AN ABSTRACT OF THE THESIS OF

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(Name) (Degree)

in MATHEMATICS presented on
(Major) (Date)

Title: A PREDICTOR-CORRECTOR METHOD FOR SOLVING THE DIFFUSION EQUATION FOR WATER MOVEMENT IN UNSATURATED SOIL

Abstract approved: [redacted]
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The study of water movement in unsaturated soils by using appropriate diffusion equations has attracted considerable attention in recent years.

In this study, a numerical technique is developed for solving a generalized, dimensionless diffusion equation by the use of a digital computer. Diffusivity and capillary conductivity equations derived by Brooks and Corey [2] have been used instead of measured data; thus, the dimensionless diffusion equation can be applied to many soils and many kinds of fluids.

In this thesis, the diffusion equation has been solved for the one-dimensional, vertical case, where drainage was permitted from an initially saturated soil.

Some dimensionless curves of water content distribution in soil

profiles at various selected times are presented. Outflow data calculated as a function of time is also included. The computer solutions using the Brooks-Corey approximations seem to fit the experimental data very well. It therefore seems that the diffusion equation solved here is a satisfactory model for water movement in unsaturated soil.

An ALGOL program is included in the Appendix.

A Predictor-Corrector Method for Solving the Diffusion Equation for Water Movement in Unsaturated Soil

by

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A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Master of Arts

June 1968

APPROVED:

[redacted]

Professor of Mathematics

in charge of major

[redacted]

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Date thesis is presented November 30, 1967

Typed by Clover Redfern for Benedict Loil-Chong NG

ACKNOWLEDGMENTS

I wish to extend my sincere gratitude and appreciation to Dr. Ronald B. Guenther for his invaluable advice and guidance throughout this thesis; it could not have been accomplished otherwise.

Thanks is also extended to Dr. Darrel Aufenkamp, the director of Computer Center at Oregon State University for the computer time on the CDC 3300 which was made available for this thesis.

I am also indebted to Dr. John Wolfe and Dr. R. H. Brooks for their lectures which provide me with a solid background to make this study possible.

To my parents and friends for their efforts to bolster my morale during the discouraging moments.

Special thanks go to the Western Regional Research Project W-51, and Oregon State Agricultural Experiment Station, by which this research was supported in part.

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LIST OF SYMBOLS

Symbol	Definition	Dimension
D	Diffusivity, $D(\theta) = K(\theta)\frac{\partial\Phi}{\partial\theta}$	L^2T^{-1}
f	Drainable porosity, the ratio of fluid drained to the volume of porous medium.	none
g	Acceleration due to gravity.	$L\ T^{-2}$
$\hat{g}$	Acceleration gravity vector.	$L\ T^{-2}$
H	Fluid pressure head, $H = p/\rho g$.	$F\ L^{-2}$
K	Unsaturated capillary conductivity, $K = f(\theta)$.	$L\ T^{-1}$
K_o	The capillary conductivity of the medium at saturation.	$L\ T^{-1}$
L	Length of soil profile.	L
p	Pressure.	$F\ L^{-2}$
p_b	Bubbling Pressure--the pressure at the height of capillary fringe depth.	$F\ L^{-2}$
p_c	Capillary pressure--a property of porous medium related to capillary fringe height.	$F\ L^{-2}$
Q	Accumulated outflow volume per unit area at any time, t.	L
v	Liquid volume flux.	$L\ T^{-1}$
$\hat{v}$	Liquid volume flux vector.	$L\ T^{-1}$
z	Vertical coordinate direction.	L

Symbol	Definition	Dimension
θ	Water content of medium, volume of fluid/ volume of porous medium.	none
θ'	Water content of medium, weight of fluid/ weight of porous medium.	none
μ	Viscosity.	$F\ L^{-2}T$
ρ	Fluid density.	$F\ T^{2}L^{-4}$
ρ'	Bulk density of the porous medium.	$F\ L^{-3}$
ϕ	Porosity, the ratio of the total pore volume to the volume of the medium.	none
ϕ_e	Effective porosity--the ratio of effective pore space to the volume of the medium.	none
div	Divergence operator.	L^{-1}
Δ	A difference.	none
λ	Pore-size distribution index.	none
Φ	Total potential, $\Phi = H + z$.	L

A PREDICTOR-CORRECTOR METHOD FOR SOLVING THE DIFFUSION EQUATION FOR WATER MOVEMENT IN UNSATURATED SOIL

I. INTRODUCTION

In the past, a mathematical treatment of ground water movement has been restricted to only a few soils. In these cases many simplifying assumptions have been made which limit either the types of boundary conditions which can be imposed or the soils are assumed to satisfy certain idealized conditions which are almost never encountered in the field. This has been caused in part by inadequate experimental tools, only a limited number of mathematical models of ground water movement, and the inadequate application of numerical techniques using digital computers to these problems. The methods developed in this thesis are applicable to many soils and to many kinds of fluids. By introducing the pore-size distribution index defined by Brooks and Corey [2], and by making use of Darcy's law, a non-linear diffusion equation governing the flow is derived. The equation is reduced to dimensionless units and is solved numerically by means of a predictor-corrector finite difference scheme, which was introduced by Douglas and Jones [8]. The advantages of this predictor-corrector method over various other finite difference methods are that it is algebraically explicit, it is second order accurate in both the space and time variables, and it is unconditionally stable.

The solution of the outflow as a function of time is also included; in general, it can be obtained through the solution of the diffusion equations.

Experimental results are also compared with the computed values to determine how well the mathematical model describes the movement of water in an unsaturated soil.

II. THE WORK OF DARCY AND DARCY'S LAW

Darcy's Experiment

Darcy's law has been used by many soil scientists and petroleum engineers to describe the flow of fluids through a homogeneous, porous medium, and from it diffusion equations are derived. For a more complete discussion of the implications of Darcy's law and a derivation of it from the Navier-Stokes equations, see Hubbert [13].

Henri Darcy, a French hydraulic engineer, was comissioned by the city of Dijon to design a water purification system for the city. Between October 29 and November 2, 1855 and February 17 to February 18, 1856, he conducted a series of experiments to try to determine how large a filter (in this case a bed of sand) would be needed to filter a certain quantity of water per day. The experiment and the results are discussed in the appendix of his book Les Fontaines Publiques De La Ville De Dijon. The results of these experiments led Darcy to the formulation of an empirical "law", which states that the flow rate is proportional to the difference in heights of properly placed manometers above a reference level divided by the thickness of the bed of sand. This relationship has since come to be known as Darcy's law and is the fundamental equation in the study of the flow of fluids through porous media. It turns out that Darcy's law is analogous to Fourier's law in the theory of heat conduction, Fick's or

Nernst's law in the theory of diffusion, and Ohm's law in the theory of electricity.

A schematic drawing of Darcy's experiment is shown in Figure 2..1. A homogeneous filter bed of length ℓ is bounded by horizontal plane areas of equal size A. Both these areas are congruent so that corresponding points could be connected by vertical straight lines. The filter bed is percolated by an incompressible fluid. If open manometer tubes are attached at the upper and lower boundaries of the filter bed, the liquid rises to the heights h_2 and h_1, respectively, above an arbitrary datum level. By adjustment of the inlet the outlet valves, the water was made to flow downward through the sand at a series of successively increasing rates. For each rate a reading of the manometers was taken and recorded as a pressure difference in meters of water above the bottom of the sand. The results of these series, using different sands are shown graphically in Figure 2. 2. In each instance it is seen that the total rate of discharge increases linearly with the drop of the two equivalent water manometers.

As stated above, the results of his experiment led Darcy to conclude that the volume of water crossing a unit area perpendicular to the flow in unit time is given by

$$v = - k \frac{h_2 - h_1}{\ell} . \tag{2.1}$$

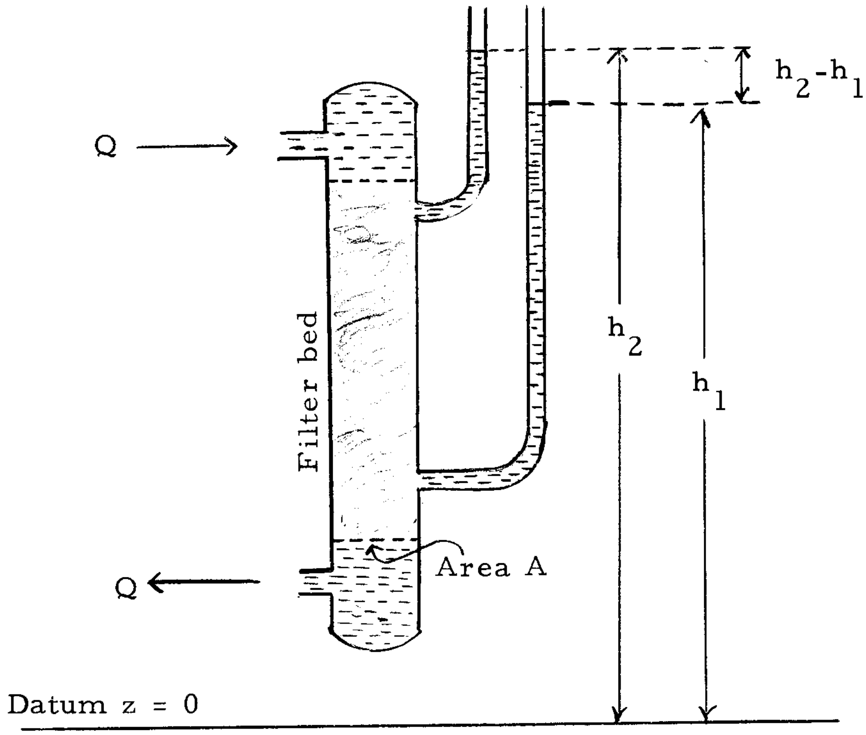

Figure 2.1. Darcy's filtration experiment.

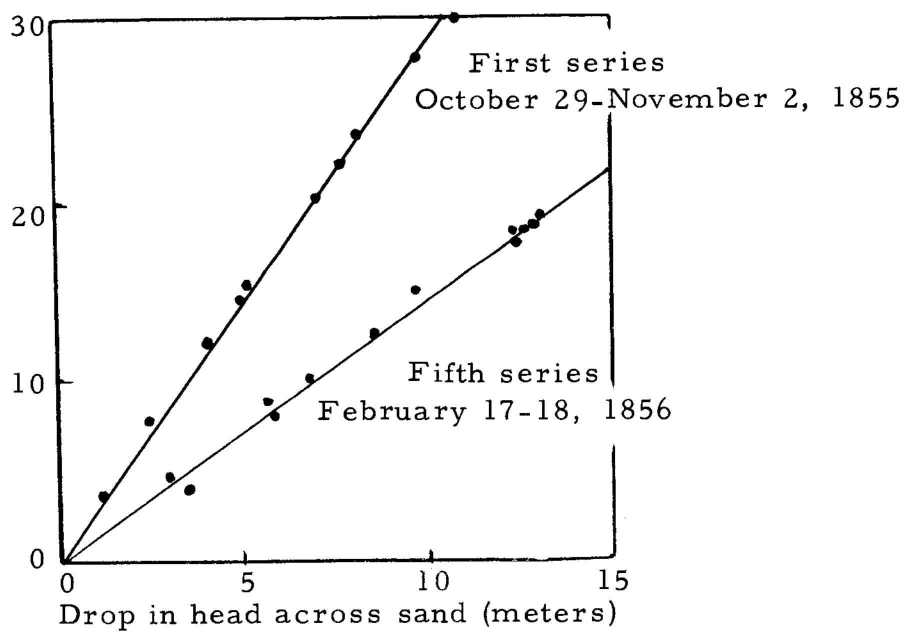

Figure 2.2. Graphs compiled from Darcy's tabular data on his experiments of Oct. 29 to Nov. 2, 1855, and of Feb. 17-18, 1856, showing linear relation between flow rate and differences in heights of equivalent water manometers.

Here k is a proportionality factor, ℓ is the thickness of the sand, h_1 and h_2 the heights above a reference level of the water in manometers terminating above and below the sand, respectively. (2. 1) is the empirical form of Darcy's law as he stated it.

