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Abstract approved: ______________________________

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In this dissertation we will extend three foundational results of one-relator group theory. Magnus' Freiheitssatz [15] states that if $\mathcal{P} = (X, t : r)$ is a one-relator presentation and the relator r strictly involves the generator t, then the free group with basis X embeds in the group $G(\mathcal{P})$. Lyndon's Identity Theorem [14] describes the module structure of the relation module of a one-relator presentation. The Identity Theorem allows one to construct an Eilenberg-Maclane space of type $K(G, 1)$ for $G(\mathcal{P})$. Brodskiĭ [4] showed that each torsion free one-relator group is locally indicable.

Howie [9, 10, 11] then generalized all three of these results to the setting of one-relator products $(A * B)/r$ of locally indicable groups A and B. Another approach to generalizing the Freiheitssatz and Identity Theorem is to consider multi-relator presentations. Anshel [1] made the first big step when she proved an extension of the Freiheitssatz for a class of two relator presentations in 1990. Bogley [2] extended Anshel's Freiheitssatz to a class of multi-relator presentations and proved an analogue of the Identity Theorem for these presentations.

We will continue in this generalization by considering relative presentations. A relative presentation is a triple $\mathcal{P} = (A, X : R)$ where A is a group, X is a set, and R is a set of words in the free product $A * F(X)$ where $F(X)$ is the free group with basis X. The group presented by the relative presentation $\mathcal{P}$ is the quotient group $(A*F(X))/N$ where N is the normal closure of R. We say that the Freiheitssatz holds for the relative presentation $\mathcal{P}$ if the natural map of A into $G(\mathcal{P})$ is injective. Our results concern the case when the coefficient group A is locally indicable. Following Anshel [1] and Bogley [2], we formulate hypothesis on a relative presentation $\mathcal{P}$ under which i) the Freiheitssatz holds for $\mathcal{P}$, ii) there exists an analogue to the Identity theorem by construction of a $K(G(\mathcal{P}), 1)$, and iii) the group $G(\mathcal{P})$ is locally indicable.

In attempts to simplify the conditions necessary for future generalizations, we explored the theory of polygons of groups and a natural question arose. In conclusion, we will discuss the natural question of when the colimit of the graph of groups represented by an edge of a polygon of groups embeds in the colimit of the polygon of groups.

Local Indicability and Relative Presentations of Groups

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Julia D. Fredericks, Author

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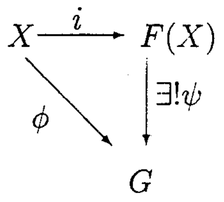
Local Indicability and Relative Presentations of Groups

1 DEFINITIONS

1.1 Combinatorial Group Theory

Let F be a group and X be a subset of F. We say that F is a *free group with basis* X, denoted $F(X)$, if for any group G and function $\phi : X \longrightarrow G$, there exists a unique homomorphism $\psi : F(X) \longrightarrow G$ such that $\psi \circ i = \phi$ where i is the inclusion of X into $F(X)$.



A word w in the alphabet X is a finite sequence $w = x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x_n^{\epsilon_n}$ where each $x_i \in X$ and $\epsilon_i \in \{-1, 1\}$. A cancelling pair in w occurs when $x_i = x_{i+1}$ and $\epsilon_i = -\epsilon_{i+1}$ for some $i \in \{1, \dots, n-1\}$. One can reduce the word w by removing any cancelling pairs. A word is *freely reduced* if it contains no cancelling pairs.

For every set X, there exists a free group $F(X)$ with basis X. Define

$$F(X) = \{w : w \text{ is a reduced word in the alphabet } X\}.$$

Let w and w' be two reduced words in the alphabet X such that $w = x_1^{\epsilon_1} \dots x_n^{\epsilon_n}$ and $w' = y_1^{\tau_1} \dots y_m^{\tau_m}$, where the $x_i, y_j \in X$, and the $\epsilon_i, \tau_j \in \{-1, 1\}$. Define the binary operation $w * w'$ to be the reduction of the word $x_1^{\epsilon_1} \dots x_n^{\epsilon_n} y_1^{\tau_1} \dots y_m^{\tau_m}$. The set $F(X)$ under the operation $*$ is a free group with basis X [13, Theorem I(2.1)].

A *presentation* $\mathcal{P}$ is a pair $\mathcal{P} = (X : R)$ where X is a set and R is a set of words in the free group $F(X)$. The elements of the set X are called *generators*, and the elements of R are called *relators*. The *normal closure* of a set R in a group G, denoted $\ll R \gg$, is the smallest normal subgroup of G that contains the subgroup generated by R. An alternate definition of this subgroup is the set

$$N = \{\prod_{i=1}^{n} g^{-1}rg : g \in A * F(X),\ r \in R\},$$

i.e. it is the subgroup generated by the relators and their conjugates. The group presented by $\mathcal{P}$, denoted $G(\mathcal{P})$, is the quotient group $F(X)/ \ll R \gg$.

If a group G has presentation $\mathcal{P} = (X : R)$, we can define a homomorphism $\phi : G \longrightarrow H$ to any group H by defining $\phi(x)$ for each $x \in X$ such that $\phi(r) = 1$ in H. In fact, every map from G to H is determined by an assignment of the basis X.

Theorem 1.1 *[16] For every group G, there exists a presentation $\mathcal{P}$ such that*

$$G \cong G(\mathcal{P}).$$

Proof: For the generating set take

$$X = \{g : g \in G\}.$$

Consider the homomorphism $\phi : F(X) \longrightarrow G$ induced by the identity function from $X = G$ to G. Let the set of relators be defined as

$$R = \{w : w \in ker\,\phi\}.$$

Then by the First Isomorphism Theorem, $G \cong G(\mathcal{P})$. □

Let A and B be groups with presentations $\mathcal{P} = (X : R)$ and $\mathcal{Q} = (Y : S)$ respectively. The *free product* of A and B, denoted $A * B$, is defined to be the group presented by the presentation $(X, Y : R, S)$. Elements of the free product $A * B$ are words of the form $w = a_1 b_1 a_2 b_2 \ldots a_n b_n$ where each $a_i \in A$ and each $b_j \in B$. The length of the word w, denoted $|w|$, is $2n$.

Theorem 1.2 *[16, Lemma IV(4.1)] The free product $A * B$ is uniquely determined by the groups A and B. Moreover, $A * B$ is generated by two subgroups $\overline{A}$ and $\overline{B}$ which are isomorphic to A and B respectively, and such that $\overline{A} \cap \overline{B} = 1$.*

This theorem indicates that the free product $A * B$ is independent of the choice of presentation of the groups A and B.

Now suppose that there exist a group C and isomorphisms $\alpha : C \longrightarrow C_A \leq A$ and $\beta : C \longrightarrow C_B \leq B$ where C_A is a subgroup of the group A and C_B is a subgroup of the group B. Let $A = G(\mathcal{P})$ where $\mathcal{P} = (X : R)$ and $B = G(\mathcal{Q})$ where $\mathcal{Q} = (Y : S)$. We define the free product of A and B amalgamated along C, denoted $A *_C B$, to be the group presented by the presentation $\mathcal{R} = (X, Y : R,\ S,\ \alpha(c)\beta(c)^{-1}\ \forall\ c \in C)$. Groups of this form are referred to as *free products with amalgamation.* Every element of the free product with amalgamation $A *_C B$ can be represented by a word of the form $w = a_1 b_1 \ldots a_n b_n$ where each $a_i \in A - C_A$ and each $b_j \in B - C_B$.

Theorem 1.3 *[16, Theorem IV(4.3)] The inclusions of A and B into the free product with amalgamation $A *_C B$ are injective.*

A *relative presentation* is a triple of the form $\mathcal{P} = (A, X : R)$ where A is a group, X is a set, and R is a set of words in the free product of the group A and the free group $F(X)$. The group presented by this relative presentation $G(\mathcal{P})$ is the quotient group $A * F(X)/ \ll R \gg$. Note that if the group A is trivial, then the relative presentation $(A, X : R)$ is equivalent to the ordinary group presentation $(X : R)$.

A *generalized presentation* is a triple $(A, B : R)$ where A and B are groups and R is a set of words in the free product $A * B$. The group presented by a generalized presentation is the quotient group $(A * B)/N$ where N is the normal closure of in the free product $A * B$.

1.2 Topological Models for Groups

An CW-complex is a topological space X with the structure

$$X^{(0)} \subseteq X^{(1)} \subseteq \ldots \subseteq X^{(n)} \subseteq X^{(n+1)} \subseteq \ldots \subseteq \bigcup_{n \geq 0} X^{(n)} = X$$

where $X^{(0)}$ is discrete and, for $n \geq 0$, each $X^{(n+1)}$ is obtained from $X^{(n)}$ by attaching $(n+1)$-cells. In other words, we are given an induced family of spherical maps

$$\dot{\phi}_\alpha \colon S^n_\alpha \longrightarrow X^{(n)}$$

and $X^{(n+1)}$ is the quotient space obtained from the disjoint union

$$(\dot{\bigcup}_\alpha B^{n+1}_\alpha) \mathbin{\dot{\cup}} X^{(n)}$$

by identifying all points $z \in S^n_\alpha \subseteq B^{n+1}_\alpha$ with their image $\dot{\phi}_\alpha\ (z) \in X^{(n)}$. The map $\dot{\phi}_\alpha$ extends to a map $\phi_\alpha : B^{n+1}_\alpha \longrightarrow X^{(n+1)}$. The set

$$c^{n+1}_\alpha = \phi_\alpha(B^{(n+1)}_\alpha - S^n_\alpha)$$

is an *open* $(n+1)$-cell in $X^{(n+1)}$. The map $\dot{\phi}_\alpha$ is called the *attaching map* for the $(n+1)$-cell c^{n+1}_α, and the map ϕ_α is called the *characteristic map* for the $(n+1)$-cell c^{n+1}_α.

The subspace $X^{(n)}$ of the CW-complex X is called the *n-skeleton* of X. The topological space X is defined to have the weak topology with respect the the collection $\{X^{(n)} : n \geq 0\}$. With this topology, the topological space X is a compactly generated, paracompact, Hausdorff space, and locally contractible hence semi-locally simply connected [22, Chapter II, Section 1]. Also, the *CW*-complex X has a universal cover (i.e. simply connected covering space) $\widetilde{X}$ [18, Chapter 14, Section 5, Theorem 5.3].

We say that a path-connected topological space K is *aspherical* if every spherical map $S^n \to K$ of the n-sphere into the space K for $n \geq 2$ can be extended to a map $B^{n+1} \to K$ of the (n+1)-ball into the space K. A topological space X is *contractible* if it has the homotopy type of a point. It is *acyclic* if it has trivial reduced homology in all dimensions.

Theorem 1.4 *The following are equivalent for a CW-complex X.*

1) The complex X is aspherical;

2) The universal cover $\widetilde{X}$ of X is contractible;

3) The universal cover $\widetilde{X}$ of X is acyclic.

Sketch of Proof: To show that (1) implies (2), build a contraction map to a point in $\widetilde{X}$ inductively. Start with a map on the 0-cells to a fixed point of $\widetilde{X}$ by using the path-connectedness of the universal cover. Then extend this map to a map on the 1-cells by using the fact that the universal cover is simply connected. You can continue to extend at each dimension since X is aspherical. To show that (2) implies (1), just lift any map of the n-sphere S^n into X to the universal cover $\widetilde{X}$. The implication (3) implies (2) is a consequence of the Hurewicz homomorphisms and (2) implies (3) is because the reduced homology of a point is trivial. □

Any map $f : X \longrightarrow Y$ between CW-complexes is *cellular* if $f(X^{(n)}) \subseteq Y^{(n)}$ where $X^{(n)}$ denotes the n-skeleton of the complex X. The next theorem is called the Relative Cellular Approximation Theorem.

Theorem 1.5 *[22, Corollary II(4.6)] Let $f : X \longrightarrow Y$ be a map of CW-complexes and suppose that A is a sub-complex of X such that $f|_A : A \longrightarrow Y$ is cellular. Then there exists a map $g : X \longrightarrow Y$ such that*

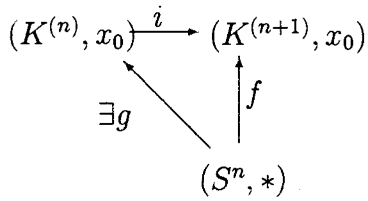
i) the map g is homotopic to the map f,

ii) The map g is cellular, and

iii) the map $g|_A = f|_A$.

Lemma 1.6 *For $n \geq 3$, if $K^{(n)}$ is a n-complex, and a $(n+1)$-complex $K^{(n+1)}$ is constructed by attaching $(n+1)$-cells to $K^{(n)}$, then $\pi_i K^{(n+1)} \cong \pi_i K^{(n)}$ for $i \in \{1, \ldots, n-1\}$.*

Proof: Let the map $i : K^{(n)} \longrightarrow K^{(n+1)}$ be the inclusion of the CW-complex $K^{(n)}$ into the CW-complex $K^{(n+1)}$. Then it suffices to show, for every $i \in \{1, \ldots, n-1\}$ that the induced map $i_{\#} : \pi_i(K^{(n)}) \longrightarrow \pi_i(K^{(n+1)})$ is an isomorphism. Let $[f] \in \pi_i(K^{(n+1)})$. Every element of the group $\pi_i(K^{(n+1)})$ can be represented by a based map $f : (S^i, *) \longrightarrow (K^{(n+1)}, x_0)$ where the point $*$ is a basepoint for S^i and x_0 is a 0-cell of $K^{(n+1)}$. By the Relative Cellular Approximation Theorem, there exists a based map $g : (S^i, *) \longrightarrow (K^{(i)}, x_0) \subseteq (K^{(n)}, x_0)$ that is homotopic to the map f.



But the map g represents an element $[g]$ of the group $\pi_i(K^{(n)})$ and since the map g is homotopic to the map f, $i_{\#}([g]) = [f]$ so we can conclude that the homomorphism $i_{\#}$ is surjective.

Now consider an element $[g] \in ker\, i_{\#}$. The element $[g]$ is represented by a based map $g : (S^i, *) \longrightarrow (K^{(n)}, x_0)$. Since $i_{\#}([g])$ is trivial in $\pi_i(K^{(n+1)})$, there exists an extension of the based map $g \circ i = f : (S^i, *) \longrightarrow (K^{(n+1)}, x_0)$ to a based map $\overline{f} : (B^i, *) \longrightarrow (K^{(n+1)}, x_0)$ where B^i is the i-ball. By the Relative Cellular Approximation Theorem, the based map $\overline{f}$ is homotopic to a map $\overline{g} : (B^i, *) \longrightarrow (K^{(i)}, x_0) \subseteq (K^{(n)}, x_0)$. But this map is homotopic to an extension of g to the i-ball, therefore the element $[g]$ is trivial in the group $\pi_i(K^{(n)})$ and the homomorphism $i_{\#}$ is injective. □

1.2.1 $K(G,1)$-complexes

Given a group G and an ordinary group presentation $\mathcal{P} = (X : R)$ for G, we can build a 2-complex $K(\mathcal{P})$ such that $\pi_1(K(\mathcal{P})) \cong G$ in the following manner. Take a single 0-cell which we will call the basepoint of $K(\mathcal{P})$. To form the one skeleton of $K(\mathcal{P})$, denoted $K^{(1)}$, attach an oriented 1-cell to the base point of $K(\mathcal{P})$ for each generator $x \in X$ by glueing its boundary to the base point. The one-skeleton $K^{(1)}$ is a one point union of oriented circles that is in one-to-one correspondence with the generators of the presentation.

