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Let φ be a convex function on R^n which belongs to $\mathcal{O}_M$ and satisfies a certain growth condition at infinity. Then $e^{-\langle z,\cdot\rangle-\varphi}$ belongs to $\mathscr{S}$ for each $z \in C^n$. If $\mathscr{F}$ is a space of distributions, let $\mathscr{F}_\varphi$ be the space of distributions T such that $e^\varphi T$ is in $\mathscr{F}$. If F is the Laplace transform of $T \in \mathscr{S}'_\varphi$, then F is an entire function and $F(z) = \langle e^\varphi T, e^{-\langle z,\cdot\rangle-\varphi}\rangle$ and F satisfies an estimate of the form

$$|F(z)| \leq C(1+|z|)^N e^{\varphi^*(-\xi)}$$

where ξ is the real part of z and where φ^* is the conjugate function of φ. Similar estimates and the converse statements are obtained for the spaces $\mathscr{S}_\varphi$, $\mathcal{O}_{M,\varphi}$ and $\mathscr{L}^2_\varphi$. In the $\mathscr{L}^2_\varphi$ case, rather restrictive hypotheses are made on φ. Finally, the differentiability condition of φ at the origin is relaxed.

Paley-Wiener Theorems with Convex Weight Functions

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PALEY-WIENER THEOREMS WITH CONVEX WEIGHT FUNCTIONS

CHAPTER I

INTRODUCTION

The Laplace transform F of a function f is formally given by

$$F(z) = (2\pi)^{-n/2}\int e^{-\langle z,x\rangle} f(x)dx.$$

More explicitly, if T is a distribution such that $e^{-\langle \xi,\cdot\rangle}T$ is temperate for any ξ belonging to a nonempty convex subset Γ of R^n, then the Laplace transform of T is defined to be the Fourier transform of the distribution $e^{-\langle \xi,\cdot\rangle}T$. According to [5], if Γ is open, then the Laplace transform of T is a holomorphic function in the cylinder $\Gamma + iR^n$ bounded by a polynomial in each cylinder with compact base in Γ. Conversely, every holomorphic function F in $\Gamma + iR^n$ such that for each compact subset K of Γ F is bounded in $K + iR^n$ by a polynomial is the Laplace transform of a unique distribution. The Paley-Wiener theorem states that if T is a distribution with compact support K, then the Laplace transform is an entire function F satisfying

$$|F(z)| \leq C(1+|z|)^N e^{H(-\xi)}$$

where N is the order of T and $z = \xi + i\eta$ and H is the support function of the convex hull of K; conversely, if F is an entire function with the estimate $|F(z)| \leq C(1+|z|)^N e^{H(-\xi)}$ where $z = \xi + i\eta$

and H is the support function of some compact convex subset of R^n, then F is the Laplace transform of a distribution supported by K. In this paper we prove some analogous results, where H is replaced by a convex function other than a support function.

Let φ be a convex function with a growth condition and some continuity condition. We study the conjugates of such functions in chapter III. Denote by $\mathscr{S}'_\varphi$ the space of distributions T such that $e^\varphi T$ is a temperate distribution. Then $\mathscr{S}'_\varphi \subseteq \mathcal{O}'_C$ (the convolution operators) and by the result in [5] the Laplace transform of $T \in \mathscr{S}'_\varphi$ is an entire function F. In chapter IV we shall show $F(z) = \langle e^\varphi T, e^{-\langle \xi, \cdot \rangle - \varphi} \rangle$. The main problem of this paper is to characterize F by an estimate involving a polynomial in $|z|$ with a weight factor involving the conjugate φ^* of φ. Explicitly, if $T \in \mathscr{S}'_\varphi$ and if F is the Laplace transform of T, then there exist constants C and N such that

$$|F(z)| \leq C(1+|z|)^N e^{\varphi^*(-\xi)}$$

where $z = \xi + i\eta \in C^n$. Conversely, if F is an entire function satisfying the inequality above, then there exists a unique distribution $T \in \mathscr{S}'_\varphi$ such that the Laplace transform of T is F. The converse has been proved by K. Hayakawa in [3] in case $n=1$ and for special φ in higher dimensions (though not in general). Here we shall investigate analogous results where the space $\mathscr{S}'$ is replaced by $\mathscr{S}$, $\mathcal{O}_M$ and $\mathcal{L}^2$. We shall also carry out some considerations when the C^∞ condition on φ at the origin is removed.

CHAPTER II

LAPLACE TRANSFORM

For the purpose of notation and terminology only, the first section is a list of various topological vector spaces of functions and distributions that appear in this paper. No detailed description of these ingredients is attempted. References may be found in say [5] or [6]. The second section reviews L. Schwartz's definition of the Laplace transform.

(i) Various Spaces

Unless otherwise noted, the underlying space shall always be R^n, i.e., $C^\infty = C^\infty(R^n)$, $\mathcal{D}' = \mathcal{D}'(R^n)$, etc.

$\mathcal{D}_K$: C^∞ functions with support contained in a compact subset K of R^n. The topology is defined by the system of seminorms

$$\|\varphi\|_{K,m} = \max_{|\alpha|\le m} \sup |D^\alpha \varphi(x)|$$

$m = 0,1,2, \ldots$.

$\mathcal{D}$: C^∞ functions with compact support, topologized by the requirement that a seminorm p on $\mathcal{D}$ is continuous if and only if the restriction of p on $\mathcal{D}_K$ is continuous for each compact set K in R^n.

$\mathcal{D}'$: dual space of $\mathcal{D}$, space of distributions.

$\mathcal{E}$: C^∞ functions, topologized with the seminorms

$$\|f\|_{K,m} = \sum_{|\alpha|\le m} \sup_K |D^\alpha f(x)|$$

where K runs over compact sets in R^n and $m = 0,1,2,\ldots$.

$\mathcal{E}'$: dual space of $\mathcal{E}$, space of distributions with compact support.

$\mathcal{S}$: C^∞ functions which together with their derivatives decrease at infinity more rapidly than any polynomial of $|x|^{-1}$, with seminorms

$$\|f\|_N = \sum_{|\alpha|\leq N} \sup |(1+|x|^2)^N D^\alpha f(x)| .$$

$\mathcal{S}'$: dual space of $\mathcal{S}$, temperate distributions.

$\mathcal{O}_M$: C^∞ functions whose derivatives increase more slowly than some polynomial; precisely, $\varphi \in \mathcal{O}_M$ if and only if φ is C^∞ and satisfies one of the following equivalent conditions.

(1) The map: $T \rightarrow \varphi T$ from $\mathcal{S}'$ into $\mathcal{D}'$ is a continuous linear map from $\mathcal{S}'$ into $\mathcal{S}'$.

(2) The map: $f \rightarrow \varphi f$ from $\mathcal{S}$ into $\mathcal{E}$ is a continuous linear map from $\mathcal{S}$ into $\mathcal{S}$.

(3) For each multi-index α there exists a polynomial P such that

$|D^\alpha \varphi(x)| \leq |P(x)|$

for all $x \in R^n$.

These functions are also called <u>multiplication</u> <u>operators</u>. The topology consists of the seminorms

$\|\varphi\|_{f,\alpha} = \sup_x |f(x) D^\alpha f(x)|$

where $f \in \mathcal{S}$. In view of (2) $\|\varphi\|_{f,\alpha}$ is always finite.

$\mathcal{O}_c'$: distributions decreasing more rapidly than any polynomial of $|x|^{-1}$; precisely, $u \in \mathcal{O}_c'$ if and only if for each

$k \geq 0$ there exist C^∞ functions f_α, where $|\alpha| \leq$ some integer ℓ depending on k, such that $u = \sum_{|\alpha| \leq \ell} D^\alpha f_\alpha$ and for all α with $|\alpha| \leq \ell$,

$$\lim_{|x| \to \infty} (1+|x|)^k |f_\alpha(x)| = 0 .$$

We shall use the fact that if $f \in \mathscr{S}$ and $T \in \mathscr{S}'$ then $fT \in \mathcal{O}_c'$; this may easily be shown with the theorem on page 236 of [2] and the Leibnitz formula. Recall also that the Fourier transform is an isomorphism of $\mathcal{O}_M$ onto $\mathcal{O}_c'$.

(ii) The Space $\mathscr{S}'(\Gamma)$ and the Laplace Transform

Let Γ be a nonempty convex set in R^n . Define $\mathscr{S}'(\Gamma) = \left\{T \in \mathscr{D}' : e^{-\langle \xi, \cdot \rangle} T \in \mathscr{S}' \text{ for each } \xi \in \Gamma\right\}$. When $\Gamma = R^n$, $\mathscr{S}'(\Gamma)$ is not to be confused with $\mathscr{S}'$. Note that $\mathscr{S}'(\Gamma)$ corresponds to $\mathscr{S}'$ when $\Gamma = \{0\}$ and that $\Gamma_1 \subseteq \Gamma_2$ implies $\mathscr{S}'(\Gamma_1) \supseteq \mathscr{S}'(\Gamma_2)$. The space $\mathcal{O}_c'(\Gamma)$ may be defined in a similar way. We have

$$\mathcal{O}_c'(\Gamma) \subseteq \mathscr{S}'(\Gamma) \subseteq \mathcal{O}_c'(\mathring{\Gamma}) .$$

Hence $\mathcal{O}_c'(\Gamma)$ and $\mathscr{S}'(\Gamma)$ are identical if Γ is open.

Let $T \in \mathscr{S}'(\Gamma)$. Consider $(e^{-\langle \xi, \cdot \rangle} T)^\wedge$ where $\xi \in \Gamma$. The Fourier transform here is in $\mathscr{S}'_\eta$; for instance, when T is a function f with compact support,

$$(e^{-\langle \xi, \cdot \rangle} T)^\wedge(\eta) = (2\pi)^{-n/2} \int e^{-\langle \xi + i\eta, x \rangle} f(x) dx .$$

<u>Definition</u>. The map: $\xi \rightarrow (e^{-\langle \xi, \cdot \rangle} T)^\wedge$ from Γ into $\mathscr{S}'$ is called the <u>Laplace transform</u> of $T \in \mathscr{S}'(\Gamma)$.

<u>Theorem 2.1</u>. Let Γ be a convex open set in R^n . The Laplace transform of $T \in \mathscr{S}'(\Gamma)$ is a C^∞ map from Γ into $(\mathcal{O}_M)_\eta$ and moreover is a holomorphic function in $\xi + i\eta \in \Gamma + iR^n$ and is bounded by a polynomial in η where ξ varies in a compact subset of Γ . Conversely, if F is a holomorphic function in the cylinder $\Gamma + iR^n$ such that for each compact subset K of Γ , $|F(\xi+i\eta)|$ with $\xi \in K$ is bounded by a polynomial in η , then F is the Laplace transform of a unique distribution $T \in \mathscr{S}'(\Gamma)$.

<u>Proof</u>. See [5], proposition 6 chapter VIII.

CHAPTER III

CONJUGATE FUNCTIONS

When we consider the Laplace transform on $\mathcal{S}'_\varphi$, there are various conditions that φ needs to satisfy. However, in order to study the conjugate function (in the sense of L. C. Young), we begin with convexity and a semicontinuity. Later in this chapter we shall see the effect on its conjugate when φ is further restricted by some increasing condition. Most theorems in this chapter will be useful in the next.

(i) A Basic Theorem

Definition. Denote by $\mathcal{C}$ the space of all lower semicontinuous convex functions on R^n with values in $(-\infty,+\infty]$ but not identically $+\infty$. If $\varphi \in \mathcal{C}$, the conjugate function φ^* of φ is defined by

$$\varphi^*(\xi) = \sup_x [\langle \xi,x\rangle - \varphi(x)] .$$

We now show that φ^* is indeed the "conjugate" of φ , i.e., $\varphi^{**} = \varphi$.

Lemma 3.1. If $\varphi \in \mathcal{C}$, then $\varphi^* \in \mathcal{C}$. Moreover if $\varphi(x_o) < \infty$, then for each $\epsilon > 0$ there exists

some $\eta \in R^n$ such that $\varphi^*(\eta) \leq \epsilon + \langle \eta,x_o\rangle - \varphi(x_o)$.

Proof. Let $A = \left\{(x,t) \in R^{n+1} : \varphi(x) \leq t\right\}$.

Since φ never takes on $-\infty$, A is a proper subset of R^{n+1} ; since φ is not identically $+\infty$, A is nonempty. Furthermore A is convex and closed. For, if (x,t) , $(y,s) \in A$ then

$$\varphi(\lambda x + (1-\lambda)y) \le \lambda\varphi(x) + (1-\lambda)\,\varphi(y)$$
$$\le \lambda t + (1-\lambda)s \quad \text{for} \quad \lambda \in [0,1]$$

and this shows the convexity. Suppose $(x,t) \notin A$. Then $\varphi(x) > t$. Choose $\delta > 0$ so that $t + \delta < \varphi(x)$ and let $U = \left\{y \in R^n : \varphi(y) > t + \delta\right\}$. Since φ is lower semicontinuous, U is open and clearly $U \times (-\infty, t+\delta)$ is disjoint from A . Then every element not in A has a neighborhood disjoint from A , i.e., A is closed.