A More Useful Form of Darcy's Law

Now consider every point in three-dimensional space, there must exit a particular value of a scalar quantity h, defined as the height above a standard elevation datum of the water column in a manometer terminated at the given point. The set of such values then gives rise to a scalar field in the quantity h with a corresponding family of surfaces, h = constant. In such a scalar field, water will flow in the direction perpendicular to the surfaces, h = constant, and at a rate given by

$$\hat{v} = - k \text{ grad } h. \tag{2.2}$$

Observe that this is just the differential form of Darcy's law stated in (2. 1) which would be obtained by formally passing to the limit as ℓ tends to zero.

In view of the experimental data, k is a constant depending on the properties of the fluid and of the porous medium, that is

$$k \sim (\rho/\mu) d^2, \tag{2.3}$$

which $\sim$ meaning that k is proportional to $(\rho/\mu)d^2$, and ρ is the density of the fluid, μ is the viscosity of the fluid and d is a specified length such as the mean grain diameter, which characterizes the size scale of the pore structure of the sand.

Substituting (2. 3) into (2. 2) gives

$$v = k'(\rho/\mu)d^2(-\text{grad } h), \tag{2.4}$$

where k' is a new proportionality factor coming from (2. 3) and containing all other soil and fluid properties not yet discussed.

Now consider the pressure at any point (x, y, z) in the flow system

$$p = \rho g(h-z)$$

where z is an elevation above some reference level and h is the height of fluid in the manometer.

It follows that

$$h = p/(\rho g) + z \tag{2.5}$$

and

$$-\text{grad } h = -1/(\rho g) \text{ grad } p - \text{grad } z$$

or

$$-g \text{ grad } h = -(1/\rho) \text{ grad } p - g \text{ grad } z \tag{2.6}$$

where grad z is a unit vector directed upward.

Let $\hat{g}$ be a vector of magnitude g and directed downward, then (2.6) becomes

$$(2.7) \qquad - g \text{ grad } h = - (1/\rho) \text{ grad } p + \hat{g},$$

which means the resultant force, $-$ grad h, of the fluid is proportional to the gradient of the fluid pressure and the force exerted upon a unit mass of fluid by the force of gravity. Hence, the fluid flow is related to the force of gravity. Introducing the gravitational term into (2.4) gives

$$(2.8) \qquad v = Nd^2(\rho/\mu)(-g \text{ grad } h),$$

where $N = k'/g$, a final factor of proportionality.

A dimensional analysis of (2.8) shows that N is dimensionless, and it must be related to the shape of the passages through which the flow occurs.

If we assume g is a constant we can write

$$- g \text{ grad } h = - \text{ grad } (gh).$$

Let

$$(2.9) \qquad \Phi' = gh$$

where Φ' is the potential energy per unit mass of the fluid at some point z.

Substituting the definition of Φ' into (2. 8) we obtain

$$v = - Nd^2(\rho/\mu)\ \text{grad}\,\Phi', \tag{2.10}$$

where Φ' has the dimension of L^2T^{-2}.

Equation (2. 10) is also usually called Darcy's law, and we shall use this form to derive our diffusion equation.

Although Darcy performed his experiment in a saturated medium, it is permissible to apply Darcy's law in the case of an unsaturated medium. In an unsaturated medium there are pores which are wholly or partly filled with air. For a partially saturated medium, curved interfaces exist between air and water. If we assumed that all the air space is replaced by a solid, and the liquid phase is interconnected then the medium can be viewed as a saturated medium, this does not in itself invalidate Darcy's law. See Klute [15].

III. DERIVATION OF DIFFUSION EQUATION

The Dimensional Case

Fairly exhaustive treatments of fluid flow through porous media are given by Collins [5], Polubarinova-Kochina [20], Muskat [19], and Scheidgger [22], but they are not broad enough to cover the problems of fluid flow in unsaturated media. It is therefore necessary to derive our diffusion equation for unsaturated water movement.

The discussion of capillary conduction of fluid through porous media has been given by many authors, see, e. g. Buckingham [3], Richards [21], and Gardner et al [11]. Buckingham defined a term, capillary conductivity and denoted it by the letter K, as the volume of water crossing a unit area perpendicular to the flow in a unit of time when there is a unit potential[1/] gradient across the soil. He also recognized that the capillary conductivity term was not a constant but a function of water content θ, where θ is the ratio of the volume of fluid occupying the pores to the bulk volume of the medium. K has a dimension of LT^{-1} and θ of L^3/L^3.

We return to (2. 10) and define a new potential Φ,

$$\Phi = \Phi'/g$$

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[1] Potential here refers to the specific work (work per unit mass) that must be done to pull an infinitestimal increment of water content from the soils.

to obtain

$$\hat{v} = - K \operatorname{grad} \Phi \tag{3.1}$$

where $K = Nd^2(\rho/\mu)g$.

A dimensional inspection shows that K has the dimensions of LT^{-1} in (3.1) and is in agreement with the dimension of capillary conductivity K. Therefore

$$\hat{v} = - K(\theta) \operatorname{grad} \Phi \tag{3.2}$$

since K is a function of water content, θ.

In this thesis the total potential,[2/] Φ, will be considered as the sum of two components, the capillary and the gravitational potentials. There may be other potentials, such as osmotic and adsorptive potentials, depending upon whether or not the pertinent force fields are present and act upon the water in the porous medium, but these will be neglected.

Let

$$\Phi = H + z \tag{3.3}$$

where $H = p/(\rho g)$ is the pressure head of fluid and z, the height above an arbitrary reference level. The elevation z, is taken

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[2] See footnote 1.

positive in the upward direction. The fluid pressure head in an unsaturated medium is usually called the capillary suction head or capillary pressure head.

Substituting (3. 3) into (3. 2), one gets

$$\hat{v} = - K(\theta) \operatorname{grad} (H+z), \tag{3.4}$$

or letting $\hat{v} = (v_x, v_y, v_z)$ be the components of the velocity in the x-direction, y-direction and the z-direction or vertical direction, respectively we find

$$v_z = - K(\theta) \frac{\partial H}{\partial z} - K(\theta) \tag{3.5}$$

The capillary suction head is given by $-\frac{1}{g}\int \frac{dp}{\rho}$, provided that the fluid density ρ depends on p alone or is constant. Buckingham [3] noted that the capillary suction head is a function of water content, i. e. $H = H(\theta)$; therefore, (3. 5) can be rewritten in the form

$$v_z = - K(\theta) \frac{\partial H}{\partial \theta} \frac{\partial \theta}{\partial z} - K(\theta). \tag{3.5}$$

Defining the diffusivity function $D(\theta)$ by the equation

$$D(\theta) = K(\theta) \frac{\partial H}{\partial \theta} \tag{3.6}$$

one has from (3. 5)

(3.7) $$v_z = - D(\theta) \frac{\partial \theta}{\partial z} - K(\theta).$$

Observing that $\frac{\partial \Phi}{\partial x} = \frac{\partial H}{\partial x}$ and $\frac{\partial \Phi}{\partial y} = \frac{\partial H}{\partial y}$, (3.4) reduces to

(3.8) $$v_x = - D(\theta) \frac{\partial \theta}{\partial x}$$ for flow in the x direction and,

(3.9) $$v_y = - D(\theta) \frac{\partial \theta}{\partial y}$$ for flow in the y direction.

The equation of continuity states that the time rate of change of fluid density ρ is equal to the negative of the divergence of the mass flux $\rho \hat{v}$, i.e.

(3.10) $$\frac{\partial \rho}{\partial t} = - \operatorname{div} (\rho \hat{v}),$$

which expresses a material balance with no source or sink, or, in other words, the accumulation equals the flow in minus the flow out, and it holds for every point in the fluid. Thus, the fluid density is changing with time. This type of flow is referred to as transient flow.

In the steady state case the fluid density, ρ is independent of time and (3.10) simplifies to

(3.11) $$\operatorname{div} (\rho \hat{v}) = 0.$$

In a porous medium, not all the space per unit volume is available for fluid flow; therefore, (3.10) may be written as

(3. 12) $$\frac{\partial}{\partial t}(\rho'\theta') = - \text{div}\,(\rho\hat{v})$$

where ρ' is the bulk density of the soil and θ' is the water content on a weight basis, i. e. θ' = (weight of water)/(bulk weight of the porous medium).

From (3. 12)

(3. 13) $$\frac{\partial}{\partial t}(\rho'\theta') = - \text{div}(\rho\hat{v}) = -(\text{grad}\,\rho)\hat{v} - \rho\,\text{div}\,\hat{v}$$

and when ρ is identically a constant, corresponding to the flow of a homogeneous fluid, grad $\rho = 0$ and

(3. 14) $$\frac{\partial}{\partial t}(\rho'\theta') = - \rho\,\text{div}\,\hat{v}.$$

Since $\theta = \rho'\theta'/\rho$ where θ is water content on a volume basis, i. e. θ = (volume of water)/(bulk volume of the porous medium), we can write (3. 14) as

(3. 15) $$\frac{\partial\theta}{\partial t} = - \text{div}\,\hat{v}$$

Putting equations (3. 7), (3. 8) and (3. 9) into (3. 15) gives

(3. 16) $$\frac{\partial\theta}{\partial t} = \text{div}\,D(\theta)\,\text{grad}\,\theta + \frac{\partial K(\theta)}{\partial z},$$

which is the general equation for the theoretical transient flow of a homogeneous fluid in an isotropic porous medium.

The study is limited to one-dimensional vertical isotropic flow in cartesian co-ordinates. For these conditions (3. 16) becomes

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left[D(\theta)\frac{\partial \theta}{\partial z}\right] + \frac{\partial K(\theta)}{\partial z} \tag{3.17}$$

where θ is the water content (L^3/L^3), t the time (T), D the diffusivity (L^2T^{-1}), z the distance (L) and K the capillary conductivity (LT^{-1}).

Unfortunately (3. 17) is non-linear and no closed form solution has been found. We therefore use a numerical technique to solve (3. 17). It is desirable to write (3. 17) in dimensionless form by introducing some scaling factors so the solutions can be applied to all soils and fluids for which these equations hold. For this, we follow closely the paper by Corey et al [6].

The Dimensionless Case

Before we change (3. 17) into a dimensionless form, it is necessary to define a few terms here.

The porosity, denoted by ϕ, is defined by

$$\phi = \frac{\text{total pore volume}}{\text{volume of porous medium}} .$$

The pores in a porous system may be interconnected or non-interconnected. The flow of fluid, is possible only if at least part of the pore space is interconnected. The interconnected part of the pore system is called the effective pore space of the porous medium.

The effective porposity, denoted by ϕ_e, is defined by

$$\phi_e = \frac{\text{effective pore space}}{\text{volume of porous medium}} .$$

The capillary pressure, p_c, is defined as the pressure difference between non-wetting fluid (usually air in drainage problem) and the wetting fluid

$$p_c = p_{nw} - p_w .$$

For a soil-liquid system, $p_c = p_a - p_w$, where p_a is the air pressure and p_w is the liquid pressure. Since the non-wetting pressure is considered to be constant throughout the system and equal to the atmospheric pressure, we have $p_c = -p_w$ if atmospheric pressure is selected as the pressure reference level.

The bubbling pressure, p_b, is approximately the smallest capillary pressure on the drainage cycle at which a continuous non-wetting phase exits in a porous medium. It received its name from the fact that experimentally it was found to be very close to the capillary pressure at which the first gas flow can be observed.

The effective water content, denoted by θ_e is defined by $\theta_e = \theta - \theta_r$, where θ_r is the residual water content at which the capillary conductivity is assumed to approach zero.

Let us define the pore diameter (denoted by δ) at any one point within the pore space to be the diameter of the largest sphere which contains this point and remains wholly within the pore space. Thus, to each point of the pore space a "diameter" could be attached rigorosly; and the pore size distribution is then determined by that fraction α of the total pore space has a pore diameter between δ and $\delta + \Delta\delta$.

The negative of the slope of the curve θ_e/ϕ_e as a function of $p_c/(\rho g)$ is designated as λ, and it is called the pore-size distribution index of the medium. It characterizes the pore size distribution of a medium. For a more detailed discussion, see Brooks and Corey [2].

To write (3. 17) in non-dimensional form we choose a system of standard units and denote them zero subscripts.

These standard units will be:

$K_o = K$ the capillary conductivity of the medium at saturation.

$L_o = p_b/(\rho g)$ the bubbling pressure head.

$\theta_o = \phi_e$ the effective porosity.

