INEQUALITIES FOR THE NUMBER
OF INTEGERS IN A SUM OF
SETS OF GAUSSIAN INTEGERS

by

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INEQUALITIES FOR THE NUMBER OF INTEGERS IN A SUM OF SETS OF GAUSSIAN INTEGERS

INTRODUCTION

Let A be a set of positive integers, and for any positive integer n denote by $A(n)$ the number of integers of A which are not greater than n. Then the Schnirelmann density of A is defined (15, p.65) to be the quantity

$$\alpha = \operatorname*{glb}_{n} \frac{A(n)}{n}.$$

Thus the set of all positive integers would have Schnirelmann density 1, the set of all odd positive integers would have Schnirelmann density ½, and the set of all even positive integers would have Schnirelmann density 0.

Besicovitch (1, p. 246) introduced the density

$$\alpha^{*} = \operatorname*{glb}_{n} \frac{A(n)}{n+1},$$

and Erdős (5, p. 66) the density

$$\alpha_{1} = \operatorname*{glb}_{n \geqq k} \frac{A(n)}{n+1}$$

where k is the smallest positive integer not contained in A. (Erdős assumed $k > 1$, but we omit this requirement.)

For any two sets of positive integers A and B, define the <u>sum set</u>

$$C = A+B = \{ a, b, a+b \mid a \in A, b \in B \} .$$

Schnirelmann (15, p. 652) proved that if α, β, γ are the Schnirelmann densities of A, B, C, respectively, then $\gamma \geqq \alpha + \beta - \alpha\beta$, and Landau (8, p. 57) proved that $\alpha + \beta \geqq 1$ implies $\gamma = 1$.

In a famous paper by Mann it is shown (10, p. 523-526) that

$$\gamma \geqq \min (1, \alpha+\beta) .$$

This result is usually referred to as the $\alpha+\beta$ Theorem. In the same paper is proved (10, p. 526-527) a result which implies that for any positive integer n not in C we have

$$C(n) \geqq \alpha^* (n+1) + B(n) ,$$

and this inequality can be strengthened, by application of a result in a later paper (11, p. 250-252), to the relation

$$C(n) \geqq \alpha_1 (n+1) + B(n) \tag{1}$$

for any positive integer n not in C.

In a still more recent paper (12, p. 911-912) Mann proved a theorem which implies, in our notation, that for any positive integer n either $C(n) = n$ or there exists a positive integer m not in C such that $m \leqq n$ and

(2) $$\frac{C(n)+1}{n+1} \geqq \frac{A(m)+B(m)+1}{m+1} .$$

This inequality is a strengthening of the inequality

(2.1) $$\frac{C(n)}{n} \geqq \frac{A(m)+B(m)}{m}$$

which Mann proved in order to obtain the $\alpha + \beta$ Theorem.

In this thesis we will be concerned with attempts to extend the above definitions and theorems to sets of <u>Gaussian integers</u>, that is, numbers of the form $x + yi$ where x and y are real integers. The sets discussed above were subsets of the set of positive integers; in our work we will consider subsets of the set

$$Q = \{x + yi \mid x \text{ and } y \text{ are non-negative real integers, } x + y > 0\} .$$

For two subsets A and S of Q we will let $A(S)$ denote the number of Gaussian integers in $A \cap S$. In particular, then, $Q(S)$ is just the number of elements in S. Whenever we use the notation $A(S)$ the set S will consist of all Gaussian integers in a given bounded region of the complex plane and therefore will be finite. For any two subsets A and B of Q we define the sum set $C = A+B$ as we did for sets of real integers. The notation $A-B$

will occasionally be used to denote the set of all elements of A which are not in B.

Very little work has been done in this area; to the author's knowledge, Cheo (3, p.2) is the only one to have extended the concept of Schnirelmann density to the Gaussian integers. His definition is as follows: Let $x_0 + y_0 i$ be any number in Q, and let S be the set of all $x + yi$ in Q such that $x \leq x_0$, $y \leq y_0$. Then for any subset A of Q,

$$\alpha_c = \operatorname*{glb}_{s} \frac{A(S)}{Q(S)} .$$

We will refer to α_c as the <u>Cheo density</u> of A and will discuss the theorems obtained by Cheo for this density, as well as prove some similar theorems, in Chapters 1 and 2. These theorems also involve modifications of the Cheo density.

Cheo was able to show by means of an example that the $\alpha+\beta$ Theorem is not valid for his density, but a result analogous to (2.1) for subsets of Q may still be true. Cheo's example, at least, does not furnish a counterexample. This result would be the statement that if $x_0 + y_0 i$ is any element of Q which is not in C and S is the set of all $x + yi$ in Q with $x \leq x_0$, $y \leq y_0$ then there exists an element $x_1 + y_1 i$ in S which is not in C such that if T is the set of all $x + yi$ in Q with $x \leq x_1$ and $y \leq y_1$ then

$$(2.2) \qquad \frac{C(S)}{Q(S)} \geq \frac{A(T) + B(T)}{Q(T)} .$$

Inequality (2.1) implies the $\alpha+\beta$ Theorem for sets of real positive integers, but the analogous inequality (2.2) does not imply the $\alpha+\beta$ Theorem for subsets of Q.

While Cheo's definition of density is very natural and simple, it proves to be a somewhat difficult one with which to work. Accordingly, we will modify it in the following way.

<u>Definition 1</u>. Let $x_1 + y_1 i, \cdots, x_t + y_t i$ be t numbers in Q, $t \geq 1$, for which $0 \leq x_t < \cdots < x_1$ and $y_t > \cdots > y_1 \geq 0$ if $t > 1$. Let R_s be the set of all $x + yi$ in Q for which $x \leq x_s$ and $y \leq y_s$, $s = 1, \cdots, t$, and let $R = R_1 \cup \cdots \cup R_t$ (see Figure 1). Then for any subset A of Q we define the <u>density</u> of A to be the quantity

$$\alpha = \operatorname*{glb}_R \frac{A(R)}{Q(R)} .$$

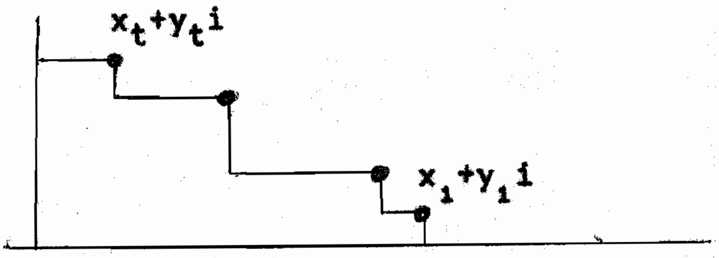

Figure 1

We will look upon this density as the extension of the Schnirelmann density to subsets of Q. It will also be necessary to have an extension of the Erdős density, which we define as follows.

Definition 2. If A is any subset of Q the modified density of A is the quantity

$$\alpha_1 = \text{glb} \frac{A(R)}{Q(R)+1} ,$$

taken over all sets R of the type described in Definition 1 for which $A(R) < Q(R)$.

Whether or not the $\alpha+\beta$ Theorem is valid for the density of Definition 1 is still a matter of conjecture; however, it is shown in Chapter 3 that

$$C(R) \geq \alpha_1 [Q(R) + 1] + B(R) \tag{3}$$

for every set R of the type described in Definition 1 for which the $x_1 + y_1i, \cdots, x_t + y_ti$ are not in C. This extension of the inequality (1) is the main result of the thesis.

The reader will notice that Theorem 1.1 and the theorem of Cheo's quoted in Chapter 1 both require that all numbers ji, $j = 1, 2, \cdots$, shall be contained in the sets A and B, and that this same condition on B is needed in the theorem of Cheo's which is quoted at the end of Chapter 2. The arguments used to prove these theorems are essentially one-dimensional, and these strong hypotheses are required in order to carry out these arguments. The proofs of Theorems 2.1, 2.2, and 3.1 are two-dimensional, and no such restrictions need be placed on A and B.

It appears that the methods used to prove the theorems of Chapters 2 and 3 can be equally well applied to sets of lattice points in n-dimensional space if density is suitably defined. The amount of exposition required is then greatly increased, of course, and we have omitted all such work from this thesis.

The method used to prove inequality (3) above can also be used to give a new proof of (1). This proof, plus a second new proof, is presented in the Appendix.

CHAPTER I

A "CHEO-TYPE" THEOREM

Let A and B be two subsets of Q, $C = A+B$, and let the Cheo density of C be γ_C (see Introduction, p.4). Let A_j be the set of all numbers $a + ji$ such that $a + ji$ is in A, $j = 0, 1, 2, \cdots$, and let $A_j(x)$ be the number of elements $a + ji$ of A_j with $a \leq x$ where x is any non-negative integer. Similarly define B_j, $B_j(x)$, C_j, and $C_j(x)$. Let S be the set of all $x + yi$ in Q with $x \leq x_0$, $y \leq y_0$ for an element $x_0 + y_0 i$ of Q.

Cheo (3, p. 10) has proved the following theorem: Let $\alpha' = \operatorname*{glb}_{s} \dfrac{A(S)}{Q(S)+1}$, $\beta' = \operatorname*{glb}_{n} \dfrac{\beta_0 + \cdots + \beta_n}{n+1}$ where

$$\beta_j = \operatorname*{glb}_{x} \frac{B_j(x)}{x+1} \text{ for } j > 0 \text{ and } \beta_0 = \operatorname*{glb}_{x} \frac{B_0(x)}{x}.$$ If

(1) $\alpha' + \beta_j \leq 1$, $j = 0, 1, 2, \cdots$, (2) $1 \geq \alpha' + \beta' \geq \beta_0$, and (3) ji is in A, B for all $j = 1, 2, \cdots$, then $\gamma_C \geq \alpha' + \beta'$.

The method used by Cheo to prove this theorem involves (among other things) using one-dimensional arguments on the sets A_j, B_j, and C_j and then summing the inequalities thus obtained over all j such that ji is contained in a set S of the type described above. Theorem 1.1 below is somewhat similar to Cheo's; our proof uses the inequality (2) in the Introduction and is simpler than his.

<u>Theorem 1.1</u>. If there is a positive integer not contained in C_o let k be the smallest such integer, let $\gamma^* = \text{glb} \frac{C(S)}{Q(S)+1}$ taken over all sets S of the type described above for which $x_o \geq k$, and let $\gamma' = \min(\gamma_c, \gamma^*)$. If there is no positive integer not contained in C_o let $\gamma' = \gamma_c$. For all $j = 0, 1, 2, \cdots$ let $\alpha_j = \underset{x}{\text{glb}} \frac{A_j(x)}{x+1}$, $\beta_j = \underset{x}{\text{glb}} \frac{B_j(x)}{x+1}$. Let $\alpha' = \underset{n}{\text{glb}} \frac{\alpha_o + \cdots + \alpha_n}{n+1}$, $\beta' = \underset{n}{\text{glb}} \frac{\beta_o + \cdots + \beta_n}{n+1}$. If (1) $\alpha_o + \beta_j \leq 1$ and $\alpha_j + \beta_o \leq 1$ for all $j = 0,1,2, \ldots$, and (2) ji is in A, B for all $j = 1, 2, \cdots$, then $\gamma' \geq \alpha' + \beta'$.

