AN ANALYTIC APPROACH TO TEMPERATURE DISTRIBUTION IN A THIN-WALLED COMBUSTION CHAMBER

by

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INTRODUCTION

A practical (or non-academic) heat conduction problem consists of two things, viz., a physical situation and a mathematical description of that physical situation. It is somewhat misleading to consider only the physical and mathematical descriptions; the transition from physical to mathematical is most often difficult and sometimes exceedingly laborious. It therefore seems apparent that a detailed description of a heat conduction problem should include three things: (1) a physical justification of the problem, (2) a logical transition from the physical to the mathematical description, and (3) the mathematical description.

In order to introduce the problem contained in this paper, Item (1) above should be dwelt upon. Consider the following physical situation: Air is blown through and over the outside of a thin-walled metal duct. About midway through the duct, fuel is added to the air; the fuel evaporates and burning is somehow initiated. The burning is presumed to be violent, i.e., burning occurs in a high-temperature, high-pressure, low-velocity air stream.

Now consider the thin-walled duct. A quasi-discontinuous heat transfer takes place along its inside surface. Over the leading inside half of the duct, there will be heat transfer from the duct to the air stream. Over the aft inside half, there will be heat transfer from the hot combustion products to the duct wall; the transition is nearly discontinuous. Over the outside surface of the duct, there is a near-steady heat transfer from the duct to the air stream.

The system described is shown schematically in Figures 1 and 2. Over approximately three quarters of its bounding surface, heat is being taken away from the duct since over this surface the body temperature is greater than the environment temperature (see Figure 1). Over the remaining quarter of the surface, heat is being added to the duct from the hot combustion products. Representative dimensions are as indicated in Figure 1.

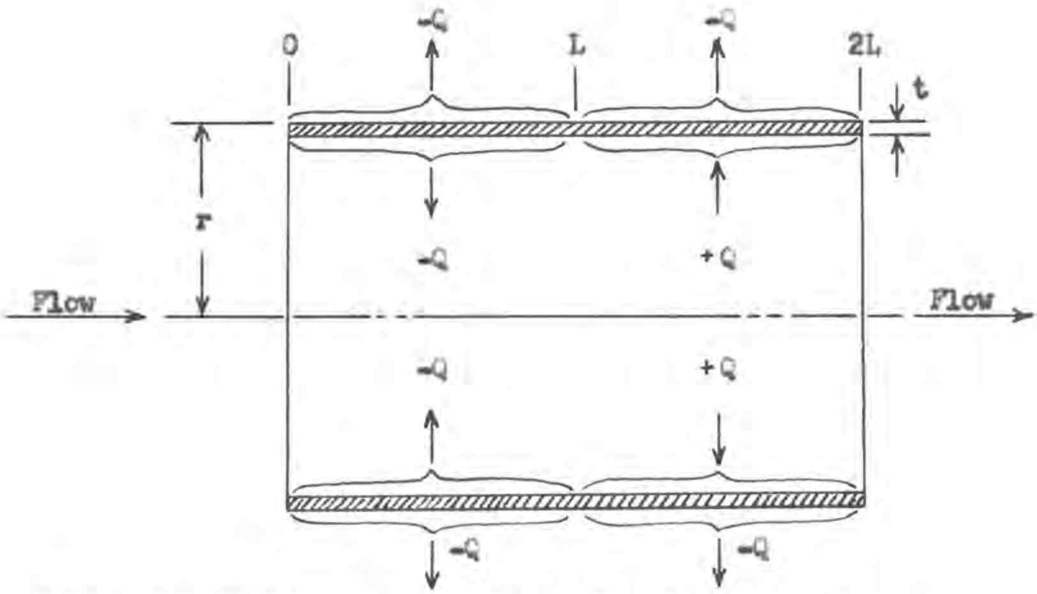

Figure 1. Sectional schematic of the physical problem showing duct, flow direction, etc. Fuel is added in the vicinity of Station (0) and burning starts at Station (L). +Q indicates heat flow to duct, -Q heat flow from duct. Typical dimensions are: r = 12", t = 1/4", 2L = 48".

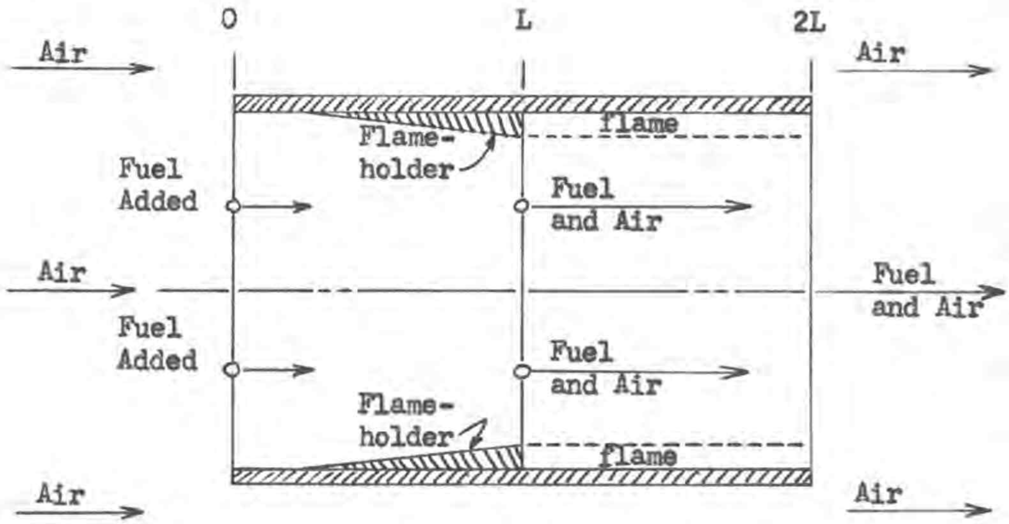

Figure 2. Sectional schematic showing duct, annular flame stabilizer, flow direction, etc. Fuel is added in the vicinity of Station (0) and burning initiated at Station (L).

In such a flow system, flame is usually stabilized in the wake of a bluff body. The flame-stabilizing apparatus is ordinarily referred to as a flameholder. An annular or "wall" flameholder is considered here and is shown in section in Figure 2. Unless the fuel used is a monopropellant (i.e., needs no oxygen to burn), it is ordinarily injected into the air stream through nozzles upstream of the combustion zone. This is also indicated in Figure 2.

The scope of the problem is not restricted to heat addition in ducts. A similar problem arises in fuel-rod and moderator designs for nuclear-reactor piles where the rod and/or moderator are half submerged in, e.g., a boiling liquid. The similarity is especially evident if the moderator and/or fuel is in the shape of a thin circular layer.

Another refinement of the same problem occurs in embedded steam pipes when the designer wishes to restrict heat flow to one direction.

This is sometimes done by covering only half the circumference of the pipe with insulation. In any event, the nature of the problem is not restricted to jet burners although that is the situation to be treated here.

After a mathematical model of such systems is formulated, analysis of the model usually proceeds with the formulation of differential equations describing heat flow in terms of physical characteristics, space variables, and time. The general differential equation of heat conduction was formulated by Jacques Fourier over 100 years ago. For constant conductivity and an absence of sources and sinks it is:

$$\frac{C\gamma}{k}\frac{\partial\Theta}{\partial t} = \mathrm{Div}\left\{\mathrm{Grad}\,\Theta\right\} \quad \text{.................................(A)}$$

where: Θ = temperature or temperature function, °F

t = time, secs.

C = specific heat, $\frac{\mathrm{Btu}}{\mathrm{lb._m{-}^\circ F}}$

γ = density, $\frac{\mathrm{lb._m}}{\mathrm{ft.}^3}$

k = thermal conductivity, $\frac{\mathrm{Btu}}{\mathrm{sec.{-}ft.{-}^\circ F}}$

The divergence and gradient may be expressed in any convenient space variables. The exact formulation of the problem also includes conditions to be imposed on the boundary of the mathematical model and sometimes includes an initial condition (or condition at time zero).

Furthermore, heat conduction problems are customarily categorized according to whether or not the heat flow from point to point is a function of time. If heat flow is not a function of time, Equation (A) reduces to:

$$\mathrm{Div}\left\{\mathrm{Grad}\,\Theta\right\} = 0 \qquad \text{(B)}$$

and is ordinarily called a steady-state problem. If heat flow is a function of time, Equation (A) holds and is ordinarily called a transient problem.

The mathematical model chosen to represent a given physical system is (as mentioned previously) never exact. Certain simplifying assumptions are customarily made in order to make the problem mathematically tractable. The boundary and initial conditions imposed on the Fourier heat conduction equation are most often approximations of the actual system conditions.

In the system which has been described, combustion gas temperatures are often as high as 2,000°F. Structural failures at the trailing edge and juncture of the flameholder and duct are commonplace occurrences. Furthermore, severe vibrations are often induced in the duct wall. In order to analyze whether the structural failures and/or induced vibrations are thermodynamic in origin, the temperature distribution and temperature gradients in the wall of the duct need to be known.

The overall objective of this paper is to find a useful analytical approach to the temperature distribution in such a system. Both the steady-state and transient problems are attempted.

SUMMARY

A thin-walled circular duct with airflow through and over it and heat addition over the aft inside half is considered. The formal statement of the problem in the rectangular region $0 \leq x \leq 2L$, $0 \leq y \leq t$ is:

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$$

such that:

$$\left.\frac{\partial \theta}{\partial x}\right|_{0,y} = 0$$

$$\left.\frac{\partial \theta}{\partial x}\right|_{2L,y} = 0$$

$$\left.\frac{\partial \theta}{\partial y}\right|_{x,0} = r\,\theta\Big|_{x,0}$$

$$\left.\frac{\partial \theta}{\partial y}\right|_{x,t} = \begin{cases} -r\,\theta\Big|_{x,t} & , 0 \leq x < L \\ -r_1\left(\theta\Big|_{x,t} - k\right) , & L < x \leq 2L \end{cases}$$

An approximate solution is expanded in a finite portion of an infinite series as follows:

$$\theta(x,y) = C_o\,(1 + ry) + \sum_{n=1}^{N} C_n \operatorname{Cos} \lambda_n x \left\{ e^{-\lambda_n y} + \frac{\lambda_n + r}{\lambda_n - r}\, e^{\lambda_n y} \right\}$$

$$\lambda_n = \frac{n\pi}{2L}$$

The form of the solution appears to be convenient for computations with either a desk calculator or computer; approximate values of the

coefficients C_n are shown to be given by the solution of an $(N + 1)$ by $(N + 1)$ segment of an infinite system of equations.

It is furthermore shown that the corresponding infinite series solution converges everywhere in the rectangular region except at two exceptional points where no proof is attempted. It is also shown that the convergence is uniform for $0 \leq y < t$.

Typical system parameters are calculated in Appendix A; these values are used in a numerical example problem. The error of the results of the example problem appears to be on the order of $\pm 10^{\circ}F$ in a $500^{\circ}F$ range.

Several exact solutions of the transient problem (temperature zero at time zero) are attempted. Two of the attempted solutions are unsuccessful. The use of a LaPlace integral transform seems to yield a solution, but it is in such a complicated form that it does not appear useful for calculations.

It seemed appropriate under these circumstances to search for either local solutions or bounds of the transient temperature. The expression:

$$\frac{\Theta}{1500} = \frac{1 + r(a - y)}{1 + ra} - \sum_{1}^{\infty} \frac{2\,(\beta_n + r^2)\, e^{-\alpha \beta_n^2 t} \operatorname{Sin} \beta_n y}{\beta_n\,(r + a\,(\beta_n^2 + r^2))}$$

$$\beta_n \operatorname{Cot} \beta_n a + r = 0;\ n = 1, 2, 3, \ldots$$

appears to be an approximate solution of the transient problem near the trailing edge.

Two problems which appear to bound the transient temperature distribution along the aft half of the duct are pointed out; no analysis of either of the problems is attempted.

Several approaches to the exact solution of the transient problem are pointed out and one (combined Fourier and LaPlace transform) is recommended for further study. Some recommendations concerning thermal stresses and thermally induced vibrations are also noted.

THEORETICAL DISCUSSION

It was mentioned previously that the general differential equation of heat conduction was derived by Jacques Fourier over 100 years ago. This equation has been derived in several ways in the literature.

Boelter, et al. (1, p. III-7 to III-9) derive Fourier's heat conduction equation using a heat balance on an arbitrary volume. The derivation includes the use of Green's theorem and is rather elegant. Sokolnikoff and Redheffer (12, p. 414) use Gauss's divergence theorem in a similar derivation on an arbitrary volume.

Both Ingersoll (7, p. 12) and Wylie (17, p. 206) use a heat balance on a rectangular parallelepiped to derive Fourier's heat conduction equation. Although these derivations are less elegant than those of Boelter and Sokolnikoff, they are more physically meaningful. The following derivation therefore follows closely those given by Ingersoll and Wylie.

Several preliminary observations should be accepted before continuing to the actual derivation. These are:

1. Energy can neither be created nor destroyed.
2. Extensive experiments have shown that heat energy flows in the direction of decreasing temperature.
3. Heat is defined as energy in transition due to a temperature difference; the quantity of heat required to produce a given temperature potential in a body is proportional to the mass of the body and the temperature potential.

4. The idea of heat flux. Imagine two parallel planes each of area A and distance Δx apart from each other. The temperature on each face is a constant and there is a temperature drop ΔT between. From observation (2) above, heat energy will flow from the hotter to the colder plane; the quantity of heat ΔQ transferred in time Δt is:

$$\Delta Q = - k \frac{\Delta T}{\Delta x} A \, (\Delta t).$$

k is a material constant called the thermal conductivity with units $\frac{\text{Btu}}{\text{hr.-ft.}^2\text{-}^\circ\text{F/ft.}}$.

Rearranging and taking the limit...

$$\frac{1}{A} \frac{\Delta Q}{\Delta t} = - k \frac{\Delta T}{\Delta x}$$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \left(\frac{1}{A} \frac{\Delta Q}{\Delta t} \right) = \lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \left(- k \frac{\Delta T}{\Delta x} \right)$$

$$\frac{1}{A} \frac{\partial Q}{\partial t} = - k \frac{\partial T}{\partial x}$$

The term on the left is commonly called the heat flux W, i.e.,

$$W = - k \frac{\partial T}{\partial x}; \; W = \text{Heat Flux}, \frac{\text{Btu}}{\text{hr.-ft.}^2}$$

Consider the infinitesimal volume in Figure 3.

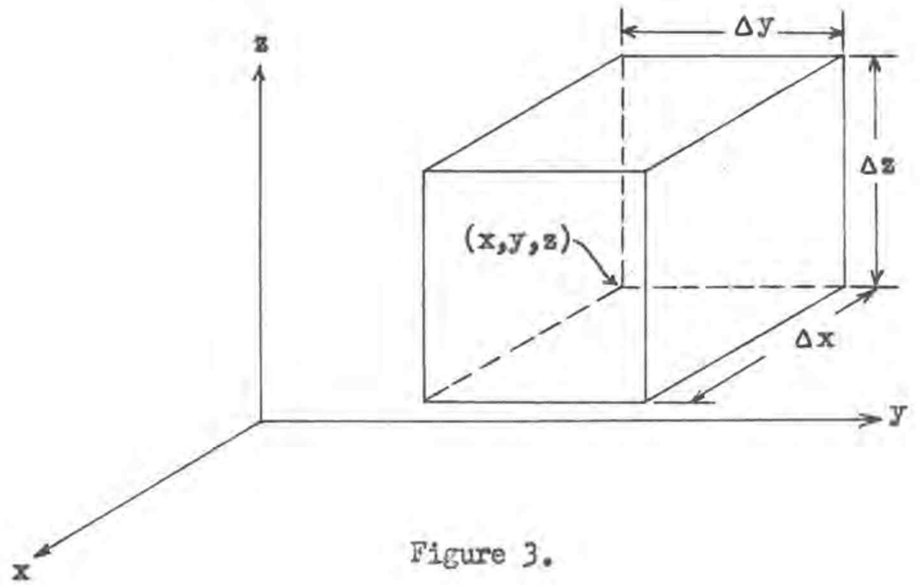

Figure 3.

There are no heat sources or sinks in this volume and it is assumed to be both homogeneous and isotropic with respect to all three space variables, temperature, and time.

A number of heat transfer modes take place in this volume as follows:

1. Heat energy is carried out of or into all six faces of the cube by reason of the temperature gradient normal to the faces. Most generally, this heat transfer may be assumed to take place by conduction.
2. Heat energy is stored within the infinitesimal cube if the temperature is a function of time.

Using idea (1) above, viz., energy can neither be created nor destroyed gives:

Heat stored = heat in by conduction......................(1)

This is a heat balance on the infinitesimal volume.

Using idea (3) above, the left term in equation (1) is

$$\text{Heat stored} = \Delta Q = C_p \Delta m \Delta T; \quad \Delta Q = \text{quantity of heat, Btu}$$

$$C_p = \text{specific heat, } \frac{\text{Btu}}{\text{lb.}_m\,^{\circ}\text{F}}$$

$$\Delta m = \text{mass of cube, lb.}_m$$

$$\Delta T = \text{temperature difference, } ^{\circ}\text{F}$$

$$\text{Heat stored} = C_p\,(\gamma \Delta x \Delta y \Delta z)\Delta T; \quad \Delta m = \gamma \Delta x \Delta y \Delta z$$

$$\gamma = \text{density, } \frac{\text{lb.}_m}{\text{ft.}^3}$$

Alternatively, for an infinitesimal increment of time Δt,

$$\frac{\text{Heat Stored}}{\text{Unit Time}} = C_p \gamma \, \Delta x \Delta y \Delta z \, \frac{\Delta T}{\Delta t} \qquad \text{(2)}$$

Turning attention now to the right-hand term of equation (1), the amount of heat flowing into the rear face of the infinitesimal cube in the x-direction per unit time is:

$$\frac{\Delta Q}{\Delta t} = -\,kA \left.\frac{\partial T}{\partial x}\right|_{\substack{x \\ y + 1/2\Delta y \\ z + 1/2\Delta z}}$$

$$\frac{\Delta Q}{\Delta t} = -\,k \Delta y \Delta z \left.\frac{\partial T}{\partial x}\right|_{\substack{x \\ y + 1/2\Delta y \\ z + 1/2\Delta z}}$$

The notation $\left.\frac{\partial T}{\partial x}\right|_{\substack{x \\ y + 1/2\Delta y \\ z + 1/2\Delta z}}$ indicates that the derivative is taken at the point $(x,\ y + 1/2\Delta y,\ z + 1/2\Delta z)$. The negative sign indicates that if the gradient $\frac{\partial T}{\partial x}$ is negative, heat flow is in the positive x-direction.

