AN ABSTRACT OF THE THESIS OF

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(Name) (Degree)

in MECHANICAL ENGINEERING presented on Dec. 11, 1973
(Major) (Date)

Title: ANALYSIS OF HEAT TRANSFER, INCLUDING AXIAL CONDUCTION, FOR LAMINAR TUBE FLOW WITH ARBITRARY CIRCUMFERENTIAL WALL HEAT FLUX

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Abstract approved: ______________________
Dr. James R. Welty

This thesis is concerned with the analysis of heat transfer in a tube with forced flow under conditions of an arbitrary variation of wall heat flux both axially and circumferentially. This total study is separated into two distinct problems which are presented separately.

The first is the case of a Newtonian fluid in laminar flow with allowance made for the inclusion of axial heat conduction, viscous heat dissipation and heat generation. Secondly, the problem of laminar flow of a non-Newtonian fluid is considered. Axial conduction is not included in this problem since it is likely negligible in those cases where non-Newtonian effects are significant.

Heretofore, no general method has been available for obtaining solutions to these problems. Analytical results are given in such generality and completeness that many of the previously reported

work in the heat transfer literature in laminar tube flow are limiting cases of the present work.

In the first problem, the solution is expanded in a power series form that accounts for any arbitrary variation of wall heat flux around the circumference that can be expressed in terms of a Fourier series expansion. Substitution of this series into the energy equation leads to an eigenvalue problem. The first 12 eigenvalues and eigenfunctions have been obtained numerically. The resulting eigenfunctions are not orthogonal and therefore the power series expansion coefficients cannot be obtained by the usual analytical schemes. A least squares method was used to determine these coefficients.

For the limiting problem of uniform wall heat flux around the circumference with the inclusion of axial conduction, the eigenfunctions and eigenvalues are in excellent agreement with previously reported work; however, two additional considerations were made to correct errors made in the heat transfer literature. The first was the determination of coefficients of the non-orthogonal power series expansion and, second was the inclusion of the nonvanishing axial conduction term at the tube entrance which was not included in earlier asymptotic expressions for the temperature. Both of these considerations are included in the numerical procedures in this work.

The problem where wall heat flux varies circumferentially but axial fluid conduction is neglected is another limiting case of the present work. For the special case of uniform wall heat flux, the eigenfunctions, eigenvalues, and expansion coefficients agree well with those in the existing literature.

The same analytical techniques were employed for the second problem. The resulting eigenfunctions for this problem are orthogonal, therefore the power series expansion coefficients were determined by utilizing the orthogonality property of the eigenfunctions. For the special case of power-law pseudo-plastic fluids with uniform wall heat flux the eigenfunctions, eigenvalues, and the expansion coefficients are in excellent agreement with previously reported values.

Finally, by an illustrative example, it was concluded that circumferential wall heat flux variation has a pronounced effect in both Newtonian and non-Newtonian heat transfer results.

Analysis of Heat Transfer, Including Axial Conduction, for Laminar Tube Flow with Arbitrary Circumferential Wall Heat Flux

by

Mohammad Faghri

A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

Completed December 1973

Commencement June 1974

APPROVED:

Redacted for privacy

Professor of Mechanical Engineering
in charge of major

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Date thesis is presented December 11, 1973

Typed by Clover Redfern for Mohammad Faghri

To my father, to whom I owe my

undying gratitude for his

encouragement, sacrifice,

and advice.

ACKNOWLEDGMENTS

I would like to take this opportunity to express my sincere appreciation to my major professor and thesis advisor, Dr. James R. Welty, who guided, aided, and encouraged me in every aspect of this work. Over the past three years, I have come to admire his patience and high scientific ability.

I am indebted to Dr. Robert E. Wilson, Dr. Lorin R. Davis, Dr. Charles E. Smith, and Dr. Robert W. Thresher of the department of Mechanical Engineering and Dr. Mysore N. L. Narasimhan, Dr. John W. Lee and Dr. Ronald B. Guenther of the department of Mathematics who were of great help to the author, both during his graduate work, and during the course of this study.

I wish to thank my brother Amir, who assisted me with the numerical and graphical methods in this work.

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1. NEWTONIAN PROBLEM

1.1 Introduction

1.1.1 Literature Review

Research in the area of laminar forced convection to conducting fluids (i.e., liquid metals) in ducts, either with uniform wall temperature or uniform wall heat flux has been relatively sparse. This is because of the complexities encountered in the analysis of such problems.

Most work dealing with the asymptotic analysis of conducting fluids has involved problems with the uniform wall temperature. For this case the fully-developed Nusselt number is affected by axial conduction; i.e., $Nu_t = f(Pe)$. This is demonstrated by the work of Pahor and Strnad [41] who determined the asymptotic Nusselt number in pipe flow as a function of the Peclet modulus. Labuntsov [27], by a power series method, arrived at similar results. In a later work, Pahor and Strnad [42] extended their work to consider a semi-infinite parallel plate geometry. They expressed their solution in terms of confluent hypergeometric functions and gave expressions for the relation $Nu_t = f(Pe)$ for very high and very low Peclet numbers. Ash and Heinbockel [2] improved and generalized the work of Pahor et al., and also considered the non-orthogonality of the eigenfunctions in the determination of the expansion coefficients.

For the case of uniform wall heat flux, the axial conduction has no effect on the asymptotic Nusselt number since the fully-developed fluid temperature increases linearly. This was verified experimentally by Petukhov and Yushin [45], Eckert and Peterson [10] and Emery and Bailey [11] who independently showed the asymptotic Nusselt number for the case of uniform wall heat flux in conducting fluids to be 4.36 in contrast to the previous experimental work by Johnson, Hartnett and Clabaugh [23] who obtained values as low as 1 for the Nusselt number.

The case of slug flow with a thermally developing temperature profile has been the subject of a number of papers in the axial conduction literature. In this simplified case one can achieve a separable solution to the energy equation with the inclusion of the axial conduction term. This leads to a simple ordinary differential equation in the direction perpendicular to flow for which the eigenfunctions are orthogonal. Hence the constants for the series expansion are obtained in the usual analytical manner. Wilson [72], as early as 1904, solved this problem for parallel plates subject to a step change in wall temperature. Poppendeik and Harrison [48] explored both analytically and experimentally the same problem for parallel plates and tubes. Schneider [55] examined the analogous problem in his classical paper for both parallel plates and tubes under conditions of finite wall resistance, and for both a uniform and a step discontinuity in the ambient temperature. The calculated mean fluid temperatures, Nusselt numbers, and thermal entry lengths were compared with corresponding predictions for the case of no axial conduction, and it was suggested that the effect of axial conduction was negligible for Peclet numbers larger than 100. Taitel, Bentwich and Tamir [66] investigated the role of upstream and downstream boundary conditions on the heat or (mass) transfer for a two-dimensional channel. Solutions were obtained in the form of Fourier integrals the inversions of which were carried out numerically. As an example, three types of thermal boundary conditions were considered. In all cases there was a central heating section with the fluid flowing into an insulated semi-infinite conduit. The situations prevailing upstream of this section varied. It was found that these situations have a substantial influence when the Peclet number is low, and the heating section is short.

Extensions of the axial conduction problems with the assumption of a parabolic velocity profile were achieved by various methods. In this case it can be verified that the classical method using separation of variables fails; nevertheless, one may assume a series expansion as a product of an exponential function and some unknown function of radius. Substitution of this series into the energy equation leads to an ordinary differential equation in the radial direction. This equation is no longer a usual "Sturm-Liouville" type differential equation, in fact it is a more "generalized Sturm Liouville system" [60] with non-orthogonal eigenfunctions. Hence, the determination of expansion coefficients becomes extremely difficult.

To overcome the difficulties encountered in the evaluation of the expansion coefficients, Millsaps and Pohlhausen [38] and later Singh [61] expanded the solution to the ordinary differential equation in the radial direction as an infinite series of Bessel functions for the case of pipe flow with uniform wall termperature. The same idea was utilized by Agrawal [1] for parallel plates subject to a step change in wall temperature. He represented the eigenfunctions by a Fourier sine series. The major drawback in the above method is directly related to the computational difficulties encountered in evaluating the eigenvalues, since it requires solving determinants of infinite order. Furthermore, the higher eigenvalues are very hard to evaluate; therefore, only the first five of them were reported.

Jones [24] examined the case of tube flow in a study that was similar to that of Singh [61], but he considered a step change in wall temperature and solved the governing differential equation by Laplace transformation.

Schmidt and Zeldin [54] used a finite difference technique and considered both parallel plates and tubes subjected to the condition of uniform wall temperature. They presented local and average Nusselt numbers and mean fluid temperatures as functions of Peclet number. It was observed that the values of fully-developed Nusselt numbers are increased as the Peclet number is decreased. This trend is in agreement with that predicted by Labuntsov [27]. These authors also established a criterion which is useful for predicting the conditions under which axial conduction may be ignored.

Nelson, Rust, and Iachetta [40] obtained a numerical solution for heat conducting and heat generating fluids flowing between isothermal parallel plates. Their results indicated that axial conduction increases or decreases the Nusselt number depending on the heat source strength.

The literature that has been discussed so far concerns either a uniform or a step change in the wall temperature. We shall now discuss the very few papers that consider either a uniform or a step change in wall heat flux.

Hsu [19] solved the problem for a tube with fully-developed velocity profile and uniform wall heat flux. He expressed his solution in terms of an infinite series and the radial ordinary differential equation resulting from his analysis was integrated numerically by Runge-Kutta methods to obtain the eigenvalues and corresponding eigenfunctions. He made two errors in his analysis of the problem.

First, he determined the coefficients of his non-orthogonal expansion by assuming the eigenfunctions to be orthogonal with respect to a known weighting function and neglected the non-orthogonal cross terms. Secondly, the non-vanishing axial conduction term at the tube entrance was not included in the asymptotic expression for the temperature solution. Pirkle and Sigillito [46,47] mentioned the errors made by Hsu, but they did not suggest any remedy for the first error and they did not include any numerical correction for the coefficients of expansion.

Hennecke [17] used a finite difference technique and considered a tube geometry under the conditions of both a uniform step change in wall temperature and wall heat flux. He showed that axial conduction upstream of the heated section plays a decisive role in the heat transfer. Recently, Hsu [21] analyzed the same problem that was solved by Hennecke and considered a step change in wall heat flux for both parallel plates and tubes. He developed a series solution and the eigenvalues and the corresponding eigenfunctions were obtained numerically. Also a set of orthogonal eigenfunctions were constructed from non-orthogonal eigenfunctions utilizing the Gram-Schmidt orthonormalization procedure. The expansion coefficients were then determined using the constructed orthogonal eigenfunctions. His results are in excellent agreement with those obtained numerically by Hennecke. In another publication, Hsu [20] analyzed the flow through an annulus having an adiabatic inner wall and an outer wall subjected to a step change in heat flux, utilizing similar mathematical techniques.

The simultaneous development of thermal and velocity profiles for ordinary fluids was extended to liquid metals for a tube geometry and uniform wall heat flux by McMordie and Emery [34]. These authors solved the governing momentum and energy differential equations numerically employing the finite difference methods. This problem was further extended by Loc [28] to consider the time-dependent heat transfer phenomenon in the entrance region of a circular tube. The solution is based on three different computing schemes: a numerical method, and two analog methods.

The integral method was extended to problems with axial conduction by Taitel and Tamir [65]. They demonstrated that heat (or mass) transfer solutions can be obtained in closed form fashion and with satisfactory accuracy. Results for the Graetz problem [8, 56] and other problems with axial diffusion were reported.

Literature in the area of non-uniform or arbitrary variation of wall heat flux or wall temperature axially or circumferentially in liquid metals is indeed sparse and with the exception of the paper by Burchill, Jones, and Stein [9], there exists no other published work. These authors examined a symmetrical annular space with arbitrary axial variation of heat flux at the walls of a section between two infinitely long adiabatic inlet and outlet sections. They considered turbulent flow with a slug flow velocity profile.

1.1.2 Present Investigation

The objective of this investigation is to solve analytically the problem of heat transfer in a circular tube with an arbitrary circumferential wall heat flux for the case of a developing temperature profile including the effect of axial conduction.

The solution is expanded in a power series form that accounts for any arbitrary variation of heat flux around the circumference that can be expressed in terms of a Fourier expansion. Substitution of this series into the energy equation leads to an eigenvalue problem. The first 12 eigenvalues and the corresponding eigenfunctions have been obtained numerically. The resulting eigenfunctions are not orthogonal and therefore the power series expansion coefficients cannot be obtained by usual analytical schemes. A least squares method was used in this work to determine these coefficients. The final solution was then generalized for any arbitrary variation of wall heat flux in the axial direction.

For the limiting problem of uniform wall heat flux around the circumference and a finite Peclet number, the eigenfunctions and eigenvalues reduces to those of Hsu's [19]; however, two additional considerations are made here, the first being the determination of coefficients of the non-orthogonal power series expansion and second, the consideration of the nonvanishing axial conduction term at the tube entrance which was not considered by Hsu. Both of these considerations are included in the numerical procedures in this paper.