<u>Proof</u>: Let $x_o + y_o i$ be an element of Q with $x_o \geq k$, S the set of all $x + yi$ in Q with $x \leq x_o$

and $y \leq y_0$. Let $\mathcal{R}[A_j]$ be the set of all positive integers a such that $a + ji$ is in A. Similarly define $\mathcal{R}[B_j]$ and $\mathcal{R}[C_j]$. Since ji is in A_j, B_j, and C_j if $j > 0$ we have $\mathcal{R}[A_j](x_0) = A_j(x_0)-1$ for all $j > 0$ and $\mathcal{R}[A_0](x_0) = A_0(x_0)$, with like relationships for the B_j and C_j. Also, the fact that ji is in A_j and B_j for $j > 0$ and that $C_0 = A_0 + B_0$ implies that $\mathcal{R}[C_j] \supseteq \mathcal{R}[A_j] + \mathcal{R}[B_0]$ and $\mathcal{R}[C_j] \supseteq \mathcal{R}[A_0] + \mathcal{R}[B_j]$ for all $j = 0, 1, 2, \cdots$.

Then for $j > 0$, applying Mann's 1960 theorem (see Introduction, page 3) either $C_j(x_0) = x_0 + 1$ and

$$\frac{C_j(x_0)}{x_0 + 1} = 1 \geq \alpha_0 + \beta_j ,$$

$$\frac{C_j(x_0)}{x_0 + 1} \geq \alpha_j + \beta_0 ,$$

or there exists a positive integer m not in $\mathcal{R}[C_j]$ such that $m \leq x_0$ and

$$\frac{\mathcal{R}[C_j](x_0)+1}{x_0 + 1} \geq \frac{\mathcal{R}[A_0](m) + \mathcal{R}[B_j](m)+1}{m + 1} ,$$

or

$$\frac{C_j(x_0)}{x_0 + 1} \geq \frac{A_0(m) + B_j(m)}{m + 1} \geq \alpha_0 + \beta_j .$$

Likewise,

$$\frac{C_j(x_o)}{x_o+1} \geq \alpha_j + \beta_o .$$

Therefore, in either case,

$$\text{(4)} \qquad C_j(x_o) \geq \frac{x_o+1}{2} (\alpha_o + \beta_o + \alpha_j + \beta_j) .$$

Applying the same theorem to C_o, and noting that $x_o \geq k$ implies $C_o(x_o) < x_o$, we have

$$\frac{C_o(x_o)+1}{x_o+1} \geq \frac{A_o(m)+B_o(m)+1}{m+1} \geq \alpha_o + \beta_o + \frac{1}{m+1} ,$$

$$\begin{aligned} \text{(5)} \qquad C_o(x_o) &\geq (x_o+1)(\alpha_o+\beta_o) + \frac{x_o+1}{m+1} - 1 \\ &\geq (x_o+1)(\alpha_o+\beta_o) \\ &= \frac{x_o+1}{2}(\alpha_o+\beta_o+\alpha_o+\beta_o) . \end{aligned}$$

We now add the inequalities (4) and (5) to obtain

$$\begin{aligned} C_o(x_o) + C_1(x_o) + \cdots + C_{y_o}(x_o) &= C(S) \\ &\geq \frac{(x_o+1)(y_o+1)}{2}(\alpha_o+\beta_o) + \frac{x_o+1}{2}(\alpha_o+\cdots+\alpha_{y_o} \\ &\qquad + \beta_o+\cdots+\beta_{y_o}) \\ &\geq \frac{(x_o+1)(y_o+1)}{2}(\alpha'+\beta') + \frac{x_o+1)(y_o+1)}{2}(\alpha'+\beta') \\ &= [Q(S)+1](\alpha'+\beta') . \end{aligned}$$

Therefore,

$$\frac{C(S)}{Q(S)+1} \geq \alpha' + \beta'$$

for all sets S of the type used in defining γ^*, and

$$\gamma^* \geq \alpha' + \beta' .$$

Now for sets S of the type used in defining γ^* we clearly have

$$\frac{C(S)}{Q(S)} \geq \frac{C(S)}{Q(S)+1} \geq \alpha' + \beta' .$$

Suppose $x_0 + y_0 i$ is in Q, $1 \leq x_0 < k$. The inequality (4) obtained for $j > 0$ is still valid. For $j = 0$ we now have

$$\frac{C_0(x_0)}{x_0} = 1 \geq \alpha_0 + \beta_0 ,$$

$$C_0(x_0) \geq \frac{x_0}{2} (\alpha_0 + \beta_0 + \alpha_0 + \beta_0) .$$

From (4),

$$C_j(x_0) \geq \frac{x_0 + 1}{2} (\alpha_0 + \beta_0 + \alpha_j + \beta_j) ,$$

$j > 0$. By adding these inequalities we obtain

$$C_0(x_0) + \cdots + C_{y_0}(x_0) = C(S)$$

$$\geq \frac{(x_0+1)(y_0+1)-2}{2} (\alpha_0+\beta_0) + \frac{x_0+1}{2} (\alpha_0+\cdots+\alpha_{y_0}+\beta_0+\cdots+\beta_{y_0})$$

$$\geq \frac{(x_0+1)(y_0+1)-2}{2} (\alpha'+\beta') + \frac{(x_0+1)(y_0+1)}{2} (\alpha' + \beta')$$

$$= \frac{Q(S) - 1}{2} (\alpha' + \beta') + \frac{Q(S) + 1}{2} (\alpha' + \beta')$$

$$= Q(S) (\alpha' + \beta') .$$

Hence,

$$\frac{C(S)}{Q(S)} \geq \alpha' + \beta'$$

for all sets S such that $x_o \geq 1$. Finally, if $x_o = 0$ then $C(S) = y_o = Q(S)$ and

$$\frac{C(S)}{Q(S)} = 1 \geq \alpha' + \beta' .$$

This gives us

$$\gamma_c \geq \alpha' + \beta'$$

and, therefore,

$$\gamma' \geq \alpha' + \beta'$$

which completes the proof.

It is obvious that the β' of Theorem 1.1 is always less than or equal to the β' of Cheo's theorem. We will now show that this is also true of the α''s. Accordingly, let S be any set of the type described in the first paragraph of this chapter.

Then

$$\frac{A(S)}{Q(S)+1} = \frac{A_o(x_o) + A_1(x_o) + \cdots + A_{y_o}(x\ )}{(x_o+1)\ (y_o+1)}$$

$$\geq \frac{\alpha_o(x_o+1) + \alpha_1(x_o+1) + \cdots + \alpha_{y_o}(x_o+1)}{(x_o+1)\ (y_o+1)}$$

$$= \frac{\alpha_0 + \alpha_1 + \cdots + \alpha_{y_0}}{y_0 + 1}$$

$$\geq \alpha' ,$$

where the $\alpha_0, \alpha_1, \cdots, \alpha_{y_0}, \alpha'$ are those defined in the statement of Theorem 1.1. Since this is true for all these sets S, it follows that the α' of Cheo's theorem is greater than or equal to the α' of Theorem 1.1. Consequently, if $\gamma_c = \gamma'$ then Theorem 1.1 cannot give a stronger result than Cheo's theorem.

The two theorems are not really comparable when $\gamma' < \gamma_c$, but the following example illustrates the results for one such case.

<u>Example</u>: Let $A = B = \{3k+1, 3k+ji \mid k = 0,1,2,\cdots$ and $j = 1,2, \cdots\}$. Then $C = \{3k+1, 3k+2, 3k+ji, 3k+1+ji \mid k = 0, 1, 2, \cdots$ and $j = 1, 2, \cdots \}$. We see that $\gamma_c = 2/3$ and $\gamma' = 1/2$.

For Cheo's theorem the $\alpha' = 1/4$, $\beta' = 1/3$, and $\gamma_c > \alpha' + \beta'$. For Theorem 1.1 the $\alpha' = \beta' = 1/4$, and $\gamma' = \alpha' + \beta'$.

CHAPTER II

ANALOGUES OF TWO THEOREMS OF SCHNIRELMANN AND LANDAU FOR GAUSSIAN INTEGERS

It was mentioned in the Introduction that if A and B are two sets of positive integers, $C = A + B$, α, β, γ the Schnirelmann densities of A, B, C, respectively, then $\alpha + \beta \geq 1$ implies $\gamma = 1$, and, in any case, $\gamma \geq \alpha + \beta - \alpha\beta$. Cheo (3, p. 6) was able to establish the first of these theorems for his density, and we will now prove both of them for the density defined in Definition 1 of the Introduction. The proof of Theorem 2.1 is essentially that given by Cheo.

<u>Theorem 2.1</u>. Let A and B be two subsets of Q, $C = A+B$, and α, β, γ the densities of A, B, C, respectively. If $\alpha + \beta \geq 1$ then $\gamma = 1$.

<u>Proof</u>: We know $\gamma \leq 1$. Hence, assume $\gamma < 1$. Now $\gamma = \operatorname{glb}_R \frac{C(R)}{Q(R)} < 1$ implies there exists a set R_0 of the type used in defining the density such that $C(R_0) < Q(R_0)$, which in turn implies that there exists a number $x_0 + y_0 i$ contained in Q but not in C. We

may let R_0 be the set of all $x + yi$ in Q such that $x \leq x_0$, $y \leq y_0$. Then for any $x + yi$ in R_0 either $x + yi$ is in A, or $x + yi = (x_0+y_0 i) - (b_1+b_2 i)$ for some $b_1+ b_2 i$ in $B \cap R_0$, or neither, but never both. Since $x_0+ y_0 i$ is not in C it cannot be in A. Also, $x_0+ y_0 i \neq (x_0+y_0 i) - (b_1+ b_2 i)$ for any $b_1+b_2 i$ in B since B does not contain 0. Hence, we have

$$A(R_0) + B(R_0) \leq Q(R_0) - 1,$$

and

$$\alpha + \beta \leq \frac{A(R_0) + B(R_0)}{Q(R_0)} < 1$$

which is a contradiction. Therefore, $\gamma = 1$.

<u>Theorem 2.2</u>. Let A and B be two subsets of Q, $C = A + B$, and α, β, γ the densities of A, B, C, respectively. Then $\gamma \geq \alpha + \beta - \alpha\beta$.

<u>Proof</u>: If 1 or i is missing from A then $\alpha = 0$ and the theorem is obvious. Hence, we assume 1 and i are in A. Also, if $A = Q$ then $\gamma = \alpha = 1$, and again the theorem is obvious. We will, therefore, assume that there exists a set R_0 of the type used in defining the density such that $A(R_0) < Q(R_0)$.