The heat gained through the front face in the x-direction is found by the same sort of reasoning to be:

$$\frac{\Delta Q}{\Delta t} = k\,\Delta y\,\Delta z\,\frac{\partial T}{\partial x}\Bigg|_{\substack{x+\Delta x\\ y+1/2\,\Delta y\\ z+1/2\,\Delta z}}$$

A similar line of reasoning holds in the orthogonal y- and z-directions to give:

<u>Heat in at</u>

$$y \quad = -\,k\,\Delta x\,\Delta z\,\frac{\partial T}{\partial y}\Bigg|_{\substack{y\\ x+1/2\,\Delta x\\ z+1/2\,\Delta z}}$$

$$y+\Delta y = \quad k\,\Delta x\,\Delta z\,\frac{\partial T}{\partial y}\Bigg|_{\substack{y+\Delta y\\ x+1/2\,\Delta x\\ z+1/2\,\Delta z}}$$

$$z \quad = -\,k\,\Delta x\,\Delta y\,\frac{\partial T}{\partial z}\Bigg|_{\substack{z\\ x+1/2\,\Delta x\\ y+1/2\,\Delta y}}$$

$$z+\Delta z = \quad k\,\Delta x\,\Delta y\,\frac{\partial T}{\partial z}\Bigg|_{\substack{z+\Delta z\\ x+1/2\,\Delta x\\ y+1/2\,\Delta y}}$$

Turning now to the heat balance described by equation (1), there is...

$$C_p \gamma \Delta x \Delta y \Delta z \frac{\partial T}{\partial t} = k \Delta y \Delta z \left(\left.\frac{\partial T}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial T}{\partial x}\right|_{x} \right) +$$

$$+ k \Delta x \Delta z \left(\left.\frac{\partial T}{\partial y}\right|_{y+\Delta y} - \left.\frac{\partial T}{\partial y}\right|_{y} \right) +$$

$$+ k \Delta x \Delta y \left(\left.\frac{\partial T}{\partial z}\right|_{z+\Delta z} - \left.\frac{\partial T}{\partial z}\right|_{z} \right)$$

Dividing by the volume $\Delta x \Delta y \Delta z$, we have alternatively...

$$C_p \gamma \frac{\Delta T}{\Delta t} = k \left\{ \frac{\left.\frac{\partial T}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial T}{\partial x}\right|_{x}}{\Delta x} + \frac{\left.\frac{\partial T}{\partial y}\right|_{y+\Delta y} - \left.\frac{\partial T}{\partial y}\right|_{y}}{\Delta y} + \right.$$

$$\left. + \frac{\left.\frac{\partial T}{\partial z}\right|_{z+\Delta z} - \left.\frac{\partial T}{\partial z}\right|_{z}}{\Delta z} \right\}$$

Taking the limit in all variables gives...

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0 \\ \Delta t \to 0}} C_p \gamma \frac{\Delta T}{\Delta t} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0 \\ \Delta t \to 0}} k \left\{ \frac{\left.\frac{\partial T}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial T}{\partial x}\right|_{x}}{\Delta x} + \right.$$

$$+ \frac{\left.\frac{\partial T}{\partial y}\right|_{y+\Delta y} - \left.\frac{\partial T}{\partial y}\right|_{y}}{\Delta y} +$$

$$\left. + \frac{\left.\frac{\partial T}{\partial z}\right|_{z+\Delta z} - \left.\frac{\partial T}{\partial z}\right|_{z}}{\Delta z} \right\}$$

The limit on the left is independent of x, y, and z, just as the limits on the right depend only upon one space coordinate. Further, the limit of a constant times a variable is the constant times the limit of the variable. Using these facts,

$$C_p \gamma \lim_{\Delta t \to 0} \frac{\Delta T}{\Delta t} = k \left\{ \lim_{\Delta x \to 0} \frac{\left.\frac{\partial T}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial T}{\partial x}\right|_{x}}{\Delta x} + \right.$$

$$+ \lim_{\Delta y \to 0} \frac{\left.\frac{\partial T}{\partial y}\right|_{y+\Delta y} - \left.\frac{\partial T}{\partial y}\right|_{y}}{\Delta y} +$$

$$\left. + \lim_{\Delta z \to 0} \frac{\left.\frac{\partial T}{\partial z}\right|_{z+\Delta z} - \left.\frac{\partial T}{\partial z}\right|_{z}}{\Delta z} \right\}$$

Proceeding to the limit indicated gives

$$C_p \gamma \frac{\partial T}{\partial t} = k \left\{ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right\}$$

$$\frac{\partial T}{\partial t} = \frac{k}{C_p \gamma} \left\{ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right\}$$

The material constant $\frac{k}{C_p \gamma}$ is sometimes abbreviated as α and was termed by Kelvin as the "thermal diffusivity." Using this abbreviation,

$$\frac{\partial T}{\partial t} = \alpha \left\{ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Equation (3) is one form of Fourier's heat conduction equation. In order to facilitate calculations on certain geometries, it may be transformed to other coordinate systems by use of the chain rule for partial derivatives. These transformations may be found in almost any advanced calculus text (17, p. 589) and will not be discussed here.

Analogous to boundary conditions imposed on the solution of ordinary differential equations, it is necessary to express formulae describing initial and boundary conditions on a partial differential equation. The solution of the differential equation is customarily formulated in a form containing unknown constants. These constants are generally determined by imposing initial and/or boundary conditions on the formal solution.

The initial condition is a formulation of the temperature at that time arbitrarily taken as zero. In general, the temperature at time zero would be some function of the space coordinates. The initial condition might then be formulated as:

$$T\Big|_{t=0} = f(x,y,z)$$

A stricter interpretation of this initial condition is "The solution of Fourier's heat conduction equation must be such that its limit as time approaches zero is f(x,y,z)." Formulated strictly mathematically, the initial condition would be

$$\lim_{t \to 0} (T(x,y,z,t)) = f(x,y,z)$$

In the general case, Fourier's heat conduction equation is solved for a surface or a volume. For such a case, certain conditions would need to be met on a bounding line or surface. Such surface conditions generally fall into one of five categories, as follows:

1. Prescribed Surface Temperature. This boundary condition supposes that one of the boundaries is maintained at some prescribed temperature. This boundary condition is rather easy to work with (it has been studied extensively) but it is ordinarily difficult to specify a temperature on a given surface for most physical problems.

2. No Heat Flux Across a Surface. This condition arises most generally on a surface that is well insulated. In such a case, there would be little heat flux; as an approximation, it might be assumed that there was no heat flow. This condition is prescribed by:

$$\left.\frac{\partial T}{\partial n}\right|_{\text{Surface}} = 0; \; n = \text{an outward drawn normal to the surface.}$$

3. Prescribed Flux Across the Surface. This boundary condition is useful, e.g., when a body is poorly insulated. Under these circumstances, the heat flux from the body to the insulation might be stipulated. The heat flux would in general be some function of space variables and time and would be formulated as:

$$\left.\frac{\partial T}{\partial N}\right|_{\text{Surface}} = f(x,y,z,t)$$

4. Newton's Law of Cooling. If heat transfer takes place by radiation and/or convection into some surrounding medium, the heat flux may be assumed proportional to the temperature difference between body and medium. In fact, many investigators (11, p. 204) in the heat-transfer field derive empirical relations for heat transfer coefficients on the supposition that this is true. Experience has shown it to be a good approximation for convective heat transfer, especially if the temperature difference between body and medium is not large. The mathematical formulation of the boundary condition is:

$$\left.\frac{\partial T}{\partial n}\right|_{\text{Surface}} = - r \left\{ T\Big|_{\text{Surface}} - T\Big|_{\substack{\text{Surrounding}\\ \text{Medium}}} \right\}$$

where: $r = h/k$

h = heat transfer coefficient, $\dfrac{\text{Btu}}{\text{hr.-ft.}^2\text{-}^\circ\text{F}}$

k = thermal conductivity of body, $\dfrac{\text{Btu}}{\dfrac{\text{hr.-ft.}^2\text{-}^\circ\text{F}}{\text{ft.}}}$

Due account must be taken of the sign depending upon the direction of heat transfer.

5. The Mutual Surface of Separation Between Two Media of Conductivities K_1 and K_2. Along the interface between the two media, heat energy is neither created

nor destroyed and the heat flux must therefore be constant at that surface, i.e.,

$$K_1 \frac{\partial T_1}{\partial N} = K_2 \frac{\partial T_2}{\partial N}; \; N = \text{An outward drawn normal to the surface of separation.}$$

Once a problem has been reduced to a mathematical model, several fundamental approaches might be used in the solution of the mathematical model. The division between approaches is rather arbitrary; as often as not, one or more methods complement a basic approach. By way of a gross survey, approaches might be broken into five categories, viz., experimental methods, analytical procedures, graphical methods, numerical procedures, and analogies or models.

Experimental techniques are generally employed when a few specific temperatures in a region need to be known. If a fine network of temperatures is needed, equipment and installation costs would be prohibitive and the results subject to much error.

On some occasions, it has been useful to establish temperature distributions by the use of "thermal paints." A thermal paint changes color when raised to a specified temperature and may be used to bracket the temperature on some given surface. These paints have a tendency to peel off under service and usually indicate temperatures within, say, a $\pm 50^{\circ}F$ range.

Analytical procedures are usually desirable in attacking a heat conduction problem since they are ordinarily the least expensive from an equipment standpoint. Analytical procedures usually fall into one of two classifications: (1) the classical method of separation of variables and (2) operational transform methods. These methods may be found in any mathematical physics text (12, pp. 455-504).

Graphical methods are sometimes employed for crude approximations of temperature distributions in relatively simple systems (11, pp. 44-51). Such methods afford at best an estimate unless a great deal of time is expended; they seem to have little application for the purposes of this paper.

Numerical methods of analysis have been extensively developed during the past ten years; the digital computer has had a marked impact in the area of heat conduction. Digital computers are especially useful in solving systems of equations which sometimes arise in heat conduction problems. The systems of equations are ordinarily handled with some iterative scheme; convergence criteria for such iterative methods have attracted much attention.

The most serious drawback of a digital computer lies in the fact that it is rather expensive. Many small concerns in industry cannot afford such equipment and must resort to analytical procedures or rent computer time.

Some work has been done in the area of analogies and models of heat conduction phenomena. Such analogies might be made with either electrical or hydraulic systems. Analog computers have been of some use in such electrical analogies, but are expensive in comparison with digital computers. For that reason, they have largely been overshadowed by the digital computer in the area of heat conduction.

LITERATURE SURVEY

A. Steady State

The general problem of heat conduction in a two-dimensional region was studied by Jacques Fourier many years ago. In fact, his study on the conduction of heat in a semi-infinite strip led him to the expansion of unity, viz.,

$$1 = \frac{4}{\pi}\left\{ \text{Cos } x - \frac{1}{3}\text{ Cos } 3x + \frac{1}{5}\text{ Cos } 5x - \ldots\right\} \;;\; -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

This prompted Fourier to consider expansion of an arbitrary function in a trigonometric series.

A survey of the more elementary problems is given in Carslaw and Jaeger (2, pp. 142-152). Problems somewhat similar to the one presented in this paper are treated on pages 146-149 of Carslaw and Jaeger. They are all attacked in the classical manner.

Problems which give rise to infinite systems of equations are treated in Kantorovich and Krylov (8, pp. 54-68). Two examples of such problems are treated. Properties of infinite systems of equations are also outlined on pages 20-44.

B. Transient Case

Thiruvenkatachar treats a transient problem similar to the one presented here by treating it as a two-region problem (15, pp. 255-262). The problem lends itself to cylindrical coordinates and he attacks it with the LaPlace transform.

Tranter (16, pp. 104-110) treats the subject of combined use of relaxation methods and integral transforms. Such methods appear to have some use in the present problem; the method will be dwelt upon in the Results section.

Under some circumstances, relaxation-type procedures may be used in transient, two-dimensional problems. In general, only elementary problems have been solved in this manner.

One notable exception to this is given in the literature (4, pp. 1155-1161). Hellman, et al., treat a rather complicated system by numerical methods. It appears that the solutions obtained leave much to be desired from the standpoint of accuracy but the problem appears unassailable by any technique other than numerical.

A more or less generalized finite-difference method of attack for transient two-dimensional problems is given by Liebman (10, pp. 129-135). He treats both one- and two-dimensional transient problems and gives a rather simple example of a two-dimensional problem.

RESULTS AND CALCULATION

STEADY STATE PROBLEM

Before proceeding to a formal problem statement, the assumptions underlying the mathematical model should be dwelt upon. The physical problem was described in the Introduction section and will not be repeated here.

At first glance, the problem seems to lend itself to cylindrical coordinates. Fourier's heat conduction equation in cylindrical coordinates is:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} = K \frac{\partial T}{\partial t}$$

T = temperature, °F

r = radius space variable, ins.

ϕ = angular space variable, radians

z = axial space variable, ins.

We are considering steady state so that $\frac{\partial T}{\partial t} = 0$ and there is no temperature variation in the ϕ-direction so $\frac{\partial^2 T}{\partial \phi^2} = 0$; Fourier's equation reduces to $\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0$.

Recall now that each of the terms $\frac{\partial^2 T}{\partial x^2}$, $\frac{\partial^2 T}{\partial y^2}$, and $\frac{\partial^2 T}{\partial z^2}$ in Fourier's heat conduction equation (3) resulted from considering heat flow in the x, y, and z directions respectively. From a similar derivation on a cylindrical wedge (1, p. III-10), it can be shown that the terms $\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}$ in the last equation arise from considerations of heat flow in the r-direction. Moreover, the term $\frac{1}{r} \frac{\partial T}{\partial r}$ arises due to the curvature of the duct; this may be seen from a comparison of the last equation with the heat conduction equation for a two-dimensional flat plate, which is $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$.

Now consider the heat conduction in the r-direction. It is well known (1, p. IIa-8) that for thin circular shells, i.e., where $\frac{\text{outside radius}}{\text{inside radius}} \leq 2$, the heat flow in the radial direction is given within four percent error by using an equivalent flat plate area or by neglecting the curvature of the shell.

Typical dimensions for the system considered here show that the ratio $\frac{\text{outside radius}}{\text{inside radius}}$ is much less than 2. Hence the curvature of the plate, i.e., the term $\frac{1}{r} \frac{\partial T}{\partial r}$, may legitimately be neglected leaving:

$$\frac{\partial^2 T}{\partial r^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

This is LaPlace's equation and might just as well be expressed in the more familiar form:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0;$$

T = temperature, °F

x = space variable, in.

y = space variable, in.

The problem statement must include appropriate conditions on the boundary. For reference purposes, a section of the duct-wall is shown in x-y coordinates in Figure 4.

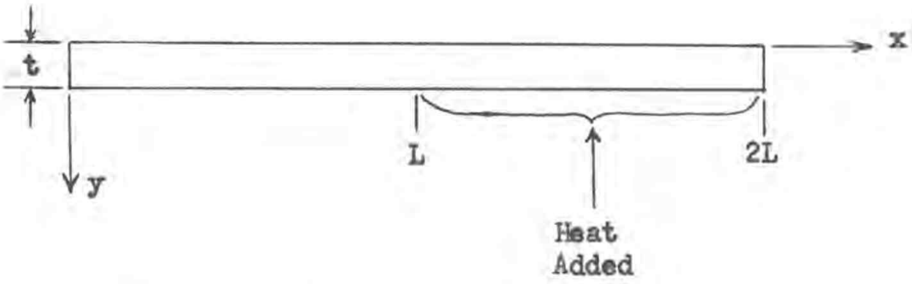

Figure 4. Cross-section of the duct wall in x,y coordinates.

All the heat added to the rectangle is added on the surface $y = t$ in $L < x \leq 2L$. The length of the rectangle is (as mentioned previously) roughly 100 times the width. Most of the heat lost in the interval $0 \leq x < L$ will be from the surfaces $y = 0$ and $y = t$ since they have a much greater heat-transfer area. As a result, the heat flux at the surface $x = 0$ is very small so that:

$$\left.\frac{\partial T}{\partial x}\right|_{x = 0} = 0$$

Heat will be lost from the surface $y = 0$ by convection and radiation into the air stream along that surface. The air stream will increase slightly in temperature from the left to the right end. Assume along this surface that Newton's law of cooling holds and that the temperature rise of the air stream is small enough to be negligible. Then:

$$\left.\frac{\partial T}{\partial y}\right|_{y = 0} = r\Theta\Big|_{y = 0} \quad ;$$

r = combined radiative and convective heat transfer coefficient, in.$^{-1}$.

Θ = temperature difference between surface and air stream, °F.

Consider heat transfer along the surface $x = 2L$. Two factors tend to produce heat flow in the x-direction along $x = 2L$. They are: (1) the influence of the heat transfer discontinuity at $x = L$ and (2) a temperature difference (if any) between the duct and ambient fluid. Using the same line of reasoning as along $x = 0$, the influence of the discontinuity will probably be negligible. It appears that the heat energy added along $y = t$ near $x = 2L$ will tend to flow straight across the metal since this is the path of least resistance. Furthermore, it appears that the gas temperature along $x = 2L$ is on the same order of magnitude as the duct temperature so that convective and radiant heat losses from this surface are minimized. These considerations point to the fact that the heat transfer along $x = 2L$ may be described by:

$$\left.\frac{\partial T}{\partial x}\right|_{x = 2L} = 0$$

The boundary condition along the surface $y = t$ is much harder to describe. If it is assumed that the temperature of the hot combustion products is constant over the interval $L < x \leq 2L$ and that an "average" velocity may be ascribed to the hot gases in that interval, then Newton's law of cooling may be used to describe the boundary condition. A detailed analysis of the heat transfer coefficients is carried out in Appendix A; the boundary condition is:

$$\left.\frac{\partial T}{\partial y}\right|_{y = t} = \begin{cases} -r\Theta, & 0 \le x < L \\ -r_1 (\Theta - k), & L < x \le 2L \end{cases}$$

r = combined radiative and convective heat transfer coefficient in $0 \le x < L$, in.$^{-1}$

r_1 = combined radiative and convective heat transfer coefficient in $L < x \le 2L$, in.$^{-1}$

k = a constant temperature, viz., the difference in temperature between hot gases and air stream, $^{\circ}F$.