Suppose $x_o \in R^n$ and $\varphi(x_o) = t_o < \infty$. Let $\epsilon > 0$. Then $(x_o, t_o - \epsilon) \notin A$. By the separation theorem [1] there exists a linear functional L on R^{n+1} such that $L(x_o, t_o - \epsilon) < \inf_A L(x,t)$. Here L must be of the form $L(x,t) = \langle \xi, x\rangle + st$ where $\xi \in R^n$, $s \in R$ and $|\xi| + |s| \neq 0$. Thus $\langle \xi, x_o\rangle + st_o - s\epsilon < \langle \xi, x\rangle + st$ for each $(x,t) \in A$. Taking $x = x_o$ and $t \ge t_o$, we obtain $st_o - s\epsilon < st$ for each $t \ge t_o$. Thus $s > 0$. Now let $\eta = -\xi/s$. Then $-\langle \eta, x_o\rangle + t_o - \epsilon < -\langle \eta, x\rangle + t$. If we set $t = \varphi(x)$, we then have $\langle \eta, x_o\rangle - \varphi(x_o) + \epsilon > \langle \eta, x\rangle - \varphi(x)$ if $\varphi(x) < \infty$. Of course this inequality continues to hold if $\varphi(x) = +\infty$. Thus

$$\varphi^*(\eta) = \sup_x \left[\langle \eta, x\rangle - \varphi(x)\right]$$
$$\le \epsilon + \langle \eta, x_o\rangle - \varphi(x_o) \ .$$

In particular, $\varphi^*(\eta) < +\infty$ and so φ^* is not identically $+\infty$. Being the supremum of a family of affine functions φ^* is lower semicontinuous and convex as well. Hence $\varphi^* \in \mathcal{C}$.

Lemma 3.2. Let A be convex set in R^n. If $x_1 \in \bar{A}$ (closure of A) and $x_1 \notin A$, then there exists $x_o \in A$ such that for $x_t = (1-t)x_o + tx_1$ where $t \geq 0$, $x_t \in A$ if $0 \leq t < 1$ and $x_t \notin \bar{A}$ if $t > 1$.

Proof. Since A is convex, there is an affine subspace H such that $\bar{A} \subseteq H$ and A has nonempty interior relative to H. Since $A \neq \bar{A}$, $\dim H \geq 1$. We shall work entirely in H; so we may assume $H = R^n$, i.e., $\text{int } A \neq \emptyset$.

Choose any $x_o \in \text{int } A$. Suppose $0 < t_o \leq 1$ and $x_{t_o} \notin \text{int } A$. By convexity $x_{t_o} \in \bar{A}$ and thus $x_{t_o} \in$ boundary of A. By the supporting hyperplane theorem [1] there exists $\xi \in R^n$ such that $\langle \xi, x \rangle \leq \langle \xi, x_{t_o} \rangle$ for each $x \in \bar{A}$. In particular, taking $x = x_t$, we have

$$t\langle \xi, x_1 - x_o \rangle \leq t_o \langle \xi, x_1 - x_o \rangle, \quad 0 \leq t \leq 1 .$$

If $0 < t_o < 1$, we must have $\langle \xi, x_1 - x_o \rangle = 0$. But then $\langle \xi, x \rangle = \langle \xi, x_o \rangle$ implies that x_o lies in the supporting hyperplane, which contradicts $x_o \in \text{int } A$. Thus $t_o = 1$ and $\langle \xi, x_1 - x_o \rangle \neq 0$. By the inequality above, we have $\langle \xi, x_1 - x_o \rangle > 0$. Now

$$\langle \xi, x_t \rangle = (1-t)\langle \xi, x_o - x_1 \rangle + \langle \xi, x_1 \rangle$$
$$> \langle \xi, x_1 \rangle \quad \text{if} \quad t > 1 .$$

Thus $x_t \notin \bar{A}$ if $t > 1$. Now the proof is complete.

Theorem 3.3. If $\varphi \in \mathcal{C}$, then $\varphi^{**} = \varphi$.

Proof. Since $\varphi^*(\xi) \geq \langle \xi, x\rangle - \varphi(x)$,

$$\varphi^{**}(x) = \sup_{\xi} \left[\langle \xi, x\rangle - \varphi^*(\xi)\right] \leq \varphi(x) .$$

We have $\varphi^{**} \leq \varphi$.

Suppose $\varphi(x_o) < +\infty$. If $\epsilon > 0$, then by lemma 3.1 there is $\eta \in R^n$ such that $\langle \eta, x_o\rangle - \varphi^*(\eta) \geq \varphi(x_o) - \epsilon$. Thus

$$\begin{aligned}\varphi^{**}(x_o) &= \sup_{\xi} \left[\langle x_o, \xi\rangle - \varphi^*(\xi)\right] \\ &\geq \varphi(x_o) - \epsilon .\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, $\varphi^{**}(x_o) \geq \varphi(x_o)$.

Now let $A = \left\{x \in R^n : \varphi(x) < +\infty\right\}$. Then A is a convex non-empty set and we have shown $\varphi^{**} = \varphi$ on A . By lemma 3.1 $\varphi^* \in \mathcal{C}$ and so $\varphi^*(\eta) < +\infty$ for some $\eta \in R^n$. Let $\varphi^*(\eta) = c$. Then $\langle \eta, x\rangle - \varphi(x) \leq c$ for each $x \in R^n$ and hence if

$$\psi(x) = \varphi(x) + c - \langle \eta, x\rangle ,$$

then $\psi \in \mathcal{C}$ and $\psi \geq 0$. Now $\psi^*(\xi) = \sup \left[\langle \xi+\eta, x\rangle - c - \varphi(x)\right]$

$$= -c + \varphi^*(\xi+\eta)$$

and so

$$\begin{aligned}\psi^{**}(x) &= \sup_{\xi} \left[\langle \xi, x\rangle + c - \varphi^{**}(\xi+\eta)\right] \\ &= \sup_{\xi} \left[\langle \xi-\eta, x\rangle + c - \varphi^*(\xi)\right] \\ &= \varphi^{**}(x) + c - \langle \eta, x\rangle .\end{aligned}$$

Since $\psi(x) = \varphi(x) + c - \langle \eta, x\rangle$ and $\psi \geq 0$, once we show $\psi^{**} = \psi$, we have $\varphi^{**} = \varphi$. Therefore we may assume $\varphi \geq 0$, which we now do.

Define $\theta(x)$ to be 0 if $x \in \bar{A}$ and $+\infty$ if $x \notin \bar{A}$. Obviously $\theta \in \mathcal{C}$ and $\theta \le \varphi$. It follows that $\varphi^* \le \theta^*$ and in turn $\theta^{**} \le \varphi^{**}$. Since $\theta^*(\xi) = \sup_{x \in \bar{A}} \langle \xi, x \rangle$, $\theta^*(t\xi) = t\theta^*(\xi)$ if $t \ge 0$ (where $0 \cdot \infty = 0$). Suppose $x_o \notin \bar{A}$. By the separation theorem there exists $\xi_o \in R^n$ such that $\langle \xi_o, x_o \rangle > \sup_{x \in \bar{A}} \langle \xi_o, x \rangle$, i.e., $\langle \xi_o, x_o \rangle > \theta^*(\xi_o)$. Then

$$\begin{aligned} \theta^{**}(x_o) &= \sup_{\xi} [\langle \xi, x \rangle - \theta^*(\xi)] \\ &= \sup_{\xi} \sup_{t>0} t\,[\langle \xi, x \rangle - \theta^*(\xi)] \\ &\ge \sup_{t>0} t\,[\langle \xi_o, x \rangle - \theta^*(\xi_o)] \\ &= +\infty \quad \text{since } \langle \xi_o, x \rangle - \theta^*(\xi_o) > 0 . \end{aligned}$$

Thus $\varphi^{**}(x_o) \ge \theta^{**}(x_o) = +\infty = \varphi(x_o)$. We have shown $\varphi = \varphi^{**}$ on A and $\varphi = \varphi^{**} = +\infty$ on $R^n \sim \bar{A}$. Finally, suppose $x_1 \in \bar{A}$ but $x_1 \notin A$. Then $\varphi(x_1) = +\infty$. By lemma 3.2 there exists $x_o \in A$ such that if $x_t = (1-t)\, x_o + tx_1$ where $t \ge 0$, then $x_t \in A$ if $0 \le t < 1$ and $x_t \notin \bar{A}$ if $t > 1$. Therefore $\varphi^{**}(x_t) = \varphi(x_t) < +\infty$ if $0 \le t < 1$, $\varphi^{**}(x_1) \le \varphi(x_1) = +\infty$, and $\varphi^{**}(x_t) = \varphi(x_t) = +\infty$ if $t > 1$. It remains to show $\varphi^{**}(x_1) = +\infty$. Suppose $\varphi^{**}(x_1) < +\infty$. If $0 \le t \le 1$, then

$$\begin{aligned} \varphi^{**}(x_t) &\le (1-t)\, \varphi^{**}(x_o) + t\varphi^{**}(x_1) \\ &\le \max\left\{\varphi^{**}(x_o), \varphi^{**}(x_1)\right\} \\ &= c . \end{aligned}$$

Therefore $[0,1) \subseteq \left\{t : \varphi(x_t) \le c\right\}$. Since φ is lower semicontinuous, the set $\left\{t : \varphi(x_t) \le c\right\}$ is closed. Then $\varphi(x_1) < +\infty$, a contradiction.

Hence $\varphi^{**}(x_1) = +\infty$. Now the proof is complete.

(ii) Examples

(1) Let $\varphi(x) = c$. Then

$$\varphi^*(\xi) = \begin{cases} -c & \text{if} \quad \xi = 0 \\ +\infty & \text{if} \quad \xi \neq 0. \end{cases}$$

(2) Let $\varphi(x) = c + \langle \eta, x \rangle$. Then

$$\varphi^*(\xi) = \begin{cases} -c & \text{if} \quad \xi = \eta \\ +\infty & \text{if} \quad \xi \neq \eta. \end{cases}$$

(3) Let $\varphi(x) = a|x|$ where $a \geq 0$. Then

$$\varphi^*(\xi) = \begin{cases} 0 & \text{if} \quad |\xi| \leq a \\ +\infty & \text{if} \quad |\xi| > a. \end{cases}$$

(4) If $\rho \in \mathbf{C}(R)$ and if ρ is decreasing on $(-\infty, 0]$, then

$$\rho^*(\tau) = \sup_{t \geq 0} [\tau t - \rho(t)] \quad \text{for} \quad \tau \geq 0 .$$

Indeed, if $\tau \geq 0$, then $\tau t - \rho(t)$ is increasing on $(-\infty, 0]$.

(5) Let $f \in \mathbf{C}(R)$ and suppose f is increasing on $[0, \infty)$. If $\varphi(x) = f(|x|)$, then $\varphi \in \mathbf{C}(R^n)$. If in addition, f is decreasing on $(-\infty, 0]$, then $\varphi^*(\xi) = f^*(|\xi|)$. Note that if f is not decreasing on $(-\infty, 0]$, this last part may be false. For example, let $f(t) = at$

where $a > 0$ and let $\varphi(x) = f(|x|)$. Then

$$\varphi^*(\xi) = \begin{cases} 0 & \text{if} \quad |\xi| \le a \\ +\infty & \text{if} \quad |\xi| > a \text{ , and} \end{cases}$$

$$f^*(|\xi|) = \begin{cases} 0 & \text{if} \quad |\xi| = a \\ +\infty & \text{if} \quad |\xi| \ne a \text{ .} \end{cases}$$

(6) Let A be a nonempty subset of R^n . <u>The support function</u> H_A of A is defined by $H_A(\xi) = \sup_{x \in A} \langle \xi, x \rangle$. Clearly $H_A(0) = 0$ and so H_A is not identically $+\infty$. For each x , the function $\xi \rightarrow \langle \xi, x \rangle$ is linear and hence continuous and convex. As the supremum of such functions, H_A is lower semicontinuous and convex. Hence $H_A \in \mathcal{C}$.