Theorem 1.7 *[19, Lemma II(2.1)] The fundamental group of $K^{(1)}$ is isomorphic to the free group $F(X)$.*

Now, for every $r \in R$, label the boundary of a 2-cell, denoted c_r^2, by the word $r \in F(X) \cong \pi_1(K^{(1)})$ and use this label as a map to attach the boundary of each 2-cell to the one-skeleton $K^{(1)}$. The 2-complex $K(\mathcal{P})$ is the union of $K^{(1)}$

and these 2-cells. This 2-complex will be referred to as the *standard 2-complex* associated to the presentation $(X : R)$.

Theorem 1.8 *[19, Theorem II(2.3)] The fundamental group of $K(\mathcal{P})$ is isomorphic to the group $G(\mathcal{P})$.*

Let G be a group. A CW-complex Y is said to be a $K(G,1)$-complex if it satisfies the following conditions:

1) Y is connected.

2) $\pi_1(Y) \cong G$.

3) Y is aspherical.

Theorem 1.9 *[20] For every group G there exists a $K(G,1)$-complex.*

Proof: Let $\mathcal{P}$ be a presentation such that $G \cong G(\mathcal{P})$ and start with the standard 2-complex associated to $\mathcal{P}$, denoted $K(\mathcal{P})$. This complex will be equal to the 2-skeleton of our $K(G,1)$-complex K. Construct $K^{(3)}$ by attaching a 3-cell to $K(\mathcal{P})$ for every map from the n-sphere S^n into the complex $K(\mathcal{P})$. Then, by the Relative Cellular Approximation Theorem, each map from S^n to $K^{(n+1)}$ is null-homotopic in $K^{(n+1)}$ and by Lemma 1.6, $\pi_1(K^{(3)}) \cong G$. Now proceed inductively. Given $K^{(n)}$ build $K^{(n+1)}$ by attaching an $(n+1)$-cell for every map of S^n into $K^{(n)}$. Then each map of the n-sphere S^n into $K^{(n+1)}$ is null-homotopic by the Relative Cellular Approximation Theorem. By Lemma 1.6, $\pi_1(K^{(n)}) \cong G$ for every $n \geq 3$. Let $K = \bigcup_{n\geq 3} K^{(n)}$. Consider any map from S^k into K for $k \geq 1$. By the Relative Cellular Approximation Theorem, this map is homotopic to map of S^k into $K^{(n)}$ for some $n \geq 3$. This implies that the fundamental group $\pi_1(K)$ is isomorphic to the group G and that K is aspherical, therefore the CW-complex K is a $K(G,1)$-complex. □

1.2.2 Homology and Cohomology of a Group

The next theorem shows that homotopy invariants for a $K(G,1)$-complex are group invariants which enables us to define the homology and cohomology of any group G.

Theorem 1.10 *[22, Theorem V(7.2)] The homotopy type of a $K(G,1)$-complex depends only on G, i.e., if X and Y are $K(G,1)$-complexes and $\theta : \pi_1 X \longrightarrow \pi_1 Y$ is an isomorphism, then there exists a map $f : X \longrightarrow Y$ such that $f_{\#} = \theta : \pi_1 X \longrightarrow \pi_1 Y$ and f is a homotopy equivalence.*

Before we define the homology and cohomology of a group G, we need to define the augmented cellular chain complex for a CW-complex. The augmented cellular chain complex for a CW-complex K is a sequence

$$\ldots \longrightarrow C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_3} C_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

that is defined in the following manner. For $i \geq 1$, each C_i is defined to be the group $H_i(K^{(i)}, K^{(i-1)})$ where $K^{(i)}$ denotes the i-skeleton of the CW-complex K. The group $H_i(K^{(i)}, K^{(i-1)})$ is free abelian with basis in correspondence with the i-cells of K. The module C_0 is defined to be $C_0 = H_0(K^{(0)}$. The map $\epsilon : C_0 \longrightarrow \mathbb{Z}$ sends each generator to the element $1 \in \mathbb{Z}$. The boundary maps are defined by compositions

$$\begin{array}{ccccccc}
 & H_i(K^{(i)}) & & & & H_{i-2}(K^{(i-2)}) & \\
 & \searrow & & & \nearrow & & \\
\ldots \longrightarrow & H_i(K^{(i)}, K^{(i-1)}) & \longrightarrow & H_{i-1}(K^{(i-1)}, K^{(i-2)}) & \longrightarrow & \ldots & \\
 & & \searrow & \nearrow & & & \\
 & & & H_{i-1}(K^{(i-1)}) & & &
\end{array}$$

The horizontal sequence is the cellular chain complex that we are defining. The downward and upward sequences are the long exact homology sequences for the pairs $(K^{(i)}, K^{(i-1)})$ and $(K^{(i-1)}, K^{(i-2)})$ respectively.

Now consider the augmented cellular chain complex for the universal cover $\widetilde{K}$ of a CW-complex K. The 0-cells of $\widetilde{K}$ are in one-to-one correspondence with the elements of the group $\pi_1(K)$. Each n-cell of $\widetilde{K}$ corresponds to a lift of an n-cell of K at a specific 0-cell of $\widetilde{K}$. We can define a group action of $\pi_1(K)$ on the augmented cellular chain complex of the universal cover as follows. For any $h \in \pi_1(K)$, let the element h take the n-cell lifted at vertex $g \in \pi_1(K)$ to the n-cell lifted at vertex $h * g \in \pi_1(K)$.

Let G be a group defined multiplicatively. Then $\mathbb{Z}G$ is the free $\mathbb{Z}$-module generated by the elements of G with multiplication induced by the multiplication of the group G. One can check that it is a ring and it is often referred to as the *integral group ring* of G. A $\mathbb{Z}G$-module consists of an abelian group A and a homomorphism from $\mathbb{Z}G$ to the automorphism group of A. In other words, a $\mathbb{Z}G$-module is an abelian group A with a G-action on A. Using the action of $\pi_1(K)$ on the augmented cellular chain complex of the universal cover, we see that $C_i(\widetilde{K})$ is a $\mathbb{Z}G$-module where $G = \pi_1(K)$.

Let K be a $K(G,1)$-complex for G. Let $\widetilde{K}$ denote the universal cover of the complex K and $C_*\widetilde{K}$ denote the augmented cellular chain complex of the universal cover $\widetilde{K}$. The *homology of the group G with coefficients in M* is defined by

$$H_i(G; M) := H_i(C_*\widetilde{K} \otimes_G M).$$

The *cohomology of G with coefficients in M* is defined by

$$H^i(G; M) := H^i(Hom_G(C_*\widetilde{K}, M)).$$

These definitions of homology and cohomology of a group use a topological model. You can also define these invariants in a purely algebraic fashion. Let R be a ring. An R-module P is said to be *projective* if whenever M is an R-module and $\phi : M \longrightarrow P$ is a surjective R-module homomorphism, then $M = ker\,\phi \oplus P$. All free R-modules are projective.

A resolution of an R-module M over the ring R is an exact sequence of R-modules of the form

$$\ldots \to P_2 \to P_1 \to M \to 0.$$

An abbreviation for the resolution is to write $\epsilon : P \to M$. If each P_i is a projective(free) module, it is a projective(free) resolution.

Let G be a group and $\epsilon : P \to M$ be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ and M be a $\mathbb{Z}G$-module. The *homology of the group G with coefficients in M* is defined by

$$H_i(G; M) := H_i(P \otimes_G M).$$

The *cohomology of G with coefficients in M* is defined by

$$H^i(G; M) := H^i(Hom_G(F, M)).$$

Let K be a $K(G, 1)$ for a group G, and let $\widetilde{K}$ denote the universal cover of K which is contractible since K is aspherical. The augmented cellular chain complex of $\widetilde{K}$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ ([5, Proposition I(4.2)]). Hence this definition of homology and cohomology of groups is equivalent to the previous definition. See Chapters I, II, and III of [5] for more details on these definitions.

1.2.3 A $K(C_e, 1)$-complex

Let e be a positive integer. We will now give an explicit construction of a $K(G_e, 1)$-complex where G_e is the cyclic group of order e. Start with the standard 2-complex K associated to the presentation $(x : x^e)$ and consider the augmented cellular chain complex of the universal cover $\widetilde{K}$ of K.

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

The universal cover $\widetilde{K}$ has e 0-cells. The 0-cells are connected by e 1-cells to form a circle that is subdivided into e sections. You then attach e 2-cells by lifting the 2-cell of K at each 0-cell. Therefore $C_i = \mathbb{Z}G_e$ for $i = 0, 1, 2$. Pick a basic lift of the 0-cell of K as the base point for $\widetilde{K}$. This 0-cell will represent the generator 1 in $C_0 = \mathbb{Z}G_e$. Now choose the lift of the 1-cell of K at the base point of $\widetilde{K}$ to be the basic 1-cell of $\widetilde{K}$. This 1-cell will represent the generator 1 in $C_1 = \mathbb{Z}G_e$. Similarly, lifting the 2-cell of K at the basepoint of $\widetilde{K}$ will give a 2-cell that represents the generator 1 of $C_2 = \mathbb{Z}G_e$. The boundary map ∂_1 takes the basic 1-cell to the element $x - 1$ times the basic 0-cell. Let $N = 1 + x + \cdots + x^{e-1}$ be the norm element. The boundary map ∂_2 takes the basic 2-cell to the element N times the basic 1-cell. Now our cellular chain complex has the following form:

$$\mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{x-1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0.$$

Let x be an element of $\mathbb{Z}G$ for some group G. The set of all elements $z \in \mathbb{Z}G$ such that $z * x = 0$ in $\mathbb{Z}G$ is called the *annihilator* of x and denoted $Ann_{\mathbb{Z}G_e} x$.

Lemma 1.11 *(1)* $Ann_{\mathbb{Z}G_e}(x-1) = \mathbb{Z}G_e * N$

(2) $Ann_{\mathbb{Z}G_e} N = \mathbb{Z}G_e * (x-1)$.

Proof: Since the universal cover is simply connected, $H_1(\widetilde{K}) = 0$, therefore

$$ker\,\partial_1 = im\,\partial_2.$$

The the submodule $ker\,\partial_1$ is the annihilator $Ann_{\mathbb{Z}G_e}(x-1)$. By the definition of ∂_2, we know the submodule $im\,\partial_2$ is all multiples of N in $\mathbb{Z}G_e$. Therefore we can conclude that $Ann_{\mathbb{Z}G_e}(x-1) = \mathbb{Z}G_e * N$.

Now we will show that $Ann_{\mathbb{Z}G_e}N = \mathbb{Z}G_e * (x-1)$. It is clear that the submodule $\mathbb{Z}G_e*(x-1)$ is contained in $Ann_{\mathbb{Z}G_e}N$ so assume that $\xi \in Ann_{\mathbb{Z}G_e}N$. We will show that $\xi \in \mathbb{Z}G_e * (x-1)$. Write ξ such that

$$\xi = \sum_{i=0}^{e-1} n_i x^i.$$

Then

$$(\sum_{i=0}^{e-1} n_i x^i) * N = 0$$

by our assumption. However, $x^i * N = N$ for every $i \in \{0, \ldots, e-1\}$, so our equation becomes

$$\sum_{i=0}^{e-1} n_i * N = 0$$

which implies that

$$\sum_{i=0}^{e-1} n_i = 0.$$

Now consider the following calculation.

$$\begin{aligned}
\xi &= n_0 + n_1x + n_2x^2 + \ldots + n_{e-1}x^{e-1} \\
&= n_0 - n_0x + n_0x + n_1x + \sum_{i=2}^{e-1} n_ix^i \\
&= n_0(1-x) + (n_0+n_1)x + \sum_{i=2}^{e-1} n_ix^i \\
&= n_0(1-x) + (n_0+n_1)x - (n_0+n_1)x^2 + (n_0+n_1)x^2 + n_2x^2 + \sum_{i=3}^{e-1} n_ix^i \\
&= (n_0 + (n_0+n_1)x)(1-x) + (n_0+n_1+n_2)x^2 + \sum_{i=3}^{e-1} n_ix^i \\
&\vdots \\
&= \left(\sum_{i=0}^{e-2}\left(\sum_{j=0}^{i} n_j\right)x^j\right)(1-x)
\end{aligned}$$

The previous observation that $\sum_{i=0}^{e-1} n_i = 0$ is the condition necessary to end this process. This calculation shows that $\xi = g * (x-1)$ where

$$g = -\left(\sum_{i=0}^{e-2}\left(\sum_{j=0}^{i} n_j\right)x^j\right).$$

Therefore $\xi \in \mathbb{Z}G_e * (x-1)$ and $Ann_{\mathbb{Z}G_e}N = \mathbb{Z}G_e * (x-1)$. □

Go back to our cellular chain complex of the universal cover $\widetilde{K}$ and note that $ker\,\partial_2 = H_2(\widetilde{K}) \cong \pi_2(K)$. We just showed that $ker\,\partial_2$ is the set of all multiples of N. To extend our sequence, we need to attach a 3-cell to K such that the boundary of a basic lift of this 3-cell corresponds to the element $x-1$ time the basic 2-cell in C_2. Since $Ann_{\mathbb{Z}G_e}N = \mathbb{Z}G_e * (x-1)$, the extended sequence will be exact.

$$C_3 \xrightarrow{x-1} C_2 \xrightarrow{N} C_1 \xrightarrow{x-1} C_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

Now continue this process by attaching a cell in each dimension with boundary maps alternating between N and $(x-1)$. This construction will produce a CW-complex K such that $\pi_1(K) \cong G_e$ by Lemma 1.6 and the cellular chain complex of the universal cover of K will be exact. The exactness of the sequence implies that the universal cover has trivial reduced homology, therefore it is acyclic. By Theorem 1.4, the CW-complex K is aspherical, therefore it is a $K(G_e, 1)$-complex. Note that K is an infinite dimensional CW-complex.

We can use this construction to compute the homology of the cyclic group of order e. In fact, we find that

$$H_{2n+1}(\mathbb{Z}/e\mathbb{Z}) \cong \mathbb{Z}/e\mathbb{Z}$$

for every $n \geq 0$.

Corollary 1.12 *If X is a finite dimensional aspherical CW-complex, then $\pi_1(X)$ is torsion free.*

Proof: If $H \leq G$ is a finite subgroup of G, then a $K(G, 1)$-complex would have to contain a $K(H, 1)$-complex as a sub-complex. Then either H is trivial, or it contains a cyclic subgroup. If it contains a cyclic subgroup, a $K(H, 1)$-complex would be infinite dimensional, therefore H must be the trivial subgroup. □

In the general setting, building $K(G, 1)$-complexes is very difficult and many tools are employed to construct aspherical spaces. The next theorem, referred to as *Whitehead Amalgamation*, is one such tool.