(3.18)

$$t_o = \frac{\theta_o L_o}{K_o}$$

$$D_o = \frac{L_o K_o}{\lambda \theta_o}$$

Scaling Equation (3.17) yields

$$\frac{\theta_o \partial(\theta/\theta_o)}{\partial t} = \frac{\partial}{L_o \partial(z/L_o)}\left[\frac{D}{D_o} D_o \frac{\theta_o \partial(\theta/\theta_o)}{L_o \partial(z/L_o)} + \left(\frac{K}{K_o}\right) K_o\right],$$

or

$$\frac{\theta_o \partial(\theta/\theta_o)}{\partial t} = \frac{\partial}{L_o \partial(z/L_o)}\left[\frac{D}{D_o} \frac{K_o}{\lambda} \frac{\partial(\theta/\theta_o)}{\partial(z/L_o)} + \left(\frac{K}{K_o}\right) K_o\right],$$

i.e.

$$\frac{\partial(\theta/\theta_o)}{\partial(tK_o/L_o\theta_o)} = \frac{\partial}{\lambda \partial(z/L_o)}\left[\left(\frac{D}{D_o}\right) \frac{\partial(\theta/\theta_o)}{\partial(z/L_o)} + \lambda\left(\frac{K}{K_o}\right)\right]$$

and finally

(3.19) $$\frac{\partial(\theta/\phi_e)}{\partial(t/t_o)} = \frac{\partial}{\lambda \partial(z\rho g/p_b)}\left[\frac{D}{D_o} \frac{\partial(\theta/\phi_e)}{\partial(z\rho g/p_b)} + \lambda\left(\frac{K}{K_o}\right)\right]$$

where $D = D(\theta)$ and $K = K(\theta)$.

The notation of Eq. (3. 19) is further simplified by letting

$$\theta. = \theta_e/\phi_e$$

$$z. = z\rho g/P_b$$

(3.20) $$t. = t/t_o$$

$$D. = D/D_o$$

$$K. = K/K_o$$

$$L. = L/L_o$$

where the dots denote scaled variables. Since $\frac{\partial\theta}{\partial t} = \frac{\partial\theta_e}{\theta t}$ and $\frac{\partial\theta}{\partial z} = \frac{\partial\theta_e}{\partial z}$, then applying (3.20), (3. 19) becomes

(3.21) $$\frac{\partial\theta.}{\partial t.} = \frac{1}{\lambda}\frac{\partial}{\partial z.}[D.\frac{\partial\theta.}{\partial z.} + \lambda K.]$$

where $D. = D.(\theta.)$ and $K. = K.(\theta.)$

It is to be noted that at complete saturation, the effective water content, θ_e is equal to the effective porosity, hence $\theta.$ at saturation is equal to 1.

<u>Methods for Determining Water Diffusivity and Capillary Conductivity</u>

The capillary conductivity can be determined by the equation

(3.22) $$K(\theta) = -\frac{v_z}{\partial\Phi/\partial z}$$

where the potential gradient $\frac{\partial \Phi}{\partial z}$ is easily measured in the laboratory by using a tensiometer or some other pressure measuring devices. The volume flux, v_z, is simply the rate of flow at some θ. By repeating the same procedure for different water contents, a capillary conductivity-water content curve can be obtained.

Since

$$D(\theta) = K(\theta)\frac{\partial H}{\partial \theta}$$

the diffusitivity function is equal to the capillary conductivity times the slope of the water-pressure head curve, taken at some water content. We then obtain a diffusivity-water content curve. By standard curve fitting techniques, e.g. least squares, we can find an equation that will adequately describe this curve.

Brooks and Corey [2] found that for many porous media the functional relationships between capillary conductivity and effective water content could be expressed by the following equations:

(3. 23) $$\frac{\theta_e}{\phi_e} = \left(\frac{p_b}{p_c}\right)^{\lambda} \qquad \text{for } p_c \geq p_b$$

and

(3. 24) $$\frac{K}{K_o} = \left(\frac{p_b}{p_c}\right)^{2+3\lambda} \qquad \text{for } p_c \geq p_b.$$

Substituting (3. 23) into (3. 24) gives

(3.25) $$\frac{K}{K_o} = (\frac{\theta_e}{\phi_e})^{(2+3\lambda)/\lambda} ,$$

and finally

(3.26) $$K_. = \theta_.^{(2+3\lambda)/\lambda} .$$

From (3.23) we have

$$(\frac{\theta_e}{\phi_e})^{\frac{1}{\lambda}} = \frac{p_b}{p_c}$$

and

(3.27) $$\frac{p_c}{\rho g} = \frac{p_b}{\rho g}(\frac{\theta_e}{\phi_e})^{-\frac{1}{\lambda}} .$$

Taking the partial derivative in (3.27) of both sides with respect to θ_e yields

(3.28) $$\frac{\partial(p_c/\rho g)}{\partial\theta_e} = -\frac{1}{\lambda}\frac{p_b}{\rho g}(\frac{1}{\phi_e})^{-\frac{1}{\lambda}}\theta_e^{-(1+\frac{1}{\lambda})} .$$

The pressure p in (3.3) referred to the wetting phase and since $p_w = -p_c$, (3.28) then becomes

(3.29) $$\frac{\partial(p/\rho g)}{\partial\theta_e} = \frac{1}{\lambda}\frac{p_b}{\rho g}(\frac{1}{\phi_e})^{-\frac{1}{\lambda}}\theta_e^{-(1+\frac{1}{\lambda})}$$

Substituting (3.25) and (3.29) into (3.6), we get

$$D(\theta_e) = \frac{K_o}{\phi_e^{(2+3\lambda)/\lambda}} \theta_e^{(2+3\lambda)/\lambda} \frac{p_b}{\lambda \rho g} (\frac{1}{\phi_e})^{-\frac{1}{\lambda}} \theta_e^{-(1+\frac{1}{\lambda})} .$$

or

(3.30) $$D(\theta_e) = \frac{K_o p_b}{\lambda \phi_e \rho g} (\frac{\theta_e}{\phi_e})^{(1+2\lambda)/\lambda}$$

or in non-dimensional form

$$\frac{D\phi_e \lambda}{K_o p_b/(\rho g)} = (\frac{\theta_e}{\phi_e})^{(1+2\lambda)/\lambda}$$

i.e.

$$\frac{D}{D_o} = (\frac{\theta_e}{\phi_e})^{(1+2\lambda)/\lambda} ,$$

and finally

(3.31) $$D_. = \theta_.^{(1+2\lambda)/\lambda} .$$

For the condition of equilibrium obtained by letting $t \to \infty$, there is no flow and Darcy's equation is then

(3.32) $$0 = - K\nabla \Phi = K\frac{\partial}{\partial z}[-p_c/(\rho g)+z]$$

or

$$d(p_c/(\rho g)) = dz.$$

Integrating both sides gives

(3.33) $$p_c/(\rho g) = z + c$$

where c is a constant.

If z is measured upward from the water table we can write (3.33) as

$$p_c/(\rho g) = z, \tag{3.34}$$

which means that the static condition is reached when the capillary potential as a consequence of drainage, has increased until they are equal to gravitational potential.

Recalling (3.23) and applying (3.34) we have

$$\frac{\theta_e}{\phi_e} = \left(\frac{p_b/(\rho g)}{p_c/(\rho g)}\right)^{\lambda} = \left(\frac{p_b/(\rho g)}{z}\right)^{\lambda} = \left(\frac{z}{p_b/(\rho g)}\right)^{-\lambda} \tag{3.35}$$

or in non-dimensional form

$$\theta_{.} = z_{.}^{-\lambda} \qquad \text{for } t_{.} = \infty \tag{3.36}$$

Equation (3.36) will provide the quickest method to obtain a water content distribution curve at equilibrium. The curve of (3.36) will be plotted along with other unsteady water content distribution curves for the purpose of comparison.

Equations (3.17), (3.25), (3.30) will yield a solution of dimensional form whereas Equations (3.21), (3.26), (3.31) yield a non-dimensional solution in terms of scaled variables. Both solutions will be identical provided the conversion using Equations (3. 18) and (3. 20) has been made.

Generally speaking capillary conductivity is not a single-valued function of the water content but is affected by hystersis. For example the values of K will be different if at the outset we start with a dry soil and let water imbibe, or a wet soil and let it drain to a given water content. These hystersis effects generally complicate the mathematical analysis and are difficult to handle. In this paper, hystersis is not considered. For problems involving only wetting or only draining of the porous body, hystersis does not exist since the water content is constantly increasing or decreasing; thus, the capillary conductivity and diffusivity coefficients can be assumed to be a single-valued function of the water content. See Miller and Miller [18].

IV. INITIAL AND BOUNDARY CONDITIONS AND PREDICTION OF OUTFLOW

The vertical drainage through a homogeneous porous medium has received comparatively little attention chiefly due to the fact that the transient state is unsolvable analytically with present mathematical tools; however, great advances have been made for the steady state case.

The system chosen for the study was a vertically oriented, cylindrical column of a homogeneous, isotropic, porous medium confined within impermeable walls, and the fluid concerned is homogeneous and incompressible. Flow within this column takes place in the vertical direction only. The working equation is the non-dimensional Equation (3. 21)

$$\text{(3. 21)} \qquad \frac{\partial \theta.}{\partial t.} = \frac{1}{\lambda} \frac{\partial}{\partial z.} [D. \frac{\partial \theta.}{\partial z.} + \lambda K.],$$

together with Equations (3. 26) and (3. 31) of capillary conductivity and diffusivity functions respectively,

$$\text{(3. 26)} \qquad K. = \theta.^{(2+3\lambda)/\lambda}$$

$$\text{(3. 27)} \qquad D. = \theta.^{(1+2\lambda)/\lambda}$$

where the dots indicate the scaled variables as described by (3. 20) and (3. 18).

Initially the soil is assumed to be saturated to the surface, or at $\theta_* = 1$. At time $t_* = 0$, the soil starts to drain and we assume no infiltration or evaporation occurring at the top boundary takes place for all $t_* \geqq 0$. When a static equilibrium condition was obtained at $t_* = \infty$, the total water-moving potential, Φ, was constant at all depths and water movement ceased. In most soils there is a region of uniform water content above the water table at all times; this region is called the capillary fringe by Childs [4], where the water table has been defined by Richards [21] as the locus of points at atmospheric pressure. Complete drainage was assumed above the capillary fringe and complete saturation below.

Gardner [12] indicated that since the capillary fringe depth does not drain, it may be treated as an impendance to flow and hence the height of the fringe is subtracted from the length of the column. Therefore the distance to the top of capillary fringe is considered instead of the distance to the base of the column throughout the drainage.

Taylor [23] defined drainable proosity, f, as the volume fraction of pore space water which can be drained from a soil under prescribed conditions, i.e.,

$$f = \frac{\text{volume of fluid drained}}{\text{volume of porous medium}} . \tag{4.1}$$

During drainage, f is not a constant, but it is related to the water table depth z, and time t. We rewrite (4.1) as

$$f = \frac{\text{vol. of fluid at time} = 0 - \text{vol. of fluid at time} = t}{\text{volume of porous medium}} . \tag{4.2}$$

In a more appropriate form (4.2) is written:

$$f(z, t) = \theta(0, 0) - \theta(z, t) \tag{4.3}$$

where $\theta(0, 0)$ and $\theta(z, t)$ are water content at $(0, 0)$ and at (z, t) repsectively.

Integrating (4.3) gives

$$\int_{L_o}^{L} f(z, t_n)dz = Q_n \tag{4.4}$$

where L is the total length of soil profile and L_o is the bubbling pressure head at time t_n, and Q_n is the total volume of water per unit area drained.

Equation (4.4) can be rewritten in the form

$$Q_n = \int_{L_o}^{L} \theta(0, 0)dz - \int_{L_o}^{L} \theta(z, t_n)dz \tag{4.5}$$

where $\theta(0, 0)$ is the initial condition, a constant, denoted by S. Therefore,

$$Q_n = S(L-L_o) - \int_{L_o}^{L} \theta(z, t_n)dz. \tag{4.6}$$

This is the equation for outflow as a function of time.

For the dimensionless case, $\theta.(0, 0)$ and L_o are taken as one, and (4.6) becomes

$$Q._n = (L. -1) - \int_1^{L.} \theta.(z., t._n)dz. \; . \tag{4.7}$$

Now we want to find the standard unit that is related to the scaled variable $Q.$.

