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The classical Hilbert transform is defined on $L^2(R)$. In the first part of this thesis we extend the definition of Hilbert transform to $L^2(R^n)$ and show that the Hilbert transform of a real-valued function u in $L^2(R^n)$ is the boundary value of a certain conjugate Poisson integral of u. This result generalizes a well-known classical formula for Hilbert transform on $L^2_R(R)$. In the second part of this thesis, the definition of Hilbert transform is further generalized to $\mathcal{D}'_{L^2}(R^n)$ and results similar to those for $L^2(R^n)$ are obtained.

GENERALIZED HILBERT TRANSFORMS ASSOCIATED TO CONES IN R^n

by

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This thesis is dedicated to my deceased father, Shao-Chiu Chang, and to my mother, Su-Wen T'ang Chang.

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GENERALIZED HILBERT TRANSFORMS ASSOCIATED TO CONES IN R^n

CHAPTER I

INTRODUCTION

The classical Hilbert transform H on $L^2(R)$ may be defined by

$$(Hu)^\wedge(t) = -i \frac{t}{|t|} \hat{u}(t), \quad \text{for } u \text{ in } L^2(R).$$

Clearly H is an isometry of $L^2(R)$ and $H^2 = -I$, $H^* = -H$. It is also easy to see that H is a real operator, i.e., H commutes with conjugation. On real functions, H may be characterized as follows:

Let u_o be a real valued function in $L^2(R)$. Then there exists a real valued function v_o in $L^2(R)$ such that if $F_o = u_o + iv_o$, then supp $\hat{F}_o \subseteq [0,\infty)$. In this case $v_o = Hu_o$. Moreover, there exists a unique holomorphic function F in the upper half plane Ω such that F is in $H^2(\Omega)$ and

$$F_o(x) = L^2\text{-lim } F(x+iy) \quad \text{as } y \downarrow 0.$$

In this case F is just the Cauchy integral of F_o, and if we set F = u + iv, where u and v are real, and compute v explicitly, we obtain

$$v_o(x) = \underset{y\downarrow o}{L^2\text{-lim}}\ v(x,y)$$

$$= \underset{y\downarrow o}{L^2\text{-lim}}\ \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(x-t)}{(x-t)^2+y^2}\, u_o(t)\, dt,$$

i.e., formally at least v_o is given by the singular integral

$$v_o(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u_o(t)}{x - t}\, dt.$$

In general this integral diverges, but it does exist in the principle value sense for almost all x.

Thus there are two aspects to the Hilbert transform, which may be described briefly as

(A) the connection with singular integrals, and

(B) the connection with boundary values of holomorphic functions.

Corresponding to these aspects there are at least two ways of generalizing the Hilbert transform to R^n.

(A) <u>The Riesz transforms</u>.

One way to generalize the Hilbert transform to R^n is to introduce the Riesz transforms R_j, $j = 1,2, \cdots\cdots ,n$. These transforms are defined by

$$(R_j u)^\wedge(\xi) = -i \frac{\xi_j}{|\xi|} \hat{u}(\xi), \quad \text{for } u \text{ in } L^2(R^n).$$

Clearly R_j is a bounded operator on $L^2(R^n)$ and we have

$$\sum_{j=1}^{n} R_j^2 = -I \quad \text{and} \quad R_j^* = -R_j, \quad j = 1,2,\cdots\cdots,n.$$

The Riesz transforms are singular integral operators and in a certain sense are the prototypes for all singular integral operators.

(B) Hilbert Transforms Associated To A Cone.

Let Γ be a closed convex cone in R^n. We assume Γ to be salient so that the dual cone Γ^* has nonempty interior. Now we define

$$h_\Gamma(\xi) = \begin{cases} 1 & \text{if } \xi \text{ is in } \Gamma^*-\{0\} \\ -1 & \text{if } -\xi \text{ is in } \Gamma^*-\{0\} \\ 0 & \text{otherwise.} \end{cases}$$

Then we define the Hilbert transform H_Γ associated to Γ by

$$(H_\Gamma u)^\wedge(\xi) = -i\, h_\Gamma(\xi)\, \hat{u}(\xi), \quad \text{for } u \text{ in } L^2(R^n).$$

Then H_Γ is a bounded real operator in $L^2(R^n)$, $H_\Gamma^* = -H_\Gamma$ and $-H_\Gamma^2$ is an orthogonal projection. This Hilbert transform is related to boundary values of holomorphic functions in much the same way as in the one-dimensional case. It is this Hilbert transform which we will study here.

In order to prepare our way, we briefly discuss cones in R^n and Fourier transforms and convolutions in chapter II, and some distribution theory in chapter IV. In chapter III, we study the Hilbert transform H_Γ on $L^2(R^n)$ and show that the Hilbert transform of a real-valued function u in $L^2(R^n)$ is the boundary value of a certain conjugate Poisson integral of u. In chapter V, we extend the definition of Hilbert transform to $\mathcal{D}'_{L^2}(R^n)$ and show that we have results similar to those obtained for $L^2(R^n)$.

The Hilbert transform on $L^2(R^1)$ is studied, for example, in Titchmarsh [2], Hewitt [1], Zygmund [1], Weiss [1], Butzer-Trebels [1] and Butzer-Nessel [1]. The Hilbert transform on distributions in one-dimensional space may be found in Horvath [1], Tillmann [1] and [2], Newcomb [1], Lauwerier [1], Beltrami-Wohlers [1] and [2], and Güttinger [1].

CHAPTER II

PRELIMINARY FOR CHAPTER III

(i) Convex Cones

A subset Γ of R^n is called a cone if tx is in Γ whenever x is in Γ and $t > 0$. If Γ is a cone, then the set

$$\Gamma^* = \{\xi \in R^n \mid (\xi,x) \geq 0 \text{ for each } x \in \Gamma\}$$

is a closed convex cone and is called the dual cone of Γ.

Theorem 2.1. $\Gamma^{***} = \Gamma^*$ and Γ^{**} is the smallest closed convex cone which contains Γ.

Proof. Clearly $\Gamma \subseteq \Gamma^{**}$ and therefore $(\Gamma^{**})^* \subseteq \Gamma^*$ and $\Gamma^* \subseteq (\Gamma^*)^{**}$, hence $\Gamma^{***} = \Gamma^*$. Suppose A is a closed convex cone and $\Gamma \subseteq A \subseteq \Gamma^{**}$. Then $A^* = \Gamma^*$ by the first part. If x is not in A, by the separation theorem we can find ξ in R^n so that $(\xi,x) < (\xi,y)$ for each y in A. Since A is a cone, we have $(\xi,x) < t(\xi,y)$ for each y in A and $t > 0$. Thus $(\xi,y) \geq 0$ for each y in A and $(\xi,x) < 0$. The first condition implies that ξ is in $A^* = \Gamma^*$, and therefore x is not in Γ^{**}. Thus $A \supseteq \Gamma^{**}$, therefore $\Gamma^{**} = A$. Now if A is any closed convex cone and $\Gamma \subseteq A$, by the above $A \cap \Gamma^{**} = \Gamma^{**}$, i.e., $\Gamma^{**} \subseteq A$. Q.E.D.

Definition. A cone Γ in R^n is said to be salient if Γ^{**} contains no subspaces other than $\{0\}$.

Lemma 2.2. If Γ is a cone, then Γ is salient if and only if Γ^* has nonempty interior.

Proof. A subspace L of R^n is clearly a cone and $L^* = L^\perp$. Suppose Γ is not salient. Then there is a subspace $L \neq \{0\}$ such that $L \subset \Gamma^{**}$. Thus $\Gamma^{***} = \Gamma^* \subset L^\perp$ which implies Γ^* has empty interior. Conversely, suppose Γ^* has empty interior. Let L be the subspace of R^n spanned by Γ^*. Then $L \neq R^n$, since Γ^* is convex and has empty interior. But then $\{0\} \neq L^\perp \subset \Gamma^{**}$ and so Γ is not salient. Q.E.D.

Lemma 2.3. Let Γ be a convex cone and for each x in Γ define

$$\delta(x) = \inf\{(\xi,x) \mid \xi \text{ is in } \Gamma^* \text{ and } |\xi| = 1\}.$$

Then $\delta(x)$ is the distance from x to the boundary of Γ.

Proof. Let d be the distance from x to the boundary of Γ and choose x_o in the boundary of Γ such that $|x-x_o| = d$. Let H be a supporting hyperplane of Γ at x_o with normal η such that $|\eta| = 1$, $(\eta,x_o) = a \geq 0$ and $(\eta,y) \geq a$ for each y in Γ. It follows that η is in Γ^*. Since 0 is in $\overline{\Gamma}$ we also have $a = 0$. Thus

$$\delta(x) \leq (\eta,x) = (\eta,x-x_o) = \text{dist}(x,H) \leq |x-x_o| = d.$$

Conversely, suppose ξ is in S^{n-1}. Then $x-d\xi$ is on the sphere with center at x and radius d. So we have $x-d\xi$ is in $\overline{\Gamma} = \Gamma^{**}$, and hence if ξ is in Γ^*, we have $(\xi,x-d\xi) \geq 0$. Thus $(\xi,x) \geq d$ which implies $\delta(x) \geq d$. Q.E.D.

An immediate consequence of the above lemma will be useful to us in chapter III.

Corollary 2.4. Suppose Γ is a convex cone. Then x lies in the interior of Γ if and only if there exists $\delta > 0$ such that $(\xi, x) \geq \delta|\xi|$ for each ξ in Γ^*.

(ii) Fourier Transforms and Convolutions

If $1 \leq p \leq \infty$, we denote by $L^p(R^n)$ the Banach space of complex-valued L^p functions on R^n relative to Lebesgue measure. As usual, we identify functions which are equal almost everywhere. We denote by $L^p_R(R^n)$ the real Banach space of real-valued functions in $L^p(R^n)$.

If u is in $L^1(R^n)$, we define the Fourier transform $\hat{u}(\xi)$ or $\mathfrak{F}[u(x);\xi]$ of $u(x)$ by

$$\hat{u}(\xi) = \int_{R^n} e^{-i(\xi,x)} u(x)\, dx$$

and the inverse Fourier transform $\tilde{u}(\xi)$ or $\mathfrak{F}^{-1}[u(x);\xi]$ of $u(x)$ by

$$\tilde{u}(\xi) = (2\pi)^{-n} \int_{R^n} e^{i(\xi,x)} u(x)\, dx .$$

Remark. For any function u on R^n we define $\check{u}(x) = u(-x)$.

We note that if u is in $L^1(R^n)$, then $\tilde{u} = (2\pi)^{-n}\, \check{\hat{u}}$, $\hat{\bar{u}} = \overline{\hat{\check{u}}}$ and $\vee$ commutes with $\wedge$, $\sim$ and $-$.

Theorem 2.5. If u is in $L^1(R^n)$ then $\hat{u}$ is a bounded uniformly continuous function which vanishes at ∞ and $\|\hat{u}\|_\infty \leq \|u\|_{L^1}$.

Proof. This theorem is the Riemann-Lebesgue lemma. See, for example, Bochner and Chandrasekharan [1] p.57.

Theorem 2.6. If u is in $L^1(R^n)$ and $\hat{u}$ is in $L^1(R^n)$ then $u = \tilde{\hat{u}} = \hat{\tilde{u}}$.

Proof. This is the Fourier inversion theorem. See, for example, Bochner and Chandrasekharan [1] p. 65.

Theorem 2.7. If u is in $L^1(R^n) \cap L^2(R^n)$, then $\hat{u}$ is in $L^2(R^n)$ and $\|\hat{u}\|_{L^2} = (2\pi)^{n/2} \|u\|_{L^2}$ (Plancherel formula).

Both $\mathfrak{F}$ and $\mathfrak{F}^{-1}$ extend uniquely to bounded linear operators of $L^2(R^n)$ onto itself and these extensions are inverses of each other . If u and v are in $L^2(R^n)$ then we have

(i) $(\hat{u},\hat{v}) = (2\pi)^n (u,v)$ (Parseval's formula),

(ii) $\hat{\hat{u}} = (2\pi)^n \check{u}$

(iii) $\int_{R^n} u(x)\, \hat{v}(x)\, dx = \int_{R^n} \hat{u}(\xi)\, v(\xi)\, d\xi .$

In particular $(2\pi)^{-n/2}\,\mathfrak{F}$ is an isometry of $L^2(R^n)$ onto $L^2(R^n)$.

Proof. This is the L^2-theory of the Fourier transform. See, for example, Bochner and Chandrasekharan [1] p. 120.

If $1 \leq p \leq 2$ then $L^p(R^n) \subseteq L^1(R^n)+L^2(R^n)$. Thus if u is in $L^p(R^n)$ we may write $u = u_1+u_2$ with u_j in $L^j(R^n)$, $j = 1,2$. In this case we define $\hat{u} = \hat{u}_1+\hat{u}_2$. It is easy to check that $\hat{u}$ is well-defined and we have the following result.

Theorem 2.8. If $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p \leq 2$ and u is in $L^p(R^n)$, then $\hat{u}$ is in $L^q(R^n)$ and moreover

$$\|\hat{u}\|_{L^q} \leq (2\pi)^{n/q}\, \|u\|_{L^p} .$$

Proof. The proof depends on the M. Riesz-Thorin interpolation (or convexity) theorem. See Weiss [2] p. 168ff, Zygmund [1] Vol.II p. 254ff, Katznelson [1] p. 141ff and Donoghue [1] p. 260. The inequality in the theorem, due to Titchmarsh [1], is called the Titchmarsh inequality and is also referred to as Hausdorff-Young inequality.

Theorem 2.9. If p, q, r are extended real numbers such that $1 \leq p, q, r \leq \infty$ and $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$ and if f is in $L^p(R^n)$ and g is in $L^q(R^n)$, then the integral

$$f * g(x) = \int_{R^n} f(x-y)\; g(y)\, dy$$

converges for almost all x and defines a function $f * g$ in $L^r(R^n)$ called the <u>convolution</u> of f and g. Moreover we have $f * g = g * f$ a.e. and

$$\|f * g\|_r \leq \|f\|_p \; \|g\|_q \qquad \text{(Young's inequality)}$$

In case $r = +\infty$, the integral converges for each x and $f * g$ is a bounded uniformly continuous function.

<u>Proof</u>. See Zygmund [1] Vol.II p. 37 or Hewitt and Stromberg [1] p. 414.