The problem where wall heat flux varies circumferentially but axial fluid conduction is neglected is another limiting case of the present work. The first 12 eigenvalues and eigenfunctions and expansion coefficients are included for values of the parameter p ranging from $p = 0$ (i.e., the case of constant wall heat flux condition) to $p = 5$ (up to fifth harmonic variation in the circumferential wall heat flux). For the special case of uniform wall heat flux ($p = 0$), the eigenfunctions, eigenvalues, and expansion coefficients agree well with values reported by Siegel, Sparrow, and Hallman [59] and Hsu [18].

Finally, a simple result has been obtained for a cosine heat flux variation around the tube periphery which illustrates all the limiting cases and shows how simultaneous influence of circumferential wall heat flux and axial fluid conduction may have a pronounced effect on the wall temperature in a liquid metal cooled reactor.

1.2 Formulation of Problem

1.2.1 Energy Equation and Boundary Conditions

The problem to be considered is represented schematically in Figure 1.1. We consider a viscous conducting fluid flowing in steady, laminar, incompressible fashion through the tube of constant radius, r_0. The wall heat flux varies circumferentially according to the general function, $q(\phi)$.

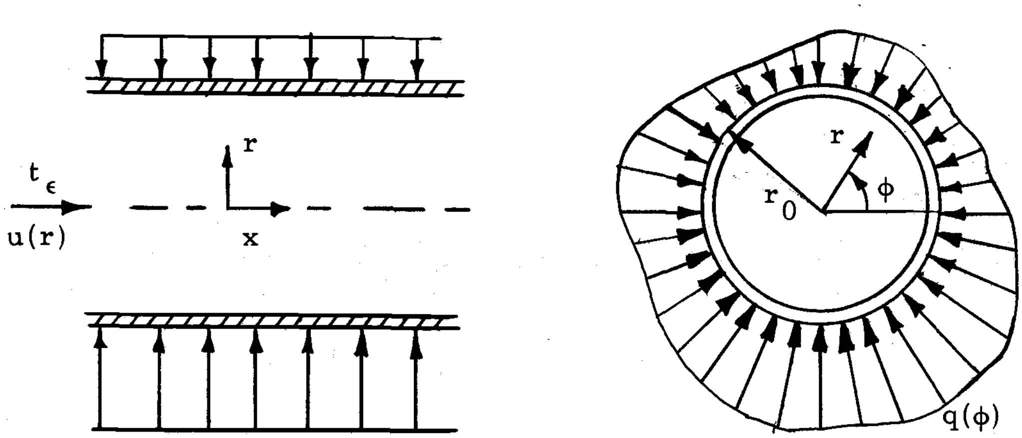

Figure 1.1. Physical model and coordinate system.

The applicable form of the energy equation is

$$\rho c_p u_x \frac{\partial t}{\partial x} = k\left(\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{1}{r^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial x^2}\right) + \frac{\mu}{g_c J}\left(\frac{\partial u_x}{\partial r}\right)^2 + Q \qquad (1.1)$$

where ρ is density, c_p is the specific heat, k is the thermal conductivity, μ is the coefficient of viscosity, u_x is the axial velocity, $t(x, r, \phi)$ is the local fluid temperature, J is conversion factor from mechanical to thermal units, g_c is a conversion factor, i.e., 32.174 $lb_m\ ft/(lb_f\ sec^2)$, and Q is the heat generation rate per unit volume.

It is assumed throughout this work that the physical properties of the fluid are constants. Since these properties vary with temperature, the results hold for relatively small difference in the temperatures.

The axial velocity, u_x, for the case of steady, laminar, fully-developed flow is

$$u_x = 2v\left[1-\left(\frac{r}{r_0}\right)^2\right] \tag{1.2}$$

where the mean velocity, v, is constant. The boundary conditions on t are as follows:

$$t(0, r, \phi) = t_\epsilon \tag{1.3a}$$

$$t(\infty, r, \phi) = t_{fd} \tag{1.3b}$$

$$k\frac{\partial t}{\partial r}(x, r_0, \phi) = q(\phi) \tag{1.3c}$$

$$t(x, 0, \phi) = \text{finite} \tag{1.3d}$$

$$t(x, r, \phi) = t(x, r, \phi+2\pi) \tag{1.3e}$$

i.e., t is single-valued

$$\frac{\partial t}{\partial \phi}(x, r, \phi) = \frac{\partial t}{\partial \phi}(x, r, \phi+2\pi) \tag{1.3f}$$

i.e., t is continuous

where the fluid temperature at the entrance of the heated section is t_ϵ and $t_{fd}(x, r, \phi)$ represents the fully-developed temperature distribution.

A solution is sought to Equation (1.1) subject to conditions specified in Equations (1.2) and (1.3a-f) for $t(x, r, \phi)$ and for the pertinent heat transfer parameters, the Nusselt number, $Nu(x, \phi)$, and the wall temperature.

Equation (1.1) can be represented in terms of the following dimensionless variables:

$$\theta = \frac{t-t_\epsilon}{\overline{q}2r_0/\pi k} \tag{1.4a}$$

$$x+ = \frac{x/r_0}{Re\, Pr} \tag{1.4b}$$

$$r+ = \frac{r}{r_0} \tag{1.4c}$$

$$u+ = \frac{u_x}{v} \tag{1.4d}$$

where

$$\overline{q} = \int_0^{2\pi} q(\phi)d\phi \tag{1.4f}$$

with the requirement that $\overline{q} \neq 0$.

Performing the necessary transformations we obtain for the energy equation

$$\frac{u+}{2}\frac{\partial\theta}{\partial x+} = \frac{\partial^2\theta}{\partial r+^2} + \frac{1}{r+}\frac{\partial\theta}{\partial r+} + \frac{1}{r+^2}\frac{\partial^2\theta}{\partial\phi^2} + \frac{1}{\mathrm{Re\ Pr}}\frac{\partial^2\theta}{\partial x^2}$$

$$+ \frac{\pi}{2g_cJ}\frac{\mu v^2}{\overline{q}r_0}\left(\frac{\partial u+}{\partial r+}\right)^2 + \frac{Qr_0}{\overline{q}}\frac{\pi}{2} \tag{1.5}$$

and for the velocity distribution

$$u+ = 2(1-r+^2) \tag{1.6}$$

With u+ from Equation (1.6) substituted into (1.5) the complete form of the energy equation becomes

$$(1-r+^2)\frac{\partial\theta}{\partial x+} = \frac{\partial^2\theta}{\partial r+^2} + \frac{1}{r+}\frac{\partial\theta}{\partial r+} + \frac{1}{r+^2}\frac{\partial^2\theta}{\partial\phi^2} + \frac{1}{\mathrm{Re\ Pr}}\frac{\partial^2\theta}{\partial x+^2} + Kr+^2 + Q' \tag{1.7}$$

where

$$K = 16\pi\left(\frac{v^2}{2g_cJc_p}\right)\left(\frac{k}{\overline{q}r_0}\right)\mathrm{Pr} \tag{1.7a}$$

and

$$Q' = \frac{Qr_0}{\overline{q}}\frac{\pi}{2} \tag{1.7b}$$

The boundary conditions, in terms of our dimensionless variable are:

$$\theta(0, r+, \phi) = 0 \tag{1.8a}$$

$$\theta(\infty, r+, \phi) = \theta_{fd}(\infty, r+, \phi) \tag{1.8b}$$

$$\frac{\partial\theta}{\partial r+}(x+, 1, \phi) = \frac{q(\phi)}{\bar{q}}\frac{\pi}{2} \tag{1.8c}$$

$$\theta(x+, 0, \phi) = \text{finite} \tag{1.8d}$$

$$\theta(x+, r+, \phi) = \theta(x+, r+, \phi+2\pi) \tag{1.8e}$$

$$\frac{\partial\theta}{\partial\phi}(x+, r+, \phi) = \frac{\partial\theta}{\partial\phi}(x+, r+, \phi+2\pi) \tag{1.8f}$$

1.2.2 Elimination of the Heat Source and the Dissipation Terms

These terms can be eliminated from the energy equation by the following linear transformation

$$\theta = \Theta + \hat{f}(r+) \tag{1.9}$$

where Θ is thermal distribution when heat source and dissipation terms are neglected. The substitution of Equation (1.9) into Equations (1.7) and (1.8) yields the following problems in $\hat{f}$ and Θ :

$$\hat{f}'' + \frac{1}{r+}\hat{f}' + Kr+^2 + Q' = 0 \tag{1.10}$$

subject to the boundary conditions

$$\hat{f}'(1) = 0 \tag{1.11a}$$

$$\hat{f}(0) = \text{finite} \tag{1.11b}$$

and

$$(1-r+^2)\frac{\partial\Theta}{\partial x+} = \frac{\partial^2\Theta}{\partial r+^2} + \frac{1}{r+}\frac{\partial\Theta}{\partial r+} + \frac{1}{r+^2}\frac{\partial^2\Theta}{\partial\phi^2} + \frac{1}{\text{Re Pr}}\frac{\partial^2\Theta}{\partial x+^2} \tag{1.12}$$

subject to the boundary conditions

$$\Theta(0, r+, \phi) = 0 \tag{1.13a}$$

$$\Theta(\infty, r+, \phi) = \Theta_{fd}(\infty, r+, \phi) \tag{1.13b}$$

$$\frac{\partial\Theta}{\partial r+}(x+, 1, \phi) = \frac{q(\phi)}{\overline{q}}\frac{\pi}{2} \tag{1.13c}$$

$$\Theta(x+, 0, \phi) = \text{finite} \tag{1.13d}$$

$$\Theta(x+, r+, \phi) = \Theta(x+, r+, \phi+2\pi) \tag{1.13e}$$

$$\frac{\partial\Theta}{\partial\phi}(x+, r+, \phi) = \frac{\partial\Theta}{\partial\phi}(x+, r+, \phi+2\pi) \tag{1.13f}$$

Equation (1.10) may now be solved directly with boundary conditions (1.11a, b) incorporated yielding

$$\hat{f} = -\frac{K}{16}r+^4 - \frac{Q'}{4}r+^2 + \text{constant}\ \underline{1}/ \tag{1.14}$$

where the constant of integration is still undetermined.

As shown above, the viscous dissipation and heat generation terms can be easily eleminated by the linear transformation given by Equation (1.9). We will disregard the contributions of these terms in futher discussion of the present work.

$\underline{1}/$The constant is determined to be $\frac{Q'}{12} + \frac{K}{96}$.

1.2.3 Fully-Developed and Entry Length Differential Equations and Boundary Conditions

We seek an exact solution, $\Theta(x+, r+, \phi)$, satisfying Equation (1.12) and the associated boundary conditions given by Equations (1.13a-f). Equation (1.12) is a linear differential equation. By experience with heat conduction problems of similar form, a solution can be obtained having the form

$$\Theta+(x+, r+, \phi) = \Theta(x+, r+, \phi) - \Theta_{fd}(x+, r+, \phi) \tag{1.15}$$

in which $\Theta_{fd}(x+, r+, \phi)$ is the asymptotic solution obtained far downstream where the temperature profile is fully developed, and $\Theta+$ is the entry region solution.

Combining Equations (1.12), (1.13), and (1.15), and noting that for the case of a fully-developed temperature profile, $\partial\Theta_{fd}/\partial x+ =$ constant, we obtain differential equations and associated boundary conditions for the two regions as follows:

$$\frac{\partial^2\Theta_{fd}}{\partial r+^2} + \frac{1}{r+}\frac{\partial\Theta_{fd}}{\partial r+} + \frac{1}{r+^2}\frac{\partial^2\Theta_{fd}}{\partial\phi^2} = (1-r+^2)\frac{\partial\Theta_{fd}}{\partial x+} \tag{1.16}$$

$$\frac{\partial\Theta_{fd}}{\partial r+}(x+, 1, \phi) = \frac{q(\phi)}{\overline{q}}\frac{\pi}{2} \tag{1.16a}$$

$$\Theta_{fd}(x+, 0, \phi) = \text{finite} \tag{1.16b}$$

$$\Theta_{fd}(x+, r+, \phi) = \Theta_{fd}(x+, r+, \phi+2\pi) \tag{1.16c}$$

$$\frac{\partial \Theta_{fd}}{\partial \phi}(x+, r+, \phi) = \frac{\partial \Theta_{fd}}{\partial \phi}(x+, r+, \phi+2\pi) \tag{1.16d}$$

$$\frac{\partial^2 \Theta+}{\partial r+^2} + \frac{1}{r+}\frac{\partial \Theta+}{\partial r+} + \frac{1}{r+^2}\frac{\partial^2 \Theta+}{\partial \phi^2} = (1-r+^2)\frac{\partial \Theta+}{\partial x+} - \frac{1}{Re\ Pr}\frac{\partial^2 \Theta+}{\partial x+^2} \tag{1.17}$$

$$\Theta+(0, r+, \phi) = -\Theta_{fd}(0, r+, \phi) \tag{1.17a}$$

$$\Theta+(\infty, r+, \phi) = 0 \tag{1.17b}$$

$$\Theta+(x+, 0, \phi) = \text{finite} \tag{1.17c}$$

$$\frac{\partial \Theta+}{\partial r+}(x+, 1, \phi) = 0 \tag{1.17d}$$

$$\Theta+(x+, r+, \phi) = \Theta+(x+, r+, \phi+2\pi) \tag{1.17e}$$

$$\frac{\partial \Theta+}{\partial \phi}(x+, r+, \phi) = \frac{\partial \Theta+}{\partial \phi}(x+, r+, \phi+2\pi) \tag{1.17f}$$

Equation (1.16) was solved by Reynolds [49] for the fully-developed temperature profile with axial conduction neglected. He considered an arbitrary variation of heat flux that was symmetrical about an axis through the center of the pipe. A solution was then obtained for the case of a tube with constant heat flux over a portion of its circumference, insulated over the remainder, and then generalized by superposition to obtain a solution for an arbitrary heat flux, $q(\phi)$.