Note $H_A(t\xi) = tH_A(\xi)$ for $t \ge 0$ (where $0 \cdot \infty = 0$) . From this homogeneity and writing

$$H_A^*(x) = \sup_{\xi} \left\{ \sup_{t > 0} [\langle x, t\xi \rangle - H_A(t\xi)] \right\} ,$$

we have

$$H_A^*(x) = \begin{cases} 0 & \text{if} \quad \langle \xi, x \rangle \le H_A(\xi) \quad \text{for each} \quad \xi \in R^n \\ +\infty & \text{if} \quad \langle \xi, x \rangle > H_A(\xi) \quad \text{for some} \quad \xi \in R^n \text{ .} \end{cases}$$

Moreover, we can show that the set

$$\left\{ x \in R^n : H_A^*(x) = 0 \right\} = \left\{ x \in R^n : \langle \xi, x \rangle \le H_A(\xi) \text{ for each } \xi \in R^n \right\}$$

is the closed convex hull of A . To do so, let A_H be the set above and B the closed convex hull of A . If $x \in A$, then $\langle x, \xi \rangle \le H_A(\xi)$ for each $\xi \in R^n$. Hence $A \subseteq A_H$. As an intersection of closed half spaces, A_H is closed and convex. Therefore $B \subseteq A_H$. Since B is the closure of the set of convex combinations of points in A, the function H_B coincides with H_A. Let $x_o \in R^n \sim B$. It suffices to show $x_o \notin A_H$. Let $d = \inf_{y \in B} |x_o - y|$. By means of orthogonal projection, we can find a unique $y_o \in B$ such that $d = |x_o - y_o|$. Let $\xi = x_o - y_o$. If $y \in B$ and $0 < t < 1$, then $ty_o + (1-t)y_o \in B$ and so

$$|\xi|^2 = d^2 \le |x_o - ty_o - (1-t)y|^2$$
$$= |t\xi + (1-t)(x_o - y)|^2 .$$

Thus $(1-t^2)|\xi|^2 \le (1-t)^2|x_o - y|^2 + 2t(1-t)\langle \xi, x_o - y\rangle$. Cancelling $1-t$ and letting $t \to 1$, we obtain

$$|\xi|^2 \le \langle \xi, y_o - y\rangle = |\xi|^2 + \langle \xi, y_o - y\rangle .$$

Hence $\langle \xi, y_o - y\rangle \ge 0$ for each $y \in B$. Since $\langle \xi, y_o\rangle \ge \langle \xi, y\rangle$ for each $y \in B$,

$$H_A(\xi) = H_B(\xi) = \sup_{y \in B} \langle \xi, y\rangle = \langle \xi, y_o\rangle .$$

Finally,

$$\langle \xi, x_o\rangle = \langle \xi, \xi + y_o\rangle$$
$$= \langle \xi, y_o\rangle + |\xi|^2$$
$$= H_A(\xi) + d^2$$
$$> H_A(\xi) .$$

Therefore $x_o \notin A_H$.

(7) The process in (6) may be reversed. Let $H \in \mathcal{C}$ and suppose $H(t\xi) = tH(\xi)$ for each $\xi \in R^n$ and $t \geq 0$ (again $0 \cdot \infty = 0$). Then there exists a nonempty closed convex set A such that

$$H^*(x) = \begin{cases} 0 & \text{if} \quad x \in A \\ +\infty & \text{if} \quad x \notin A \end{cases} .$$

Here the set A of course is the set A_H given in (6). To see that it is nonempty, let ξ_o be the point on the unit sphere S^{n-1} where, by virtue of lower semicontinuity, H attains its minimum. By homogeneity, $|\xi|\ H(\xi_o) \leq H(\xi)$ for all $\xi \in R^n$. If $H(\xi_o) = +\infty$, then $H = +\infty$ and so $A = R^n$. If $H(\xi_o) < +\infty$, then let $x_o = \xi_o H(\xi_o)$ and thus, for any $\xi \in R^n$

$$\begin{aligned} \langle x_o, \xi \rangle &= H(\xi_o)\langle \xi_o, \xi \rangle \\ &\leq H(\xi_o)|\xi| \\ &\leq H(\xi) . \end{aligned}$$

Hence $x_o \in A$.

By theorem 3.3,
$$\begin{aligned} H(\xi) &= H^{**}(\xi) \\ &= \sup_{x \in A} \langle \xi, x \rangle \\ &= H_A(\xi) . \end{aligned}$$

Hence $H \in \mathcal{C}$ is the support function of a closed convex nonempty set A if and only if $H(t\xi) = tH(\xi)$, $\xi \in R^n$, $t \geq 0$.

(8) Let $\varphi(x) = \frac{1}{p}|x|^p$ where $p > 1$. Then $\varphi \in C$ and $\varphi^*(\xi) = \frac{1}{q}|\xi|^q$ where $\frac{1}{p} + \frac{1}{q} = 1$. It is obvious that $\varphi \in C$ and $\varphi^*(0) = 0$. To compute φ^*, fix $\xi \in R^n$, $\xi \neq 0$, and consider $g(x) = \langle \xi, x \rangle - \frac{1}{p}|x|^p$. Note that $g(x) \leq |x|(|\xi| - \frac{1}{p}|x|^{p-1}) \to -\infty$ as $|x| \to +\infty$. Hence g has a maximum. Now $g(0) = 0$ and if $x \neq 0$ then $\frac{\partial g}{\partial x_j}(x) = \xi_j - x_j|x|^{p-2}$. Thus $dg(x) = 0$ only if $x|x|^{p-2} = \xi$. At this point, we have

$$g(x) = \langle x|x|^{p-2}, x \rangle - \frac{1}{p}|x|^p$$
$$= (1 - \frac{1}{p})|x|^p$$

and then from $|x|^p = (|x|^{p-1})^{p/(p-1)} = |\xi|^{1/(1-p)} = |\xi|^q$. We have $g(x) = \frac{1}{q}|\xi|^q > 0$, which shows that $\xi = x|x|^{p-2}$ is the maximum.

(iii) <u>Growth Conditions and Convexity</u>

<u>Theorem 3.4</u>. Let $\varphi \in C$. Then $\varphi^*(\xi) < +\infty$ for each $\xi \in R^n$ if and only if

$$\liminf_{|x| \to +\infty} \frac{\varphi(x)}{|x|} = +\infty .$$

<u>Proof</u>. Suppose $\varphi^*(\xi) < +\infty$ for each $\xi \in R^n$. Since φ^* is convex by lemma 3.1, it follows that φ^* is continuous. If $\lambda > 0$, let m_λ be the maximum of φ^* on the sphere $|\xi| = \lambda$. By theorem 3.3,

$\varphi(x) \geq \langle \xi, x \rangle - \varphi^*(\xi)$ for each $\xi \in R^n$. Taking $\xi = \frac{\lambda x}{|x|}$ we obtain $\varphi(x) \geq \lambda|x| - m_\lambda$. Thus $\liminf_{|x| \to \infty} \frac{\varphi(x)}{|x|} \geq \lambda$ where $\lambda > 0$ is arbitrary. This proves the "only if" part.

Suppose

$$\liminf_{|x| \to +\infty} \frac{\varphi(x)}{|x|} = +\infty .$$

Let $\lambda > 0$. Then for $|x| > R_\lambda > 0$ we have $\varphi(x) \geq \lambda|x|$. Since φ is lower semicontinuous, there is some constant A_λ such that for $|x| \leq R_\lambda$ we have $\varphi(x) \geq A_\lambda$. Choose any number $a_\lambda \geq 0$ such that $\lambda R_\lambda - a_\lambda \leq A_\lambda$. Then $|x| > R_\lambda$ implies $\varphi(x) \geq \lambda|x| \geq \lambda|x| - a_\lambda$ and $|x| \leq R_\lambda$ implies

$$\begin{aligned} \varphi(x) &\geq A_\lambda \\ &\geq \lambda R_\lambda - a_\lambda \\ &\geq \lambda|x| - a_\lambda . \end{aligned}$$

Hence, if $|\xi| \leq \lambda$,

$$\begin{aligned} \varphi^*(\xi) &= \sup_x [\langle \xi, x \rangle - \varphi(x)] \\ &\leq \sup_x [(|\xi| - \lambda)|x| + a_\lambda] \\ &\leq a_\lambda . \end{aligned}$$

<u>Theorem 3.5</u>. Let $\varphi \in C$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\epsilon > 0$. Then

$$\liminf_{|x| \to +\infty} \frac{\varphi(x)}{|x|^p} \geq \frac{1}{p} \epsilon^p$$

if and only if

$$\limsup_{|\xi| \to +\infty} \frac{\varphi^*(\xi)}{|\xi|^q} \leq \frac{1}{q} \epsilon^{-q} .$$

Proof. Assume

$$\liminf_{|x|\to+\infty} \frac{\varphi(x)}{|x|^p} \geq \frac{1}{p}\epsilon^p \quad .$$

Let $0 < \lambda < \epsilon$. As in the proof of theorem 3.4, the lower semi-continuity implies $\varphi(x) \geq \frac{1}{p}\lambda^p|x|^p - a$ where $a \geq 0$ is some constant depending on λ . But

$$\langle \xi,x\rangle = \langle \lambda^{-1}\xi,\lambda x\rangle \leq \frac{1}{p}\lambda^p\ |x|^p + \frac{1}{q}\lambda^{-q}\ |\xi|^q$$

by the fact $\langle \xi,x\rangle \leq \varphi^*(\xi) + \varphi(x)$ and by example (8). Therefore $\langle \xi,x\rangle - \varphi(x) \leq \frac{1}{q}\lambda^{-q}\ |\xi|^q + a$. Then $\varphi^*(\xi) \leq \frac{1}{q}\lambda^{-q}\ |\xi|^q + a$ and so

$$\limsup_{|\xi|\to+\infty} \frac{\varphi^*(\xi)}{|\xi|^q} \leq \frac{1}{q}\lambda^{-q} \quad .$$

Letting $\lambda \uparrow \epsilon$, we complete the "only if" part.

Assume

$$\limsup_{|\xi|\to+\infty} \frac{\varphi^*(\xi)}{|\xi|^p} \leq \frac{1}{q}\epsilon^{-q} \quad .$$

Let $0 < \lambda < \epsilon$. Then for all large ξ , $\varphi^*(\xi) \leq \frac{1}{q}\lambda^{-q}\ |\xi|^q$. In particular, φ^* is finite in the complement of a ball and thus by convexity is finite everywhere. Then φ^* is continuous and so for some suitable constant a depending on λ , we have $\varphi^*(\xi) \leq \frac{1}{q}\lambda^{-q}\ |\xi|^q + a$ for each $\xi \in R^n$. Therefore

$$\begin{aligned}\varphi(x) &= \varphi^{**}(x)\\ &= \sup_\xi[\langle x,\xi\rangle - \varphi^*(\xi)]\\ &\geq -a + \lambda^{-q} \sup_\xi[\langle \xi,\lambda^q x\rangle - \frac{1}{q}|\xi|^q]\\ &= -a + \lambda^{-q}\frac{1}{p}|\lambda^q x|^p \quad \text{(by example 8)}\\ &= -a + \lambda^{-q+pq}\frac{1}{p}|x|^p\\ &= -a + \lambda^p \frac{1}{p}|x|^p .\end{aligned}$$

Thus

$$\limsup_{|x|\to+\infty} \frac{\varphi(x)}{|x|^p} \geq \frac{1}{p}\lambda^p .$$ Letting $\lambda \uparrow \epsilon$ completes the proof.

Corollary 3.6. Let $\varphi \in C$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\liminf_{|x|\to+\infty} \frac{\varphi(x)}{|x|^p} = +\infty$$

if and only if

$$\limsup_{|\xi|\to+\infty} \frac{\varphi^*(\xi)}{|\xi|^q} \leq 0 .$$

(iv) Differentiability and Convexity

Lemma 3.7. If $\varphi \in C$ is finite everywhere, then there exists a map $h: R^n \to R^n$ such that $\varphi^*\big(h(x)\big) + \varphi(x) = \langle h(x),x\rangle$ for each $x \in R^n$.

Proof. By theorem 3.4,

$$\liminf_{|\xi|\to+\infty} \frac{\varphi^*(\xi)}{|\xi|} = +\infty .$$

Fix $x \in R^n$. For large $r > 0$, we have $\varphi^*(\xi) \geq (|x| + 1)|\xi|$ for $|\xi| \geq r$; so $\langle \xi,x \rangle - \varphi^*(\xi) \leq -|\xi| \leq -r$ for $|\xi| \geq r$. Now if we choose r large enough so that $-r < \varphi(x)$, we obtain

$$\varphi(x) = \sup_{|\xi| \leq r} [\langle \xi,x \rangle - \varphi^*(\xi)] .$$

Since φ^* is lower semicontinuous, the function $\xi \to \langle \xi,x \rangle - \varphi^*(\xi)$ is upper semicontinuous and hence achieves its maximum on the ball $|\xi| \leq r$. Let $h(x)$ be a point where the maximum is achieved. Then $\varphi(x) = \langle h(x),x \rangle - \varphi^*\big(h(x)\big)$. The maximum cannot be $-\infty$; for, otherwise, $\varphi^* = +\infty$. Hence $\varphi^*\big(h(x)\big)$ is finite and the result follows.

<u>Remark</u>. If we set $\eta = h(x)$, then by the lemma

$$\begin{aligned}\langle \eta,y \rangle - \varphi(y) &\leq \varphi^*(\eta)\\ &= \langle \eta,x \rangle - \varphi(x) \quad \text{for each } y \in R^n .\end{aligned}$$

Equivalently, $\langle \eta, y - x \rangle \leq \varphi(y) - \varphi(x)$ for each $y \in R^n$. Such an η is called a <u>subgradient</u> of φ at x . Note that η is a subgradient of φ at x if and only if $\varphi^*(\eta) + \varphi(x) = \langle \eta,x \rangle$.

If φ is differentiable at x , then the gradient of φ at x is denoted by $d\varphi(x)$, i.e.,

$$d\varphi(x) = \left(\frac{\partial\varphi}{\partial x_1}(x) , \frac{\partial\varphi}{\partial x_2}(x) , \ldots , \frac{\partial\varphi}{\partial x_n}(x)\right) .$$

Theorem 3.8. Let $\varphi \in \mathcal{C}$ be finite everywhere. If φ is differentiable at x^o, then $d\varphi(x^o)$ is a subgradient of φ at x^o. In particular, $\varphi^*\left(d\varphi(x^o)\right) = \langle d\varphi(x^o), x^o\rangle - \varphi(x^o)$.