The Fourier transform converts convolution into multiplication. This statement has a wide range of applicability when we pass to distributions. For the present, we have the following results.

<u>Proposition 2.10</u>. If f and g are in $L^1(R^n)$ then $f * g$ is in $L^1(R^n)$ and $(f * g)^\wedge = \hat{f}\,\hat{g}$.

<u>Proof</u>. See, for example, Bochner and Chandrasekharan [1] p. 58.

<u>Theorem 2.11</u>. If f and g are in $L^2(R^n)$ then

$$f * g = (\hat{f}\,\hat{g})^\sim$$

<u>Proof</u>. If f is in $L^1(R^n)$ then

$$(\mathcal{T}_x f)^\wedge(\xi) = e^{-i(x,\xi)}\,\hat{f}(\xi)$$

where $\mathfrak{T}_x f(y) = f(y-x)$. Since $|e^{-i(x,\xi)}| = 1$ this property also holds for f in $L^2(R^n)$. Then by theorem 2.7 we have

$$f * g(x) = \int_{R^n} \mathfrak{T}_x \check{f}(y)\ g(y)\ dy$$

$$= \int_{R^n} (\mathfrak{T}_x \check{f})^{\wedge}(y)\ g(y)\ dy$$

$$= \int_{R^n} (\mathfrak{T}_x \check{f})^{\sim}(\xi)\ \hat{g}(\xi)\ d\xi$$

$$= \int_{R^n} (\mathfrak{T}_{-x} f)^{\tilde{\vee}}(\xi)\ \hat{g}(\xi)\ d\xi$$

$$= (2\pi)^{-n} \int_{R^n} (\mathfrak{T}_{-x} f)^{\wedge}(\xi)\ \hat{g}(\xi)\ d\xi$$

$$= (2\pi)^{-n} \int_{R^n} e^{i(x,\xi)}\ \hat{f}(\xi)\ \hat{g}(\xi)\ d\xi$$

$$= (\ \hat{f}\ \hat{g}\ )^{\sim} \qquad \text{Q.E.D.}$$

The main references to Fourier transforms are the books by Bochner [4], Wiener [1], Titchmarsh[2], Carleman [1], Bochner and Chandrasekharan [1], Zygmund [1], Hewitt [1], Goldberg [1], Weiss [1], and Katznelson [1].

CHAPTER III

THE HILBERT TRANSFORM ON $L^2(R^n)$

Let Γ be a closed convex salient cone, so Γ^* has nonempty interior. We define the function h by

$$h(\xi) = \begin{cases} 1, & \text{if } \xi \text{ is in } \Gamma^*-\{0\} \\ -1, & \text{if } -\xi \text{ is in } \Gamma^*-\{0\} \\ 0, & \text{otherwise.} \end{cases}$$

Now we define the <u>Hilbert transform</u> H_Γ (or simply H if no confusion can arise) associated with Γ by

$$(Hu)^\wedge(\xi) = -i\ h(\xi)\ \hat{u}(\xi) \quad \text{for } u \text{ in } L^2(R^n).$$

<u>Theorem 3.1</u>. H is a bounded linear operator on $L^2(R^n)$. In fact $\|H\| \leq 1$. Moreover we have

(a) the adjoint H^* of H is $-H$,

(b) $Q = -H^2$ is an orthogonal projection in $L^2(R^n)$ and $(Qu)^\wedge(\xi) = |h(\xi)|\ \hat{u}(\xi)$ for u in $L^2(R^n)$,

(c) $H = QH = HQ$.

<u>Proof</u>. $(Hu)^\wedge(\xi) = -i\,h(\xi)\ \hat{u}(\xi)$ is in $L^2(R^n)$ for u in $L^2(R^n)$. By theorem 2.7 we have Hu is in $L^2(R^n)$, and

$$\begin{aligned} \|Hu\|_{L^2} &= (2\pi)^{-n/2}\ \|\widehat{Hu}\|_{L^2} \\ &= (2\pi)^{-n/2}\ \|i\ h(\xi)\ \hat{u}(\xi)\|_{L^2} \\ &\leq (2\pi)^{-n/2}\ \|\hat{u}\|_{L^2} \\ &= \|u\|_{L^2}\ . \end{aligned}$$

Moreover, we have following:

(a) For every u and v in $L^2(R^n)$, we have

$$
\begin{aligned}
(Hu,v) &= (2\pi)^{-n}\,(\widehat{Hu},\hat{v}) \\
&= (2\pi)^{-n}\,(-ih(\xi)\hat{u}(\xi),\hat{v}(\xi)) \\
&= (2\pi)^{-n}\,(\hat{u}(\xi),ih(\xi)\hat{v}(\xi)) \\
&= (2\pi)^{-n}\,(\hat{u},-\widehat{Hv}) \\
&= (u,-Hv) \\
&= (u,H^{*}v)\ .
\end{aligned}
$$

Therefore $H^{*} = -H$.

(b) Let $Q = -H^2$. For each u in $L^2(R^n)$, we have

$$
\begin{aligned}
(Qu)^{\wedge}(\xi) &= (-H^2u)^{\wedge}(\xi) \\
&= -(H(Hu))^{\wedge}(\xi) \\
&= ih(\xi)(Hu)^{\wedge}(\xi) \\
&= h^2(\xi)\hat{u}(\xi) \\
&= |h(\xi)|\ \hat{u}(\xi)
\end{aligned}
$$

$$
\begin{aligned}
(Q^2u)^{\wedge}(\xi) &= |h(\xi)|\ (Qu)^{\wedge}(\xi) \\
&= |h(\xi)|^2\ \hat{u}(\xi) \\
&= |h(\xi)|\ \hat{u}(\xi) \\
&= (Qu)^{\wedge}(\xi)\ .
\end{aligned}
$$

Hence $Q^2 = Q$ by theorem 2.7. We also have

$$Q^{*} = (-H^2)^{*} = -(H^{*})^2 = -(-H)^2 = -H^2 = Q.$$

Therefore Q is an orthogonal projection.

(c) Since $Q = -H^2$, it is obvious that $QH = HQ$.

Also we have, for every u in $L^2(R^n)$,

$$\begin{aligned}(QHu)^\wedge(\xi) &= |h(\xi)|\ (Hu)^\wedge(\xi) \\ &= |h(\xi)|\ (-ih(\xi))\ \hat{u}(\xi) \\ &= -i\ h(\xi)\ \hat{u}(\xi) \\ &= (Hu)^\wedge(\xi)\ .\end{aligned}$$

By theorem 2.7, we have QHu = Hu for every u in $L^2(R^n)$. Therefore H = QH = HQ . Q.E.D.

<u>Definition</u>. Suppose u is in $L^1_{loc}(R^n)$. We define the <u>support</u> of u as the complement of the largest open set on which u vanishes almost everywhere. We denote the support of u by supp u.

To see that the definition makes sense, i.e., the existence of the largest open set, let $\{U_\alpha\}_{\alpha\in A}$ be the family of all open sets (suitably indexed) such that u vanishes almost everywhere in U_α for each α in A, and let $\mathfrak{u} = \bigcup_{\alpha\in A} U_\alpha$. Since $\mathfrak{u}$ is σ-compact there is a countable subset N of A such that $\mathfrak{u} = \bigcup_{\alpha\in N} U_\alpha$. Thus u vanishes almost everywhere in $\mathfrak{u}$.

Let im T be the image of a transformation T and ker T be the kernel of T.

<u>Theorem 3.2</u>. The Hilbert transform H associated with Γ has the following properties.

(a) H is an isometry of im Q onto im Q .

(b) ker H = ker Q = $(\text{im } Q)^{\perp}$.

(c) im H = im Q = $\{u \in L^2(R^n) \mid \text{supp } \hat{u} \subseteq \Gamma^* \cup (-\Gamma^*)\}$.

Proof. According to theorem 2.7 and 3.1, we have

$$\|HQu\|_{L^2} = (2\pi)^{-n/2} \|\widehat{HQu}\|_{L^2}$$

$$= (2\pi)^{-n/2} \|\widehat{Hu}\|_{L^2}$$

$$= (2\pi)^{-n/2} \|-i\ h(\xi)\ \hat{u}(\xi)\|_{L^2}$$

$$= (2\pi)^{-n/2} \|\,|h(\xi)|\ \hat{u}(\xi)\|_{L^2}$$

$$= (2\pi)^{-n/2} \|\widehat{Qu}\|_{L^2}$$

$$= \|Qu\|_{L^2} .$$

Now if v is in im Q, then $v = Q^2 v = H(-QHv)$ and so H maps im Q onto im Q isometrically. In particular im Q $\subseteq$ im H whence QH = H implies im Q = im H. Now $Q = -H^2$ implies ker H $\subseteq$ ker Q and H = HQ implies ker Q $\subseteq$ ker H. Finally u is in im Q if and only if Qu = u, i.e., $\hat{u} = |h|\hat{u}$ a.e. Thus u is in im Q if and only if $\hat{u} = 0$ a.e. outside the closed set $\Gamma^* \cup (-\Gamma^*)$. Q.E.D.

Definition. A linear operator T on $L^2(R^n)$ is called a real operator if it commutes with conjugation, i.e., T is real if $T\bar{u} = \overline{Tu}$ for each u in $L^2(R^n)$.

A real operator clearly maps real functions to real functions and hence induces a linear operator on $L^2_R(R^n)$, which is the real Hilbert space of real-valued functions in $L^2(R^n)$.

Proposition 3.3. H and Q are real operators on $L^2(R^n)$.

Proof. From the remark preceding the theorem 2.5 we have following equalities, for u in $L^2(R^n)$,

$$(\overline{Hu})^\wedge(\xi) = \overline{(Hu)^\wedge}^{\vee}(\xi) = \overline{(Hu)^\wedge(-\xi)}$$

$$= \overline{-i\ h(-\xi)\ \hat{u}(-\xi)}$$

$$= \overline{-i\ (-h(\xi))\ \hat{u}(-\xi)}$$

$$= \overline{i\ h(\xi)\ \hat{u}(-\xi)}$$

$$= -i\ h(\xi)\ \overline{\hat{u}}^{\vee}(\xi)$$

$$= -i\ h(\xi)\ \hat{\bar{u}}(\xi)$$

$$= (H\bar{u})^\wedge(\xi).$$

Therefore $\overline{Hu} = H\bar{u}$ for every u in $L^2(R^n)$. Also we have $Q\bar{u} = -H^2\bar{u} = -H(H\bar{u}) = -H(\overline{Hu}) = -\overline{H(Hu)} = \overline{-H(Hu)} = \overline{Qu}$, for every u in $L^2(R^n)$. Q.E.D.

We denote the characteristic function of a set A by χ_A . Thus

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \text{ is in } A \\ 0 & \text{otherwise.} \end{cases}$$

Now let W be the bounded linear operator in $L^2(R^n)$ defined by $W = \frac{1}{2}(Q + iH)$.

Proposition 3.4. We have $(Wu)^\wedge = \chi_{\Gamma^*}\hat{u}$ a.e. for each u in $L^2(R^n)$. In particular W is an orthogonal projection in $L^2(R^n)$ and $\operatorname{im} W = \{ u \in L^2(R^n) \mid \operatorname{supp} \hat{u} \subseteq \Gamma^* \}$.

Proof.
$$\begin{aligned}(Wu)^\wedge(\xi) &= \tfrac{1}{2} \widehat{(Q + iH)u}(\xi) \\ &= \tfrac{1}{2} ((Qu)^\wedge(\xi) + i(Hu)^\wedge(\xi)) \\ &= \tfrac{1}{2} (|h(\xi)|\hat{u}(\xi) + i(-i)h(\xi)\hat{u}(\xi)) \\ &= \tfrac{1}{2} (|h(\xi)| + h(\xi)) \hat{u}(\xi) \\ &= \chi_{\Gamma^*}(\xi) \hat{u}(\xi) \quad \text{a.e.}\end{aligned}$$

By theorem 3.1, we have the following equalities:

$$\begin{aligned}W^2 &= \tfrac{1}{4} (Q + iH)^2 \\ &= \tfrac{1}{4} (Q^2 - H^2 + 2iHQ) \\ &= \tfrac{1}{4} (Q + Q + 2iH) \\ &= \tfrac{1}{2} (Q + iH) \\ &= W.\end{aligned}$$

$$\begin{aligned}W^* &= \tfrac{1}{2} (Q + iH)^* \\ &= \tfrac{1}{2} (Q^* + (iH)^*) \\ &= \tfrac{1}{2} (Q - iH^*) \\ &= \tfrac{1}{2} (Q + iH) \\ &= W.\end{aligned}$$

Therefore W is an orthogonal projection in $L^2(R^n)$.

Thus u is in im W if and only if $u = Wu$ if and only if $\hat{u} = \chi_{\Gamma^*}\hat{u}$ a.e. if and only if $\hat{u} = 0$ a.e. outside the closed set Γ^*. Q.E.D.

Lemma 3.5. If u_o and v_o are in $L^2_R(R^n)$, then the following statements are equivalent.

(a) $W(u_o+iv_o) = u_o + iv_o$.

(b) $Qu_o - Hv_o = 2u_o$ and $Qv_o + Hu_o = 2v_o$.

(c) $Qu_o = u_o$ and $Hu_o = v_o$.

Proof. (a) <=> (b) Consider following equalities:

$$W(u_o+iv_o) = \tfrac{1}{2}\,(Q + iH)(u_o+iv_o)$$
$$= \tfrac{1}{2}\,[(Qu_o - Hv_o) + i(Qv_o + Hu_o)].$$

Therefore $W(u_o+iv_o) = u_o + iv_o$ if and only if

$$Qu_o - Hv_o = 2u_o \quad\text{and}\quad Qv_o + Hu_o = 2v_o .$$

(b) <=> (c) If $Qu_o - Hv_o = 2u_o$ and $Qv_o + Hu_o = 2v_o$, then, applying H on first equality and using theorem 3.1, we have $2Hu_o = HQu_o - H^2v_o = Hu_o + Qv_o = 2v_o$ which implies $Hu_o = v_o$. Apply this fact to first equality, we have $2u_o = Qu_o - Hv_o = Qu_o - H(Hu_o) = Qu_o + Qu_o = 2Qu_o$ which implies $Qu_o = u_o$. Conversely, if $Qu_o = u_o$ and $Hu_o = v_o$, then $Qu_o - Hv_o = Qu_o - H(Hu_o) = 2Qu_o = 2u_o$ and $Qv_o + Hu_o = Q(Hu_o) + Hu_o = 2Hu_o = 2v_o$. Q.E.D.

