the following way.

Substituting equations (60) and (61), into equations (54) and (55), the dynamic equations for the d-c motors can be shown by:

$$e_1(s) = \frac{R_1}{K_{T1}} Td_1(s) + Ke_1 \, s \, \theta_1(s) \quad (69)$$

$$e_2(s) = \frac{R_2}{K_{T2}} Td_2(s) + Ke_2 \, s \, \theta_2(s) \quad (70)$$

from which

$$Td_1(s) = \frac{K_{T1}}{R_1}\left[E(s)K_1 - n_1 Ke_1 s \theta_0(s)\right] \quad (71)$$

$$Td_2(s) = \frac{K_{T2}}{R_2}\left[E(s)K_2 - n_2 Ke_2 s \theta_0(s)\right] \quad (72)$$

Hence, substituting equations (56) and (57), into the operational equation of spur gears (8), we obtained the following equation:

$$n_1Td_1(s) + n_2Td_2(s) = n_1J_{M1}s^2\theta_1(s)+n_2J_{M2}s^2\theta_2(s)=n_1T_1(s)+n_2T_2(s)$$

$$= (n_1^2J_{M1} + n_2^2J_{M2} + J_L)s^2\theta_0(s) \quad (73)$$

$$= J_{ML}\ s^2\theta_0(s) = T_{ML}(s) \quad (74)$$

where

$$J_{ML} = n_1^2J_{M1} + n_2^2J_{M2} + J_L$$

From equations (71), (72) and (73), the resultant equation is obtained as follows:

$$\frac{n_1K_{T1}}{R_1}\left[E(s)K_1 \quad - n_1Ke_1s\theta_0(s)\right] +$$

$$\frac{n_2K_{T2}}{R_2}\left[E(s)\ K_2 \quad - n_2Ke_2s\theta_0(s)\right]$$

$$= (J_L + n_1^2J_{M1} + n_2^2J_{M2})\ s^2\theta_0(s) \quad (75)$$

Therefore, the open-loop transfer function of the completely parallel control system A becomes the same equation as given by equation (68).

APPENDIX III

DERIVATION OF THE OPEN-LOOP TRANSFER FUNCTION OF THE COMPLETELY PARALLEL CONTROL SYSTEM B

The operational equations for differential gears are given by:

$$\theta_0 = \frac{\theta_1}{n_1} + \frac{\theta_2}{n_2} \qquad (9)$$

$$T_L = n_1 T_1 = n_2 T_2 \qquad (10)$$

The torque equation of load is, in this case, shown as:

$$T_L(s) = J_L s^2 \theta_0(s) \qquad (76)$$

$$= J_L s^2 \left(\frac{\theta_1(s)}{n_1} + \frac{\theta_2(s)}{n_2}\right) \qquad (77)$$

According to equation (10), equation (77) can be written as follows:

$$T_1(s) = \frac{1}{n_1} J_L\, s^2\left(\frac{\theta_1(s)}{n_1} + \frac{\theta_2(s)}{n_2}\right)$$

$$= \frac{J_L}{n_1^2}\, s^2\, \theta_1(s) + \frac{J_L}{n_1 n_2}\, s^2 \theta_2(s) \qquad (78)$$

Similarly,

$$T_2(s) = \frac{J_L}{n_2^2}\, s^2 \theta_2(s) + \frac{J_L}{n_1 n_2}\, s^2 \theta_1(s) \qquad (79)$$

Therefore, the equations for the developed torque by two motors are given by:

$$Td_1(s) = J_{M1} s^2 \theta_1(s) + T_1(s)$$

$$= \left(J_{M1} + \frac{J_L}{n_1^2}\right) s^2 \theta_1(s) + \frac{J_L}{n_1 n_2}\, s^2 \theta_2(s)$$

$$= J_{TM1}\ s^2\theta_1(s) + \frac{J_L}{n_1 n_2}\ s^2\theta_2(s) \qquad (80)$$

and

$$Td_2(s) = J_{TM2}\ s^2\theta_2(s) + \frac{J_L}{n_1 n_2}\ s^2\theta_1(s) \qquad (81)$$

Furthermore, from equations (60) and (61), in Appendix II, the above two equations can be shown as:

$$Ia_1(s) = \frac{1}{K_{T1}}\left[J_{TM1}\ s^2\theta_1(s) + \frac{J_L}{n_1 n_2}\ s^2\theta_2(s)\right] \qquad (82)$$

$$Ia_2(s) = \frac{1}{K_{T2}}\left[J_{TM2}\ s^2\theta_2(s) + \frac{J_L}{n_1 n_2}\ s^2\ \theta_1(s)\right] \qquad (83)$$