In this paper we utilize a Fourier series approach. The formulation of this complete problem includes effects of both axial conduction and a developing temperature profile.

1.3 Discussion of Solution

1.3.1 The Fully-Developed Temperature

For the case of a fully-developed temperature profile we have the condition

$$\frac{\partial \Theta_{fd}}{\partial x} = \frac{d\Theta_m}{dx}\bigg|_{x+\to\infty} = \text{constant} \tag{1.18}$$

An energy balance, for a tube with wall heating as shown in Figure 1.1, yields the following expression

$$\pi r_0^2 \rho v c \frac{dt_m}{dx} dx + \pi r_0^2 \frac{dq_{axial}}{dx} dx = dx \int_0^{2\pi} q(\phi) r_0 d\phi \tag{1.19}$$

The axial heat flux and fluid temperature are related according to

$$q_{axial} = -\frac{1}{\pi r_0^2} \int_0^{2\pi} \int_0^{r_0} k \frac{\partial t}{\partial x} r dr d\phi$$

and it follows that

$$\frac{dq_{axial}}{dx} = -\frac{k}{\pi r_0^2} \frac{d}{dx} \int_0^{2\pi} \int_0^{r_0} \frac{\partial t}{\partial x}(x, r, \phi) r dr d\phi \tag{1.20}$$

Combining Equations (1.4f), (1.19) and (1.20) and solving for $\frac{dt_m}{dx}$ we obtain

$$\frac{dt_m}{dx} = \frac{\partial t_{fd}}{\partial x} = \frac{\bar{q}}{\pi r_0 \rho v c} + \frac{k}{\rho v c \pi r_0^2} \frac{d}{dx} \int_0^{2\pi} \int_0^{r_0} \frac{\partial t}{\partial x} \, r dr d\phi \qquad (1.21)$$

This equation may put in nondimensional form by introducing

$$Pe = Re\ Pr = \frac{2 \rho v c r_0}{k}$$

and other dimensionless variables defined in Equations (1.4a-d) to yield

$$\frac{d\Theta_m}{dx+} = \frac{\partial \Theta_{fd}}{\partial x+} = 1 + \frac{2}{\pi Pe^2} \frac{d}{dx+} \int_0^{2\pi} \int_0^{1} \frac{\partial \Theta}{\partial x+}(x+, r+, \phi) r+ dr+ d\phi \qquad (1.22)$$

Integrating Equation (1.22) for Θ_{fd} from zero to $x+$, we obtain

$$\Theta_{fd} = x+ + \frac{2}{\pi Pe^2} \left[\int_0^{2\pi} \int_0^{1} \frac{\partial \Theta}{\partial x+}(x+, r+, \phi) r+ dr+ d\phi \right]_0^{x+} + f+(r+, \phi) \qquad (1.23)$$

We now use Equation (1.15) and let $x+ \to \infty$; Equation (1.23) thus becomes

$$\Theta_{fd} = x+ + \frac{2}{\pi Pe^2} \left[\int_0^{2\pi} \int_0^{1} \frac{\partial \Theta+}{\partial x+}(\infty, r+, \phi) r+ dr+ d\phi - \int_0^{2\pi} \int_0^{1} \frac{\partial \Theta+}{\partial x+}(0, r+, \phi) r+ dr+ d\phi \right] + f+(r+, \phi) \qquad (1.24)$$

The first integral in this equation is zero by the boundary condition given by Equation (1.17b). Thus our expression for Θ_{fd} becomes

$$\Theta_{fd} = x+ - \frac{2}{\pi Pe^2} \int_0^{2\pi} \int_0^1 \frac{\partial \Theta+}{\partial x+} (0, r+, \phi) r+ dr+ d\phi + f+(r+, \phi) \qquad (1.25)$$

The form of the function $f+(r+, \phi)$ is expressed as a differential equation obtained by substituting Equation (1.25) into (1.16). Simplification of this result yields the following equation and boundary conditions in $f+(r+, \phi)$

$$\frac{\partial^2 f+}{\partial r+^2} + \frac{1}{r+} \frac{\partial f+}{\partial r+} + \frac{1}{r+^2} \frac{\partial^2 f+}{\partial \phi^2} = (1 - r+^2) \qquad (1.26)$$

$$\frac{\partial f+}{\partial r+} (1, \phi) = \frac{q(\phi)}{\overline{q}} \frac{\pi}{2} \qquad (1.26a)$$

$$f+(0, \phi) = \text{finite} \qquad (1.26b)$$

$$f+(r+, \phi) = f+(r+, \phi + 2\pi) \qquad (1.26c)$$

$$\frac{\partial f+}{\partial \phi} (r+, \phi) = \frac{\partial f+}{\partial \phi} (r+, \phi + 2\pi) \qquad (1.26d)$$

To eliminate the difficulty arising from the non-homogeneity in Equation (1.26), we express the function $f+$ as the sum of two functions in the form

$$f+(r+, \phi) = F(r+, \phi) + W(r+) \qquad (1.27)$$

and include $(1 - r+^2)$ in the formulation of the one-dimensional, $W(r+)$, problem. The two problems which result are now

$$\frac{\partial^2 F}{\partial r_+^2} + \frac{1}{r_+}\frac{\partial F}{\partial r_+} + \frac{1}{r_+^2}\frac{\partial^2 F}{\partial \phi^2} = 0 \tag{1.30}$$

$$\frac{\partial F}{\partial r_+}(1, \phi) = \frac{q(\phi)}{\overline{q}}\frac{\pi}{2} - \frac{1}{4} \tag{1.30a}$$

$$F(0, \phi) = \text{finite} \tag{1.30b}$$

$$F(r_+, \phi) = F(r_+, \phi + 2\pi) \tag{1.30c}$$

$$\frac{\partial F}{\partial \phi}(r_+, \phi) = \frac{\partial F}{\partial \phi}(r_+, \phi + 2\pi) \tag{1.30d}$$

and

$$\frac{1}{r_+}\frac{d}{dr_+}\left(r_+\frac{dW}{dr_+}\right) = (1 - r_+^2) \tag{1.31a}$$

$$W(0) = \text{finite} \tag{1.31b}$$

Equation (1.31a) may be solved directly with boundary condition (1.31b) incorporated, yielding

$$W = \left[\frac{r_+^2}{4} - \frac{r_+^4}{16}\right] + \text{constant} \tag{1.32}$$

where the constant is still undetermined.

A product solution for $F(r_+, \phi)$ is assumed of the form

$$F(r_+, \phi) = R(r_+)\Phi(\phi) \tag{1.33}$$

which allows the variables in Equation (1.30) to be separated yielding

$$r_+^2\frac{R''}{R} + r_+\frac{R'}{R} = -\frac{\Phi''}{\Phi} = +\lambda^2 \tag{1.34}$$

The requirement that the equation in Φ, i.e., in the homogeneous direction, be a characteristic-value problem dictates a positive sign for λ^2.

The resulting equation and boundary conditions are now

$$\frac{d^2\Phi}{d\phi^2} + \lambda^2\Phi = 0 \tag{1.35}$$

$$\Phi(\phi) = \Phi(\phi+2\pi) \tag{1.36a}$$

$$\frac{\partial\Phi}{\partial\phi}(\phi) = \frac{\partial\Phi}{\partial\phi}(\phi+2\pi) \tag{1.36b}$$

and

$$r+^2 \frac{d^2R}{dr+^2} + r+ \frac{dR}{dr+} - \lambda^2 R = 0 \tag{1.37}$$

$$R(0) = \text{finite} \tag{1.38a}$$

The solution to Equation (1.35) is

$$\Phi = A \cos \lambda\phi + B \sin \lambda\phi \tag{1.39}$$

The physics of the problem requires that Φ be single-valued. This condition, which is expressed by (1.36a) can be satisfied when the circular functions of Equation (1.39) have a common period 2π. The same requirement also serves to determine the permissible values of the separation constant

$$\lambda = n \quad \text{where} \quad n = 0, 1, 2, \ldots \tag{1.40}$$

Thus Equation (1.39) becomes

$$\Phi = A_0 + A_n \cos n\phi + B_n \sin n\phi \tag{1.41}$$

where $n = 1, 2, 3, \ldots$

It is clear that Φ, besides being single-valued, is continuous and thus automatically satisfies the second boundary condition given by (1.36b).

Equation (1.37) is an Euler equation and has the solution

$$R = Cr+^{\lambda} + Dr+^{-\lambda} \tag{1.42}$$

Combining Equations (1.40) and (1.42), we obtain

$$R = Cr+^{n} + Dr+^{-n} \tag{1.43}$$

Boundary condition (1.38a), requires the coefficient D be zero. Equation (1.43) may now be expressed in the following form.

$$R = \begin{cases} C_0 & \text{when } n = 0 \\ C_n r+^{n} & \text{when } n = 1, 2, 3, \ldots \end{cases} \tag{1.45}$$

Combining Equations (1.33), (1.41), and (1.45) we obtain

$$F(r+, \phi) = C_0 A_0 + \sum_{n=1}^{\infty} C_n r+^{n} (A_n \cos n\phi + B_n \sin n\phi) \tag{1.46}$$

and with the constants combined this becomes

$$F(r+, \phi) = a_0 + \sum_{n=1}^{\infty} r+^n (a_n \cos n\phi + b_n \sin n\phi) \tag{1.47}$$

Finally, the non-homogeneous boundary condition (1.30a) is expressed as

$$\frac{\partial F}{\partial r+}(1, \phi) = \sum_{n=1}^{\infty} n r+^{n-1} (a_n \cos n\phi + b_n \sin n\phi)\Big|_{r+ = 1}$$

$$= \frac{q(\phi)}{\bar{q}} \frac{\pi}{2} - \frac{1}{4}$$

or

$$\sum_{n=1}^{\infty} n(a_n \cos n\phi + b_n \sin n\phi) = \frac{q(\phi)}{\bar{q}} \frac{\pi}{2} - \frac{1}{4}$$

which allows the Fourier coefficients to be evaluated in the usual manner.

The completed solution for $f+(r, \phi)$ from Equation (1.26) may now be summarized

$$f+(r+, \phi) = \left(\frac{r+^2}{4} - \frac{r+^4}{16}\right) + \sum_{n=1}^{\infty} r+^n (a_n \cos n\phi + b_n \sin n\phi) + A_0 \tag{1.48}$$

where

$$a_n = \frac{1}{2n} \int_0^{2\pi} \frac{q(\phi)}{\bar{q}} \cos n\phi \, d\phi \tag{1.49a}$$

$$b_n = \frac{1}{2n} \int_0^{2\pi} \frac{q(\phi)}{\bar{q}} \sin n\phi \, d\phi \tag{1.49b}$$

$$A_0 = \text{constant} \qquad \text{(still unknown)} \tag{1.49c}$$

We may now express the fully-developed temperature profile by combining Equations (1.25) and (1.48) to obtain

$$\Theta_{fd}(x+, r+, \phi) = x+ + \frac{r+^2}{4} - \frac{r+^4}{16} - \frac{2}{\pi Pe^2} \int_0^{2\pi} \int_0^1 \frac{\partial \Theta+}{\partial x+}(0, r+, \phi) r+ dr+ d\phi$$

$$+ \sum_{n=1}^{\infty} r+^n (a_n \cos n\phi + b_n \sin n\phi) + A_0 \tag{1.50}$$

where a_n and b_n are given by Equations (1.49a, b). A_0 [2/] is still to be determined. The complete solution for Θ_{fd} awaits a knowledge of $\Theta+$ so that the integral in Equation (1.50) might be determined.

The fully-developed portion to the problem has been solved to this point by a direct use of Fourier series methods. In the next section we shall proceed to solve the entry portion of the problem.

[2/] This constant is determined to be $\frac{-7}{96}$.