Using the transformation $\Theta = T - T_{ambient}$, the formal statement of the problem is:

$$\frac{\partial^2\Theta}{\partial x^2} + \frac{\partial^2\Theta}{\partial y^2} = 0 \quad \text{.......................................(4)}$$

$$\left.\frac{\partial\Theta}{\partial x}\right|_{0,y} = 0 \quad \text{.......................................(5)}$$

$$\left.\frac{\partial\Theta}{\partial x}\right|_{2L,y} = 0 \quad \text{.......................................(6)}$$

$$\left.\frac{\partial\Theta}{\partial y}\right|_{x,0} = r\Theta\Big|_{x,0} \quad \text{.......................................(7)}$$

$$\left.\frac{\partial\Theta}{\partial y}\right|_{x,t} = \begin{cases} -r\Theta\Big|_{x,t} & , 0 \le x < L \\ -r_1\left\{\Theta\Big|_{x,t} - k\right\} & , L < x \le 2L \end{cases} \quad \text{....................(8)}$$

The expression:

$$\Theta(x,y) = C_o\ (1+ry)+\sum_{n=1}^{\infty} C_n\ \text{Cos}\,\lambda_n x \left\{e^{-\lambda_n y} + \frac{\lambda_n+r}{\lambda_n-r}\, e^{\lambda_n y}\right\} \ldots\ldots(9)$$

$$\lambda_n = \frac{n\pi}{2L};\ n = 1,\ 2,\ 3,\ldots$$

satisfies (4), (5), (6), and (7). It remains then to adjust the coefficients C_n to satisfy boundary condition (8). Using equation (9), boundary condition (8) gives:

$$C_o r + \sum_{n=1}^{\infty} C_n \lambda_n\ \text{Cos}\,\lambda_n x \left\{\frac{\lambda_n+r}{\lambda_n-r}\, e^{\lambda_n t} - e^{-\lambda_n t}\right\} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

$$= \begin{cases} -rC_o\ (1+rt) - r\displaystyle\sum_{n=1}^{\infty} C_n\ \text{Cos}\,\lambda_n x \left\{e^{-\lambda_n t} + \frac{\lambda_n+r}{\lambda_n-r}\, e^{\lambda_n t}\right\} & ,\ 0 \leq x < L \\ -r_1 C_o(1+rt) - r_1\displaystyle\sum_{n=1}^{\infty} C_n\ \text{Cos}\,\lambda_n x \left\{e^{-\lambda_n t} + \frac{\lambda_n+r}{\lambda_n-r}\, e^{\lambda_n t}\right\} + r_1 k, & L < x \leq 2L \end{cases}$$

The sequence of functions $\left\{\text{Cos}\,\lambda_n x\right\}$ is orthogonal in the interval (0, 2L) but not in (0, L) or (L, 2L). The expressions on the right of equation (10) are therefore non-orthogonal due to their interval.

Since this last expression is a function only of x by reason of the terms $\left\{\text{Cos}\,\lambda_n x\right\}$, we may rid the expression of the x space-variable according to the method of Fourier. Multiplying equation (10) by $\text{Cos}\,\lambda_m x dx$ where m = 0, 1, 2,... on the assumption that termwise integration is legitimate (see Appendix C), integration in the appropriate intervals gives:

$$C_o r \int_0^{2L} \cos \lambda_m x dx +$$

$$+ \sum_{n=1}^{\infty} C_n \lambda_n \left\{ \frac{\lambda_n + r}{\lambda_n - r} e^{\lambda_n t} - e^{-\lambda_n t} \right\} \int_0^{2L} \cos \lambda_n x \cos \lambda_m x dx =$$

$$= - r C_o (1 + rt) \int_0^{L} \cos \lambda_m x dx + r_1 (k - C_o (1 + rt)) \int_L^{2L} \cos \lambda_m x dx -$$

$$- r \sum_{n=1}^{\infty} C_n \left\{ e^{-\lambda_n t} + \frac{\lambda_n + r}{\lambda_n - r} e^{\lambda_n t} \right\} \int_0^{L} \cos \lambda_n x \cos \lambda_m x dx -$$

$$- r_1 \sum_{n=1}^{\infty} C_n \left\{ e^{-\lambda_n t} + \frac{\lambda_n + r}{\lambda_n - r} e^{\lambda_n t} \right\} \int_L^{2L} \cos \lambda_n x \cos \lambda_m x dx \text{............} (11)$$

Rearranging equation (11) gives:

$$-r_1 (k - C_o (1 + rt)) \int_L^{2L} \cos \lambda_m x dx +$$

$$+ C_o r \int_0^{2L} \cos \lambda_m x dx + r C_o (1 + rt) \int_0^{L} \cos \lambda_m x dx +$$

$$+ \sum_{n=1}^{\infty} C_n \lambda_n \left\{ \frac{\lambda_n + r}{\lambda_n - r} e^{\lambda_n t} - e^{-\lambda_n t} \right\} \int_0^{2L} \cos \lambda_n x \cos \lambda_m x dx =$$

$$= - r \sum_{n=1}^{\infty} C_n \left\{ e^{-\lambda_n t} + \frac{\lambda_n + r}{\lambda_n - r} e^{\lambda_n t} \right\} \int_0^{L} \cos \lambda_m x \cos \lambda_n x dx -$$

$$-r_1 \sum_{n=1}^{\infty} C_n \left\{ e^{-\lambda_n t} + \frac{\lambda_n + r}{\lambda_n - r} e^{\lambda_n t} \right\} \int_L^{2L} \cos \lambda_m x \cos \lambda_n x dx \text{.............} (12)$$

The terms on the left of equation (12) depend only upon m since $\int_0^{2L} \cos\lambda_m x \cos\lambda_n x dx$ is not zero only when m = n. Arbitrarily, call the term in brackets $\beta_{m,n}$. The evaluation of the integrals is carried out in Appendix B; the result is:

$$\underline{\underline{m=0}},\ C_0\left\{rL(3+rt)+r_1L(1+rt)\right\}+$$

$$+\frac{2(r-r_1)L}{\pi}\sum_{n=1}^{\infty} C_n\left\{e^{-\lambda_n t}+\frac{\lambda_n+r}{\lambda_n-r}e^{\lambda_n t}\right\}\frac{\sin\frac{n\pi}{2}}{n}=r_1kL\text{........}(13)$$

$$\underline{\underline{m\neq 0}},\ C_m\left(\frac{m\pi}{2}\right)\left\{\frac{\lambda_m+r}{\lambda_m-r}e^{\lambda_m t}-e^{-\lambda_m t}\right\}+$$

$$\frac{2L}{\pi}\left\{(r-r_1)(1+rt)\ C_0+r_1k\right\}\frac{\sin\frac{m\pi}{2}}{m}=\beta_{m,n}\text{..............}(14)$$

$$\underline{\underline{m\neq n}},\ \beta_{m,n}=-\frac{(r-r_1)L}{\pi}\sum_{\substack{n=1\\ n\neq m}}^{\infty} C_n\left\{e^{-\lambda_n t}+\frac{\lambda_n+r}{\lambda_n-r}e^{\lambda_n t}\right\}\cdot$$

$$\cdot\left\{\frac{\sin(m+n)\frac{\pi}{2}}{m+n}+\frac{\sin(m-n)\frac{\pi}{2}}{m-n}\right\}\text{...}(15)$$

$$\underline{\underline{m=n}},\ \beta_{m,n}=-\frac{(r+r_1)L}{2}\ C_m\left\{e^{-\lambda_m t}+\frac{\lambda_m+r}{\lambda_m-r}e^{\lambda_m t}\right\}\text{.................}(16)$$

Equations (13), (14), (15), and (16) represent an infinite system of equations of the form:

$$C_m=\sum_{n=0}^{\infty} D_{m,n}\ C_n+B_m;\ m=0,1,2,3,\ldots$$

It is shown in Appendix C that the solution of a finite segment of the system of equations (13), (14), (15), (16) gives approximate solutions of the coefficients C_n. It is also shown there that the use of these approximate values of C_n give an approximate solution to the problem posed.

An approximate solution of the problem posed is then given by:

$$\Theta(x,y) = C_o\,(1+ry) + \sum_{n=1}^{N} C_n \operatorname{Cos} \lambda_n x \left\{ e^{-\lambda_n y} + \frac{\lambda_n + r}{\lambda_n - r} e^{\lambda_n y} \right\} \quad \ldots(17)$$

Where the C_n are solutions of a segment of system (13), (14), (15), and (16) and N is a finite number.

Consider the expression $\frac{\lambda_n + r}{\lambda_n - r}$ in the summand of equation (17). This function is not well behaved; if some value of $\lambda_n = \frac{n\pi}{2L}$ is close to the value of the heat transfer function r, $\frac{\lambda_n + r}{\lambda_n - r}$ becomes very large. Schematically, it behaves as shown in Figure 5.

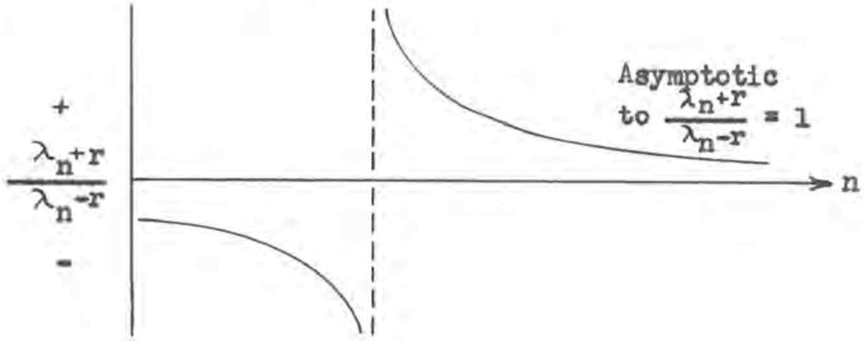

Figure 5. Schematic of the function $\frac{\lambda_n + r}{\lambda_n - r}$.

With the heat transfer function r derived in Appendix A, the break in the curve occurs in $9 < n < 10$. In order to arrive at a reasonably good approximation of the temperature, the break in $\frac{\lambda_n^{+r}}{\lambda_n^{-r}}$ would at least need to be bracketed. As a first approximation, it might therefore be assumed that twenty terms of the infinite system (13), (14), (15), and (16) are required.

The analysis of this system of twenty equations in twenty unknowns C_n may be handled rather easily by noting that the terms on the main diagonal, i.e., terms for which $m = n$, are invariably the largest terms in each equation. The finite system of twenty equations is in shorthand notation as follows:

$$A_{0,0}\,C_0 + A_{0,1}\,C_1 + A_{0,2}\,C_2 + \cdots + A_{0,19}\,C_{19} = B_0$$

$$A_{1,0}\,C_0 + A_{1,1}\,C_1 + A_{1,2}\,C_2 + \cdots + A_{1,19}\,C_{19} = B_1$$

$$A_{2,0}\,C_0 + A_{2,1}\,C_1 + A_{2,2}\,C_2 + \cdots + A_{2,19}\,C_{19} = B_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$A_{19,0}\,C_0 + A_{19,1}\,C_1 + A_{19,2}\,C_2 + \cdots + A_{19,19}\,C_{19} = B_{19}$$

The $A_{m,n}$ are known coefficients and the B_m are also known. Using a superscript to indicate the iterative order of the approximation and noting again that the principal terms are on the main diagonal, neglect the terms to the right of the main diagonal and write:

$$C_0^{(1)} = \frac{B_0}{A_{0,0}}$$

$$C_1^{(1)} = \frac{B_1 - A_{1,0}\, C_0^{(1)}}{A_{1,1}}$$

$$C_2^{(1)} = \frac{B_2 - A_{2,0}\, C_0^{(1)} - A_{2,1}\, C_1^{(1)}}{A_{2,2}}$$

....

$$C_n^{(1)} = \frac{B_n - A_{n,0}\, C_0^{(1)} - A_{n,1}\, C_1^{(1)} - \dots - A_{n,n-1}\, C_{n-1}^{(1)}}{A_{n,n}}$$

As a second approximation, use the complete zeroth equation to arrive at $C_0^{(2)}$ and carry out the same operations previously indicated:

$$C_0^{(2)} = \frac{B_0 - A_{0,1}\, C_1 - A_{0,2}\, C_2 - \dots - A_{0,19}\, C_{19}}{A_{0,0}}$$

$$C_1^{(2)} = \frac{B_1 - A_{1,0}\, C_0^{(2)}}{A_{1,1}}$$

$$C_2^{(2)} = \frac{B_2 - A_{2,0}\, C_0^{(2)} - A_{2,1}\, C_1^{(2)}}{A_{2,2}}$$

....

$$C_n^{(2)} = \frac{B_n - A_{n,0}\, C_0^{(2)} - A_{n,1}\, C_1^{(2)} - \dots - A_{n,n-1}\, C_{n-1}^{(2)}}{A_{n,n}}$$

The procedure may be repeated until the desired accuracy of results is obtained.

Using this iterative procedure in conjunction with the typical constants from Appendix A, the system of equations (13), (14), (15), and (16) was solved for C_0, C_1, ..., C_{19}. It was found that the second iteration produced no change in the fourth decimal place except for C_0; the process was accordingly stopped after two iterations. The results, i.e., $C_0^{(2)}$, $C_1^{(2)}$, ..., $C_{19}^{(2)}$ are given in Appendix D along with several plots of the temperatures obtained by using equation (17) with twenty terms.

In order to estimate the error in the temperatures given in Appendix D, consider the region of interest from a physical standpoint. The typical thickness or y-dimension is one-fourth inch as compared to a typical total length or x-dimension of 48 inches. The only heat transfer to the left end, i.e., to x = 0, is by means of conduction along the plate in the minus x-direction (see Figure 6).

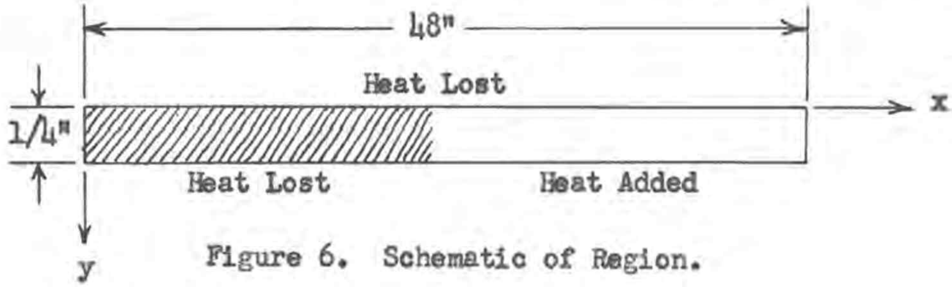

Figure 6. Schematic of Region.

There is a much greater tendency for the heat to escape from surfaces y = 0 and y = t in the single cross-hatched region because these surfaces present a far greater area for convective heat transfer than the cross-sectional area presents for conduction along the plate.

These considerations point to the fact that the temperature at the left end of the region should be close to the ambient temperature. The series solution (17) using the twenty coefficients in Appendix D gives a temperature range 311.2°F to 313.3°F or temperatures about 13°F above ambient along x = 0 (recall that the ambient temperature assumed was 300°F). The temperature at the left end therefore checks the physical considerations remarkably well.

In order to estimate the temperature on the right end of the region, consider an element taken out of the region in Figure 6 at $x = 2L$, i.e., at the right end. Such a rod is shown in Figure 7; since the derivatives $\frac{\partial \Theta}{\partial x} = \frac{\partial \Theta}{\partial z} \triangleq 0$ on the right end, the rod's curved surface is effectively insulated.

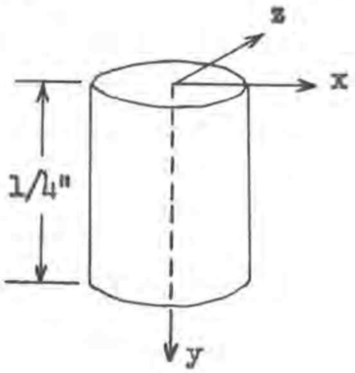

Figure 7. Element taken from body in the vicinity of $x = 2L$.

The heat flow in this rod is therefore one dimensional and for steady state,

$$\frac{\partial^2 \Theta}{\partial y^2} = 0 \quad \text{...}(18)$$

$$\left.\frac{\partial \Theta}{\partial y}\right|_{y=t} = r_1 \left(1500 - \left.\Theta\right|_{y=t}\right) \quad \text{..........................}(19)$$

$$\left.\frac{\partial \Theta}{\partial y}\right|_{y=0} = r \left.\Theta\right|_{y=0} \quad \text{..................................}(20)$$

where boundary conditions (19) and (20) coincide with those used in the actual problem at the right end. $\Theta = T - 300^{\circ}F$ as before.

The solution of this problem is straightforward; it is:

$$\Theta(y) = \frac{1500\ r_1}{r_1 + r\ r_1\ t + r}\ (1 + ry) \qquad (21)$$

At the points $y = 0$ and $y = t$, equation (21) gives:

$\Theta(0) = 446°F$

$\Theta(t) = 515°F$

so that the temperatures are 746°F and 815°F respectively.

The solution of equation (21) is shown superimposed on the result of the series solution (17) using twenty coefficients in Appendix D, Figure 3D. It may be seen there that the slope of the two curves is nearly the same and that the temperatures differ by only a few degrees.

The trends exhibited in the temperature plots in Appendix D may also be checked by physical considerations. Referring again to Figure 6, the heat added along $y = t$ in the interval $L < x \leq 2L$ will have more tendency to conduct straight across the metal duct than it will to be conducted along the plate in the x-direction except in the vicinity of $x = L$. The temperature on the left half of the duct should be close to ambient since there is a much larger surface area for heat convection away from the plate than there is for heat conduction along the plate in the minus x-direction.

These physical considerations lead to the conclusion that the temperature distribution in the x-direction (i.e., along any $y =$ constant) should be:

1. about ambient in $0 \leq x < L$ except within several thicknesses of $x = L$.