Proof. By lemma 3.7 and the remark above, $\langle \eta, x-x^o\rangle \le \varphi(x) - \varphi(x^o)$ for each $x \in R^n$, where η is a subgradient of φ at x^o. Then, if $x \neq x^o$,

$$\frac{1}{|x-x^o|}\langle \eta - d\varphi(x^o), x-x^o\rangle$$

$$\le \frac{1}{|x-x^o|}\left[\varphi(x) - \varphi(x^o) - \sum_{j=1}^{n} \frac{\partial\varphi}{\partial x_j}(x^o)(x_j - x_j^o)\right] .$$

Let $\xi \in S^{n-1}$ and let $x = t\xi + x^o$ and define $g(t) = \varphi(t\xi + x^o)$. Note $g(0) = \varphi(x^o)$ and

$$g'(0) = \sum_{j=1}^{n} \frac{\partial\varphi}{\partial x_j}(x^o)\xi_j .$$

Then the inequality above becomes

$$\langle \eta - d\varphi(x^o), \xi\rangle \le \frac{1}{|t|}[g(t) - g(0) - g'(0)t] , \quad t \neq 0 .$$

As $t \to 0$ we obtain $\langle \eta - d\varphi(x^o), \xi\rangle \le 0$ for each $\xi \in S^{n-1}$. It follows that $\eta = d\varphi(x^o)$.

Example (9). In example (8) we have, for $x \neq 0$, $h(x) = x|x|^{p-2} = d\varphi(x)$. Indeed, $\varphi^*(x|x|^{p-2}) = \frac{1}{q}|x|^{pq-q}$

$$= \frac{1}{q}|x|^p = \langle |x|^{p-2}x, x\rangle - \frac{1}{p}|x|^p$$

$$= \langle h(x), x\rangle - \varphi(x) .$$

Remark. A function φ of class $C^2(R^n)$ is said to be strictly convex on R^n if the matrix

$$\left[\frac{\partial^2\varphi}{\partial x_j \partial x_k}(x)\right] ,$$

called the Hessian of φ, is positive definite, i.e., if

$$\sum_{j,k} \frac{\partial^2\varphi}{\partial x_j \partial x_k}(x)\, \xi_j \xi_k > 0$$

for each $x \in R^n$ and each $\xi \in R^n$, $\xi \neq 0$. In this case, if $\lambda(x)$ is the lowest eigenvalue of the Hessian, then we can easily see that λ is a positive continuous function and

$$\sum_{j,k} \frac{\partial^2\varphi}{\partial x_j \partial x_k}(x)\, \xi_j \xi_k \geq \lambda(x)\, |\xi|^2 .$$

Lemma 3.9. If $\varphi \in C^2(R^n)$ is strictly convex, then $d\varphi: R^n \to R^n$ is one-to-one and $\varphi^*\big(d\varphi(x)\big) = \langle d\varphi(x), x\rangle - \varphi(x)$ for each $x \in R^n$.

Proof. Let $x^o, x' \in R^n$, $x^o \neq x'$, and let $x^t = (1-t)x^o + tx'$ and define $g(t) = \varphi(x^t)$. Then

$$g'(t) = \sum_j \frac{\partial\varphi}{\partial x_j}(x^t)(x'_j - x^o_j)$$

$$= \langle d\varphi(x^t), x' - x^o\rangle \quad \text{and}$$

$$g''(t) = \sum_{j,k} \frac{\partial^2\varphi}{\partial x_j \partial x_k}(x^t)(x'_j - x^o_j)(x'_k - x^o_k)$$

$$\geq \lambda(x^t)|x' - x^o|^2 > 0 .$$

So $g'(t)$ is strictly increasing on $[0,1]$ and hence $g'(0) < g'(1)$, i.e., $\langle d\varphi(x^o), x'-x^o\rangle < \langle d\varphi(x'), x'-x^o\rangle$. Hence $d\varphi(x^o) \neq d\varphi(x')$ and this shows that $d\varphi$ is one-to-one. The last part is theorem 3.8.

<u>Theorem 3.10</u>. If $k \geq 2$, $\varphi \in C^k(R^n)$ is strictly convex and if

$$\liminf_{|x| \to \infty} \frac{\varphi(x)}{|x|} = +\infty ,$$

then (1) $d\varphi : R^n \to R^n$ is a C^{k-1} diffeomorphism <u>onto</u> R^n ;

(2) $d\varphi^* = (d\varphi)^{-1}$

(3) $\varphi^* \in C^k(R^n)$

(4) φ^* is strictly convex.

<u>Proof</u>.

(1) The Jacobian of $d\varphi$ at x is

$$\det\left[\frac{\partial^2\varphi}{\partial x_j \partial x_k}(x)\right] \geq \big(\lambda(x)\big)^n > 0 \quad .$$

By the inverse function theorem and lemma 3.9, $d\varphi$ is a C^{k-1} diffeomorphism onto an open subset of R^n . By lemma 3.9, $\langle d\varphi(x), y-x\rangle \leq \varphi(y) - \varphi(x)$ for each $x,y \in R^n$. Taking $y = 0$ yields $\varphi(x) - \varphi(0) \leq \langle d\varphi(x), x\rangle$ and thus $\frac{\varphi(x)}{|x|} \leq \frac{\varphi(0)}{|x|} + |d\varphi(x)|$. By hypothesis, $|d\varphi(x)| \to \infty$ as $|x| \to \infty$. Hence $d\varphi$ is proper, i.e., $(d\varphi)^{-1}K$ is compact whenever K is compact. Thus $d\varphi$ has closed image as well. Therefore $d\varphi$ is onto. By lemma 3.8,

$$\varphi^*\big(d\varphi(x)\big) = \langle d\varphi(x), x\rangle - \varphi(x) \quad ,$$

which shows $\varphi^* \in C^{k-1}(R^n)$.

(2) Let $\eta \in R^n$. By (1) $d\varphi(x) = \eta$ for some $x \in R^n$ and then by lemma 3.9 $\varphi^*(\eta) = \langle \eta, x \rangle - \varphi(x)$. Equivalently $\varphi(x) = \langle \eta, x \rangle - \varphi^*(\eta)$. Since $\varphi = \varphi^{**}$, x is a subgradient of φ^* at η . Now the same proof of theorem 3.8 shows that $d\varphi^*(\eta) = x$.

(3) By (1) and (2), $d\varphi^* \in C^{k-1}(R^n)$. Thus $\varphi^* \in C^k(R^n)$.

(4) We have $x = d\varphi^*\big(d\varphi(x)\big)$, i.e.,

$$x_j = \frac{\partial \varphi^*}{\partial \xi_j} \big(d\varphi(x)\big) .$$

Hence differentiation gives

$$\delta_{ij} = \sum_k \frac{\partial^2 \varphi^*}{\partial \xi_j \partial \xi_k} \; \frac{\partial \varphi}{\partial x_j \partial x_k} .$$

Then

$$\left[\frac{\partial^2 \varphi^*}{\partial \xi_j \partial \xi_k} (\xi) \right] = \left[\frac{\partial^2 \varphi}{\partial x_j \partial x_k} (x) \right]^{-1} .$$

Since the inverse of a positive definite matrix is positive definite and since $d\varphi$ maps R^n onto R^n, φ^* is strictly convex.

Example 10. Consider the same φ as in examples (8) and (9): $\varphi(x) = \frac{1}{p} |x|^p$, $p > 1$. For $x \neq 0$, φ is strictly convex; because the Hessian H of φ has positive eigenvalues for $x \neq 0$ and therefore is positive definite. To compute the eigenvalues of

$$H = \left[\frac{\partial^2 \varphi}{\partial x_j \partial x_k} (x) \right] ,$$

we have

$$\frac{\partial^2 \varphi}{\partial x_j \partial x_k}(x) = |x|^{p-2} x_j + (p-2)|x|^{p-4} x_j x_k .$$

Choose an orthonormal basis as follows: $v_1 = x^t \neq 0$, $v_j \perp x^t$, $j = 2,\dots,n$, where x^t = transpose of x . Now H is of the form $\alpha I + \beta x^t x$. Then for $j \geq 2$, $Hv_j = \alpha v_j + \beta(x^t x)v_j$

$$= \alpha v_j + \beta x^t (xv_j)$$

$$= \alpha v_j$$

and

$$Hv_1 = \alpha x^t + \beta(x^t x)x^t$$

$$= \alpha x^t + \beta x^t (xx^t) = (\alpha + \beta|x|^2)x^t .$$

Hence $\alpha = |x|^{p-2}$ is an eigenvalue of multiplicity $n-1$ and $\alpha + \beta|x|^2 = (p-1)|x|^{p-2}$ an eigenvalue of multiplicity 1.

We have $d\varphi(x) = x|x|^{p-2}$, $x \neq 0$, and $d\varphi^*(\xi) = \xi|\xi|^{q-2}$, $\xi \neq 0$. Clearly $d\varphi$ is a C^∞ diffeomorphism of $R^n - (0)$ onto itself. Furthermore, $d\varphi\left(d\varphi^*(\xi)\right) = d\varphi(\xi|\xi|^{q-2}) = \xi|\xi|^{q-2}|\xi|^{(p-2)(q-1)}$, but $q-2+(p-2)(q-1) = pq-q-p = 0$, so $d\varphi\left(d\varphi^*(\xi)\right) = \xi$. By symmetry $d\varphi^*\left(d\varphi(x)\right) = x$. Thus $d\varphi^* = (d\varphi)^{-1}$. Thus the hypotheses of theorem 3.10 in so far as conclusion (2) is concerned are overly restrictive.

CHAPTER IV

LAPLACE TRANSFORM ON $\mathscr{S}'_\varphi$

(i) Preliminary Lemmas

Lemma 4.1. Let φ be a lower semicontinuous convex function on R^n such that $\varphi(0) < +\infty$ and that

$$\liminf_{|x|\to\infty} \frac{\varphi(x)}{|x|^p} > 0$$

for some $p > 1$. Then $\varphi^*(\xi) < +\infty$ for each $\xi \in R^n$. Moreover, for each integer $m \geq 0$ there exist constants C and N such that

$$(1 + |x|^2)^m e^{\langle \xi, x\rangle - \varphi(x)} \leq C(1 + |\xi|^2)^N e^{\varphi^*(\xi)}$$

for each $x, \xi \in R^n$.

Proof. The increasing condition here easily implies that in theorem 3.4 as may be seen as follows:

$$\liminf_{|x|\to+\infty} \frac{\varphi(x)}{|x|} = \sup_r \inf_{|x|\geq r} \frac{\varphi(x)}{|x|^p} |x|^{p-1}$$

$$\geq \sup_r \inf_{|x|\geq r} \frac{\varphi(x)}{|x|^p} \cdot \sup_r \inf_{|x|\geq r} |x|^{p-1}$$

$$= +\infty .$$

Thus by theorem 3.4, φ^* is finite everywhere. As in Section (iii) of chapter III, the lower semicontinuity together with the increasing condition implies that there are constants $a,b > 0$ such that

$\varphi(x) \geq b|x|^p - a$. Then $\langle \xi,x\rangle - \varphi(x) + m \log(1 + |x|^2)$

$$\leq a + (|\xi| + 2m - b|x|^{p-1})\ |x|$$

$$\leq 0 \text{ if } |x| \geq r(\xi) \text{ where}$$

$$r(\xi) = \max\left\{a, \left(\frac{1+2m+|\xi|}{b}\right)^{1/(p-1)}\right\}.$$

Now $\varphi^*(\xi) = \sup_x [\langle \xi,x\rangle - \varphi(x)] \geq -\varphi(0)$ and therefore for $|x| \geq r(\xi)$ we have

$$(1+|x|^2)^m\, e^{\langle \xi,x\rangle - \varphi(x)} \leq 1 \leq e^{\varphi(0)} \cdot e^{\varphi^*(\xi)} .$$

If $|x| \leq r(\xi)$, from the definition of $r(\xi)$ we can find constants C_1 and $N > 0$ so that $(1+|x|^2)^m \leq C_1(1+|\xi|^2)^N$. Since

$$\langle \xi,x\rangle - \varphi(x) \leq \varphi^*(\xi) \text{ for all } x,\xi \in R^n ,$$

we obtain

$$(1+|x|^2)^m\, e^{\langle \xi,x\rangle - \varphi(x)} \leq (C_1 + e^{\varphi(0)})(1+|\xi|^2)^N\, e^{\varphi^*(\xi)} .$$

<u>Lemma 4.2</u>. Let φ be a lower semicontinuous convex function on R^n, not identically $+\infty$ and with

$$\liminf_{|x|\to\infty} \frac{\varphi(x)}{|x|} = +\infty .$$

Then $\varphi^*(\xi) < +\infty$ for each $\xi \in R^n$. Moreover, for each compact set K in R^n and each integer $m \geq 0$ there is a constant $C_{m,K} > 0$ such that $(1+|x|^2)^m\, e^{\langle \xi,x\rangle - \varphi(x)} \leq C_{m,K}$ for each $\xi \in K$ and each $x \in R^n$.