Let H be the set of all $x + yi$ which are in R_0 but not in A. By the choice of R_0, H is not empty. We want to partition H into disjoint, non-empty subsets $L_1, \cdots, L_r$, $r \geq 1$, of the following type: For each L_j, $j = 1, \cdots, r$, there exists a number $a_j + a_j'i$ in $A \cap R_0$, and a set L_j' contained in R_0 such that L_j' is of the type used in defining the density, and $L_j = \{(x + yi) + (a_j + a_j'i) \mid x + yi \in L_j'\}$. We will speak of L_j as being "based on" the number $a_j + a_j'i$. Note that $a_j + a_j'i$ is not in L_j.

In order to effect this partition of H we let a_1' be the smallest real integer for which there exists a real integer x such that $x + a_1'i$ is in $A \cap R_0$, but either $(x + 1) + a_1'i$ or $x + (a_1' + 1)i$ is in H. Let a_1 be the smallest such x. (The existence of $a_1 + a_1'i$ follows from the fact that 1 and i are in A. If $x + yi$ is in R_0 and either $y < a_1'$ or $y = a_1'$ and $x < a_1$, then $x + yi$ is in A.) Now let $\ell_{1,1} + \ell_{1,1}'i$ be that number in H such that (1) $a_1 \leq \ell_{1,1}$, $a_1' \leq \ell_{1,1}'$, (2) for any $x + yi \neq a_1 + a_1'i$ in Q with $a_1 \leq x \leq \ell_{1,1}$ and $a_1' \leq y \leq \ell_{1,1}'$ we have $x + yi$ in H, (3) for any

$(\ell_{1,1}+m)+(\ell'_{1,1}+n)i$ in Q such that $m \geq 0$, $n \geq 0$, $\max(m,n) > 0$, there exists $x+yi$ in Q but not in H with $a_1 \leq x \leq \ell_{1,1}+m$, $a'_1 \leq y \leq \ell'_{1,1}+n$. If there is more than one such $\ell_{1,1}+\ell'_{1,1}i$ in H pick that one for which $\ell'_{1,1}$ is minimal, and let $\ell_{2,1}+\ell'_{2,1}i$ be that number in H such that $\ell_{2,1}+\ell'_{2,1}i$ satisfies the conditions (1), (2), (3), $\ell_{2,1}+\ell'_{2,1}i \neq \ell_{1,1}+\ell'_{1,1}i$, and $\ell'_{2,1}$ minimal. Then we will have $\ell'_{2,1} > \ell'_{1,1} \geq a'_1$, since if $\ell'_{2,1} = \ell'_{1,1}$ either $\ell_{1,1}+\ell'_{1,1}i =$ $(\ell_{2,1}+m)+(\ell'_{2,1}+n)i$ with $m > 0$, $n = 0$, and there does not exist $x+yi$ not in H for which $a_1 \leq x \leq \ell_{2,1}+m$ and $a'_1 \leq y \leq \ell'_{2,1}+n$, which contradicts the choice of $\ell_{2,1}+\ell'_{2,1}i$, or else $\ell_{2,1}+\ell'_{2,1}i = (\ell_{1,1}+m)+(\ell'_{1,1}+n)i$ with $m > 0$ and $n = 0$, and we have a contradiction on the choice of $\ell_{1,1}+\ell'_{1,1}i$. Also, we will have $a_1 \leq \ell_{2,1} < \ell_{1,1}$ for the same reason. We continue, designating numbers $\ell_{1,1}+\ell'_{1,1}i$, $\ell_{2,1}+\ell'_{2,1}i$, $\cdots$ in this manner as long as possible, and call the last of these numbers $\ell_{n_1,1}+\ell'_{n_1,1}i$. Let the set $L_{k,1} =$ $\{x+yi \mid x+yi \in Q,\ x+yi \neq a_1+a'_1i,\ a_1 \leq x \leq \ell_{k,1}, a' \leq y \leq \ell'_{k,1}\}$.

Then the set $L_1 = L_{1,1} \cup L_{2,1} \cup \cdots \cup L_{n_1,1}$ is of the same type as the L_j described above.

If $H - L_1$ is not empty let a_2' be the smallest real integer for which there exists a real integer x such that $x + a_2'i$ is in $A \cap R_0$, but either $(x+1) + a_2'i$ or $x + (a_2' + 1)i$ is in $H - L_1$. Let a_2 be the smallest such x and form the set L_2 based on $a_2 + a_2'i$ in the same way as was done for L_1, substituting $H-L_1$ for H. We continue forming sets $L_1, L_2, \cdots$ in this manner as long as possible. Call the last set formed L_r.

Suppose there exists a number $h_1 + h_2 i$ in $H - (L_1 \cup \cdots \cup L_r)$. We know that $h_2 \geq a_1'$, so there are two possibilities:

(i) There exists a set L_j based on $a_j + a_j'i$ such that $a_j \leq h_1$, $a_j' \leq h_2$. Let j be minimal. Because of the way L_j was formed, the set $M = \{x+yi \mid x+yi \in Q,\ x+yi \neq a_j + a_j'i,\ a_j \leq x \leq h_1,\ a_j' \leq y \leq h_2\} \in R_0$ must contain at least one point of A. (If not, we would have $h_1 + h_2 i$ in L_j.) Let a_k' be the smallest real integer for which there exists a real integer x such that $x + a_k'i$ is in $A \cap M$

and for any $x' + y'i \neq x + a_k'i$ in Q with $x \leq x' \leq h_1$, $a_k' \leq y' \leq h_2$, we have $x' + y'i$ in H. Let a_k be the smallest such x. Then eventually our method of forming the sets $L_1, L_2, \cdots$ would lead us to form a set L_k based upon $a_k + a_k'i$ which would include $h_1 + h_2 i$, and we would not have $h_1 + h_2 i$ in $H - (L_1 \cup \cdots \cup L_r)$.

(ii) We have $a_j > h_1$ whenever $h_2 \geq a_j'$ for all the $a_j + a_j'i$ upon which the L_j are based. Then there exists a number $x+yi$ in $A \cap R_0$ such that $x \leq h_1$, $y \leq h_2$ (i, for example, since $h_2 > 0$ by choice of a_1'). Again, among these numbers we can find $a_k + a_k'i$ just as in (i) which must eventually be chosen as the base for a set L_k which would include $h_1 + h_2 i$.

Hence, the set $L_1 \cup \cdots \cup L_r$ exhausts H.

Figure 2 illustrates sets R_0 and A such that $1, i \in A$ and $A(R_0) < Q(R_0)$ for which the set H has been partitioned in the manner described above. The dots represent points of A and the circles, points of H.

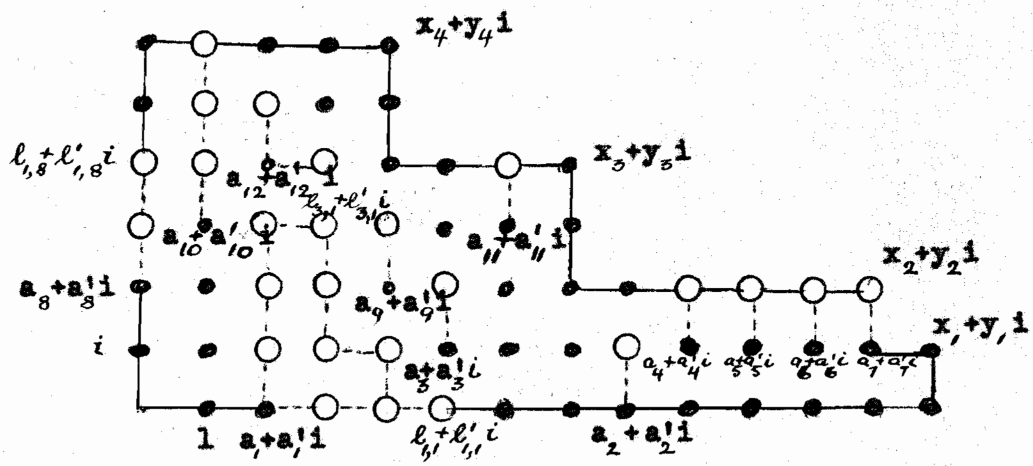

Figure 2

Now, for each $j = 1, \cdots, r$, we note that the set $L'_j = \{(x-a_j) + (y-a'_j)i \mid x+yi \in L_j\}$, and that $Q(L_j) = Q(L'_j)$. If $b_1 + b_2 i$ is in $B \cap L'_j$, then $(a_j + a'_j i) + (b_1 + b_2 i)$ is in $C \cap L_j$ and, therefore, not in A. Hence,

$$C(R_0) \geq A(R_0) + B(L'_1) + \cdots + B(L'_r)$$

$$\geq A(R_0) + \beta[Q(L'_1) + \cdots + Q(L'_r)].$$

We also have

$$Q(R_0) - A(R_0) = Q(H) = Q(L'_1) + \cdots + Q(L'_r),$$

so we substitute to get

$$C(R_0) \geq A(R_0) + \beta[Q(R_0) - A(R_0)]$$

$$= A(R_0)\,[1-\beta] + \beta[Q(R_0)]$$

$$\geq \alpha[Q(R_o)]\,[1-\beta] + \beta[Q(R_o)]\ .$$

We divide both members of this inequality by $Q(R_o)$ to obtain

$$\frac{C(R_o)}{Q(R_o)} \geq \alpha + \beta - \alpha\beta\ .$$

Since this relationship holds for any R_o such that $A(R_o) < Q(R_o)$ it must also hold for any R_o such that $C(R_o) < Q(R_o)$.

Also, $1 \geq \alpha$ and $1 \geq \beta$ implies $(1-\alpha)(1-\beta) \geq 0$, or $1 \geq \alpha + \beta - \alpha\beta$. Hence, if $C(R_o) = Q(R_o)$ then

$$\frac{C(R_o)}{Q(R_o)} \geq \alpha + \beta - \alpha\beta\ .$$

Since this relationship holds for every set R_o of the type used in defining the density we must have

$$\gamma \geq \alpha + \beta - \alpha\beta\ .$$

Although Cheo did not prove $\gamma_c \geq \alpha_c + \beta_c - \alpha_c\beta_c$ for all sets A, B, $C = A+B$, he did prove the following theorem: Let A and B be subsets of Q, $C = A + B$, and let β_c and γ_c be the Cheo densities of B and C. Let A_o be the set of all real integers in A, and $\alpha_o = \operatorname{glb}_x \dfrac{A_o(x)}{x}$. If B contains all numbers ji, $j = 1, 2, \cdots,$ then

$$\gamma_c \geq \alpha_o + \beta_c - \alpha_o\beta_c.$$

Since $\alpha_o \geq \alpha_c$ and $\beta_c \leq 1$ this conclusion, of course, implies $\gamma_c \geq \alpha_c + \beta_c - \alpha_c\beta_c$.