Substituting equations (82) and (83), into the electric equations of motors,

$$e_1(s) = Ia_1 R_1(s) + Ke_1 s\theta_1(s) \qquad (84)$$

$$e_2(s) = Ia_2 R_2(s) + Ke_2 s\theta_2(s) \qquad (85)$$

we obtain the following two equations:

$$E(s) = \frac{R_1}{K_1 K_{T1}}\left[J_{TM1}\ s^2\theta_1(s) + \frac{J_L}{n_1 n_2}\ s^2\theta_2(s)\right] + \frac{Ke_1 s}{K_1}\ \theta_1(s)$$

$$= \left(\frac{R_1 J_{TM1}}{K_1 K_{T1}}\ s^2 + \frac{Ke_1}{K_1}\ s\right)\theta_1(s) + \frac{R_1 J_L}{K_1 K_{T1} n_1 n_2}\ s^2\theta_2(s) \qquad (86)$$

$$E(s) = \frac{R_2 J_L}{K_2 K_{T2} n_1 n_2}\ s^2\theta_1(s) + \left(\frac{R_2 J_{TM2}}{K_2 K_{T2}}\ s^2 + \frac{Ke_2}{K_2}\ s\right)\theta_2(s) \qquad (87)$$

in which

$$e_1(s) = E(s)K_1$$

$$e_2(s) = E(s)K_2$$

Therefore, $\theta_1(s)$ and $\theta_2(s)$ are:

$$\theta_1(s) = \frac{E(s)\left\{\left[\left(\frac{R_2 J_{TM2}}{K_2 K_{T2}} - \frac{J_L R_1}{K_1 K_{T1} n_1 n_2}\right) s^2\right] + \frac{Ke_2}{K_2}\, s\right\}}{\Delta} \tag{88}$$

$$\theta_2(s) = \frac{E(s)\left\{\left[\left(\frac{R_1 J_{TM1}}{K_1 K_{T1}} - \frac{J_L R_2}{K_2 K_{T2} n_1 n_2}\right) s^2\right] + \frac{Ke_1}{K_1}\, s\right\}}{\Delta} \tag{89}$$

where

$$\Delta = \frac{R_1 R_2}{K_1 K_2 K_{T1} K_{T2}} \left(J_{TM1} J_{TM2} - \frac{J_L^2}{n_1^2 n_2^2}\right) s^2 \left[s^2 + \frac{R_1 J_{TM1} K_{T2} Ke_2 + R_2 J_{TM2} K_{T1} Ke_1}{R_1 R_2 \left(J_{TM1} J_{TM2} - \frac{J_L^2}{n_1^2 n_2^2}\right)}\, s + \frac{K_{T1} K_{T2} Ke_1 Ke_2}{R_1 R_2 \left(J_{TM1} J_{TM2} - \frac{J_L^2}{n_1^2 n_2^2}\right)}\right] \tag{90}$$

From the operational equation (9)

$$\theta_0 = \frac{\theta_1}{n_1} + \frac{\theta_2}{n_2}$$

$\theta_0(s)$ becomes

$$\theta_0(S) = \frac{E(S)\left[\frac{n_1 R_1}{K_1 K_{T1}}\left(J_{TM1} - \frac{J_L}{n_2^2}\right) - \frac{n_2 R_2}{K_2 K_{T2}}\left(J_{TM2} - \frac{J_L}{n_2^2}\right)\right] S + \left(\frac{n_1 K_{e1}}{K_1} + \frac{n_2 K_{e2}}{K_2}\right)}{\frac{n_1 n_2 R_1 R_2}{K_1 K_2 K_{T1} K_{T2}}\left(J_{TM1} J_{TM2} - \frac{J_L^2}{n_1^2 n_2^2}\right) S \left[S^2 + \frac{K_{T2} K_{e2} R_1 J_{TM1} + K_{T1} K_{e1} R_2 J_{TM2}}{R_1 R_1 \left(J_{TM1} J_{TM2} - J_L^2 / n_1^2 n_2^2\right)} S + \frac{K_{T1} K_{T2} K_{e1} K_{e2}}{R_1 R_2 \left(J_{TM1} J_{TM2} - \frac{J_L^2}{n_1^2 n_2^2}\right)}\right]} \tag{91}$$