1.3.2 The Thermal Entry Length

Consideration will now be given to the problem of solving for $\Theta+(x+, r+, \phi)$ as posed in Equation (1.17). The classical separation of variables method fails when the axial conduction term is retained in the governing energy equation. Nevertheless, a series expansion of the convenient form

$$\Theta+(x+, r+, \phi) = \sum_{n=1}^{\infty} \sum_{p=0}^{\infty} e^{-\lambda_{np}^2 x+} R_{np}(r+)(a_{np} \cos p\phi + b_{np} \sin p\phi) \tag{1.51}$$

may be assumed; it may be noted that boundary condition, Equation (1.17b), is satisfied by this expression. Substituting Equation (1.51) in (1.17) and simplifying, we see that $\Theta+$ satisfies the equation and the boundary conditions provided that $R_{np}(r+)$ is the solution of the following ordinary differential equation and its associated boundary conditions

$$\frac{1}{r+} \frac{d}{dr+}\left(r+ \frac{dR_{np}}{dr+}\right) + \left[\lambda_{np}^2\left(1 - r+^2 + \frac{\lambda_{np}^2}{Pe^2}\right) - \frac{p^2}{r+^2}\right] R_{np} = 0 \tag{1.52}$$

$$\frac{dR_{np}(1)}{dr+} = 0 \tag{1.53a}$$

$$R_{np}(0) = \text{finite} \tag{1.53b}$$

The terms, λ_{np}, and R_{np} are, respectively, the eigenvalues and eigenfunctions of the above equation; p is an integer parameter. For $p = 0$, Equation (1.52) reduces to the characteristic equation for the case with no circumferential variation. The Peclet number, Pe, is also a parameter in Equation (1.52). For $Pe = \infty$, Equation (1.52) reduces to the limiting problem of no axial conduction. Therefore, Equation (1.52) is a general characteristic equation to a variety of heat transfer problems.

Mathematically, Equation (1.52) is not a usual "Sturm-Liouville" type differential equation, in fact it is a more generalized Sturm-Liouville differential equation with non-orthogonal eigenfunctions. Singh [60] showed the orthogonality relationships for the "generalized Sturm-Liouville" system; however, his analysis is in error. Therefore, the orthogonality relationships for the generalized Sturm-Liouville problem are not as yet available in the literature.

1.3.2.1 Analysis of the Eigenvalue Problem. The eigenfunctions of Equation (1.52) cannot be expressed in terms of simple functions. Thus we are forced to employ a power series to obtain

$$R_{np}(r+) = \sum_{i=0}^{\infty} b_{i;np}(r+)^{i+p} \tag{1.54}$$

It is easily found that the coefficients $b_{i;np}$ satisfy

$$b_{i;np} = \frac{\lambda_{np}^2 \left\{ b_{i-4} - \left[1+\left(\frac{\lambda_{np}}{Pe}\right)^2\right] b_{i-2} \right\}}{i(i+p)} \tag{1.55}$$

where

$$b_{i;np} = \text{zero} \quad \text{if} \quad (i-4) \quad \text{and} \quad (i-2) < 0$$

$$b_{i;np} = 1 \qquad \text{if} \quad i = \text{zero}.$$

Every coefficient $b_{i;np}$ is equal to zero whenever i is odd, so Equations (1.54) and (1.55) become:

$$R_{np}(r+) = \sum_{i=0}^{\infty} b_{2i;np} (r+)^{2i+p} \tag{1.56}$$

$$b_{2i;np} = \frac{\lambda_{np}^2 \left\{ b_{2i-4} - \left[1+\left(\frac{\lambda_{np}}{Pe}\right)^2\right] b_{2i-2} \right\}}{2i(2i+p)} \tag{1.57}$$

The eigenvalues are determined by the equation

$$\sum_{i=0}^{\infty} b_{2i;np}(2i+p) = 0 \tag{1.58}$$

following from (1.56) and boundary condition (1.53a).

1.3.2.2 Determination of an Integral. With the present knowledge of the form of $\Theta+$, the integral in Equation (1.50) may be determined as follows

$$\int_0^{2\pi}\int_0^1 \frac{\partial\Theta+}{\partial x+}(0, r+, \phi)r+dr+d\phi$$

$$= -\int_0^{2\pi}\int_0^1 \sum_{n=1}^{\infty}\sum_{p=0}^{\infty} \lambda_{np}^2 R_{np}(r+)(a_{np} \cos p\phi + b_{np} \sin p\phi) \quad r+dr+d\phi$$

$$= -2\pi \sum_{n=1}^{\infty} a_{n0}\lambda_{n0}^2 \int_0^1 R_{n0}(r+)r+dr+ \tag{1.59}$$

which follow from the orthogonality properties of the sine and cosine functions.

The solution to the fully-developed portion is now expressed in the following form by combining Equations (1.50) and (1.59).

$$\Theta_{fd}(x+, r+, \phi) = x+ + \frac{r+^2}{4} - \frac{r+^4}{16} - \frac{7}{96}$$

$$+ \frac{4}{Pe^2} \sum_{n=1}^{\infty} a_{n0}\lambda_{n0}^2 \int_0^1 R_{n0}(r+)r+dr+$$

$$+ \sum_{n=1}^{\infty} r+^n(a_n \cos n\phi + b_n \sin n\phi) \tag{1.60}$$

<u>1.3.2.3 Determination of Expansion Coefficients</u>. Condition (1.17a) is used to determine the coefficients of expansion in Equation (1.51), i.e., a_{np} and b_{np}. Substitution yields

$$\sum_{n=1}^{\infty} \sum_{p=0}^{\infty} R_{np}(r+)[a_{np} \cos p\phi + b_{np} \sin p\phi] = -\Theta_{fd}(0, r+, \phi) \tag{1.61}$$

Combining Equations (1.60) and (1.61), we obtain

$$\sum_{n=1}^{\infty} a_{n0}\left[R_{n0}(r+) + \frac{4\lambda_{n0}^2}{Pe^2}\int_0^1 R_{n0}(r+)r+dr+\right]$$

$$+ \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} R_{np}(r+)(a_{np} \cos p\phi + b_{np} \sin p\phi)$$

$$= \frac{7}{96} - \frac{r+^2}{4} + \frac{r+^4}{16} - \sum_{n=1}^{\infty} r+^n(a_n \cos n\phi + b_n \sin n\phi) \tag{1.62}$$

We next define a parameter, $\hat{\Theta}_{fd}$, and three expansions of the form

$$\hat{\Theta}_{fd}(r+,\phi) \equiv \frac{7}{96} - \frac{r+^2}{4} + \frac{r+^4}{16} - \sum_{n=1}^{\infty} r+^n[a_n \cos n\phi + b_n \sin n\phi] \tag{1.63a}$$

$$A_0(r+) \equiv \sum_{n=1}^{\infty} a_{n0}\left[R_{n0}(r+) + \frac{4\lambda_{n0}^2}{Pe^2}\int_0^1 R_{n0}(r+)r+dr+\right] \tag{1.63b}$$

$$A_p(r+) \equiv \sum_{n=1}^{\infty} a_{np}R_{np}(r+) \tag{1.63c}$$

$$B_p(r+) \equiv \sum_{n=1}^{\infty} b_{np}R_{np}(r+) \tag{1.63d}$$

We now combine Equations (1.62) and (1.63) to obtain

$$\hat{\Theta}_{fd}(r+, \phi) = A_0(r+) + \sum_{p=1}^{\infty} [A_p(r+) \cos n\phi + B_p(r+) \sin n\phi] \qquad (1.64)$$

Equation (1.64) is a complete Fourier series expansion of $\hat{\Theta}_{fd}$. Therefore

$$A_0(r+) = \frac{1}{2\pi} \int_0^{2\pi} \hat{\Theta}_{fd} d\phi \qquad (1.65a)$$

$$A_p(r+) = \frac{1}{\pi} \int_0^{2\pi} \hat{\Theta}_{fd} \cos p\phi \, d\phi \qquad (1.65b)$$

$$B_p(r+) = \frac{1}{\pi} \int_0^{2\pi} \hat{\Theta}_{fd} \sin p\phi \, d\phi \qquad (1.65c)$$

Combining Equations (1.63) and (1.65) we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} \hat{\Theta}_{fd}(r+, \phi) d\phi = \sum_{n=1}^{\infty} a_{n0} \left[R_{n0}(r+) + \frac{4\lambda_{n0}^2}{Pe^2} \int_0^1 R_{n0}(r+) r+ dr+ \right] \qquad (1.66a)$$

$$\frac{1}{\pi} \int_0^{2\pi} \hat{\Theta}_{fd}(r+, \phi) \cos p\phi \, d\phi = \sum_{n=1}^{\infty} a_{np} R_{np}(r+) \qquad (1.66b)$$

$$\frac{1}{\pi} \int_0^{2\pi} \hat{\Theta}(r+, \phi) \sin p\phi \, d\phi = \sum_{n=1}^{\infty} b_{np} R_{np}(r+) \qquad (1.66c)$$

In order to determine the coefficients a_{n0}, a_{np}, and b_{np} we must be able to expand an arbitrary function of $r+$, say, $g(r+)$ in terms of the eigenfunctions of the characteristic Equation (1.52). As mentioned before, the eigenfunctions of Equation (1.52) are not orthogonal with respect to any known weighting function on the interval of integration. Therefore, we proceed as follows: Let Equations (1.66a-c) be expressed in the general form

$$g(r+) = \sum_{n=1}^{N} a_n R_n(r+) \cong y(r+) \tag{1.67}$$

Equation (1.67) represents the integral appearing in the left side of Equations (1.66a-c) with $g(r+)$ expressed in terms of a finite series of N terms, symbolized as $y(r+)$. In this work $N = 12$.

In place of satisfying Equation (1.67) at n points (point matching) it is often preferable to require that $y(r+)$ and $g(r+)$ agree as well as possible (in some sense) over a domain D of greater extent. This method to satisfy Equation (1.67) involves the minimization of the integral of the square of the error in D (i.e., least squares). More generally, it is required that the squared error be multiplied by the weight $\omega(r+)$ before the integration. We now require that the squared error meet the condition

$$E(a_1, a_2, \dots, a_N) = \int_0^1 \omega(r+)\left[g(r+) - \sum_{n=1}^{N} a_n R_n(r+)\right]^2 dr+ = \text{minimum}$$

Expanding yields

$$E(a_1, a_2, \dots, a_N) = \int_0^1 \omega(r+)\left[g^2(r+) - 2g(r+)\sum_{n=1}^{N} a_n R_n(r+) + \sum_{n=1}^{N}\sum_{m=1}^{N} a_n a_m R_n R_m\right] dr+$$

$$= \text{minimum}$$

Minimizing with respect to each coefficient a_r, we require that

$$\frac{\partial}{\partial a_r}\int_0^1 \omega(r+)\left[g^2(r+) - 2g(r+)\sum_{n=1}^{N} a_n R_n(r+) + \sum_{n=1}^{N}\sum_{m=1}^{N} a_n a_m R_n R_m\right] dr+ = 0$$

Interchanging the order of differentiation and integration and expanding we have

$$\sum_{n=1}^{N} a_n \int_0^1 \omega(r+)R_n(r+)R_r(r+)dr+ = \int_0^1 \omega(r+)g(r+)R_r(r+)dr+$$

$$r = 1, 2, \dots, N$$

which, in expanded form, becomes

$$\begin{bmatrix} \int_0^1 \omega(r+)R_1^2(r+)dr+ & \cdots & \int_0^1 \omega(r+)R_1(r+)R_N(r+)dr+ \\ \int_0^1 \omega(r+)R_2(r+)R_1(r+)dr+ & \cdots & \int_0^1 \omega(r+)R_2(r+)R_N(r+)dr+ \\ \vdots & & \vdots \\ \int_0^1 \omega(r+)R_N(r+)R_1(r+)dr+ & \cdots & \int_0^1 \omega(r+)R_N^2(r+)dr+ \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}$$

$$= \begin{bmatrix} \int_0^1 \omega(r+)g(r+)R_1(r+)dr+ \\ \int_0^1 \omega(r+)g(r+)R_2(r+)dr+ \\ \vdots \\ \int_0^1 \omega(r+)g(r+)R_N(r+)dr+ \end{bmatrix} \tag{1.69}$$

For any arbitrary variation of circumferential wall heat flux, $g(r+)$ is known. The next and most time-consuming step is the evaluation of the integrals for the coefficients of the matrix in Equation (1.69). Note that these coefficients are symmetric, making the computation somewhat easier. In this work, we choose $\omega(r+) = r+(1-r+^2)$, the weighting function when axial conduction is absent. Finally, the simultaneous equations (1.69) are solved numerically to obtain the expansion coefficients.