Corollarly 3.6. Let u_o be in $L^2_R(R^n)$. There exists v_o in $L^2_R(R^n)$ such that $\operatorname{supp}(u_o+iv_o)^\wedge \subseteq \Gamma^*$ if and only if $Qu_o = u_o$.

Moreover, in this case we have $v_o = Hu_o$.

Proof. By proposition 3.4, if $\text{supp}(u_o+iv_o)^\wedge \subseteq \Gamma^*$ then u_o+iv_o is in im W and so $W(u_o+iv_o) = u_o + iv_o$ which implies $Qu_o = u_o$ and $Hu_o = v_o$ by the above lemma. Conversely, suppose $Qu_o = u_o$. Let $v_o = Hu_o$. By the above lemma, we have $W(u_o+iv_o) = u_o + iv_o$ which implies $u_o + iv_o$ is in im W. Therefore $\text{supp}(u_o+iv_o)^\wedge \subseteq \Gamma^*$. Q.E.D.

Suppose ξ is in R^n and $\alpha = (\alpha_1, \cdots\cdots, \alpha_n)$ with $\alpha_j \geq 0$, $j = 1,2,\cdots\cdots,n$. We define $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and

$$\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}\cdots\cdots\xi_n^{\alpha_n} .$$

Now let Γ_o denote the interior of Γ. Thus we have following proposition.

Proposition 3.7. If y is in Γ_o , then $\xi^\alpha\, e^{-(\xi,y)}\, \chi_{\Gamma^*}(\xi)$ belongs to $L^p(R^n)$ as a function of ξ, for $p \geq 1$.

Proof. By corollary 2.4 there exists $\delta > 0$ such that $(\xi,y) \geq \delta|\xi|$ for each ξ in Γ^*. Thus

$$\int_{R^n} |\xi^\alpha\, e^{-(\xi,y)}\, \chi_{\Gamma^*}(\xi)|^p\, d\xi$$

$$\leq \int_{R^n} |\xi|^{p|\alpha|}\, e^{-p\delta|\xi|}\, d\xi$$

$$= |S^{n-1}| \int_0^\infty r^{p|\alpha|+n-1}\, e^{-p\delta|r|}\, dr < \infty$$

where $|S^{n-1}|$ denotes the area of the unit sphere S^{n-1}.

If Γ_0 is nonempty, then the proposition 3.7 implies that the integral

$$K(z) = (2\pi)^{-n} \int_{\Gamma^*} e^{i(z,\xi)}\, d\xi$$

converges absolutely for z in $\Omega = R^n + i\Gamma_0 \subseteq \mathbb{C}^n$. The function K is called the Cauchy kernel of the tube Ω or of the cone Γ . It was first studied by S. Bochner (see Bochner [2]). If y is in Γ_0 , it is convenient to introduce the function K_y defined by $K_y(x) = K(x + iy)$, x in R^n.

We note that

$$K_y(x) = \mathfrak{F}^{-1}[e^{-(y,\xi)}\chi_{\Gamma^*}(\xi);x]$$

$$= (e^{-(y,\xi)}\chi_{\Gamma^*}(\xi))^{\sim}(x) .$$

Theorem 3.8. If Γ_0 is not empty then the Cauchy kernel K is a holomorphic function in the tube Ω . We may compute the derivatives of K by differentiating under the integral sign.

Proof. By lemma 2.3, if y is in Γ and ξ is in Γ^*, then $(\xi,y) \geq \delta(y)|\xi|$ where $\delta(y)$ is the distance from y to the boundary of Γ . Now let A be any compact subset of Γ_0 . By continuity of the distance function, there exists $\delta > 0$ such that $(\xi,y) \geq \delta|\xi|$ for ξ in Γ^* and y in A. Therefore

$$|\xi^\alpha e^{i(x+iy,\xi)}\chi_{\Gamma^*}(\xi)| \leq |\xi|^{|\alpha|}e^{-\delta|\xi|}$$

for ξ,x in R^n and y in A. Since A was an arbitrary compact subset of Γ_0, by Fubini's theorem, we conclude that

$K(x+iy)$ is a C^∞ function of (x,y) in $R^n \times \Gamma_\circ$ and that

$$D_x^\alpha D_y^\beta K(x+iy)$$

$$= (2\pi)^{-n}\, i^{|\alpha|+2|\beta|} \int_{R^n} \xi^{\alpha+\beta}\, e^{i(x+iy,\xi)} \chi_{\Gamma^*}(\xi)\, d\xi \quad .$$

In particular $\frac{\partial K}{\partial x_j} + i\frac{\partial K}{\partial y_j} = 0, \quad j = 1,2,\cdots\cdots,n,$

i.e., the Cauchy-Riemann equations hold, and therefore K is holomorphic. Q.E.D.

Corollary 3.9. If y is in $\Gamma_\circ$, then K_y is in $C^\infty(R^n)$ and

$$D^\alpha K_y(x) = i^{|\alpha|}\, (\xi^\alpha\, e^{-(y,\xi)}\, \chi_{\Gamma^*}(\xi))^\sim(x) \quad .$$

Proof. The corollary is immediate by proposition 3.7 and the proof of theorem 3.8. Q.E.D.

We define, for $1 \leq p \leq \infty$,

$$\mathcal{D}_{L^p} = \{u \in C^\infty(R^n) \mid D^\alpha u \in L^p(R^n) \quad \text{for each } \alpha \}$$

and provide it with the topology defined by the system of semi-norms of the form, for u in $\mathcal{D}_{L^p}$,

$$\|u\|_k = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p} , \quad k = 0,1,2,\cdots\cdots\cdots \quad .$$

$\mathcal{D}_{L^p}$ is a Frechet space, and if $1 < p < \infty$, then $\mathcal{D}_{L^p}$ is even reflexive according to the theorem of Mackey-Arens (see Schwartz [1] p.200). It is customary to denote $\mathcal{D}_{L^\infty}$ by $\mathcal{B}$ and to define

$$\mathcal{B}^\circ = \{u \in \mathcal{B} \mid D^\alpha u \text{ vanishes at } \infty \text{ for each } \alpha\} .$$

We recal that $C_c^\infty(R^n)$ is dense in $\mathcal{D}_{L^p}$ for $1 \leq p < \infty$ and also dense in $\mathcal{B}^\circ$. But $C_c^\infty(R^n)$ is not dense in $\mathcal{B}$.

<u>Proposition 3.10</u>. If y is in $\Gamma_\circ$, then K_y is in $\mathcal{D}_{L^p}$ for $p \geq 2$. Moreover K_y is in $\mathcal{B}^\circ$.

<u>Proof</u>. Corollary 3.9 and proposition 3.7 together with theorem 2.8 (applied to ~ rather than ∧) imply K_y is in $\mathcal{D}_{L^p}$ for $p \geq 2$. Corollary 3.9 and proposition 3.7 together with theorem 2.5 (applied to ~ rather than ∧) imply K_y is in $\mathcal{B}^\circ$. Q.E.D.

<u>Remark</u>. Since $D^\alpha K_y$ is in $L^2(R^n)$ by proposition 3.10, it now makes sense to speak of the Fourier transform of $D^\alpha K_y$. In view of corollary 3.9 and theorem 2.8 we have

$$(D^\alpha K_y)^\wedge(\xi) = i^{|\alpha|}\, \xi^\alpha\, e^{-(y,\xi)}\, \chi_{\Gamma^*}(\xi) .$$

Suppose now $F_\circ$ is in $L^2(R^n)$. We define the <u>Cauchy Integral</u> $F = KF_\circ$ of $F_\circ$ by

$$F(z) = \int_{R^n} K(z-t)\, F_\circ(t)\, dt, \quad \text{for } z \text{ in } R^n + i\Gamma_\circ .$$

This Cauchy integral was also studied by S. Bochner (see

Bochner [2]). If we let $F_y(x) = F(x+iy)$, then we clearly have $F_y = K_y * F_\circ$ which shows that F is well-defined and F_y is a bounded continuous function (theorem 2.9). Moreover, by theorem 2.11 we have $F_y = (\hat{K}_y \hat{F}_\circ)^\sim$ which we may write explicitly as

$$F(z) = (2\pi)^{-n} \int_{\Gamma^*} e^{i(z,\xi)} \hat{F}_\circ(\xi)\, d\xi$$

where the integral converges absolutely for each z in $\Omega = R^n + i\Gamma_\circ$. In particular since $(WF_\circ)^\wedge = \hat{F}_\circ$ a.e. on Γ^*, we conclude $KWF_\circ = KF_\circ$.

<u>Lemma 3.11</u>. If $F_\circ$ is in $L^2(R^n)$ and $F = KF_\circ$, then

$$\sup_{y\in\Gamma_\circ} \int_{R^n} |F(x+iy)|^2\, dx = \|WF_\circ\|^2_{L^2} .$$

<u>Proof</u>. By the remark following proposition 3.10, we have

$$\hat{K}_y(\xi) = e^{-(y,\xi)} \chi_{\Gamma^*}(\xi) .$$

$\hat{K}_y(\xi)$ is bounded, since $(y,\xi) > 0$ for y in $\Gamma_\circ$ and ξ in Γ^* . Since $F_y = (\hat{K}_y \hat{F}_\circ)^\sim$ and $\hat{K}_y$ is bounded, we see that F_y is in $L^2(R^n)$ and moreover by theorem 2.7, $\hat{F}_y = \hat{K}_y \hat{F}_\circ$ and $\|F_y\|^2_{L^2} = (2\pi)^{-n} \|\hat{F}_y\|^2_{L^2}$. Since $(y,\xi) > 0$ for ξ in Γ^*, we have $|\hat{F}_y| = |\hat{K}_y \hat{F}_\circ| \le |\chi_{\Gamma^*} \hat{F}_\circ| = |(WF_\circ)^\wedge| \in L^2(R^n)$.

Clearly $\hat{F}_y(\xi) = \hat{K}_y(\xi)\, \hat{F}_\circ(\xi)$ converges to $(WF_\circ)^\wedge(\xi)$ as y converges to 0 in Γ_o and therefore by the Lebesgue dominated convergence theorem we have

$$\|\hat{F}_y\|_{L^2}^2 \longrightarrow \|(WF_o)^\wedge\|_{L^2}^2 = (2\pi)^n \|WF_o\|_{L^2}^2$$

as $y \longrightarrow 0$, y in Γ_o . Hence

$$\sup_{y\in\Gamma_o} \int_{R^n} |F(x+iy)|^2 \, dx = \lim_{y\to o, y\in\Gamma_o} \|F_y\|_{L^2}^2$$

$$= \lim_{y\to o, y\in\Gamma_o} (2\pi)^{-n} \|\hat{F}_y\|_{L^2}^2$$

$$= (2\pi)^{-n} \|\widehat{WF_o}\|_{L^2}^2$$

$$= \|WF_o\|_{L^2}^2 . \qquad \text{Q.E.D.}$$

Let $\Omega = R^n + i\Gamma_o$. A function F defined and holomorphic on the tube Ω is said to belong to the space $H^2(\Omega)$, if

$$|||\, F \,|||_2 = \sup_{y\in\Gamma_o} \left(\int_{R^n} |F(x+iy)|^2 \, dx\right)^{\frac{1}{2}} < \infty.$$

<u>Lemma 3.12</u>. The Cauchy integral is a bounded linear map of $L^2(R^n)$ into $H^2(\Omega)$ where $\Omega = R^n + i\Gamma_o$. Moreover

$$|||\, KF_o \,|||_2 = \|WF_o\|_{L^2} \leq \|F_o\|_{L^2}$$

for each F_o in $L^2(R^n)$ and so in particular $\ker K = \ker W$.

<u>Proof</u>. Once we show that $F = KF_o$ is holomorphic in Ω, then lemma 3.11 implies $|||\, KF_o \,|||_2 = \|WF_o\|_{L^2}$ and therefore

$KF_\circ = 0$ if and only if $WF_\circ = 0$, i.e., ker K = ker W. To see that F is holomorphic, we first note that

$$F(x+iy) = (2\pi)^{-n} \int_{\Gamma^*} e^{i(x+iy,\xi)} \hat{F}_\circ(\xi)\, d\xi$$

As in the proof of theorem 3.8, if A is any compact subset of $\Gamma_\circ$, there exists $\delta > 0$ such that

$$|\xi^\alpha e^{i(x+iy,\xi)} \chi_{\Gamma^*}(\xi) \hat{F}_\circ(\xi)|$$

$$\leq |\xi|^{|\alpha|} e^{-\delta|\xi|} |\hat{F}_\circ(\xi)|$$

and so by Fubini's theorem F(x+iy) is a C^∞ function of (x,y) in $R^n+\Gamma_\circ$ and

$$D_x^\alpha D_y^\beta F(x+iy)$$

$$= (2\pi)^{-n} i^{|\alpha|+2|\beta|} \int_{R^n} \xi^{\alpha+\beta} e^{i(x+iy,\xi)} \chi_{\Gamma^*}(\xi) \hat{F}_\circ(\xi)\, d\xi$$

In particular $\dfrac{\partial F}{\partial x_j} + i\dfrac{\partial F}{\partial y_j} = 0$, $j = 1,\ldots,n$,

i.e., the Cauchy Riemann equations hold, and therefore F is holomorphic. Q.E.D.

<u>Theorem 3.13</u>. The Cauchy integral maps $L^2(R^n)$ onto $H^2(\Omega)$ and is an isometry of im W onto $H^2(\Omega)$.

<u>Proof</u>. In view of lemma 3.12 and $KWF_\circ = KF_\circ$, it suffices to show that if F is in $H^2(\Omega)$, then there exists $F_\circ$ in $L^2(R^n)$ such that $F = KF_\circ$, i.e.,

$$F(x + iy) = (2\pi)^{-n} \int_{\Gamma^*} e^{i(x+iy,\xi)} \hat{F}_o(\xi)\, d\xi .$$

That such an F_o exists is a theorem of S. Bochner (see Bochner [2]). Q.E.D.

Corollary 3.14. If F_o is in $L^2(R^n)$, then $F = KF_o$ is in $H^2(\Omega)$ and each element of $H^2(\Omega)$ is of this form. Moreover

$$WF_o(x) = L^2\text{-}\lim_{y\in\Gamma_o, y\to o} F(x + iy) .$$

In particular F_o in $L^2(R^n)$ is the L^2-boundary value along the edge of Ω of some F in $H^2(\Omega)$ if and only if $WF_o = F_o$, i.e., $\text{supp } \hat{F}_o \subseteq \Gamma^*$.