Defining Q as before,

$$Q = \int \theta_e dz$$

we then have

$$\frac{Q}{\theta_o z_o} = \int \frac{\theta_e}{\theta_o} \frac{dz}{z_o}$$

and

$$Q. = \int \theta. dz.$$

where

$$Q. = \frac{Q}{\theta_o z_o} = \frac{Qg\rho}{\phi_e p_b} \; . \tag{4.8}$$

V. THE PREDICTOR-CORRECTOR METHOD

§ 1. In recent papers by Aschroft [1], Wassmuth [24], Liakopoulos [17], Jensen [14], etc. on their fluid flow problems, the finite difference methods used give rise to a system of nonlinear algebraic equations at each time step. In order to solve these nonlinear equations, various assumptions have to be made restricting their applicability. The predictor-corrector method proposed by Douglas and Jones [8] avoids these nonlinear problems since it gives rise to a system of linear equations and is unconditionally stable.

In their paper, Douglas and Jones describe the predictor-corrector method as follows:

Consider the general parabolic differential equation

$$\frac{\partial^2 u}{\partial x^2} = F(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}), \quad 0 < x < 1, \quad 0 < t \leq T, \tag{5.1}$$

$$u(x, 0), u(0, t), u(1, t) \text{ specified}, \quad 0 < x < 1, \quad 0 < t \leq T. \tag{5.2}$$

The following notation will be convenient

(5.3)

$$h = N^{-1}, \qquad k = TM^{-1}, \qquad \text{N and M being positive integers}$$

$$x_i = ih, \qquad t_n = nk \;, \qquad U_{in} = U(x_i, t_n),$$

$$\Delta_x^2 U_{in} = h^{-2}(U_{i+1,n} - 2U_{in} + U_{i-1,n}),$$

$$\delta_x U_{in} = (2h)^{-1}(U_{i+1,n} - U_{i-1,n}),$$

and

$$(5.4) \qquad \|U\|^2 = \sum_{i=1}^{N-1} U_i^2 h$$

denote the L_2 norm of a vector defined on x_i, $i=1,\ldots,N-1$ and let

$$\|U\|_1^2 = \sum_{i=1}^{N} U_i^2 h.$$

Now, the predictor of (5.1) is

$$(5.5) \qquad \Delta_x^2 U_{in+\frac{1}{2}} = F(x_i, t_{n+\frac{1}{2}}, U_{in}, \delta_x U_{in}, (U_{in+\frac{1}{2}} - U_{in})/(k/2))$$

for $i=1,2,\ldots,N-1$, and the corrector is

$$(5.6) \qquad \frac{1}{2}\Delta_x^2(U_{in+1}+U_{in}) = F(x_i, t_{n+\frac{1}{2}}, U_{in+\frac{1}{2}}, \frac{1}{2}\delta_x(U_{in+1}+U_{in}), (U_{in+1}-U_{in})/k).$$

By the Taylor's theorem, it is obvious that the predictor has a local error of $O(h^2+k)$ and the corrector has a local error of $O(h^2+k^2)$. Clearly, if F has the form

$$(5.7) \qquad F = f_1(x,t,u)\frac{\partial u}{\partial t} + f_2(x,t,u)\frac{\partial u}{\partial x} + f_3(x,t,u).$$

the system (5.5), (5.6) combined with the data

(5.8) $$U_{i0} = u_{i0}, \quad U_{0m} = u_{0m}, \quad U_{Nm} = u_{Nm},$$

leads to linear algebraic equations.

§ 2. The object of this section is to prove the convergence of the solutions given by the predictor corrector method to the solution of the differential equation as $k, h \to 0$, where the mesh ratio $r = k/h^2$ is held fixed. The proof follows very closely the arguement given by Douglas and Jones [8], but we are able to obtain some simplifications to their proof.

We shall first prove five lemmas which will be used in the convergent proof. Lemma 2 is due to Lees [16]. However, his proof is difficult to follow and consequently we shall include a simpler proof of this result.

Lemma 1. Let C, K be arbitrary, non-negative, real numbers. Then for any integer $r \geq 1$.

$$1 + CK \sum_{i=1}^{r} \exp(CKi) \leq \exp((r+1)CK).$$

Proof: This lemma follows by induction.

For, letting $r = 1$, we have,

$$1 + CKe^{CK} \leq e^{CK} + CKe^{CK} = e^{CK}(1+CK) \leq e^{CK}e^{CK} = e^{2CK},$$

so the lemma is true for $r = 1$. Assume that it is true for r. Then for $r + 1$, we have

$$1+CK\sum_{i=1}^{r+1}\exp(CKi) = 1 + CK\sum_{i=1}^{r}\exp(CKi) + CK\exp(CK(r+1))$$

$$\leq \exp(CK(r+1)) + CK\exp(CK(r+1))$$

$$= \exp(CK(r+1))(1+CK) \leq \exp(CK(r+1))\exp(CK)$$

$$= \exp(CK(r+2)).$$

Lemma 2. Let $\{\alpha_i\}$, $\{\beta_i\}$, $i = 1, 2, \ldots, n$ be sets of non-negative real numbers such that $\beta_i \leq \beta_j$ for $i \leq j$. Let C, K be positive constants. If

$$\alpha_j \leq \beta_i + CK\sum_{i=1}^{j-1}\alpha_i, \quad j = 1, 2, \ldots, n,$$

then

$$\alpha_j \leq \beta_j \exp(CKj), \quad j = 1, 2, \ldots, n.$$

Proof: Use induction on j.

If $j = 1$, the lemma says that if $\alpha_1 \leq \beta_1$, then $\alpha_1 \leq \beta_1 e^{CK}$, which is obviously true. Assume the lemma holds for j. Then for $j + 1$, making use of Lemma 1, we find

$$\alpha_{j+1} \le \beta_{j+1} + CK\sum_{i=1}^{j}\alpha_i \le \beta_{j+1} + CK\sum_{i=1}^{j}\beta_i \exp(CKi)$$

$$\le \beta_{j+1}(1+CK\sum_{i=1}^{j}\exp(CKi)) \le \beta_{j+1}\exp(CK(j+1)).$$

For the convenience of the reader, we also recall the inequalities:

Cauchy's inequality $2|ab| \le \epsilon |a|^2 + \frac{1}{\epsilon}|b|^2$, for arbitrary $\epsilon > 0$,

where a and b are real numbers.

Schwarz inequality $|\sum_{i=1}^{n} a_i b_i| \le (\sum_{i=1}^{n} a_i^2)^{\frac{1}{2}}(\sum_{i=1}^{n} b_i^2)^{\frac{1}{2}}$,

where a_i's and b_i's are real numbers.

Using the notation of (5. 3) and (5. 4), we state the following lemmas where a_i's denote real numbers.

<u>Lemma 3.</u> $\sum_i a^2_{i+1, n+\frac{1}{2}} \le \sum_i a^2_{i, n+\frac{1}{2}}$.

<u>Lemma 4.</u> $h\sum_{i=1}^{N} 1 = 1$

<u>Lemma 5.</u> $\left|\sum_i h^{\frac{1}{2}} a_{i+1,n+\frac{1}{2}} h^{\frac{1}{2}} a_{in+\frac{1}{2}}\right| \le \|a_{n+\frac{1}{2}}\|^2$

<u>Lemma 6.</u> $\sum_i h a_{in} a_{in+\frac{1}{2}} \le \frac{1}{2\epsilon} \|a_{in}\|^2 + \frac{\epsilon}{2} \|a_{in+\frac{1}{2}}\|^2$

<u>Lemma 7.</u> $\sum_i h^{\frac{1}{2}} (\delta_x a_{in}) h^{\frac{1}{2}} O(h^2+k^2) \le \frac{1}{2} \|\delta_x a_n\|^2 + O(h^2+k^2)^2$.

Lemma 3 and 4 are immediate. To prove Lemma 5, observe that

$$\left|\sum_i h^{\frac{1}{2}} a_{i+1,n+\frac{1}{2}} h^{\frac{1}{2}} a_{in+\frac{1}{2}}\right| \le \left|(\sum_i h a_{in+\frac{1}{2}})^{\frac{1}{2}} (\sum_i h a^2_{in+\frac{1}{2}})^{\frac{1}{2}}\right| = \|a_{n+\frac{1}{2}}\|^2 .$$

For Lemma 6, we note that

$$\sum_i h a_{in} a_{in+\frac{1}{2}} \le \sum_i h\left(\frac{1}{2\epsilon} a^2_{in} + \frac{\epsilon}{2} a^2_{in+\frac{1}{2}}\right) = \frac{1}{2\epsilon} \|a_{in}\|^2 + \frac{\epsilon}{2} \|a_{in+\frac{1}{2}}\|^2 .$$

Finally, Lemma 7 follows from the inequalities

$$\sum_i h^{\frac{1}{2}} (\delta_x a_{in}) h^{\frac{1}{2}} O(h^2+k^2) \le \left(\sum_i h(\delta_x a_{in})^2\right)^{\frac{1}{2}} \left(\sum_i h O(h^2+k^2)^2\right)^{\frac{1}{2}}$$

$$\le \frac{1}{2} \|\delta_x a_n\|^2 + \frac{1}{2} \sum_i h O(h^2+k^2)^2$$

$$= \frac{1}{2} \|\delta_x a_n\|^2 + O(h^2+k^2)^2 .$$

We now assume that the function F and the data are such that there exists a solution $u(x, t)$ to the problem which is four times continuously differentiable with respect to x and twice with respect to t in $0 \leq x \leq 1$, $0 \leq t \leq T$. Sufficient conditions for this may be found in the monograph of Friedman [10].

In particular, we assume that the function $F(x, t, u_1, u_2, u_3)$ is defined for all $0 \leq x \leq 1$, $0 \leq t \leq T$, $-\infty < u_1, u_2, u_3 < \infty$ and is bounded there. Further, the functions $\frac{\partial F}{\partial u_i}$ for $i = 1, 2, 3$, exist, are continuous and are bounded in this region.

Let u be the solution to the differential equation (5.1) satisfying (5.2). Then from Taylor's theorem and the predictor (5.5) and the corrector (5.6) follows

(5.9)

$$\Delta_x^2 u_{in+\frac{1}{2}} + O(h^2) = F(x_i, t_{n+\frac{1}{2}}, u_{i,n}, \delta_x u_{in} + O(h^2), (u_{in+\frac{1}{2}} - u_{in})/(k/2) + O(k)),$$

and

(5.10)

$$\frac{1}{2}\Delta_x^2(u_{in+1} + u_{in}) + O(h^2) = F(x_i, t_{n+\frac{1}{2}}, u_{i,n+\frac{1}{2}}, \frac{1}{2}\delta_x(u_{in+1} + u_{in}) + O(h^2), (u_{in+1} - u_{in})/k + O(k^2)).$$

To prove a convergent theorem, it will be sufficient to derive a difference equation satisfied by the error

(5.11) $$e = u - U.$$

Then, e satisfies the difference equation

$$\Delta_x^2 e_{i,n+\frac{1}{2}} + O(h^2) \tag{5.12}$$

$$= F(x_i, t_{n+\frac{1}{2}}, u_{i,n}, \delta_x u_{in} - O(h^2), (u_{in+\frac{1}{2}} - u_{in})/(k/2) + O(k))$$

$$- F(x_i, t_{n+\frac{1}{2}}, U_{in}, \delta_x U_{in}, (U_{i,n+\frac{1}{2}} - U_{in})/(k/2)).$$

By the mean value theorem,

(5.13)

$$\Delta_x^2 e_{in+\frac{1}{2}} = \frac{\partial F^*}{\partial u_1} e_{in} + \frac{\partial F^*}{\partial u_2} \delta_x e_{in} + \frac{\partial F^*}{\partial u_3} \frac{e_{in+\frac{1}{2}} - e_{in}}{k/2} + O(h^2 + k).$$

where the asterisk indicates that the partial derivatives are evaluated at intermediate points called for by the mean value theorem. By the same token, the corrector is

$$\frac{1}{2} \Delta_x^2 (e_{in+1} + e_{in}) = \frac{\partial F^{**}}{\partial u_1} e_{in+\frac{1}{2}} + \frac{1}{2} \frac{\partial F^{**}}{\partial u_2} \delta_x (e_{in+1} + e_{in}) \tag{5.14}$$

$$+ \frac{\partial F^{**}}{\partial u_3} \frac{e_{in+1} - e_{in}}{k} + O(h^2 + k^2),$$

(5.15) $\qquad e = 0 \qquad$ initially and on the boundary.