2. about the same as the right-end temperature in $L < x \leq 2L$ except within several thicknesses of $x = L$.

3. in the vicinity of $x = L = 24"$, there should be a sharp rise in temperature from roughly ambient to the right-end temperature.

Three temperature plots along $y = 0$, $y = 0.12"$, and $y = t = 0.25"$ are given in Appendix D as Figure 1D. All three plots exhibit the characteristics outlined from physical considerations. Temperatures in the left half of the duct are nearly ambient, in the right half nearly the same as the right-end temperature, and there is a sharp rise in temperature at the middle of the duct.

Two temperature plots along $x = L = 24"$ and $x = 2L = 48"$ are given in Appendix D as Figures 2D and 3D. Both plots show a quasi-linear characteristic as they should. A temperature plot along $x = 0"$ has not been included since the error in the temperatures given by equation (17) appears to be on the order of five times as great as the actual variation of temperature in the vicinity of $x = 0"$.

Barring the exceptional point $x = 22"$, the results of the analysis appear to deviate $\pm 10^{\circ}F$ from a mean curve. A more refined analysis considering perhaps thirty equations in thirty unknowns would probably reduce this deviation considerably.

The point $x = 22''$ appears to have an exceptionally large error. No matter what value of y is taken, the temperature at $x = 22''$ could be no less than ambient and the series solution (17) with $N = 19$ gives the temperature here as roughly 50°F below ambient.

It was pointed out in Appendix C that there was a likelihood of a many-valued solution at (L, t). It is well known that a finite number of terms in a trigonometric expansion cannot accurately represent a many-valued function and this fact is ordinarily manifested as Gibb's phenomenon or an oscillation in the region of the non-unique solution. Although the evidence is superficial, the error at $x = 22''$ appears to be due to Gibb's phenomenon.

Several isothermal lines are plotted in Figure 4D, Appendix D. Such a plot is sometimes convenient for visualization of the direction of heat flow.

As a matter of interest, it should be mentioned that the steady-state problem just considered may also be treated as a two-region problem. Separate the given region into the parts $0 \leq x \leq L$ and $L \leq x \leq 2L$ and match the temperature function and its first derivative along the common face $x = L$. This approach gives rise to an infinite system of equations just as the non-orthogonal series approach did and therefore seems to have no inherent advantage other than perhaps being slightly more elegant.

TRANSIENT PROBLEM

The assumptions used in the transient problem are the same as those for the steady-state problem insofar as the space coordinates are concerned. It is only necessary to treat the assumption(s) used in the initial condition.

In any high-velocity flow system, wall friction becomes appreciable. An adiabatic wall assumes a temperature between the bulk temperature of the stream and the stagnation or total temperature. The wall in the given system has flow over both the inside and outside and there is therefore little or no heat transfer across the wall during cold flow or prior to the time burning is initiated. This indicates that the wall will come to a temperature close to the adiabatic wall temperature during the soaking period. The adiabatic wall temperature is given by:

$$T_{AW} = T_m + N_{RF}\,(T_S - T_m);$$

T_{AW} = Adiabatic wall temperature, °F

N_{RF} = Recovery factor

T_S = Stagnation temperature, °F

T_m = Bulk temperature, °F

At a Mach Number of 0.5 as is supposed here, there is very little difference between the bulk and stagnation temperatures and the adiabatic wall temperature is thus very nearly the same as the bulk stream temperature.

If it is assumed that there is a negligible difference between bulk stream temperature and adiabatic wall temperature at time zero, the transient problem reduces to:

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = k\,\frac{\partial\theta}{\partial t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(22)$$

$$\left.\frac{\partial \Theta}{\partial x}\right|_{0,y,t} = 0 \quad \ldots\ldots(23)$$

$$\left.\frac{\partial \Theta}{\partial x}\right|_{2L,y,t} = 0 \quad \ldots\ldots(24)$$

$$\left.\frac{\partial \Theta}{\partial y}\right|_{x,0,t} = r\,\Theta\Big|_{x,0,t} \quad \ldots\ldots(25)$$

$$\left.\frac{\partial \Theta}{\partial y}\right|_{x,a,t} = \begin{cases} -\,r\,\Theta\Big|_{x,a,t}, & 0 \leq x < L \\ -\,r_1\left(\Theta\Big|_{x,a,t} - T\right), & L < x \leq 2L \end{cases} \quad \ldots\ldots(26)$$

$$\Theta\Big|_{x,y,0} = 0 \quad \ldots\ldots(27)$$

where: k = (thermal diffusivity, $\frac{\text{in.}^2}{\text{sec.}})^{-1}$

T = a constant

t = time, secs.

Thickness in the y-direction is indicated as "a" rather than "t" to avoid confusing it with the time t.

The transient problem as stated above was first attacked by the classical method of separation of variables. Using such an approach leads to serious difficulties; superficial inspection of the problem shows that this leads to a situation with n equation and $(n+1)$ unknowns. No approach to this difficulty seemed available.

It seemed feasible that this difficulty might be overcome by treatment as a two-region problem, i.e., by splitting the rectangular region into the two parts $0 \le x \le L$ and $L \le x \le 2L$ and matching the temperature function and its derivative on the common boundary $x = L$. The same difficulty arose here as with the classical approach.

Both of these attacks seem to break down because of the impossibility of satisfying both space and time conditions simultaneously. It therefore seemed promising to use the LaPlace transform to temporarily rid the problem of the time-variable.

By defining $\bar{\Theta}(x,y,s) = \int_0^\infty e^{-st}\,\Theta(x,y,t)\,dt$, the problem as posed reduces to:

$$\frac{\partial^2\bar{\Theta}}{\partial x^2} + \frac{\partial^2\bar{\Theta}}{\partial y^2} - ks\bar{\Theta} = 0 \quad \text{(28)}$$

$$\left.\frac{\partial\bar{\Theta}}{\partial x}\right|_{0,y} = 0 \quad \text{(29)}$$

$$\left.\frac{\partial\bar{\Theta}}{\partial x}\right|_{2L,y} = 0 \quad \text{(30)}$$

$$\left.\frac{\partial\bar{\Theta}}{\partial y}\right|_{x,0} = r\bar{\Theta}\Big|_{x,0} \quad \text{(31)}$$

$$\left.\frac{\partial\bar{\Theta}}{\partial y}\right|_{x,t} = \begin{cases} -r\bar{\Theta} & , 0 \le x < L \\ -r_1\left(\bar{\Theta} - \frac{T}{s}\right), & L < x \le 2L \end{cases} \quad \text{(32)}$$

The transformed problem may be attacked by the classical method of separation of variables. The expression

$$\bar{\Theta}(x,y,s) = E_{0,s}\,(1+ry) +$$

$$+\sum_{n=1}^{\infty} E_{n,s} \cos\frac{n\pi x}{2L} \left\{ \frac{\sqrt{ks+\alpha_n^2}+r}{\sqrt{ks+\alpha_n^2}-r}\, e^{\sqrt{ks+\alpha_n^2}\,y} + e^{-\sqrt{ks+\alpha_n^2}\,y} \right\} \quad \ldots\ldots(33)$$

satisfies (28), (29), (30), and (31) above. The use of boundary condition (32) leads to an infinite system of equations as before except that the s-variable appears in the system. Without going into great detail, the system of equations which gives the coefficients $E_{n,s}$ (which are now functions of s) is:

<u>For m = 0,</u>

$$E_{0,s}\left\{ rL\,(3+ra) + r_1L\,(1+ra)\right\} +$$

$$+\frac{2(r-r_1)L}{\pi}\sum_{n=1}^{\infty} E_{n,s}\left\{ \frac{\beta_{n,s}+r}{\beta_{n,s}-r}\, e^{\beta_{n,s}a} + e^{-\beta_{n,s}a}\right\} \frac{\sin\frac{n\pi}{2}}{n} = \frac{r_1TL}{s} \ldots(34)$$

<u>For m ≠ 0,</u>

$$E_{m,s}\beta_{m,s}\,L\left\{ \frac{\beta_{m,s}+r}{\beta_{m,s}-r}\, e^{\beta_{m,s}a} - e^{-\beta_{m,s}a}\right\} -$$

$$-\frac{2L}{\pi}\left\{ (r-r_1)(1+ra)\,E_{0,s} + \frac{r_1T}{s}\right\} \frac{\sin\frac{m\pi}{2}}{m} = \gamma_{m,n} \ldots\ldots\ldots\ldots\ldots\ldots(35)$$

where $\beta_{m,s} = \sqrt{ks+\alpha_m^2}$ and $\alpha_m = \frac{m\pi}{2L}$, m = 1, 2, 3,...

The $\gamma_{m,n}$ are given by:

For $m \neq n$,

$$\gamma_{m,n} = -\frac{(r-r_1)L}{\pi} \sum_{\substack{n=1 \\ n\neq m}}^{\infty} E_{n,s} \left\{ \frac{\beta_{n,s}+r}{\beta_{n,s}-r} e^{\beta_{n,s}a} + e^{-\beta_{n,s}a} \right\} \cdot$$

$$\cdot \left\{ \frac{\operatorname{Sin}(m+n)\frac{\pi}{2}}{m+n} + \frac{\operatorname{Sin}(m-n)\frac{\pi}{2}}{m-n} \right\} \ldots (36)$$

For $m = n$,

$$\gamma_{m,m} = -\frac{(r+r_1)L}{2} E_{m,s} \left\{ \frac{\beta_{m,s}+r}{\beta_{m,s}-r} e^{\beta_{m,s}a} + e^{-\beta_{m,s}a} \right\} \ldots\ldots\ldots\ldots\ldots (37)$$

The similarity of this system of equations to the steady-state system obviates a detailed discussion of their derivation.

The similarity of this system to the steady-state system further indicates that a finite segment of the resulting infinite system might again be used to obtain an approximation to the LaPlacian functions $E_{n,s}$ in terms of s. Theoretically, the $E_{n,s}$ could then be expressed as functions of s and placed in equation (33) so that the inversion might be performed.

Following this line of reasoning, a four-by-four block of the system (34), (35), (36), (37) was reduced to four equations in four unknowns. The first of these equations is:

$$E_{1,s}\sqrt{ks+\alpha_1^2}\;L\left\{\frac{\sqrt{ks+\alpha_1^2}+r}{\sqrt{ks+\alpha_1^2}-r}\,e^{\sqrt{ks+\alpha_1^2}\,a}-e^{-\sqrt{ks+\alpha_1^2}\,r}\right\}=$$

$$=-\frac{(r+r_1)L}{2}\,E_{1,s}\left\{\frac{\sqrt{ks+\alpha_1^2}+r}{\sqrt{ks+\alpha_1^2}-r}\,e^{\sqrt{ks+\alpha_1^2}\,a}+e^{-\sqrt{ks+\alpha_1^2}\,a}\right\}-$$

$$-\frac{2}{3}\;\frac{(r-r_1)L}{\pi}\,E_{2,s}\left\{\frac{\sqrt{ks+\alpha_2^2}+r}{\sqrt{ks+\alpha_2^2}-r}\,e^{\sqrt{ks+\alpha_2^2}\,a}+e^{-\sqrt{ks+\alpha_2^2}\,a}\right\}$$

where $\alpha_n = \frac{n\pi}{2L}$.

Considering that the $E_{n,s}$ are functions of s, the reduction of four such equations in four unknowns looms as a most formidable task. The inversion of the result would be even more formidable. Another consideration is this: Even if approximate expressions for the $E_{n,s}$ could be determined and the inversion performed, the error of the results would probably be intolerable judging from the steady-state solution.

In order to get an accurate solution to this problem, twenty or more equations in twenty or more unknowns would probably need to be reduced. With proper manpower and facilities, such a solution might be accomplished. The circumstances under which this paper was prepared forbade such an undertaking.

It should be noted that (in theory at least) the problem is solved; it seems more appropriate, however, to search for either a numerical or approximate approach to the transient problem.

Consider the region in the vicinity of $x = 2L$, i.e., at the trailing edge of the duct. The transient response of the trailing edge can be found rather simply by noting that the temperature gradient is zero in the x-direction. If an element such as that shown in Figure 8 is taken out of the trailing edge, it will therefore be effectively insulated in the xz-plane.

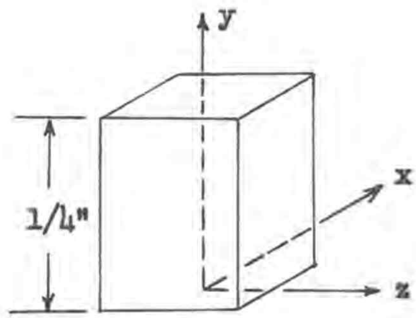

Figure 8. Element taken from right end of region. The y-axis has been reversed for convenience.

The combustion process is more stable near the trailing edge since it has had time to proceed to near completion. It seems reasonable under these circumstances to assume that the temperature of the face $y = 0$ is held constant since the heat absorbed by the trailing edge will not influence the gas temperature considerably. As before, it might be assumed that the heat transfer from the face $y = a$ follows Newton's law of cooling.

The considerations outlined on page 39 again lead to the conclusion that the initial temperature of the element will be constant and of the same magnitude as the gas temperature.

The problem then is a transient, one-dimensional problem and its formal statement is as follows:

$$\alpha \frac{\partial^2 T}{\partial y^2} = \frac{\partial T}{\partial t} \quad \ldots\ldots (34)$$

$$\left.\frac{\partial T}{\partial y}\right|_{a,t} = -r \left(T \Big|_{a,t} - 300 \right) \quad \ldots\ldots (35)$$

$$T \Big|_{0,t} = 1800 \quad \ldots\ldots (36)$$

$$T \Big|_{y,0} = 300 \quad \ldots\ldots (37)$$

where: T = temperature, °F

α = thermal diffusivity, $\frac{\text{in.}^2}{\text{sec.}}$

Others as before

The transformation $\Theta = T - 300$ reduces the problem to:

$$\alpha \frac{\partial^2 \Theta}{\partial y^2} = \frac{\partial \Theta}{\partial t} \quad \ldots\ldots (38)$$

$$\left.\frac{\partial \Theta}{\partial y}\right|_{a,t} = -r\Theta \Big|_{a,t} \quad \ldots\ldots (39)$$

$$\Theta \Big|_{0,t} = 1500 \quad \ldots\ldots (40)$$

$$\Theta \Big|_{y,0} = 0 \quad \ldots\ldots (41)$$

The result is well known; it may be found in Carslaw and Jaeger (5, p. 105). The result is:

$$\frac{\Theta}{1500} = \frac{1+r(a-y)}{1+ra} - \sum_{n=1}^{\infty} \frac{2(\beta_n + r^2)\, e^{-\alpha \beta_n^2 t} \sin \beta_n y}{\beta_n \left\{ r + a(\beta_n^2 + r^2) \right\}} \quad \text{.........(42)}$$

The β_n's are the positive roots of:

$$\beta_n \operatorname{Cot} \beta_n a + r = 0 \quad \text{.........(43)}$$

This approximate result should apply in the region of the duct near the trailing edge.

The transient temperature distribution near the center of the duct, i.e., near $x = L$, is not nearly as amenable to analysis as that near the trailing edge. In fact, the analysis in this region leads to the same problems as were described at the beginning of this section. A method of bounding the transient solution in $L < x \leq 2L$ will be described; a detailed analysis will not be attempted.

Consider the region $L < x \leq 2L$. The part of this region near $x = 2L$ has a rather small temperature gradient in the x-direction during steady-state. Since this same gradient is zero at time zero, it is possible to bracket or bound the temperature response by considering Figure 7 and the following two problems:

Problem 1,

$$K \quad \frac{\partial \Theta}{\partial t} = \frac{\partial^2 \Theta}{\partial y^2} + \phi(y) \quad \text{.........(44)}$$

$$\left.\frac{\partial \Theta}{\partial y}\right|_{0,t} = r \left.\Theta\right|_{0,t} \quad \text{.........(45)}$$

$$\left.\frac{\partial \Theta}{\partial y}\right|_{a,t} = r_1 \left(k - \left.\Theta\right|_{a,t}\right) \quad \text{.........(46)}$$

$$\left.\Theta\right|_{y,0} = 0 \quad \text{.........(47)}$$

Problem 2,

$$K \frac{\partial \Theta}{\partial t} = \frac{\partial^2 \Theta}{\partial y^2} \quad \text{.......................................}(48)$$

$$\left.\frac{\partial \Theta}{\partial y}\right|_{0,t} = r\Theta\Big|_{0,t} \quad \text{.......................................}(49)$$

$$\left.\frac{\partial \Theta}{\partial y}\right|_{a,t} = r_1 \left(k - \Theta\Big|_{a,t}\right) \text{.......................................}(50)$$

$$\Theta\Big|_{y,0} = 0 \text{.......................................}(51)$$

Problem 1 treats the transient problem in $L < x \leq 2L$ with a function $\phi(y)$ replacing the heat stored in the element due to change of temperature gradient in the x-direction. The function $\phi(y)$ may be chosen knowing the temperature gradients in the x-direction at steady-state.

Problem 2 treats the transient problem as though $\phi(y) = 0$, as it would be at time zero. In both cases, the change of temperature gradient in the x-direction is treated as though there were a heat source in the element.

One other alternative method of approximating the transient temperature distribution is to resort to numerical methods. This problem is difficult to treat directly by numerical methods because of its three-dimensional character (two space variables and time).

It is possible, however, that an integral transform other than the LaPlace transform might be used in conjunction with numerical methods. This subject is treated in some detail in Tranter (16, p. 104) although his treatment does not cover the present case.

Consider the transient problem (22), (23), (24), (25), (26), and (27). It is possible to eliminate the x space-variable by using a finite Fourier integral transform defining

$$\bar{\Theta}(P, y, t) = \int_0^{2L} \Theta(x, y, t) \text{ Cos } \frac{P\pi x}{2L} \text{ dx} \qquad (52)$$

An iterative technique may now be used to solve the one-dimensional transient problem approximately and the values of $\bar{\Theta}(P)$ thus derived may be inverted using the well-known inversion formula

$$\Theta = \frac{1}{\pi} \bar{\Theta}(0) + \frac{2}{\pi} \sum_{P=1}^{\infty} \bar{\Theta}(P) \text{ Cos Px} \qquad (53)$$

Furthermore, it may be useful to use an approach involving both a LaPlace and a finite Fourier cosine transform. In this manner, it appears that the transient problem might be reduced to an ordinary differential equation with certain conditions to be met. Such a technique is by no means well known and might prove to be both thought-provoking and rewarding.