<u>Proof</u>. Again by theorem 3.4, φ^* is finite everywhere. Choose $R > 0$ so that $\xi \in K$ implies $|\xi| \leq R - m$. Again by the lower semicontinuity and the growth condition there is $a > 0$ such that

$$\varphi(x) \geq R|x| - a ,\ x \in R^n .$$

Then for $\xi \in K$, $x \in R^n$,

$$\langle \xi, x \rangle - \varphi(x) \leq a + (|\xi| - R)|x|$$
$$\leq a - m|x| \ .$$

Thus for $\xi \in K$, $x \in R^n$, we have

$$(1+|x|^2)^m e^{\langle \xi, x \rangle - \varphi(x)} \leq (1+|x|^2)^m e^{a-m|x|}$$
$$\leq 2^m e^a \ .$$

Definition. A function φ on R^n is said to satisfy hypothesis (H) if

(1) $\varphi \in \mathcal{O}_M$

(2) φ is convex

(3) $\liminf_{|x| \to \infty} \frac{\varphi(x)}{|x|} = +\infty$.

The function φ is said to satisfy hypothesis (H′) if φ satisfies (1), (2), and the stronger (3′)

$$\liminf_{|x| \to \infty} \frac{\varphi(x)}{|x|^p} > 0 \ ,$$

for some $p > 1$. So (H′) implies (H) .

Lemma 4.3. If φ satisfies (H) , then $e^{\langle z, \cdot \rangle - \varphi}$ belongs to $\mathcal{S}$ for each $z \in C^n$ and is a holomorphic function of z with values in $\mathcal{S}$. Moreover, for each compact set K in R^n and each continuous seminorm p on $\mathcal{S}$, there are constants C , N such that $p(e^{\langle z, \cdot \rangle - \varphi}) \leq C(1+|\eta|^2)^N$ for each $\xi \in K$ where $z = \xi + i\eta$ and N does not depend on K .

Proof. Since $\varphi \in \mathcal{O}_M$,

$$(1+|x|^2)^k \left| D^\alpha \left(e^{\langle z,x\rangle-\varphi(x)}\right)\right|$$

$$\leq C(1+|z|)^N (1+|x|^2)^m e^{\langle \xi,x\rangle-\varphi(x)} \qquad (*).$$

By lemma 4.2, $e^{\langle z,\cdot\rangle-\varphi} \in \mathcal{S}$ and in addition the last part follows. To prove the holomorphicity, define $f: C^n \to \mathcal{S}$ by $f(z) = e^{\langle z,\cdot\rangle-\varphi}$, and let

$$h(w,z) = f(w) - f(z) - \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(z)\,(w_j - z_j) .$$

It suffices to show

$$\lim_{w \to z} \frac{h(w,z)}{|w - z|} = 0 \quad \text{in } \mathcal{S} .$$

Now $h(w,z)(x)$

$$= e^{\langle z,x\rangle-\varphi(x)} \left(e^{\langle w-z,x\rangle} - 1 - \langle w-z,x\rangle\right) .$$

We shall make use of the following inequalities for $\tau \in C$:

$e^\tau - 1 = \tau \int_0^1 e^{\tau t}\,dt$ implies

$|e^\tau-1| \leq |\tau|(1 + e^{Re\tau})$;

$e^\tau - 1 - \tau = \tau \int_0^1 (e^{\tau t} - 1)dt$ implies

$|e^\tau-1-\tau| \leq 2|\tau|^2(1 + e^{Re\tau})$.

It follows that if we restrict w to be in, say, the ball of radius $\frac{1}{2}$ and center at z , then

$$|e^{\langle w-z,x\rangle} - 1 - \langle w - z,x\rangle|$$

$$\leq C(1+|x|^2)\ |w-z|^2\ (1 + e^{|x|/2}) ,$$

$$\left|\frac{\partial}{\partial x_j} (e^{\langle w-z,x\rangle} - 1 - \langle w-z,x\rangle)\right|$$

$$= |w_j-z_j|\ |e^{\langle w-z,x\rangle} - 1|$$

$$\leq |w-z|^2\ (1+|x|^2)(1 + e^{\frac{1}{2}|x|}) ,$$

and for $|\beta| \geq 2$,

$$|D^\beta(e^{\langle w-z,x\rangle} - 1 - \langle w-z,x\rangle)|$$

$$\leq C_\beta(1 + e^{\frac{1}{2}|x|})\ |w-z|^2$$. Thus for any β ,

$$|D^\beta(e^{\langle w-z,x\rangle} - 1 - \langle w-z,x\rangle|$$

$$\leq C_\beta(1+|x|^2)(1 + e^{\frac{1}{2}|x|})\ |w-z|^2 \quad \text{whenever } |w-z| \leq \frac{1}{2} . \qquad (**)$$

Now for each fixed $z \in C^n$, we can find, as in lemma 4.2 and before, some constant a such that

$$\varphi(x) \geq (|z| + 1)|x| - a .$$

Then $\mathrm{Re}\langle z,x\rangle - \varphi(x) \leq a - |x|$. Hence by (*)

$$(1+|x|^2)^k \left|D^\alpha(e^{\langle z,x\rangle-\varphi(x)})\right|$$

$$\leq C_{k,\gamma}(1+|x|^2)^m\ e^{-|x|}$$. Finally by Leibnitz formula and (**),

$$\left|(1+|x|^2)^k \, D^\alpha h(w,z)\right|$$

$$\leq C'_{k,\alpha}(1+|x|^2)^{m+1} \, (e^{-|x|} + e^{-\frac{1}{2}|x|}) \, |w-z|^2$$

$$\leq C''_{k,\alpha} \, |w-z|^2 \ . \quad \text{Therefore}$$

$$\lim_{w \to z} \frac{h(w,z)}{|w-z|} = 0 \quad \text{in } \mathscr{S} \ .$$

Lemma 4.4. If φ satisfies (H′), then $e^{\langle z,\cdot\rangle - \varphi}$ belongs to $\mathscr{S}$ for every $z \in C^n$ and is a holomorphic function in z with values in $\mathscr{S}$. Moreover, for each continuous seminorm p on $\mathscr{S}$ there exist constants C and N such that

$$p(e^{\langle z,\cdot\rangle - \varphi}) \leq C(1+|z|)^N \, e^{\varphi^*(-\xi)}$$

for $z \in C^n$ where $z = \xi + i\eta$.

Proof. The first part follows from lemma 4.3. For the last part, since $\varphi \in \mathcal{O}_M$, $(1+|x|^2)^k \left|D^\alpha(e^{\langle z,x\rangle - \varphi(x)})\right|$

$$\leq C'(1+|z|)^{N'}(1+|x|^2) \, e^{\langle \xi,x\rangle - \varphi(x)}$$

$$\leq C''(1+|z|)^{N'}(1+|\xi|^2)^{N''} \, e^{\varphi^*(-\xi)}$$

$$\leq C(1+|z|)^N \, e^{\varphi^*(-\xi)}$$

where we have used lemma 4.1.

(ii) The Space $\mathscr{S}'_\varphi$ and Laplace Transform

Definition. Suppose φ satisfies (H) .

Define $\mathscr{S}'_\varphi = \left\{ T \in \mathscr{D}' : e^{\varphi} T \in \mathscr{S}' \right\}$.

If $z \in C^n$ and $T \in \mathscr{S}'_\varphi$, then $e^{-\langle z,\cdot\rangle}T = e^{-\langle z,\cdot\rangle-\varphi}(e^{\varphi}T)$ and by lemma 4.3 this is the product of a function in $\mathscr{S}$ with a distribution in $\mathscr{S}'$. Thus $e^{-\langle z,\cdot\rangle}T \in \mathcal{O}'_c$ for any $z \in C^n$ and $T \in \mathscr{S}'_\varphi$. In particular, taking $z=0$, we have $\mathscr{S}'_\varphi \subseteq \mathcal{O}'_c$. By theorem 2.1 (with $\Gamma = R^n$ and $\mathcal{O}'_c(\Gamma) = \mathscr{S}'(\Gamma)$), the Laplace transform of $T \in \mathscr{S}'_\varphi$ is an entire function F and $F(\xi+i\eta) = (e^{-\langle \xi,\cdot\rangle}T)^\wedge(\eta)$.

<u>Theorem 4.5</u>. If φ satisfies hypothesis (H), $T \in \mathscr{S}'_\varphi$ and if F is the Laplace transform of T, then $F(z) = \langle e^{\varphi}T, e^{-\langle z,\cdot\rangle-\varphi}\rangle$. Moreover there exists a constant N such that for each compact set K in R^n there is a constant C_K such that

$$|F(z)| \leq C_K(1+|\eta|)^N \quad \text{where} \quad z = \xi + i\eta,\ \xi \in K .$$

<u>Proof</u>. Since $e^{\varphi}T \in \mathscr{S}'$, there exists a continuous seminorm p on $\mathscr{S}$ so that

$$|\langle e^{\varphi}T, u\rangle| \leq p(u) \ , \ u \in \mathscr{S} . \tag{1}$$

By lemma 4.3 the last part follows from the first and (1). To prove the first part, choose $\theta \in \mathscr{D}$ with $\theta(x) = 1$ for $|x| \leq 1$ and define $\theta_k(x) = \theta(k^{-1}x)$, $k=1,2,\ldots$. By Leibnitz formula we can show $\theta_k u \rightarrow u$ in $\mathscr{S}$ for each $u \in \mathscr{S}$. (The proof is exactly the same way as proving that $\mathscr{D}$ is dense in $\mathscr{S}$; see, for example, [2] page 135.) Therefore

$$\theta_k e^{\varphi}T \rightarrow e^{\varphi}T \quad \text{weak}^* \text{ in } \mathscr{S}' . \tag{2}$$

Now let $G(z) = \langle e^{\varphi}T, e^{-\langle z,\cdot\rangle - \varphi}\rangle$,

$$G_k(z) = \langle \theta_k e^{\varphi}T, e^{-\langle z,\cdot\rangle - \varphi}\rangle \ .$$

Note that G and G_k are holomorphic functions by lemma 4.3. Thus by (2)

$$G_k(z) \to G(z) \quad \text{for each} \quad z \in C^n \ . \tag{3}$$

Since $|D^{\alpha}\theta_k| \le k^{-|\alpha|} \sup |D^{\alpha}\theta|$, clearly there exists a continuous seminorm q on $\mathscr{S}$ such that $p(\theta_k u) \le q(u)$ for each k and each $u \in \mathscr{S}$. Thus by (1)

$$|G(z)| \le p(e^{-\langle z,\cdot\rangle - \varphi}) \ ,$$

$$|G_k(z)| \le p(\theta_k e^{-\langle z,\cdot\rangle - \varphi}) \le q(e^{-\langle z,\cdot\rangle - \varphi}) \ .$$

Since $\max(p,q)$ is a continuous seminorm on $\mathscr{S}$, by lemma 4.3 for each $\xi \in R^n$ there is a constant C_{ξ} such that

$$\left.\begin{aligned} &|G(z)| \le C_{\xi}(1+|\eta|)^N \ , \\ &|G_k(z)| \le C_{\xi}(1+|\eta|)^N \ , \quad \text{where} \quad z = \xi + i\eta \ . \end{aligned}\right. \tag{4}$$

We will show $G = F$. If $u \in \mathscr{S}$, then $|u(\eta)| \le C(1+|\eta|)^{-N-n-1}$. Hence by (3), (4) and the Lebesque dominated convergence theorem,

$$G_k(\xi + i\eta) \to G(\xi + i\eta) \quad \text{weak}^* \quad \text{in} \ \mathscr{S}'_{\eta} \ .$$

Now let F_k be the Laplace transform of $\theta_k e^{\varphi}T$. Then

$$F_k(\xi + i\eta) = (e^{-\langle \xi,\cdot\rangle}\theta_k e^{\varphi}T)^{\wedge}(\eta) \ .$$

Since by (2)

$$e^{-\langle \xi,\cdot\rangle}\theta_k e^{\varphi}T \to e^{-\langle \xi,\cdot\rangle} e^{\varphi}T \quad \text{weak}^* \text{ in } \mathcal{S}' ,$$

by continuity of Fourier transform we have

$$F_k(\xi + i\eta) \to F(\xi + i\eta) \quad \text{weak}^* \text{ in } \mathcal{S}'_\eta .$$

On the other hand, since $\theta_k T \in \mathcal{E}'$,

$$\begin{aligned} G_k(z) &= \langle \theta_k e^{\varphi}T, e^{-\langle z,\cdot\rangle - \varphi}\rangle \\ &= \langle \theta_k e^{-\langle \xi,\cdot\rangle}T, e^{-i\langle \eta,\cdot\rangle}\rangle \\ &= (e^{-\langle \xi,\cdot\rangle}\,\theta_k T)^\wedge(\eta) = F_k(z) . \end{aligned}$$

Thus $G_k = F_k$. Then (5) and (6) imply for each ξ and for almost all η , $F(\xi + i\eta) = G(\xi + i\eta)$. By continuity $F(\xi + i\eta) = G(\xi + i\eta)$ for all ξ and η .