Therefore, the open-loop transfer function of the competely parallel control system B is obtained as follows:

$$G(S) = \frac{\theta_0(S)}{E(S)} = \frac{\left[\frac{R_1}{K_{01}K_{T1}} J_{M1} + \frac{R_2}{K_{02}K_{T2}} J_{M2}\right]S + \left(\frac{K_{e1}}{K_{01}} + \frac{K_{e2}}{K_{02}}\right)}{\frac{R_1R_2}{K_{01}K_{02}K_{T1}K_{T2}}\left(J_{TM1}J_{TM2} - \frac{J_L}{n_1^2 n_2^2}\right)S\left[S^2 + \frac{K_{T2}K_{e2}R_1J_{TM1} + K_{T1}K_{e1}R_2J_{TM2}}{R_1R_2\left(J_{TM1}J_{TM2} - \frac{J_L^2}{n_1^2 n_2^2}\right)}S + \frac{K_{T1}K_{T2}K_{e1}K_{e2}}{R_1R_2\left(J_{TM1}J_{TM2} - \frac{J_L^2}{n_1^2 n_2^2}\right)}\right]} \qquad (92)$$

where $K_1 = n_1K_{01}$

$K_2 = n_2K_{02}$

$J_{M1} = J_{TM1} - \frac{J_L}{n_1^2}$

$J_{M2} = J_{TM2} - \frac{J_L}{n_2^2}$

APPENDIX IV

THE MOTOR TIME CONSTANT IN DESIGN TERMS

The time constant of the motor and load is converted to design terms as follows: (9, p. 1093-1094)

$$Tm = \frac{RJ}{K_T Ke} \tag{29}$$

Total motor armature circuit resistance R:

$$R = \frac{\rho LZB_{ca}}{Ia\ a_A}\left[1 + \frac{\pi}{P_M \cos\theta}\left(\frac{D'}{L}\right) + (1.2\ K_c)\left(\frac{Bccp}{Bca}\right)\left(\frac{Pc}{P_M}\right)\right] \text{ ohms} \tag{93}$$

where Z = Total number of armature conductor

Bca = Armature conductor copper density in ampere per square inch

Ia = Armature current in amperes

a_A = Number of armature circuit

Bpf = Pole face flux density in lines per square inch

Total motor armature and load inertia J:

$$J = J_M + J_L = 6 \times 10^{-6}\ K_J\ D'^4\ L\left(1 + \frac{J_L}{J_M}\right)$$

Pound-feet per second squared (94)

in which $J_M = 6 \times 10^{-6}\ K_J D'^4 L$ Pound-feet per second squared (95)

where J_M and J_L are motor armature and load inertia referred to the motor shaft.

Motor torque constant, K_T:

$$K_T = 0.308 \times 10^{-8}\ \frac{B_{PF}\ D'\tau\ LZ}{a_A}$$

Pound-feet per amperes (96)

where τ = pole arc/pole pitch

d = armature slot depth in inches

Motor generated voltage, Ke:

$$Ke = 0.5 \times 10^{-8} \times \frac{B_{PF} D' \tau LZ}{aA} \quad \text{Volts per radian per second} \tag{97}$$

Motor time constant, Tm:

If equations (94), (95), (96), (97) are substituted into equation (93), the motor and load time constant becomes

$$Tm = 8.5 \times 10^4 \; K_J(1 + \frac{J_L}{J_M}) \times \left[1 + \frac{\pi}{P_M \cos\theta} (\frac{D'}{L}) + (1.2\; K_C)(\frac{Bccp}{Bca})(\frac{Pc}{P_M}) \right] / \; K_S B^2{}_{PF} \tau^2 \; (\frac{d}{D'}) \left[1 - 2(\frac{d}{D}) - \frac{B_{PF}}{K_i B_T} \right] \tag{98}$$

In order to make the motor time constant to a minimum the denominator of equation (98) should be a maximum. This can be done by use of the following values which are obtained from the design consideration:

$$B_{PF} = \tfrac{1}{2} K_i \; B_T \tag{99}$$

$$\frac{d}{D'} = 1/8 \tag{100}$$

Pole force density B_{PF} = 66.500 lines per square inch which is high by normal design standards, but which is theoretically possible.

With a sufficient number of armature slots the ratio of pole arc to pole pitch can be made equal to 0.7. The substituion of these values into equation (98) gives a minimum value of time constant at 75 degrees centigrades of

$$(Tm)\ \min = 1.25 \times 10^{-3} \frac{K_J}{K_S} \left(1 + \frac{J_L}{J_M}\right) \left[1 + \frac{\pi}{P_M \cos\theta} \left(\frac{D'}{L}\right) + (1.2\ K_C) \left(\frac{Bccp}{Bca}\right) \left(\frac{Pc}{P_M}\right) \right] \text{seconds} \qquad (101)$$