CONCLUSIONS AND RECOMMENDATIONS

An approximate solution of the steady-state problem embodied in equations (4), (5), (6), (7), and (8) is:

$$\Theta(x,y) = C_0\,(1+ry) + \sum_{n=1}^{N} C_n \cos \lambda_n x \left\{ e^{-\lambda_n y} + \frac{\lambda_n + r}{\lambda_n - r}\, e^{\lambda_n y} \right\} \quad \ldots\ldots(54)$$

$$\lambda_n = \frac{n\pi}{2L}$$

Approximate values of the coefficients C_n are given by the solution of a finite N-dimensional segment of the infinite system of equations (13), (14), (15), and (16).

A simple iterative procedure for the system of equations (13), (14), (15), and (16) is given. Although this procedure is intended for calculations on a desk calculator, the same procedure or more exact procedures may be easily programmed on a digital computer.

The proof of the validity of the solution (54) does not presuppose any magnitudes of system constants and the results are therefore extendable to other similar systems.

Typical system constants are outlined in Appendix A and an example is worked out in some detail. Using $N = 19$ in equation (54) and the iterative procedure outlined previously, the example problem solutions appear to deviate from a mean curve on the order of $\pm 10^{\circ}$F. Since the range of temperatures considered is about 500°F, the deviation appears to be within tolerable limits.

It therefore seems appropriate to conclude that the solution (54) of the steady-state problem (4), (5), (6), (7), and (8) is not only accurate and useful but suitable for desk calculator techniques.

Several approaches to the exact solution of the transient problem (22), (23), (24), (25), (26), and (27) were discussed and it does not appear that any of them are very useful. It seemed appropriate therefore to search for either an approximate solution or at least some way of indicating the bounds on the solution.

At the extreme right end of the region, i.e., near $x = 2L$, the well-known solution (42) should give a good idea of the transient temperature response of the trailing edge of the duct. Over the aft half of the duct, the transient response appears to be bounded by solutions of the two problems (44), (45), (46), (47), and (48), (49), (50), (51). No analysis of these bounds was attempted.

It is recommended that a study be made of the thermal stresses which result from the steady-state temperature distribution. The solution of this thermal stress problem would be highly useful not only to the jet-propulsion industry but also to the nuclear industry.

It is further recommended that a study be made of the possibility of thermally induced vibrations in a duct (or plate) of the sort considered here. Such a study would also be useful to the industries just mentioned.

It furthermore appears that it may be possible to solve the transient problem by the use of both a LaPlace and Fourier transform. By using a LaPlace transform to rid the problem of the time variable and a finite Fourier cosine transform to lose the x space-variable, it may be possible to reduce the transient problem to a rather complex ordinary differential equation. Although there are many difficulties inherent in such a method, the approach might be fruitful; it is recommended that this approach be attempted.

The only check on the validity of the steady-state solution posed has been through analysis. Although this analytical verification indicates the mathematical solution to be correct, it should be noted that such verification does not extend to the physical problem. It therefore appears that an experimental verification of the analytical results of the steady-state problem would be appropriate.

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APPENDICES

APPENDIX A: ANALYSIS OF HEAT TRANSFER COEFFICIENTS

Humble, et al. (6, p. 348) derive an empirical relationship for a heat transfer coefficient that seems appropriate along $y = t$ in the interval $0 \leq x < L$. Humble's relationship is derived using data in the freestream Reynold's number range $10^4 \leq (Re)_F \leq 10^6$. The bulk Reynold's number of the leading inside half of the duct and the outside of the duct is on the order of 10×10^6. However, no more suitable relationship can be found; turning to Humble's formula:

$$\frac{hD}{k_F} = 0.034 \left(\frac{\rho_F V_B D}{\mu_F}\right)^{0.8} \left(\frac{(C_p)_F \mu_F}{k_F}\right)^{0.4} \left(\frac{L}{D}\right)^{-0.1} \qquad \text{.............(1A)}$$

Subscripts: B indicates bulk properties of the air stream

F indicates freestream conditions

Symbology: h = heat transfer coefficient, $\dfrac{\text{Btu}}{\text{hr.-ft.}^2\text{-}^\circ\text{F}}$

D = characteristic diameter, ft.

k_F = freestream thermal conductivity, $\dfrac{\text{Btu}}{\text{hr.-ft.-}^\circ\text{F}}$

ρ_F = freestream air density, $\dfrac{\text{lb.}_m}{\text{ft.}^3}$

V_B = bulk velocity, $\dfrac{\text{ft.}}{\text{hr.}}$

μ_F = freestream viscosity, $\dfrac{\text{lb.}_m}{\text{ft. sec.}}$

$(C_p)_F$ = freestream specific heat, $\dfrac{\text{Btu}}{\text{lb.}_m\ ^\circ\text{R}}$

L = characteristic length, ft.

In order to define the heat transfer coefficient, some initial properties must be known. These values should be representative of the physical situation; choose the following parameters:

(a) M_B = Mach number = 0.5

(b) T_T = Total air stream temperature = 800°R

(c) P_B = Static pressure = 3 atmospheres

(d) D = 2 ft., L = 4 ft.

(e) $k_F = 0.0204 \dfrac{\text{Btu}}{\text{hr.-ft.-}^\circ\text{R}}$

These parameters fix all the properties in the equation of Humble, i.e., in equation (1A).

The Mach number fixes the ratio $\dfrac{T_{static}}{T_{total}}$

$$\frac{T}{T_T} = 0.952$$

$$T_F = 0.952(800)^\circ R = 302^\circ F$$

The assumed value of P_B coupled with the temperature gives:

$$\rho_B = \frac{P_B}{RT_B} = \frac{44.1(144)}{53.4(762)} = 0.156 \ \frac{\text{lb.}_m}{\text{ft.}^3}$$

The Mach number fixes the velocity...

$$V \simeq M\,(49.1)\sqrt{T} = 0.5(49.1)(800)^{1/2} = 691 \text{ fps}$$

The viscosity is a function of temperature only...

$$\mu_{B \atop 300^\circ F} = 1.60\ (10^{-5})\ \frac{\text{lb.}_m}{\text{ft. sec.}}$$

The dimensionless groups in equation (1A) are

$$(Re)_F = \frac{\rho D V}{\mu} = \frac{0.156(2)(691)}{1.60\ (10^{-5})} = 13.5\ (10^6)$$

$$\left(\frac{L}{D}\right)_F^{-0.1} \simeq 1.0$$

$$(Pr)_F = \frac{C_p \mu}{k} = \frac{0.24\ (1.60(10^{-5}))}{0.0204}\ (3600) = 0.677$$

The heat transfer coefficient is then:

$$h = \frac{k_F}{D}\ (0.034)\ (Re)_F^{0.8}\ (Pr)_F^{0.4}\ \left(\frac{L}{D}\right)^{-0.1}$$

$$= \frac{0.0204}{2}\ (0.034)(135(10^5))^{0.8}\ (0.677)^{0.4}$$

$$h = 150\ \frac{\text{Btu}}{\text{hr.-ft.}^2\text{-}^\circ\text{F}}$$

An order-of-magnitude check of this heat transfer coefficient may be arrived at by the Martinelli analogy of transfer of heat and momentum:

$$(Re)_B = 13.5\ (10^6)$$

$$(Pr)_B = 0.68$$

$$(Nu)_B \simeq 10^4;\ (8,\ p.\ 213)$$

$$= \frac{hD}{k}$$

$$h = \frac{k}{D}\ (10^4)$$

$$= \frac{0.0204}{2}\ 10^4\ \frac{\text{Btu}}{\text{hr.-ft.}^2\text{-}^\circ\text{F}}$$

$$h = 102\ \frac{\text{Btu}}{\text{hr.-ft.}^2\text{-}^\circ\text{F}},\ \text{check}$$

The heat transfer coefficient given by equation (1A) thus appears to be a good estimate on the leading inside half of the duct.

Consider now the heat transfer coefficient along the outside surface of the duct, i.e., along $y = 0$ in the interval $0 \leq x \leq 2L$. In such a region, McAdams (11, p. 242) recommends the following relationship for the heat transfer coefficient based on log-mean temperature difference:

$$0.87\left(\frac{D_2}{D_1}\right)^{0.53} = \frac{1}{0.023}\left(\frac{D_E G}{\mu_B}\right)^{0.2}\left(\frac{h_L}{C_{p_B} G}\right)\left(\frac{C_p \mu}{k}\right)^{2/3}$$

so that:

$$h_L = 0.020\ C_{p_B} G \left(\frac{D_2}{D_1}\right)^{0.53}\left(\frac{D_E G}{\mu_B}\right)^{-0.2}\left(\frac{C_p \mu}{k}\right)^{-2/3} \quad \text{..............(2A)}$$

where: h_L = heat transfer coefficient based on log-mean temperature difference, $\frac{\text{Btu}}{\text{hr.-ft.}^2\text{-}^\circ\text{F}}$

D_2 = outside diameter of annular cooling air, in.

D_1 = inside diameter of annulus, in.

D_E = equivalent diameter of annulus, in.

G = flow rate, $\frac{\text{lb.}_m}{\text{ft.}^2\text{-sec.}}$

Others as before.

The subscript B indicates bulk or freestream conditions. Take D_1 = 24 inches as before and let D_2 = 26 inches.

$$\left(\frac{D_2}{D_1}\right)^{0.53} = \left(\frac{26}{24}\right)^{0.53} \simeq 1.0$$

$$D_E = D_2 - D_1 = 2 \text{ in.}$$

$$G = \rho_B V = 0.156(691)\ \frac{\text{lb.}_m}{\text{ft.}^2\text{-sec.}} = 108\ \frac{\text{lb.}_m}{\text{ft.}^2\text{-sec.}}$$

$$k_B = \frac{0.0204\ \frac{\text{Btu}}{\text{ft.-hr.-}^\circ\text{R}}}{3600\ \frac{\text{sec.}}{\text{hr.}}} = 5.66\ (10^{-6})\ \frac{\text{Btu}}{\text{ft.-sec.-}^\circ\text{R}}$$

Equation (2A) then gives $h_L = 149 \dfrac{\text{Btu}}{\text{hr.-ft.}^2\text{-}^\circ\text{R}}$.

Although the latter heat transfer coefficient is based upon the log-mean temperature difference and is a bulk coefficient, it should be fairly representative of the local heat transfer coefficients along the outside edge y = 0 of the duct.

Since this analysis is not concerned with exact values but with representative values, assume that the heat transfer coefficients along the two surfaces just analyzed are the same, i.e., assume that:

$$h = 150 \frac{\text{Btu}}{\text{hr.-ft.}^2\text{-}^\circ\text{R}}; \qquad \begin{array}{l} y = 0,\ 0 \le x \le 2L \\ y = t,\ 0 \le x < L \end{array}$$

Consider the heat transfer in the combustion zone, i.e., along $y = t$, $L < x \le 2L$. There is some evidence (13, p. 49) that the bulk velocity of the combustion products in the wake of a flameholder is nearly as high as the freestream velocity four or five baffle widths downstream. Since the baffle width is on the order of one and one-half inches, assume that the following bulk velocities exist:

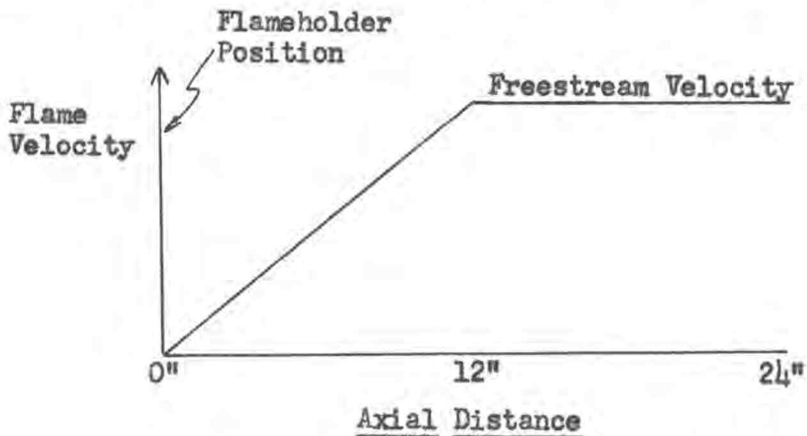


Figure 1A. Assumed velocity distribution in the wake of the flameholder.

From Figure 1A, the average velocity is:

$$V_{ave} = \frac{\frac{1}{2} V_{F.S.} + V_{F.S.}}{2} = \frac{3}{4} V_{F.S.};$$ F.S. indicates freestream

$$V_{ave} = 0.75\ (691)\ f_{PS}$$

$$= 519\ \frac{ft.}{sec.}$$

Assume that the temperature in the wake is constant due to the violent mixing and that it has a value of 1800°F.

The pressure distribution in the wake of a flameholder is complex, especially during burning. A qualitative look (14, p. 31) indicates a low-pressure region in the immediate wake, a small rise in pressure for perhaps a baffle width, and a gradual decay further downstream. Suppose then that the average pressure in the wake of the flameholder is 90 percent of the freestream pressure, i.e.,

$$P_{ave} = 0.90\ (44.1)\ \text{psia} = 39.7\ \text{psia}$$

Suppose further that there is complete combustion with C_8H_{18} as fuel. Such a situation is unlikely, but should lead to a reasonable approximation of the heat-transfer coefficient. This assumption gives:

$$C_8H_{18} + \frac{25}{2}\ O_2 + \frac{79}{21}\left(\frac{25}{2}\right) N_2 \rightarrow 8CO_2 + 9H_2O + \frac{79}{21}\left(\frac{25}{2}\right) N_2$$

$$C_8H_{18} + 12\tfrac{1}{2}\ O_2 + 47\ N_2 \rightarrow 8CO_2 + 9H_2O + 47\ N_2 \ldots\ldots\ldots\ldots\ldots\ldots (3A)$$

The gas constant R, $\frac{ft.-lb.}{lb._m-{}^{\circ}R}$ for the products of combustion is given by:

$$(R)_{\text{products}} = \frac{n_1}{n} R_1 + \frac{n_2}{n} R_2 + \frac{n_3}{n} R_3 \text{.............................(4A)}$$

where $n_{1,2,3}$ = No. of moles of (1) CO_2, (2) H_2O, and (3) N_2.

n = total No. of moles

$= n_1 + n_2 + n_3$

$R_{1,2,3}$ = gas constant for (1) CO_2, (2) H_2O, and (3) N_2.

Using the assumed chemical reaction (3A) in (4A):

$$(R)_{\text{products}} \simeq \frac{8}{64}(55.2) + \frac{9}{64}(85.8) + \frac{47}{64}(55.2);$$ $R_{1,2,3}$ are for one atmosphere pressure.

$$\simeq 59.5 \ \frac{\text{ft.-lb.}}{\text{lb.}_m\text{-}^\circ\text{R}}$$

The perfect gas equation then gives:

$$(\rho)_{\text{products}} = \frac{P}{RT}; \text{ symbols as before.}$$

$$= \frac{(39.7)(144)}{59.5\ (2260)} \ \frac{\text{lb.}_m}{\text{ft.}^3}$$

$$= 0.0425 \ \frac{\text{lb.}_m}{\text{ft.}^3}$$

Using the polycomponent Wassiljewa method (9, p. 14) to determine the fluid's thermal conductivity:

$$k_F = \frac{k_1}{1 + A_1\left(\frac{1-y_1}{y_1}\right)} + \frac{k_2}{1 + A_2\left(\frac{1-y_2}{y_2}\right)} + \frac{k_3}{1 + A_3\left(\frac{1-y_3}{y_3}\right)} \text{ ....(5A)}$$

where by definition:

$$A_1\ (1-y_1) = y_2\ A_{1-2} + y_3\ A_{1-3} \text{...............................(6A)}$$

$$A_2\ (1-y_2) = y_1\ A_{2-1} + y_3\ A_{2-3} \text{...............................(7A)}$$

$$A_3\ (1-y_3) = y_1\ A_{3-1} + y_2\ A_{3-2} \text{...............................(8A)}$$

The notation in equations (5A), (6A), (7A), and (8A) is as follows:

Subscript: (1) indicates CO_2

(2) indicates H_2O

(3) indicates N_2

i-j indicates gas (i) mixed with gas (j)