Proof. The first part is only a restatement of theorem 3.13. In the proof of lemma 3.11 we have that F_y is in $L^2(R^n)$, and $\hat{F}_y$ converges to $\widehat{WF_o}$ in $L^2(R^n)$ as y converges to 0 in Γ_o , therefore F_y converges to WF_o in L^2 as y converges to 0 in Γ_o by Parseval's formula. In particular, if F_o is in $L^2(R^n)$ and

$$F_o(x) = L^2\text{-}\lim_{y\to o, y\in\Gamma_o} F(x + iy) \quad \text{for some } F \text{ in } H^2(\Omega),$$

then we have $F = KF_o'$ for some F_o' in $H^2(\Omega)$. Hence

$$F_o(x) = L^2\text{-}\lim_{y\to o, y\in\Gamma_o} F(x + iy) = WF_o'(x) .$$

Therefore $F_o = WF_o'$ a.e. and so $WF_o = F_o$ since W is a projection. Conversely, if $WF_o = F_o$, taking $F = KF_o$, we have

$$F_\circ(x) = WF_\circ(x) = \underset{y\to o,\, y\in\Gamma_\circ}{L^2\text{-lim}}\, F(x + iy) \quad . \qquad \text{Q.E.D.}$$

Remark. The existence of boundary values of $H^p(\Omega)$ functions in dimensions > 1 was first considered by E. M. Steim, G. Weiss and M. Weiss (see Stein, E. M., Weiss, G. and Weiss, M. [1]).

Let Re F be the real part of F and Im F be the imaginary part of F.

Theorem 3.15. Let $u_\circ$ be in $L^2_R(R^n)$. Then there exists F in $H^2(\Omega)$ such that

$$u_\circ(x) = \underset{y\to o,\, y\in\Gamma_\circ}{L^2\text{-lim}}\, \text{Re } F(x + iy) \ ,$$

if and only if $u_\circ = Qu_\circ$. Moreover, in this case, if $v_\circ = Hu_\circ$, then

$$v_\circ(x) = \underset{y\to o,\, y\in\Gamma_\circ}{L^2\text{-lim}}\, \text{Im } F(x + iy)$$

and if $F(x + iy) = u(x,y) + iv(x,y)$ where u and v are real then

$$u(x,y) = \int_{R^n} p(x-t,y)\, u_\circ(t)\, dt$$

and

$$v(x,y) = \int_{R^n} q(x-t,y)\, u_\circ(t)\, dt$$

where $p(x,y) = 2\text{ Re } K(x+iy)$ is called Poisson kernel and $q(x,y) = 2\text{ Im } K(x+iy)$ is called conjugate Poisson kernel.

Proof. Let u_o be in $L^2_R(R^n)$. If there exists F in $H^2(\Omega)$ such that

$$u_o(x) = \underset{y\to o, y\in\Gamma_o}{L^2\text{-lim}} \operatorname{Re} F(x+iy), \text{ then let}$$

$$v_o(x) = \underset{y\to o,\ y\in\Gamma_o}{L^2\text{-lim}} \operatorname{Im} F(x+iy).$$

From corollary 3.14, we have $F = KF_o$ for some F_o in $L^2(R^n)$ and $WF_o(x) = \underset{y\to o, y\in\Gamma_o}{L^2\text{-lim}} F(x+iy) = u_o(x) + iv_o(x)$.

Therefore $W(u_o + iv_o) = W^2F_o = WF_o = u_o + iv_o$ which, by lemma 3.5, is equivalent to $Hu_o = v_o$ and $Qu_o = u_o$. Conversely, if $Qu_o = u_o$, then, letting $v_o = Hu_o$, we have $W(u_o + iv_o) = u_o + iv_o$ by lemma 3.5. Therefore if we let $F = K(u_o + iv_o)$, we have $u_o(x) = \underset{y\to o, y\in\Gamma_o}{L^2\text{-lim}} \operatorname{Re} F(x+iy)$ by corollary 3.14. Now if $F(x+iy) = u(x,y) + iv(x,y)$ where u and v are real, we consider following equalities

$$F = KF_o = KWF_o = K(u_o+iv_o) = K(Qu_o+iHu_o) = 2KWu_o = 2Ku_o .$$

Hence
$$u(x,y) = \int_{R^n} 2 \operatorname{Re} K(z-t)\, u_o(t)\, dt$$

$$= \int_{R^n} p(x-t,y)\, u_o(t)\, dt$$

and
$$v(x,y) = \int_{R^n} 2 \operatorname{Im} K(z-t)\, u_o(t)\, dt$$

$$= \int_{R^n} q(x-t,y)\, u_o(t)\, dt . \qquad \text{Q.E.D.}$$

Lemma 3.16. If y is in $\Gamma_\circ$, then

$$p(x,y) = (2\pi)^{-n} \int_{R^n} e^{i(x,\xi)} e^{-|(y,\xi)|} |h(\xi)| \, d\xi$$

and
$$q(x,y) = -i \ (2\pi)^{-n} \int_{R^n} e^{i(x,\xi)} e^{-|(y,\xi)|} h(\xi) \, d\xi,$$

hence
$$\hat{p}_y(\xi) = e^{-|(y,\xi)|} |h(\xi)| .$$

and
$$\hat{q}_y(\xi) = -i \ e^{-|(y,\xi)|} h(\xi) .$$

Proof. Consider following equalities

$$K(x + iy) = (2\pi)^{-n} \int_{\Gamma^*} e^{i(x+iy,\xi)} \, d\xi$$

$$= (2\pi)^{-n} \int_{R^n} \chi_{\Gamma^*}(\xi) \, e^{i(x,\xi)} e^{-(y,\xi)} \, d\xi$$

$$= (2\pi)^{-n} \int_{R^n} \chi_{\Gamma^*}(\xi) \, e^{i(x,\xi)} e^{-|(y,\xi)|} \, d\xi ,$$

$$\overline{K(x + iy)} = (2\pi)^{-n} \int_{\Gamma^*} \overline{e^{i(x+iy,\xi)}} \, d\xi$$

$$= (2\pi)^{-n} \int_{R^n} \chi_{\Gamma^*}(\xi) \, e^{-i(x,\xi)} e^{-|(y,\xi)|} \, d\xi$$

$$= (2\pi)^{-n} \int_{R^n} \chi_{\Gamma^*}(-\xi) \, e^{i(x,\xi)} e^{-|(y,\xi)|} \, d\xi .$$

Therefore $p(x,y) = 2 \operatorname{Re} K(x+iy) = K(x+iy) + \overline{K(x+iy)}$

$$= (2\pi)^{-n} \int_{R^n} e^{i(x,\xi)} e^{-|(y,\xi)|} (\chi_{\Gamma^*}(\xi)+\chi_{\Gamma^*}(-\xi)) \, d\xi$$

$$= (2\pi)^{-n} \int_{R^n} e^{i(x,\xi)} \, e^{-|(y,\xi)|} \, |h(\xi)| \, d\xi .$$

Similarly $q(x,y) = \text{Im}\, K(x+iy) = -\frac{i}{2}\,(K(x+iy) - \overline{K(x+iy)})$

$$= -i \, (2\pi)^{-n} \int_{R^n} e^{i(x,\xi)} \, e^{-|(y,\xi)|} \, (\chi_{\Gamma^*}(\xi) - \chi_{\Gamma^*}(-\xi)) \, d\xi$$

$$= -i \, (2\pi)^{-n} \int_{R^n} e^{i(x,\xi)} \, e^{-|(y,\xi)|} \, h(\xi) \, d\xi .$$

Since $p(x,y) = (e^{-|(y,\xi)|} \, |h(\xi)|)^{\sim}(x)$ and $q(x,y) = (-i \, e^{-|(y,\xi)|} \, h(\xi))^{\sim}(x)$ are in $L^2(R^n)$, we have $\hat{p}_y(\xi) = e^{-|(y,\xi)|} \, |h(\xi)|$, and $\hat{q}_y(\xi) = -i \, e^{-|(y,\xi)|} \, h(\xi)$.

Q.E.D.

<u>Theorem 3.17</u>. If u_o is in $L^2_R(R^n)$, then

$$Hu_o = \underset{y\to o,\, y\in\Gamma_o}{L^2\text{-lim}} \; q_y * Qu_o$$

$$= \underset{y\to o,\, y\in\Gamma_o}{L^2\text{-lim}} \; q_y * u_o .$$

Therefore

$$Hu_o(x) = \underset{y\to o,\, y\in\Gamma_o}{L^2\text{-lim}} \int_{R^n} q(x-t,y) \, u_o(t) \, dt$$

<u>Proof</u>. By theorem 2.11, we have $q_y * Qu_o = (\hat{q}_y \, \widehat{Qu_o})^{\sim}$, and $q_y * u_o = (\hat{q}_y \, \hat{u}_o)^{\sim}$. From lemma 3.16, we know that $\hat{q}_y$ is bounded. Thus $\hat{q}_y \, \widehat{Qu_o}$ and $\hat{q}_y \, \hat{u}_o$ are in $L^2(R^n)$ and so by theorem 2.7, we have

$$(q_y * Qu_\circ)^\wedge(\xi) = \hat{q}_y(\xi)\ (Qu_\circ)^\wedge(\xi)$$

$$= -i\ e^{-|(y,\xi)|}\ h(\xi)\ |h(\xi)|\ \hat{u}_\circ(\xi)$$

$$= -i\ e^{-|(y,\xi)|}\ h(\xi)\ \hat{u}_\circ(\xi)$$

$$= \hat{q}_y(\xi)\ \hat{u}_\circ(\xi)$$

$$= (q_y * u_\circ)^\wedge(\xi).$$

While $(Hu_\circ)^\wedge(\xi) = -i\ h(\xi)\ \hat{u}_\circ(\xi)$ and

$|(q_y * Qu_\circ)^\wedge(\xi) - (Hu_\circ)^\wedge(\xi)| \leq 2\ |(Hu_\circ)^\wedge(\xi)| \in L^2(R^n)$,

we have , by dominated convergence theorem,

$$\|(q_y * Qu_\circ) - Hu_\circ\|_{L^2}^2 = (2\pi)^{-n}\ \|(q_y * Qu_\circ)^\wedge - (Hu_\circ)^\wedge\|_{L^2}^2 \longrightarrow 0$$

as $y \longrightarrow 0$, y in $\Gamma_\circ$. Q.E.D.

<u>Examples</u>.

(A) Suppose $n = 1$ and $\Gamma = [0,\infty)$. In this case H is the classical Hilbert transform and Ω is the upper half plane. We note $\Gamma^* = [0,\infty)$, so $\Gamma^* \cup (-\Gamma^*) = R$ and therefore $Q = I$. Hence by theorem 3.2, H is an automorphism of $L^2(R)$. By theorem 3.15, we have that if $u_\circ$ is in $L^2_R(R)$, then there exists a unique F in $H^2(\Omega)$ such that

$$u_\circ(x) = L^2\text{-}\lim_{y \downarrow 0} \operatorname{Re} F(x + iy) \text{ , and in this case}$$

$$Hu_\circ(x) = L^2\text{-}\lim_{y \downarrow 0} \operatorname{Im} F(x + iy) \text{ .}$$

For the Cauchy kernel $K(z)$, we have

$$K(z) = \frac{1}{2\pi} \int_0^\infty e^{iz\xi}\, d\xi = \frac{-1}{2\pi i z} .$$

Hence $\quad q(x,y) = 2 \operatorname{Im} K(x+iy) = \operatorname{Im}\left(\frac{-1}{\pi i(x+iy)}\right)$

$$= \operatorname{Im}\left(\frac{1}{\pi}\,\frac{y+ix}{x^2+y^2}\right) = \frac{x}{\pi(x^2+y^2)} .$$

Therefore, by theorem 3.17, we have

$$Hu_o(x) = L^2\text{-}\lim_{y \downarrow 0} \int_{-\infty}^{\infty} q(x-t,y)\, u_o(t)\, dt$$

$$= L^2\text{-}\lim_{y \downarrow 0} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x-t}{(x-t)^2+y^2}\, u_o(t)\, dt$$

which is a well-known classical formula.

(B) Let $k > 0$ and

$$\Gamma = \{\, y \in R^3 \mid y_3 \geq k(y_1^2+y_2^2)^{\frac{1}{2}} \,\} .$$

Then $\quad \Gamma^* = \{\, \xi \in R^3 \mid \xi_3 \geq \frac{1}{k}(\xi_1^2+\xi_2^2)^{\frac{1}{2}} \,\} .$

Suppose y is in Γ_o, we have

$$K(iy) = (2\pi)^{-3} \int_{\Gamma^*} e^{-(y,\xi)}\, d\xi$$

$$= \frac{(k/(2\pi))^2}{[y_3^2-k^2(y_1^2+y_2^2)]^{3/2}} .$$

Now, for z in $\mathbb{C}^3$, define $u = k^2(z_1^2+z_2^2) - z_3^2$.