Pirkle and Sigillito [47], by an alternative method, suggested that a_{n0} in the expansion (1.66a) be determined approximately for sufficiently large Peclet numbers by

$$a_{n0} \simeq \frac{1}{4}\,\frac{\displaystyle\int_0^1 r^+\left[2\left(\frac{\lambda_{n0}}{Pe}\right)^2+1-r^{+2}\right]\left[R_{n0}(r^+)\frac{4\lambda_{n0}^2}{Pe^2}\int_0^1 R_{n0}(r^+)r^+dr^+\right]\left[\frac{7}{24}-r^{+2}+\frac{r^{+4}}{4}\right]dr^+}{\displaystyle\int_0^1 r^+\left[2\left(\frac{\lambda_{n0}}{Pe}\right)^2+1-r^{+2}\right]\left[R_{n0}(r^+)+\frac{4\lambda_{n0}^2}{Pe^2}\int_0^1 R_{n0}(r^+)r^+dr^+\right]^2dr^+} \tag{1.70a}$$

We generalize their results to determine the coefficients of expansion of Equations (1.66b, c) in the following forms

$$a_{np} \simeq \frac{1}{\pi}\,\frac{\displaystyle\int_0^1\int_0^{2\pi} r^+\left[2\left(\frac{\lambda_{np}}{Pe}\right)^2+1-r^{+2}\right]\left[\cos p\phi\right]\left[\hat{\Theta}_{fd}\right]\left[R_{np}(r^+)\right]d\phi dr^+}{\displaystyle\int_0^1 r^+\left[2\left(\frac{\lambda_{np}}{Pe}\right)^2+1-r^{+2}\right]R_{np}^2(r^+)dr^+} \tag{1.70b}$$

$$b_{np} \simeq \frac{1}{\pi}\,\frac{\displaystyle\int_0^1\int_0^{2\pi} r^+\left[2\left(\frac{\lambda_{np}}{Pe}\right)^2+1-r^{+2}\right]\left[\sin p\phi\right]\left[\hat{\Theta}_{fd}\right]\left[R_{np}(r^+)\right]d\phi dr^+}{\displaystyle\int_0^1 r^+\left[2\left(\frac{\lambda_{np}}{Pe}\right)^2+1-r^{+2}\right]R_{np}^2(r^+)dr^+} \tag{1.70c}$$

It is verified by the method of least squares (i.e., Equation (1.69)) that Equations (1.70a, b, c) are valid for Peclet numbers greater than 100 at every axial position. Since the main effect of axial conduction occurs with low Peclet numbers, such an approximation to the coefficients of expansion is not valid when axial conduction is of importance.

1.3.3 Complete Solution

At this point the solution to the thermal entrance region is completed. We may now add this solution to the fully developed portion, using Equations (1.15), (1.51), and (1.60) to obtain the complete solution as follows:

$$\Theta(x+, r+, \phi) = x+ + \frac{r+^2}{4} - \frac{r+^4}{16} + \sum_{n=1}^{\infty} r+^n [a_n \cos n\phi + b_n \sin n\phi]$$

$$+ \frac{4}{Pe^2} \sum_{n=1}^{\infty} a_{n0} \lambda_{n0}^2 \int_0^1 R_{n0}(r+) r+ dr+ + \sum_{n=1}^{\infty} a_{n0} e^{-\lambda_{n0}^2 x+} R_{n0}(r+)$$

$$+ \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} e^{-\lambda_{np}^2 x+} R_{np}(r+)[a_{np} \cos p\phi + b_{np} \sin p\phi]$$

$$+ \text{constant } \underline{3}/ \qquad (1.71)$$

where

$$a_n = \frac{1}{2n} \int_0^{2\pi} \frac{q(\phi)}{\bar{q}} \cos n\phi \, d\phi$$

$$b_n = \frac{1}{2n} \int_0^{2\pi} \frac{q(\phi)}{\bar{q}} \sin n\phi \, d\phi$$

3/ This constant is determined to be $\frac{-7}{96}$.

and a_{n0}, a_{np}, and b_{np} are obtained exactly by the numerical solution of the simultaneous Equations (1.69), and approximately for sufficiently large Peclet numbers and far away from the entrance (i.e., $x+ > .04$) by Equations (1.70a, b, c). The eigenfunctions and eigenvalue, R_{np} and λ_{np} respectively are obtained from Equations (1.56) and (1.58). If viscous dissipation and heat generation are also present, we may use Equations (1.9), (1.14), and (1.71) to express the most complete solution to the thermal profile as

$$\theta(x+, r+, \phi) = \Theta + \hat{f}(r+) = x+ - (1+K)\frac{r+^4}{16}$$

$$+ (1-Q')\frac{r+^2}{4} + \sum_{n=1}^{\infty} r+^n[a_n \cos n\phi + b_n \sin n\phi]$$

$$+ \frac{4}{Pe^2}\sum_{n=1}^{\infty} a_{n0}\lambda_{n0}^2 \int_0^1 R_{n0}(r+)r+dr+ + \sum_{n=1}^{\infty} a_{n0} e^{-\lambda_{no}^2 x+} R_{n0}(r+)$$

$$+ \sum_{n=1}^{\infty}\sum_{p=1}^{\infty} e^{-\lambda_{np}^2 x+} R_{np}(r+)[a_{np} \cos p\phi + b_{np} \sin p\phi]$$

$$+ \text{constant}$$ [4/]

[4/] The constant is determined to be $= \frac{-7}{96} + \frac{Q'}{12} + \frac{K}{96}$

1.3.4 Calculation of the Average Mean Fluid Temperature

In this section we determine the average mixed mean temperature of the fluid by two alternative methods. First the average mixed mean fluid temperature is obtained by integrating Equation (1.22) from 0 to x+. Second the definition of mixed mean fluid temperature is employed and Equation (1.71) is integrated over the flow cross-section. Finally, by comparing these two methods, the unknown constant in Equation (1.71) is determined.

Integrating Equation (1.22) for Θ_{mean} from 0 to x+ and employing Equations (1.15), and (1.18), we obtain

$$\Theta_{mean}(x+) = x+ + \frac{2}{\pi Pe^2}\left[\int_0^{2\pi}\int_0^1 \frac{\partial\Theta+}{\partial x+}(x+, r+, \phi)r+ \, dr+ \, d\phi - \int_0^{2\pi}\int_0^1 \frac{\partial\Theta+}{\partial x+}(0, r+, \phi)r+ \, dr+ \, d\phi\right] \qquad (1.72)$$

The last integral in Equation (1.72) was obtained previously, and is given by the Equation (1.59). The first integral in this equation is obtained by the same procedure. Substituting yields

$$\Theta_{mean}(x+) = x+ + \frac{4}{Pe^2} \sum_{n=1}^{\infty} a_{n0} \lambda_{n0}^2 \int_0^1 R_{n0}(r+) r+ \, dr+$$

$$- \frac{4}{Pe^2} \sum_{n=1}^{\infty} a_{n0} \lambda_{n0}^2 e^{-\lambda_{n0}^2 x+} \int_0^1 R_{n0}(r+) r+ \, dr+ \qquad (1.73)$$

It is apparent from this expression that $\Theta_m(x+)$ does not vary linearly with $x+$. However, far from the entrance region, the last term in Equation (1.73) goes to zero and Θ_{mean} varies linearly with $x+$ i.e.,

$$\left. \frac{d\Theta_{mean}}{dx+} \right|_{x+ \to \infty} = 1$$

For the special case where axial conduction is absent (i.e., $Pe \to \infty$), the last two terms of Equation (1.73) go to zero and Θ_{mean} varies linearly with $x+$. It is worth noting that the circumferential variation part of the wall heat flux does not have any effect on the fluid mean temperature. This point will become clear when we derive the mean fluid temperature by integrating Equation (1.71). Finally by comparing Equation (1.73) with Equation (1.24) of Hsu [19], we observe that Hsu made an error in his analysis, and did not include the exponential term that appears in Equation (1.73).

The average mean fluid temperature is defined by

$$\Theta_{mean} = \frac{\int_0^{2\pi}\int_0^1 \left[2v(1-r+^2)\Theta(x+, r+, \phi)\right] r+\ dr+\ d\phi}{\int_0^{2\pi}\int_0^1 \left[2v(1-r+^2)\right] r+\ dr+\ d\phi}$$

Simplifying, we obtain

$$\Theta_{mean} = \frac{2}{\pi}\int_0^{2\pi}\int_0^1 \left[r+(1-r+^2)\Theta(x+, r+, \phi)\right] r+\ dr+\ d\phi \qquad (1.74)$$

Substituting Equation (1.71) into (1.74) and carrying out the integration yields

$$\Theta_{mean} = x+ - \frac{7}{96} + \frac{4}{Pe^2}\sum_{n=1}^{\infty} a_{n0}\lambda_{n0}^2 \int_0^1 R_{n0}(r+)r+\ dr+$$
$$+ 4\sum_{n=1}^{\infty} a_{n0}e^{-\lambda_{n0}^2 x+}\int_0^1 r+(1-r+^2)R_{n0}(r+)dr+ + \text{constant} \qquad (1.75)$$

When axial conduction is not present, it can be proved that the last integral appearing in Equation (1.75), i.e.,

$$\int_0^1 r+(1-r+^2)R_{n0}(r+)dr+$$

reduces to zero; however, when axial conduction is present this integral does not reduce to zero. To evaluate this integral, we substitute for $r+(1-r+^2)R_{n0}(r+)$ from the characteristic Equation (1.52) and carry out the resulting integration by parts to obtain

$$\int_0^1 r+(1-r+^2)R_{n0}dr+ = -\frac{\lambda_{n0}^2}{Pe^2}\int_0^1 r+R_{n0}(r+)dr+ \tag{1.76}$$

Combining Equations (1.76) and (1.75) we obtain

$$\begin{aligned}\Theta_{mean} &= x+ + \frac{7}{96} + \frac{4}{Pe^2}\sum_{n=1}^{\infty} a_{n0}\lambda_{n0}^2\int_0^1 R_{n0}(r+)r+\,dr+ \\ &\quad - \frac{4}{Pe^2}\sum_{n=1}^{\infty} a_{n0}\lambda_{n0}^2 e^{-\lambda_{n0}^2 x+}\int_0^1 r+R_{n0}(r+)dr+ + \text{constant}\end{aligned} \tag{1.77}$$

By comparing Equations (1.73) and (1.77), we find the unknown constant to be $-\frac{7}{96}$.

1.3.5 Calculation of Nusselt Number

The Nusselt number is defined as

$$Nu(x,\phi) = \frac{2h(x,r_0,\phi)r_0}{k} = \frac{k\frac{\partial t}{\partial r}(x,r_0,\phi)}{t_w-t_m}\,\frac{2r_0}{k}$$

or equivalently

$$Nu(x,\phi) = \frac{q(\phi)2r_0}{(t_w-t_m)k} \tag{1.78}$$

where $t_w(x, \phi) = t(x, r_0, \phi)$, the wall temperature, and $t_m(x)$ is the mixed mean temperatures.

It will be convenient to represent the heat flux distribution in the form

$$q(\phi) = \bar{q}\, f(\phi) \tag{1.78a}$$

where $\bar{q}$ is given by

$$\bar{q} = \int_0^{2\pi} q(\phi)\, d\phi \tag{1.78b}$$

and $f(\phi)$ is a specified angular variation. With this specification on $q(\phi)$, Equation (1.78) reduces to

$$Nu(x+, \phi) = \pi f(\phi) \left[\frac{\bar{q} 2 r_0 / k\pi}{t_w - t_m} \right] \tag{1.79}$$

Now, expressing Equation (1.71) in terms of mean fluid temperature by combining with Equation (1.73) we obtain

$$\begin{aligned} \frac{t - t_m}{\bar{q} 2 r_0 / k\pi} &= -\frac{7}{96} + \frac{r+^2}{4} - \frac{r+^4}{16} + \sum_{n=1}^{\infty} r+^n [a_n \cos n\phi + b_n \sin n\phi] \\ &+ \frac{4}{Pe^2} \sum_{n=1}^{\infty} a_{n0} \lambda_{n0}^2 e^{-\lambda_{n0}^2 x+} \int_0^1 R_{n0}(r+) r+ dr+ + \sum_{n=1}^{\infty} a_{n0} e^{-\lambda_{n0}^2 x+} R_{n0}(r+) \\ &+ \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} e^{-\lambda_{np}^2 x+} R_{np}(r+) [a_{np} \cos p\phi + b_{np} \sin p\phi] \end{aligned} \tag{1.80}$$

When the wall heat flux is specified, the wall temperature is the unknown quantity that is usually of most practical interest. It is found by evaluating Equation (1.80) at $r+ = 1$ to yield

$$\frac{t_w - t_m}{\overline{q}\, 2r_0/k\pi} = \frac{11}{96} + \sum_{n=1}^{\infty} (a_n \cos n\phi + b_n \sin n\phi)$$

$$+ \frac{4}{Pe^2} \sum_{n=1}^{\infty} a_{n0} \lambda_{n0}^2 e^{-\lambda_{n0}^2 x+} \int_0^1 R_{n0}(r+) r+ \, dr+$$

$$+ \sum_{n=1}^{\infty} a_{n0} R_{n0}(1) e^{-\lambda_{n0}^2 x+}$$

$$+ \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} e^{-\lambda_{np}^2 x+} R_{np}(1)[a_{np} \cos p\phi + b_{np} \sin p\phi] \tag{1.81}$$

Finally, we solve for $Nu(x+, \phi)$ by using Equations (1.79) and (1.81) to obtain

$$Nu(x+,\phi) = \pi f(\phi) \Bigg\{ \frac{11}{96} + \sum_{n=1}^{\infty} (a_n \cos n\phi + b_n \sin n\phi)$$

$$+ \frac{4}{Pe^2} \sum_{n=1}^{\infty} a_{n0} \lambda_{n0}^2 e^{-\lambda_{n0}^2 x+} \int_0^1 R_{n0}(r+) r+ dr+ + \sum_{n=1}^{\infty} a_{n0} R_{n0}(1) e^{-\lambda_{n0}^2 x+}$$

$$+ \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} e^{-\lambda_{np}^2 x+} R_{np}(1)[a_{np} \cos p\phi + b_{np} \sin p\phi] \Bigg\}^{-1} \tag{1.82}$$

1.3.6 Axial Non-Uniform Wall Heat Flux

The temperature solution obtained for uniform axial heat input can be used to generate solutions for any arbitrary specified axial variation of wall heat flux, using superposition. This is possible because of the linearity of the energy differential equation. Following the approach used by Siegel, Sparrow, and Hallman [59], for any arbitrary axial heat flux variation of the form $\hat{Q}(x+)$, the thermal distribution is

$$\frac{t-t_\epsilon}{r_0/k} = \int_0^{x+} \hat{Q}(\zeta)\,\frac{\partial\Theta}{\partial x+}\,(x+-\zeta, r+, \phi)\,d\zeta \tag{1.82*}$$

where Θ is the solution for uniform axial input.