Theorem 1.13 Whitehead Amalgamation *[23] Suppose that Z is a union of connected aspherical CW-complexes X and Y, which intersect in a connected aspherical sub-complex $X \cap Y$. If the inclusions of $X \cap Y$ into X and Y induce monomorphisms on fundamental groups, then Z is aspherical.*

2 ONE-RELATOR THEORY

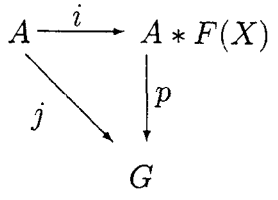
In this chapter, we will present some fundamental results of one-relator group theory by Magnus, Lyndon and Brodskiĭ. James Howie has generalized each of these results to the setting of one-relator products. The principal results of this dissertation, presented in Chapter 3, continue to generalize these results to the setting of multi-relator groups.

2.1 The Freiheitssatz

Given a presentation of the form $\mathcal{P} = (x_1, x_2, \ldots, x_n, t : r)$, one says that the relator r strictly involves t if it is not conjugate to a word involving only the x_i's. In 1930, Magnus proved the following result, which is referred to as the (One Relator) Freiheitssatz.

Theorem 2.1 *[15] Given a presentation of the form $\mathcal{P} = (x_1, x_2, \ldots, x_n, t : r)$ where r is a word in the generators that strictly involves t, the inclusion induced homomorphism from the free group on basis $\{x_1, \ldots, x_n\}$ into $G(\mathcal{P})$ is injective.*

We can restate this property for generalized presentations in the following way. Given the relative presentation $\mathcal{P} = (A, X : R)$, we say that the *Freiheitssatz holds for $\mathcal{P}$* if the inclusion induced homomorphism $j : A \longrightarrow G(\mathcal{P})$ is injective.



Definition 1 *A group G is said to be* **locally indicable** *if every non-trivial, finitely generated subgroup of G admits a surjection onto the integers.*

Howie generalized Magnus' Freiheitssatz to the setting of one-relator products of locally indicable groups in 1981.

Theorem 2.2 *[9, Theorem 4.3] Suppose $G = (A * B)/N$, where A and B are locally indicable groups and N is the normal closure in the free product $A * B$ of a cyclically reduced word r of length at least 2. Then the canonical maps $A \to G$ and $B \to G$ are injective.*

The conditions on the relator r ensure that it is not conjugate in $A * B$ to an element of A or B.

2.2 The Identity Property

Another property I will work with in this dissertation is the Identity Property. Consider the ordinary group presentation $\mathcal{P} = (X : r)$ with just one relator r. Let N be the normal closure of the element r in the free group $F(X)$. The abelianization of the subgroup N, denoted N^{ab}, is the quotient group $N/[N,N]$ where $[N,N]$ is the subgroup of N generated by the commutators $\{n_1^{-1}n_2^{-1}n_1n_2 : n_1, n_2 \in N\}$ of the group N.

Let $G = G(\mathcal{P})$. We can define a G-action on N^{ab} which is induced by conjugation in $F(X)$. The element $g = wN \in G(\mathcal{P})$ will act on $n[N,N] \in N^{ab}$ by

$$g * n[N,N] = wnw^{-1}[N,N]$$

Recall that a $\mathbb{Z}G$-module is an abelian group with a G-action. Therefore, N^{ab} is a $\mathbb{Z}G$-module. Moreover, since N is generated by the element r and its conjugates, N^{ab} is generated as a $\mathbb{Z}G$-module by the element $r[N, N]$ determined by the relators r of $\mathcal{P}$. The *relation module* for the presentation $\mathcal{P}$ is defined to be the $\mathbb{Z}G$-module N^{ab}.

Let $r = q^e$, where e is maximal and q is the root of the relator r in $F(X)$. Note that the element q commutes with r in $F(X)$, therefore the element $qN \in G$ acts trivially on $r[N, N]$ giving us the relation $(qN - 1) * r[N, N] = 0$ in the relation module N^{ab}. In 1950, Lyndon proved the following theorem which he referred to as the *Simple Identity Theorem.*

Theorem 2.3 *[14] Given a one-relator presentation $(X : r)$ for a group G, the relation module is a $\mathbb{Z}G$-module generated as a $\mathbb{Z}G$-module by the element $r[N, N]$ and with defining relation $(qN - 1) * r[N, N] = 0$ where q is the root of the relator r in $F(X)$, i.e. the kernel of the surjective module homomorphism*

$$\pi : \mathbb{Z}G \longrightarrow N^{ab}$$

induced by $\pi(1) = r[N, N]$ is given by $ker\, \pi = \mathbb{Z}G * (q - 1)$.

We can now generalize this property to the setting of relative presentations as follows. For a relative presentation $\mathcal{P} = (A, X : R)$, let N be the normal closure of the set of relators R in the free product $A * F(X)$. Then define N^{ab} to be the abelianization of the group N. If $G = G(\mathcal{P})$, then define a G-action on N^{ab} that is induced by conjugation in $A * F(X)$. Define the action of $g = wN \in G(\mathcal{P})$ on the element $n[N, N] \in N^{ab}$ by

$$g * n[N, N] = wnw^{-1}[N, N]$$

Under this action, the abelian group is a $\mathbb{Z}G$-module. Moreover, it is generated as a $\mathbb{Z}G$-module by the set of elements $\{r[N, N] : r \in R\}$ which are determined by the relators of the presentation $\mathcal{P}$. We will define the *relation module* for the relative presentation $\mathcal{P}$ to be the $\mathbb{Z}G$-module N^{ab}.

Since the G-action is induced by conjugation, if an element $g \in A * F(X)$ and a relator $r \in R \subset A * F(X)$ commute in $A * F(X)$, then $gN \in G(\mathcal{P})$ will act trivially on $r[N,N]$. Therefore $(gN - 1) * r[N,N] = 0$ which gives us a relation for N^{ab}.

Lemma 2.4 *Let $w \in A * B$ be a cyclically reduced word in the free product of free product length at least 2. Write $w = q^e$ where q is not a proper power. Then the centralizer of the word w in the free product $A * B$ is the cyclic subgroup generated by q.*

Proof: By Corollary 4.1.6 in [16] we know that the centralizer of the word w is a cyclic subgroup, say the subgroup generated by s. Since $w = q^e$, the element q will commute with w, so q is in the centralizer, i.e. $q = s^n$ for some $n \geq 1$. But we assumed that q is not a proper power, therefore $n = 1$ and the centralizer of w is the cyclic subgroup generated by q. □

For every relator $r \in R$, write $r = q_r^{e(r)}$ where q_r is the root of the relator r. By the previous lemma, the centralizer of r is generated by q_r, so we will always have the relations, $(q_r - 1) * r[N,N] = 0$ when our relators are proper powers. These relations are often referred to as the *trivial relations.* The relative presentation $\mathcal{P} = (A, X : R)$ is said to have the *Identity Property* if, as a $\mathbb{Z}G$-module, N^{ab} is generated by the set $\{r[N,N] : r \in R\}$ and the set of trivial relations

$$\{(q_r - 1)[N,N] : r \in R\}$$

are defining relations, i.e., the kernel of the surjective module homomorphism

$$\pi : \bigoplus_{r\in R} \mathbb{Z}G \longrightarrow N^{ab}$$

given by $\pi(1_r) = r[N, N]$ is given by $ker\, \pi = \bigoplus_{r\in R} \mathbb{Z}G * (q_r - 1)$. Lyndon also proved that the Identity Property also held for a special class of multi-relator presentations that are referred to as staggered presentations.

2.2.1 Computing Cohomology and Homology of a Group

The identity property allows Lyndon to construct a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ which he used to compute the homology and cohomology of one-relator group G. Let G be a one-relator group. This implies that there exists a presentation of the form $\mathcal{P} = (X : r)$ where r is a single relator. Construct the standard 2-complex $K(\mathcal{P})$ for the presentation $\mathcal{P}$.

Now consider the augmented cellular chain complex of the universal cover $\widetilde{K}$ of the 2-complex K associated to $G(\mathcal{P})$.

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

One notes that each $\mathbb{Z}G$-module $C_i = \mathbb{Z}G$ for $i \in \{0, 1, 2\}$. Recall from the definition of the augmented cellular complex that the boundary map ∂_2 factors through the group $H_1(\widetilde{K}^{(1)})$. The group $H_1(\widetilde{K}^{(1)}) \cong (\pi_1(\widetilde{K}^{(1)}))^{ab}$. We will show that $H_1(\widetilde{K}^{(1)}) \cong N^{ab}$ by showing that $\pi_1(\widetilde{K}^{(1)}) \cong N$. Consider the following diagram.

$$\begin{array}{ccc} \widetilde{K}^{(1)} & \longrightarrow & \widetilde{K} \\ p\downarrow & & \downarrow p \\ K^{(1)} & \longrightarrow & K \end{array}$$

A path in the one-skeleton of the universal cover $\widetilde{K}^{(1)}$ is a loop in $\widetilde{K}^{(1)}$ if and only if its image under p is trivial in $\pi_1(K)$. Therefore,

$$p_*(\pi_1(\widetilde{K}^{(1)})) = ker\,(i_* : \pi_1 K^{(1)} \to \pi_1 K) = N.$$

Since p is injective, $\pi_1(\widetilde{K}^{(1)}) \cong N$. Thus, $H_1(\widetilde{K}^{(1)}) \cong N^{ab}$. Now we will use the information provided by the Identity Theorem to extend the augmented cellular chain complex to a resolution of $\mathbb{Z}$ over $\mathbb{Z}G_e$.

Theorem 2.5 *[12, Proposition 1] If the presentation $\mathcal{P} = (X : R)$ has the Identity Property, then the image of the root q of the relator r in $G(\mathcal{P})$ has order e where $r = q^e$.*

From this theorem, we can conclude that the cyclic subgroup of order e is isomorphic to the subgroup Q generated by the element q in the group G.

To form the free resolution, start with the resolution of the cyclic group of order e with generator q, denoted Q, given in [5].

$$\ldots \xrightarrow{q-1} \mathbb{Z}Q \xrightarrow{N} \mathbb{Z}Q \xrightarrow{q-1} \mathbb{Z}Q \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

where $N = 1 + q + \ldots + q^{e-1}$ is the *norm element.*

Now apply the functor $\mathbb{Z}G \otimes_Q -$. The module $\mathbb{Z}G$ is a free right $\mathbb{Z}Q$-module since $Q \leq G$, so this functor is exact.

$$\ldots \longrightarrow \mathbb{Z}G \otimes_Q \mathbb{Z}Q \longrightarrow \mathbb{Z}G \otimes_Q \mathbb{Z}Q \longrightarrow \mathbb{Z}G \otimes_Q \mathbb{Z}Q \xrightarrow{\delta} \mathbb{Z}G \otimes_Q \mathbb{Z} \longrightarrow 0$$

But $\mathbb{Z}G \otimes_Q \mathbb{Z} = \mathbb{Z}[G/Q] \cong N^{ab}$ in Lyndon's case. So this is a resolution of N^{ab}. In the multi-relator case, $N^{ab} \cong \bigoplus_{r \in R} \mathbb{Z}[G/Q_r]$. We can form a resolution of N^{ab} by resolving each summand and then using the direct sum of all the resolutions.

We will use the resolution of the relation module N^{ab} to extend the augmented cellular complex that we began with. The Identity Property implies that $ker\,\partial_2 = \mathbb{Z}G * (q-1)$ where the relator $r = q^e$. We also know that

$$ker\, \partial_2 = H_2(\widetilde{K}) \cong \pi_2(K).$$

Therefore the $ker\, \partial_2$ is equal to the image of the homomorphism δ from the resolution of N^{ab} so we can extend the augmented cellular chain complex with the resolution of N^{ab}. This extension will give you a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

This construction allowed Lyndon to prove that a one-relator group has finite cohomological dimension if and only if it is torsion free, and that if it is torsion free, the cohomological dimension is at most 2.

2.2.2 The Pre-Aspherical Model

Now let's examine the connection between the Identity Property and asphericity. Given a group G, let $(X : R)$ be an ordinary group presentation for G. Build a topological space as follows. Take one 0-cell, and for each $x \in X$ attach an oriented circle, S^1_x to the 0-cell to form the one skeleton of K, denoted $K^{(1)}$. Note, $\pi_1 K^{(1)} \cong F(X)$ by Theorem 1.7

For every $r \in R$, write $r = q_r^{e(r)}$ where $e(r)$ is maximal and q_r is the root of the relator r. Let $\alpha(r) : D^{(1)}_{e(r)} = S^1_{e(r)} \longrightarrow K^{(1)}$ be a loop in $K^{(1)}$ representing the word q_r as an element of $F(X) \cong \pi_1 K^{(1)}$. This map defines a way to attach D_e to $K^{(1)}$. Let D be the one-point union of $D_{e(r)}$ for each $r \in R$ and α be the map induced by the set of maps $\{\alpha(r) : r \in R\}$ from D to $K^{(1)}$. Let K be the union of $K^{(1)}$ and D where for every $r \in R$, $S^1_{e(r)} \subseteq D$ is identified with its image under α in $K^{(1)}$. The complex K is called the *pre-aspherical model* and was first introduced by Dyer and Vasquez in [7]. The two skeleton of K, denoted $K^{(2)}$ is equivalent to the standard 2-complex associated to the given presentation of G.

Theorem 2.6 *[7] Suppose that $\alpha : S^1 \longrightarrow K^1$ is a loop in a 1-complex K^1, and that e is a positive integer. Let K be the union of K^1 and D_e, where each element of $D_e^1 = S^1$ is identified with its image in K^1 under α. If α does not represent a proper power in $\pi_1 K^1$, then K is aspherical.*

Theorem 2.6 is equivalent to Lyndon's Identity theorem. In fact, the free resolution constructed as a consequence of the Identity Property gives instructions on how to build the pre-aspherical model since it can be viewed as the cellular chain complex of the universal cover of a $K(G,1)$.

We can generalize the pre-aspherical model to relative presentations and generalized presentations. To build the pre-aspherical model $K(\mathcal{P}) = K$ for a generalized presentation $\mathcal{P} = (A, B : R)$, start with the one point union of a $K(A,1)$-complex K_A and a $K(B,1)$ complex K_B, denoted $K_A \vee K_B$. Now for every $r \in R$, there exists a based loop $\alpha(r) : S^1 \to (K_A \vee K_B)^{(1)}$ that represents q_r where $r = q_r^{e(r)}$ and q_r is the root of the relator r. Let the CW-complex D_e be the $K(G,1)$-complex for the cyclic group of order e. Attach the CW-complex $D = \bigvee_{r \in R} D_{e(r)}$ to $(K_A \vee K_B)^{(1)}$ by $\alpha = \bigvee_{r \in R} \alpha(r)$ and let pre-aspherical model K equal this complex.