We cannot remove the requirement that ji be in B for $j = 1, 2, \cdots$ by means of an argument like that used to establish Theorem 2.2, since it would be necessary that the sets L_j' be of the type S used in defining the Cheo density, and thus H would have to be partitioned into sets $L_1, \cdots, L_r$ such that for each L_j there exists a number $a_j + a_j'i$ in $A \cap R_0$ and a number $\ell_j + \ell_j'i$ in H for which $a_j \leq \ell_j$, $a_j' \leq \ell_j'$, and

$$L_j = \{x+yi \mid x+yi \in R_0,\ a_j \leq x \leq \ell_j,\ a_j' \leq y \leq \ell_j',\ x+yi \neq a_j+a_j'i\}.$$

This is not always possible, as the reader can easily verify by means of the example shown in Figure 3. Again, the dots are points of A while the circles are points of H. The set R_0 is the set of all $x+yi$ in Q with $x \leq x_0$ and $y \leq y_0$.

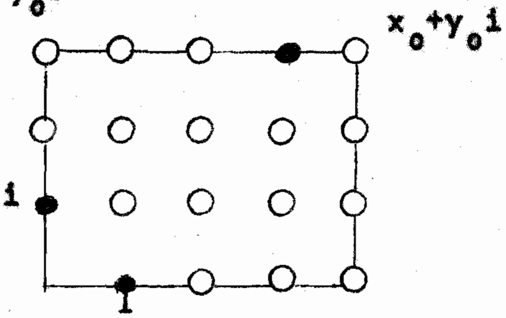

Figure 3

Schnirelmann (15, p. 652-653) extended the result $\gamma \geq \alpha + \beta - \alpha\beta$ for sets of positive integers to that given in Corollary 2.1 below, then used this result to prove the statement of Corollary 2.2 for sets of positive integers (15, p. 655). Our proofs are essentially the same as his.

<u>Corollary 2.1</u>. Let $A_1, \cdots, A_n$ be any n subsets of Q, $n \geq 2$, and define the sum set $A_1 + \cdots + A_n = \{a_1 + \cdots + a_n \mid a_1 + \cdots + a_n \neq 0, \text{ each } a_i \in A_i \cup \{0\}, i = 1, \cdots, n\}$. (This addition is clearly commutative and associative, and if $n = 2$ is equivalent to the sum set defined in the Introduction.) Let $d(A_1 + \cdots + A_n)$ be the density of $A_1 + \cdots + A_n$ and α_i the density of A_i, $i = 1, \cdots, n$. Then

$$1 - d(A_1 + \cdots + A_n) \leq (1-\alpha_1)\cdots(1-\alpha_n).$$

<u>Proof</u>: If $n = 2$ Theorem 2.2 implies

$$d(A_1 + A_2) \geq \alpha_1 + \alpha_2 - \alpha_1\alpha_2,$$

or

$$1 - d(A_1 + A_2) \leq 1 - \alpha_1 - \alpha_2 + \alpha_1\alpha_2$$

$$= (1 - \alpha_1)(1 - \alpha_2).$$

Assume that for some integer $k \geq 2$ we have

$$1 - d(A_1 + \cdots + A_k) \leq (1-\alpha_1) \cdots (1 - \alpha_k).$$

Then

$$1 - d(A_1 + \cdots + A_k + A_{k+1}) = 1 - d([A_1 + \cdots + A_k] + A_{k+1})$$

$$\leq (1 - d(A_1 + \cdots + A_k))\,(1 - \alpha_{k+1})$$

$$\leq (\,1 - \alpha_1) \cdots (1-\alpha_k)\,(1-\alpha_{k+1})\ ,$$

and the proof is complete.

Let k be a real integer, $k \geq 1$. We will call A a <u>basic set</u> of Q of order k if $A + \cdots + A$ with k summands (or kA) equals Q where k is minimal.

<u>Corollary 2.2</u>. If the density α of A is positive then A is a basic set of Q.

<u>Proof</u>: There exists an integer $n \geq 1$ such that $(1-\alpha)^n \leq \frac{1}{2}$. Then Corollary 2.1 implies that

$$1 - d(nA) \leq (1-\alpha)^n \leq \tfrac{1}{2}\ ,$$

and

$$d(nA) \geq \tfrac{1}{2}\ .$$

From Theorem 2.1, $d(nA) + d(nA) \geq 1$ implies $d(nA + nA) = 1$, or $2nA = Q$. Note that the order of A is less than or equal to $2n$.

CHAPTER III

ANALOGUE OF A THEOREM OF MANN FOR GAUSSIAN INTEGERS

If A and B are two sets of positive integers, $C = A + B$, and if n is any positive integer not in C, then Mann has proved a result that implies that $C(n) \geq \alpha_1(n+1) + B(n)$ where α_1 is the Erdős density of A (see page 2). We will now extend this theorem to sets of Gaussian integers. Accordingly, throughout the remainder of this chapter it is to be understood that A and B are subsets of Q and $C = A + B$. Also, α_1 is the modified density of A described in Definition 2 of the Introduction. Our theorem is then the following:

<u>Theorem 3.1</u>. Let R be any set of the type used to define α_1 for which the numbers $x_1 + y_1 i, \cdots, x_t + y_t i$ are not in C. Then

$$C(R) \geq \alpha_1 \, [Q(R) + 1] + B(R) .$$

We will need to make frequent reference to sets of two special types. For convenience we will define them here.

<u>Definition 3</u>. A set S will be said to be of type S^1 if it satisfies the following conditions 2, 3, and either 1, 1', or 1".

1. There exist in Q numbers $X_1 + Y_1 i$, $X_2 + Y_2 i$, $x_1 + y_1 i$, $\cdots$, $x_u + y_u i$, $x_1' + y_1' i$, $\cdots$, $x_v' + y_v' i$, $u, v \geq 1$, such that $X_1 \leq x_1' < \cdots < x_v' = X_2$, $Y_1 = y_1' > \cdots > y_v' \geq Y_2$, $X_1 = x_1 < \cdots < x_u \leq X_2$, $Y_1 \geq y_1 > \cdots > y_u = Y_2$, and if for any $x_m + y_m i$, $m = 2, \cdots, u$, there exists $x_n' + y_n' i$ with $1 \leq n \leq v-1$ such that $x_m > x_n'$, then $y_{m-1} \leq y_{n+1}'$ (or, equivalently, if for any $x_m + y_m i$, $m = 1, \cdots, u-1$, there exists $x_n' + y_n' i$ with $2 \leq n \leq v$ such that $y_m > y_n'$, then $x_{m+1} \leq x_{n-1}'$). Note that the requirement that $x_u + y_u i$ be in Q implies that if $u = 1$ then $\max(X_1, Y_2) > 0$. Let $T = \{x+yi \mid x+yi \in Q,\ X_1 \leq x \leq X_2,\ Y_1 \geq y \geq Y_2\}$. If $v > 1$, let

$T_1' = \{x+yi \mid x+yi \in Q,\ x_1' < x \leq x_2',\ y_1' \geq y > y_2'\}$,

$T_2' = \{x+yi \mid x+yi \in Q, x_2' < x \leq x_3' ,\ y_1' \geq y > y_3'\}$, $\cdots$,

$T_{v-1}' = \{x+yi \mid x+yi \in Q,\ x_{v-1}' < x \leq x_v',\ y_1' \geq y > y_v'\}$.

If $u > 1$ let $T_1 = \{x+yi \mid x+yi \in Q,\ x_1 \leq x < x_2,\ y_1 > y \geq y_u\}$,

$T_2 = \{x+yi \mid x+yi \in Q,\ x_2 \leq x < x_3,\ y_2 > y \geq y_u\}$, $\cdots$,

$T_{u-1} = \{x+yi \mid x+yi \in Q,\ x_{u-1} \leq x < x_u,\ y_{u-1} > y \geq y_u\}$.

Then if $u = v = 1$, $S = T$. If $u = 1$, $v > 1$, then $S = T - (T'_1 \cup \cdots \cup T'_{v-1})$. If $u > 1$, $v = 1$, then $S = T - (T_1 \cup \cdots \cup T_{u-1})$. If $u > 1$, $v > 1$, then

$S = T - (T_1 \cup \cdots \cup T_{u-1} \cup T'_1 \cup \cdots \cup T'_{v-1})$.

1'. There exist numbers $X_1 + Yi$, $X_2 + Yi$ in Q with $X_1 < X_2$ such that $S = \{x+yi \mid x+yi \in Q,$ $y = Y,\ X_1 \leq x \leq X_2\}$.

1". There exist numbers $X + Y_1 i$, $X + Y_2 i$ in Q with $Y_1 > Y_2$ such that $S = \{x+yi \mid x+yi \in Q,\ x = X,$ $Y_1 \geq y \geq Y_2\}$.

2. $B(S) \geq 1$, $Q(S) - C(S) \geq 1$.

3. If $b_1 + b_2 i$ is in $B \cap S$ and $g_1 + g_2 i$ is in S but not in C then $(g_1 + g_2 i) - (b_1 + b_2 i)$ is in Q. That is, $g_1 \geq b_1$ and $g_2 \geq b_2$ with strict inequality in at least one of these cases.

Figure 4 illustrates a set S of the type described in requirement 1 above, S being those points of Q which are on and within the solid lines. The set T consists of all points of Q in the entire rectangle. The reader will note that S is a set $R - R'$ where $R' \subset R$ and R', R are sets of the type used to define the density.

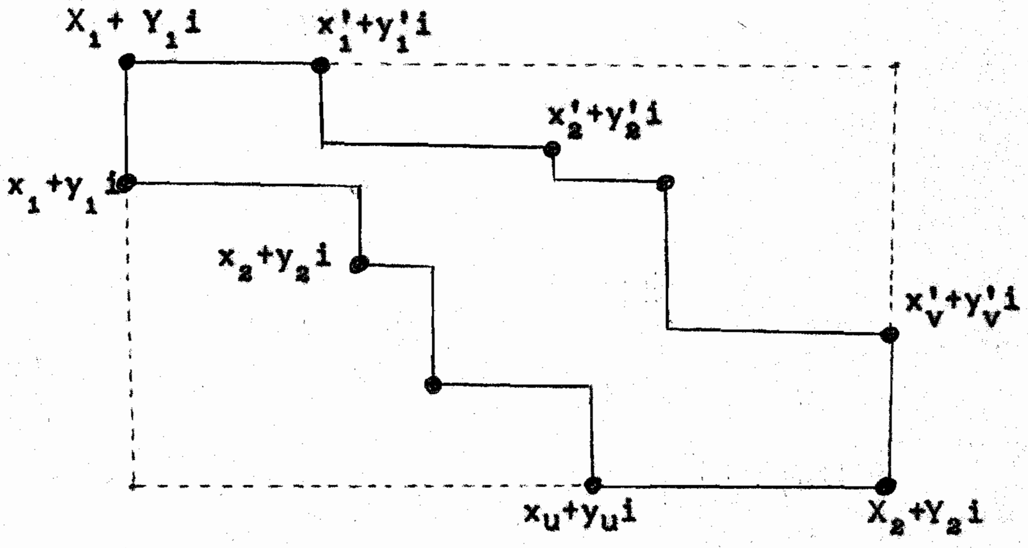

Figure 4

<u>Definition 4</u>. A set S will be said to be of type S^0 if it satisfies the following conditions 1, 2, 3.