Multiplying (5.13) by $k/2$ and by letting $r = k/h^2$, we have

$$\frac{r}{2}(e_{i+1,n+\frac{1}{2}} - 2e_{in+\frac{1}{2}} + e_{i-1,n+\frac{1}{2}})$$

$$= \frac{k}{2}\frac{\partial F^*}{\partial u_1} e_{in} + \frac{k}{2}\frac{\partial F^*}{\partial u_2}\delta_x e_{in} + \frac{\partial F^*}{\partial u_3}(e_{in+\frac{1}{2}} - e_{in}) + kO(h^2+k).$$

Simplifying, we obtain

(5.16)

$$(r+\frac{\partial F^*}{\partial u_3})e_{in+\frac{1}{2}}$$

$$= \frac{r}{2}(e_{i+1,n+\frac{1}{2}} + e_{i-1,n+\frac{1}{2}}) + (\frac{\partial F^*}{\partial u_3} - \frac{k}{2}\frac{\partial F^*}{\partial u_1})e_{in} - \frac{k}{2}\frac{\partial F^*}{\partial u_2}\delta_x e_{in} + kO(h^2+k).$$

Multiplying (5.16) by $he_{in+\frac{1}{2}}$ and summing over i, we obtain

$$(r+\frac{\partial F^*}{\partial u_3})\,\Sigma h e^2_{i,n+\frac{1}{2}} = \frac{r}{2}(\Sigma\, h e_{i+1,n+\frac{1}{2}} e_{i,n+\frac{1}{2}} + \Sigma\, h e_{i-1,n+\frac{1}{2}} e_{i,n+\frac{1}{2}})$$

$$+ (\frac{\partial F^*}{\partial u_3} - \frac{k}{2}\frac{\partial F^*}{\partial u_1})\,\Sigma\, h e_{in} e_{in+\frac{1}{2}} - \frac{k}{2}\frac{\partial F^*}{\partial u_2}\Sigma\, h(\delta_x e_{in})e_{in+\frac{1}{2}}$$

$$+ \Sigma\, h e_{in+\frac{1}{2}} kO(h^2+k).$$

Using Lemmas 5, 6, and 7; $\left|\frac{\partial F}{\partial u_i}\right| \leq A$, for $i = i, 2, 3$; and $\frac{\partial F}{\partial u_3} \geq a > 0$, we find that

(5.17)

$$(r+a)\|e_{in+\frac{1}{2}}\|^2 \le \frac{r}{2}\{\|e_{n+\frac{1}{2}}\|^2+\|e_{n+\frac{1}{2}}\|^2\} + (a-kC_1)\{\frac{1}{2\epsilon}\|e_{in}\|^2+\frac{\epsilon}{2}\|e_{i,n+\frac{1}{2}}\|^2\}$$

$$- kC_2\{\frac{1}{2\epsilon}\|\delta_x e_{in}\|^2+\frac{\epsilon}{2}\|e_{in+\frac{1}{2}}\|^2\} + \frac{\epsilon}{2}\|e_{in+\frac{1}{2}}\|^2 + C_3 k(h^2+k),$$

or equivalently,

$$\|e_{in+\frac{1}{2}}\|^2 \le A(\|e_n\|^2+k\|\delta_x e_{in}\|^2+k(h^2+k)). \tag{5.18}$$

Next, (5.14) can be written in the form

$$\frac{\partial F^{**}}{\partial u_3}\,\frac{e_{in+1}-e_{in}}{k} = \frac{1}{2}\,\Delta_x^2(e_{in+1}+e_{in}) + G_{in}, \tag{5.19}$$

where

$$|G_{in}| \le A_1[\,|e_{in+\frac{1}{2}}|+\frac{1}{2}|\delta_x e_{in+1}|+\frac{1}{2}|\delta_x e_{in}|\,] + O(h^2+k^2). \tag{5.20}$$

Squaring (5.20) we obtain

(5.21)

$$|G_{in}|^2 \le A_1^2[\,|e_{in+\frac{1}{2}}|^2+\frac{1}{4}|\delta_x e_{in+1}|^2+\frac{1}{4}|\delta_x e_{in}|^2+|e_{in+\frac{1}{2}}||\delta_x e_{in+1}|$$

$$+ |e_{n+\frac{1}{2}}||\delta_x e_{n+1}|+\frac{1}{2}|\delta_x e_{n+1}||\delta_x e_n|\,] + 2A_1|e_{n+\frac{1}{2}}|O(h^2+k^2)$$

$$+ A_1|\delta_x e_{n+1}|O(h^2+k^2) + A|\delta_x e_{in}|O(h^2+k^2) + O(h^2+k^2)^2.$$

Multiplying (5.21) by h, summing over i, and applying Lemmas 3, 4, 5, 6 and 7 (5.21) becomes

(5.22)

$$\|G_{in}\|^2 \le A_1^2\{\|e_{n+\frac{1}{2}}\|^2+\frac{1}{4}\|\delta_x e_{n+1}\|^2+\frac{1}{4}\|\delta_x e_n\|^2+\frac{1}{2}\|e_{in+\frac{1}{2}}\|^2+\frac{1}{2}\|\delta_x e_{n+1}\|^2$$

$$+\frac{1}{2}\|e_{in+\frac{1}{2}}\|^2+\frac{1}{2}\|\delta_x e_{n+1}\|^2+\frac{1}{4}\|\delta_x e_{n+1}\|^2+\frac{1}{4}\|\delta_x e_n\|^2\}$$

$$+ A_1\|e_{n+\frac{1}{2}}\|^2 + O(h^2+k^2)^2 + \frac{1}{2}A_1\|\delta_x e_{n+1}\|^2 + O(h^2+k^2)^2$$

$$+\frac{1}{2}A_1\|\delta_x e_n\|^2 + O(h^2+k^2).$$

Simplifying (5.22) gives

$$\|G_{in}\|^2 \le A_2\{\|e_{n+\frac{1}{2}}\|^2 + \|\delta_x e_{n+1}\|^2 + \|\delta_x e_n\|^2\} + O(h^2+k^2)^2. \tag{5.23}$$

Substituting (5.18) into (5.23) yields

$$\|G_{in}\|^2 \le A_3[\|e_n\|^2+k\|\delta_x e_n\|^2+k(h^2+k)+\|\delta_x e_{n+1}\|^2+\|\delta_x e_n\|^2]+O(h^2+k^2)^2,$$

or equivalently,

$$\|G_n\|^2 \le A_4[\|e_n\|^2+\|\delta_x e_n\|^2+\|\delta_x e_{n+1}\|^2] + O(h^2+k^2)^2. \tag{5.24}$$

Now, an application of an energy estimate of Lees [16, Lemma 2] to (5.19) yields for all sufficiently small k,

$$\|\Delta_x e_{n+1}\|_1^2 \le A_5\sum_{m=0}^{n}\|G_m\|^2 k, \tag{5.25}$$

where

(5.26) $$\Delta_x e_{in} = h^{-1}(e_{in} - e_{i-1,n}).$$

Observe that

$$\delta_x e_{in} = \frac{1}{2}\Delta_x e_{i+1,n} + \frac{1}{2}\Delta_x e_{in},$$

so that squaring both sides, multiplying by h, and summing over i, we obtain,

(5.27)

$$\Sigma h \delta_x e_{in}^2 = \frac{1}{4}\Sigma h \Delta_x e_{i+1,n}^2 + \frac{1}{4}\Sigma h \Delta_x e_{in}^2 + \frac{1}{2}\Sigma h \Delta_x e_{i+1,n} \Delta_x e_{in}.$$

Applying Lemma 5 to (5.27) yields

(5.28) $$\|\delta_x e_{in}\|^2 \le \frac{1}{4}\{\|\Delta_x e_{i+1,n}\|^2 + \|\Delta_x e_{in}\|^2 + 2\|\Delta_x e_{in}\|^2\}.$$

Since $e_{i+1,n} \le e_{i,n}$, therefore (5.28) is reduced to

(5.29) $$\|\delta_x e_n\|^2 \le \|\Delta_x e_{in}\|^2$$

or

(5.30) $$\|\delta_x e_n\| \le \|\Delta_x e_{in}\| \le \|\Delta_x e_n\|_1.$$

We also examine that

$$e_{in} = h\sum_{j=0}^{i} \Delta_x e_{j,n},$$

so that squaring both sides gives,

$$e_{in}^2 = h^2\Big(\sum_{j=0}^{i} \Delta_x e_{jn}\Big)^2 \leq \sum_{j=0}^{i} h\,|\Delta_x e_{jn}|^2 \sum_{j=0}^{i} h\,|\Delta_x e_{jn}|^2 \leq \|\Delta_x e_{jn}\|^2 \leq \|\Delta_x e_{jn}\|_1^2$$

or

(5.31) $$\|e_{in}\|^2 \leq \ell\,\|\Delta_x e_{in}|_1^2 \ .$$

Substituting (5.30) and (5.31) into (5.24), we obtain

(5.32) $$\|G_n\|^2 \leq A_6[\ell\,\|\Delta_x e_{in}\|_1^2 + \|\Delta_x e_n\|_1^2 + \|\Delta_x e_{n+1}\|_1^2] + O(h^2+k^2)^2 .$$

From (5.25) we obtain

(5.33) $$\|\Delta_x e_{n+1}\|_1^2 \leq A_6 \sum_{m=1}^{n+1} \|\Delta_x e_m\|^2 k + O(h^2+k^2)^2 .$$

Applying Lemma 2, yields

(5.34) $$\|\Delta_x e_{n+1}\|_1^2 \leq O(h^2+k^2)^2 \exp\,(A_6 k(n+2)\,);$$

therefore, as $nk \leq T$,

(5.35) $$\|\Delta_x e_{n+1}\|_1 \leq O(h^2+k^2)\,, \qquad 0 \leq n \leq M-1 .$$

As

(5.36)
$$\max_{\substack{0 < i \leq N \\ 0 \leq n \leq M}} |e_{in}| \leq \sqrt{\ell} \max_{\substack{0 < i \leq N \\ 0 \leq n \leq M}} \|\Delta_x e_n\|_1 ,$$

then

(5.37)
$$\max_{i, n} |e_{in}| \leq O(h^2+k^2),$$

which is the desired result.

§ 3. We shall now apply the predictor-corrector method to the dimensionless diffusion equation (3.21)

(3.21)
$$\frac{\partial\theta}{\partial t} = \frac{1}{\lambda} \frac{\partial}{\partial z}[D(\theta) \frac{\partial\theta}{\partial z}] + \frac{\partial K(\theta)}{\partial z} .$$

The predictor for (3.21) is

(5.38)
$$\frac{\theta_{i, j+\frac{1}{2}} - \theta_{i, j}}{k/2} = \frac{1}{\lambda} D(\theta_{i-\frac{1}{2}, j}) \frac{\theta_{i-1, j+\frac{1}{2}} - \theta_{i, j+\frac{1}{2}}}{h^2} - \frac{1}{\lambda} D(\theta_{i+\frac{1}{2}, j}) \frac{\theta_{i, j+\frac{1}{2}} - \theta_{i+1, j+\frac{1}{2}}}{h^2} + \frac{K(\theta_{i-1, j}) - K(\theta_{i+1, j})}{2h} .$$

Since

$$\frac{K(\theta_{i-1, j}) - K(\theta_{i+1, j})}{2h} = \frac{K(\theta_{i-\frac{1}{2}, j}) - K(\theta_{i+\frac{1}{2}, j})}{h}$$

we may rewrite the predictor as follows

(5.39)

$$-\frac{r}{2\lambda}D(\theta_{i-\frac{1}{2},j})\theta_{i-1,j+\frac{1}{2}}+[1+\frac{r}{2\lambda}D(\theta_{i-\frac{1}{2},j})+\frac{r}{2\lambda}D(\theta_{i+\frac{1}{2},j})]\theta_{i,j+\frac{1}{2}}-\frac{r}{2\lambda}D(\theta_{i+\frac{1}{2},j})\theta_{i+1,j+\frac{1}{2}}$$

$$= \theta_{i,j} + \frac{r'}{2}[K(\theta_{i-\frac{1}{2},j}) - K(\theta_{i+1,j})]$$

where $r = k/h^2$ and $r' = k/h$.