Note that for each relator $r \in R$, the CW-complex K has a 2-cell $c_r^2 \subseteq D_{e(r)}$ that is attached along path $\beta(r) : S^1 \to K^{(1)}$ which traverses the path $\alpha(r)$ $e(r)$ times. By the Seifert Van Kampen Theorem, $\pi_1(K) \cong G = (A * F(X))/U$ where $U = \ll R \gg$. If B is a free group, the generalized presentation is equivalent to a relative presentation so we can build a pre-aspherical model for the relative presentation since a $K(G,1)$-complex for a free group $F(X)$ is a one point union of circles that is in one-to-one correspondence with the basis X.

Go back to the setting of $G = (A * B)/N$ where N is the normal closure of the single relator r in the free product $A * B$. Write $r = q^e$ where q is the root of the relator r.

Theorem 2.7 *[11, Theorem 1] Let $(A, B : r)$ be a one-relator generalized presentation in which the relator r has free product at least 2 and A and B are locally indicable groups. Then the pre-aspherical model of this generalized presentation is aspherical.*

This theorem is equivalent to saying that the generalized presentation $\mathcal{P}$ has the identity property when A and B are locally indicable groups and r is a word of free product length at least 2. Once again, if B is a free group, this reduces to the case of relative presentations.

2.3 Locally Indicable Groups

Recall that a group is locally indicable if every non-trivial, finitely generated subgroup admits a surjection onto the integers. In 1980, Brodskǐi announced the following theorem.

Theorem 2.8 *[4] Torsion-free 1-relator groups are locally indicable.*

Other examples of locally indicable groups include free groups and knot groups [10]. The following two theorems show that one can construct further examples of locally indicable groups using free products and one-relator presentations.

Theorem 2.9 *If the groups A and B are locally indicable, then their free product $A * B$ is locally indicable.*

Sketch of Proof: Let H be a non-trivial finitely generated subgroup of $A * B$. The Kurosh Subgroup Theorem [16, Corollary IV(4.9.1)] implies that H is the free product of a free group F and the intersection of H with conjugates of the subgroups A and B.

$$H = F * (*_i g_i A_i g_i^{-1}) * (*_j h_j B_j h_j^{-1})$$

The Grushko-Neumann Theorem [16, IV(1)] then implies that since H is finitely generated, the total number of factors is finite and each A_i and B_j is finitely generated. Since H is non-trivial, at least one free factor must be non-trivial. Any non-trivial free factor will be a finitely generated subgroup of a locally indicable group, therefore admit a surjection onto the integers. Then by first projecting H onto this factor, then following by this surjection, we see that H admits a surjection onto the integers and the free product $A * B$ is locally indicable. □

Theorem 2.10 *[10] Let A and B be locally indicable groups, and let G be the quotient of the free product $A * B$ by the normal closure of a cyclically reduced word r of length at least 2. Then the following are equivalent:*

i) G is locally indicable;

ii) G is torsion-free;

iii) r is not a proper power in $A * B$.

3 NEW RESULTS FOR SEMI-STAGGERED PRESENTATIONS

3.1 Results for Ordinary Group Presentations

In 1990, Anshel [1] published a Freiheitssatz statement for a class of two relator groups. She extended Magnus' approach to the one-relator case by developing what she termed an *independence hypothesis* for a two relator presentation of the form $\mathcal{P} = (X, y, z : R, S)$ and proving that the inclusion of the free group with basis X into the group $G(\mathcal{P})$ is injective. Her methods, like Magnus', were combinatorial. This theorem was a first step in attempting to generalize 1-relator group theory to multi-relator groups. In 1991, by interpreting Anshel's conditions in a topological setting, Bogley proved that a larger class of multi-relator presentations that includes Anshel's two relator presentation have the Identity Property. He also extended her Freiheitssatz to this class of multi-relator groups.

Let $\mathcal{P} = (A, X : R)$ be a relative presentation where A is a group, X is a set, and R is a set of cyclically reduced words representing elements in $A * F(X)$ where $F(X)$ is the free group with basis X. Let $G(\mathcal{P}) = (A * F(X))/U$ where $U = \ll R \gg$. Also, let $H = (A * F(X))/N$ where $N = \ll A \cup R \gg$. Thus H is obtained from G by "killing" the (normal closure) of A.

After cyclic permutation, we can assume that each $r \in R$ has the cyclically reduced form

$$r = x_1 a_1 x_2 a_2 \dots x_n a_n$$

where x_i is a word in $F(X)$, $a_i \in A$, and $i \geq 1$. Now define P_r to be the subset of cosets of N

$$P_r = \{x_1N,\ x_1x_2N,\ \ldots,\ x_1 \ldots x_nN = 1N\}.$$

The set P_r is the set of initial segments of the relator r modulo A. Let $\Pi = \bigcup_{r \in R} P_r \subseteq H$. If Ω is a subset of H, define $\Omega^\bullet = \Omega - \{1N\}$. Now we are ready for the definition of a semi-staggered presentation.

Definition 2 *A relative presentation $\mathcal{P} = (A, X : R)$ is* ***semi-staggered*** *if the following three conditions are satisfied:*

(S1) $P_r^\bullet \neq \emptyset$ for every $r \in R$;

(S2) There exists linear orderings on R and $\Pi^\bullet$ such that if $r, s \in R$ and $r < s$ then $\min P_r^\bullet < \min P_s^\bullet$ and $\max P_r^\bullet < \max P_x^\bullet$;

(S3) $\Pi^\bullet$ is a basis for a free subgroup of H.

Anshel and Bogley's results are stated in the following theorem.

Theorem 3.1 *If $\mathcal{P}$ is semi-staggered in A as defined above and A is a free group, then*

1) [1] the Freiheitssatz holds for $\mathcal{P}$, and

2) [2] the presentation $\mathcal{P}$ has the Identity Property.

3.2 New Results

In this dissertation, I will generalize the arguments of Anshel and Bogley to prove the following theorems.

Theorem 3.2 *If the relative presentation $\mathcal{P} =< A, X : R >$ is semi-staggered and A is a locally indicable group, then the pre-aspherical model of $\mathcal{P}$ is aspherical.*

Theorem 3.3 *(Freiheitssatz) If the relative presentation* $\mathcal{P} =< A, X : R >$ *is semi-staggered and* A *is a locally indicable group, then the inclusion of* A *into* $G(\mathcal{P})$ *is an injection.*

Theorem 3.4 *If the relative presentation* $\mathcal{P} =< A, X : R >$ *is semi-staggered,* A *is a locally indicable group, and no relator is a proper power, then the subgroup* N/U *of the group* $G(\mathcal{P})$ *is locally indicable.*

Corollary 3.5 *If in addition to the assumptions made in Theorem 3.4 the group* H *is locally indicable, then the group* $G(\mathcal{P})$ *is locally indicable.*

Proof: Consider the short exact sequence

$$1 \longrightarrow N \xrightarrow{i} G(\mathcal{P}) \xrightarrow{q} H = G(\mathcal{P})/N \rightarrow 1.$$

Let F be a non-trivial finitely generated subgroup of $G(\mathcal{P})$. If the image $q(F)$ is not the trivial subgroup in H, then it is a non-trivial finitely generated subgroup of H and since H is locally indicable, there exists a surjective homomorphism $\phi : q(F) \rightarrow \mathbb{Z}$ and $\phi \circ q : F \rightarrow \mathbb{Z}$ is a surjective homomorphism from F to $\mathbb{Z}$. If $F \subseteq \ker q$, then F is a non-trivial finitely generated subgroup of N which is locally indicable so F admits a surjection onto $\mathbb{Z}$. □

3.3 Connection to Identity Property

For a relative presentation, we define the relation module to be the abelianization of the normal closure of the set of relators, denoted N^{ab}. For every $r \in R$, write $r = q_r^{e(r)}$ where q_r is the root of the relator r.

Corollary 3.6 *(Identity Theorem) If the relative presentation* $\mathcal{P} = (A, X : R)$ *is semi-staggered and* A *is a locally indicable group, then the relation module* N^{ab} *is generated by the set* $\{r[N, N] : r \in R\}$ *and the set of trivial relations*

$$\{(q_r - 1)[N, N] : r \in R\}$$

are defining relations, i.e. the kernel of the surjective homomorphism

$$\pi : \bigoplus_{r\in R} \mathbb{Z}G \longrightarrow N^{ab}$$

given by $\pi(1_r) = r[N, N]$ *is given by* $ker\, \pi = \bigoplus_{r\in R} \mathbb{Z}G * (q_r - 1)$.

Proof: Let K be the pre-aspherical model for the semi-staggered presentation $\mathcal{P} = (A, X : R)$. The CW-complex K has the structure

$$K = K(A, 1) \vee (\bigvee_{x\in X} S^1_x) \cup_\alpha (\bigcup_{r\in R} D_r)$$

where D_r is a $K(G, 1)$ for the cyclic group of order $e(r)$ with attaching map α. Let $L = K(A, 1) \vee (\bigvee_{x\in X} S^1_x) \cup_\alpha (\bigcup_{r\in R} D^{(2)}_r)$. The CW-complex L is a sub-complex of K. If none of the relators are proper powers, then $L = K$.

Let $p : \widetilde{L} \to L$ be the universal cover of L. Let $Z = K(A, 1) \cup L^{(1)}$ which is a sub-complex of L. Then $\overline{Z} = p^{-1}(Z)$ is a connected sub-complex of $\widetilde{L}$.

$$\begin{array}{ccc} \overline{Z} = p^{-1}(Z) & \longrightarrow & \widetilde{L} \\ p \downarrow & & \downarrow p \\ Z & \longrightarrow & L \end{array}$$

Consider the exact homology sequence for the pair $(\widetilde{L}, \overline{Z})$.

$$H_2(\overline{Z}) \to H_2(\widetilde{L}) \to H_2(\widetilde{L}, \overline{Z}) \to H_1(\overline{Z}) \to H_1(\widetilde{L})$$

Since $\widetilde{L}$ is simply connected, $H_1(\widetilde{L}) = 0$. The group $H_1(\overline{Z})$ is defined to be the abelianization of $\pi_1(\overline{Z})$. A path is a loop in the universal cover if and only if its image under p is trivial in L, so $\pi_1(\overline{Z}) \cong N$, hence $H_1(\overline{Z}) \cong N^{ab}$ which is the relation module for the relative presentation $\mathcal{P}$. The group $H_2(\widetilde{L}, \overline{Z})$ is the free abelian group on generators corresponding to the 2-cells of $\widetilde{L} - \overline{Z}$ since this space contains no cells of higher dimension. Therefore, $H_2(\widetilde{L}, \overline{Z}) \cong \bigoplus_{r\in R} \mathbb{Z}G$. Lastly, the group $H_2(\widetilde{L}) \cong \pi_2(\widetilde{L}) \cong \pi_2(L)$. Under these observations, our exact sequence becomes the following:

$$\pi_2(L) \xrightarrow{\partial_3} \bigoplus_{r\in R} \mathbb{Z}G \xrightarrow{\partial_2} N^{ab} \xrightarrow{\partial_1} 0.$$

To show that the relative presentation $\mathcal{P}$ has the Identity Property, it suffices to show that $ker\,\partial_2 = \bigoplus_{r\in R} \mathbb{Z}G * (q_r - 1)$. Since K is aspherical by Theorem 3.2, and K is built from L by adding only cells of dimension 3 and higher, $\pi_2 L$ is generated by the attaching maps of the 3-cells that you add. These are the 3-cells needed to build the D_r for each $r \in R$. For every relator $r = q_r^{e(r)}$, we attach a 3-cell to L necessary for the construction of a $K(G,1)$ for the cyclic group of order $e(r)$. In Chapter 1, Section 2, we saw that these attaching maps correspond to the elements $q_r - 1$, so $\pi_2(L)$ is generated by the boundary of these maps. The $ker\,\partial_2 \cong \pi_2(L)$, therefore the $ker\,\partial_2 = \bigoplus_{r\in R} \mathbb{Z}G * (q_r - 1)$. □

3.4 Constructing Examples

To construct examples of semi-staggered presentations $\mathcal{P} = (A, X : R)$, we will start with the group $H \cong G(\mathcal{P})/U$ where U is the normal closure of the set $A \cup R$. For H, one must choose a group who has a free subgroup with a basis Π. The first step is to linearly order Π. Now let A be any locally indicable group and your basis for the free subgroup of H be the set $\Pi = \{b_1, b_2, b_3, b_4, \dots\}$ with indicated linear ordering. Construct your relators in the following manner:

$$r_1 = b_1\, a_{1\,1}\, b_2\, a_{1\,2}\, b_3\, a_{1\,3} \dots b_k\, a_{1\,k}$$
$$r_2 = b_l\, a_{2\,1}\, b_{l+1}\, a_{2\,2} \dots b_m\, a_{2\,m-l}$$
$$\vdots$$

where each relator has free product length at least 2, $2 \le l \le k$, and each a_{ij} is an element of the group A. One can continue this process of "staggering" the basis elements to build a set of relators. Note, if you start with an infinite basis for the free subgroup of H, you can build an infinite number of relators. The presentation $\mathcal{P} = (A, \Pi : r_i)$ is semi-staggered. At this point, none of the relators are proper powers. One can construct a new semi-staggered presentation by replacing any non-empty subset of the relators $S \leq \{r_i\}$ by the set

$$\{s^{e_s} : e_s \geq 2, s \in S\}$$

which adds proper powers to the presentation. In the case where none of the relators are proper powers, if you choose H to be a locally indicable group, then by Corollary 3.5, the group $G(\mathcal{P})$ is locally indicable.

4 PROOFS OF THEOREMS 3.2 - 3.4

4.1 A Preliminary Lemma

To prove Theorem 3.4 we will need the following lemma that shows that a direct product of locally indicable groups is locally indicable.

Lemma 4.1 *Let K and $\{K_\alpha : \alpha \in \mathcal{A}\}$ be CW-complexes such that the complex $K = \bigcup_{\alpha \in A} K_\alpha$, every compact sub-complex of K is contained in K_α for some $\alpha \in \mathcal{A}$, and for every α, $\beta \in \mathcal{A}$ there exists a γ such that $K_\alpha \cup K_\beta \subseteq K_\gamma$. If $\pi_1 K_\alpha$ is locally indicable for each K_α, then $\pi_1 K$ is locally indicable.*

Proof: Let K and $\{K_\alpha : \alpha \in \mathcal{A}\}$ be as above and let H be a finitely generated subgroup of $\pi_1(K)$. Assume that H does not admit a surjection onto the integers. We will show that H is trivial in $\pi_1(K)$. Since H is finitely generated, there exist $x_1, \dots , x_n$ in $\pi_1(K)$ such that $H =< x_1, \dots , x_n >$. The complex K is a union of sub-complexes K_α and each compact sub-complex of K is contained in some K_α. Therefore, we can find an element $N \in \mathcal{A}$ and elements x_i' in $\pi_1(K_N)$ such that the homomorphism induced by the inclusion of K_N into K sends each x_i' to the element x_i in $\pi_1(K)$.

Since H does not admit a surjection onto the integers, the abelianization of H, denoted H^{ab}, is finite. Then, for every i, there exists an integer e_i such that

$$x_i^{e_i} \in [H, H] \leq \pi_1(K).$$

Say that $x_i^{e_i} = w_i$ where w_i is a product of commutators of H. Then $x_i^{-e_i} w_i = 1$ in $\pi_1 K$, i. e. $x_i^{-e_i} w_i$ is a trivial loop in K so, without loss of generality, we can assume that there exists a disk map $d_i : B^2 \longrightarrow K$ such that the boundary of d_i is equal to $x_i^{-e_i} w_i$.