If $B(x,y) = x_3y_3 - k^2(x_1y_1+x_2y_2)$, then

$\operatorname{Re} u = B(y,y) - B(x,x)$ and $\operatorname{Im} u = -2\, B(x,y)$. Suppose $\operatorname{Im} u = 0$ and y is in $\Gamma_\circ$. Then $B(y,y) > 0$ implies $y_3 > 0$. If $x = 0$, then $\operatorname{Re} u = B(y,y) > 0$. If $x \neq 0$, then $B(x,y) = 0$ and $B(y,y) > 0$ imply

$$|x_3| y_3 = k^2 |x_1 y_1 + x_2 y_2|$$

$$\leq k^2 (x_1^2 + x_2^2)^{\frac{1}{2}} (y_1^2 + y_2^2)^{\frac{1}{2}}$$

$$< k (x_1^2 + x_3^2)^{\frac{1}{2}} y_3 \quad .$$

Thus $|x_3| < k(x_1^2 + x_3^2)^{\frac{1}{2}}$ which implies $B(x,x) < 0$, and so $\operatorname{Re} u > 0$. We conclude that if z is in $R^n + i\Gamma_\circ$ and $u = k^2(z_1^2 + z_2^2) - z_3^2$, then $u \neq 0$ and $-\pi < \arg u < \pi$. Hence we may define $u^{3/2}$ by taking the determination which is positive when u is real and positive. Then $u^{3/2}$ is a holomophic function on $R^n + i\Gamma_\circ$ which agrees with $(y_3^2 - k^2(y_1^2 + y_2^2))^{3/2}$ on $i\Gamma_\circ$. Thus

$$K(z) = \frac{(k/(2\pi))^2}{[k^2(z_1^2 + z_2^2) - z_3^2]^{3/2}} , \quad \text{for } z \text{ in } R^n + i\Gamma_\circ \ .$$

The conjugate Poisson Kernel $q_y(x) = 2 \operatorname{Im} K(x+iy)$ is rather unpleasant to compute. However if we let y converge to 0, we have, formally,

$$q_\circ(x) = \begin{cases} -\frac{1}{2}\left(\frac{k}{\pi}\right)^2 [x_3^2 - k^2(x_1^2 + x_2^2)]^{-3/2} , & \text{if } x \text{ is in } \Gamma \cup (-\Gamma) \\ 0 , & \text{otherwise.} \end{cases}$$

Now $q_\circ$ is not locally integrable along the boundary of $\Gamma \cup (-\Gamma)$. Since we would expect $Hu_\circ = L^2\text{-}\lim_{y \to 0} q_y * u_\circ$ to be given by an appropriate finite part of $q_\circ * u_\circ$, it appears that H is not a singular integral operator in the usual sense.

(C) Let S be the space of 2 x 2 real symmetric matrices. Then $E : \mathbb{C}^3 \longrightarrow S+iS$ defined by

$$E(z) = \begin{pmatrix} z_3+z_1 & z_2 \\ z_2 & z_3-z_1 \end{pmatrix}$$

is an isomorphism. Let P be the positive 2 x 2 real symmetric matrices. It is easy to see that $E^{-1}(P) = \Gamma_\circ$ where $\Gamma = \{ x \in R^3 \mid x_3 \geq (x_1^2+x_2^2)^{\frac{1}{2}} \}$. Note $\Gamma^* = \Gamma$, and $tr(E(x)E(\xi)) = 2(x,\xi)$. Thus if B is in S, then $tr(AB) > 0$ for each A in P if and only if B is also in P. Bochner defined the Cauchy kernel of P to be, for A in S+iP,

$$K(A) = (2\pi)^{-3} \int_P e^{\frac{1}{2}i\, tr(AB)}\, dV_B$$

where dV_B is the Lebergue measure on R^3 (see Bochner [2]). By the previous example, if z is in $R^3+i\Gamma_\circ$, then

$$K(E(z)) = (2\pi)^{-3} \int_{\Gamma^*} e^{i(z,\xi)}\, d\xi$$

$$= (2\pi)^{-2} (z_1^2 + z_2^2 - z_3^2)^{-3/2}$$

is the Cauchy kernel of Γ.

CHAPTER IV

PRELIMINARY FOR CHAPTER V

In chapter V, we shall generalize the ideas in chapter III to $\mathcal{D}'_{L^2}$, the dual of $\mathcal{D}_{L^2}$. In this chapter, we present a quick review of distribution theory. The book by L. Schwartz [1] is highly recommended.

For each compact subset K of R^n, let

$$\mathcal{D}_K = \{ u \in C^\infty(R^n) \mid \text{supp } u \subseteq K \}$$

and provide $\mathcal{D}_K$ with the locally convex topology defined by the system of seminorms

$$|||u|||_{K,m} = \max_{|\alpha| \le m} \sup |D^\alpha u| , \qquad m = 0,1,2, \ldots\ldots .$$

It is obvious that $\mathcal{D}_K$ is Fréchet. If Ω is an open subset of R^n , we denote by $\mathcal{D}(\Omega)$ the space $C_c^\infty(\Omega)$ topologized by the requirement that a seminorm p on $\mathcal{D}(\Omega)$ is continuous if and only if its restriction to $\mathcal{D}_K$ is continuous for each compact set K in Ω.

The dual space of $\mathcal{D}(\Omega)$, denoted by $\mathcal{D}'(\Omega)$, is called the space of distributions in Ω. Since a linear functional T is continuous if and only if $p(u) = |\langle T,u \rangle|$ is a continuous seminorm, we have following proposition.

Proposition 4.1. A linear functional T on $\mathcal{D}(\Omega)$ is a distribution if and only if for each compact subset K of Ω, there exists a constant $C_K > 0$ and an integer $m_K \geq 0$ such that, for each u in $\mathcal{D}_K$,

$$|\langle T,u\rangle| \leq C_K \max_{|\alpha|\leq m_K} \sup |D^\alpha u|.$$

The space $\mathcal{D}(\Omega)$ is not metrizable (see Donoghue [1] p. 100), but it is an inductive limit of metrizable spaces $\mathcal{D}_K$ (see Schwartz [1] p. 66 or Yosida [1] p. 28) (in fact, it is an LF-space, i.e., a countable strict inductive limit of Fréchet spaces) and therefore we have the following useful criterion .

Proposition 4.2. A linear functional T on $\mathcal{D}(\Omega)$ is a distribution if and only if $\langle T,u_\nu\rangle$ converges to 0 whenever $\{u_\nu\}_{\nu\geq 1}$ is a sequence converging to 0 in $\mathcal{D}(\Omega)$.

Proof. See Donoghue [1] p. 100.

To make use of this criterion we need to identify the 0-convergent sequences. In this connection we have following proposition.

Proposition 4.3. A sequence $\{u_\nu\}_{\nu\geq 1}$ in $\mathcal{D}(\Omega)$ converges to 0 if and only if there exists a compact subset K of Ω such that supp $u_\nu \subseteq K$ for each ν, and we have $D^\alpha u_\nu$ converges to 0 uniformly on K for each multi-index α.

Proof. See Yosida [1] p. 28 or Donoghue [1] p. 99.

Let f be in $L^1_{loc}(\Omega)$ and define

$$\langle f,u\rangle = \int_{R^n} f(x)\, u(x)\, dx, \quad \text{for every } u \text{ in } \mathcal{D}(\Omega) .$$

It is obvious that f is in $\mathcal{D}'(\Omega)$ by proposition 4.2.

If T is in $\mathcal{D}'(\Omega)$, we define $D^\alpha T$ by duality,

$$\langle D^\alpha T,u\rangle = (-1)^{|\alpha|} \langle T,D^\alpha u\rangle , \quad \text{for every } u \text{ in } \mathcal{D}(\Omega) .$$

Thus a distribution, and so a locally integrable function, has distribution derivatives of all orders. In the case of function, if the classical derivatives exists they may differ from the distribution derivatives. However, if f is in $C^k(\Omega)$, then $D^\alpha f$ in the classical and distributional sense is the same, for $|\alpha| \leq k$, as may be seen by integration by parts. Conversely, by a regularization technique one may show that if f is in $C(\Omega)$ and the distribution derivative $D_j f$ is a continuous function in Ω, then $D_j f$ exists in the classical sense and agrees with $D_j f$ in the distribution sense (see Donoghue [1] p. 96).

Let X be a locally convex linear topological space and X' be its dual space. The weak* topology on X' is the topology of convergence at each point of X and thus is defined by the family of semi-norms p of the form

$$p(T) = |\langle T,x\rangle|, \quad \text{for every } T \text{ in } X' ,$$

where x is an element of X. The strong topology on X' is the topology of uniform convergence on bounded sets of X and thus is defined by the family of semi-norms p of the form

$$p(T) = \sup_{x\in B} |\langle T,x\rangle|, \quad \text{for every } T \text{ in } X' ,$$

where B is a bounded set in X. All distributional spaces treated will be provided with the strong topology.

If f is in $C^\infty(\Omega)$, we define fT, T in $\mathcal{D}'(\Omega)$, by the duality

$$\langle fT,u\rangle = \langle T,fu\rangle, \quad \text{for every } u \text{ in } \mathcal{D}(\Omega) ,$$

and also define $\check{T}$, T in $\mathcal{D}'(\Omega)$, by the duality

$$\langle \check{T},u\rangle = \langle T,\check{u}\rangle, \quad \text{for every } u \text{ in } \mathcal{D}(\Omega) .$$

We say that a distribution T in $\mathcal{D}'(\Omega)$ vanishes in an open set U of Ω if and only if $\langle T,u\rangle = 0$ for every u in $\mathcal{D}(\Omega)$ with supp u in U. The support of T, denoted by supp T, is defined as the complement of the largest open set on which T vanishes. To see that this definition makes sense, let $\{U_i\}_{i\in A}$ be the family of all open sets on which T vanish.

Let $\mathcal{U} = \bigcup_{i \in A} U_i$ and u in $\mathcal{D}(\mathcal{U})$. We construct a partition of unity $\{\alpha_i\}_{i \in N}$ subordinate to the covering $\{U_i\}_{i \in N}$ of $\mathcal{U}$. Then $u = \sum_{i \in N} \alpha_i u$ is a finite sum and so $\langle T,u \rangle = \sum_{i \in N} \langle T, \alpha_i u \rangle = 0$ since supp $\alpha_i u$ is in some U_i with i in N. If f is in $L^1_{loc}(R^n)$, then its support as a distribution is same as its support as a function.

Let $\mathcal{E}(\Omega) = \{ f \in C^\infty(\Omega) \}$ and provide it with the topology defined by the system of semi-norms,

$$|||\, f \,|||_{K,p} = \sup_{\substack{|\alpha| \le p \\ x \in K}} |D^\alpha f(x)| \ , \quad \text{for } f \text{ in } \mathcal{E}(\Omega) \ ,$$

where K is a compact subset of Ω and p is a positive number.

<u>Theorem 4.4</u>. The set of all distributions in Ω with compact support may be naturally identified with the space $\mathcal{E}'(\Omega)$, the dual space of $\mathcal{E}(\Omega)$, i.e., a distribution has compact support if and only if it extends (uniquely) to a continuous linear functional on $\mathcal{E}(\Omega)$.

<u>Proof</u>. See Schwartz [1] p.89 or Yosida [1] p.64 or Donoghue [1] p.104.

A function f in $C^\infty(R^n)$ is said to be in $\mathcal{S}$, if

$\sup_{x \in R^n} |x^\beta D^\alpha f(x)| < \infty$, for each pair of multi-indices α and β.

We provide $\mathcal{S}$ with the topology defined by the system of semi-norms,

$$|||\, f \,|||_k = \sum_{|\alpha|\leq k} \sup_{x\in R^n} |(1+|x|^2)^k D^\alpha f(x)|.$$

It is obvious that $\mathcal{D}$ is dense in $\mathcal{S}$ and that $\mathcal{S}$ is dense in $L^p(R^n)$, for $p \geq 1$.

Theorem 4.5. The Fourier transform $\mathfrak{F}$ and the inverse Fourier transform $\mathfrak{F}^{-1}$ establish two mutual inverse automorphismes on $\mathcal{S}$.

Proof. See Schwartz [1] p.249, Yosida [1] p.147 or Donoghue [1] p.140.

Let $\mathcal{S}'$ be the dual space of $\mathcal{S}$ and call it the space of temperate distributions.

Let f(x) be a bounded function and define

$$\langle f,u\rangle = \int_{R^n} f(x)\ u(x)\ dx, \quad \text{for every } u \text{ in } \mathcal{S}.$$

That f is in $\mathcal{S}'$ follows by the inequality

$$\begin{aligned}\langle f,u\rangle &= \int_{R^n} f(x)\ u(x)\ dx\\ &= \int_{R^n} \frac{f(x)}{(1+|x|^2)^n}\,(1+|x|^2)^n\ u(x)\ dx\\ &\leq C|||\, u \,|||_n \ .\end{aligned}$$

In fact, we have the following theorem.

Theorem 4.6. A distribution T is temperate if and only if it is a derivative (in distributional sense) of a continuous function of slow increase i.e., a function which is the product of $(1+|x|^2)^{k/2}$ by a bounded continuous function $f(x)$: $T = D^\alpha((1+|x|^2)^{k/2} f(x))$.

Proof. See Schwartz [1] p.239.

Note that if f is in $L^1_{loc}(R^n) \cap \mathcal{S}'$, the quantity $\langle f,u\rangle$, for u in $\mathcal{S}$, may not equal the integral $\int_{R^n} f(x)u(x)dx$, since the latter may fail to exist. For example: $f(x) = e^x \cos e^x$ is the derivative of the bounded function $\sin e^x$. Since the distributional derivative and the usual derivative are same for C^∞ functions, we have that $f(x) = e^x \cos e^x$ is in $L^1_{loc} \cap \mathcal{S}'$ by theorem 4.6. But the integral $\int_{-\infty}^{\infty} e^x \cos e^x\, u(x)\, dx$ does not exist in general for u in $\mathcal{S}$.

Remark. If T is in $\mathcal{S}'$, then by theorem 4.6, for each u in $\mathcal{D}$, we have

$$\langle T,u\rangle = \langle D^\alpha((1+|x|^2)^{k/2} f(x)),u(x)\rangle$$
$$= (-1)^{|\alpha|} \langle (1+|x|^2)^{k/2} f(x),D^\alpha u(x)\rangle$$

$$= (-1)^{|\alpha|} \int_{R^n} (1+|x|^2)^{k/2} f(x) D^\alpha u(x)\, dx$$

where f is a bounded continuous function. Clearly the last integral converges for u in $\mathcal{S}$ and depends continuously on u. Thus the integral actually gives $\langle T,u\rangle$ for u in $\mathcal{S}$.

For T in $\mathcal{S}'$, we define $\hat{T}$ and $\tilde{T}$ by dualities, for u in $\mathcal{S}$, $\langle \hat{T},u\rangle = \langle T,\hat{u}\rangle$ and $\langle \tilde{T},u\rangle = \langle T,\tilde{u}\rangle$. These definitions make sense, since $\wedge$ and $\sim$ are automorphisms of $\mathcal{S}$.

Theorem 4.7. The Fourier transform $T \longrightarrow \hat{T}$ and inverse Fourier transform $T \longrightarrow \tilde{T}$ establish two mutually inverse automophisms on $\mathcal{S}'$ with respect to weak* topology or the strong topology.

Proof. See Schwartz [1] p.251, Yosida [1] p.152, or Donoghue [1] p.144.

Theorem 4.8. Let T be in $\mathcal{S}'$, then

$$(D^\alpha T)^\wedge = (i\xi)^\alpha \hat{T} \text{ , and } D^\alpha \hat{T} = ((-ix)^\alpha T)^\wedge$$

are also in $\mathcal{S}'$.