The corrector is

(5.40)

$$\frac{\theta_{i,j+1}-\theta_{i,j}}{k} = \frac{1}{\lambda}D(\theta_{i-\frac{1}{2},j+\frac{1}{2}})\frac{\theta_{i-1,j+1}-\theta_{i,j+1}}{2h^2} - \frac{1}{\lambda}D(\theta_{i+\frac{1}{2},j+\frac{1}{2}})\frac{\theta_{i,j+1}-\theta_{i+1,j+1}}{2h^2}$$

$$+\frac{1}{\lambda}D(\theta_{i-\frac{1}{2},j+\frac{1}{2}})\frac{\theta_{i-1,j}-\theta_{i,j}}{2h^2} - \frac{1}{\lambda}D(\theta_{i+\frac{1}{2},j+\frac{1}{2}})\frac{\theta_{i,j}-\theta_{i+1,j}}{2h^2}$$

$$+\frac{1}{2}\{\frac{K(\theta_{i-1,j+\frac{1}{2}})-K(\theta_{i+1,j+\frac{1}{2}})}{2h} + \frac{K(\theta_{i-1,j+\frac{1}{2}})-K(\theta_{i+1,j+\frac{1}{2}})}{2h}\}.$$

Similarly, the last term can be replaced by $(K(\theta_{i-\frac{1}{2},j+\frac{1}{2}})-K(\theta_{i+\frac{1}{2},j+\frac{1}{2}}))/h$, and after rearranging, we have

(5.41)

$$-\frac{r}{2\lambda}D(\theta_{i-\frac{1}{2},j+\frac{1}{2}})\theta_{i-1,j+1} + [1+\frac{r}{2\lambda}D(\theta_{i-\frac{1}{2},j+\frac{1}{2}})+\frac{r}{2\lambda}D(\theta_{i+\frac{1}{2},j+\frac{1}{2}})]\theta_{i,j+1}$$

$$-\frac{r}{2\lambda}D(\theta_{i+\frac{1}{2},j+\frac{1}{2}})\theta_{i+1,j+1} = \frac{r}{2\lambda}D(\theta_{i-\frac{1}{2},j+\frac{1}{2}})\theta_{i-1,j} + [1-\frac{r}{2\lambda}D(\theta_{i-\frac{1}{2},j+\frac{1}{2}})$$

$$-\frac{r}{2\lambda}D(\theta_{i+\frac{1}{2},j+\frac{1}{2}})]\theta_{i,j} + \frac{r}{2\lambda}D(\theta_{i+\frac{1}{2},j+\frac{1}{2}})\theta_{i+1,j} + r'[K(\theta_{i-\frac{1}{2},j+\frac{1}{2}})-K(\theta_{i+\frac{1}{2},j+\frac{1}{2}})].$$

The column is initially saturated with water, i.e.,

(5.42) $$\theta(z, 0) = \theta_0 = \theta_{i,0}$$

where θ_0 is the initial water content.

The bottom boundary is held at saturation at all time thus

(5.43) $$\theta(L, t) = \theta_s = \theta_{n,j}$$

where θ_s is water content at saturation.

Wassmuth, [24] treated the top boundary by the following equation for the first soil block, as shown in Figure 5.1:

Infiltration (in flow) - percolation (outflow) = storage.

Since this paper assumes no flux across the top boundary, infiltration is then equal to zero; for the predictor scheme, we have

(5.44) $$O - [\frac{1}{\lambda}D(\theta_{\frac{1}{2},j})\frac{\theta_{0,j+\frac{1}{2}} - \theta_{1,j+\frac{1}{2}}}{h} + K(\theta_{\frac{1}{2},j})]k/2 = \frac{h}{2}[\theta_{0,j+\frac{1}{2}} - \theta_{0,j}].$$

$\frac{\Delta z}{2}$ was used instead of Δz because $\theta_{0,t}$ is considered to be the center of the first soil block. Rearranging (5.44), we obtain

(5.45) $$[1 + \frac{r}{\lambda}D(\theta_{\frac{1}{2},j})]\theta_{0,j+\frac{1}{2}} - \frac{r}{\lambda}D(\theta_{\frac{1}{2},j})\theta_{1,j+\frac{1}{2}} = \theta_{0,j} - r'k(\theta_{1,j}).$$

For the corrector scheme we have for the top boundary

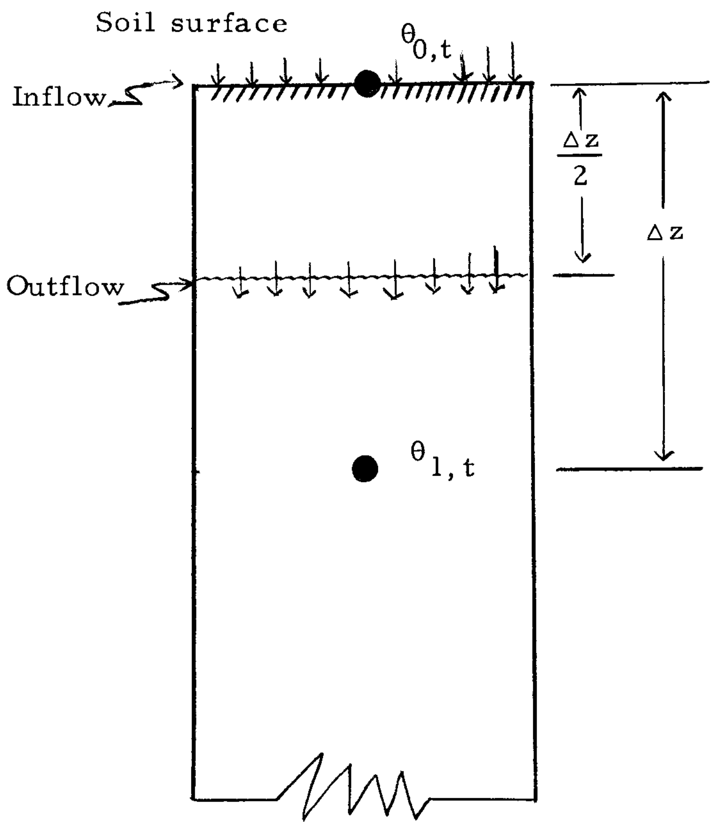


Figure 5.1. Boundary condition at the top of the soil column assuming infiltration equal to zero.

(5.46)

$$O - [\frac{1}{\lambda}D(\theta_{\frac{1}{2},j+\frac{1}{2}})\frac{\theta_{0,j+1}-\theta_{1,j+1}}{2h} + \frac{1}{\lambda}D(\theta_{\frac{1}{2},j+\frac{1}{2}})\frac{\theta_{0,j}-\theta_{1,j+1}}{2h} + K(\theta_{\frac{1}{2},j+\frac{1}{2}})]k$$

$$= \frac{h}{2}[\theta_{0,j+1}-\theta_{0,j}].$$

Rearranging (5.46) we find

(5.47)
$$[1+\frac{r}{\lambda}D(\theta_{\frac{1}{2},j})]\theta_{0,j+1} - \frac{r}{\lambda}D(\theta_{\frac{1}{2},j})\theta_{1,j+1}$$

$$= [1-\frac{r}{\lambda}D(\theta_{\frac{1}{2},j+\frac{1}{2}})]\theta_{0,j} + \frac{r}{\lambda}D(\theta_{\frac{1}{2},j+\frac{1}{2}})\theta_{1,j} - 2r'K(\theta_{\frac{1}{2},j+\frac{1}{2}}).$$

The predictor, equations (5.39) and (5.45) form $N - 1$ linear equations in $N - 1$ unknowns. Similarly, the corrector, equations (5.41) and (5.47) also form $N - 1$ linear equations in $N - 1$ unknowns. Both systems are tri-diagonal and are solved by using recursive formulae obtained by means of Gaussian elimination. A complete explanation can be found e.g. in Forsythe and Wasow [9]. In the computation, $\theta_{i\pm\frac{1}{2},j}$ is the average of two successive water contents at z and $z \pm \Delta z$.

VI. DISCUSSION AND RESULTS

The finite difference Equations (5. 39), (5. 41), (5. 45) and (5. 47) were programmed on a CDC 3300 computer for six different soils characterized by choosing the pore size distribution index, λ having values of 0. 167, 0. 667, 1. 333, 2. 000, 3. 333, 6. 000, with ten different scaled profile depths, namely $L_{.} = 1.2$, 1. 6, 1. 8, 2. 0, 3. 0, 5. 0, 7. 0, 9. 0, 15. 0 and 21. 0.

Solutions to Equation (3. 31) are of the form

$$\theta_{.} = f(z_{.}, t_{.}, \lambda) \tag{6.1}$$

and, in addition to this solution, outflow from the soil profile as a function of time was also obtained from Equation (3. 43).

An Algol program is attached in the appendix. The predictor-corrector method is unconditionally stable for all values of Δz and Δt. However, when the water content changes rapidly it becomes necessary to use small time and space increments. At any rate, if we choose the product of Δz and λ within 0. 6, no discrepency has been found. Most solutions take one to five minutes computer time depending on the value of Δz chosen.

Solutions in the form of (6. 1) were plotted in Figures 6. 1 and 6. 2. Figure 6. 1 shows the water content distributions as a function of scaled elevation above the water table for $L_{.} = 2$ and for three different values of λ. The scaled elevation on each profile begins with 1.00 since

$$\theta_{.}(z_{.}, t_{.}) = 1.00 \quad \text{for all} \quad 0 \leq z_{.} \leq 1.00.$$

The curve $t_{.} = \infty$ represented the equilibrium value of water content distribution, and was obtained by Equation (3.36). The numerical solution converges to the equilibrium case. This is a good indication that the diffusion function using Brooks-Corey relations of permeability, saturation and capillary pressure is a good approximation.

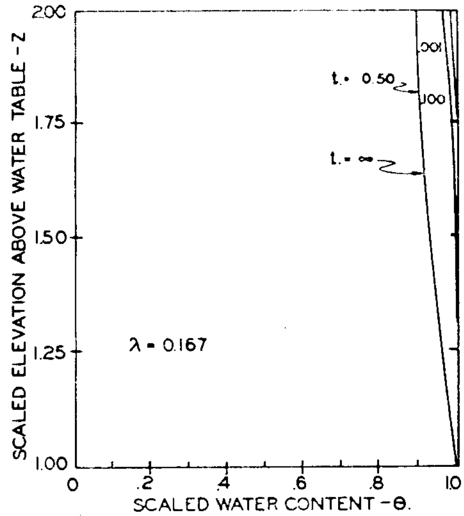
Figure 6.2 gives the same plot except for a different scaled profile, $L_{.} = 7$. The smaller the value of λ, the greater capacity for the soil to hold water and therefore the static condition is reached much earlier as can be seen from Figures 6.1 and 6.2. By the same reasoning, longer soil columns take longer time to drain other things being equal.

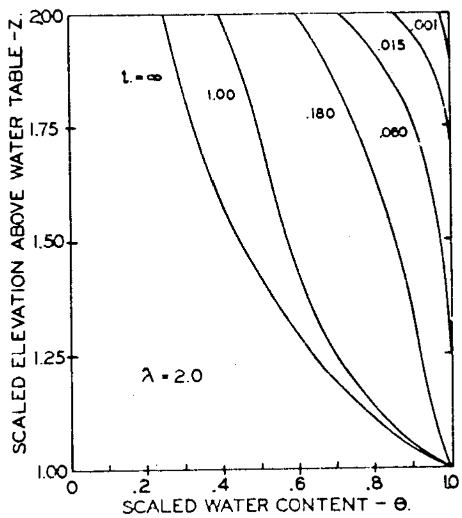
Figure 6.3 shows the accumulative outflow as a function of time for various soils ranging $\lambda = 0.167$ to $\lambda = 6$. For a given profile depth, the greater the value of λ, the greater the time for drainage to occur, primarily due to the lesser water holding capacity. It is interesting to know that it is possible to determine the pore size distribution index from the outflow data for a given profile depth. It is believed that this fact is first observed in this paper.