<u>Theorem 4.6</u>. If φ satisfies (H′) , $T \in \mathcal{S}'_\varphi$ and F is the Laplace transform of T , then F is an entire function and

$$F(z) = \langle e^{\varphi}T, e^{-\langle z,\cdot\rangle - \varphi}\rangle .$$

Moreover, there exist constants C and N such that

$$|F(z)| \le C(1+|z|)^N e^{\varphi^*(-\xi)}$$

for each $z \in C^n$ where $z = \xi + i\eta$.

<u>Proof</u>. The first part follows from theorem 4.5. Since $e^{\varphi}T \in \mathcal{S}'$ there is a continuous seminorm p on $\mathcal{S}$ such that $|\langle e^{\varphi}T, u\rangle| \le p(u)$

for each $u \in \mathscr{S}$. In particular, $|F(z)| \leq p(e^{-\langle z,\cdot\rangle - \varphi})$, so the last part follows from lemma 4.4.

Remark. We have been unable to prove a converse to theorem 4.6. In the case $n = 1$ or when φ has the special form

$$\varphi(x) = \varphi_1(x_1) + \ldots + \varphi_n(x_n)$$

a converse has been proved by K. Hayakawa [3]. The general case seems to be quite difficult. In certain other function spaces, namely $\mathscr{S}$, $\mathcal{O}_M$ and $\mathcal{L}^2$, we have been able to establish analogues of theorem 4.6 and also its converse. We now proceed to describe these results.

CHAPTER V

A PALEY-WIENER THEOREM FOR $\mathscr{S}_\varphi$

Let φ be a convex function on R^n such that $\varphi \in \mathcal{O}_M$ and for some $p > 1$ $\liminf_{|x| \to \infty} \frac{\varphi(x)}{|x|^p} > 0$. Denote $\mathscr{S}_\varphi = \left\{ u \in C^\infty : e^\varphi u \in \mathscr{S} \right\}$.

<u>Theorem 5.1</u>. If F is the Laplace transform of $u \in \mathscr{S}_\varphi$, then for each multi-index α and each integer N there exists a constant C such that

$$|D^\alpha F(z)| \leq C(1+|z|)^{-N} e^{\varphi^*(-\xi)}$$

where $z \in C^n$ and $z = \xi + i\eta$. Conversely if F is an entire function on C^n satisfying the estimates above, then there exists $u \in \mathscr{S}_\varphi$ such that F is the Laplace transform of u .

<u>Proof</u>. By theorem 4.6, we have

$$\begin{aligned} F(z) &= \int e^{\varphi(x)} u(x) e^{-\langle z,x\rangle - \varphi(x)} dx \\ &= \int u(x) e^{-\langle z,x\rangle} dx . \end{aligned}$$

Then for any β

$$\begin{aligned} z^\beta D_z^\alpha F(z) &= \int u(x) e^{-\langle z,x\rangle} (-x)^\alpha z^\beta dx \\ &= \int D^\beta [u(x)(-x)^\alpha] e^{-\langle z,x\rangle} dx . \end{aligned}$$

By Leibnitz formula,

$$D^{\beta}[u(x)(-x)^{\alpha}] = D^{\beta}[e^{\varphi(x)}u(x)(-x)^{\alpha}e^{-\varphi(x)}]$$

$$= \sum_{\gamma \leq \beta} h_{\gamma}(x)\ e^{-\varphi(x)}$$

where
$$h_{\gamma}(x) = \frac{\beta!}{(\beta-\gamma)!\,\gamma!} P_{\gamma,\beta}(x)\ D^{\gamma}[e^{\varphi(x)}\ u(x)(-x)^{\alpha}]$$

where
$$P_{\gamma,\beta} = e^{\varphi}D^{\beta-\gamma}e^{-\varphi} \in \mathcal{O}_M \quad .$$

Then $h_{\gamma} \in \mathcal{S}$. Since

$$\mathcal{S} \subseteq \mathcal{L}' \quad \text{and} \quad |z^{\beta}D^{\alpha}F(z)|$$

$$\leq \sum_{\gamma \leq \beta} \int |h_{\gamma}(x)|\ e^{-\varphi(x)-\langle \xi, x\rangle}\ dx$$

$$\leq e^{\varphi^{*}(-\xi)} \sum_{\gamma \leq \beta} \int |h_{\gamma}(x)|\ dx \quad ,$$

we have shown the first part.

For the converse, let

$$u(x) = (2\pi)^{-n/2} \int F(\xi + i\eta)\ e^{\langle \xi + i\eta, x\rangle}\ d\eta \ .$$

By the estimate for F, differentiation under the integral sign is allowed and it shows $u \in C^{\infty}$. Furthermore, using the Cauchy-Riemann equations,

$$\frac{\partial F}{\partial \xi_k} = -\ i\ \frac{\partial F}{\partial \eta_k} \quad ,$$

and integrating by parts, we have for each k ,

$$(2\pi)^{n/2} \frac{\partial u}{\partial \xi_k}(x) = -i \int \frac{\partial F}{\partial \eta_k}(\xi + i\eta)\, e^{\langle \xi+i\eta, x\rangle}\, d\eta$$

$$+ \int F(\xi + i\eta)\, x_k\, e^{\langle \xi+i\eta, x\rangle}\, d\eta$$

$$= 0 .$$

So u does not depend upon ξ . Let

$$I = (1+|x|^2)^m D^\alpha (e^\varphi u)(x)$$

$$= (2\pi)^{-n/2} \int (1+|x|^2)^m P_\alpha(x,z)\, e^{\varphi(x)+\langle z,x\rangle} F(z)\, d\eta$$

where

$$P_\alpha(x,z) = e^{-\varphi(x)-\langle z,x\rangle} D_x^\alpha e^{\varphi(x)+\langle z,x\rangle} .$$

Then $P_\alpha(x,z)$ is a polynomial in z with coefficients in $\mathcal{O}_M$ (as functions of x). From $(1+|x|^2)^m e^{i\langle \eta,x\rangle} = (1 - \Delta_\eta)^m e^{i\langle \eta,x\rangle}$ and integration by parts, we have

$$I = (2\pi)^{-n/2} \int e^{\varphi(x)+\langle z,x\rangle} (1 - \Delta_\eta)^m [P_\alpha(x,z)F(z)]\, d\eta .$$

Now

$$(1 - \Delta_\eta)^m [P_\alpha(x,z)F(z)]$$

$$= \sum_{|\beta|\leq 2m} Q_\beta(x,z)\, D_z^\beta F(z)$$

where each Q_β (also depending on α and m) is a polynomial in z with coefficients in $\mathcal{O}_M$. Thus

$$|I| \leq C_o \sum_{|\beta| \leq 2m} \int e^{\varphi(x)+\varphi^*(-\xi)+\langle \xi,x\rangle} |Q_\beta(x,z)| (1+|z|)^{-N} \, d\eta \ .$$

Now for each x , I is independent of ξ and by lemma 3.7, for each x we may choose ξ so that $\varphi(x) + \varphi^*(-\xi) + \langle \xi,x\rangle = 0$; moreover if N is large enough,

$$\left|Q_\beta(x,z)\right| \ (1+|z|)^{-N} \leq C_1(1+|x|^2)^{m'}(1+|\eta|)^{-n-1}$$

where m' may be chosen to depend only on α (but not on m). Then $|I| \leq C_{m,\alpha}(1+|x|^2)^{m'}$ for each x . Since m' does not depend on m , if we replace m by $m+m'$, we have

$$\sup_x \left| (1+|x|^2)^m \, D^\alpha(e^\varphi u)(x)\right| < \infty \quad ,$$

i.e., $e^\varphi u \in \mathscr{S}$. Finally from the definition of u the Fourier transform of $e^{-\langle \xi,\cdot\rangle}u$ is $\eta \longrightarrow F(\xi + i\eta)$, or equivalently F is the Laplace transform of u .

CHAPTER VI

A PALEY-WIENER THEOREM FOR $\mathcal{O}_{M,\varphi}$

Let φ be a convex function on R^n such that $\varphi \in \mathcal{O}_M$ and for some $p > 1$,

$$\liminf_{|x| \to \infty} \frac{\varphi(x)}{|x|^p} > 0 \quad .$$

Recall from theorem 3.8 $\varphi^*(d\varphi(x)) = \langle d\varphi(x), x\rangle - \varphi(x)$ for each $x \in R^n$.

Theorem 6.1. Let F be an entire function on C^n and suppose for each integer $N \geq 0$, there are constants C_N and M_N such that

$$|F(\xi + i\eta)| \leq C_N(1 + |\eta|^2)^{-N} \, (1 + |\xi|^2)^{M_N} \, e^{\varphi^*(-\xi)} \quad . \quad (*)$$

Then there exists $u \in C^\infty(R^n)$ such that $e^\varphi u \in \mathcal{O}_M$ and F is the Laplace transform of u .

Conversely, if $e^\varphi u \in \mathcal{O}_M$ and if F is the Laplace transform of u , then F is an entire function and for each integer $N \geq 0$ F satisfies (*).

Proof. Assume (*) and let

$$u(x) = (2\pi)^{-n/2} \int F(\xi + i\eta) \, e^{\langle \xi + i\eta, x\rangle} \, d\eta \quad .$$

By the continuity of φ^* , we may differentiate under the integral sign. Thus, as we have shown in chapter V, the integral is independent of ξ.

Now

$$(e^{\varphi}u)(x) = (2\pi)^{-n/2}\int F(z)\ e^{\varphi(x)+\langle z,x\rangle}\ d\eta \quad ,$$

where $z = \xi + i\eta$. Again we may differentiate under the integral sign and so

$$D^{\alpha}(e^{\varphi}u)(x) = (2\pi)^{-n/2}\int F(z)\ P_{\alpha}(x,z)\ e^{\varphi(x)+\langle z,x\rangle}\ d\eta$$

where $P_{\alpha}(x,z)$ is a polynomial in z with coefficients in $\mathcal{O}_M$ (as functions of x). The integral is independent of ξ and so we may set $\xi = -d\varphi(x)$. Then

$$\begin{aligned} &D^{\alpha}(e^{\varphi}u)(x) \\ &= (2\pi)^{-n/2}\int F(-d\varphi(x)+i\eta)\ P_{\alpha}(x,-d\varphi(x)+i\eta)\ \cdot \\ &\qquad\qquad e^{-\varphi^{*}(-d\varphi(x))+i\langle\eta,x\rangle}\ d\eta \ . \end{aligned}$$

Thus

$$\begin{aligned} &\left|D^{\alpha}(e^{\varphi}u)(x)\right| \\ &\le C_N\int(1+|\eta|^2)^{-N}(1+|d\varphi(x)|^2)^{M_N}\ \cdot \\ &\qquad |P_{\alpha}(x,-d\varphi(x)+i\eta)|\ d\eta \ . \end{aligned}$$

Since $\varphi \in \mathcal{O}_M$ and the coefficients of P_{α} are in $\mathcal{O}_M$, we have

$$\left|D^{\alpha}(e^{\varphi}u)(x)\right|$$

$$\leq C_N\, C\int(1+|\eta|^2)^{-N}\ (1+|x|^2)^{M_N}(1+|\eta|^2)^{L}\ d\eta$$

where C and M depend on φ, α and M_N and L depends only on α (say $L = \frac{|\alpha|}{2}$) . Thus, if we choose N large enough, $D^{\alpha}e^{\varphi}u$ is bounded by a polynomial in x , i.e., $e^{\varphi}u \in \mathcal{O}_M$. Clearly F is the Laplace transform of u .