1. There exist numbers $x_1 + y_1 i, \cdots, x_t + y_t i$ in Q, $t \geq 1$, such that if $t > 1$ then $x_t < \cdots < x_1$ and $y_t > \cdots > y_1$. Let $S_r = \{x+yi \mid x+yi \in Q,\ x \leq x_r,\ y \leq y_r\}$, $r = 1, \cdots, t$. Then $S = S_1 \cup \cdots \cup S_t$. (In other words, S is of the same type as the set R used to define α and α_1.)

2. $Q(S) - C(S) \geq 1$.

3. If $b_1 + b_2 i$ is in $B \cap S$ and $g_1 + g_2 i$ is in S but not in C then $(g_1 + g_2 i) - (b_1 + b_2 i)$ is in Q.

The reader will note that a set S may satisfy conditions 1,1', or 1",2,3 of Definition 3 and also conditions 1, 2, 3 of Definition 4. Thus S may be of both type S^0 and type S^1 .

We will now prove three lemmas which will be needed for the proof of Theorem 3.1.

Lemma 3.1. Let S be a set of type S^1 satisfying requirement 1 of Definition 3, and let S_j be the set of all $x+yi$ in S such that $x \geq x_j$, $y \geq y_j$, $j = 1, \cdots, u$. Let S_j' be the set of all $(x+yi) - (x_j + y_j i)$ for which $x+yi$ is in S_j and $x+yi \neq x_j+y_j i$. Let $S' = S_1' \cup \cdots \cup S_u'$. (Clearly, $S = S_1 \cup \cdots \cup S_u$.) Then $Q(S') \leq Q(S) - 1$.

Proof: If $u = 1$ then S' is just a translation of all points of S except one, and $Q(S') = Q(S) - 1$. Assume that for some $k \geq 1$ we have $Q(S') \leq Q(S) - 1$ whenever $u \leq k$, and consider $u = k+1$ (see Figure 5).

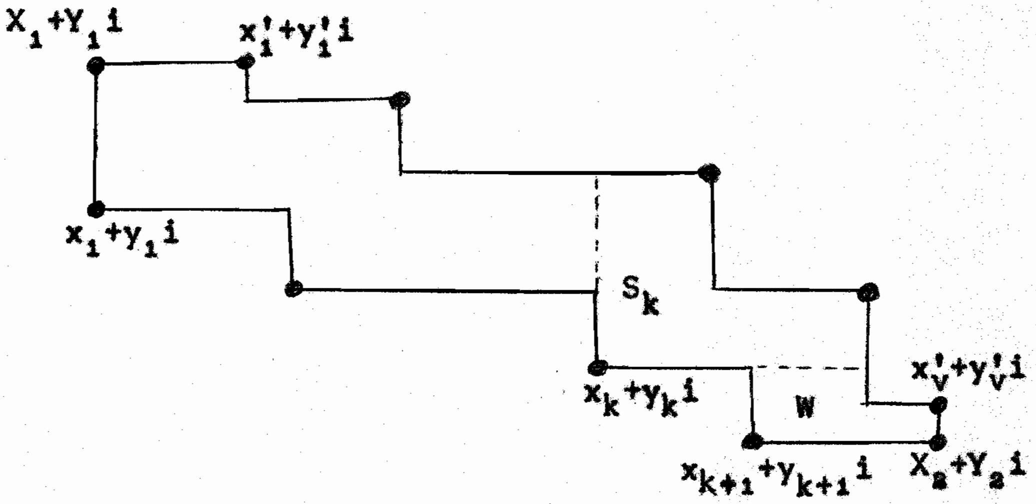

FIGURE 5

Define sets W, S^*_{k+1}, S^{**}_{k+1} as follows:

$$W = \{x+yi \mid x+yi \in S, \quad y_{k+1} \leq y < y_k\},$$

$$S^*_{k+1} = \{(x+yi) + (y_k - y_{k+1})i \mid x+yi \in S_{k+1}\},$$

$$S^{**}_{k+1} = \{(x+yi) - (x_{k+1} - x_k) \mid x+yi \in S^*_{k+1}\}.$$

Then

$$\begin{aligned} Q(S_k') + 1 + Q(W) &= Q(S_k) + Q(W) \\ &= Q(S_k \cup S_{k+1}) \\ &= Q(S_k \cup S^*_{k+1}) \\ &\geq Q(S_k \cup S^{**}_{k+1}) \\ &= Q(S_k' \cup S_{k+1}') + 1, \end{aligned}$$

and

$$Q(S_k') + Q(W) \geq Q(S_k' \cup S_{k+1}').$$

If $k = 1$ then $Q(S'_k \cup S'_{k+1}) = Q(S')$. If $k > 1$ then

$$Q(S'_1 \cup \cdots \cup S'_k) + Q(W)$$

$$= Q(S'_1 \cup \cdots \cup S'_{k-1}) + Q(S'_k) - Q([S'_1 \cup \cdots \cup S'_{k-1}] \cap S'_k) + Q(W)$$

$$\geq Q(S'_1 \cup \cdots \cup S'_{k-1}) + Q(S'_k \cup S'_{k+1}) - Q([S'_1 \cup \cdots \cup S'_{k-1}] \cap [S'_k \cup S'_{k+1}])$$

$$= Q(S'_1 \cup \cdots \cup S'_k \cup S'_{k+1})$$

$$= Q(S').$$

From the induction hypothesis

$$Q(S'_1 \cup \cdots \cup S'_k) \leq Q(S_1 \cup \cdots \cup S_k) - 1$$

$$= Q(S) - Q(W) - 1.$$

Therefore,

$$Q(S) \geq Q(S'_1 \cup \cdots \cup S'_k) + Q(W) + 1$$

$$\geq Q(S') + 1,$$

or

$$Q(S') \leq Q(S) - 1.$$

Lemma 3.2. If S is any set of type S^1 then

$$C(S) \geq \alpha_1[Q(S)] + B(S).$$

Proof: Assume S satisfies requirement 1 of Definition 3.

We know the set $B \cap S$ is not empty. Let b_a^0 be the largest real integer for which there exists a real integer b_1^0 such that $b_1^0 + b_2^0 i \in B \cap S$ and $b_1^0 + b_2^0 = \max\{b_1 + b_2 | b_1 + b_2 i \in B \cap S\}$. (That is, if $b_1' + b_2' i \in B \cap S$, $b_1' + b_2' i \neq b_1^0 + b_2^0 i$, $b_1' + b_2' = \max\{b_1 + b_2\}$, then $b_2^0 > b_2'$.) Likewise, let g_2^0 be the largest real integer for which there exists a real integer g_1^0 such that $g_1^0 + g_2^0 i$ is in S but not in C and $g_1^0 + g_2^0 = \max\{g_1 + g_2 \mid g_1 + g_2 i \in S,\ g_1 + g_2 i \notin C\}$.

Define the set S' as in Lemma 3.1. Let $B(S) = r \geq 1$, $Q(S) - C(S) = g_C \geq 1$, and $Q(S') - A(S') = g_A$. The set of all $(g_1^0 + g_2^0 i) - (b_1 + b_2 i)$ with $b_1 + b_2 i$ in $B \cap S$ gives r numbers in Q (Definition 3, requirement 3) which are not in A, since suppose $(g_1^0 + g_2^0 i) - (b_1 + b_2 i) = a_1 + a_2 i \in A$. Then $g_1^0 + g_2^0 i = (a_1 + a_2 i) + (b_1 + b_2 i) \in C$ which is a contradiction. We also show that these r numbers are in S': Since $b_1 + b_2 i$ is in S, there exists an S_j as defined in Lemma 3.1 such that $b_1 + b_2 i$ is in S_j. This implies $x_j \leq b_1$, $y_j \leq b_2$. But $b_1 \leq g_1^0$ and $b_2 \leq g_2^0$, so $g_1^0 + g_2^0 i$ is also in S_j. Let $b_1 + b_2 i = (x_j + m) + (y_j + n)i$. Then $g_1^0 - m \geq b_1 - m = x_j$, $g_2^0 - n \geq b_2 - n = y_j$, so $(g_1^0 - m) + (g_2^0 - n)i$ is in S_j. Hence

$$0 \neq (g_1^0 + g_2^0 i) - (b_1 + b_2 i) = [(g_1^0 - m) + (g_2^0 - n)i]$$

$- (x_j + y_j i)$ which is in S_j' and, therefore, in S'.

Likewise, the (possibly empty) set of all $(g_1 + g_2 i) - (b_1^0 + b_2^0 i)$ with $g_1 + g_2 i$ in S but not in C and $g_1 + g_2 i \neq g_1^0 + g_2^0 i$ gives $g_C - 1$ numbers which are in S' but not in A. We must show that these two sets are disjoint. Hence, suppose that for some $g_1 + g_2 i \neq g_1^0 + g_2^0 i$ (and, therefore, $b_1 + b_2 i \neq b_1^0 + b_2^0 i$) we have

$$(g_1^0 + g_2^0 i) - (b_1 + b_2 i) = (g_1 + g_2 i) - (b_1^0 + b_2^0 i).$$

Then

$$g_1^0 + b_1^0 = g_1 + b_1,$$

$$g_2^0 + b_2^0 = g_2 + b_2.$$

We add these equalities to obtain

$$(g_1^0 + g_2^0) + (b_1^0 + b_2^0) = (g_1 + g_2) + (b_1 + b_2).$$

The method of choosing $g_1^0 + g_2^0 i$ and $b_1^0 + b_2^0 i$ implies that we must have $g_1^0 + g_2^0 = g_1 + g_2$ and $b_1^0 + b_2^0 = b_1 + b_2$. But this, in turn, implies $g_2^0 > g_2$ and $b_2^0 > b_2$. Hence, $g_2^0 + b_2^0 \neq g_2 + b_2$, and we have obtained a contradiction.

Therefore, the two sets are disjoint,

$$g_A \geq g_C - 1 + r.$$

$$Q(S) - g_c \geq Q(S) - g_A - 1 + r,$$

and

$$Q(S) - g_c \geq Q(S') - g_A + Q(S) - Q(S') - 1 + r.$$

We recall that $Q(S)-Q(S')-1 \geq 0$ (Lemma 3.1), and note that S' is a set of the type R used to define α_1. Hence, we may write

$$\begin{aligned} C(S) &\geq A(S') + [Q(S)-Q(S')-1] + B(S) \\ &\geq \alpha_1[Q(S')+1] + \alpha_1[Q(S)-Q(S')-1] + B(S) \\ &= \alpha_1[Q(S)] + B(S). \end{aligned}$$

Thus, the Lemma is established for any set S of type S^1 which satisfies requirement 1. If S satisfies requirement 1' we may let $u=1$ and $x_1+y_1 i = X_1 + Yi$ in the above, or if S satisfies requirement 1" we may let $u=1$ and $x_1+y_1 i = X + Y_2 i$. The proof will then be the same, except that now $Q(S) - Q(S') - 1 = 0$ and Lemma 3.1 is not needed.