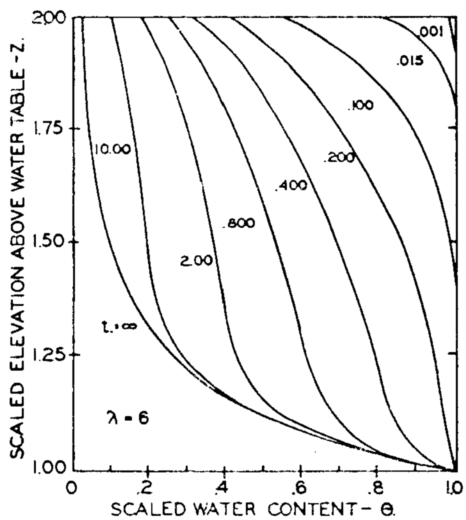
Figure 6.4 shows the accumulative outflow as a function of time for various profile depths. An increase in the profile depth has the effect of greater time for drainage to occur. The experimental data is presented by circles, and the theoretical curves by solid lines.

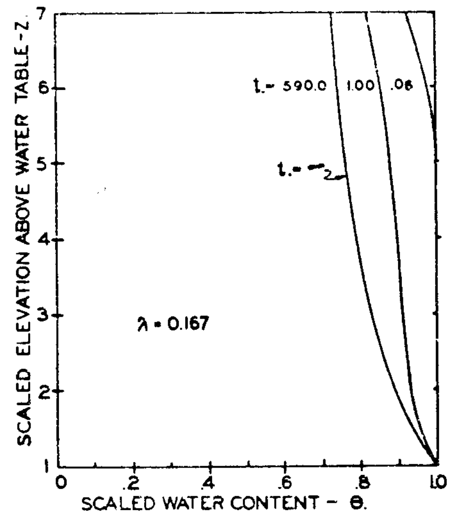
Figure 6.5 shows another plot of outflow curve for $L_\cdot = 3.75$ and $\lambda = 2.567$. Good agreement is shown between the theory and the experiment from Figures 6.4 and 6.5.

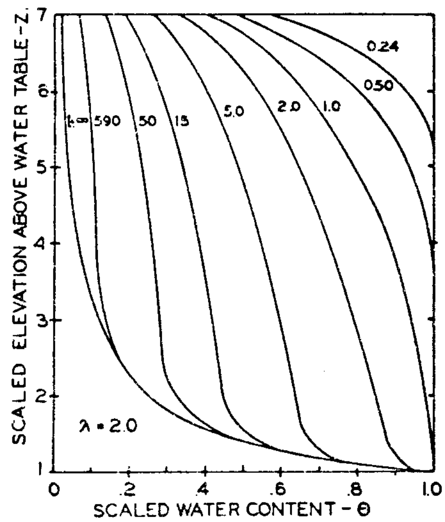
Initially the accumulative outflow was linear for small value of $t_\cdot$ and having a slope of unity, and this has been predicted by the capillary tube theory, see Corey et al [9]. The accumulative outflow becomes nonlinear after t becomes large. As t becomes very large, or at the equilbirium value, the outflow ceases and approaches a maximum constant, see Figures 6.3, 6.4 and 6.5.

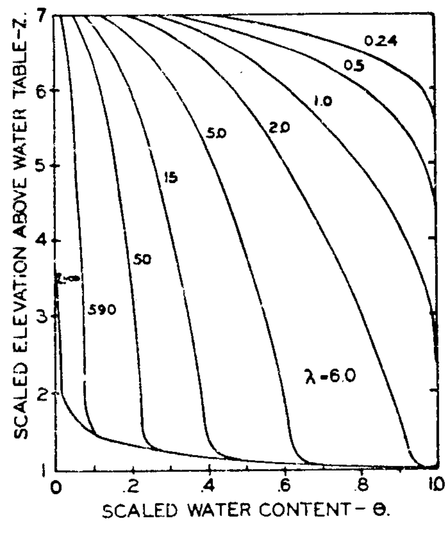

SCALED ELEVATION ABOVE WATER TABLE - Z
2.00
1.75
1.50
1.25
1.00
0
.2
.4
.6
.8
1.0
SCALED WATER CONTENT - θ.
.001
t.= 0.50
.100
t.= ∞
λ = 0.167


SCALED ELEVATION ABOVE WATER TABLE - Z.
2.00
1.75
1.50
1.25
1.00
0
.2
.4
.6
.8
1.0
SCALED WATER CONTENT - θ.
.001
.015
t.= ∞
1.00
.180
.060
λ = 2.0


SCALED ELEVATION ABOVE WATER TABLE - Z.
2.00
1.75
1.50
1.25
1.00
0
.2
.4
.6
.8
1.0
SCALED WATER CONTENT - θ.
.001
.015
.100
.200
.400
.800
10.00
2.00
t.= ∞
λ = 6


SCALED ELEVATION ABOVE WATER TABLE - Z.
SCALED WATER CONTENT - Θ.
t = 590.0
1.00
.06
λ = 0.167


SCALED ELEVATION ABOVE WATER TABLE - Z.
SCALED WATER CONTENT - Θ
0.24
0.50
1.0
2.0
5.0
15
50
590
λ = 2.0


SCALED ELEVATION ABOVE WATER TABLE - Z.
SCALED WATER CONTENT - Θ.
0.24
0.5
1.0
2.0
5.0
15
50
590
λ = 6.0

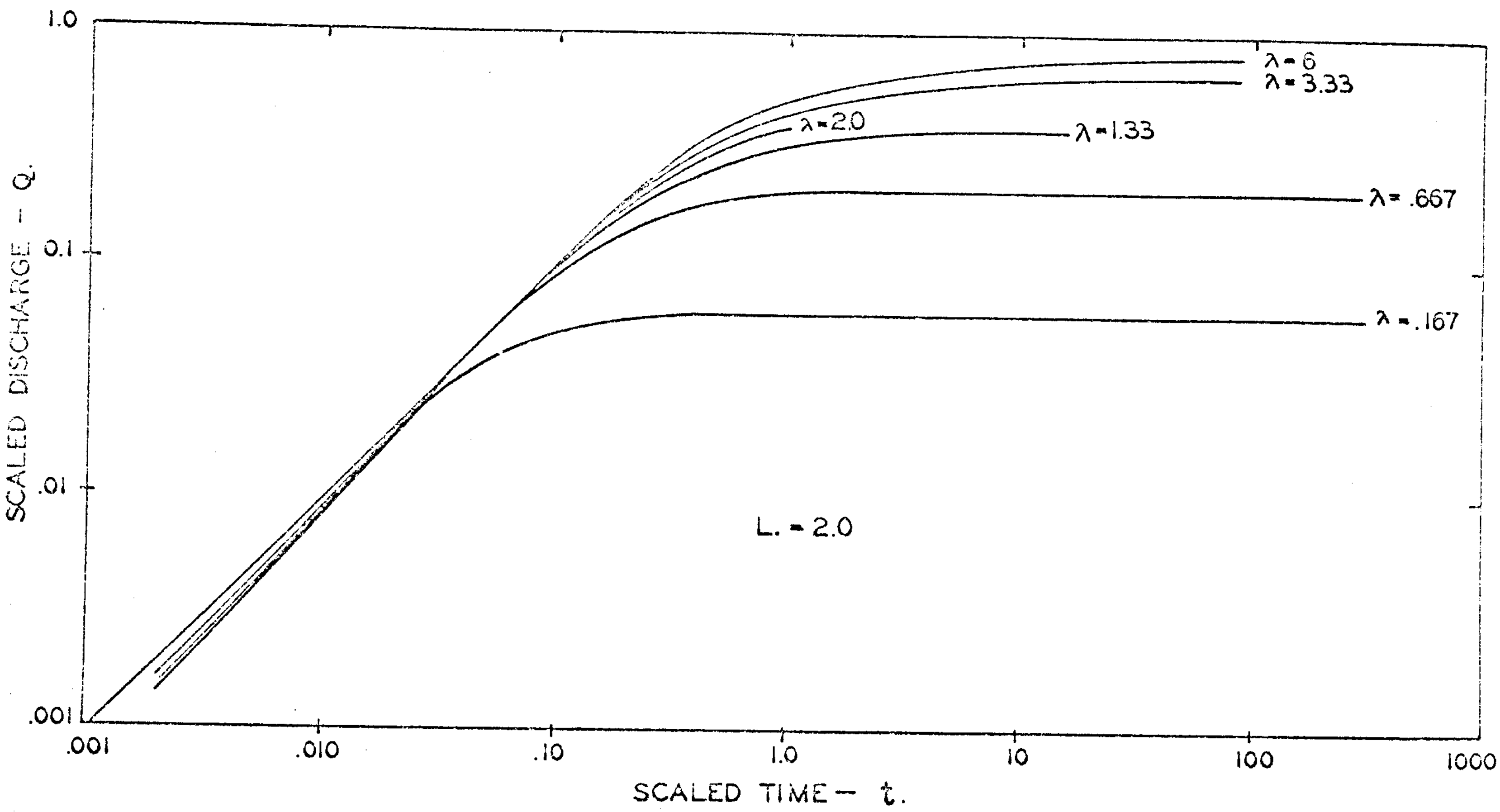


Figure 6.3. Scaled Discharge as a Function of Scaled Time for Drainage from A Scaled Column Length of 2.0 and for Various Pore-Size Distribution Indexes.

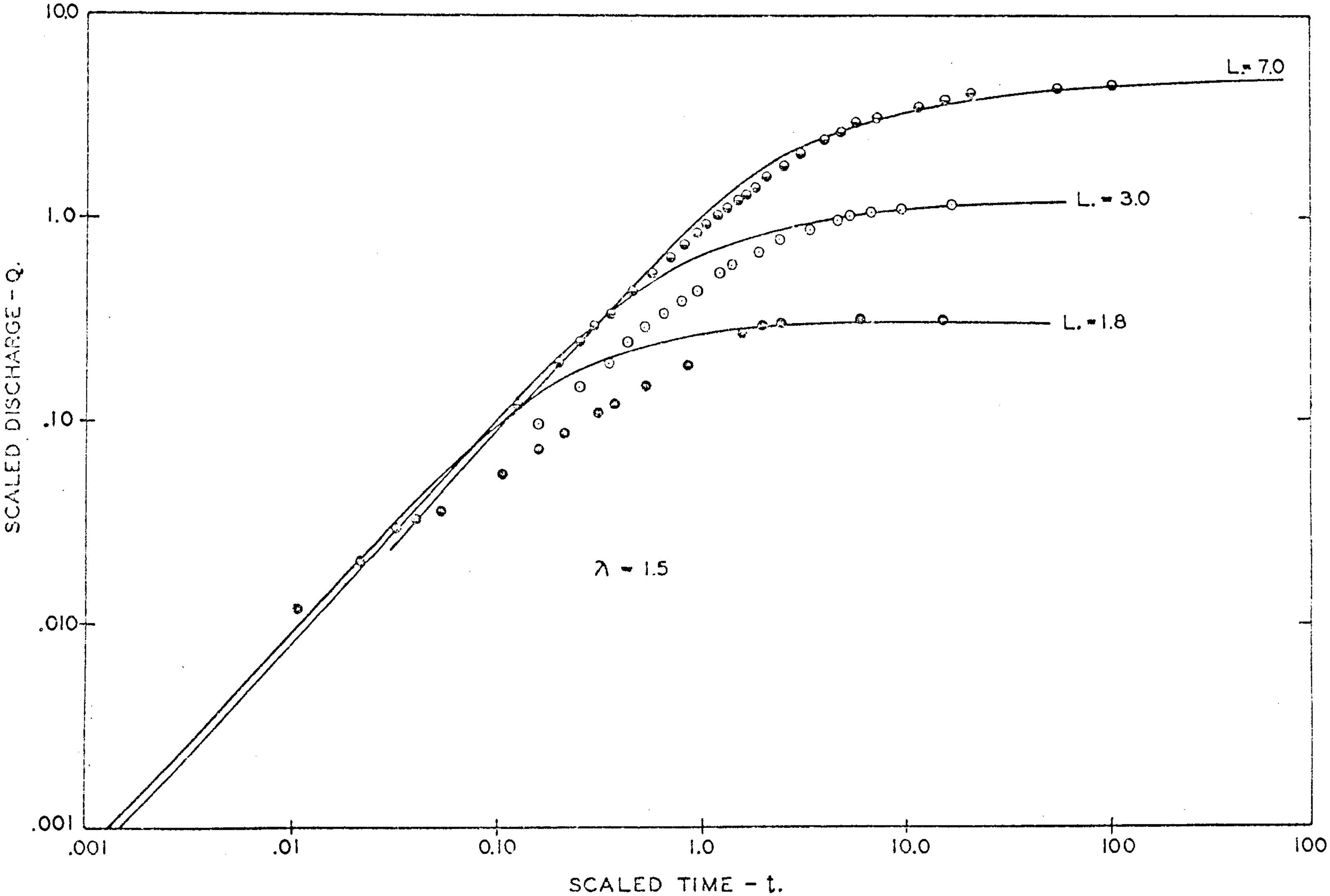


Figure 6.4. Theoretical Scaled discharge as a function of scaled time compared with experimental data.