Substitution for Θ from Equation (1.71) into Equation (1.82*) yields

$$\frac{t-t_\epsilon}{r_0/k} = \int_0^{x+} \Bigg\{ 4-4\sum_{n=1}^{\infty} a_{n0}\lambda_n^2 R_{n0}(r+)e^{-\lambda_{n0}^2(x+-\zeta)}$$

$$-4\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} e^{-\lambda_{np}^2(x+-\zeta)}\lambda_{np}^2 R_{np}(r+)$$

$$\times\,[a_{np}\cos p\phi + b_{np}\sin p\phi]\Bigg\}\,\hat{Q}(\zeta)\,d\zeta$$

1.3.7 Limiting Solution for $Pe \to \infty$ (No Axial Fluid Conduction)

For the limiting case where axial conduction is not of importance, the Peclet number goes to infinity, and Equations (1.71), (1.81), and (1.82) simplify to

$$\Theta = x+ - \frac{7}{96} + \frac{r+^2}{4} - \frac{r+^4}{16} + \sum_{n=1}^{\infty} r+^n[a_n \cos n\phi + b_n \sin n\phi]$$

$$+ \sum_{n=1}^{\infty} a_{n0} R_{n0}(r+) e^{-\lambda_{n0}^2 x+}$$

$$+ \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} e^{-\lambda_{np}^2 x+} R_{np}(r+)(a_{np} \cos p\phi + b_{np} \sin p\phi) \qquad (1.83)$$

$$\frac{t_w - t_m}{\overline{q} 2r_0/k\pi} = \frac{11}{96} + \sum_{n=1}^{\infty} (a_n \cos n\phi + b_n \sin n\phi)$$

$$+ \sum_{n=1}^{\infty} a_{n0} R_{n0}(1) e^{-\lambda_{n0}^2 x+}$$

$$+ \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} e^{-\lambda_{np}^2 x+} R_{np}(1)[a_{np} \cos p\phi + b_{np} \sin p\phi] \qquad (1.84)$$

$$Nu(x+, \phi) = \pi f(\phi) \left\{ \frac{11}{96} + \sum_{n=1}^{\infty} (a_n \cos n\phi + b_n \sin n\phi) \right.$$

$$+ \sum_{n=1}^{\infty} a_{n0} R_{n0}(1) e^{-\lambda_{n0}^2 x+} \tag{1.85}$$

$$\left. + \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} e^{-\lambda_{np}^2 x+} R_{np}(1)[a_{np} \cos p\phi + b_{np} \sin p\phi] \right\}^{-1}$$

where λ_{np}, R_{np} are the eigenvalues and eigenfunctions of the following "Sturm Liouville" system.

$$\frac{d}{dr+}\left(r+ \frac{dR_{np}}{dr+}\right) + \left[r+(1-r+^2)\lambda_{np}^2 - \frac{p^2}{r+}\right] R_{np} = 0 \tag{1.86}$$

$$\frac{dR_{np}}{dr+}(1) = 0, \qquad R_{np}(0) = \text{finite}$$

As the eigenfunctions R_{np} of Equation (1.86) form a complete orthogonal set in the interval (0, 1) with respect to the weighting function, $\omega = r+(1-r+^2)$, we have the following orthogonal property.

$$\int_0^1 r+(1-r+^2) R_{np}(r+) R_{mp}(r+) dr+ = 0 \qquad np \neq mp \tag{1.87}$$

The coefficients of expansion in Equations (1.83), (1.84) and (1.85) (i.e., a_{n0}, b_{n0}, b_{np}) are calculated exactly by the following relationships which are obtained after utilizing the orthogonal property of the eigenfunctions given by Equation (1.87).

$$a_{n0} = +\frac{1}{4}\,\frac{\int_0^1 r^+(1-r^{+2})\left(\frac{7}{24}-r^{+2}+\frac{r^{+4}}{4}\right)R_{n0}(r^+)dr^+}{\int_0^1 r^+(1-r^{+2})R_{n0}^2(r^+)dr^+} \tag{1.88}$$

$$a_{np} = +\frac{1}{\pi}\,\frac{\int_0^1\int_0^{2\pi} r^+(1-r^{+2})\cos p\phi\ \hat{\Theta}_{fd}\ R_{np}(r^+)d\phi\ dr^+}{\int_0^1 r^+(1-r^{+2})R_{np}^2(r^+)dr^+} \tag{1.89}$$

$$b_{np} = +\frac{1}{\pi}\,\frac{\int_0^1\int_0^{2\pi} r^+(1-r^{+2})\sin p\phi\ \hat{\Theta}_{fd}\ R_{np}(r^+)d\phi\ dr^+}{\int_0^1 r^+(1-r^{+2})R_{np}^2(r^+)dr^+} \tag{1.90}$$

1.4 Special Examples

1.4.1 Cosine Heat Flux Variation Around the Tube Periphery

As an illustrative case, a cosine circumferential heat flux distribution of the form $q(\phi) = q_{av}\ (1 + b\cos p\phi)$ is considered. A functional relationship of this form is of special interest. Furthermore, the simultaneous effects of circumferential wall heat flux variation and axial conduction on the convection process are investigated. The following cases are considered.

1.4.1.1 Asymptotic Examples. Letting $x+ \to \infty$, Equations (1.71), (1.81), and (1.82) reduce to

$$\Theta(x+, r+, \phi) = -\frac{7}{96} + x+ + \frac{r+^2}{4} - \frac{r+^4}{16} + \frac{4}{Pe^2} \sum_{n=1}^{\infty} a_{n0} \lambda_{n0}^2 \int_0^1 R_{n0}(r+) r+ \, dr+$$

$$+ \sum_{n=1}^{\infty} r+^n [a_n \cos n\phi + b_n \sin n\phi] \qquad (1.91)$$

$$\frac{t_w - t_m}{\bar{q}\, 2r_0 / k\pi} = \frac{11}{96} + \sum_{n=1}^{\infty} (a_n \cos n\phi + b_n \sin n\phi) \qquad (1.92)$$

$$Nu(\phi) = \frac{\pi f(\phi)}{\frac{11}{96} + \sum_{n=1}^{\infty} (a_n \cos n\phi + b_n \sin n\phi)} \qquad (1.93)$$

Solving for a_n, b_n, $\bar{q}$, and $f(\phi)$ using Equations (1.49a, b) and (1.78a, b) we obtain the following coefficients.

$$a_p = \frac{b}{4p}, \quad b_n = 0, \quad f(\phi) = \frac{1 + b\cos\phi}{2\pi}, \quad \bar{q} = 2\pi q_{av}. \qquad (1.94)$$

Substituting for a_n, b_n, $\bar{q}$, and $f(\phi)$ from (1.94) into (1.91), (1.92), and (1.93) we obtain

$$\frac{t-t_\epsilon}{q_{av}r_o/k} = -\frac{7}{24} + 4x+ + r+^2 - \frac{r+^4}{4} + \frac{br+^p}{p}\cos p\phi$$

$$+ \frac{8}{Pe^2}\sum_{n=1}^{\infty} a_{n0}\lambda_{n0}^2 \int_0^1 R_{n0}(r+)r+\, dr+ \qquad (1.95)$$

$$\frac{t_w - t_m}{q_{av}r_0/k} = \frac{11}{24} + \frac{b}{p}\cos p\phi \qquad p = 1, 2, 3, \ldots \qquad (1.96)$$

$$Nu(\phi) = \frac{1+b\cos\phi}{\frac{11}{48} + \frac{b}{2p}\cos p\phi} \qquad p = 1, 2, 3, \ldots \qquad (1.97)$$

For the special case with $p = 1$, Equation (1.97) becomes

$$Nu(\phi) = \frac{1+b\cos\phi}{\frac{11}{48} + \frac{b}{2}\cos\phi} \qquad (1.98)$$

which is the solution obtained by Reynolds [49].

Equation (1.97) demonstrates that axial conduction does not influence the asymptotic Nusselt number. This was verified experimentally by Petukhov and Yushin [45], Eckert and Peterson [10], and Emery and Bailey [11]. It is interesting to note that axial conduction does influence the local fluid temperature given by Equation (1.95).

For the case of uniform wall heat flux, $b = 0$, Equations (1.95), (1.97) reduce to

$$\frac{t-t_\epsilon}{q_{av}r_0/k} = -\frac{7}{24} + 4x+ + r+^2 - \frac{r+^4}{4}$$

$$+ \frac{8}{Pe^2}\sum_{n=1}^{\infty} a_{n0}\lambda_{n0}^2 \int_0^1 R_{n0}(r+)r+\, dr+ \qquad (1.99)$$

and

$$Nu = \frac{48}{11} \qquad (1.100)$$

Finally, for the case where axial conduction is not present, Peclet number goes to infinity and Equations (1.99) and (1.100) yield

$$\frac{t-t_\epsilon}{q_{av}r_0/k} = 4x+ + r+^2 - \frac{r+^4}{4} - \frac{7}{24} \qquad (1.101)$$

$$Nu = \frac{48}{11} \qquad (1.102)$$

which are the asymptotic values given by Siegel, Sparrow, Hallman [57] and Kays [25].

1.4.1.2 Thermal-Entry-Length Examples for Peclet Number of Infinity. Equation (1.63a) is reduced to the following form using Equation (1.94)

$$\hat{\Theta}_{fd} = \frac{7}{96} - \frac{r+^2}{4} + \frac{r+^4}{16} - \frac{br+^p}{4p}\cos p\phi \qquad (1.103)$$

Now, the coefficients a_{n0}, a_{np}, and b_{np} are obtained after substitution for $\hat{\Theta}_{fd}$ from Equation (1.103) into Equations (1.89) and (1.90).

$$a_{n0} = \frac{\int_0^1 r+(1-r+^2)\left(\frac{7}{96} - \frac{r+^2}{4} + \frac{r+^4}{16}\right)R_{n0}(r+)dr+}{\int_0^1 r+(1-r+^2)R_{n0}^2(r+)dr+} \tag{1.104a}$$

$$a_{np} = \frac{-b\int_0^1 (1-r+^2)(r+)^{p+1}R_{np}(r+)dr+}{4p\int_0^1 r+(1-r+^2)R_{np}^2(r+)dr+} \tag{1.104b}$$

$$b_{np} = 0$$

The numerators of (1.104a, b) are simplified by substituting for $(r+)^{p+1}(1-r+^2)R_{np}(r+)$ from the characteristic Equation (1.86) and integrating twice by parts to obtain the following general equations for the expansion coefficients.

$$\hat{a}_{n0} \equiv 4a_{n0} = \frac{-R_{no}(1)}{\lambda_{n0}^2\int_0^1 r+(1-r+^2)R_{n0}^2(r+)dr+} \tag{1.105a}$$

$$\hat{a}_{np} \equiv \frac{4p}{b}\, a_{np} = \frac{-R_{np}(1)}{\lambda_{np}^2\int_0^1 r+(1-r+^2)R_{np}^2(r+)dr+} \tag{1.105b}$$

The above equations were used to evaluate the expansion coefficients with axial conduction absent (i.e., Peclet number $\rightarrow \infty$).