Let $T \in \mathcal{A}$ so that $K_N \subseteq K_T$ and K_T supports each disk map d_i. Since $K_N \subseteq K_T$, the image of x_i' under the inclusion induced homomorphism $i_\# : \pi_1(K_N) \longrightarrow \pi_1(K_T)$ is an element x_i'' of $\pi_1(K_T)$. Let H_T be the subgroup of $\pi_1(K_T)$ that is generated by the x_i'. Note that this is a finitely generated subgroup of the locally indicable group $\pi_1(K_T)$. Since K_T supports each disk map d_i, each generator $(x_1'')^{e_i} \in [H_T, H_T]$. It follows that the abelianization of H_T is finite, therefore there exists no surjective homomorphism from H_T onto the integers. Since $\pi_1(K_T)$ is locally indicable, we conclude that H_T is the trivial group. The inclusion of K_T into K induces a surjective homomorphism from H_T onto the subgroup H of $\pi_1(K)$, therefore H must be trivial. $\square$

4.2 Topological Models for the Proof

Now we will build the pre-aspherical model $K(\mathcal{P}) = K$ for a relative presentation $\mathcal{P} = (A, X : R)$, we start with the CW-complex $K(A,1) \vee (\bigvee_{x \in X} S^1_x)$. Note

$$K^{(1)} \subseteq K(A,1) \vee (\bigvee_{x \in X} S^1_x).$$

Now for every $r \in R$, there exists a based loop $\alpha(r) : S^1 \to K^{(1)}$ that represents the root q_r of the relator r, i.e. $r = q_r^{e(r)}$ and $e(r)$ is maximal. We say that q_r is the root of the relator r. Let the CW-complex D_e be a $K(C_e, 1)$-complex for the cyclic group C_e of order e. Attach the complex $D = \bigvee_{r \in R} D_{e(r)}$ to the 1-skeleton $K^{(1)}$ by the induced map $\alpha = \bigvee_{r \in R} \alpha(r)$ and let K equal the resulting complex. Note that for each $r \in R$, the complex K has a 2-cell $c_r^2 \subseteq D_{e(r)}$ that is attached along path $\beta(r) : S^1 \to K^{(1)}$ which traverses the path $\alpha(r) : S^1 \longrightarrow K^{(1)}$ $e(r)$ times. By the Seifert Van Kampen Theorem, $\pi_1(K) \cong G(\mathcal{P}) = (A * F(X))/U$ where $U = \ll R \gg$.

Recall that $N = \ll A \cup R \gg_{A*F(X)}$. Let $p : \overline{K} \to K$ be the regular covering of K such that $p_\sharp(\pi_1(\overline{K})) = N/U \trianglelefteq \pi_1 K = G(\mathcal{P})$.

The 0-cells of $\overline{K}$ are in one-to-one correspondence with $H = (A * F)/N$, so we can choose a labelling of the 0-cells by elements of the group H. At each vertex of $\overline{K}$, there will be a lift of each 1-cell of K. For every $x \in X$, the lift of S^1_x at the vertex $h = wN$ will be a 1-cell of $\overline{K}$ with initial vertex wN and terminal vertex wxN. Let $T = p^{-1}(\bigvee_{x \in X} S^1_x)$.

At each vertex there will also be a lift of $K(A,1) \subseteq K$. We will call the lift of the sub-complex $K(A,1)$ of K at the vertex $h \in H$ the "rose" at vertex h, denoted $V(h)$. Then $p^{-1}(K(A,1)) = \dot{\bigcup}_{h \in H} V(h)$. In fact $p^{-1}(K(A,1)) = K(A,1) \times H$ where H represents the discrete set of 0-cells of $\overline{K}$. We will examine more carefully the lift, for each $r \in R$, of the sub-complex $D_{e(r)}$ of K.

Note, for the remainder of the paper, we assume that the relative presentation $\mathcal{P} = (A, X : R)$ is semi-staggered.

Lemma 4.2 *For every $r \in R$ and for every $h \in H$, the loop $\alpha(r)$ lifts at 0-cell h in $\overline{K}$ to a loop $\alpha(r,h)$ in $\overline{K}^{(1)}$. Moreover, the image of $\alpha(r,h)$ is contained in $T \cup (\bigcup_{k \in hP_r} V(k))$ and $\alpha(r,h)$ strictly involves at least one 1-cell from each rose $V(k)$ for every $k \in hP_r$. The loop $\alpha(r,h)$ does not represent a proper power in $\pi_1 \overline{K}^{(1)}$.*

Proof: The path $\beta(r)$ lifts to a path $\beta(r,h)$ in $\overline{K}$ which begins at $h = wN$ and traverses a path in T that covers the non-empty path x_1 and ends at vertex wx_1N. Then it travels a lift of a_1 in the rose $V(wx_1N)$. For $m = 1, \ldots, n-1$, $\beta(r,h)$ travels from $wx_1 \ldots x_m N$ to $wx_1 \ldots x_m x_{m+1} N$ and lifts to an essential loop in the rose $V(wx_1 \ldots x_{m+1} N)$ covering a_{m+1}.

This implies that $im(\beta(r,h)) \subseteq T \cup (\bigcup_{k \in hP_r} V(k))$ and strictly involves at least one 1-cell from each rose $V(k)$ for each $k \in hP_r$. Since $im(\alpha(r,h)) = im(\beta(r,h))$, we also know that $im(\alpha(r,h))$ involves at least one 1-cell from each rose $V(k)$ for each $k \in hP_r$. Now we will show that $\alpha(r,h)$ lifts to a loop at $h = wN \in H$.

Since $r = q_r^{e(r)}$, it suffices to show that $q_r \in N$. Note that $q_rN \in P_r$ and $(q_rN)^e = q_r^{e(r)}N = rN = 1N$. However, by assumption, the subgroup generated by Π is free, hence torsion free so $q_rN = 1N$. Therefore $q_r \in N$ and $\alpha(r)$ lifts to a loop at h. We will show that $\alpha(r,h)$ is not a proper power in $\pi_1\overline{K}^{(1)}$ by way of contradiction. If $\alpha(r,h)$ was a proper power it would transverse a loop $\gamma(r,h) : S^1 \longrightarrow \overline{K}$ at least 2 times. Since the covering map p is continuous, the image of $\gamma(r,h)$ under p would be a loop γ in K. Then the image of $\alpha(r,h)$ under p would transverse the loop γ at least 2 times. But the image of $\alpha(r,h)$ is α which is not a proper power, therefore we have a contradiction and conclude that $\alpha(r,h)$ is not a proper power. □

Now for $r \in R$, let

$$[r] = \{p \in \Pi^\bullet : min\, P_r^\bullet \leq p \leq max\, P_r^\bullet\} \cup \{1N\} \subseteq \Pi.$$

For $h \in H$, let

$$\overline{K}(r,h) = D_{e(r)} \cup T \cup (\textstyle\bigcup_{k \in h[r]} V(k))$$

where $S^1_{e(r)} \subseteq D_{e(r)}$ is identified with its image in

$$T \cup (\textstyle\bigcup_{k \in hP_r} V(k)) \subseteq T \cup (\textstyle\bigcup_{k \in h[r]} V(k))$$

by $\alpha(r,h)$. This identification is well-defined by Lemma 4.2. If $e(r) = 1$, then D_e is a single 2-cell, denoted $c^2(r,h)$ attached by $\alpha(r,h)$.

In particular, $\overline{K}(r,h)$ has a single 2-cell outside $T \cup (\bigcup_{h[r]} V(k))$ with attaching map $\beta(r,h)$. Moreover, the sub-complex $\overline{K}(r,h)$ of $\overline{K}$ contains the lifts at h of all k-cells in $D_{e(r)} \subseteq K$ for $k \geq 2$.

Lemma 4.3 *For every $k \in hP_r^\bullet$, the inclusion induced homomorphisms*

$$\pi_1(T \cup (\textstyle\bigcup_{k \neq l \in hP_r^\bullet} V(l))) \longrightarrow \pi_1(\overline{K}(r,h)) \text{ and}$$
$$\pi_1(V(k)) \longrightarrow \pi_1(\overline{K}(r,h))$$

are injective.

Proof: By Lemma 4.2, the attaching map $\beta(r,h)$ for the 2-cell $c^2(r,h)$ strictly involves the rose $V(k)$. By the condition $(S1)$ of the definition of a semi-staggered presentation, the attaching map $\beta(r,h)$ also strictly involves the rose $V(l)$ for some $k \neq l \in hP_r^\bullet$. The Seifert Van Kampen theorem implies that

$$(\pi_1(\overline{K}(r,h)) \cong \pi_1(V(k)) * \pi_1(T \cup (\textstyle\bigcup_{k \neq l \in hP_r^\bullet} V(l))))/ \ll r \gg$$

The group $\pi_1(V(k))$ is locally indicable since it is isomorphic to the group A. The group $\pi_1(V(k)) * \pi_1(T \cup (\bigcup_{k \neq l \in hP_r^\bullet} V(l))$ is locally indicable since it is a free product of a free group and locally indicable groups. Since $\beta(r,h)$ strictly involves $V(k)$ and $V(l)$, the relator r has free product length at least 2. Therefore, Theorem 2.2 implies that the inclusion of each factor into $\pi_1(\overline{K}(r,h))$ is injective. □

For $h \in H$, let $\overline{K}(h) = \bigcup_{r \in R} \overline{K}(r,h)$. Note that if $g \in H = Aut(p)$, then $g\overline{K}(r,h) = \overline{K}(r,gh)$ and so $g\overline{K}(h) = \overline{K}(gh)$ and $\overline{K} = \bigcup_{h \in H} \overline{K}(h)$.

4.3 The Proofs of Theorems 3.2 - 3.4

The method of proof used is to construct the covering space $\overline{K}$ as a union of smaller pieces. The following lemmas will show that the conclusions hold for each of these pieces. Then compact supports, covering space properties, and Lemma 4.1 will provide the final step to prove Theorems 3.2 - 3.4.

Lemma 4.4 *Let $h \in H$. Then*

(1) $\overline{K}(h)$ is aspherical;

(2) if $r \in R$, then the inclusion of $\overline{K}(r,h)$ into $\overline{K}(h)$ induces a monomorphism of fundamental groups; and

(3) if $e(r) = 1$ for every $r \in R$, then $\pi_1(\overline{K}(h))$ is locally indicable.

Proof: To show that $\overline{K}(h)$ is aspherical, you consider maps of the n-sphere S^n into $\overline{K}(h)$. The image of each of these maps is a compact set in $\overline{K}(h)$. Compact supports says that each compact set in $\overline{K}(h)$ is contained in a finite sub-complex of $\overline{K}(h)$. Moreover, every finite sub-complex is contained in $X = T \cup (\bigcup_{k \in h\Pi} V(k)) \cup (\bigcup_{i=1}^{n} c^2(r(i),h))$ where each $r(i) \in R$. If X is aspherical for every finite subset $\{r(1), \dots, r(n)\}$ then we can conclude that $\overline{K}(h)$ is aspherical. Therefore it suffices to show $(1)'$ X is aspherical. A similar simplification can be made for part (2). Let $[f]$ be an element in the kernel of the inclusion induced homomorphism $i_{\#} : \pi_1(\overline{K}(r,h)) \longrightarrow \pi_1(\overline{K}(h))$. To show this map is injective, we need to show that $[f]$ is trivial. The element $[f]$ is represented by a map $f : S^1 \longrightarrow \overline{K}$. Since $[f] \in ker\, i_{\#}$, there exists an extension of f to a map $g : B^2 \longrightarrow \overline{K}(h)$. By compact supports, this map g is supported in a finite sub-complex of $\overline{K}(h)$, therefore by a complex X as before for some choice of finite subset of R. If the inclusion induced homomorphism on fundamental groups from $\overline{K}(r,h)$ into X is injective, it would imply that the element $[f]$ is trivial in $\pi_1(\overline{K}(r,h))$. Therefore, it suffices to show that $(2)'$ the inclusion induced homomorphism from $\pi_1(\overline{K}(r,h))$ into $\pi_1(X)$ is injective. For part (3), if we show that the collection of complexes X for each finite subset of relators satisfies the conditions of Lemma 4.1, it will suffice to show that $(3)'$ if $e(r) = 1$

for every $r \in R$, then $\pi_1(X)$ is locally indicable. To see that the conditions of Lemma 4.1 are upheld, note that every finite sub-complex is contained in a complex X for some finite subset of relators. Also, the union of two such complexes corresponds to the complex associated to the union of the two finite subsets of relators, which is a finite subset of R. Therefore the conditions for the Lemma are upheld. We will now prove $(1)'$, $(2)'$, and $(3)'$ by induction on the number n of relators.

For $n = 1$, the Lemma 4.2 provides that $\alpha(r(1), h)$ does not represent a proper power in $\pi_1(\overline{K}(r(1), h)^{(1)})$. It follows that $X = \overline{K}(r(1), h)$ is aspherical by Theorem 2.7. The result $(2)'$ is trivial in the case $n = 1$.

If $e(r) = 1$, then $\overline{K}(r, h) = T \cup (\bigcup_{k \in hP_r} V(k)) \cup c^2(r, h)$. By $(S1)$ and Lemma 4.2, there exists $k_0 \in hP_r$ such that $\alpha(r, h)$ strictly involves at least one 1-cell of $V(k_0)$. Consider the following decomposition of $\overline{K}(r, h)$:

$$\overline{K}(r, h) = (T \cup (\bigcup_{k_0 \neq k \in hP_r} V(k))) \cup V(k_0) \cup c^2(r, h)$$

By the Seifert Van Kampen Theorem,

$$\pi_1(\overline{K}(r, h)) = \pi_1(T \cup (\bigcup_{k_0 \neq k \in hP_r} V(k)) * \pi_1 V(k_0) / \ll r \gg$$

But $\pi_1(V(k_0))$ is locally indicable because it is isomorphic to A which is locally indicable by assumption. The group $\pi_1(T \cup (\bigcup_{k_0 \neq k \in hP_r} V(k))$ is locally indicable because it is a free product of a free group with locally indicable groups. By $(S1)$, the relator r is a word of at least length 2 in the free product and by assumption is not a proper power. Therefore, by Theorem 2.10, $\pi_1 \overline{K}(r, h)$ is locally indicable. This completes the $n = 1$ case.

Now suppose $n > 1$. Without loss of generality, we may assume that

$$r(1) < r(2) < \ldots < r(n)$$

in the ordering on R. Set $Y = \bigcup_{m=1}^{n-1} \overline{K}(r(m), h)$ so that $X = Y \cup \overline{K}(r(n), h)$. The complexes Y and $\overline{K}(r(n), h)$ are aspherical by our inductive hypothesis. Also, if $e(r_i) = 1$ for every $i \in \{1, \dots, n\}$, $\pi_1 Y$ and $\pi_1 \overline{K}(r, h)$ are locally indicable by our inductive hypothesis.