<u>Lemma 3.3</u>. If S is any set of type S^0 then

$$C(S) \geq \alpha_1[Q(S)+1] + B(S).$$

<u>Proof</u>: (i) Suppose $B(S)=0$. Then $C(S)=A(S) \geq \alpha_1[Q(S)+1] + B(S)$.

(ii) Suppose $B(S) \geq 1$. Then define $b_1^0 + b_2^0 i$, $g_1^0 + g_2^0 i$ as in the proof of Lemma 3.2. Let $Q(S)-A(S)=g_A$,

$B(S) = r \geq 1$, $Q(S) - C(S) = g_c \geq 1$. Again, the two sets $\{g_1^0 + g_2^0 i - (b_1 + b_2 i) \mid b_1 + b_2 i \in B \cap S\}$ and $\{(g_1 + g_2 i) - (b_1^0 + b_2^0 i) \mid g_1 + g_2 i \in S,\ g_1 + g_2 i \notin C,\ g_1 + g_2 i \neq g_1^0 + g_2^0 i\}$ give us $g_c - 1 + r$ distinct numbers not in A which now will be in S. The number $g_1^0 + g_2^0 i$ is not in the first of these sets since $0 \notin B \cap S$. If we assume $g_1^0 + g_2^0 i = (g_1 + g_2 i) - (b_1^0 + b_2^0 i)$ contained in the second set, then

$$g_1^0 = g_1 - b_1^0 ,$$

$$g_2^0 = g_2 - b_2^0 ,$$

and

$$g_1 + g_2 = g_1^0 + g_2^0 + b_1^0 + b_2^0$$

$$> g_1^0 + g_2^0 ,$$

which is impossible. Therefore, $g_1^0 + g_2^0 i$ is in neither of these two sets. However, $g_1^0 + g_2^0 i$ is in S but not in C, hence not in A.

Since S is now a set of the type R used to define α_1 we have

$$g_A \geq g_c + r,$$

$$Q(S) - g_c \geq Q(S) - g_A + r,$$

$$C(S) \geq A(S) + B(S)$$

$$\geq \alpha_1[Q(S) + 1] + B(S) .$$

We are now ready to prove Theorem 3.1. Let R be any set of the type described in the statement of the theorem. We will use induction on the number of elements of R which are not in C, such elements being referred to in the following as gaps of C in R.

(i) Suppose there is just one gap of C in R. Then we must have $t=1$ and $x_1 + y_1 i$ is that gap (see Figure 6). If $b_1 + b_2 i \in B \cap R$ then $(x_1+y_1 i)-(b_1+b_2 i)$ is in Q. Therefore, R is a set of type S^0 and we may apply Lemma 3.3.

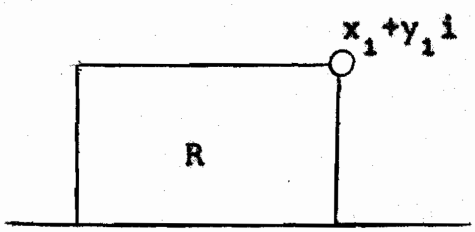

Figure 6

(ii) Suppose there are two gaps of C in R. (Consideration of this case is not necessary for the proof, but is included for greater clarity.) Then we may have $t=1$, in which case $x_1+y_1 i$ is one of the gaps and the other is a number $g_1 + g_2 i$ in $R_1 = R$ (see Figure 7a), or we may have $t=2$ and the two gaps are $x_1+y_1 i$ and $x_2+y_2 i$ (see Figure 7b).

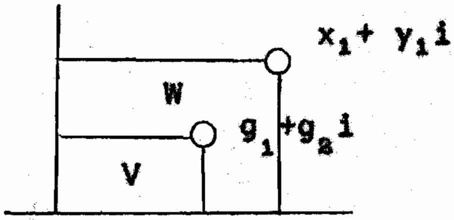

Figure 7a

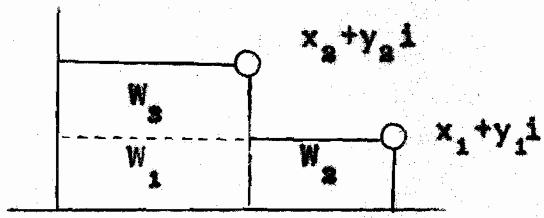

Figure 7b

In the case illustrated in Figure 7a let $V=\{x+yi \mid x+yi \in Q,$ $x \leq g_1,\ y \leq g_2,$ and let $W = R-V$. If $B(W) = 0$ then R is of type S^0. If $B(W) > 0$ then W is a set of type S^1 and V is of type S^0 . Hence,

$$C(R) = C(V) + C(W)$$
$$\geq \alpha_1[Q(V) + 1] + B(V) + \alpha_1[Q(W)] + B(W)$$
$$= \alpha_1[Q(R) + 1] + B(R).$$

In the case illustrated in Figure 7b let

$$W_1 = \{x+yi \mid x+yi \in Q,\ x \leq x_2,\ y \leq y_1\},$$
$$W_2 = \{x+yi \mid x+yi \in Q,\ x_2 < x \leq x_1,\ y \leq y_1\},$$
$$W_3 = \{x+yi \mid x+yi \in Q,\ x \leq x_2,\ y_1 < y \leq y_2\}.$$

If $B(W_2) = B(W_3) = 0$ then R is of type S^0. If $B(W_j) > 0$ where $j=2$ or 3 then W_j is of type S^1 and $W_k \cup W_1$ is of type S^0 where $W_k = W_2$ if $j=3$ and $W_k = W_3$ if $j=2$.

(iii) Assume the theorem is true for any set of the proper type in which there are less than k gaps of C for an integer $k \geq 3$. Let R be a set of the proper type in which there are k gaps of C, of which t are the points $x_1+y_1 i, \cdots, x_t+y_t i, 1 \leq t \leq k$.

If $B(R) = 0$ then R is a set of type S^0 and we are done. Hence, assume $B(R) \geq 1$.

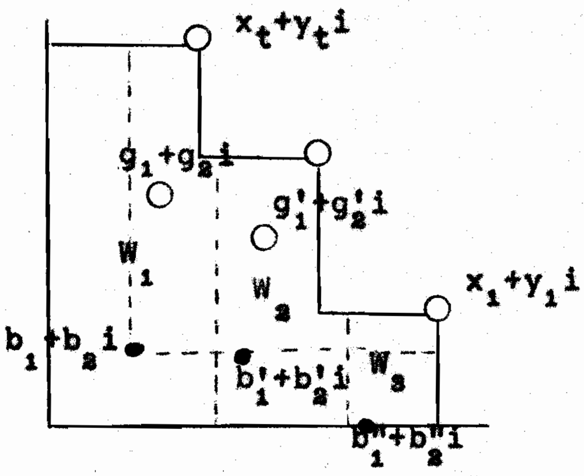

Figure 8a

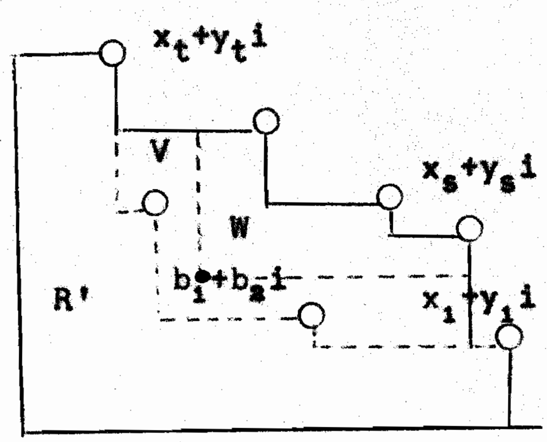

Figure 8b

Let b_2 be the largest real integer for which there exists a real integer x such that $x+b_2 i \in B \cap R$, and let b_1 be the largest such x. The set

$$W = \{x+yi \mid x+yi \in R,\ x \geq b_1,\ y \geq b_2\}$$

then contains precisely one element of B. Also, $b_1+b_2 i$

is in $R = R_1 \cup \cdots \cup R_t$ (for notation, see Definition 1 of the Introduction); hence, there exists an integer s such that $1 \leq s \leq t$ and $b_1 + b_2 i$ is in R_s. This implies that $x_s + y_s i$ is in W, that is, $Q(W) - C(W) \geq 1$.

Case 1. $Q(W) - C(W) = k$, or all k gaps of C in R are in W (see Figure 8a). This implies that $b_1 \leq x_t$ and $b_2 \leq y_1$. If $B(R) = 1$ then R is a set of type S^0 and we are done. If $B(R) > 1$ then there exist numbers $b_1^* + b_2^* i$ in $B \cap R$ with $b_1^* + b_2^* i \neq b_1 + b_2 i$, $b_2^* \leq b_2$. Let g_1 be the smallest real integer for which there exists a real integer g_2 such that $g_1 + g_2 i$ is a gap of C in W. If $b_1^* \leq g_1$ for all $b_1^* + b_2^* i$ in $B \cap R$ then again R is a set of type S^0 and we are done.

Otherwise, let b_1' be the smallest real integer for which there exists an integer b_2' such that $b_1' + b_2' i \in R \cap B$ and $b_1' > g_1$. Then the set

$$W_1 = \{x+yi \mid x+yi \in R,\ x < b_1'\}$$

is of type S^0. (Note that W_1 contains $b_1 + b_2 i$ and $g_1 + g_2 i$, so $B(W_1) \geq 1$, $Q(W_1) - C(W_1) \geq 1$.)