For the case where the circumferential heat flux varies according to $q(\phi) = q_{av}(1+b\cos\phi)$, the only non-zero coefficients are

$\hat{a}_{n0}$, and $\hat{a}_{n1}$. The expressions for fluid temperature, wall temperature, and Nusselt number are obtained by simplifying Equations (1.83), (1.84), and (1.85) using (1.94).

$$\frac{t-t_\epsilon}{q_{av}r_0/k} = 4x+ + r+^2 - \frac{r+^4}{4} - \frac{7}{24} + \sum_{n=1}^{\infty} \hat{a}_{n0}R_{n0}(r+)e^{-\lambda_{n0}^2 x+}$$

$$+ b\cos\phi\left[r+ + \sum_{n=1}^{\infty} \hat{a}_{n1}R_{n1}(r+)e^{-\lambda_{n1}^2 x+}\right] \tag{1.106}$$

$$\frac{t_w-t_m}{q_{av}r_0/k} = \frac{11}{24} + \sum_{n=1}^{\infty} \hat{a}_{n0}R_{n0}(1)e^{-\lambda_{n0}^2 x+}$$

$$+ b\cos\phi\left[1 + \sum_{n=1}^{\infty} \hat{a}_{n1}R_{n1}(1)e^{-\lambda_{n1}^2 x+}\right] \tag{1.107}$$

$$Nu(\phi, x+) = \frac{2(1+b\cos\phi)}{\frac{11}{24} + \sum_{n=1}^{\infty} \hat{a}_{n0}R_{n0}(1)e^{-\lambda_{n0}^2 x+} + b\cos\phi\left[1+\sum_{n=1}^{\infty} \hat{a}_{n1}R_{n1}(1)e^{-\lambda_{n1}^2 x+}\right]} \tag{1.108}$$

For the case of uniform wall heat flux, $b = 0$, the only non-zero expansion coefficients are $\hat{a}_{n0}$. Equations (1.106), (1.107), and (1.108) thus reduce to

$$\frac{t-t_\epsilon}{q_{av}r_0/k} = 4x+ - \frac{7}{24} + r+^2 - \frac{r+^4}{4} + \sum_{n=1}^{\infty} \hat{a}_{n0}R_{n0}(r+)e^{-\lambda_{n0}^2 x+} \tag{1.109}$$

$$\frac{t_w - t_m}{q_{av} r_0 / k} = \frac{11}{24} + \sum_{n=1}^{\infty} \hat{a}_{n0} R_{n0}(1) e^{-\lambda_{n0}^2 x+} \tag{1.110}$$

$$Nu(x+) = \frac{2}{\frac{11}{24} + \sum_{n=1}^{\infty} \hat{a}_{n0} R_{n0}(1) e^{-\lambda_{n0}^2 x+}} \tag{1.111}$$

which are the expressions obtained by Siegel, Sparrow, and Hallman [59].

1.4.1.3 Thermal-Entry-Length Examples for Finite Peclet Number. For this case Equations (1.66a, b, c) reduce to the following expressions after substituting for $\hat{\Theta}_{fd}(r+, \phi)$ from Equation (1.103) and expressing a_{n0} and a_{np} in terms of $\hat{a}_{n0}$ and $\hat{a}_{np}$:

$$\frac{7}{24} - r+^2 + \frac{r+^4}{4} = \sum_{n=1}^{\infty} \hat{a}_{n0} \left[R_{n0} + \frac{4\lambda_{n0}^2}{Pe^2} \int_0^1 R_{n0}(r+) r+ \, dr+ \right] \tag{1.112}$$

$$- \frac{(r+)^p}{p} = \sum_{n=1}^{\infty} \hat{a}_{np} R_{np}(r+) \tag{1.113}$$

For the case where the circumferential heat flux varies according to $q(\phi) = q_{av}(1+b \cos \phi)$, the only non-zero coefficients are $\hat{a}_{n0}$, and $\hat{a}_{n1}$. Equation (1.113) for $p = 1$ becomes

$$-r+ = \sum_{n=1}^{\infty} \hat{a}_{n1} R_{n1}(r+) \tag{1.114}$$

In Equations (1.112) and (1.114), we need to expand $(\frac{7}{24} - r+^2 + \frac{r+^4}{4})$ and $(-r+)$ in terms of the non-orthogonal eigenfunctions of the characteristic Equation (1.52). These equations are of the same form as Equation (1.67), where $g(r+) = (\frac{7}{24} - r+^2 + \frac{r+^4}{4})$ in Equation (1.112), and $g(r+) = (-r+)$ in Equation (1.114). Therefore, the procedure outlined for the solution of Equation (1.67) (i.e., least squares) given by the expression (1.69) can be utilized to obtain $\hat{a}_{n0}$, $\hat{a}_{n1}$.

For sufficiently large Peclet number Equations (1.70a, b, c) reduce to the following expressions by using Equation (1.103) for $\hat{\Theta}_{fd}$.

$$\hat{a}_{n0} \simeq \frac{\int_0^1 r+\left[2(\frac{\lambda_{n0}}{Pe})^2+1-r+^2\right]\left[R_{n0}(r+)+\frac{4\lambda_{n0}^2}{Pe^2}\int_0^1 R_{n0}(r+)r+dr+\right]\left[\frac{7}{24}-r+^2+\frac{r+^4}{4}\right]dr+}{\int_0^1 r+\left[2(\frac{\lambda_{n0}}{Pr})^2+1-r+^2\right]\left[R_{n0}(r+) + \frac{4\lambda_{n0}^2}{Pe^2}\int_0^1 R_{n0}(r+)r+dr+\right]^2 dr+} \tag{1.115}$$

$$\hat{a}_{np} \simeq -\frac{1}{p}\,\frac{\int_0^1 (r+)^{p+1}\left[2(\frac{\lambda_{np}}{Pe})^2+1 - r+^2\right]R_{np}(r+)dr+}{\int_0^1 r+\left[2(\frac{\lambda_{np}}{Pe})^2+1 - r+^2\right]R_{np}^2(r+)dr+} \qquad p > 0 \tag{1.116}$$

The determination of the coefficients of expansion for finite Peclet number was outlined above. We now proceed to determine the expressions for fluid temperature, wall temperature, and Nusselt number for the special problem where $q(\phi) = q_{av}(1+b\cos\phi)$ and the Peclet number is finite.

Substituting for a_n, b_n, $\overline{q}$, and $f(\phi)$ from Equation (1.94) in (1.71), (1.81), and (1.82), we obtain

$$\frac{t-t_\epsilon}{q_{av}r_0/k} = 4x+ - \frac{7}{24} + r+^2 - \frac{r+^4}{4} + \frac{4}{Pe^2}\sum_{n=1}^{\infty}\hat{a}_{n0}\lambda_{n0}^2\int_0^1 R_{n0}(r+)r+\,dr+$$

$$+ \sum_{n=1}^{\infty}\hat{a}_{n0}R_{n0}(r+)e^{-\lambda_{n0}^2 x+} + b\cos\phi\left[r+ + \sum_{n=1}^{\infty}\hat{a}_{n1}R_{n1}(r+)e^{-\lambda_{n1}^2 x+}\right] \qquad (1.117)$$

$$\frac{t_w-t_m}{q_{av}r_0/k} = \frac{11}{24} + \frac{4}{Pe^2}\sum_{n=1}^{\infty}\hat{a}_{n0}\lambda_{n0}^2 e^{-\lambda_{n0}^2 x+}\int_0^1 R_{n0}(r+)r+\,dr+$$

$$+ \sum_{n=1}^{\infty}\hat{a}_{n0}R_{n0}(1)e^{-\lambda_{n0}^2 x+} + b\cos\phi\left[1 + \sum_{n=1}^{\infty}\hat{a}_{n1}R_{n1}(1)e^{-\lambda_{n1}^2 x+}\right] \qquad (1.118)$$

$$\mathrm{Nu}(x+, \phi) = 2(1 + b\cos\phi)\left\{\frac{11}{24} + \sum_{n=1}^{\infty} \hat{a}_{n0} R_{n0}(1) e^{-\lambda_{n0}^2 x+} + \frac{4}{\mathrm{Pe}^2}\sum_{n=1}^{\infty} \hat{a}_{n0}\lambda_{n0}^2 e^{-\lambda_{n0}^2 x+}\int_0^1 R_{n0}(r+)r+\,dr+ + b\cos\phi\left[1 + \sum_{n=1}^{\infty} \hat{a}_{n1} R_{n1}(1) e^{-\lambda_{n1}^2 x+}\right]\right\}^{-1} \tag{1.119}$$

For the special case of a finite Peclet number and uniform wall heat flux, $b = 0$, and $\hat{a}_{n0}$ are the only non-zero expansion coefficients. Equations (1.117), (1.73), (1.118) and (1.119) reduce to

$$\frac{t - t_\epsilon}{q_{av} r_0/k} = 4x+ - \frac{7}{24} + r+^2 - \frac{r+^4}{4} + \sum_{n=1}^{\infty} \hat{a}_{n0} R_{n0}(r+) e^{-\lambda_{n0}^2 x+} + \frac{4}{\mathrm{Pe}^2}\sum_{n=1}^{\infty} \hat{a}_{n0}\lambda_{n0}^2 \int_0^1 R_{n0}(r+)r+\,dr+ \tag{1.120}$$

$$\frac{t_m - t_\epsilon}{q_{av} r_0/k} = 4x+ + \frac{4}{\mathrm{Pe}^2}\sum_{n=1}^{\infty} \hat{a}_{n0}\lambda_{n0}^2 \int_0^1 R_{n0}(r+)r+\,dr+ - \frac{4}{\mathrm{Pe}^2}\sum_{n=1}^{\infty} \hat{a}_{n0}\lambda_{n0}^2 e^{-\lambda_{n0}^2 x+}\int_0^1 R_{n0}(r+)r+\,dr+ \tag{1.121}$$

$$\frac{t_w - t_m}{q_{av} r_0 / k} = \frac{11}{24} + \frac{4}{Pe^2} \sum_{n=1}^{\infty} \hat{a}_{n0} \lambda_{n0}^2 e^{-\lambda_{n0}^2 x+} \int_0^1 R_{n0}(r+) r+ \, dr+$$

$$+ \sum_{n=1}^{\infty} \hat{a}_{n0} R_{n0}(1) e^{-\lambda_{n0}^2 x+} \tag{1.122}$$

$$Nu(x+, \phi) = 2 \left\{ \frac{11}{24} + \sum_{n=1}^{\infty} \hat{a}_{n0} R_{n0}(1) e^{-\lambda_{n0}^2 x+} \right.$$

$$\left. + \frac{4}{Pe^2} \sum_{n=1}^{\infty} \hat{a}_{n0} \lambda_{n0}^2 e^{-\lambda_{n0}^2 x+} \int_0^1 R_{n0}(r+) r+ \, dr+ \right\}^{-1} \tag{1.123}$$

Equation (1. 123) for Nusselt number is in excellent agreement with Equation (25) of Hsu [19]; however, Hsu made an error in his analysis and obtained the following expressions for the fluid temperature and mean temperature in place of the Equations (1. 120) and (1. 121) of this work.

$$\frac{t - t_\epsilon}{q_{av} r_0 / k} = 4x+ - \frac{7}{24} + r+^2 - \frac{r+^4}{4} + \sum_{n=1}^{\infty} \hat{a}_{n0} R_{n0}(r+) e^{-\lambda_{n0}^2 x+}$$

$$\frac{t_m - t_\epsilon}{q_{av} r_0 / k} = 4x+ - \frac{4}{Pe^2} \sum_{n=1}^{\infty} \hat{a}_{n0} \lambda_{n0}^2 e^{-\lambda_{n0}^2 x+} \int_0^1 R_{n0}(r+) r+ \, dr+$$

1.5 Results

1.5.1 Numerical Determination of the Eigenvalues, Eigenfunctions, and Expansion Coefficients

Using the CDC 6400, the first 12 eigenvalues and eigenfunctions of the characteristic Equation (1.52) for $p = 0, 1, 2$ and Peclet numbers of 5, 10, 20, 30, 50, 100 have been obtained. The resulting eigenvalues and eigenfunctions are used to determine the expansion coefficients. For low Peclet numbers (i.e., $Pe = 5, 10, 20$), the expansion coefficients are obtained by the method of least squares, and for sufficiently high Peclet numbers (i.e., $Pe = 30, 50, 100$) expressions (1.115) and (1.116) are used to evaluate these coefficients.