A = Wassiljewa constant

y = mole fraction

Lenoir's data (9, p. 11) has been extrapolated in Figure 2A to arrive at the coefficients A_{i-j}. Using equations (6A), (7A), and (8A) with appropriate values from Figure 2A gives:

$$A_1\,(1-y_1) = \frac{9}{64}\,(0.38) + \frac{47}{64}\,(0.55) = 0.46$$

$$A_2\,(1-y_2) = \frac{8}{64}\,(0.595) + \frac{47}{64}\,(0.73) = 0.61$$

$$A_3\,(1-y_3) = \frac{8}{64}\,(0.82) + \frac{9}{64}\,(0.87) = 0.22$$

Extrapolation of thermal conductivity data in McAdams (11, p. 457) to 1800°F gives:

$$k_1 = 0.0486\,\frac{\text{Btu}}{\text{hr.-ft.-}^\circ\text{F}}$$

$$k_2 = 0.059\,\frac{\text{Btu}}{\text{hr.-ft.-}^\circ\text{F}}$$

$$k_3 = 0.049\,\frac{\text{Btu}}{\text{hr.-ft.-}^\circ\text{F}}$$

So that equation (5A) gives:

$$k_F = \frac{0.0486}{1+\frac{0.46(64)}{8}} + \frac{0.059}{1+\frac{0.61(64)}{9}} + \frac{0.049}{1+\frac{0.22(64)}{47}}$$

$$= 0.0591\,\frac{\text{Btu}}{\text{hr.-ft.-}^\circ\text{F}}$$

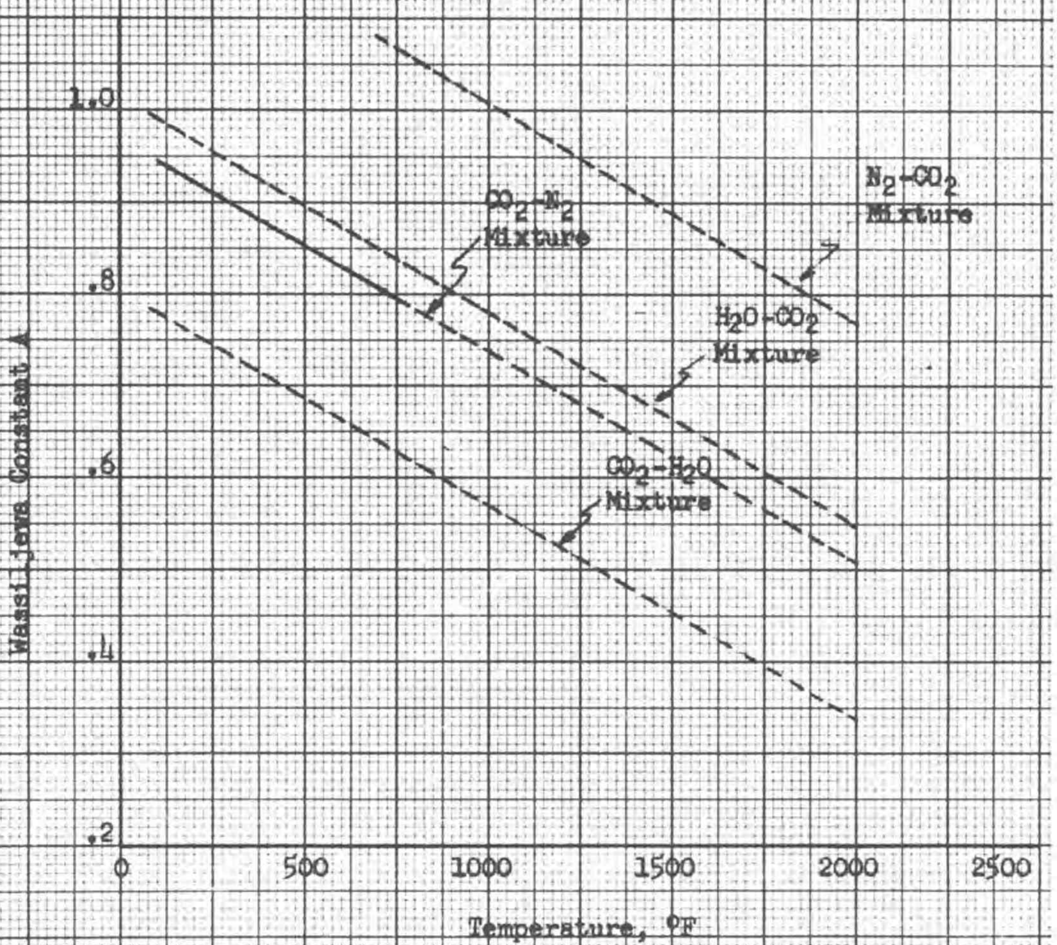


Figure 2A. Curve of Wassiljewa constants extrapolated from data in (9, p. 11). A dashed line indicates that the data is extrapolated.

Viscosities for several representative industrial gases are given by Hirschfelder, et al. (5, p. 933). Taking a mixture of 85 percent N_2, 10.8 percent CO_2, 2 percent O_2, and 2.2 percent H_2 as representative gives:

$$\mu_F = 0.000486 \text{ poise}$$

$$= 32.7\ (10^{-6})\ \frac{\text{lb.}_m}{\text{ft.-sec.}}$$

Extrapolating data given by McAdams (11, p. 464) gives for the specific heat of the combustion gases:

$$(C_p)_F = y_1\ C_{p_1} + y_2\ C_{p_2} + y_3\ C_{p_3}; \text{ Subscripts and notation as in equation (5A).}$$

$$= \frac{8}{64}\ (0.309) + \frac{9}{64}\ (0.57) + \frac{47}{64}\ (0.288)$$

$$= 0.331\ \frac{\text{Btu}}{\text{lb.}_m\text{-}^\circ\text{F}}$$

Before proceeding, two things should be noted: (1) equation (1A) gives the heat transfer coefficient in terms of bulk flow conditions, and (2) the data used in deriving equation (1A) was taken in a flow system with only one gas, viz., air.

In the interval $L < x \leq 2L$ being treated here, there is a striated flow; there is airflow in the core and combustion product flow along the inner wall of the duct. It seems apparent that heat transfer to the duct in this region is controlled principally by the combustion products due to their high temperature. However, it does not appear to be quite legitimate to use properties of the combustion products in equation (1A) since they are as much local as bulk conditions.

Moreover, it also appears to be slightly illegitimate to extend an equation derived on the basis of homogeneous flow to a situation with striated flow.

No heat transfer coefficient correlation appears to be available for a complicated flow system as we have here. On the assumption that it is legitimate to use somewhat localized flow conditions and treat striated flow by using equation (1A), we obtain:

$$h = 0.034\left(\frac{k_F}{D}\right)\left(\frac{\rho_F V_B D}{\mu_F}\right)^{0.8}\left(\frac{C_{p_F}\mu_F}{k_F}\right)^{0.4}\left(\frac{L}{D}\right)^{-0.1}$$

$$= 0.034\left(\frac{0.0591}{2}\right)\left(\frac{0.0425(519)2}{32.7\ (10^{-6})}\right)^{0.8}\left(\frac{0.331(32.7)10^{-6}}{0.0591}\right)^{0.4}$$

$$= 68.0\ \frac{\text{Btu}}{\text{lb.}_m\text{-}^{\circ}\text{F}};\ y = t,\ L < x \leq 2L$$

The modified heat transfer coefficient r is defined as

$$r = \frac{h}{k}$$

where r = modified heat transfer coefficient, in.$^{-1}$

h = heat transfer coefficient, $\frac{\text{Btu}}{\text{hr.-in.}^2\text{-}^{\circ}\text{F}}$

k = thermal conductivity of the duct, $\frac{\text{Btu}}{\text{hr.-in.-}^{\circ}\text{F}}$

McAdams (11, p. 447) lists data on thermal conductivities of various steels; this data indicates that for a steel duct, k is on the order of 20 $\frac{\text{Btu}}{\text{hr.-ft.}^2\text{-}^{\circ}\text{F}}$. Using this value gives:

$$r = \frac{h}{k} = \frac{150 \dfrac{\text{Btu}}{\text{hr.-ft.}^2\text{-}^{\circ}\text{F}}}{20 \dfrac{\text{Btu}}{\text{hr.-ft.-}^{\circ}\text{F}}} \left(\frac{\text{ft.}}{12\ \text{in.}}\right)$$

$$r = \frac{0.625}{\text{in.}};\ \begin{array}{l} y = 0,\ 0 \leq x \leq 2L \\ y = t,\ 0 \leq x < L \end{array}$$

$$r_1 = \frac{h}{k} = \frac{68.0}{20\ (12)}$$

$$r_1 = \frac{0.283}{\text{in.}};\ y = t,\ L < x \leq 2L$$

APPENDIX B: EVALUATION OF INTEGRALS

In the evaluation of the integrals introduced previously, there are three cases to consider, viz.,

(1) $m \neq n,\ m \neq 0$

(2) $m = n,\ m = 0$

(3) $m = 0$

Necessary integrals are broken down into these three categories in the work that follows.

It will be helpful to recall that the characteristic values (eigen values) of the problem λ_m are given by:

$$\lambda_m = \frac{m\pi}{2L};\ m = 0, 1, 2, \ldots$$

The necessary integrals are as follows:

For $m \neq n,\ m \neq 0$,

$$\int_0^L \operatorname{Cos}\lambda_m x \operatorname{Cos}\lambda_n x\,dx = \left.\frac{\operatorname{Sin}(\lambda_m + \lambda_n)x}{2(\lambda_m + \lambda_n)}\right|_0^L + \left.\frac{\operatorname{Sin}(\lambda_m - \lambda_n)x}{(\lambda_m - \lambda_n)2}\right|_0^L$$

$$= \frac{L}{\pi}\left\{\frac{\operatorname{Sin}(m+n)\frac{\pi}{2}}{m+n} + \frac{\operatorname{Sin}(m-n)\frac{\pi}{2}}{m-n}\right\}$$

$$\int_L^{2L} \operatorname{Cos}\lambda_m x \operatorname{Cos}\lambda_n x\,dx = -\frac{L}{\pi}\left\{\frac{\operatorname{Sin}(m+n)\frac{\pi}{2}}{m+n} + \frac{\operatorname{Sin}(m-n)\frac{\pi}{2}}{m-n}\right\}$$

$$\int_0^L \operatorname{Cos}\lambda_m x\,dx = \frac{2L}{m\pi}\operatorname{Sin}\frac{m\pi}{2}$$

$$\int_L^{2L} \operatorname{Cos}\lambda_m x\,dx = -\frac{2L}{m\pi}\operatorname{Sin}\frac{m\pi}{2}$$

$$\int_0^{2L} \operatorname{Cos}\lambda_m x\,dx = 0$$

For m = n, m $\neq$ 0,

$$\int_0^L \cos\lambda_m x \cos\lambda_n x dx = \int_0^L \cos^2\lambda_m x dx = \frac{1}{\lambda_m}\int_0^L \cos^2\lambda_m x\,(\lambda_m dx)$$

$$= \frac{2L}{m\pi}\left\{\frac{m\pi x}{4L}\Big/_0^L + \frac{1}{4}\sin\frac{m\pi x}{L}\Big/_0^L\right\}$$

$$= \frac{L}{2}$$

$$\int_0^{2L} \cos\lambda_m x \cos\lambda_n x dx = L$$

$$\int_L^{2L} \cos\lambda_m x \cos\lambda_n x dx = \frac{L}{2}$$

For m = 0,

$$\int_0^{2L} \cos\lambda_n x \cos\lambda_m x dx = \int_0^{2L} \cos\lambda_n x dx = \frac{\sin\lambda_n x}{\lambda_n}\Big/_0^{2L} = 0$$

$$\int_0^{2L} \cos\lambda_m x dx = \int_0^{2L} dx = 2L$$

$$\int_0^{L} \cos\lambda_m x dx = L$$

$$\int_L^{2L} \cos\lambda_m x dx = L$$

$$\int_0^{L} \cos\lambda_n x \cos\lambda_m x dx = \frac{\sin\lambda_n x}{\lambda_n}\Big/_0^{L} = \frac{2L}{n\pi}\sin\frac{n\pi}{2}$$

$$\int_L^{2L} \cos\lambda_n x \cos\lambda_m x dx = \frac{\sin\lambda_n x}{\lambda_n}\Big/_L^{2L} = -\frac{2L}{n\pi}\sin\frac{n\pi}{2}$$

APPENDIX C: PARTIAL PROOF OF STEADY-STATE PROBLEM

Equations (13), (14), (15), and (16) in the Results section describe an infinite system of equations, i.e., an infinite number of equations in an infinite number of unknowns. This system of equations may be written:

For $m = 0$,

$$C_o = -\frac{2(r-r_1)L}{\pi\left\{rL(3+rt)+r_1L(1+rt)\right\}}\sum_{n=1}^{\infty} C_n\left\{\frac{\operatorname{Sin}\frac{n\pi}{2}}{n}\right\} \cdot$$

$$\cdot\left\{e^{-\lambda_n t}+\frac{\lambda_n+r}{\lambda_n-r}\,e^{\lambda_n t}\right\}+\frac{r_1kL}{rL(3+rt)+r_1L(1+rt)}$$

For $m \neq 0$,

$$C_m\,\frac{m\pi}{2}\left\{\frac{\lambda_m+r}{\lambda_m-r}\,e^{\lambda_m t}-e^{-\lambda_m t}\right\}+\frac{(r+r_1)L}{2}\left\{\frac{\lambda_m+r}{\lambda_m-r}\,e^{\lambda_m t}+e^{-\lambda_m t}\right\}=$$

$$= -C_o\,\frac{2L}{\pi}\,(r-r_1)(1+rt)\,\frac{\operatorname{Sin}\frac{m\pi}{2}}{m}-$$

$$-\frac{(r-r_1)L}{\pi}\sum_{\substack{n=1\\ n\neq m}}^{\infty} C_n\left\{\frac{\lambda_n+r}{\lambda_n-r}\,e^{\lambda_n t}+e^{-\lambda_n t}\right\}\left\{\frac{\operatorname{Sin}\,(m+n)\frac{\pi}{2}}{m+n}+\frac{\operatorname{Sin}\,(m-n)\frac{\pi}{2}}{m-n}\right\}-$$

$$-\frac{2r_1kL}{\pi}\,\frac{\operatorname{Sin}\frac{m\pi}{2}}{m}$$

For convenience, call: $A_0 = C_0$

$$A_n = C_n\left\{\frac{\lambda_n+r}{\lambda_n-r}\,e^{\lambda_n t}+e^{-\lambda_n t}\right\}$$

For m = 0,

$$A_0 = -\frac{2(r-r_1)L}{\pi\{rL(3+rt)+r_1L(1+rt)\}}\sum_{n=1}^{\infty} A_n \frac{\mathrm{Sin}\,\frac{n\pi}{2}}{n} +$$

$$+\frac{r_1kL}{rL(3+rt)+r_1L(1+rt)} \qquad \text{(1C)}$$

For m ≠ 0,

$$A_m \left[\frac{m\pi}{2}\;\frac{\frac{\lambda_m+r}{\lambda_m-r}\,e^{\lambda_m t}-e^{-\lambda_m t}}{\frac{\lambda_m+r}{\lambda_m-r}\,e^{\lambda_m t}+e^{-\lambda_m t}} + \frac{(r+r_1)L}{2}\right] =$$

$$= -A_0\left\{\frac{2L}{\pi}(r-r_1)(1+rt)\right\}\frac{\mathrm{Sin}\,\frac{m\pi}{2}}{m} -$$

$$-\frac{(r-r_1)L}{\pi}\sum_{n=1}^{\infty} A_n\left\{\frac{\mathrm{Sin}(m+n)\,\frac{\pi}{2}}{m+n}+\frac{\mathrm{Sin}(m-n)\,\frac{\pi}{2}}{m-n}\right\} - \frac{2r_1kL}{\pi}\;\frac{\mathrm{Sin}\,\frac{m\pi}{2}}{m}$$

Call: $\Theta_m = \pi\left\{\dfrac{\frac{\lambda_m+r}{\lambda_m-r}\,e^{\lambda_m t}-e^{-\lambda_m t}}{\frac{\lambda_m+r}{\lambda_m-r}\,e^{\lambda_m t}+e^{-\lambda_m t}}\right\}$

$$\alpha = (r+r_1)L$$

So that:

$$A_m = -A_0\left\{\frac{4L}{\pi}(r-r_1)(1+rt)\right\}\frac{\mathrm{Sin}\,\frac{m\pi}{2}}{m^2\Theta_m+\alpha} -$$

$$-\frac{2(r-r_1)L}{\pi}\sum_{\substack{n=1\\ n\neq m}}^{\infty} A_n\left\{\frac{\mathrm{Sin}\,(m+n)\,\frac{\pi}{2}}{(m+n)(m\Theta_m+\alpha)}+\frac{\mathrm{Sin}\,(m-n)\,\frac{\pi}{2}}{(m-n)(m\Theta_m+\alpha)}\right\} -$$

$$-\frac{2r_1kL}{\pi}\;\frac{\mathrm{Sin}\,\frac{m\pi}{2}}{m^2\Theta_m+m\alpha} \qquad \text{(2C)}$$

We now have the system of equations:

$$A_0 = -\frac{2(r-r_1)L}{\pi\left\{rL(3+rt)+r_1L(1+rt)\right\}}\sum_{n=1}^{\infty} A_n \frac{\operatorname{Sin}\frac{n\pi}{2}}{n} +$$

$$+\frac{r_1kL}{rL(3+rt)+r_1L(1+rt)} \qquad \text{(1C)}$$

$$A_m = -A_0\left\{\frac{4L}{\pi}(r-r_1)(1+rt)\right\}\frac{\operatorname{Sin}\frac{m\pi}{2}}{m^2\Theta_m+m\alpha} -$$

$$-\frac{2(r-r_1)L}{\pi}\sum_{\substack{n=1\\ n\neq m}}^{\infty} A_n\left\{\frac{\operatorname{Sin}(m+n)\frac{\pi}{2}}{(m+n)(m\Theta_m+\alpha)}+\frac{\operatorname{Sin}(m-n)\frac{\pi}{2}}{(m-n)(m\Theta_m+\alpha)}\right\} -$$

$$-\frac{2r_1kL}{\pi}\;\frac{\operatorname{Sin}\frac{m\pi}{2}}{m^2\Theta_m+m\alpha} \qquad \text{(2C)}$$

where $m = 1, 2, 3, 4, \ldots$

This system of equations may be written in the shorthand form:

$$A_m = \sum_{n=0}^{\infty} D_{m,n}A_n + b_m \qquad \text{(3C)}$$

For the system of equations (1C) and (2C), it has been established by Kantorovich (8, p. 43) that:

If $\sum_{m,n=0}^{\infty} D_{m,n}^{\;2}$ and $\sum_{m=0}^{\infty} b_m^{\;2}$ converge, the given system (3C) has a unique solution satisfying the condition that $\sum_{n=0}^{\infty} A_n^{\;2}$ converges and these solutions are given by the method of segments.

Consider $\sum_{m=0}^{\infty} b_m^{\,2}$

$$\sum_{m=0}^{\infty} b_m^{\,2} = b_o^{\,2} + \sum_{m=1}^{\infty} b_m^{\,2}$$

For the system of equations under consideration here

$$b_o = \frac{r_1 kL}{rL(3+rt) + r_1 L(1+rt)}$$

$$b_m = \frac{2r_1 kL}{\pi} \; \frac{\operatorname{Sin} \frac{m\pi}{2}}{m^2 \Theta_m + m\,\alpha}$$

$$\sum_{m=0}^{\infty} b_m^{\,2} = \left(\frac{r_1 kL}{rL(3+rt) + r_1 L(1+rt)}\right)^2 + \left(\frac{2r_1 kL}{\pi}\right)^2 \sum_{m=0}^{\infty} \frac{(\operatorname{Sin} \frac{m\pi}{2})^2}{(m^2\Theta_m + m\alpha)^2}$$

$$= \beta + \gamma \sum_{m=1}^{\infty} \frac{1}{((2m-1)^2 \Theta m + m\alpha)^2};$$ β and γ are constants.