Proof. See Schwartz [1] p.109 or Donoghue [1] p.144 or Yosida [1] p.152.

In chapter III we introduced $\mathcal{D}_{L^p}$ and $\mathcal{B}^\circ$. Now, if

$\frac{1}{p} + \frac{1}{q} = 1$, $1 < p \leq \infty$, we denote the dual space of $\mathcal{D}_{L^q}$ by $\mathcal{D}'_{L^p}$ and the dual of $\mathcal{B}°$ by $\mathcal{D}'_{L^1}$. The fact that we have continuous inclusion with dense images, $p \leq q$,

$$\mathcal{D} \subset \mathcal{S} \subset \mathcal{D}_{L^p} \subset \mathcal{D}_{L^q} \subset \mathcal{E}$$

implies that we have canonical inclusions

$$\mathcal{E}' \subset \mathcal{D}'_{L^p} \subset \mathcal{D}'_{L^q} \subset \mathcal{S}' \subset \mathcal{D}' ,$$

continuous for any of the usual dual topologies (e.g. weak*, strong, etc.). Since $\mathcal{D}$ is not dense in $\mathcal{B}$, the dual of $\mathcal{B}$ is not a space of distribution.

If f is in $L^p(R^n)$ then for u in $\mathcal{D}_{L^q}$ with $\frac{1}{p} + \frac{1}{q} = 1$, we define

$$\langle f,u \rangle = \int_{R^n} f(x)\ u(x)\ dx .$$

It is obvious that f is in $\mathcal{D}'_{L^p}$ by Hölder's inequality. In fact we have following representation theorem.

<u>Theorem 4.9</u>. A distribution T belongs to $\mathcal{D}'_{L^p}$, if and only if it is finite sum of the derivatives of functions in $L^p(R^n)$.

<u>Proof</u>. See Schwartz [1] p.201.

<u>Remark</u>. Let T be in $\mathcal{D}'_{L^p}$, we have, by theorem 4.9,

$$\langle T,u\rangle = \langle \sum_\alpha D^\alpha f_\alpha , u\rangle$$

$$= \sum_\alpha (-1)^\alpha \langle f_\alpha , D^\alpha u\rangle$$

$$= \sum_\alpha (-1)^\alpha \int_{R^n} f_\alpha(x)\ D^\alpha u(x)\ dx$$

where f_α is in L^p and u is in $\mathcal{D}$. By continuity, this formula continues to hold for u in $\mathcal{D}_{L^p}$.

If T is in $\mathcal{D}'_{L^p}$ and α is in $\mathcal{D}_{L^q}$ with $\frac{1}{p} + \frac{1}{q} - 1 \geq 0$, the convolution $\alpha * T$ is defined by

$$\alpha * T\ (x) = \langle T_t , \alpha(x-t)\rangle = \langle T, \mathcal{T}_x \check{\alpha}\rangle$$

which is in $\mathcal{D}_{L^r}$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$ by theorem 4.9, and theorem 2.9.

CHAPTER V

HILBERT TRANSFORM ON $\mathcal{D}'_{L^2}$

In this chapter, we extend the Hilbert transform to $\mathcal{D}'_{L^2}$. Let Γ be a closed convex salient cone in R^n with nonempty interior. We recall that H_Γ is the bounded linear operator on $L^2(R^n)$ defined by

$$(H_\Gamma u)^\wedge(\xi) = -i\ h(\xi)\ \hat{u}(\xi) \ , \quad \text{for } u \text{ in } L^2(R^n) \ ,$$

where $h = \chi_{\Gamma^*} - \chi_{-\Gamma^*}$ a.e. and we recall $H_\Gamma^* = -H_\Gamma = H_{-\Gamma}$. From now on we simply write H for H_Γ . Note that since H is a real operator and $H^* = -H$, we have the relation

$$* \qquad \int_{R^n} Hu(x)\ v(x)\ dx = (Hu,\overline{v}) = (u, -H\overline{v}) = -(u,\overline{Hv})$$

$$= -\int_{R^n} u(x)\ Hv(x)\ dx$$

for each u and v in $L^2(R^n)$.

First of all, let's study H on $\mathcal{D}_{L^2}$.

<u>Lemma 5.1</u>. $\hat{\mathcal{D}}_{L^2} = \{f \in L^2(R^n) \mid p(x)f(x) \in L^2(R^n)$ for each polynomial $p(x)\}$.

<u>Proof</u>. If f is in $\mathcal{D}_{L^2}$, then $D^\alpha f$ is in $L^2(R^n)$ for each α. Hence $(D^\alpha f)^\wedge(\xi) = (i\xi)^\alpha \hat{f}(\xi)$ is in $L^2(R^n)$ for any α .

Therefore $p(x)\hat{f}(x)$ is in $L^2(R^n)$ for any polynomial $p(x)$. So $\hat{\mathcal{D}}_{L^2} \subset \{f \in L^2(R^n) \mid p(x)\hat{f}(x) \in L^2(R^n)$ for any polynomial $p(x)\}$. Conversely, suppose $p(x)f(x)$ is in $L^2(R^n)$ for any polynomial $p(x)$. Then, we see that $(1+|x|^2)^s f(x)$ is in $L^2(R^n)$ for any real s. Let $|\alpha| \leq h < 2s - \frac{n}{2}$. So

$$\frac{x^\alpha}{(1+|x|^2)^s} \text{ is in } L^2(R^n) \text{ . Then}$$

$$x^\alpha f(x) = \frac{x^\alpha}{(1+|x|^2)^s}(1+|x|^2)^s f(x) \text{ is in } L^1(R^n) \text{ for any } \alpha$$

with $|\alpha| \leq h$. Taking inverse Fourier transform, we know that $((ix)^\alpha f(x))^\sim(\xi) = D^\alpha\tilde{f}(\xi)$ is continuous. Hence $\tilde{f}$ is in $C^h(R^n)$, for $h < 2s - \frac{n}{2}$. Since s can be arbitrarily large, we have that $\tilde{f}$ is in $C^\infty(R^n)$. For any α, we have

$$\|D^\alpha\tilde{f}(\xi)\|_{L^2} = \|((ix)^\alpha f(x))^\sim(\xi)\|_{L^2}$$
$$= (2\pi)^{-n/2}\|(ix)^\alpha f(x)\|_{L^2} < \infty$$

which implies $\tilde{f}$ is in $\mathcal{D}_{L^2}$. Therefore

$\{f \in L^2(R^n) \mid p(x)f(x) \in L^2(R^n)$ for any polynomial $p(x)\}$ $\subset \hat{\mathcal{D}}_{L^2}$.

Q.E.D.

Proposition 5.2. H is a continuous linear map of $\mathcal{D}_{L^2}$ into $\mathcal{D}_{L^2}$.

Proof. If u is in $\mathcal{D}_{L^2}$, then $(Hu)^\wedge(\xi) = -i\, h(\xi)\, \hat{u}(\xi)$.

Since $\xi^\alpha \hat{u}(\xi) = i^{-|\alpha|} (D^\alpha u)^\wedge(\xi)$ is in $L^2(R^n)$ for each α, we have $(Hu)^\wedge$ is in $\hat{\mathcal{D}}_{L^2}$ by lemma 5.1. Thus Hu is in $\mathcal{D}_{L^2}$. For continuity, note that

$$\begin{aligned}
\|D^\alpha Hu\|_{L^2} &= (2\pi)^{-n/2} \|(D^\alpha Hu)^\wedge(\xi)\|_{L^2} \\
&= (2\pi)^{-n/2} \|\xi^\alpha (Hu)^\wedge\|_{L^2} \\
&\leq (2\pi)^{-n/2} \|\xi^\alpha \hat{u}(\xi)\|_{L^2} \\
&= (2\pi)^{-n/2} \|(D^\alpha u)^\wedge(\xi)\|_{L^2} \\
&= \|D^\alpha u\|_{L^2} \, .
\end{aligned}$$

Q.E.D.

Since $\mathcal{D}_{L^2} \subset \mathcal{D}'_{L^2}$, let's see how H operates on f in $\mathcal{D}_{L^2}$ as an element of $\mathcal{D}'_{L^2}$. For every u in $\mathcal{D}_{L^2}$, we have, by the remark after theorem 4.9 and formula * before lemma 5.1,

$$\begin{aligned}
\langle Hf, u\rangle &= \int_{R^n} Hf(x)\, u(x)\, dx \\
&= \int_{R^n} f(x)\, (-Hu)(x)\, dx \\
&= \langle f, -Hu\rangle \, .
\end{aligned}$$

Definition. We now define $H' : \mathcal{D}'_{L^2} \longrightarrow \mathcal{D}'_{L^2}$ to be the dual operator of $-H : \mathcal{D}_{L^2} \longrightarrow \mathcal{D}_{L^2}$. Note that H' is continuous for the various dual topologies (e.g. weak*, strong, etc.). Moreover, we have a natural inclusion $L^2(R^n) \subseteq \mathcal{D}'_{L^2}$ and according to formula * , H' is an extension of H on $L^2(R^n)$ (relative to this inclusion). Hence we will simply denote H' by H and refer it as the Hilbert transform on $\mathcal{D}'_{L^2}$ associated to the cone Γ.

If P is a polynomial with constant coefficients, then $P(-D)$ is a continuous linear map of $\mathcal{D}_{L^2}$ into $\mathcal{D}_{L^2}$ and hence the transpose of $P(-D)$ maps $\mathcal{D}'_{L^2}$ continuously into $\mathcal{D}'_{L^2}$ and clearly coincides with $P(D)$ taken in the distribution sense. Thus for T in $\mathcal{D}'_{L^2}$ and u in $\mathcal{D}_{L^2}$, we have

$$\langle P(D)T,u\rangle = \langle T,P(-D)u\rangle \ .$$

Lemma 5.3. If P is a polynomial, then H commutes with $P(D)$.

Proof. If T is in $\mathcal{D}'_{L^2}$ and u is in $\mathcal{D}_{L^2}$, then

$$\langle P(D)HT,u\rangle = -\langle T,HP(-D)u\rangle \ , \text{ and}$$

$$\langle HP(D)T,u\rangle = -\langle T,P(-D)Hu\rangle \ .$$

Now clearly

$$(P(-D)Hu)^{\wedge}(\xi) = P(-i\xi)\ (Hu)^{\wedge}(\xi)$$

$$= -i\ P(-i\xi)\ h(\xi)\ \hat{u}(\xi) \ ,$$

and since $P(-D)u$ is in $L^2(R^n)$ we have

$$(HP(-D)u)^\wedge(\xi) = -i\ h(\xi)\ (P(-D)u)^\wedge(\xi)$$

$$= -i\ h(\xi)\ P(-i\xi)\ \hat{u}(\xi)\ .$$

Therefore $P(D)HT = HP(D)T$ for every T in $\mathcal{D}'_{L^2}$. Q.E.D.

<u>Theorem 5.4</u>. If T is in $\mathcal{D}'_{L^2}$, then $\hat{T}$ and $(HT)^\wedge$ are locally integrable functions and $(HT)^\wedge(\xi) = -i\ h(\xi)\ \hat{T}(\xi)$.

<u>Proof</u>. By theorem 4.9 we have a finite number of f_α in $L^2(R^n)$ with $T = \sum_\alpha D^\alpha f_\alpha$. By above lemma $HT = \sum_\alpha D^\alpha Hf_\alpha$ and therefore

$$(HT)^\wedge(\xi) = \sum_\alpha (i\xi)^\alpha\ (Hf_\alpha)^\wedge(\xi)$$

$$= -i\ h(\xi) \sum_\alpha (i\xi)^\alpha\ \hat{f}_\alpha(\xi)$$

$$= -i\ h(\xi)\ \hat{T}(\xi)\ .$$

Q.E.D.

<u>Theorem 5.5</u>. Let $Q = -H^2 : \mathcal{D}'_{L^2} \longrightarrow \mathcal{D}'_{L^2}$. We have

(a) $Q^2 = Q$, and

(b) $H = QH = HQ$.

<u>Proof</u>. (a) For every T in $\mathcal{D}'_{L^2}$ and f in $\mathcal{D}_{L^2}$, we have

$$\langle QT,f\rangle = \langle -H^2T,f\rangle = -\langle H^2T,f\rangle = -\langle HT,-Hf\rangle$$

$$= -\langle T,H^2f\rangle = \langle T,-H^2f\rangle = \langle T,Qf\rangle\ .$$

Therefore, by theorem 3.1, we have

$$\langle Q^2T,f\rangle = \langle T,Q^2f\rangle = \langle T,Qf\rangle = \langle QT,f\rangle \ .$$

(b) Since $Q = -H^2$, we have $HQ = QH$. For every T in $\mathcal{D}'_{L^2}$ and f in $\mathcal{D}_{L^2}$, we have, by theorem 3.1,

$$\langle QHT,f\rangle = \langle HT,Qf\rangle = \langle T,-HQf\rangle = \langle T,-Hf\rangle = \langle HT,f\rangle \ .$$

Therefore $H = QH = HQ$. Q.E.D.

<u>Theorem 5.6</u>. The Hilbert transform H associated with Γ has the following properties

(a) H maps $\operatorname{im} Q$ onto $\operatorname{im} Q$,

(b) $\ker H = \ker Q = \{T \in \mathcal{D}'_{L^2} \mid \hat{T} = 0 \text{ a.e. in } \Gamma^* \cup (-\Gamma^*)\}$,

(c) $\operatorname{im} H = \operatorname{im} Q = \{T \in \mathcal{D}'_{L^2} \mid \operatorname{supp} \hat{T} \subseteq \Gamma^* \cup (-\Gamma^*)\}$.

<u>Proof</u>. By theorem 5.4, if T is in $\mathcal{D}'_{L^2}$ then

$$(HT)^\wedge(\xi) = -i\, h(\xi)\, \hat{T}(\xi) \quad \text{and} \quad (QT)^\wedge(\xi) = |h(\xi)|^2\, \hat{T}(\xi) \ .$$

Thus (b) follows, and in addition we see

$\operatorname{im} Q = \{ T \in \mathcal{D}'_{L^2} \mid \operatorname{supp} \hat{T} \subseteq \Gamma^* \cup (-\Gamma^*)\}$. Now $Q = -H^2$ implies $\operatorname{im} Q \subseteq \operatorname{im} H$. Suppose that T is in $\operatorname{im} H$, say $T = HS$. Then $QT = QHS = HS = T$ implies that T is in $\operatorname{im} Q$, hence (c) follows. If $T = QS$, then $T = Q^2S = H(-QHS)$. Thus (a) follows. Q.E.D.