Claim: Let $W = h[r(n-1)] \cap h[r(n)]$. Then

$$Y \cap \overline{K}(r(n), h) = T \cup (\textstyle\bigcup_{k \in W} V(k)).$$

Reason: From the definitions, it is clear that

$$Y \cap \overline{K}(r(n), h) = T \cup (\textstyle\bigcup_{k \in S} V(k))$$

where $S = (\bigcup_{m=1}^{n-1} h[r(m)]) \cap h[r(n)]$. It would suffice to show for each $1 \leq m \leq n-1$, that $h[r(m)] \cap h[r(n)] \subseteq h[r(n-1)]$. If $1N \neq p \in h[r(m)] \cap h[r(n)]$ then by (S2)

$$h \text{ min } P^\bullet_{r(n-1)} < h \text{ min } P^\bullet_{r(n)} \leq p \leq h \text{ max } P^\bullet_{r(m)} < h \text{ max } P^\bullet_{r(n-1)}$$

and so $p \in h[r(n-1)]$ and the claim follows.

By the claim, the intersection $Y \cap \overline{K}(r(n), h)$ is contained in $\overline{K}(r(n-1), h) \subseteq Y$. Also, recall that

$$Y \cap \overline{K}(r(n), h) = T \cup (\textstyle\bigcup_{k \in W} V(k))$$

where $W = h[r(n-1)] \cap h[r(n)]$. Therefore the inclusion of the intersection

$$Y \cap \overline{K}(r(n), h)$$

into Y is the composition

$$Y \cap \overline{K}(r(n), h) \xrightarrow{i} T \cup (\textstyle\bigcup_{k \in h[r(n-1)]} V(k)) \xrightarrow{j} \overline{K}(r(n-1), h) \xrightarrow{k} Y$$

which gives the induced composition on fundamental groups

$$\pi_1(Y \cap \overline{K}(r(n),h)) \xrightarrow{i_\#} \pi_1(T \cup (\bigcup_{k\in h[r(n-1)]} V(k))) \xrightarrow{j_\#} \pi_1(\overline{K}(r(n-1),h)) \xrightarrow{k_\#} \pi_1(Y)$$

The homomorphism $i_\#$ is injective because it can be viewed as the inclusion of a factor of a free product into a free product. The induced homomorphism $j_\#$ is injective by Theorem 2.2, and the induced homomorphism $k_\#$ is injective by our inductive hypothesis. Therefore the inclusion of the intersection $Y \cap \overline{K}(r(n),h)$ into Y induces a monomorphism on fundamental groups. By a similar argument, one can show that the inclusion of the intersection $Y \cap \overline{K}(r(n),h)$ into $\overline{K}(r(n),h)$ also induces a monomorphism on fundamental groups. Since

$$Y \cap \overline{K}(r(n),h) = T \cup (\bigcup_{k\in W} V(k))$$

is aspherical, we see that X is aspherical by Whitehead Amalgamation. The Seifert Van Kampen Theorem tells us that $\pi_1(X)$ is the free product with amalgamation

$$\pi_1(Y) *_{\pi_1(Y \cap \overline{K}(r(n),h))} \pi_1(\overline{K}(r(n),h)).$$

Therefore the induced homomorphism from $\pi_1(\overline{K}(r(n),h))$ into $\pi_1(X)$ is injective by the theory of free products with amalgamation which proves $(2)'$.

By condition $(S2)$ of a semi-staggered presentation, the map $\beta(r,h)$ associated to the 2-cell corresponding to the relator $r(n)$ that is lifted at the 0-cell h of $\overline{K}$ strictly involves a 1-cell of the rose $V(h \max P^\bullet_{r(n)})$. Moreover, by the previous claim, the rose $V(h \max P^\bullet_{r(n)})$ is not contained in the complex Y. Let the set $M = h[r(n)] - h[r(n-1)]$ and the set $M' = M - h \max P^\bullet_{r(n)}$. For $(3)'$, consider the following decomposition of X:

$$X = (Y \cup (\bigcup_{k\in M'} V(k))) \cup V(hmax\, P^\bullet_{r(n)}) \cup hc^2_{r(n)}$$

Then, by the Seifert Van Kampen Theorem,

$$\pi_1 X = \pi_1(Y \cup (\bigcup_{k\in M'} V(k))) * (\pi_1(V(h \max P^\bullet_{r(n)})))/ \ll r(n) \gg.$$

By $(S1)$, r is a word of at least length 2 in the free product and not a proper power by assumption. The group $\pi_1(Y \cup (\bigcup_{k\in M'} V(k)))$ is locally indicable by our inductive hypothesis and the fact that a free product of locally indicable groups is locally indicable, and the group $\pi_1(V(h \max P^\bullet_{r(n)}))$ is locally indicable since it is isomorphic to the group A. By Theorem 2.10, $\pi_1 X$ is locally indicable.

□

Now let Φ denote the subgroup of H that is generated by Π. By $(S3)$, Φ is a free group with basis $\Pi^\bullet$.

Lemma 4.5 *Let $h_0, h_1, \ldots, h_n$ be distinct elements of H where n is a positive integer. Then*

(1) $\bigcup_{m=0}^{n} \overline{K}(h_m)$ is aspherical;

(2) for $i = 0, \ldots, n$, the inclusion of $\overline{K}(h_i)$ into $\bigcup_{m=0}^{n} \overline{K}(h_m)$ induces a monomorphism of fundamental groups; and

(3) if $e(r) = 1$ for every $r \in R$, then $\pi_1(\bigcup_{m=0}^{n} \overline{K}(h_m))$ is locally indicable.

Proof: Partition H into the cosets of Φ. Note that if the coset

$$h\Phi \neq h'\Phi \text{ for } h, h' \in H,$$

then $\overline{K}(h) \cap \overline{K}(h') = T$. The inclusion of T into $T \cup V(k)$ induces a monomorphism on fundamental groups by the theory of free products since

$$\pi_1(T \cup V(k)) \cong \pi_1(T) * \pi_1(V(k)).$$

Then by Lemma 4.3, the inclusion of $T \cup V(k)$ for some $k \in hP_r^\bullet$ into $K(r,h)$ for any relator r and any 0-cell h of $\overline{K}$ induces a monomorphism on fundamental groups. By Lemma 4.4, the inclusion of $K(r,h)$ into $K(h)$ induces a monomorphism on fundamental groups. Therefore, since a composition of injective maps is injective, the inclusion of T into $K(h)$ induces a monomorphism on fundamental groups. Once we know that this induced homomorphism is injective, we can show that $\overline{K}(h) \cup \overline{K}(h')$ is aspherical by Whitehead Amalgamation which proves (1) for this case. Furthermore, e know that the inclusions of $\pi_1(\overline{K}(h))$ and $\pi_1(\overline{K}(h'))$ into $\pi_1(\overline{K}(h) \cup \overline{K}(h'))$ by the theory of free products with amalgamation since $\pi_1(\overline{K}(h) \cup \overline{K}(h')) \cong \pi_1(\overline{K}(h) *_{\pi_1(T)} \overline{K}(h'))$ by the Seifert Van Kampen Theorem, therefore (2) is satisfied for this case.

The free product with amalgamation structure of $\pi_1(\overline{K}(h) \cup \overline{K}(h'))$ is unfortunately not enough to show that this group is locally indicable. To see this, we must consider a collection of sub-complexes of $\overline{K}(h) \cup \overline{K}(h')$ and then apply Lemma 4.1. First assume that all relators are not proper powers. Let

$$\Sigma = \{T \cup (\bigcup_{k \in h\Pi} V(k)) \cup (\bigcup_{k \in h'\Pi} V(k)) \cup (\bigcup_{i=1}^{n} c^2(r(i),h)) \cup (\bigcup_{j=1}^{m} c^2(s(j),h')) : \\ r(1),\dots,r(n) \textit{ and } s(1),\dots,s(m) \textit{ are finite subsets of } R\}$$

be a collection of sub-complexes of $\overline{K}(h) \cup \overline{K}(h')$. We will show that the fundamental group of each of these is locally indicable.

If a sub-complex $X \in \Sigma$ has no additional 2-cells $c^2(r,h)$ or $c^2(s,h')$, then the fundamental group of X is a free product of a free group with locally indicable groups hence locally indicable. So assume that there is at least one additional 2-cell. We will proceed by induction on the number t of 2-cells. For $t = 1$, assume that the one additional 2-cell is lifted at the vertex h and is associated to the relator r. By (S1), there exists at least 2 roses that the attaching map $\beta(r,h)$ strictly involves, say $V(k_1)$ and $V(k_2)$. By the Seifert Van Kampen

Theorem, the fundamental group of the sub-complex is the free product with amalgamation

$$\pi_1(T \cup (\bigcup_{k_1 \neq k \in h\Pi \cup h'\Pi} V(k)) * \pi_1(V(k_1))/ \ll r \gg.$$

This group is locally indicable by Theorem 2.10. Now let $t > 1$. Without loss of generality, just consider the additional 2-cells associated to the vertex h. None of these 2-cells will share a rose with a 2-cell lifted at h' since h and h' are from distinct cosets of Φ. By (S2), their is a linear ordering on the 2-cells lifted at h and the roses that they use. Without loss of generality, let $r(n)$ be the maximum relator of the set $\{r(1), \ldots, r(n)\}$ and let $k_1 = h \max P^{\bullet}_{r(n)}$. Then the attaching map $\beta(r(n), h)$ strictly involves the roses $V(h)$ and $V(k_1)$ and no other additional 2-cell uses the rose $V(k_1)$. Let $M = \{k \in h\Pi \cup h'\Pi : k \neq k_1\}$. Then the Seifert Van Kampen Theorem implies that the fundamental group of this sub-complex is the one-relator free product

$$\pi_1(T \cup (\bigcup_{k \in M} V(k)) \cup (\bigcup_{i=1}^{n-1} c^2(r(i), h)) \cup (\bigcup_{j=1}^{m} c^2(s(j), h'))) * \pi_1(V(k_1)/ \ll r(n) \gg$$

which is locally indicable by Theorem 2.10 since the first factor is locally indicable by our inductive hypothesis and the second factor is isomorphic to the locally indicable group A. Therefore every sub-complex in the collection Σ has locally indicable fundamental group. Moreover, every finite sub-complex of $\overline{K}(h) \cup \overline{K}(h')$ is contain in an element of Σ and the union of two element of Σ is contained in Σ, therefore by Lemma 4.1, $\pi_i(\overline{K}(h) \cup \overline{K}(h'))$ is locally indicable if no relators are proper powers.

The lemma is now proved for the case where the intersection $\overline{K}(h) \cap \overline{K}(h') = T$. Now recall that $\overline{K}(h)$ is homeomorphic to its translate $g\overline{K}(h) = \overline{K}(gh)$ so it suffices to prove the lemma in the case where each $h_0, \ldots, h_n$ are distinct elements of the trivial coset $1\Phi = \Phi$. Under this assumption, the result is proven by induction on n.

For the case $n = 0$, all three results are consequences of Lemma 4.4. Now assume that $n > 0$. Without loss of generality, we may assume that $|h_0| \geq |h_i|\ for\ i = 1, \ldots, n$ where $|h|$ indicates the length of the element h in the free group Φ. Set $X = \bigcup_{i=0}^{n} \overline{K}(h_i)$ and $Y = \bigcup_{i=1}^{n} \overline{K}(h_i)$ so that $X = Y \cup \overline{K}(h_0)$.

Let $U = (\bigcup_{i=1}^{n} h_i \Pi) \cap h_0 \Pi$. Lemma 1 in [2] implies that this intersection U is contained in a singleton. This implies that there exists an element $k_0 \in h_0 \Pi$ such that

$$T \subseteq Y \cap \overline{K}(h_0) \subseteq T \cup V(k_0)$$

If $Y \cap \overline{K}(h_0) = T$, then the result follows by the same arguments given above. Otherwise, select $r \in R$ such that $k_0 \in h_0 P_r$. By $(S1)$ and Lemma 2, the attaching map $\beta(r, h_0)$ for the 2-cell $c^2(r, h_0)$ of $\overline{K}(r, h_0)$ strictly involves some 1-cell of a rose other than $V(k_0)$. By Lemma 4.3, the inclusion of $Y \cap \overline{K}(h_0) = T \cup V(k_0)$ into $\overline{K}(h_0, r)$ induces a monomorphism of fundamental groups. By Lemma 4.4, the inclusion of $\overline{K}(r, h_0)$ into $\overline{K}$ induces a monomorphism on fundamental groups, therefore the inclusion of $Y \cap \overline{K}(h_0)$ into $\overline{K}(h_0)$ induces a monomorphisms of fundamental groups since it is the composition of these two monomorphisms. Similarly, there exists an $m \in \{1, \ldots, n\}$ such that $k_0 \in h_m \Pi$ and the 2-cell $c^2(r, h_m)$ of $\overline{K}(r, h_m)$ strictly involves some 1-cell of the rose $V(k_0)$ and any other rose, so the inclusion of $Y \cap \overline{K}(h_0)$ into $\overline{K}(r, h_m)$ and then into $\overline{K}(h_m)$ induces a monomorphism of fundamental groups. By part 2 of the inductive hypothesis, the inclusion of $Y\overline{K}(h_m)$ into Y induces a monomorphism of fundamental groups, therefore the inclusion induced homomorphism from $\pi_1(Y \cap \overline{K}(h_0))$ into $\pi_1(Y)$ is a monomorphism.

The complexes Y and $\overline{K}(h_0)$ are aspherical by part 1 of the inductive hypothesis. By applying Whitehead Amalgamation, $X = Y \cup \overline{K}(h_0)$ is aspherical, therefore proving part 1 of the lemma. To show part 2, note that the Seifert Van Kampen Theorem implies that the group $\pi_1 X$ is a free product of $\pi_1 Y$ and $\pi_1 \overline{K}(h_0)$ with free subgroup amalgamated. In particular, $\pi_1 \overline{K}(h_0)$ embeds into $\pi_1 X$. Further, if $m = \{1, \ldots, n\}$, then by the inductive hypothesis, $\pi_1 \overline{K}(h_m)$ embeds into $\pi_1 Y$, therefore into $\pi_1 X$ by the theory of free products with amalgamation.

To show part (3), by Lemma 4.1, it suffices to show that $\pi_1 X$ is locally indicable where $X = Y \cup (\bigcup_{k \in h_0 \Pi} V(k)) \cup c^2(r(1), h_0) \cup \ldots \cup c^2(r(m), h_0)$ with $\{r(1), \ldots, r(m)\}$ being any finite subset of the set R of relators. We will show this by induction on m. For $m = 1$, let k_* be a vertex of $\overline{K}$ such that the attaching map $\beta(r, h_0)$ strictly involves the rose $V(k_*)$. Consider the decomposition $X = (Y \cup (\bigcup_{k_* \neq k \in h_0 \Pi} V(k))) \cup V(k_*) \cup h_0 c_r^2$. By the Seifert Van Kampen Theorem,

$$\pi_1 X = \pi_1 (Y \cup (\bigcup_{k_* \neq k \in h_0 \Pi} V(k)) * \pi_1 (V(k_*)))/ \ll r \gg.$$

By $(S1)$, r is a word of at least length 2 in the free product and not a proper power by assumption, so by Theorem 2.10, $\pi_1 X$ is locally indicable.