There is at least one gap of C in $R - W_1$, namely $x_1 + y_1 i$. Let g_1' be the smallest real integer for which there exists g_2' such that $g_1' + g_2'i$ is a gap of C in $R-W_1$. If we have $b_1^* \leq g_1'$ for all $b_1^* + b_2^*i$ in $B \cap (R-W_1)$ then $R-W_1 = W_2$ is of type S^1. If not, let b_1'' be the smallest real integer for which there exists a real integer b_2'' such that $b_1'' + b_2''i \in B \cap (R-W_1)$ and $b_1'' > g_1'$. Then the set

$$W_2 = \{x+yi \mid x+yi \in R-W_1, \quad x < b_1''\}$$

is of type S^1. (Note that W_2 contains $b_1' + b_2'i$ and $g_1' + g_2'i$.) We may continue this process until we have constructed $\underline{n}$ disjoint sets $W_1, W_2, \cdots, W_n$ such that $R = W_1 \cup \cdots \cup W_n$, W_1 is of type S^0; $W_2, \cdots, W_n$ are of type S^1. Then

$$\begin{aligned} C(R) &= C(W_1) + C(W_2) + \cdots + C(W_n) \\ &\geq \alpha_1[Q(W_1) + 1] + B(W_1) + \alpha_1[Q(W_2)] + B(W_2) \\ &\quad + \cdots + \alpha_1[Q(W_n)] + B(W_n) \\ &= \alpha_1[Q(R)+1] + B(R). \end{aligned}$$

Case 2. $1 \leq Q(W) - C(W) < k$ (see Figure 8b). Then $R-W$ contains at least one but no more than $k-1$ gaps of C. Let h_1' be the largest real integer for which there exists a real integer x such that $x + h_1'i$ is a gap of C in $R-W$, and let h_1 be the largest such x. (Note that $h_1 + h_1'i = x_t + y_t i$ if $b_1 > x_t$.) Let h_2' be the largest real integer, if one exists, for which $h_2' < h_1'$ and there exists $x > h_1$ such that $x + h_2'i$ is a gap of C in $R-W$, and let h_2 be the largest such x. Continue this process as long as possible. We will then have m gaps of C in $R-W$, $h_1 + h_1'i, \cdots, h_m + h_m'i$, $k-1 \geq m \geq 1$, such that $h_1' > \cdots > h_m'$ and $h_1 < \cdots < h_m$ if $m > 1$. (If $b_2 > y_1$, then $h_m + h_m'i = x_1 + y_1 i$.)

Let $R_j' = \{x+yi \mid x+yi \in Q,\ x \leq h_j,\ \ y \leq h_j'\}$, $j = 1, \cdots, m$, and let $R' = R_1' \cup \cdots \cup R_m'$. Since R' is a set of the type described in the statement of Theorem 3.1 in which the number of gaps of C is less than k, we have

$$C(R') \geq \alpha_1[Q(R') + 1\ ] + B(R')$$

by the induction hypothesis.

Let $V = R - (W \cup R')$. Because of the way in which the $h_1 + h_1'i, \cdots, h_m + h_m'i$ were chosen, there can be no gaps of C in V. Elements $b_1^* + b_2^*i$, if any exist, in $B \cap V$ will have $b_2^* \leq b_2$. Hence, either $W \cup V$ is already a set W_1 of type S^1, or we may partition $W \cup V$ into n disjoint subsets $W_1, \cdots, W_n$, each of type S^1, in precisely the same manner as was done for the set R in Case 1.

This gives us

$$\begin{aligned} C(R) &= C(R') + C(W_1) + \cdots + C(W_n) \\ &\geq \alpha_1[Q(R')+1]+B(R')+\alpha_1[Q(W_1)]+B(W_1) \\ &\quad + \cdots + \alpha_1[Q(W_n)] + B(W_n) \\ &= \alpha_1[Q(R) + 1] + B(R), \end{aligned}$$

which completes the proof.

The hypothesis of Theorem 3.1 that the numbers $x_1+y_1 i,\cdots, x_t+y_t i$ be gaps of C is analogous to the requirement in Mann's theorem that n be a gap of C. We can now see that this is somewhat more than is necessary, however; in constructing the disjoint subsets $W_1, \cdots, W_n$ of R in Case 1 and of $W \cup V$ in Case 2 we had a point $b_1+ b_2 i$ of B and a gap $g_1+ g_2 i$ of C in W_1, a point $b_1'+ b_2' i$ of B and a gap $g_1'+ g_2' i$ of C in $W_2, \cdots,$ a point $b_1^{(n-2)}+ b_2^{(n-2)} i$ of B and a gap $g_1^{(n-2)}+ g_2^{(n-2)} i$ of C in W_{n-1}, and a point $b_1^{(n-1)}+ b_2^{(n-1)} i$ of B in W_n (with $b_1^{(n-1)} > g_1^{(n-2)}$, so that $b_1^{(n-1)}+ b_2^{(n-1)} i$ could not be included in W_{n-1}). Let S be the smallest subscript such that $x_S+y_S i$ is in W, $1 \leq s \leq t$. From the method of construction of W_n we know that $x_s+y_s i$ is in W_n, $x_s \geq b_1^{(n-1)}$, and $y_s \geq b_2^{(n-1)}$. Thus the requirement that $x_1+ y_1 i, \cdots, x_t+ y_t i$ be gaps of C in R is then sufficient to insure obtaining a last set W_n of type S^1. It would be enough for our purposes, however, to require that R be so chosen that for any point $b_1+ b_2 i$ in $B \cap R$ there exist a gap $g_1+ g_2 i$ of C in R such that $b_1 \leq g_1$, $b_2 \leq g_2$, and that there exist at least one gap of C in R even if $B(R) = 0$. (This last requirement is necessary, since if $B(R) = 0$ then $C(R) = A(R)$, and we cannot conclude that $A(R) \geq \alpha_1 [Q(R) + 1]$ unless $A(R) < Q(R)$.)

Corollary 3.1. Define β_1 for the set B in the same manner as α_1 was defined for the set A. Let $\gamma_1' =$ glb $\frac{C(R)}{Q(R)+1}$ taken over all R satisfying the hypotheses of Theorem 3.1. Then

$$\gamma_1' \geq \alpha_1 + \beta_1 .$$

Proof: For any set R satisfying the hypotheses of Theorem 3.1 we have

$$C(R) \geq \alpha_1[Q(R)+1] + B(R) ,$$

and

$$\frac{C(R)}{Q(R)+1} \geq \alpha_1 + \frac{B(R)}{Q(R)+1} > \alpha_1 + \beta_1 .$$

Therefore,

$$\gamma_1' \geq \alpha_1 + \beta_1 .$$

Corollary 3.2. Let β be the density of B (Definition 1, Introduction) and $\gamma' = \text{glb}\,\frac{C(R)}{Q(R)}$, taken over all R satisfying the hypotheses of Theorem 3.1. Then

$$\gamma' \geq \alpha_1 + \beta .$$

Proof: We have

$$\begin{aligned} C(R) &\geq \alpha_1[Q(R) + 1] + B(R) \\ &\geq \alpha_1[Q(R)] + B(R), \end{aligned}$$

and

$$\begin{aligned} \frac{C(R)}{Q(R)} &\geq \alpha_1 + \frac{B(R)}{Q(R)} \\ &\geq \alpha_1 + \beta \end{aligned}$$

for any set R satisfying the hypotheses of Theorem 3.1. Therefore,

$$\gamma' \geq \alpha_1 + \beta .$$

The following example shows that γ_1' may equal $\alpha_1 + \beta_1$ and thus that the conclusion of Corollary 3.1 is, in a sense, the best possible.

<u>Example 1</u>. Let $A = B = \{1, 2, i, 1+i,$ all $x+yi$ in Q with $x \geq 5$ or $y \geq 2\}$. Then $C = \{1, 2, 3, 4, i, 1+i, 2+i, 3+i,$ all $x+yi$ in Q with $x \geq 5$ or $y \geq 2\}$, and we have $\alpha_1 = \beta_1 = 2/5$, $\gamma_1' = 4/5$.

The proof of Theorem 3.1 given here involved an induction on the number of Gaussian integers in the set R which were not in the sum set C, the induction hypothesis being applied only to a subset R' of R which was known to contain fewer of these gaps of C than did R. Mann's proof of the corresponding theorem for sets of positive integers, $C(n) \geq \alpha_1(n+1) + B(n)$ for n not in C, is carried out by induction on the number of gaps of C in the interval $I = \{x \mid 1 \leq x \leq n\}$. However, his proof involves the application of the induction hypothesis in two different ways; in one case it is applied to a smaller interval which is known to contain one less gap of C than I, and in the other case it is applied to a new set C_1 which contains C as a proper subset.

Mann's approach yields a quite simple and elegant proof for the theorem in the one-dimensional case, and it is only natural to attempt to apply similar methods to the sets of Gaussian integers. Difficulties are soon encountered, however. It appears that the extension to the

Gaussian integers of Mann's method of transforming C to the new set C_1 requires the existence in R of a gap $x_0 + y_0 i$ of C such that for any other gap $g_1 + g_2 i$ of C in R we have $g_1 \leq x_0$ and $g_2 \leq y_0$. This precludes consideration of sets R of the type described in the statement of Theorem 3.1 for which $t>1$. If we restrict ourselves to sets R for which $t=1$ we find that the case in which Mann transformed C to the new set C_1 in one dimension now yields to almost precisely the same treatment, but the other case presents difficulties which seem to the author to be insurmountable since apparently use of the induction hypothesis will require consideration of sets R' for which $\underline{t}$ may be greater than 1.

Attempts to find other ways of transforming the set C to a new set C_1 to which the induction hypothesis could be applied were also unsuccessful. Among these attempts was one which led to a new proof of the theorem for sets of positive integers, the second of the two proofs given in the Appendix to this thesis. A transformation of the type described therein can be successfully carried out for sets of Gaussian integers whenever the set R has been so chosen so that for some subscript s, $1 \leq s \leq t$, there exists a number $b_1 + b_2 i$ in B such that $x_{s+1} < b_1 \leq x_s$ (or $0 \leq b_1 \leq x_s$ if $s=t$) and

$y_{s-1} < b_2 \leq y_s$ (or $0 \leq b_2 \leq y_s$ if $s=1$), and such that there exists a gap $g_1 + g_2 i$ of C not equal to $x_s + y_s i$ with $b_1 \leq g_1$ and $b_2 \leq g_2$. It is not necessary to restrict ourselves to sets R in which $t=1$ in order to use this transformation; however, the case in which the transformation cannot be used has not yet been resolved.

There are, of course, other approaches to this theorem. A "counting process" devised by Besicovitch (1, p. 246-248) has been used in several ways to prove the theorem for the one-dimensional case (9, p. 20-29) and may perhaps be extendible to the Gaussian integers. Attempts to apply a transformation developed by Dyson (4, p. 8-14) to the theorem for the one-dimensional case have so far not been successful (9, p. 30-37), but this transformation cannot be completely dismissed as a possibility.

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APPENDIX

APPENDIX

TWO NEW PROOFS OF A THEOREM OF MANN

The investigations which led to the proof of Theorem 3.1 produced two new proofs of the corresponding theorem for sets of positive integers. The first of these is just a specialization of the proof presented in Chapter 3; however, this specialization is so much simpler that it is of interest to have it presented separately. The second proof involves the use of a new transformation on the set B. We restate the theorem:

<u>Theorem A.1</u>. Let A and B be two sets of positive integers, and $C = A+B$. Let α_1 be the Erdos density of A. Then

$$C(n) \geq \alpha_1(n+1) + B(n)$$

for any positive integer n not in C.

1. We will apply our first new method of proof to Theorem A.2 below, which is the theorem actually proved by Mann, and then show that Theorem A.2 implies Theorem A.1.