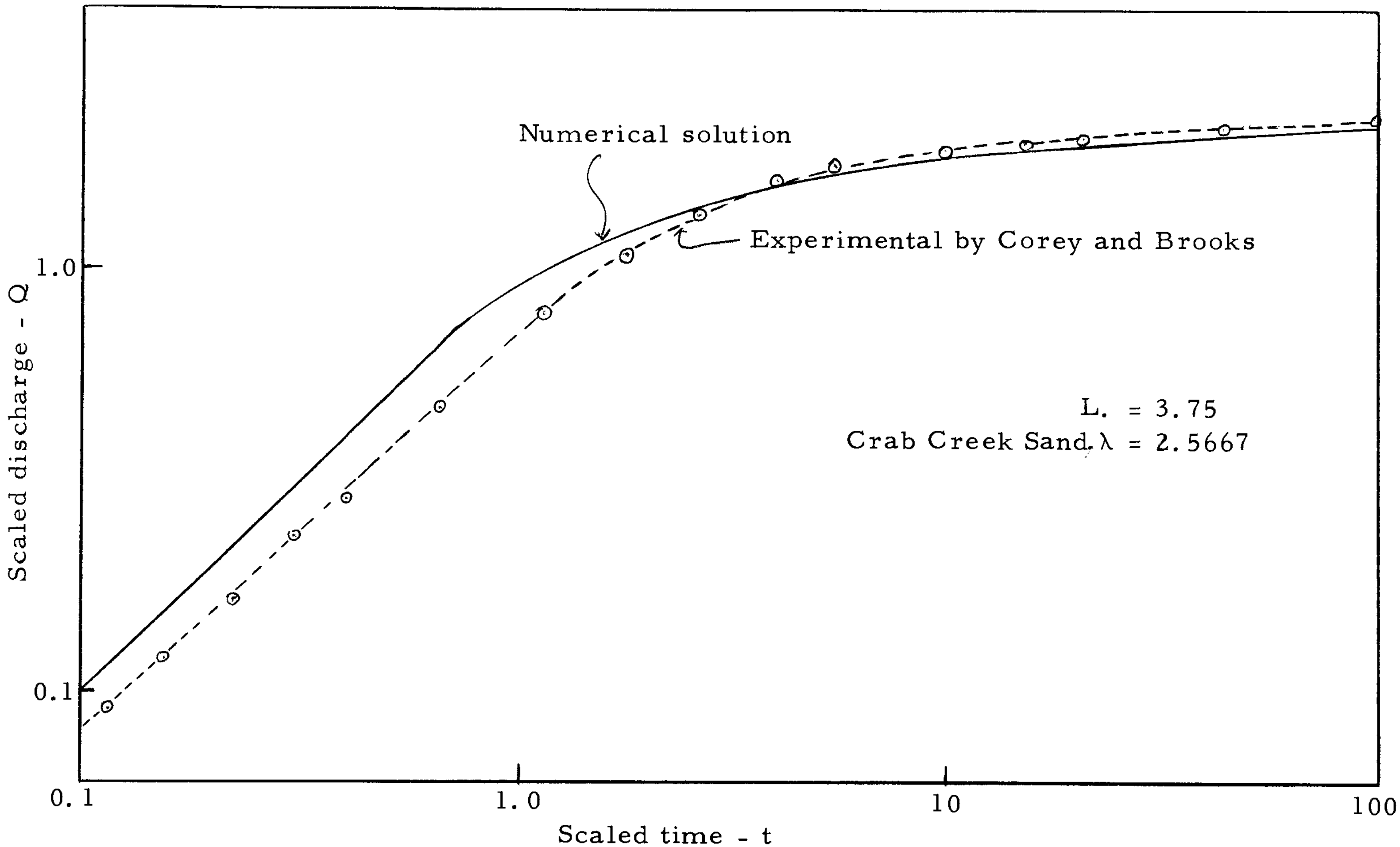


Figure 6.5. Scaled discharge as a function of scaled time for drainage from a scaled column length of 2.0.

VII. CONCLUSIONS

Soil-water distribution may be predicted by solving Equation (3. 21) for the entire unsaturated zone. A computer program has been developed and presented to facilitate its tedious computation. More than 60 sets of solutions of one-dimensional, unsteady vertical drainage problems involving ten different porous media are obtained. Most of them are being submitted elsewhere for publication.

The predictor-corrector method presented here is an excellent numerical technique for solving parabolic partial differential equation since the system of algebraic equations arised is linear. Although this method involves twice as many computations as other standard finite difference methods for each time step, it is worth the extra computing since the order of accuracy is $0(h^2+k^2)$.

It is possible to predict the pore size distribution index just from the outflow data for a given soil profile depth. Experimental data agrees extremely well with this prediction.

Further modifications must be devised for analyzing different boundary conditions such as evaporation and transpiration and for taking account of hysteresis. These results are now being extended to two dimensional drainage problems; however new difficulties arise here, e. g. the location of the water table is an unknown boundary that may vary with time, and consequently this will form the topic of another investigation.

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APPENDIX

APPENDIX

Algorithm Solving the Diffusion Equation

$$\frac{\partial\theta}{\partial t} = \frac{1}{\lambda}\left[D(\theta)\frac{\partial\theta}{\partial z}\right] + \frac{\partial K(\theta)}{\partial z}$$

```
procedure  fct (theta, k, d, lambda);
value      theta, lambda; real theta, k, d, lambda);
comment    This procedure computes  K and D where
           K = θ^(2+3λ)λ   and  D = θ^(1+2λ)/λ ;
begin      k: = theta  (3+2/lambda);
           d: = theta  (2+1/lambda);
end        fct;

procedure integration (x, n, h, sum);
value      x, n, h; array x; integers n; real h, sum;
comment This procedure computes the value of

           sum = ∫_1^L. θ. (z. , t. _n)dz.

           by means of Simpson's rule;
begin
real       se, s; integer i;
           se: = s: = 0;
           for i: = 1 step 2 until n - 1 do
               s: = s + x[i];
```

```
        for  i: = 2 step 2 until n - 2 do
             se: = se + x[i];
             sum: = (2*se+4*s + x[0] + x[n])*h/3;
end     Simpson's rule;

procedure recursive (a, b, c, y, x, n);
value   a, b, c, y, n: array a, b, c, y, x; integer n;
comment This procedure solves a triadiagonal matrix using Gauss'
        elimination.  See Forsythe and Wasow [ 9 ] for details.
        The system of equations is represented by
```

$$
\begin{array}{lr}
b_1x_1 + c_1x_2 & = y_1 \\
a_2x_1 + b_2x_2 + c_2x_3 & = y_2 \\
\quad a_3x_2 + b_3x_3 + c_3x_4 & = y_3 \\
\quad\quad \cdots\cdot \quad \cdots\cdot \quad \cdots\cdot & \\
a_{n-1}x_{n-2} + b_{n-1}x_{n-1} + c_{n-1}x_n & = y_{n-1} \\
a_nx_{n-1} + b_nx_n & = y_n
\end{array}
$$

```
        where a, b, c and y are known constants.
begin
integer i;
        c[0]: = c[0]/b[0];
        y[0]: = y[0] /b[0];
```

```
          for i: = 0 step 1 until n - 1 do
               begin
               c[i+1]: = c[i+1]/(b[i+1]-a[i+1]*c[i]);
               y[i+1]: = (y[i+1]-a[i+1]*y[i])/(b[i+1]-a[i+1]*c[i]);
               end i;
          for i: = n - 1 step -1 until 0 do
               x[i]: = y[i] - c[i] * x[i+1];
end       This solved the system;

procedure (n, tℓ, xi, xℓ, t, lambda; recursive, fct) result: (x, q);
real      tℓ, xi, xℓ, lambda, q; procedure recursive, fct;
array     x;
integer   n;
comment   The following are the input variables:
          xi, xℓ, t, lambda, n
          where n is the number of step sizes.
          θ(xℓ, k) = tℓ  for all t.
          t is the increment of each time step, xℓ the length of the
          soild column, xi, the bubbling pressure head, lambda,
          the pore size distribution index and q is the outflow as a
          function of time;
begin
real      vol, h, ee, e, dd, d, theta, k, k2, sum, qq;
```

```
array     a, b, c, x, y, u[0,n];
comment   h is the mesh size,  e is k/(h^2λ) and ee is k/h;
          q: = 0;
          h: = (xℓ-xi)/n;
          ee: = t/h;
          e: = t/(h*h*lambda);
          vol: = tℓ*xℓ;
comment   We shall now initialize the initial condition which was held
          at saturation;
          for i: = 0 step 1 until n do
              x[i]: = tℓ;
              u[n]: = x[n];
              n: = n - 1;
comment   We implement (5.45), the top boundary as follows
          where b_o is denoted by 1 + (r/λ)D(θ_{1/2,j}),
          c_o by (-r/λ)D(θ_{1/2,j}) and y_o by θ_{o,j} - r'K(θ_{1/2,j});
6:        theta: = (x[0]+x[1])/2;
          fct (theta, k, d, lambda);
          b[0]: = 1+ e*d;
          c[0]: = - e*d;
          y[0]: = x[0] - k*ee;
```

```
comment We are now ready to implement (5. 39) as follows;
        for i: = 1 step 1 until n do
            begin
               dd: = d;
               k2: = k;
               theta: = (x[i]+x[i+1] )/ 2;
               fct (tehta, k, d, lambda);
               comment from (5. 39) we have
```

$$a_i = -\frac{r}{2\lambda}D(\theta_{i-\frac{1}{2},j})$$

$$b_i = 1 + \frac{r}{2\lambda}D(\theta_{i-\frac{1}{2},j}) + \frac{r}{2\lambda}D(\theta_{i+\frac{1}{2},j})$$

$$\text{and } c_i = -\frac{r}{2\lambda}D(\theta_{i+\frac{1}{2},j});$$

```
               a[i]: = = e*dd/ 2;
               c[i]: = - e*d/ 2;
               b[i]: = 1 - a[i] - c[i];
               y[i]: = x[i] + ee*(k2-k)/ 2;
            end;
        y[n]: = y[n] - c[n]*x[n+1];
        recursive (a, b, c, y, u, n);
```

comment The array u gives the water content for the predictor scheme at $j+\frac{1}{2}$ time step. We shall now proceed to use the corrector scheme to find the water content, θ, at

```
          j+1  time step;

          theta: = (u[0]+u[1])/2;
          fct (theta, k, d, lambda);
comment   Using Equation (5.47) we have;
          b[0]: = 1 + e*d;
          c[0]: = - e*d;
          y[0]: = (1-e*d)*x[0]+e*d*x[1]-2*ee*k;
comment   The following make use of Equation (5.41);
          for i: = 1 step 1 until n do
              begin
                  dd: = d;
                  k2: = k;
                  theta: = (u[i]+u[i+1])/2;
                  fct (theta, k, d, lambda);
                  a[i]: = - e*dd/2;
                  c[i]: = - e*d/2;
                  b[i]: = 1 - a[i] - c[i];
                  y[i]: = - a[i]*x[i-1] - (b[i]-2)*x[i] - c[i]*x[i+1]
                          + ee*(k2-k);
              end Equation (5.41);
          y[n]: = y[n] - c[n]*x[n+1];
          recursive (a, b, c, y, x, n);
```

```
comment The array x returns all the values of water content at
        j+1 time step which is ready for output;
comment Now we want to use Equation (4. 7) to calculate the outflow
        as a function of time;

        integration (x, n+1, h, sum);
        qq: = vol - sum;
         q: = qq + q;
comment The value q here is the accumulative outflow as a function
        of time;

        vol: = sum;
        go to 6;
end
```