Now consider the general case. Let $m > 1$. Without loss of generality, we can assume that

$$r(1) < r(2) < \ldots < r(m)$$

under the linear ordering given by $(S2)$. By $(S1)$ and $(S2)$ there exists a $k_0 \in h_0 \Pi$ such that the attaching map for $c^2(r(m), h_0)$ strictly involves a 1-cell of $V(k_0)$ and no other 2-cell outside the rose involves it. Then consider the following decomposition of X:

$$X = (Y \cup (\bigcup_{k \neq k_0 \in h_0 \Pi} V(k)) \cup (\bigcup_{i=1}^{m-1} h_0 c_r^2(i))) \cup V(k_0) \cup h_0 c_{r(m)}^2$$

Then by Seifert Van Kampen,

$$\pi_1 X = (\pi_1(Y \cup (\bigcup_{k \neq k_0 \in h_0 \Pi} V(k)) \cup (\bigcup_{i=1}^{m-1} h_0 c_r^2(i))) * \pi_1 V(k_0)) / \ll r \gg.$$

The first factor of the free product is locally indicable by our inductive hypothesis and the second one is locally indicable since it is isomorphic to the group A. Applying Theorem 2.10, we conclude that $\pi_1 X$ is locally indicable. □

Theorem 3.2 *If the relative presentation* $\mathcal{P} =< A, X : R >$ *is semi-staggered and* A *is a locally indicable group, then the pre-aspherical model of* $\mathcal{P}$ *is aspherical.*

Proof: It suffices to show that the covering space $\overline{K}$ of K is aspherical. This follows from Lemma 4.5 by compact supports.

Theorem 3.3 *If the relative presentation* $\mathcal{P} =< A, X : R >$ *is semi-staggered and* A *is a locally indicable group, then the inclusion of* A *into* $G(\mathcal{P})$ *is an injection.*

Proof: It is enough to prove that the inclusion of $K(A, 1)$ into K induces a monomorphism of fundamental groups. Lifting through the covering p at the 0-cell $1N$, it is sufficient to prove that the inclusion of $V(1N)$ into $\overline{K}$ induces a monomorphism of fundamental groups. Let r be any element of R. By $(S1)$ and Lemma 4.2, the attaching map for the 2-cell of $\overline{K}(r, 1N)$ strictly involves a 1-cell of a rose other than $V(1N)$. By Theorem 4.3 in [9], the inclusion of $V(1N)$ into $\overline{K}(r, 1N)$ induces a monomorphism of fundamental groups. Using Lemma 4.4 and Lemma 4.5 together with compact supports, it follows that the inclusion of $V(1N)$ into $\overline{K}$ induces a monomorphism of fundamental groups.

Theorem 3.4 *If the relative presentation $\mathcal{P} =< A, X : R >$ is semi-staggered, A is a locally indicable group, and no relator is a proper power, then the subgroup $N/U \leq G(\mathcal{P})$ is locally indicable.*

Proof: Let the collection Ω of sub-complexes of $\overline{K}$ be defined to be

$$\Omega = \{\bigcup_{h \in M} \overline{K}(h) : M \text{ is a finite subset of } H\}.$$

Every finite sub-complex of $\overline{K}$ is contained in an element of Ω for some finite subset $\{h_1, \ldots, h_n\} \subseteq H$. Also, the union of any two elements of Ω is also a union of complexes $\overline{K}(h_i)$ for a finite number of elements h_i, therefore an element of the collection Ω. The result follows by applying Lemma 4.1 to the collection Ω.

5 CONCLUSION

5.1 Overview of Proofs

The downside to the methods begun by Anshel are that the hypotheses are complicated. By switching to the topological setting, Bogley was able to interpret how Anshel's conditions affected the covering space associated to the normal subgroup N/U of the fundamental group of the pre-aspherical model. Recall that for a relative presentation $\mathcal{P} = (A, X : R)$,

$$G = G(\mathcal{P}) = (A * F(X))/U$$

where $U = \ll R \gg$ and the group $H = (A * F(X))/N$, where $n = \ll A \cup R \gg$. When one examines the conditions of a semi-staggered presentation, we find that they impose conditions on how the 2-cells are attached to the covering space $\overline{K}$. The conditions fall into two classes. The first two conditions $(S1)$ and $(S2)$ are "local" conditions and regulate the how of the lifts of the 2-cells of K overlap when lifted at a fixed vertex in the covering space $\overline{K}$.

A graph of groups is a graph together with a set of groups under certain conditions. Each vertex and edge of the graph is assigned a group with the restriction that each edge group must embed in the vertex groups of its boundary. The fundamental group of the graph of groups is the colimit of the diagram, i.e. if there exists a group H and homomorphisms from each vertex and edge group to the group H, the colimit G would be the group such that these homomorphisms determined a unique homomorphism from G to H such that the diagram commuted.

The conditions (S1) and (S2) imply that we can view

$$\pi_1(\overline{K}(r(1), h) \cup \ldots \cup \overline{K}(r(n), h))$$

as the fundamental group of a tree of groups (a graph of groups where the underlying graph is a tree). Without loss of generality, assume that $r(1) < r(2) < \ldots < r(n)$ under the linear ordering given by (S2). There is a vertex for each relator $r(i)$ and the vertex group associated to the vertex $r(i)$ is $\pi_1(\overline{K}(r(i), h))$. There is an edge e_i connecting vertex $r(i)$ to vertex $r(i+1)$ for each i in $\{1, \ldots, n-1\}$. The edge group for the edge e_i is $\pi_1(\overline{K}(r(i), h) \cap \overline{K}(r(i+1), h))$. The graph of groups structure is a geometric way to view that each presentation $(A, X : r(1), \ldots, r(n))$ is staggered in the sense introduced by Lyndon in [14], so that the presentation $\mathcal{P} = (A, X : R)$ could be thought of as "locally staggered".

The last condition $(S3)$ is a "global" condition, and it determines how the lifts of the 2-cells of K overlap when lifted at distinct vertices. The condition $(S3)$ implies that the one-skeleton of the covering space $\overline{K}$, which is the Cayley graph of H, contains a tree. This fact was used when we invoked Lemma 1 from [2] in the proof of Lemma 4.5. Lemma 1 in [2] used the freeness of the subgroup Φ to predict how the lifts of 2-cells at different vertices would overlap.

In efforts to weaken the conditions of Theorems 3.2 - 3.4, one could consider implying the condition that the group H be word hyperbolic. The Cayley graph of a word hyperbolic group is "tree-like" in its spreading, but they could contain closed loops locally. To solve the local problem that these closed loops introduce, the connection of the staggered conditions to a tree of groups has lead us to try to apply the theory of polygons of groups, which is a generalization of graphs of groups. When we weaken condition (S3) to the condition that the group H is word hyperbolic and try to represent the group $\pi_1(\overline{K}(r(1), h) \cup \ldots \cup \overline{K}(r(n), h))$ as a graph of groups, the chain could now lie on a loop which is a segmented circle instead of a tree.

Polygons of groups can provide information on these closed loop graphs since each one can be pictured as a polygon.

5.2 Polygons of Groups

Gersten and Stallings [21] developed the theory of polygons of groups by generalizing graphs of groups. In that paper, they introduced non-spherical triangles of groups, however all of their results hold for any non-spherical polygons of groups.

A polygon of groups is a polygon with an assignment of groups to the vertices, edges and face of the polygon under certain restrictions. Each edge group must be a subgroup of the vertex groups of its boundary. The face group must be a subgroup of each edge group, and hence each vertex group. An example of a square of groups is shown below.

$$
\begin{array}{ccccc}
V_1 & \xleftarrow{\sigma_1} & E_{1,2} & \xrightarrow{\tau_2} & V_2 \\
\uparrow{\scriptstyle \tau_1} & & \uparrow & & \uparrow{\scriptstyle \sigma_2} \\
E_{4,1} & \longleftarrow & F & \longrightarrow & E_{2,3} \\
\downarrow{\scriptstyle \sigma_4} & & \downarrow & & \downarrow{\scriptstyle \tau_3} \\
V_4 & \xleftarrow{\tau_4} & E_{3,4} & \xrightarrow{\sigma_3} & V_3
\end{array}
$$

The V_i are the vertex groups. The group $E_{i,j}$ is the edge group and is a subgroup of V_i and V_j. The group F is the face group and is a subgroup of each of the edge groups and hence a subgroup of each of the vertex groups. Let σ_i denote the inclusion of $E_{i,j}$ into V_i and τ_j denote the inclusion of $E_{i,j}$ into V_j.

We will label the parts of the polygon by the name of the group associated to each part.

To define the measure of an angle of the polygon, consider the inclusion induced map

$$p_j : E_{i,j} *_F E_{j,k} \longrightarrow V_j$$

where $E_{i,j} *_F E_{j,k}$ is the free product of $E_{i,j}$ and $E_{j,k}$ amalgamated along the subgroup F. An element in the free product with amalgamation $E_{i,j} *_F E_{j,k}$ can always be represented by a reduced word $w = a_1b_1a_2b_2 \ldots a_nb_n$ where the $a_i \in E_{i,j}$ and the $b_i \in E_{j,k}$. The length of the word w is defined to be $2n$ and is denoted $|w|$. Note that the length will always be even. If the homomorphism p_j is injective, define the angle at V_j to be 0. Otherwise, let w_i be a non-trivial word of shortest length in $ker\, p_j$. The angle at V_j is then defined to be $\frac{2\pi}{|w_i|}$. We say a polygon of groups is *non-spherical* if the sum of the angles of the polygon is less than or equal to $(s-2)\pi$ where s is the number of sides of the polygon.

Each polygon of groups represents a new group which is the colimit of the diagram. There is a topological way to view the colimit as well the algebraic one mentioned in the preamble of this chapter. Construct a topological space K in the following manner. For each vertex group V_i, choose a $K(V_i, 1)$ that contains a $K(E_{i,j}, 1)$ and a $K(E_{k,i}, 1)$ as a sub-complex. This choice is possible since if H is a subgroup of a group G, there exists a presentation $\mathcal{P}$ for G that contains a sub-presentation $\mathcal{R} \subseteq \mathcal{P}$ such that $H = G(\mathcal{R})$ [16, Theorem II(2.6)]. Use this presentation for G to build the standard 2-complex associated to the presentation $\mathcal{P}$. We can extend this 2-complex to a $K(G, 1)$-complex and it will contain a sub-complex that is a $K(H, 1)$-complex for the subgroup H. For each edge group $E_{i,j}$, take a $K(E_{i,j}, 1) \times I$ where I is the closed unit interval. Attach the "edge spaces" to the "vertex spaces" by identifying $K(E_{i,j}, 1) \times \{0\}$ with its image under the given inclusions of the edge groups into the adjacent vertex groups in $K(V_i)$ and $K(E_{i,j}, 1) \times \{1\}$ with its image in $K(V_j, 1)$. Then take a $K(F, 1) \times P$ for the face group F where P is a 2-cell in the shape of the polygon. By a similar argument as above, we can find a copy of the $K(F, 1)$-complex as a sub-complex of each edge complex. Attach $K(F, 1) \times P$ to the existing space by identifying $K(F, 1) \times \partial P$ with its image in the existing space as defined by the give inclusions of F into all edge and vertex groups. This construction forms the complex K. The fundamental group of K is isomorphic to the colimit of the polygon of groups.

The theory of polygons of groups is an extension of Bass-Serre theory, for graphs of groups. Graphs of groups assign groups to the vertices and edges of the graph with the condition that each edge group embed in the vertex groups of its boundary. Each sub-complex of the one-skeleton of the polygon of groups is a graph of groups. For example, each edge

$$V_i \longleftarrow E_{i,j} \longrightarrow V_j$$

represents a graph of groups. The colimit of this graph of groups is the free product of V_i and V_j amalgamated along the subgroup $E_{i,j}$. This graph of groups is a standard example from Bass-Serre theory and is discussed in [17]. An important result for graphs of groups is that there exists a tree which the colimit of the graph of groups acts on such that the orbit graph is isomorphic to the original graph of groups [17]. In polygons of groups, Gersten and Stallings have generalized this result. For each polygon of groups Γ, there exists a contractible 2-complex L called the *building* for Γ such that the colimit G of Γ acts on L and the orbit graph G/L is isomorphic to Γ [21].

The term building comes from the theory of Coxeter groups. Tits showed that the two descriptions of buildings are the same under certain conditions.

Theorem 5.1 *[6, Theorem 4.2] For non-spherical polygons of groups, the building described by Gersten and Stallings is a building in the Coxeter sense of the definition if and only if the vertex links of the polygon all have diameter π.*

5.3 Results and Conjectures

In 1991, Gersten and Stallings proved the following theorem.

Theorem 5.2 *[21] For any non-spherical polygon of groups, the natural maps of the vertex groups into the colimit of the polygon G are injective.*

In [21], the proof is given for the case of a non-spherical triangle of groups, but the arguments hold for any non-spherical polygon of groups.

5.3.1 A Natural Question

In attempting to apply this theory to semi-staggered presentations, we arrived at the question, under what conditions would the colimit of an edge of the polygon (when viewed as a graph of groups) embeds in the colimit of the polygon. Precisely stated, the question is under what circumstances is the natural map $V_i *_{E_{i,j}} V_j \longrightarrow G$ from the free product with amalgamation of two adjacent vertex groups to the colimit of the polygon G be injective? An equivalent question is under what conditions will the tree that the colimit of an edge of the polygon of groups acts on embed in the building L of the polygon of groups. In the investigation of this question, we have limited ourselves to consider only non-spherical polygons of groups. This is partly because with spherical examples, the vertex groups need not necessarily embed in the colimit, which can be the trivial in any event. Also, in our attempts to apply this theory to semi-staggered presentations, we would also need to know that the vertex groups embed in the colimit. The case for triangles of groups seems to be a special case and a first step has been made.

Theorem 5.3 *Let Δ be a triangle of groups. If the inclusion of the colimit of an edge of Δ into the colimit of Δ is injective, then either*

i) the measure of the angle opposite that edge is equal to 0, or

ii) the measure of an adjacent angle is π.

Proof: Let G be the colimit of the triangle of groups Δ. The colimit of the edge of the triangle when viewed as a graph of groups is the free product with amalgamation $V_i *_{E_{i,j}} V_j$. Without loss of generality, assume that the natural map $f : V_1 *_{E_{1,2}} V_2 \longrightarrow G$ is injective and consider the following commutative diagram.

$$\begin{array}{ccc} E_{2,3} *_F E_{3,1} & \xrightarrow{\sigma_2 * \tau_1} & V_2 *_{E_{1,2}} V_1 \\ \downarrow{\scriptstyle p_3} & & \downarrow{\scriptstyle f} \\ V_3 & \xrightarrow[j]{} & G \end{array}$$

The map f is injective by assumption. If $\sigma_2 * \tau_1$ were injective, p_3 would be injective by commutativity of the diagram, and hence the angle opposite edge $E_{1,2}$ would have measure 0.