Conversely, suppose $e^{\varphi}u \in \mathcal{O}_M$ and let F be the Laplace transform of u . Then

$$F(\xi + i\eta) = (e^{-\langle \xi,\cdot\rangle}u)^{\wedge}(\eta)$$

and so

$$\eta^{\alpha}F(\xi + i\eta) = i^{-|\alpha|}[D^{\alpha}(e^{-\langle \xi,\cdot\rangle}u)]^{\wedge}(\eta) \quad .$$

Since

$$D^{\alpha}(e^{-\langle \xi,\cdot\rangle}u) = \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}(-\xi)^{\alpha-\beta}(D^{\beta}u)e^{-\langle \xi,\cdot\rangle} \quad ,$$

$$\eta^{\alpha}F(\xi + i\eta) =$$

$$i^{-|\alpha|}\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\int(-\xi)^{\alpha-\beta}\ D^{\beta}u(x)\ e^{-\langle \xi+i\eta,x\rangle}\ dx \quad .$$

If we write $u = e^{-\varphi} e^{\varphi} u$, we have

$$D^{\beta} u = \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} D^{\gamma}(e^{\varphi} u)\, P_{\beta-\gamma}\, e^{-\varphi}$$

where $P_{\beta-\gamma} = e^{\varphi} D^{\beta-\gamma} e^{-\varphi}$ is a polynomial in derivatives of φ. Thus $P_{\beta-\gamma} \in \mathcal{O}_M$. We now have

$$\eta^{\alpha} F(\xi + i\eta)$$

$$= i^{-|\alpha|} \sum_{\beta \leq \alpha} \sum_{\gamma \leq \beta} \binom{\alpha}{\beta} \binom{\beta}{\gamma} \int (-\xi)^{\alpha-\beta} D^{\gamma}(e^{\varphi} u)(x)\, P_{\beta-\gamma}(x)$$

$$e^{-\varphi(x) - \langle \xi + i\eta, x\rangle}\, dx \quad .$$

Since $e^{\varphi} u$ is also in $\mathcal{O}_M$,

$$|\eta^{\alpha} F(\xi + i\eta)| \leq C \int (1+|\xi|^2)^L (1+|x|^2)^M e^{-\varphi(x) - \langle \xi, x\rangle}\, dx \; .$$

By lemma 4.1

$$(1+|x|^2)^{M+n+1} e^{-\varphi(x) - \langle \xi, x\rangle} \leq C'(1+|\xi|^2)^K e^{\varphi^*(-\xi)}$$

and hence

$$|\eta^{\alpha} F(\xi + i\eta)| \leq CC'(1+|\xi|^2)^{L+K} e^{\varphi^*(-\xi)} \int (1+|x|^2)^{-n-1}\, dx$$

$$\leq C''(1+|\xi|^2)^{L+K} e^{\varphi^*(-\xi)} \quad .$$

From

$$|\eta|^{2N} = \sum_{|\alpha|=N} \frac{N!}{\alpha!} \eta^{2\alpha} ,$$

we have

$$(1+|\eta|^2)^N = \sum_{|\alpha| \leq N} \frac{N!}{\alpha!\,(N-|\alpha|)!} \eta^{2\alpha} .$$

Hence, finally,

$$(1+|\eta|^2)^N |F(\xi+i\eta)|$$

$$\leq C_N (1+|\xi|^2)^{M_N} e^{\varphi^*(-\xi)} .$$

<u>Example</u>. Let $\varphi(x) = \frac{1}{2}|x|^2$ and $u(x) = e^{-(|x|^2)/2}$. Then $e^{\varphi}u = 1 \in \mathcal{O}_M$ and the Laplace transform of u is

$$F(z) = (2\pi)^{-n/2} \int e^{-\langle z,x\rangle - \frac{1}{2}|x|^2} dx$$

$$= (2\pi)^{-n/2} \int e^{-\frac{1}{2}\langle x+z,x+z\rangle + \frac{1}{2}\langle z,z\rangle} dx$$

$$= (2\pi)^{-n/2} \left(\sqrt{2\pi}\right)^n e^{\frac{1}{2}\langle z,z\rangle} = e^{\frac{1}{2}\langle z,z\rangle} .$$

So

$$|F(z)| = e^{\frac{1}{2}|\xi|^2 - \frac{1}{2}|\eta|^2} = e^{-\frac{1}{2}|\eta|^2} e^{\varphi^*(-\xi)} .$$

CHAPTER VII

A PALEY-WIENER THEOREM FOR $\mathcal{L}^2_\varphi$

Let φ be a C^2 function on R^n such that there exist positive numbers C_1 and C_2 for which

$$C_1|\eta|^2 \leq \sum_{j,k} \frac{\partial^2\varphi}{\partial x_j \partial x_k}(x)\, \eta_j \eta_k \leq C_2|\eta|^2 \tag{1}$$

for each $x \in R^n$ and each $\eta \in R^n$. Consider

$$g(t) = \varphi[y + t(x-y)] \quad \text{where} \quad t \in R.$$

By Taylor's theorem

$$g(1) = g(0) + g'(0) + \frac{1}{2} g''(s) \text{ where } 0 < s < 1 .$$

From the computation:

$$g(1) = \varphi(x),$$

$$g(0) = \varphi(y),$$

$$g'(t) = \sum_{j=1}^{n} \frac{\partial\varphi}{\partial x_j}[y + t(x-y)](x_j - y_j),$$

$$g'(0) = \langle d\varphi(y), x-y\rangle ,$$

we get

$$\varphi(x) = \varphi(y) + \langle d\varphi(y), x-y\rangle$$

$$+ \frac{1}{2}\sum_{j,k}^{n} \frac{\partial^2\varphi}{\partial x_j \partial x_k}[y + s(x-y)](x_j - y_j)(x_k - y_k) \tag{2}$$

where $0 < s < 1$ depends on x and y. In particular, when $y = 0$, we have

$$\varphi(x) = \varphi(0) + \langle d\varphi(0), x\rangle + \frac{1}{2}\sum_{j,k}^{n} \frac{\partial^2\varphi}{\partial x_j \partial x_k}(sx)x_j x_k .$$

Thus by the lower bound in (1),

$$\varphi(0) + \langle d\varphi(0), x\rangle + \frac{c_1}{2}|x|^2 \leq \varphi(x)$$

and we see that

$$\liminf_{|x| \to +\infty} \frac{\varphi(x)}{|x|} = +\infty .$$

The lower bound in (1) also shows that

$$g''(t) = \sum_{j,k}^{n} \frac{\partial^2\varphi}{\partial x_j \partial x_k}[y + t(x-y)]\ (x_j - y_j)(x_k - y_k)$$

$$\geq c_1|x - y|^2$$

and hence φ is strictly convex.

Remark. The lower bound in (1) is more restrictive than strict convexity, but we get the growth condition as well as strict convexity from it. Moreover, as we shall see, the lower bound and the upper bound in (1), the latter being very restrictive, will enable us to establish respectively the sufficiency and the necessity of the following theorem.

Theorem 7.1 Let φ be a C^2 function on R^n satisfying (1). Let F be an entire function on C^n . Then

$$\iint \left| e^{-\varphi^*(-\xi)} F(\xi + i\eta) \right|^2 d\xi d\eta < \infty \tag{3}$$

if and only if there exists

$$f \in \mathcal{L}^1_{loc}(R^n)$$

such that

$$e^{\varphi} f \in \mathcal{L}^2(R^n)$$

and F is the Laplace tranform of f .

Proof. Assume

$$e^{\varphi} f \in \mathcal{L}^2(R^n)$$

and

$$F(\xi + i\eta) = (e^{-\langle \xi, \cdot \rangle} f)^{\wedge}(\eta) \quad .$$

Then

$$f \in \mathcal{L}^2(R^n)$$

and by Plancheral formula the integral in (3) is equal to

$$\iint \left| e^{-\varphi^*(-\xi)-\langle \xi,x\rangle} f(x)\right|^2 dxd\xi$$

$$= \iint \left| e^{-\varphi^*(-\xi)-\langle \xi,x\rangle-\varphi(x)} e^{\varphi(x)} f(x)\right|^2 d\xi dx \quad .$$

Then from (2)

$$\frac{1}{2} C_1 \; |x-y|^2 \le \varphi(x) - \varphi(y) - \langle d\varphi(y),x-y\rangle$$

$$\le \frac{1}{2} C_2 \; |x-y|^2 \quad .$$

Let

$$I_x = \int \left| e^{-\varphi^*(-\xi)-\langle \xi,x\rangle-\varphi(x)}\right|^2 d\xi \quad .$$

Since $d\varphi$ is a C^1 diffeomorphism of R^n into R^n (theorem 3.10), we may change variables

$$-\xi = d\varphi(y) \quad \text{in} \quad I_x \quad ;$$

since

$$\varphi^*(d\varphi(y)) = \langle d\varphi(y),y\rangle - \varphi(y) \qquad \text{(lemma 3.9)} \quad ,$$

$$I_x = \int e^{-2[\varphi(x)-\varphi(y)-\langle d\varphi(y),x-y\rangle]} \, |J_\varphi(y)| \, dy$$

where

$$J_\varphi(y) = \det\left[\frac{\partial^2\varphi}{\partial x_j \partial x_k}(y)\right] .$$

But

$$\det\left[\frac{\partial^2\varphi}{\partial x_j \partial x_k}(y)\right] = \lambda_1(y)\lambda_2(y) \ldots \lambda_n(y) ,$$

where each λ_j is an eigenvalue of the Hessian of φ . From (1),

$$c_1 \leq \lambda_j(y) \leq c_2$$

for each $j = 1,2,\ldots,n$. Hence

$$c_1^n \leq |J_\varphi(y)| \leq c_2^n .$$

It follows that

$$c_1^n \int e^{-c_2|x-y|^2} dy \leq I_x \leq c_2^n \int e^{-c_1|x-y|^2} dy ,$$

equivalently,

$$c_1^n c_2^{-n/2} \pi^{n/2} \leq I_x \leq c_2^n c_1^{-n/2} \pi^{n/2} . \qquad (4)$$

Now

$$\iint \left| e^{-\varphi^*(-\xi)} F(\xi+i\eta) \right|^2 d\xi d\eta$$

$$= \int I_x \left| e^{\varphi(x)} f(x) \right|^2 dx$$

$$\leq c_2^n c_1^{-n/2} \pi^{n/2} \|e^{\varphi} f\|^2_{\mathcal{L}^2} < + \infty .$$

For the converse, assume (3). It implies $F \in \mathcal{L}^2_\eta(R^n)$ and so

$$F(\xi + i\eta) = \hat{g}_\xi(\eta)$$

where

$$g_\xi \in \mathcal{L}^2(R^n) .$$

By Cauchy-Riemann equations,

$$\frac{\partial}{\partial \xi_j} \hat{g}_\xi(\eta) = - i \frac{\partial}{\partial \eta_j} \hat{g}_\xi(\eta) = -x_j \hat{g}_\xi(\eta)$$

and so

$$\frac{\partial}{\partial \xi_j} [e^{\langle \xi, x \rangle} g_\xi(x)] = e^{\langle \xi, x \rangle} \left[x_j g_\xi(x) + \frac{\partial}{\partial \xi_j} g_\xi(x) \right]$$

$$= 0 \quad , \quad j = 1,2,\ldots,n \quad .$$

Since $e^{\langle \xi, \cdot \rangle} g_\xi$ is independent of ξ , $g_\xi = e^{-\langle \xi, \cdot \rangle} f$, where

$$f \in \mathcal{L}^1_{loc}(R^n)$$

and

$$F(\xi + i\eta) = (e^{-\langle \xi, \cdot \rangle} f)^\wedge(\eta) \quad ,$$

so F is the Laplace transform of f .

By the lower bound in (4) and the Plancheral formula,

$$C_1^n C_2^{-n/2} \pi^{n/2} \int \left| e^{\varphi(x)} f(x) \right|^2 dx$$

$$\leq \int I_x \left| e^{\varphi(x)} f(x) \right|^2 dx$$

$$= \iint \left| e^{-\varphi^*(-\xi) - \langle \xi, x \rangle} f(x) \right|^2 d\xi dx$$

$$= \iint \left| e^{-\varphi^*(-\xi) - \langle \xi, x \rangle} f(x) \right|^2 dx d\xi$$

$$= \iint \left| e^{-\varphi^*(-\xi)} F(\xi + i\eta) \right|^2 d\xi d\eta$$

$$< +\infty \ .$$

Hence $e^\varphi f \in \mathcal{L}^2(R^n)$.

<u>Example</u>. Let

$$\varphi(x) = e^{-|x|^2} + \frac{1}{2} \sum_{j=1}^{n} \lambda_j x_j^2$$

where $2 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then

$$\frac{\partial\varphi}{\partial x_j}(x) = -e^{-|x|^2}\, 2x_j + \lambda_j x_j$$

and

$$\frac{\partial^2\varphi}{\partial x_j \partial x_k}(x) = 4e^{-|x|^2}\, x_j x_k - 2e^{-|x|^2}\, \delta_{jk} + \lambda_j \delta_{jk} .$$

Then

$$\sum_{j,k} \frac{\partial^2\varphi}{\partial x_j \partial x_k}(x)\, \eta_j \eta_k$$

$$= e^{-|x|^2}\, 4\langle\eta,x\rangle^2 - 2e^{-|x|^2}\, |\eta|^2 + \sum_{j=1}^{n} \lambda_j \eta_j^2$$

$$\leq \left(4e^{-|x|^2}|x|^2 + \lambda_n\right)|\eta|^2$$

$$\leq (4e^{-1} + \lambda_n)|\eta|^2 \quad ; \text{ on the other hand}$$

$$\sum_{j,k} \frac{\partial^2\varphi}{\partial x_j \partial x_k}(x)\, \eta_j \eta_k \geq \left(-2e^{-|x|^2} + \lambda_1\right)|\eta|^2 .$$

Therefore we may let $C_1 = \lambda_1 - 2$ and $C_2 = 4e^{-1} + \lambda_n$.

<u>Example</u>. Let $\varphi(x) = |x|^2 + \log(1+|x|^2)$.