<u>Theorem A.2</u>. Let A and B be two sets of non-negative integers with 0 in A, 1 in B. Let C be

the set of all numbers of the form $a+b$ where a is in A, b is in B. Let α_1 be the Erdős density of A. Then

$$C(n) \geq \alpha_1 n + B(n)$$

for any positive integer n not in C. (Note: In the definition of α_1, $A(n)$ is still the number of positive integers in A which are less than or equal to n. Also, $B(n)$ and $C(n)$ are interpreted similarly.)

Proof: Let I be the set of all positive integers, and for any set S of positive integers let $A(S)$ denote the number of integers in $A \cap S$. Choose any positive integer n_0 which is not in C. Let $1 = b_1 < b_2 < \cdots$ be all the integers in B.

Let r be the maximum j such that $C(S^j)=I(S^j)$ where $S^j = \{x \mid b_1 \leq x \leq b_j\}$. Then there exists a gap n of C such that $b_r < n < b_{r+1}$. (If B is a finite set of r elements we have only $b_r < n$.)

Let n_1 be the largest gap of C such that $b_r < n_1 < b_{r+1}$ or let $n_1 = n_0$, whichever is smaller. (If b_{r+1} does not exist, then $n_1 = n_0$.) If $n_1 < n_0$ let $S_1 = \{x \mid x \in I,\ b_1 \leq x \leq b_{r+1}-1\}$. If $n_1 = n_0$ let

$S_1 = \{x \mid x \in I,\ b_1 \le x \le n_0\}$. Let $S_1' = \{x - b_1 \mid x \in S_1,$ $x > b_1\}$. The r integers $n_1 - b_1 > \cdots > n_1 - b_r$ are gaps of A in S_1'. If we let g_C be the total number of gaps of C in S_1 then the set of all integers $n - b_r$ such that n is a gap of C in S_1, $n \neq n_1$, gives $g_C - 1$ gaps of A in S_1'. These are each less than $n_1 - b_r$, hence distinct from the other r gaps. Letting g_A be the total number of gaps of A in S_1', we have thus shown

$$g_A \ge r + g_C - 1. \tag{1}$$

If k is the smallest positive integer not in A then $k \le n_1 - 1$, since $n_1 - 1$ is not in A. Therefore, k is in S_1'. Hence, from (1) we have

$$\begin{aligned} I(S_1) - g_C &\ge I(S_1) - g_A + r - 1 \\ &= I(S_1') - g_A + r . \end{aligned}$$

This is equivalent to

$$\begin{aligned} C(S_1) &\ge A(S_1') + B(S_1) \\ &\ge \alpha_1[I(S_1') + 1] + B(S_1) \\ &= \alpha_1[I(S_1)] + B(S_1). \end{aligned}$$

If $n_1 = n_0$ we are done. If $n_1 < n_0$ we can form a set S_2 as we did S_1, using b_{r+1} instead of b_1 and

b_{r+t} instead of b_r, t being the maximum h such that $C(S^h) = I(S^h)$ where $S^h = \{x \mid b_{r+1} \leq x \leq b_{r+h}\}$. We let $S_2' = \{x-b_{r+1} \mid x \in S_2, x > b_{r+1}\}$. Proceeding in the same manner as before, and noting that $k \in S_2'$ since $k \leq n_2 - b_{r+1}$ where n_2 is the gap of C in S_2 which corresponds to n_1 in S_1, we obtain

$$C(S_2) \geq \alpha_1[I(S_2)] + B(S_2).$$

Eventually we must arrive at a set S_j whose right end point is n_0. Then

$$\begin{aligned} C(n_0) &= C(S_1) + \cdots + C(S_j) \\ &\geq \alpha_1[I(S_1)] + \cdots + \alpha_1[I(S_j)] \\ &\quad + B(S_1) + \cdots + B(S_j) \\ &= \alpha_1 n_0 + B(n_0) . \end{aligned}$$

We must now show that Theorem A.2 implies Theorem A.1, which has been shown before (9, p. 9). Let A and B be two sets of positive integers, $C = A + B$, $A' = A \cup \{0\}$, $B' = B \cup \{0\}$, $C' = C \cup \{0\}$. Then C' is the set of all $a+b$ where a is in A' and b is in B'. Let B_1 be the set of all $b+1$ with b in B', C_1 the set of all $a+b+1$ with a in A' and $b+1$ in B_1. If n is a gap in C' (therefore a gap in C), then $n+1$

is a gap in C_1. The sets A', B_1, C_1 satisfy the hypotheses of Theorem A.2, and so we must have

$$C_1(n+1) \geq \alpha_1(n+1) + B(n)$$

where α_1 is the Erdős density of A' or, equivalently, A. But $C_1(n+1) = C(n) + 1$ and $B_1(n+1) = B(n) + 1$. Therefore,

$$C(n) \geq \alpha_1(n+1) + B(n)$$

which implies Theorem A.1.

2. We will apply our second new method of proof to Theorem A.1 directly. Let A and B be any two sets of positive integers, $C = A + B$. Let k_1 be the smallest positive integer not in A.

Suppose n is a positive integer not in C such that $C(n) = n-1$. Since A is a subset of C we must have $k_1 \leq n$.

If x is any positive integer such that $1 \leq x \leq n$ then either x is in B, or x is of the form $n-a$ for some a in A and less than n, or neither, but not both. The integer n is neither in B nor of the form $n-a$. Therefore,

$$\begin{aligned} C(n) = n-1 &\geq A(n) + B(n) \\ &\geq \alpha_1(n+1) + B(n) . \end{aligned}$$

Thus the theorem is established for the first gap of C, and we proceed by induction on the number of gaps of C which are less than or equal to a given gap. To be more precise, we assume that if $C(n) \geq n - (r-1)$ for some integer $r \geq 2$ and n not in C, then $C(n) \geq \alpha_1(n+1) + B(n)$.

Let $n_1 < n_2 < \cdots < n_r$ be the first r gaps of C, so that $C(n_r) = n_r - r$. We distinguish two cases.

(i) Suppose $n_r - n_{r-1} > k_1$. We have

$$(1) \qquad C(n_{r-1}) \geq \alpha_1(n_{r-1}+1) + B(n_{r-1})$$

by the induction hypothesis. Also, if $n_{r-1} < x \leq n_r$ for any integer x then either x is in B, or x is of the form $n_r - a$ for some a in A such that $1 \leq a \leq n_r - n_{r-1} - 1$, or neither, but not both. Again, n_r is neither. We have $n_r - n_{r-1} - 1 \geq k_1$, so we may write

$$(2) \qquad \begin{aligned} C(n_r) - C(n_{r-1}) &= n_r - n_{r-1} - 1 \\ &\geq A(n_r - n_{r-1} - 1) + B(n_r) - B(n_{r-1}) \\ &\geq \alpha_1(n_r - n_{r-1}) + B(n_r) - B(n_{r-1}). \end{aligned}$$

Adding the inequalities (1) and (2) gives the desired result,

$$C(n_r) \geq \alpha_1(n_r + 1) + B(n_r).$$

(ii) Suppose $n_r - n_{r-1} \leq k_1$. We may assume that B is the largest set such that $A + B = C$ in the interval from 1 to n_r or, in other words, that B consists of all integers in this interval except those of the set

$$S = \{x \mid x = n_i - a, \ 1 \leq i \leq r, \ a \in A \cup \{0\}, \ a < n_i\},$$

for if we can prove $C(n_r) \geq \alpha_1(n_r + 1) + B(n_r)$ for this B then it will also be established for any set B' such that $A + B' = C$, since necessarily B' is a subset of B and $B(n_r) \geq B'(n_r)$.

Let $k_1 < k_2 < \cdots < k_t$ be all the gaps of A which are less than n_r. All the gaps of C are also gaps of A, so $t \geq r-1 \geq 1$.

If $B(n_r) = 0$ then $C = A$ in the interval from 1 to n_r, and $C(n_r) = A(n_r) \geq \alpha_1(n_r + 1) + B(n_r)$. Hence, we assume $B(n_r) \geq 1$. If x is in B, $1 \leq x < n_r$, then there exists a positive integer y such that $x + y = n_r$. If y is in A then $x + y$ is in C. However, n_r is not in C. Therefore, y is a gap of A and there exists j such that $1 \leq j \leq t$ and $y = k_j$. This implies that the elements of B will be found among those of the set $\{n_r - k_j \mid j = 1, \cdots, t\}$. The largest element of this set is $n_r - k_1$, which we will call x_1. We are assuming $n_r - n_{r-1} < k_1$, so $x_1 = n_r - k_1 \leq n_{r-1}$. If $x_1 = n_{r-1}$ then x_1 is not in B. Suppose $x_1 < n_{r-1}$. Then there exists y_1 such that $x_1 + y_1 = n_{r-1}$, and $y_1 = n_{r-1} - x_1 < n_r - x_1 = k_1$. Therefore, y_1 is in A. If x_1 were in B we would have n_{r-1} in C. Consequently, x_1 is not in B, and the assumption $B(n_r) \geq 1$ implies $t \geq 2$. Let j be the smallest subscript such that $x_j = n_r - k_j$ is in B. (That is, x_j is the largest element of B less than n_r.)

If $x_j < n_i$ for some $i \leq r$, then there exists k_h such that $x_j + k_h = n_i$, and $k_h = n_i - x_j \leq n_r - x_j = k_j$ implies that $1 \leq h \leq j$. Hence, there can be at most j gaps of C greater than x , and we must have $n_{r-j} < x_j$.

This implies $n_{r-j} < n_r - k_{j-1} < \cdots < n_r - k_1 \leq n_{r-1}$. We know that each of the integers $n_r - k_{j-1}, \cdots, n_r - k_1$ is not in B. Let $B_1 = B \cup \{n_r - k_{j-1}, \cdots, n_r - k_1\}$, and let $C_1 = A + B_1$.

For each $n_r - k_h$, $1 \leq h \leq j - 1$, we have $n_r - k_h$ in S, and hence there exists $n_i - a$ such that

$1 \leq i \leq r$, $a = 0$ or $a \in A$, such that $n_r - k_h = n_i - a > n_{r-j}$. This implies $n_i > n_{r-j}$. We also know $i \neq r$, so n_i can only be one of the gaps $n_{r-(j-1)}, \cdots, n_{r-1}$. Since $n_i - a$ is in B_1, $a = 0$ or $a \in A$, we have n_1 in C_1. Hence, we have added at least one element, but not more than $j - 1$ elements, to C in forming C_1. Thus we have

$$B_1(n_r) - B(n_r) = j - 1 \geq C_1(n_r) - C(n_r) \geq 1,$$

and

(3) $$C(n_r) - C_1(n_r) \geq B(n_r) - B_1(n_r).$$

By the induction hypothesis,

(4) $$C_1(n_r) \geq \alpha_1(n_r + 1) + B_1(n_r) .$$

Adding (3) and (4), we obtain

$$C(n_r) \geq \alpha_1(n_r + 1) + B(n_r) .$$