If $\tau_1 * \sigma_2$ is not injective, let $x = a_1c_1a_2c_2 \dots a_nc_n$ be a word representing a non-trivial element of the $ker\,(\tau_1 * \sigma_2)$ with n minimal. Since n is minimal, we can conclude that each a_i is an element of $E_{3,1} - F$, each c_i is an element of

$E_{2,3} - F$ (see Theorem 4.4 in [16]). Since the element $x \in ker\, \tau_1 * \sigma_2$, the element x is trivial in $V_1 *_{E_{1,2}} V_2$. This fact implies that some a_i or c_j is an element of $E_{1,2}$. Without loss of generality, assume $c_j \in E_{1,2}$. The element $c_j \notin F$, therefore $c_j \in E_{1,2} - F$ which implies that there exists a non-trivial element $g \in E_{1,2} - F$ such that $c_j = g$. Recall that map $p_2 : E_{2,3} *_F E_{1,2} \longrightarrow G$ is used to compute the angle at vertex V_2. The non-trivial element $c_j g^{-1} \in ker\, p_2$ and is a word of free product length 2. This fact implies that the angle at vertex V_2 is equal to π. □

Corollary 5.4 *If Δ is a non-spherical triangle of groups and the inclusion of the colimit of an edge of Δ into the colimit of the triangle Δ is injective, then the angle opposite the edge has measure* 0.

Proof: By Theorem 5.3, either the angle opposite the edge has measure 0, or an adjacent angle has measure π. Assume that an adjacent angle has measure π. Since the triangle is non-spherical, the sum of the measure of all angles must be less than or equal to π, therefore, the other two angles of the triangle must have measure 0, therefore the angle opposite the edge will always have measure 0. □

Theorem 5.5 *Given a triangle of groups, if the angle opposite an edge of the triangle has measure* 0, *then the inclusion of the colimit of that edge of the triangle into the colimit of the triangle is injective.*

Proof: Let G be the colimit of the triangle of groups Δ. The colimit of the edge of the triangle when viewed as a graph of groups is the free product with amalgamation $V_i *_{E_{i,j}} V_j$. Without loss of generality assume that the angle opposite the edge $E_{1,2}$ has measure 0 and consider the following commutative diagram.

$$\begin{array}{ccc} E_{3,1} *_F E_{2,3} & \xrightarrow{\tau_1 * \sigma_2} & V_1 *_{E_{1,2}} V_2 \\ {\scriptstyle p_3}\downarrow & & \downarrow{\scriptstyle f} \\ V_3 & \xrightarrow[j]{} & G \end{array}$$

The fact that the angle opposite edge $E_{1,2}$ has measure 0 implies that the homomorphism p_3 is injective. Let the word $w = a_1b_1a_2b_2 \ldots a_nb_n$ be a reduced word representing an element of the $ker\, f$ such that each $a_i \in V_1$ and each $b_i \in V_2$. It would suffice to show that the word w is trivial in $V_1 *_{E_{1,2}} V_2$.

Since $w \in ker\, f$, it represents a trivial element of the colimit of the triangle G. Recall that in Section 5.1, we constructed a topological space K such that $\pi_1(K) \cong G$. This implies that each element of the group G is represented by a map

$$\alpha : S^1 \longrightarrow K$$

of the circle into the topological space K. Since the element w is trivial in G, there exists and extension

$$\overline{\alpha} : B^2 \longrightarrow K$$

of the map α to the disc.

Following the method taken by Gersten and Stallings in the proof of Theorem 5.2 we consider the natural map $r : K \longrightarrow \Delta$ when we view the triangle Δ as a 2-cell given by sending the $K(V_i, 1)$-complex to the vertex V_i, the space $K(E_{i,j}, 1) \times I$ to the edge $E_{i,j}$, and the space $K(F, 1) \times P$ where P is a 2-cell in the shape of a triangle to the face F of Δ. Let p be a point in the center of the face F and draw a perpendicular line from the point p to each edge of the triangle. Call the resulting graph the triod T of Δ.

Consider composition $r \circ \overline{\alpha}$ as a map from B^2 to the triangle Δ.

$$B^2 \xrightarrow{\overline{\alpha}} K \xrightarrow{r} \Delta$$

The inverse image of the triod T, $(r \circ \overline{\alpha})^{-1}(T)$, is a graph on the 2-ball. Each vertex of this graph is a pre-image of the point p, so has valence 3. Since the triod T separates the three vertices of the triangle Δ, we can label each region of the graph by a vertex of Δ. Note that since $r \circ \overline{\alpha}$ takes the boundary of the 2-ball into the edge $E_{1,2}$, each region touching the boundary of the 2-ball must be labelled by either vertex V_1 or vertex V_2. If we show that this graph is equivalent to a graph with no regions labelled by vertex V_3, then the graph would represent an extension of α into the sub-complex $K(V_1, 1) \cup (K(E_{1,2}, 1) \times I) \cup K(V_2, 1)$ of K which would show that the element w is trivial in $V_1 *_{E_{1,2}} V_2$.

Assume there is a region labelled by V_3. Define a loop β in the 2-ball to be the boundary of this region. Since all the regions adjacent to this region must be labelled by the vertices V_1 and V_2, the loop β represents a path that travels back and forth from the complex $K(E_{2,3}, 1) \times \{1/2\}$ to the complex $K(E_{3,1}, 1) \times \{1/2\}$. By sliding the loop β out away from the V_3 region, we get a new loop that is homotopic to β and whose image under the map $\overline{\alpha}$ represents an element c of the group $E_{2,3} *_F E_{3,1}$. The image of the disk enclosed by the new loop under $\overline{\alpha}$ is an extension of the map of the boundary, which implies that the element c is trivial in the vertex group V_3. By assumption, the map $p_3 : E_{2,3} *_F E_{3,1} \longrightarrow V_3$ is injective, so the element c is trivial in $E_{2,3} *_F E_{3,1}$. This implies that the disk bounded by the loop β can be replaced by a disk that does not contain a region labelled by V_3. Therefore we have reduced the number of regions labelled by V_3 by one, and since there will only be a finite number of them, continuing this process will produce a graph with no regions labelled V_3. $\square$

5.3.2 A Few Examples

Theorems 5.3 and 5.5 imply that the inclusion of the colimit of an edge of the triangle into the colimit of the triangle does not happen very frequently. This observation may be somewhat disappointing, but it turns out that triangles seem to be a special case, and there are many more positive outcomes for this question in polygons with $n \geq 4$ sides. Here are some examples.

Example 1: Start with a square of groups as pictured in the beginning of section 5.1 of this dissertation. Define the vertex groups to be dihedral groups with presentations

$$V_1 \text{ is presented by } (x_1, x_2 : x_1^2, x_2^2, (x_1x_2)^{m_1}),$$
$$V_2 \text{ is presented by } (x_2, x_3 : x_2^2, x_3^2, (x_2x_3)^{m_2}),$$
$$V_3 \text{ is presented by } (x_3, x_4 : x_3^2, x_4^2, (x_3x_4)^{m_3}), \text{ and}$$
$$V_4 \text{ is presented by } (x_4, x_1 : x_4^2, x_1^2, (x_4x_1)^{m_4}).$$

where $m_i \geq 2$ for $i = \{1, 2, 3, 4\}$. Then the edge groups are presented by

$$E_{1,2} \text{ is presented by } (x_2 : x_2^2),$$
$$E_{2,3} \text{ is presented by } (x_3 : x_3^2),$$
$$E_{3,4} \text{ is presented by } (x_4 : x_4^2),$$
$$E_{4,1} \text{ is presented by } (x_1 : x_1^2).$$

Let the face group be trivial. The colimit of an edge will be the group presented by the presentation

$$\mathcal{E} = (x_i, x_j, x_k : x_i^2, x_j^2, x_k^2, (x_ix_j)^{m_i}, (x_jx_k)^{m_j})$$

where (i, j, k) is one of the triples in the set $\{(1, 2, 3), (2, 3, 4), (3, 4, 1), (4, 1, 2)\}$. The colimit of the square will be the group presented by the presentation

$$\mathcal{E} = (x_1, x_2, x_3, x_4 : x_1^2, x_2^2, x_3^2, x_4^2, (x_1x_2)^{m_1}, (x_2x_3)^{m_2}, (x_3x_4)^{m_3}, (x_4x_1)^{m_4}).$$

Claim 5.6 *For each edge of the square in Example 1, the inclusion of the colimit of the edge into the colimit of the square will be injective.*

Proof: To see that this inclusion is injective, we will need employ the theory of Coxeter groups. A *Coxeter system* is a triple (W, S, m), where W is a group, S is a subset of W, and m is a function

$$m : S \times S \longrightarrow \mathbb{N} \cup \infty$$

such that

1. for all $s, t \in S$, $m(s,t) = 1$ if and only if $s = t$;
2. $m(s,t) = m(t,s)$ for all $s, t \in S$;
3. W has presentation of the form $(S : (st)^{m(s,t)}, s, t \in S)$.

The group W is called a *Coxeter group*. Let (W, S, m) be a Coxeter system and let $T \subseteq S$. Let W_T denote the Coxeter group associated to the system $(W_T, T, m|_{T\times T})$ and $\overline{W}_T$ be the subgroup of W generated by the subset T.

Theorem 5.7 *[3, Chapter IV, Section 1.8, Theorem 2(i)] The natural map $W_T \longrightarrow \overline{W}_T$ is an isomorphism.*

Note that the colimit of the square W is a Coxeter system with generating set $S = \{x_1, x_2, x_3, x_4\}$. Let $A = \{x_i, x_j, x_k\}$. Theorem 5.7 implies that W_A, which is equal to the colimit of the edge, is a subgroup of W, the colimit of the square. Therefore the inclusion of the colimit of an edge of this square into the colimit of the square is injective. □

In fact, the form of Example 1 can be generalize to any polygon with greater than 4 sides and Theorem 5.7 will imply the the colimit of the edge is a subgroup of the colimit of the polygon.

Example 2: Start with a square of groups as pictured in the beginning of section 5.1 of this dissertation. Define the vertex groups as follows:

$$V_1 \text{ is presented by } (a_1, a_2 : a_1^{-1}a_2a_1 = a_2^2)$$
$$V_2 \text{ is presented by } (a_2, a_3 : a_2^{-1}a_3a_2 = a_3^2)$$
$$V_3 \text{ is presented by } (a_3, a_4 : a_3^{-1}a_4a_3 = a_4^2)$$
$$V_4 \text{ is presented by } (a_4, a_1 : a_4^{-1}a_1a_4 = a_1^2).$$

Now define the edge groups so that $E_{i,j}$ is the group presented by $(a_j : -)$. Once again, we will let the face group be the trivial group.

The colimit of the edge $E_{i,j}$ will be the group presented by

$$\mathcal{Q}_{(i,j,k)} = (a_i, a_j, a_k : a_i^{-1}a_ja_i = a_j^2, a_j^{-1}a_ka_j = a_k^2)$$

where the triple (i, j, k) is an element of the set

$$\{(1,2,3), (2,3,4), (3,4,1), (4,1,2)\}.$$

The colimit of the square of groups is the group presented by

$$\mathcal{P} = (a_1, a_2, a_3, a_4 : a_1^{-1}a_2a_1 = a_2^2, a_2^{-1}a_3a_2 = a_3^2, a_3^{-1}a_4a_3 = a_4^2, a_4^{-1}a_1a_4 = a_1^2).$$

In [8], Higman showed that the inclusion of the group $G(\mathcal{Q}_{(i,j,k)})$ into the group $G(\mathcal{P})$ was injective by showing that

$$G(\mathcal{P}) = G(\mathcal{Q}_{(i,j,k)}) *_{<a_i,a_k>} G(\mathcal{Q}_{(k,l,i)}).$$

Therefore we can conclude that the colimit of any edge of this square of groups embeds in the colimit of the square of group.

These two examples show us that there are some interesting cases where the inclusion of the colimit of an edge of a polygon into the colimit of the polygon is injective.

BIBLIOGRAPHY

[1] I. L. Anshel. A Freiheitssatz for a class of two-relator groups. *J. Pure Appl. Algebra*, 72(3):207–250, 1991.

[2] W. A. Bogley. An identity theorem for multi-relator groups. *Math. Proc. Camb. Phil. Soc.*, 109:313–321, 1991.

[3] N. Bourbaki. *Groups et Algebrès de LIe, Chapitres 4-6.* Hermann, Paris, 1981.

[4] S. D. Brodskii. Equations over groups and groups with a single defining relator. *Uspehi Mat. Nauk. 35*, 4:183, 1980.

[5] K. S. Brown. *Cohomology of Groups.* Springer-Verlag, 1982.

[6] P. R. Brown. Buildings and non-positively curved polygons of finite groups. *New York J. Math.*, 4:237–248, 1998.

[7] E. Dyer and A. T. Vasquez. Some small aspherical spaces. *J. Austral. Math. Soc.*, 16:332–352, 1973.

[8] G. Higman. A finitely generated infinite simple group. *J. London Math. Soc.*, 26:61–64, 1951.

[9] J. Howie. On pairs of 2-complexes and systems of equations over groups. *J. Reine Angew. Math.*, 324:165–174, 1981.

[10] J. Howie. On locally indicable groups. *Math. Z.*, 180:445–461, 1982.

[11] J. Howie. Cohomology of one-relator products of locally indicable groups. *J. London Math. Soc.*, 30:419–430, 1984.

[12] J. Huebschmann. Cohomology theory of aspherical groups and of small cancellation groups. *Journal of Pure and Applied Algebra*, 14:137–143, 1979.

[13] D. L. Johnson. *Presentations of Groups.* London Mathematical Society, 1997.

[14] R. C. Lyndon. Cohomology theory of groups with a single defining relation. *Ann. of Math.*, 52:650–665, 1950.

[15] W. Magnus. Über diskontinuierliche gruppen mit einer definierenden relation. (der freiheitssatz). *J. Reine Angew. Math.*, 163:141–165, 1930.

[16] W. Magnus, A. Karrass, and D. Solitar. *Combinatorial Group Theory: Presentations of Groups in Terms of Generators and Relations.* Dover Publications, Inc., 1966.

[17] J. P. Serre. *Trees.* Springer-Verlag, 1980.

[18] A. J. Sieradski. *An Introduction to Topology and Homotopy.* PWS-KENT Publishing Company, 1992.

[19] A. J. Sieradski. Algebraic topology for two dimensional complexes. In C. Hog-Angeloni, W. Metzler, and A.J. Sieradski, editors, *Two-dimensional Homotopy and Combinatorial Group Theory*, chapter 2. Cambridge University Press, 1993.

[20] E. Spanier. *Algebraic Topology.* McGraw-Hill Book Company, 1966.

[21] J. R. Stallings. *Group Theory from a Geometrical Viewpoint (E. Ghys, A. Haefliger, and A. Verjovski, eds).* World Scientific, 1991.

[22] G. W. Whitehead. *Elements of Homotopy Theory.* Springer-Verlag, 1978.

[23] J. H. C. Whitehead. On the asphericity of regions in a 3-sphere. *Fund. Math.*, 32:149–166, 1939.