If T is in $\mathcal{D}'_{L^2}$ we define $\overline{T}$ by $\langle \overline{T}, u\rangle = \overline{\langle T, \overline{u}\rangle}$, for u in $\mathcal{D}_{L^2}$. We say that T is <u>real</u> if $T = \overline{T}$. Note if $T = \sum_\alpha D^\alpha f_\alpha$, f_α in $L^2(R^n)$, and T is real then $T = \frac{1}{2}(T + \overline{T})$ implies $T = \sum_\alpha D^\alpha g_\alpha$ where $g_\alpha = \frac{1}{2}(f_\alpha + \overline{f}_\alpha)$ are real L^2-functions. An operator L on $\mathcal{D}'_{L^2}$ is said to be <u>real</u> if it maps real distributions to real distributions. This condition is equivalent to $L\overline{T} = \overline{LT}$, for T in $\mathcal{D}'_{L^2}$.

<u>Proposition 5.7</u>. H and Q are real operators on $\mathcal{D}'_{L^2}$.

<u>Proof</u>. If T is in $\mathcal{D}'_{L^2}$ and u is in $\mathcal{D}_{L^2}$, we have, by proposition 3.3,

$$\langle H\overline{T}, u\rangle = \langle \overline{T}, -Hu\rangle = \overline{\langle T, -\overline{Hu}\rangle} = \overline{\langle T, -H\overline{u}\rangle} = \overline{\langle HT, \overline{u}\rangle} = \langle \overline{HT}, u\rangle .$$

Since $Q = -H^2$, Q is also real. Q.E.D.

Let W be the continuous linear operator on $\mathcal{D}'_{L^2}$ defined by $W = \frac{1}{2}(Q + iH)$.

<u>Proposition 5.8</u>. We have $(WT)^\wedge(\xi) = \chi_{\Gamma^*}(\xi)\,\hat{T}(\xi)$ for each T in $\mathcal{D}'_{L^2}$. In particular, $W^2 = W$ on $\mathcal{D}'_{L^2}$ and $\operatorname{im} W = \{ T \in \mathcal{D}'_{L^2} \mid \operatorname{supp} \hat{T} \subseteq \Gamma^* \}$.

Proof. If T is in $\mathcal{D}'_{L^2}$ then by theorem 5.4 we have

$$(WT)^\wedge(\xi) = \tfrac{1}{2}\,(|h(\xi)|+h(\xi))\,\hat{T}(\xi) = \chi_{\Gamma^*}(\xi)\,\hat{T}(\xi)\ .$$

Thus $W^2 = W$. In particular T is in im W if and only if WT = T, whence the last part follows. Q.E.D.

Lemma 5.9. If $U_\circ$ and $V_\circ$ are real distributions in $\mathcal{D}'_{L^2}$, then the following statements are equivalent.

(a) $W(U_\circ + iV_\circ) = U_\circ + iV_\circ$.

(b) $QU_\circ - HV_\circ = 2U_\circ$ and $QV_\circ + HU_\circ = 2V_\circ$.

(c) $QU_\circ = U_\circ$ and $HU_\circ = V_\circ$.

Proof. The proof is exactly same as that of lemma 3.5.

Corollary 5.10. Let $U_\circ$ be a real distribution in $\mathcal{D}'_{L^2}$. There exists real $V_\circ$ in $\mathcal{D}'_{L^2}$ such that supp $(U_\circ+iV_\circ)^\wedge \subseteq \Gamma^*$ if and only if $QU_\circ = U_\circ$. Moreover, in this case we have $V_\circ = HU_\circ$.

Proof. The proof is exactly same as that of corollarly 3.6.

In chapter III we met the Cauchy kernel K of the tube $\Omega = R^n+i\Gamma_\circ$ or of the cone Γ . K is holomorphic in Ω . If y is in $\Gamma_\circ$ then $(i\xi)^\alpha\, e^{-(y,\xi)}\, \chi_{\Gamma^*}(\xi)$ is in $L^p(R^n_\xi)$ for $1 \leq p \leq \infty$ and its inverse Fourier transform is just $D^\alpha K_y$

where $K_y(x) = K(x+iy)$ is in $\mathcal{D}_{L^q}$, for $q \geq 2$. Thus

$$(D^\alpha K_y)^\wedge(\xi) = (i\xi)^\alpha \, e^{-(y,\xi)} \, \chi_{\Gamma^*}(\xi) \ .$$

Since $D^\alpha K_y$ is in $L^2(R^n)$, if f is in $L^2(R^n)$ then, by theorem 2.11, we have

$$D^\alpha K_y * f = ((D^\alpha K_y)^\wedge \hat{f})^\sim \ .$$

Since $(D^\alpha K_y)^\wedge(\xi) = (i\xi)^\alpha e^{-(y,\xi)} \chi_{\Gamma^*}(\xi)$ is bounded, we have that $D^\alpha K_y * f$ is in $L^2(R^n)$ and

$$\begin{aligned}(D^\alpha K_y * f)^\wedge &= (D^\alpha K_y)^\wedge \hat{f} = (i\xi)^\alpha \, e^{-(y,\xi)} \, \chi_{\Gamma^*}(\xi) \, \hat{f}(\xi) \\ &= e^{-(y,\xi)} \, \chi_{\Gamma^*}(\xi) \, (D^\alpha f)^\wedge(\xi) \\ &= e^{-(y,\xi)} \, (WD^\alpha f)^\wedge(\xi) \ .\end{aligned}$$

Now $D^\alpha f$ is in $\mathcal{D}'_{L^2}$ and as we have already seen in chapter IV, we have

$$\begin{aligned}K_y * D^\alpha f(x) &= \langle D^\alpha f, \mathcal{T}_x \check{K}_y \rangle = (-1)^\alpha \int_{R^n} f(x) \, D^\alpha(\mathcal{T}_x \check{K}_y)(x) \, dx \\ &= \langle f, (-D)^\alpha(\mathcal{T}_x \check{K}_y) \rangle = \langle f, (-1)^\alpha (-1)^\alpha \, \mathcal{T}_x (D^\alpha K_y)^\vee \rangle \\ &= \langle f, \mathcal{T}_x (D^\alpha K_y)^\vee \rangle = D^\alpha K_y * f(x) \ .\end{aligned}$$

Now suppose T is in $\mathcal{D}'_{L^2}$, then $T = \sum_\alpha D^\alpha f_\alpha$ with f_α in $L^2(R^n)$, and so

$$(K_y * T)^\wedge(\xi) = \sum_\alpha (K_y * D^\alpha f_\alpha)^\wedge(\xi)$$

$$= \sum_\alpha (D^\alpha K_y * f_\alpha)^\wedge(\xi)$$

$$= e^{-(y,\xi)} \chi_{\Gamma^*}(\xi) \sum_\alpha (D^\alpha f_\alpha)^\wedge(\xi)$$

$$= e^{-(y,\xi)} \chi_{\Gamma^*}(\xi) \hat{T}(\xi) .$$

Hence we have proved following theorem.

Theorem 5.11. If y is in Γ_o and T is in $\mathcal{D}'_{L^2}$, then $(K_y * T)^\wedge$ is an L^2-function and we have

$$(K_y * T)^\wedge(\xi) = e^{-(y,\xi)} \chi_{\Gamma^*}(\xi) \hat{T}(\xi)$$

$$= e^{-(y,\xi)} (WT)^\wedge(\xi)$$

In particular $K_y * T = K_y * WT$.

Again if $T = \sum_\alpha D^\alpha f_\alpha$ with f_α in $L^2(R^n)$, then by above theorem

$$(D^\beta(K_y * T))^\wedge(\xi) = (i\xi)^\beta (K_y * T)^\wedge(\xi)$$

$$= \sum_\alpha (i\xi)^{\alpha+\beta} e^{-(y,\xi)} \chi_{\Gamma^*}(\xi) \hat{f}_\alpha(\xi)$$

belongs to $L^2(R^n)$ since $\xi^{\alpha+\beta} e^{-(y,\xi)} \chi_{\Gamma^*}(\xi)$ is bounded.

Above also shows that

$$D^\beta(K_y * T) = D^\beta K_y * T = K_y * D^\beta T .$$

Thus $K_y * T$ is in $\mathcal{D}_{L^2}$. $K_y * T(x)$ is called the generalized Cauchy integral of T in $\mathcal{D}'_{L^2}$, denoted by $KT(x+iy)$, and was introduced in case Γ is an octant by H. Tillmann who called it "Indikatrix" (see Tillmann [1] and [2]). Hence theorem 5.11 implies following corollary.

<u>Corollary 5.12</u>. For z in $\Omega = R^n + i\Gamma_\circ$ and T in $\mathcal{D}'_{L^2}$, we have $KT = K(WT)$.

Now if u and v are in $L^2(R^n)$ then by theorem 2.7(iii)

$$\int_{R^n} v(x)\ u(x)\ dx = \int_{R^n} \hat{v}(\xi)\ \tilde{u}(\xi)\ d\xi \ .$$

In particular we have, for u in $\mathcal{D}_{L^2}$,

$$\langle K_y * T, u\rangle = \int_{R^n} (K_y * T)(x)\ u(x)\ dx$$

$$= \int_{R^n} e^{-(y,\xi)}\ \chi_{\Gamma*}(\xi)\ \hat{T}(\xi)\ \tilde{u}(\xi)\ d\xi \ .$$

If $T = \sum_\alpha D^\alpha f_\alpha$ with f_α in $L^2(R^n)$, and u is in $\mathcal{D}_{L^2}$, then by the remark after theorem 4.9 we have

$$\langle T, u\rangle = \sum_\alpha (-1)^{|\alpha|} \int_{R^n} f_\alpha(x)\ D^\alpha u(x)\ dx$$

$$= \sum_\alpha (-1)^{|\alpha|} \int_{R^n} \hat{f}_\alpha(\xi)\ (-i\xi)^\alpha\ \tilde{u}(\xi)\ d\xi$$

$$= \sum_\alpha \int_{R^n} (D^\alpha f_\alpha)^\wedge(\xi)\ \tilde{u}(\xi)\ d\xi$$

$$= \int_{R^n} \hat{T}(\xi)\ \tilde{u}(\xi)\ d\xi\ .$$

Since the formula holds for any T in $\mathcal{D}'_{L^2}$, we have, in particular,

$$\langle WT,u\rangle = \int_{R^n} \chi_{\Gamma^*}(\xi)\ \hat{T}(\xi)\ \tilde{u}(\xi)\ d\xi\ .$$

Thus for y in $\Gamma_\circ$, T in $\mathcal{D}'_{L^2}$ and u in $\mathcal{D}_{L^2}$, we have

$$\langle K_y * T - WT,\ u\rangle = \int_{R^n} (e^{-(y,\xi)}-1)\ \chi_{\Gamma^*}(\xi)\ \hat{T}(\xi)\ \tilde{u}(\xi)\ d\xi\ .$$

Now if $T = \sum_\alpha D^\alpha f_\alpha$ with f_α in $L^2(R^n)$, then

$$\hat{T}(\xi) = \sum_\alpha (i\xi)^\alpha\ \hat{f}_\alpha(\xi)\ .$$

Since $(i\xi)^\alpha\ \tilde{u}(\xi) = (-1)^{|\alpha|}\ (D^\alpha u)^\sim(\xi)$, we have

$$\langle K_y * T-WT,u\rangle = \sum_\alpha (-1)^{|\alpha|}\int_{R^n}(e^{-(y,\xi)}-1)\chi_{\Gamma^*}(\xi)\hat{f}_\alpha(\xi)(D^\alpha u)^\sim(\xi)d\xi$$

which implies

$$|\langle K_y * T - WT,\ u\rangle| \le \sum_\alpha C_\alpha(y)\ \|(D^\alpha u)^\sim\|_{L^2}$$

$$= (2\pi)^{-n/2} \sum_\alpha C_\alpha(y)\ \|D^\alpha u\|_{L^2}$$

where $C_\alpha(y) = \left(\int_{R^n} \chi_{\Gamma^*}(\xi)\ |(e^{-(y,\xi)} - 1)\ \hat{f}_\alpha(\xi)|^2\ d\xi\right)^{\frac{1}{2}}$.

We note, by dominated convergence theorem, that $C_\alpha(y) \longrightarrow 0$

as $y \longrightarrow 0$ in Γ_o . Thus we have proved following theorem which improves that of R. Carmichael (see Carmichael [5]) who uses the weak* topology on $\mathcal{D}'_{L^2}$.

<u>Theorem 5.13</u>. If T is in $\mathcal{D}'_{L^2}$, then $K_y * T$ converges to WT in $\mathcal{D}'_{L^2}$ as y converges to 0 in Γ_o, where the convergence is in the sense of the strong topology on $\mathcal{D}'_{L^2}$.

We now define $DH^2(\Omega)$ to be the space of holomorphic functions in Ω which are finite sums of derivatives of functions in $H^2(\Omega)$ where $\Omega = R^n + i\Gamma_o$.

<u>Theorem 5.14</u>. If T is in $\mathcal{D}'_{L^2}$ and F = KT then F is in $DH^2(\Omega)$. If we define $F_y(x) = F(x+iy)$, then F_y converges to WT in $\mathcal{D}'_{L^2}$ with respect to the strong topology as y converges to 0 in Γ_o .

<u>Proof</u>. In view of theorem 5.13, we need only to prove that KT is in $DH^2(\Omega)$. Suppose $T = \sum_\alpha D^\alpha f_\alpha$ with f_α in $L^2(R^n)$. By theorem 5.11, we have

$$KT(z) = K_y * T(x)$$

$$= \sum_\alpha (e^{-(y,\xi)} \chi_{\Gamma^*}(\xi) (D^\alpha f_\alpha)^\wedge(\xi))^\sim(x)$$

$$= \sum_{\alpha} (2\pi)^{-n} \int_{R^n} e^{i(x,\xi)} e^{-(y,\xi)} \chi_{\Gamma*}(\xi) (i\xi)^{\alpha} \hat{f}_{\alpha}(\xi) d\xi$$

$$= \sum_{\alpha} D_x^{\alpha}[(2\pi)^{-n} \int_{\Gamma*} e^{i(z,\xi)} \hat{f}_{\alpha}(\xi) d\xi]$$

$$= \sum_{\alpha} D_x^{\alpha} K f_{\alpha}(z)$$

$$= \sum_{\alpha} D_z^{\alpha} K f_{\alpha}(z)$$

Thus KT is in $DH^2(\Omega)$. Q.E.D.