Now, $\Theta_m = \pi \left\{ \dfrac{\frac{\lambda_m + r}{\lambda_m - r}\, e^{\lambda_m t} - e^{-\;_m t}}{\frac{\lambda_m + r}{\lambda_m - r}\, e^{\lambda_m t} + e^{-\;_m t}} \right\}$ and $\Theta_m \to \pi$ as $m \to \infty$.

The summand of $\sum_{m=0}^{\infty} b_m^{\,2}$ is therefore on the order of $\frac{1}{m^4}$ and convergence is thus guaranteed.

Consider $\sum_{m,n=0}^{\infty} D^2_{m,n}$ and McLaurin's test...

"If a function f(x,y) is positive and steadily decreases to zero as x and y tend to infinity, then the double series $\sum_{0}^{\infty} \sum_{0}^{\infty} f(m,n)$ converges or diverges accordingly as $\int_0^{\infty} \int_0^{\infty} f(x,y)dxdy$ converges or diverges."

$D^2_{m,n}$, i.e., f(m,n), is positive and steadily decreases to zero as $m \rightarrow \infty$ and $n \rightarrow \infty$.

$$\int_0^{\infty} \int_0^{\infty} f(m,n)dmdn = \int_0^{\infty} \int_0^{\infty} D^2_{m,n}\, dmdn$$

$$= \int_0^{\infty} \left[\int_0^1 D^2_{m,0}\, dn + \int_1^{\infty} D^2_{m,n}\, dn \right] dm$$

$$= \int_0^{\infty} \left[\begin{array}{l} \displaystyle\int_0^1 \frac{16L^2}{2} (r-r_1)^2(1+rt)^2 \frac{\text{Sin}^2 \frac{m\pi}{2}}{(m^2\Theta_m + m\alpha)^2}\, dn + \\ \displaystyle+\int_1^{\infty} \frac{4(r-r_1)^2L^2}{\pi^2} \left\{ \frac{\text{Sin}^2(m+n)\frac{\pi}{2}}{(m+n)^2} + \frac{\text{Sin}^2(m-n)\frac{\pi}{2}}{(m-n)^2} \right\} dn \end{array} \right] dm$$

The first integrand in the brackets does not depend on n; the second integrand is odd in n and hence vanishes. The integral reduces to:

$$\int_0^\infty \int_0^\infty f(m,n)\ dmdn = \int_0^\infty \frac{16L^2}{\pi^2}(r-r_1)^2(1+rt)^2 \frac{\mathrm{Sin}^2\ \frac{m\pi}{2}}{(m^2\Theta_m + m\alpha)^2}\ dm$$

$$= \frac{16L^2}{\pi^2}(r-r_1)^2(1+rt)^2 \int_0^\infty \frac{\mathrm{Sin}^2\ \frac{m\pi}{2}}{(m^2\Theta_m + m\alpha)^2}\ dm$$

It is sufficient now to note that the integrand is on the order of $\frac{1}{m^4}$ and rather obviously converges. It must therefore be concluded on the basis of the theorem stated previously that:

(1) The system of equations (1C) and (2C) has a unique solution and the sum of the squares of the solutions A_n converge, i.e., $\sum_{n=0}^{\infty} A_n^2$ converges.

(2) Approximate solutions $A_n^{(n)}$ of the system of equations (1C) and (2C) are given by the method of segments, i.e., by considering only a finite number of equations in a finite number of unknowns. Concisely, $\lim_{n\to\infty} A_n^{(n)} = A_n$.

Consider now the convergence of the solution of the steady-state equation. The solution posed was as follows:

$$\Theta(x,y) = C_o\ (1+ry) + \sum_{n=1}^{\infty} C_n \cos \lambda_n x \left\{ e^{-\lambda_n y} + \frac{\lambda_n + r}{\lambda_n - r}\, e^{\lambda_n y} \right\}$$

$$C_n = \frac{A_n}{\frac{\lambda_n + r}{\lambda_n - r}\, e^{\lambda_n t} + e^{-\lambda_n t}}$$

Rewriting the solution in a form involving the alternative coefficients A_n:

$$\Theta(x,y) = A_0\ (1+ry) + \sum_{n=1}^{\infty} A_n \operatorname{Cos} \lambda_n x \left\{ \frac{\frac{\lambda_n+r}{\lambda_n-r}\, e^{\lambda_n y} + e^{-\lambda_n y}}{\frac{\lambda_n+r}{\lambda_n-r}\, e^{\lambda_n t} + e^{-\lambda_n t}} \right\}$$

Consider convergence of the solution along $y = 0$ where:

$$\Theta(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \operatorname{Cos} \frac{n\pi x}{2L} \left\{ \frac{1 + \frac{\lambda_n+r}{\lambda_n-r}}{\frac{\lambda_n+r}{\lambda_n-r}\, e^{\lambda_n t} + e^{-\lambda_n t}} \right\}$$

It has been shown that $\sum_{0}^{\infty} A_n^{\,2}$ converges. Although we have no idea how the terms $A_n^{\,2}$ behave, the series may be rearranged since it is absolutely convergent. Arrange the terms so that as n increases, $A_n^{\,2}$ decreases monotonely.

An order of magnitude estimate of the A_n's may be deduced. It is well known that $\sum_{1}^{\infty} \frac{1}{n}$ diverges and therefore: $\sum_{0}^{\infty} A_n^{\,2} < \sum_{1}^{\infty} \frac{1}{n}$. Moreover, both summands are bounded and monotone decreasing so that:

$$A_n^{\,2} < \frac{\xi_1}{n}$$

$$A_n < \frac{\xi}{\sqrt{n}}; \quad \xi = \text{a constant}$$

Furthermore, the A_n's may be considered as monotone decreasing since $A_n^{\,2}$ had this property.

The term $\mathrm{Cos}\,\frac{n\pi x}{2L}$ is bounded.

If $\sum_{1}^{\infty} \dfrac{1+\frac{\lambda_n+r}{\lambda_n-r}}{\frac{\lambda_n+r}{\lambda_n-r}e^{\lambda_n t}+e^{-\lambda_n t}}$ converges, then

$\sum_{1}^{\infty} A_n \,\mathrm{Cos}\,\frac{n\pi x}{2L}\left\{\dfrac{1+\frac{\lambda_n+r}{\lambda_n-r}}{\frac{\lambda_n+r}{\lambda_n-r}e^{\lambda_n t}+e^{-\lambda_n t}}\right\}$ converges since the latter summand is bounded by the first.

For $n > N$ where N is finite, the terms $\frac{\lambda_n+r}{\lambda_n-r}$ are always positive.

$$\sum_{1}^{\infty} \frac{1+\frac{\lambda_n+r}{\lambda_n-r}}{\frac{\lambda_n+r}{\lambda_n-r}e^{\lambda_n t}+e^{-\lambda_n t}} = \sum_{1}^{N} \frac{1+\frac{\lambda_n+r}{\lambda_n-r}}{\frac{\lambda_n+r}{\lambda_n-r}e^{\lambda_n t}+e^{-\lambda_n t}} + \sum_{N}^{\infty} \frac{1+\frac{\lambda_n+r}{\lambda_n-r}}{\frac{\lambda_n+r}{\lambda_n-r}e^{\lambda_n t}+e^{-\lambda_n t}}$$

The first sum on the right of the equality is finite and hence does not influence the convergence. For the second sum, $\frac{\lambda_n+r}{\lambda_n-r}$ is always positive. Since the exponentials are invariably positive, the following inequality may be written:

$$\frac{1+\frac{\lambda_n+r}{\lambda_n-r}}{\frac{\lambda_n+r}{\lambda_n-r}e^{\lambda_n t}+e^{-\lambda_n t}} \leq \frac{\beta}{\mathrm{Sin\,h}\,(\lambda_n t)}; \quad n > N \text{ and } \beta \text{ is a constant}$$

Replacing the summation in $1 \leq n \leq N$ above by a constant α, we therefore have:

$$\Theta(x,0) = A_0 + \sum_{1}^{\infty} A_n \operatorname{Cos} \frac{n\pi x}{2L} \left\{ \frac{1 + \frac{\lambda_n + r}{\lambda_n - r}}{\frac{\lambda_n + r}{\lambda_n - r} e^{\lambda_n t} + e^{-\lambda_n t}} \right\} \leq \alpha + \sum_{N}^{\infty} \frac{\beta}{\operatorname{Sin} h(\lambda_n t)}$$

The convergence of the latter sum is well known and since it bounds the given sum, the given sum converges along $y = 0$.

Consider the convergence of the solution in the region $0 < y < t$:

$$\Theta(x,y) = A_0 (1 + ry) + \sum_{n=1}^{\infty} A_n \operatorname{Cos} \frac{n\pi x}{2L} \left\{ \frac{\frac{\lambda_n + r}{\lambda_n - r} e^{\lambda_n y} + e^{-\lambda_n y}}{\frac{\lambda_n + r}{\lambda_n - r} e^{\lambda_n t} + e^{-\lambda_n t}} \right\}$$

As before, restrict attention to $n > N$; we may write the inequality:

$$A_n \operatorname{Cos} \frac{n\pi x}{2L} \; \frac{\frac{\lambda_n + r}{\lambda_n - r} e^{\lambda_n y} + e^{-\lambda_n y}}{\frac{\lambda_n + r}{\lambda_n - r} e^{\lambda_n t} + e^{-\lambda_n t}} \leq \frac{1}{\sqrt{n}} \left\{ \frac{\frac{\lambda_n + r}{\lambda_n - r} e^{\lambda_n y} + e^{-\lambda_n y}}{\frac{\lambda_n + r}{\lambda_n - r} e^{\lambda_n t} + e^{-\lambda_n t}} \right\}$$

$$\leq \frac{1}{\sqrt{n}} \; \frac{\beta \, e^{\lambda_n y}}{e^{\lambda_n t}} \; ; \; \beta = \text{constant}$$

since $\frac{\lambda_n + r}{\lambda_n - r} \to 1$ as $n \to \infty$ and $e^{-\lambda_n t}$ is bounded. Recalling now that $\lambda_n = \frac{n\pi}{2L}$ and that we are considering the region $0 < y < t$, we may write:

$$\frac{\beta}{\sqrt{n}} \; \frac{e^{\lambda_n y}}{e^{\lambda_n t}} = \beta \; \frac{e^{-n(t-y)\frac{\pi}{2L}}}{\sqrt{n}}$$

So that in $0 < y < t$, the solution summand is bounded by an exponential function and obviously converges.

Consider the convergence of the solution along y = t. Here we have:

$$\Theta(x,t) = A_0\ (1+rt) + \sum_{1}^{\infty} A_n \operatorname{Cos} \frac{n\pi x}{2L}$$

For convenience, call $z = \frac{\pi x}{2L}$ and rewrite the sum:

$$\Theta(x,t) = A_0\ (1+rt) + \sum_{1}^{\infty} A_n \operatorname{Cos} nz$$

Since $A_0\ (1+rt)$ is a constant and $A_n \leq \frac{\beta}{\sqrt{n}}$, consider the sum

$$\Theta(x,t) \leq \beta \sum_{1}^{\infty} \frac{\operatorname{Cos} nz}{\sqrt{n}}$$

Consider Dirichlet's test, which is as follows:

"$\sum_{1}^{\infty} A_n(z)\ B_n(z)$ is uniformly convergent if the partial sums of the series $\sum_{1}^{\infty} A_n(z)$ are uniformly bounded and if the functions $B_n(z)$ converge uniformly to 0, the convergence being monotone for each fixed x."

Now let: $A_n(z) = \operatorname{Cos} nz$

$$B_n(z) = \frac{1}{\sqrt{n}}$$

The interval $0 \leq x \leq 2L$ which is under consideration is equivalent to the interval $0 \leq z \leq \pi$. The partial sums of $\sum_{1}^{\infty} \operatorname{Cos} nz$ are uniformly bounded by $K = \dfrac{1}{\sin\frac{1}{2}\delta}$, $\delta > 0$, in the interval $0 \leq z \leq \pi$. $B_n(x) = \dfrac{1}{\sqrt{n}}$ tends monotonely and uniformly to zero since it does not depend upon x.

Therefore, $\sum_{1}^{\infty} \dfrac{\cos nz}{\sqrt{n}}$ converges uniformly in the interval $\delta < z \leq \pi$ or in $\delta < x \leq 2L$.

Now, $\sum_{1}^{\infty} A_n \cos\dfrac{n\pi x}{2L}$ is bounded by $\sum_{1}^{\infty} \beta \dfrac{1}{\sqrt{n}} \cos\dfrac{n\pi x}{2L}$ and the latter converges. It must be concluded therefore that the proposed solution:

$$\Theta(x,t) = A_0\,(1 + rt) + \sum_{1}^{\infty} A_n \cos\frac{n\pi x}{2L}$$

converges along $y = t$ in the interval $\delta < x \leq 2L$ where $\delta > 0$.

It is not difficult to show that the proposed solution is in fact uniformly convergent when $y < t$. Consider the Weierstrass m-test as follows:

> "Let $\sum_{n=1}^{\infty} U_n(x,y)$ be a series of functions all defined for sets E_1 and E_2 of values of x and y respectively. If there is a convergent series of constants $\sum_{n=1}^{\infty} M_n$ such that:
>
> $$\left| U_n(x,y) \right| \leq M_n \text{ for all } x \text{ and } y \text{ in } E_1 \text{ and } E_2,$$
>
> then the series $U_n(x)$ converges absolutely for each x and y in E_1 and E_2 and is uniformly convergent in (E_1, E_2)."

Using the bound $A_n \leq \frac{\xi}{\sqrt{n}}$ derived previously, for $y < t$ we have:

$$\left| A_n \cos\frac{n\pi x}{2L} \quad \frac{\frac{\lambda_n + r}{\lambda_n - r} e^{\lambda_n y} + e^{-\lambda_n y}}{\frac{\lambda_n + r}{\lambda_n - r} e^{\lambda_n t} + e^{-\lambda_n t}} \right| \leq \frac{\beta}{\sqrt{n}} e^{-\lambda_n E}; \ \beta \text{ is a constant}, \ E < t - y$$

and the latter converges for all $y < t$ as was shown. By Weierstrass's test, the solution is uniformly convergent in $y < t$.

According to two other well-known theorems, the term-by-term integration carried out in equation (11) is therefore legitimate and we may expect a bounded error in the region of consideration by using only a finite number of terms in equation (17) as long as $y < t$. Since we may approach $y = t$ arbitrarily close, it seems pointless to show uniform convergence along $y = t$.

Consider again the proposed solution:

$$\Theta(x,y) = C_o\,(1 + ry) + \sum_{n=1}^{\infty} C_n \cos\lambda_n x \left\{ \frac{\lambda_n + r}{\lambda_n - r} e^{\lambda_n y} + e^{-\lambda_n y} \right\}$$

subject to: $$\left.\frac{\partial\Theta}{\partial y}\right|_{x,t} = \begin{cases} -r\Theta, & 0 \leq x < L \\ -r_1(\Theta - k), & L < x \leq 2L \end{cases}$$

It is a simple matter to show that these last two expressions satisfy the originally posed problem. It follows that these expressions are indeed a solution (although not necessarily a unique solution) of the original problem.

By way of summary, it has been shown that a solution of the infinite system of equations (1C) and (2C) derived for the coefficients

C_n, n = 0, 1, 2, 3,...does exist and that this solution is unique. It was shown further that an approximate solution to the infinite system is given by the method of segments.

It was next shown that the proposed solution:

$$\Theta(x,y) = C_o\ (1+ry)+\sum_{1}^{\infty} C_n\ \text{Cos}\,\lambda_n x\ \left\{\frac{\lambda_n+r}{\lambda_n-r}\,e^{\lambda_n y}+e^{-\lambda_n y}\right\}$$

converges everywhere in the given region except at the point (0,t). The convergence at (0,t) will not be discussed further.

It is furthermore likely that the approximate solution posed is not unique at (L,t) since the boundary condition is many-valued at that point. No analysis of the uniqueness of the solution at (L,t) will be attempted.

The values of the constants (heat transfer coefficients, conductivity, etc.) have never influenced the results of this proof. The proof therefore remains valid for whatever system constants are chosen and the results of this analysis apply to similar systems.

In concise summary, it follows that the expression

$$\Theta(x,y) \cong C_o\ (1+ry)+\sum_{1}^{N} C_n\ \text{Cos}\,\frac{n\pi x}{2L}\ \left\{\frac{\lambda_n+r}{\lambda_n-r}\,e^{\lambda_n y}+e^{-\lambda_n y}\right\}$$

where the C_n are given approximately by using the method of segments on equations (1C) and (2C) and N is any finite number is evidently an approximate solution of the posed steady-state problem except at the points (0,t) and (L,t) where no proof was attempted.

APPENDIX D: NUMERICAL RESULTS

(1) Approximate values of the coefficients $C_n^{(2)}$ are given in Table 1D.

(2) Plots of temperature distribution from twenty-by-twenty system of equations along selected lines are given in Figures 1D through 4D.