Then

$$\sum_{j,k} \frac{\partial^2 \varphi}{\partial x_j \partial x_k} (x)\eta_j\eta_k = 2|\eta|^2 - 4(1+|x|^2)^{-2} \langle \eta,x\rangle^2 + 2(1+|x|^2)^{-1}|\eta|^2$$

$$\leq \left(2 + 2(1+|x|^2)^{-1}\right) |\eta|^2$$

$$\leq 4|\eta|^2 \quad .$$

Also

$$2(1+|x|^2)^{-2} \left\{(1+|x|^2)^2|\eta|^2 - 2 \langle \eta,x\rangle^2 + (1+|x|^2)|\eta|^2\right\}$$

$$\geq 2(1+|x|^2)^{-2} \left\{(1+|x|^2)^2 - 2|x|^2 + (1+|x|^2)\right\} |\eta|^2$$

$$\geq 2(1+|x|^2)^{-2} \left\{1 + 2|x|^2 + |x|^4 - 2|x|^2 + 1 + |x|^2\right\} |\eta|^2$$

$$\geq 2(1+|x|^2)^{-2} \left\{2 + |x|^2 + |x|^4\right\} |\eta|^2$$

$$\geq 2(1+|x|^2)^{-2} \left\{\frac{1}{2} (1 + 2|x|^2 + |x|^4)\right\} |\eta|^2$$

$$= |\eta|^2 \quad .$$

Thus

$$|\eta|^2 \leq \sum_{j,k} \frac{\partial^2 \varphi}{\partial x_j \partial x_k} (x) \eta_j\eta_k \leq 4|\eta|^2 \quad .$$

CHAPTER VIII

THE CASE φ IS NOT C^∞ AT THE ORIGIN

Thus far we have assumed φ to be C^∞ on R^n. In order not to exclude functions of the form

$$\frac{1}{p}|x|^p, \quad p > 1 ,$$

we need to ease such restriction.

Let φ now be a real-valued function on R^n such that

(i) φ is convex,

(ii) $\liminf_{|x| \to +\infty} \frac{\varphi(x)}{|x|^p} > 0$, $p > 1$,

(iii) φ is C^∞ on $R^n - (0)$ and for each α ,

there exist C and m such that

$$\left|D^\alpha \varphi(x)\right| \leq C(1+|x|)^m$$

for $|x| > 1$. Let θ be a C^∞ function with compact support such that $\theta(x) = 1$ on $|x| \leq 1$ and $\theta(x) = 0$ for $|x| \geq R$, where $R > 1$ of course. We consider the function $\psi = \varphi(1-\theta)$. Note

$$\psi(x) = \begin{cases} 0 & \text{if} \quad |x| \leq 1 \\ \varphi(x)-\varphi(x)\theta(x) & \text{if} \quad 1 < |x| < R \\ \varphi(x) & \text{if} \quad |x| \geq R \ . \end{cases}$$

Thus $\psi \in \mathcal{O}_M$ and

$$\liminf_{|x| \to +\infty} \frac{\psi(x)}{|x|^p} > 0 \ , \ p > 1 \ .$$

We denote still

$$\psi^*(\xi) = \sup_x \left[\langle \xi, x \rangle - \psi(x) \right] .$$

Then

$$\psi^*(\xi) \leq \sup_x \left[\langle \xi, x \rangle - \varphi(x) \right] + \sup_x \varphi(x)\, \theta(x)$$

$$= \varphi^*(\xi) + a \ , \qquad (*)$$

where a is a constant depending on φ and R . Hence ψ^* is finite everywhere and so continuous by convexity. (This is half of theorem 3.4.) We then obtain lemmas 4.1, 4.2, 4.3, 4.4 for ψ , since the proofs do not depend on the convexity. From lemma 4.4 we have $e^{-\psi - \langle z, \cdot \rangle} \in \mathcal{S}$ for each $z \in C^n$ and is a holomorphic function with values in $\mathcal{S}$.

Define

$$\mathcal{S}'_\varphi = \left\{ T \in \mathcal{D}' \ : \ e^{\psi} T \in \mathcal{S}' \right\} .$$

Note that $\mathcal{S}'_\varphi$ does not depend on the choice of θ . For, if $\theta' \in C_c^\infty$ and $\theta'(x) = 1$ for $|x| \leq 1$, then

$$e^{\varphi(1-\theta)} T = e^{\varphi(\theta - \theta')} \left(e^{\varphi(1-\theta)} T \right)$$

and we have

$$e^{\varphi(1-\theta)}T - e^{\varphi(1-\theta')}T$$

$$= \left(1 - e^{\varphi(\theta-\theta')}\right) e^{\varphi(1-\theta)}T \in \mathcal{E}' \subseteq \mathcal{S}' ,$$

since

$$1 - e^{\varphi(\theta-\theta')} \in C_c^\infty .$$

Also note that the space $\mathcal{S}'_\varphi$ is closed under differentiation. For, if $e^\psi T \in \mathcal{S}'$ then by Leibnitz formula

$$e^\psi D_j T = D_j(e^\psi T) - (D_j\psi)\, e^\psi T .$$

Since $D_j\psi \in \mathcal{O}_M$ and $D_j(e^\psi T) \in \mathcal{S}'$, we have $e^\psi D_j T \in \mathcal{S}'$. By induction, $D^\alpha T \in \mathcal{S}'_\varphi$ whenever $T \in \mathcal{S}'_\varphi$.

Now, with the same argument as in the beginning of section (ii) of chapter IV, $\mathcal{S}'_\varphi \subseteq \mathcal{O}'_c$ and so we may speak of the Laplace transform of elements of $\mathcal{S}'_\varphi$. In view of (*) theorem 4.6 takes the following form.

<u>Theorem 8.1</u>. If $T \in \mathcal{S}'_\varphi$ and if F is the Laplace transform of T, then F is an entire function and

$$F(z) = \langle e^\psi T, e^{-\langle z,\cdot\rangle - \psi}\rangle .$$

Moreover, there exist constants C and N such that

$$|F(z)| \leq C(1+|z|)^N e^{\varphi^*(-\xi)} , \quad z = \xi + i\eta \in C^n .$$

<u>Theorem 8.2</u>. If F is the Laplace transform of $u \in \mathcal{S}_\varphi$, then for each α and each integer N there exists a constant C such that

$$\left|D^\alpha F(z)\right| \leq C(1+|z|)^{-N} e^{\varphi^*(-\xi)}$$

where $z \in C^n$ and $z = \xi + i\eta$. Conversely, if F is an entire function on C^n satisfying the inequality above, then there exists $u \in \mathcal{S}_\varphi$ such that F is the Laplace transform of u.

<u>Proof</u>. Since convexity was not used in the proof of theorem 5.1, we have

$$F(z) = \int u(x) \; e^{-\langle z,x\rangle} \, dx$$

and by integration by parts and by Leibnitz formula we have for any β,

$$z^\beta D^\alpha F(z) = \sum_{\gamma \leq \beta} \int h_\gamma(x) \; e^{-\psi(x)-\langle z,x\rangle} \, dx \quad ,$$

where

$$h_\gamma(x) = C_{\gamma,\beta} \; P_{\gamma,\beta}(x) \; D^\gamma\left[e^{\psi(x)} u(x)(-x)^\alpha\right]$$

where $P_{\gamma,\beta} \in \mathcal{O}_M$. Then $h_\gamma \in \mathcal{S}$. Since $\mathcal{S} \subseteq \mathcal{L}^1$,

$$\left|z^{\beta}D^{\alpha}F(z)\right| \leq \sum_{\gamma\leq\beta}\int\left|h_{\gamma}(x)\right| e^{-\psi(x)-\langle\xi,x\rangle} dx$$

$$\leq C\, e^{\psi^{*}(-\xi)}$$

$$\leq C\, e^{a}\, e^{\varphi^{*}(-\xi)} .$$

We have shown the first part.

For the converse we let

$$u(x) = (2\pi)^{-n/2}\int F(\xi+i\eta)\, e^{\langle\xi+i\eta,x\rangle}\, d\eta .$$

Since φ^{*} is continuous, by the estimate on $D^{\alpha}F$ we may differentiate under the integral sign. As in the proof of theorem 5.1, we can show $u \in C^{\infty}$ and that u does not depend upon ξ . To show $e^{\psi}u \in \mathscr{S}$, as before, let

$$I = (1+|x|^{2})^{m}\, D^{\alpha}(e^{\psi}u)(x)$$

and we get the inequality

$$|I|$$

$$\leq C \sum_{|\beta|\leq 2m}\int e^{\psi(x)+\varphi^{*}(-\xi)+\langle\xi,x\rangle}\, |Q_{\beta}(x,z)|$$

$$(1+|z|)^{-N}\, d\eta$$

where Q_{β} is a polynomial in z with coefficients in $\mathcal{O}_{M}$.

By lemma 3.7 for each x , we may choose some ξ so that

$$\varphi^*(-\xi) + \langle \xi, x\rangle = -\varphi(x) \quad .$$

Now $\psi - \varphi = -\theta\varphi$ has compact support and so $e^{\psi-\varphi} = 1$ outside a compact set. By continuity we now have

$$|I| \leq c_{m,\alpha} \int \left| Q_\beta(x,z) \right| \; (1+|z|)^{-N} \, d\eta \quad .$$

As before, if we let N be large enough,

$$|I| \leq c_{m,\alpha}(1+|x|^2)^{m'}$$

for each x where m' does not depend on m . Replacing m by $m + m'$ we have

$$\sup_x \left| (1+|x|^2)^m \, D^\alpha e^\psi u(x) \right| < \infty \; ,$$

i.e., $e^\psi u \in \mathscr{S}_\varphi$. Hence $u \in \mathscr{S}_\varphi$.

In order for theorem 6.1 to fit our present scheme, we make another assumption on φ , namely, for each $x \neq 0$,

$$\varphi(x) + \varphi^*(d\varphi(x)) = \langle x, d\varphi(x)\rangle \; .$$

As seen in example 9, chapter III, such assumption is satisfied by functions of the type

$$\frac{1}{p}\,|x|^p \; , \; p > 1 \; .$$

Theorem 8.3. Let F be an entire function and suppose for each integer $N \geq 0$ there are constants C_N and M_N such that

$$|F(\xi + i\eta)| \leq C_N(1+|\eta|^2)^{-N} (1+|\xi|^2)^{M_N} e^{\varphi^*(-\xi)} \quad . \tag{1}$$

Then there exists $u \in \mathfrak{S}_{M,\varphi}$ such that F is the Laplace transform of u .

Conversely if $e^{\psi}u \in \mathfrak{S}_M$ and if F is the Laplace transform of u , then F is an entire function and for each integer $N \geq 0$, F satisfies (1).

Proof. Assume (1) and let

$$u(x) = (2\pi)^{-n/2} \int F(\xi + i\eta)\, e^{\langle \xi+i\eta, x\rangle}\, d\eta \quad .$$

By the continuity of ψ^* we may differentiate under the integral sign. Thus, as usual, the integral is independent of ξ . Now

$$(e^{\psi}u)(x) = (2\pi)^{-n/2} \int F(\xi + i\eta)\, e^{\psi(x)+\langle \xi+i\eta, x\rangle}\, d\eta \quad .$$

Again we may differentiate under the integral and since $\psi \in \mathfrak{S}_M$

$$D^{\alpha}(e^{\psi}u)(x)$$

$$= (2\pi)^{-n/2} \int F(\xi + i\eta) P_{\alpha}(x, \xi+i\eta) e^{\psi(x)+\langle \xi+i\eta, x\rangle}\, d\eta$$

where $P_\alpha(x,z)$ is a polynomial in z with coefficients in $\mathcal{O}_M$ (as functions of x). The integral is independent of ξ. So we may set $\xi = -d\varphi(x)$ and by $\varphi(x) + \varphi^*(d\varphi(x)) = \langle x, d\varphi(x)\rangle$, we have

$$D^\alpha(e^\psi u)(x)$$

$$= (2\pi)^{-n/2} \int F(-d\varphi(x) + i\eta)\ P_\alpha(x, -d\varphi(x) + i\eta) \cdot$$
$$e^{-\varphi^*(-d\varphi(x)) + i\langle\eta,x\rangle - \theta(x)\varphi(x)}\, d\eta \ .$$

Since $-\theta\varphi$ has compact support, by (1)

$$\left|D^\alpha(e^\psi u)(x)\right|$$

$$\leq C_N \int (1+|\eta|^2)^{-N} \ (1+|d\varphi(x)|^2)^{M_N} \left|P_\alpha(x, -d\varphi(x) + i\eta)\right| \, d\eta \ .$$

Since the coefficients of P_α are in $\mathcal{O}_M$ and the derivatives of φ are bounded by polynomials outside the unit ball, we see $e^\psi u \in \mathcal{O}_M$. Clearly F is the Laplace transform of u.

Conversely, suppose $e^\psi u \in \mathcal{O}_M$ and let F be the Laplace transform of u. As before we have

$$\eta^\alpha F(\xi + i\eta) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \int (-\xi)^{\alpha-\beta}\, D^\beta u(x)\ e^{-\langle\xi+i\eta, x\rangle}\, dx \ .$$

Writing $u = e^{-\psi}e^\psi u$, we have

$$D^\beta u = \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} D^\gamma(e^\psi u)\ P_{\beta-\gamma}(x)\ e^{-\psi}$$

where $P_{\beta-\gamma}$ is a polynomial of derivatives of ψ . Thus $P_{\beta-\gamma} \in \mathcal{O}_M$ and so $D^{\gamma}(e^{\psi}u)P_{\beta,\gamma} \leftarrow \mathcal{O}_M$. Now

$$|\eta^{\alpha}F(\xi + i\eta)| \leq C \int (1+|\xi|^2)^L \; (1+|x|^2)^M \; e^{-\psi(x)-\langle \xi,x\rangle} \; dx \; .$$

By the fact that $\varphi\theta$ has compact support and lemma 4.1, the same argument on page 42, goes through and the proof is complete.

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