<u>Theorem 5.15</u>. If F is in $DH^2(\Omega)$, then there exists T in $\mathcal{D}'_{L^2}$ such that F_y converges to T in $\mathcal{D}'_{L^2}$ with respect to the strong topology as y converges to 0 in Γ_o . Moreover WT = T and KT = F .

<u>Proof</u>. If F is in $DH^2(\Omega)$ then $F = \sum_{\alpha} D_z^{\alpha} F_{\alpha}$ with F_{α} in $H^2(\Omega)$. Now $F_{\alpha,y}(x) = F_{\alpha}(x+iy)$ defines a function $F_{\alpha,y}$ in $L^2(R^n)$. By the Cauchy-Riemann equations $F_y = \sum_{\alpha} D_x^{\alpha} F_{\alpha,y}$, whence F_y is in $\mathcal{D}'_{L^2}$ when y is in Γ_o . Now since F_{α} is in $H^2(\Omega)$, by theorem 3.13 and corollary 3.14, there exists f_{α} in $L^2(R^n)$ such that $F_{\alpha,y}$ converges to f_{α} in $L^2(R^n)$, i.e., $|\langle F_{\alpha,y} - f_{\alpha}, u\rangle| = C_{\alpha}(y) \|u\|_{L^2}$ for u in $L^2(R^n)$, where $C_{\alpha}(y)$ converges to 0 as y converges to 0 in Γ_o . Moreover $Wf_{\alpha} = f_{\alpha}$ and $F_{\alpha} = Kf_{\alpha}$, i.e., $F_{\alpha,y} = K_y * f_{\alpha}$.

Let $T = \sum_{\alpha} D^{\alpha} f_{\alpha}$. Then T is in $\mathcal{D}'_{L^2}$ and WT = T since W commutes with P(D) and $Wf_{\alpha} = f_{\alpha}$. If u is in $\mathcal{D}_{L^2}$, we have

$$\langle T,u\rangle = \sum_{\alpha} (-1)^{|\alpha|} \langle f_{\alpha}, D^{\alpha}u\rangle \text{ , and}$$

$$\langle F_y,u\rangle = \sum_{\alpha} (-1)^{|\alpha|} \langle F_{\alpha,y}, D^{\alpha}u\rangle \text{ .}$$

Thus $|\langle F_y - T, u\rangle| \leq \sum_{\alpha} C_{\alpha}(y) \|D^{\alpha}u\|_{L^2}$. Hence F_y converges to T in $\mathcal{D}'_{L^2}$ with respect to the strong topology as y converges to 0 in Γ_o . In addition

$$\begin{aligned} KT = K_y * T &= \sum_{\alpha} K_y * D^{\alpha} f_{\alpha} \\ &= \sum_{\alpha} D^{\alpha}(K_y * f_{\alpha}) \\ &= \sum_{\alpha} D^{\alpha} F_{\alpha} \\ &= F \text{ .} \end{aligned}$$

Q.E.D.

<u>Remark</u>. T in the above theorem is called the distributional boundary value of F along the edge of Ω .

Combining theorems 5.14 and 5.15 we have following corollary.

<u>Corollary 5.16</u>. T in $\mathcal{D}'_{L^2}$ is the distributional boundary value of some F in $DH^2(\Omega)$ along the edge of Ω if and only if WT = T , i.e., supp $\hat{T} \subseteq \Gamma^*$.

The concept of distributional boundary value in $\mathcal{D}'_{L^p}$ was extensively studied by Tillmann for tubes over octants (see Tillmann [1]) and by Beltrami and Wohlers in the one dimensional case (see Beltrami and Wohlers [1],[2],[3],[4]).

<u>Theorem 5.17</u>. Let $U_\circ$ be a real distribution in $\mathcal{D}'_{L^2}$. Then there exists F in $DH^2(\Omega)$ such that Re F_y converges to $U_\circ$ in $\mathcal{D}'_{L^2}$ (strong topology) as y converges to 0 in $\Gamma_\circ$, if and only if, $U_\circ = QU_\circ$. Moreover, in this case, if $V_\circ = HU_\circ$, then Im F_y converges to $V_\circ$ in $\mathcal{D}'_{L^2}$ (strong topology) as y converges to 0 in $\Gamma_\circ$, and if $F(x+iy) = u(x,y) + iv(x,y)$ where u and v are real, then $u(x,y) = p_y * U_\circ(x)$ and $v(x,y) = q_y * U_\circ(x)$ where $p_y(x) = 2 \text{ Re } K(x+iy)$ and $q_y(x) = 2 \text{ Im } K(x+iy)$.

<u>Proof</u>. If the real distribution $U_\circ$ in $\mathcal{D}'_{L^2}$ is such that, for some F in $DH^2(\Omega)$, Re F_y converges to $U_\circ$ in $\mathcal{D}'_{L^2}$ (strong topology) as y converges to 0 in $\Gamma_\circ$, by theorem 5.15 we know $F = KT$ for some T in $\mathcal{D}'_{L^2}$ and F_y converges to T in $\mathcal{D}'_{L^2}$ (strong topology) as y converges to 0 in $\Gamma_\circ$. Moreover $WT = T$ and $T = U_\circ + iV_\circ$ where $V_\circ$ is also real. Thus, by lemma 5.9, $QU_\circ = U_\circ$ and $HU_\circ = V_\circ$. Conversely, suppose $QU_\circ = U_\circ$ and let $V_\circ = HU_\circ$. Let $T = U_\circ + iV_\circ$, then T is in

$\mathcal{D}'_{L^2}$ and WT = T by lemma 5.9. By corollary 5.16, T is the boundary value of some F in $DH^2(\Omega)$ and clearly Re F_y converges to U_o and Im F_y converges to V_o in $\mathcal{D}'_{L^2}$ (strong topology). Now suppose $QU_o = U_o$, $V_o = HU_o$, $T = U_o + iV_o$ and $F = KT$ so $WT = T$ and F_y converges to T in $\mathcal{D}'_{L^2}$. Then $F = KT = K(U_o+iV_o) = K(QU_o+iHU_o) = 2\ KWU_o = 2\ KU_o$ which implies that, if $F(x+iy) = u(x,y) + iv(x,y)$, then

$$u(x,y) = 2 \text{ Re } KU_o = p_y * U_o(x) \ , \text{ and}$$

$$v(x,y) = 2 \text{ Im } KU_o = q_y * U_o(x) \ . \qquad \text{Q.E.D.}$$

Corollary 5.18. If U_o is a real distribution in $\mathcal{D}'_{L^2}$, then $q_y * U_o$ converges to HU_o in $\mathcal{D}'_{L^2}$ (strong topology) as y converges to 0 in Γ_o .

Proof. Let $T = QU_o + iHU_o = 2\ WU_o$. Then WT = T and if F = KT we have that F_y converges to T in $\mathcal{D}'_{L^2}$. Thus Im F_y converges to HU_o in $\mathcal{D}'_{L^2}$. Now Im $F_y = q_y * U_o$ is proved in the above theorem. Q.E.D.

Since $HQU_o = HU_o$, we have following corollary immediately.

Corollary 5.19. If U_o is a real distribution in $\mathcal{D}'_{L^2}$, then $q_y * QU_o \longrightarrow HU_o$ in $\mathcal{D}'_{L^2}$ (strong topology) as $y \longrightarrow 0$ in Γ_o.

BIBLIOGRAPHY

Beltrami, E. J. and Wohlers, M. R.

[1] Distributional Boundary Value Theorems and Hilbert Transforms. Arch. Rational Mech. Anal., 18 (1965), pp. 304-309.

[2] Distributions and the Boundary Values of Analytic Functions. Academic Press, 1966.

[3] Distributional Boundary Values of Functions Holomorphic in A Half Plane. J. Math. Mech., 15 (1966), pp. 137-146.

[4] The Cauchy Integral of Tempered Distributions and Some Theorems on Analytic Continuation. SIAM J. Appl. Math., 15(1967), pp. 1077-1087.

Bochner, S.

[1] Bounded Analytic Functions in Several Variables and Multiple Laplace Integrals. Amer. J. Math., 59 (1937), pp. 732-738.

[2] Group Invariance of Cauchy's Formula in Several Variables. Ann. of Math., 45 (1944), pp. 686-707.

[3] Boundary Values of Analytic Functions of Several Variables and of Almost Periodic Function. Ann. of Math., 45 (1944), pp. 708-722.

[4] Lectures on Fourier Integrals (with an author's supplement on Monotonic Functions, Stieljes Integrals, and Harmonic Analysis). (Ann. of Math. Stud. 42) Princeton Univ. Press, 1959.

Bochner, S. and Chandrasekharan, K.

[1] Fourier Transforms. (Ann. of Math. Stud. 19), Princeton Univ. Press, 1949.

Bremermann, H.

[1] Distributions, Complex Variables and Fourier Transforms. Addison-Wesley, 1965.

Butzer, P. L. and Nessel, R. J.

[1] Fourier Analysis and Approximation. Vol. I, Academic Press, 1971.

Butzer, P. L. and Trebels, W.

[1] Hilbert Transforms, Fractional Integration and Differentiation. Bull. Amer. Math. Soc., 74 (1968), pp. 106-110.

Carleman, T.

[1] L'intégral de Fourier et questions qui s'y rattachent. Publ. Sci. Inst. Mittag-Leffler 1, Uppsala 1944.

Carmichael, R. D.

[1] Distributional Boundary Values in $\mathcal{D}'_{L^p}$. Rend. Sem. Mat. Univ. Padova, 43 (1970), pp. 35-53.

[2] Distributional Boundary Values in $\mathcal{D}'_{L^p}$. II, Rend. Sem. Mat. Univ. Padova, 45 (1971), pp. 249-277.

[3] Distributional Boundary Values of Functions Analytic in Tubular Radial Domains. Indiana Univ. Math. J. (formerly J. Math. Mech.), 20 (1971), pp. 843-853.

[4] Functions Analytic in An Octant and Boundary Values of Distributions. J. Math. Anal. Appl., 33 (1971), pp. 616-626.

[5] Generalized Cauchy and Poisson Integrals and Distributional Boundary Values. SIAM J. Math. Anal., Vol. 4 No. 1, Feb. 1973, pp. 198-219.

Donoghue, W. F.

[1] Distributions and Fourier Transforms. Academic Press, 1969.

Gelfand, I. M. and Shilov, V. E.

[1] Generalized Functions. Vol. I: Properties and Operations; Vol. II: Spaces of Fundamental and Gerneralized Functions; Vol. III; Theory of Differential Equations. Academic Press, 1964.

Goldberg, R. R.

[1] Fourier Transforms. Cambridge Univ. Press, 1962.

Güttinger, W.

[1] Generalized Functions and Dispersion Relations in Physics. Fortschr. Physik, 14 (1966), pp. 483-602.

Hewitt, E.

[1] Topics in Fourier Analysis. Mimeographed Lecture Notes; Univ. Washington, Seattle, 1959.

Hewitt, E. and Stromberg, K.

[1] Real and Abstract Analysis. Springer-Verlag, 1965.

Heywood, P.

[1] On a Modification of the Hilbert Transform. J. London Math. Soc., 42 (1967), pp. 641-645.

Horváth, J.

[1] Singular Integral Operators and Spherical Harmonics. Trans. Amer. Math. Soc., 82 (1956). pp. 52-63.

[2] Topological Vector Spaces and Distributions. Addison-Wesley (1966).

Katznelson, Y.

[1] An Introduction to Harmonic Analysis. John Wiley, 1968.

Kober, H.

[1] A Note on Hilbert's Operator. Bull, Amer, Math. Soc., 48 (1942), pp. 421-426.

[2] A Note on Hilbert Transforms. J. London Math. Soc., 18 (1943), pp. 66-71.

[3] A Modification of Hilbert Transforms, The Weyl Integral and Functional Equations. J. London Math. Soc., 42 (1967), pp. 42-50.

Lauwerier, H. A.

[1] The Hilbert Problem for Generalized Functions. Arch. Rational Mech. Anal., 13 (1963), pp.157-166.

Loomis, L. H.

[1] A Note on the Hilbert Transform. Bull. Amer. Math. Soc., 52 (1946), pp. 1082-1086.

Newcomb, R. W.

[1] Hilbert Transforms — Distributional Theory. Stanford Electronics Laboratories, Technical Rep. 2250-1 (1962).

Schwartz, L.

[1] Théorie des distributions. Hermann, Paris, 1966.

Stein, E. M. and Weiss, G.

[1] Introduction to Fourier Analysis on Euclidean Space. Princeton, 1971.

Stein, E. M., Weiss, G. and Weiss, M.

[1] H^p Classes of Holomorphic Functions in Tube Domains. Proc. Nat. Acad. Sci. U. S. A., 52 (1964), pp. 1035-1039.

Tillmann, H. G.

[1] Randverteilungen analytischer Functionen und Distributionen. Math. Z., 59 (1953), pp. 61-83.

[2] Distributionen als Randverteilungen analytischer Functionen II. Math. Z., 76 (1961), pp. 5-21.

Titchmarsh, E. C.

[1] A Contribution to the Theory of Fourier Transforms. Proc. London Math. Soc., (2) 23, (1923), pp.279-289.

[2] Introduction to the Theory of Fourier Integrals. Oxford Univ. Press, (1937).

Treves, F.

[1] Topological Vector Spaces, Distributions and Kernels. Academic Press. 1967.

Weiss, G.

[1] Analisis armonico en varias variables. Teoria de los espacios H^p . (Mimeographed Lecture Notes; Cursos y Seminarios de Matemática 9) Univ. de Buenos Aires, 1960.

[2] Harmonic Analysis. In: Studies in Real and Complex Analysis. (Stud. in Math. 3; Ed. I.I. Hirschman, Jr.) Math. Ass. of America, Buffalo, N.Y.; Prentice-Hall, 1965, pp. 124-178.

Wiener, N.

[1] The Fourier Integral and Certain of its Applications. Dover Publ., 1933.

Yosida, K.

[1] Functional Analysis. Springer, 1965.

Zygmund, A.

[1] Trigonometric Series I, II, Cambridge Univ. Press, 1959 (2^{nd} ed. 1968).