Table 1D

n	$C_n^{(2)}$
0	221.8387
1	1218.9719
2	-39.0652
3	-99.3313
4	8.6593
5	23.9996
6	-2.6213
7	-6.5809
8	0.6571
9	0.8231
10	0.1213
11	1.3904
12	-0.4453
13	-2.2527
14	0.5686
15	2.5330
16	-0.6022
17	-2.5455
18	0.5779
19	2.4357

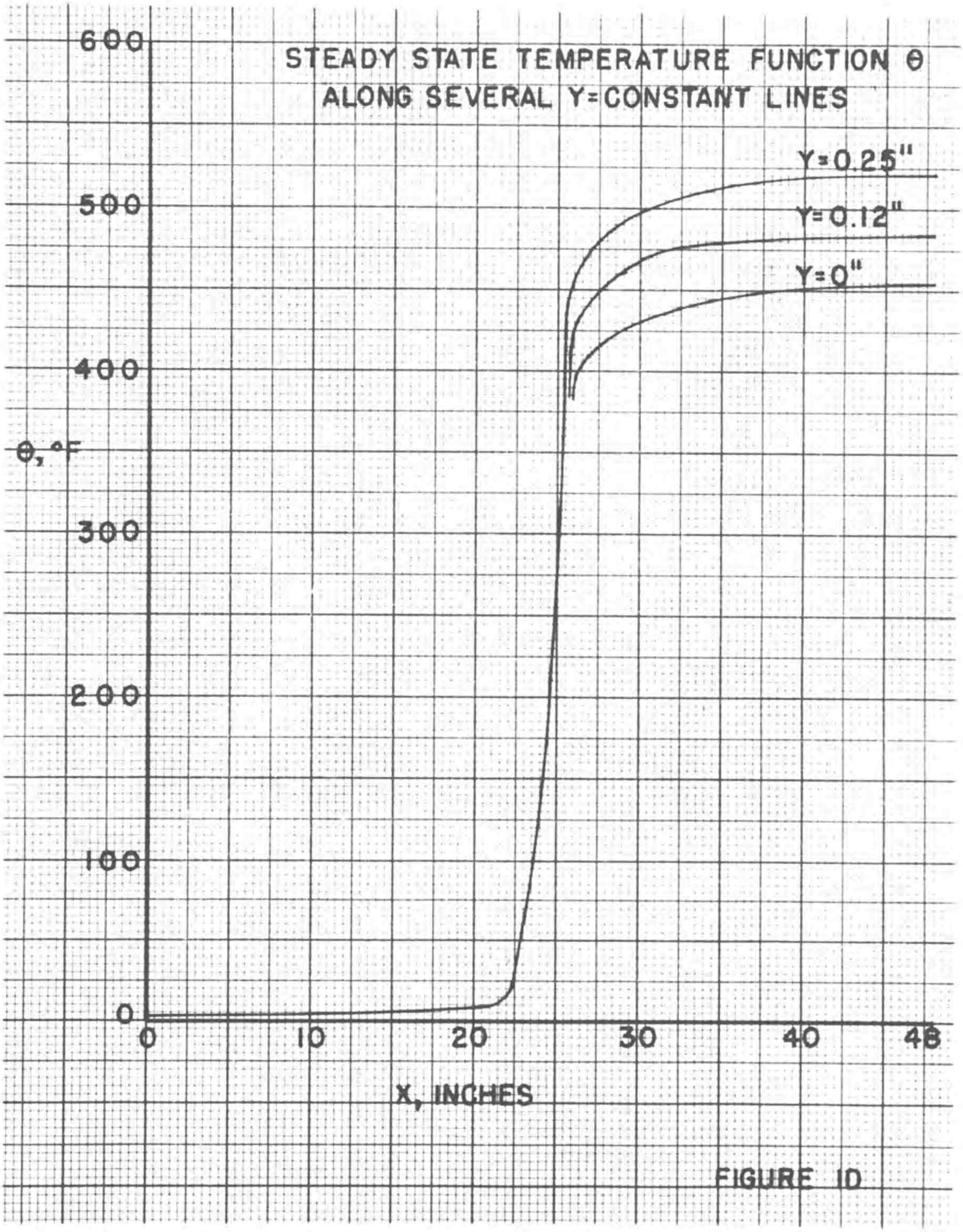
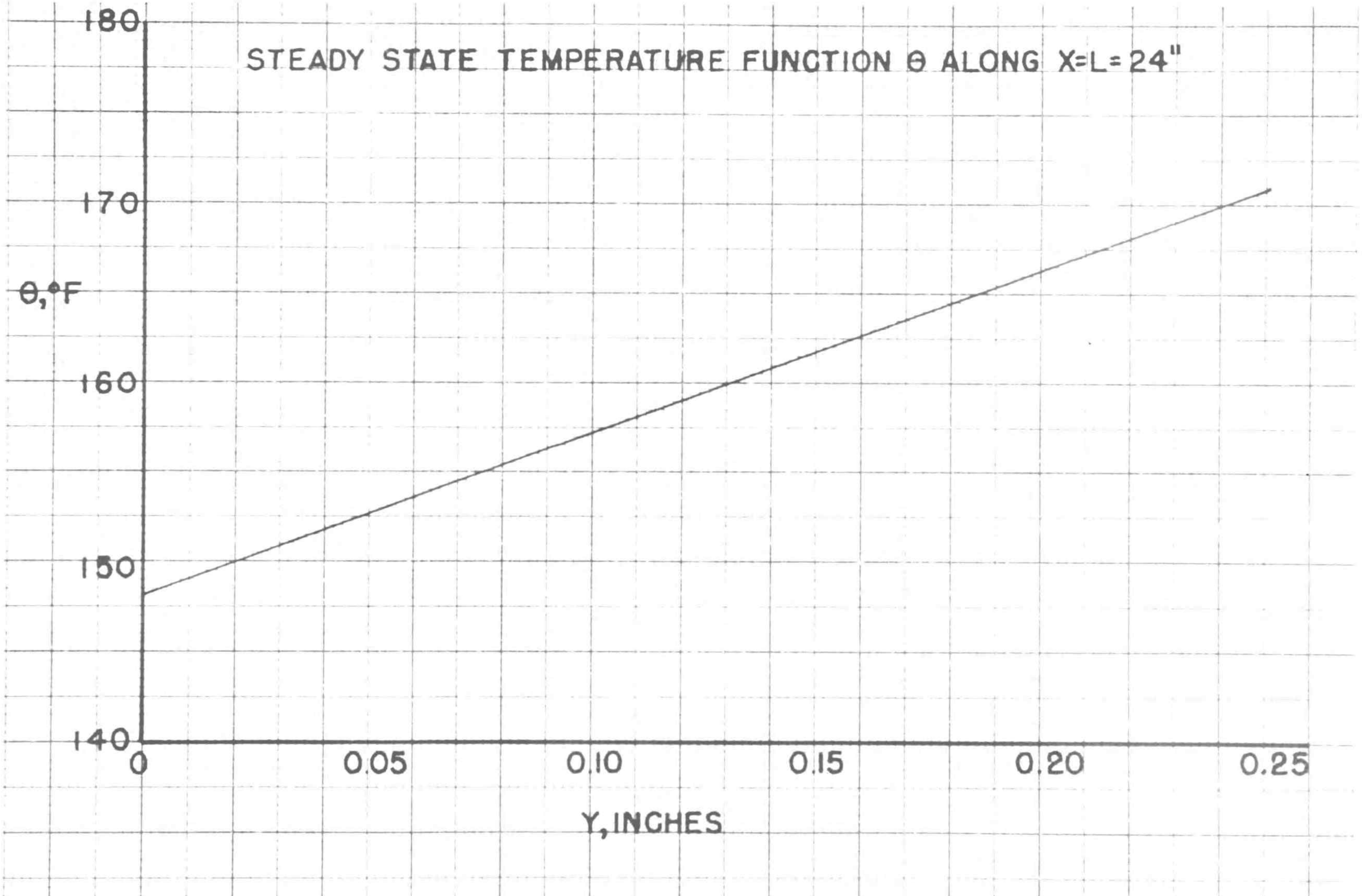
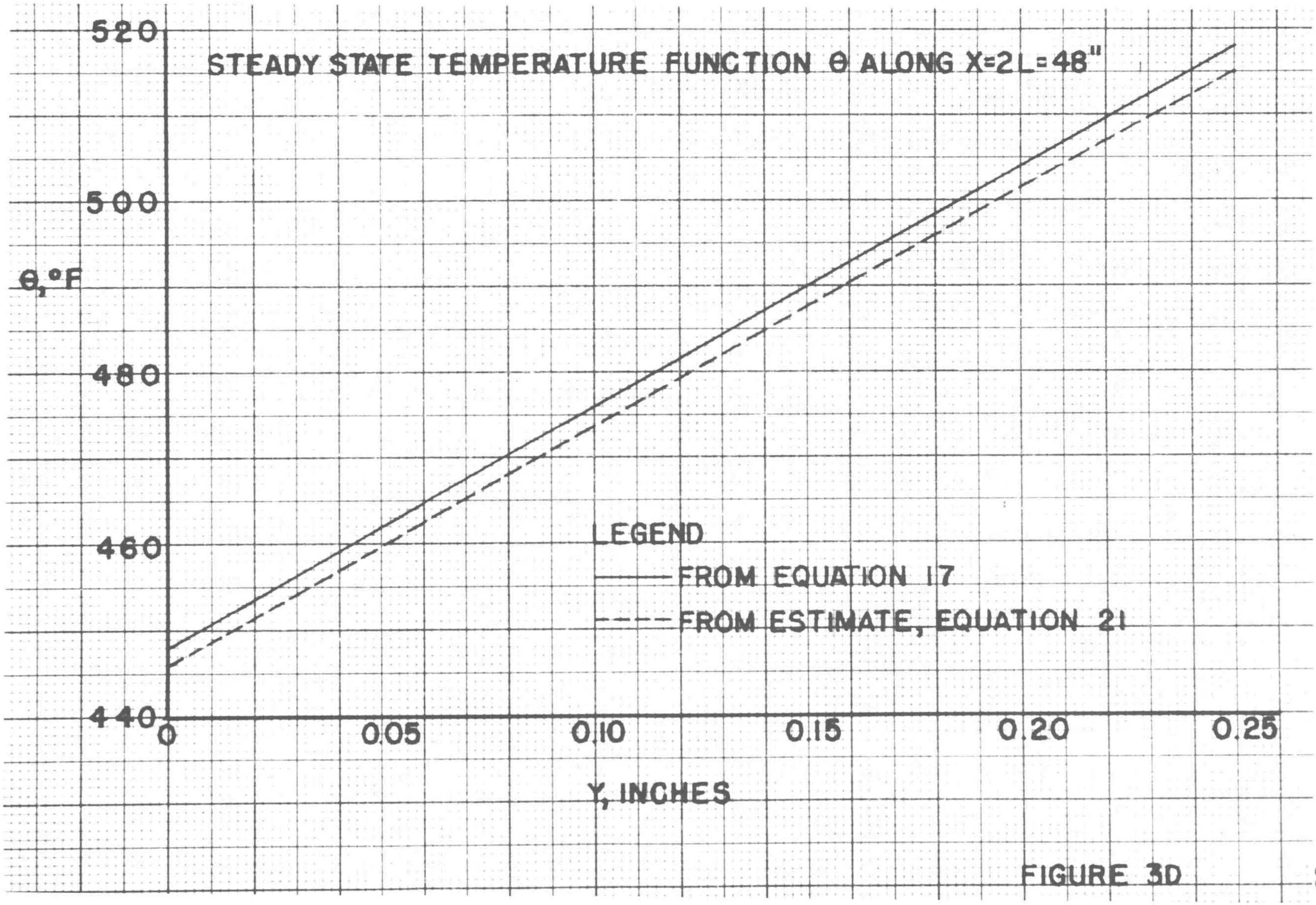

STEADY STATE TEMPERATURE FUNCTION θ
ALONG SEVERAL Y=CONSTANT LINES
Y=0.25"
Y=0.12"
Y=0"
600
500
400
300
200
100
0
θ, °F
0
10
20
30
40
48
X, INCHES

FIGURE 1D


STEADY STATE TEMPERATURE FUNCTION Θ ALONG X=L=24"
180
170
160
150
140
Θ, °F
0
0.05
0.10
0.15
0.20
0.25
Y, INCHES
FIGURE 2D



STEADY STATE TEMPERATURE FUNCTION θ ALONG X=2L=48"
θ, °F
520
500
480
460
440
0
0.05
0.10
0.15
0.20
0.25
Y, INCHES
LEGEND
FROM EQUATION 17
FROM ESTIMATE, EQUATION 21

FIGURE 3D

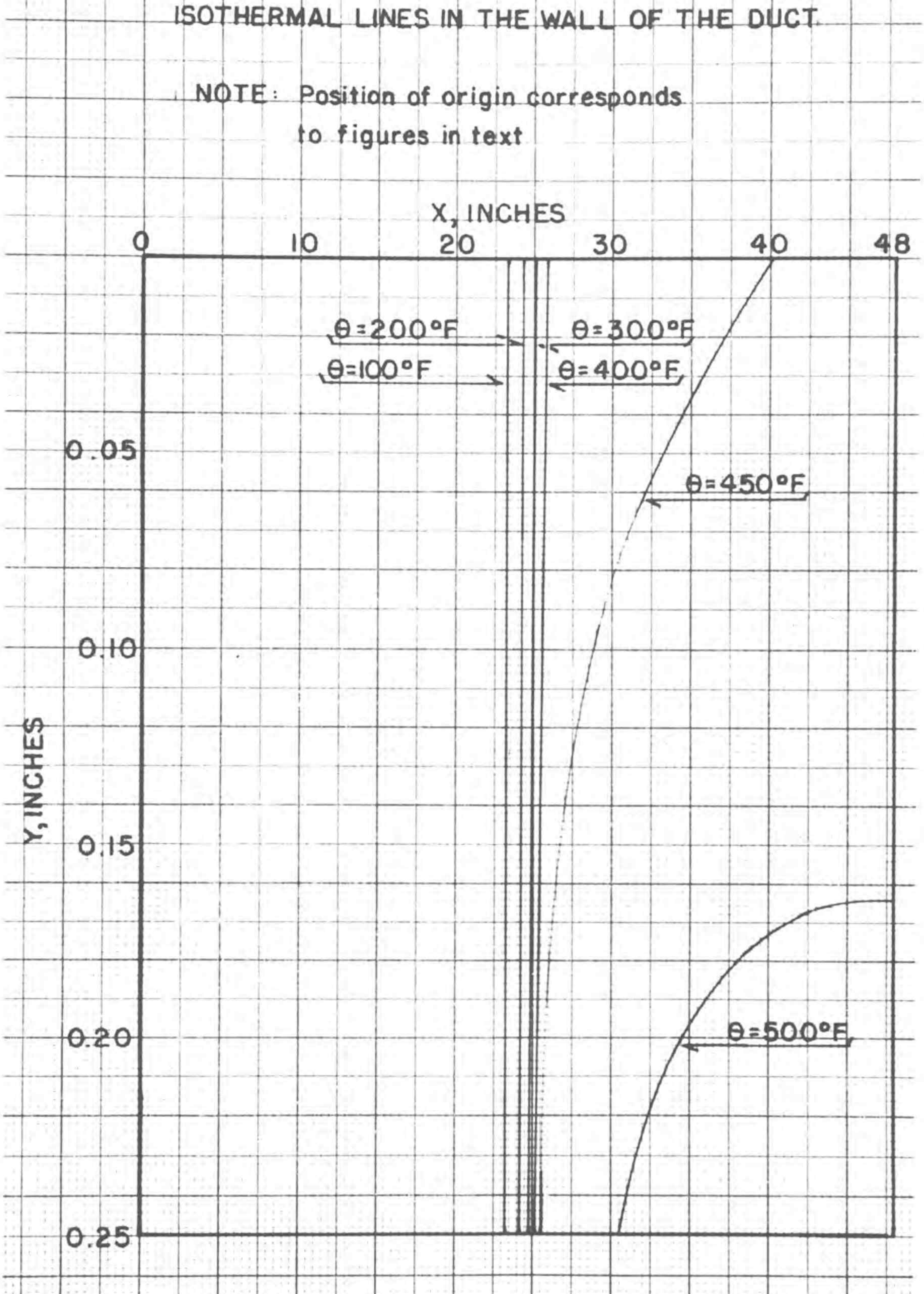

ISOTHERMAL LINES IN THE WALL OF THE DUCT
NOTE: Position of origin corresponds to figures in text
X, INCHES
0
10
20
30
40
48
θ=200°F
θ=300°F
θ=100°F
θ=400°F
θ=450°F
θ=500°F
0.05
0.10
0.15
0.20
0.25
Y, INCHES

FIGURE 4D

APPENDIX E: GLOSSARY OF TERMS

Alphabetic Letters

a = thickness of duct wall, in.

A = unknown coefficient; the symbol A represents a Wassiljewa constant in Appendix A.

b = free or standing term

B = free or standing term

C = unknown coefficient

D = known coefficient as on page 30; this symbol indicates diameter, in. throughout Appendix A.

e = base of Napierian logarithms

E = unknown coefficient

f(x,y,z) should be read "function of x, y, and z."

G = mass velocity of fluid, $\frac{\text{lb.}}{\text{hr.-ft.}^2}$

h = heat transfer coefficient, $\frac{\text{Btu}}{\text{hr.-ft.}^2\text{-}^\circ\text{F}}$

k = thermal conductivity, $\frac{\text{Btu}}{\text{hr.-ft.-}^\circ\text{F}}$; the symbol k represents a constant temperature as on page 27 and in Appendix C.

K = a constant

L = length, in.

m = a summation index; m indicates Mach number in Appendix A.

M = a constant

n = a summation index

N = a constant

p = fluid pressure, $\frac{\text{lb.}}{\text{in.}^2}$

P = Fourier transform variable

Q = heat flow, $\frac{\text{Btu}}{\text{sec.}}$

r = modified heat transfer coefficient, in.$^{-1}$

R = gas constant, $\frac{\text{ft.-lb.}}{\text{lb.}_m\text{-}^\circ\text{R}}$

s = LaPlace transform variable

t = thickness of duct wall in the steady-state problem; t indicates time, sec. in the transient problem.

T = temperature, °F except when used with the perfect gas equation where it is absolute temperature, °R.

V = velocity, $\frac{\text{ft.}}{\text{sec.}}$

x = space variable, in.

y = space variable, in.

z = space variable, in.

Greek Letters

α = thermal diffusivity, $\frac{\text{in.}^2}{\text{sec.}}$; the symbol α denotes an eigen value in the transient problem.

β = a constant for the steady-state problem; a function of s for the transient problem.

γ = density, $\frac{\text{lb.}}{\text{in.}^3}$; the symbol γ denotes a constant in the transient problem.

x should be read "change of x."

ϵ = an arbitrarily small constant

θ = temperature difference, °F

λ = an eigen value

μ = fluid viscosity, $\frac{\text{lb.}}{\text{in.-sec.}}$

ξ = a constant

ρ = fluid density, $\frac{lb._m}{in.^3}$

ϕ = angular space coordinate

Subscripts

e indicates an effective value

m indicates the m^{th} value, where m = 1, 2, 3,...

n indicates the n^{th} value, where n = 1, 2, 3,...

s indicates that the quantity is a function of the LaPlace transform variable s