For the limiting problem of uniform wall heat flux $(p = 0)$, the eigenvalues and eigenfunctions at the tube wall are in excellent agreement with the corresponding values obtained by Hsu [19]. However, Hsu made an error in the determination of the coefficients of the non-orthogonal expansion in assuming the eigenfunctions to be orthogonal with respect to a known weighting function. In this work, the expansion coefficients for the case of uniform wall heat flux are obtained from Equation (1.112) in conjunction with the least squares method and also by the approximate method of Equation (1.115).
Table 1.1 presents a comparison of the eigenvalues, eigenfunctions, and expansion coefficients with the results of Hsu. The accuracy of

Table 1.1. Comparison of eigenvalues, eigenfunctions at tube wall, and expansion coefficients for uniform wall heat flux (p = 0) and Peclet numbers of 5, 10, 20, 30, 50 and 100 with the result of Hsu [19].

n	λ_{n0}	$R_{n0}(1)$	$\hat{a}_{n0}$ (Eq. 1.116)	$\hat{a}_{n0}$ (Least Squares)	$\int_0^1 r^+ R_{n0}(r^+)dr^+$
(1) Pe = 5: Present Work					
1	3.5988876	-.4640022	.4967833	4.9325154E-01	-.0520689
2	5.2843136	.3339972	-.2670201	-2.0048448E-01	.0115080
3	6.5834339	-.2701109	.1580058	1.0508700E-01	-.0034233
4	7.6746650	.2321182	-.1019974	-6.4090176E-02	.0013057
5	8.6323615	-.2064987	.0713815	4.3044300E-02	-.0005982
6	9.4954903	.1877844	-.0531438	-3.0714883E-02	.0003129
7	10.2872754	-.1733598	.0414194	2.2742068E-02	-.0001803
8	11.0228168	.1618080	-.0334093	-1.7175562E-02	.0001117
9	11.7125652	-.1522894	.0276672	1.3031320E-02	-.0000733
10	12.3641008	.1442712	-.0233943	-9.7518042E-03	.0000503
11	12.9831251	-.1373971	.0201122	6.9646387E-03	-.0000357
12	13.5740531	.1314191	-.0175332	-4.2965751E-03	.0000262
(1) Pe = 5: Hsu					
1	3.598889	-.464000	.499297		-.0520686
2	5.284307	.333996	-.222230		.0115073
3	6.583434	-.270110	.119823		-.00342328
4	7.674666	.232117	-.0741155		.00130576
5	8.632364	-.206498	.0507140		-.000598351
6	9.495494	.187784	-.0372547		.000313020
7	10.28728	-.173359	.0287822		-.000180393
8	11.02282	.161807	-.0230710		.000111756
9	11.71257	-.152289	.0190197		-.0000732955
10	12.36410	.144271	-.0160251		.0000502644
11	12.98313	-.137397	.0137418		-.0000357868
12	13.57406	.131418	-.0119519		.0000262429
(2) Pe = 10: Present Work					
1	4.3345060	-.4837456	.4644293	4.7237908E-01	-.0748196
2	6.7407717	.3595440	-.2690251	-2.1571311E-01	.0236097
3	8.6329181	-.2891267	.1782808	1.2123166E-01	-.0082918
4	10.2294108	.2457087	-.1207206	-7.5291985E-02	.0033425
5	11.6291065	-.2165869	.0851330	5.0338981E-02	-.0015403
6	12.8875272	.1955923	-.0628566	-3.5515532E-02	.0007951
7	14.0389783	-.1796149	.0483573	2.5987096E-02	-.0004498
8	15.1061298	.1669579	-.0384892	-1.9424350E-02	.0002736
9	16.1047775	-.1566224	.0314878	1.4614688E-02	-.0001764
10	17.0464029	.1479812	-.0263388	-1.0868418E-02	.0001192
11	17.9396407	-.1406198	.0224336	7.7339315E-03	-.0000836
12	18.7911687	.1342521	-.0193966	-4.7822892E-03	.0000606

Table 1.1. Continued.

n	λ_{n0}	$R_{n0}(1)$	$\hat{a}_{n0}$ (Eq. 1.116)	$\hat{a}_{n0}$ (Least Squares)	$\int_0^1 r+ R_{n0}(r+)dr+$
(2) Pe = 10: Hsu					
1	4.33450	-.483746	.465121		-.0748186
2	6.74077	.359544	-.243045		.0236096
3	8.63292	-.289127	.146465		-.00829184
4	10.22941	.245709	-.0935418		.00334237
5	11.62911	-.216587	.0636698		-.00154025
6	12.88753	.195592	-.0459578		.000795171
7	14.03898	-.179615	.0348186		-.000449866
8	15.10614	.166958	-.0274142		.000273804
9	16.10479	-.156622	.0222471		-.000176604
10	17.04642	.147981	-.0184983		.000119493
11	17.93967	-.140620	.0156844		-.0000840654
12	18.79121	.134252	-.0135174		.0000611847
(3) Pe = 20: Present Work					
1	4.8005295	-.4912222	.4304247	4.4606273E-01	-.0895407
2	8.0019897	.3840379	-.2431798	-2.1505890E-01	.0396699
3	10.6608834	-.3172395	.1834832	1.3338288E-01	-.0183183
4	12.9605450	.2698456	-.1402822	-8.8874666E-02	.0086871
5	15.0008331	-.2359056	.1060923	6.1570983E-02	-.0043330
6	16.8440194	.2110246	-.0806391	-4.3976762E-02	.0023008
7	18.5326903	-.1921503	.0623488	3.2172446E-02	-.0013021
8	20.0971657	.1773431	-.0492739	-2.3912511E-02	.0007817
9	21.5596125	-.1653850	.0398166	1.7864440E-02	-.0004945
10	22.9366366	.1554932	-.0328417	-1.3203726E-02	.0003273
11	24.2409663	-.1471475	.0275850	9.3694540E-03	-.0002251
12	25.4825642	.1399905	-.0235374	-5.8410022E-03	.0001601
(3) Pe = 20: Hsu					
1	4.800531	-.491220	.430415		-.0895402
2	8.001987	.384036	-.234730		.0396696
3	10.66088	-.317238	.166261		-.0183180
4	12.96054	.269844	-.119841		.00868695
5	15.00083	-.235905	.0865912		-.00433295
6	16.84402	.211024	-.0636293		.00230079
7	18.53269	-.192149	.0479774		-.00130206
8	20.097163	.177342	-.0372045		.000781744
9	21.559610	-.165384	.0296261		-.000494527
10	22.936634	.155492	-.0241548		.000327290
11	24.240965	-.147147	.0201002		-.000225152
12	25.482564	.139990	-.0170199		.000160085

Table 1.1. Continued.

n	λ_{n0}	$R_{n0}(1)$	$\hat{a}_{n0}$ (Eq. 1.116)	$\int_0^1 r+ R_{n0}(r+)dr+$
(4) Pe = 30: Present Work				
1	4.9361184	-.4921947	.4175560	-.0936872
2	8.5096337	.3914558	-.2202316	.0473239
3	11.6259614	-.3320388	.1707199	-.0257199
4	14.3925222	.2872632	-.1414867	.0140391
5	16.8876479	-.2523569	.1156219	-.0077367
6	19.1655642	.2253145	-.0928546	.0043706
7	21.2658440	-.2042589	.0741678	-.0025572
8	23.2187473	.1875953	-.0595367	.0015565
9	25.0478587	-.1741355	.0483265	-.0009860
10	26.7717379	.1630417	-.0397749	.0006487
11	28.4051300	-.1537297	.0332155	-.0004419
12	29.9598780	.1457879	-.0281277	.0003105
(4) Pe = 30: Hsu				
1	4.936112	-.492193	.417538	-.0936862
2	8.509624	.391455	-.217097	.0473232
3	11.62595	-.332037	.162101	-.0257195
4	14.39252	.287262	-.128337	.0140389
5	16.88764	-.252356	.100492	-.00773653
6	19.16556	.225313	-.0778508	.00437049
7	21.26584	-.204258	.0603902	-.00255717
8	23.21874	.187594	-.0473469	.00155651
9	25.04785	-.174135	.0377057	-.000985976
10	26.77173	.163041	-.0305544	.000648680
11	28.40513	-.153729	.0251900	-.000441857
12	29.95988	.145787	-.0211041	.000310501
(5) Pe = 50: Present Work				
1	5.0173058	-.4924689	.4090588	-.0961193
2	8.8834123	.3947255	-.1967150	.0531370
3	12.4592203	-.3422159	.1449343	-.0334161
4	15.7692798	.3045142	-.1253637	.0215704
5	18.8422179	-.2735544	.1123305	-.0139016
6	21.7081511	.2472212	-.0994928	.0089239
7	24.3933444	-.2249510	.0863158	-.0057418
8	26.9198292	.2063090	-.0736674	.0037303
9	29.3063511	-.1907546	.0623035	-.0024613
10	31.5690681	.1777263	-.0525553	.0016555
11	33.7219403	-.1667241	.0444302	-.0011373
12	35.7769994	.1573384	-.0377606	.0007986

Table 1.1. Continued.

n	λ_{n0}	$R_{n0}(1)$	$\hat{a}_{n0}$ (Eq. 1.116)	$\int_0^1 r+ R_{n0}(r+)dr+$
(5) Pe = 50: Hsu				
1	5.017300	-.492467	.409055	-.0961183
2	8.883403	.394724	-.196069	.0531363
3	12.45921	-.342214	.142521	-.0334156
4	15.76927	.304513	-.120395	.0215701
5	18.84221	-.273553	.104848	-.0139013
6	21.70814	.247220	-.0902077	.00892372
7	24.39333	-.224950	.0761521	-.00574167
8	26.91982	.206308	-.0634100	.00373022
9	29.30634	-.190754	.0524720	-.00246125
10	31.56906	.177725	-.0434252	.00165546
11	33.72193	-.166723	.0361056	-.00113734
12	35.77699	.157338	-.0302437	.000798583
(6) Pe = 100: Present Work				
1	5.0546124	-.4925134	.4049395	-.0972221
2	9.0831132	.3954485	-.1813803	.0562345
3	12.9813489	-.3455282	.1190075	-.0386110
4	16.7603522	.3128848	-.0949691	.0283288
5	20.4151358	-.2884159	.0848006	-.0213526
6	23.9431293	.2682358	-.0799935	.0162288
7	27.3457161	-.2505043	.0767858	-.0123255
8	30.6270282	.2344012	-.0734708	.0093203
9	33.7925506	-.2196304	.0694380	-.0070147
10	36.8482018	.2061234	-.0646618	.0052621
11	39.7999002	-.1938678	.0593875	-.0039432
12	42.6534229	.1828318	-.0539230	.0029588
(6) Pe = 100; Hsu				
1	5.054612	-.492511	.404939	-.0972214
2	9.0830997	.395447	-.181328	.0562338
3	12.98133	-.345527	.118765	-.0386104
4	16.76034	.312883	-.0943193	.0283284
5	20.41511	-.288415	.0835024	-.0213522
6	23.94311	.268235	-.0778545	.0162285
7	27.34569	-.250503	.0737151	-.0123252
8	30.62701	.234400	-.0694983	.00932010
9	33.79253	-.219629	.0646957	-.00701453
10	36.84818	.206122	-.0593462	.00526199
11	39.79988	-.193866	.0537122	-.00394315
12	42.65341	.182830	-.0480897	.00295876

the expansion coefficients obtained by the present work for $p = 0$ (i.e., uniform wall heat flux) can be checked by comparing the exact function $(\frac{7}{24} - r+^2 + \frac{r+^4}{4})$ in (1.112) in the range of $0 \leq r+ \leq 1$ and its 12-term least squares expansion for Peclet number of 10. This is done and the results are presented in Table 1.2.

Table 1.2. Comparison of the function $(\frac{7}{24} - r+^2 + \frac{r+^4}{4})$ and its 12-term least squares expansion in the range of $0 \leq r+ \leq 1$ for uniform wall heat flux ($p = 0$) and Peclet number of 10.

$r+$	$\frac{7}{24} - r+^2 + \frac{r+^4}{4}$	$\sum_{n=1}^{12} \hat{a}_{n0}(R_{n0}(r+) + \frac{4\lambda_{n0}^2 \alpha_{n0}^*}{Pe})$
0.00	.29167	.29002
.05	.28917	.28891
.10	.28169	.28231
.15	.26929	.26889
.20	.25207	.25198
.25	.23014	.23057
.30	.20369	.20336
.35	.17292	.17285
.40	.13807	.13847
.45	.09942	.09908
.50	.05729	.05721
.55	.01204	.01251
.60	-.03593	-.03633
.65	-.08621	-.08637
.70	-.13831	-.13760
.75	-.19173	-.19228
.80	-.24593	-.24640
.85	-.30033	-.29875
.90	-.35431	-.35567
.95	-.40721	-.40911
1.00	-.45833	-.43168

$^*\alpha_{n0} = \int_0^1 R_{n0}(r+)r+ \, dr+ .$

The determination of eigenvalues, eigenfunctions, and expansion coefficients when axial conduction is included in a tube with an arbitrary circumferential wall heat flux is the main concern of the present work. For a cosine heat flux variation of the form $q(\phi) = q_{av}(1+b\cos\phi)$, the only non-zero expansion coefficients are $\hat{a}_{n0}$ and $\hat{a}_{n1}$. The eigenvalues, eigenfunctions, and expansion coefficients for $p = 1$ are presented in Table 1.3. The expansion coefficients were obtained by the method of least squares for Pe = 5, 10, 20, and by Equation (1.116) for Pe = 5, 10, 20, 30, 50, 100. The accuracy of the coefficients of expansion for $p = 1$, and Pe = 5 was checked by comparing the exact function (-r+) in Equation (1.114) and its 12-term least squares expansion. Table 1.5 shows the comparative results.

For cases involving variation of wall heat flux in the form $\cos p\phi$ or $\sin p\phi$ $(p \neq 0, 1)$, additional coefficients, $\hat{a}_{np}$ and $\hat{b}_{np}$, are required. For $p = 2$, the related coefficients are given in Table 1.4. The expansion coefficients were obtained by the approximate Equation (1.116).

In Figures 1.2 and 1.3 the eigenvalues for $p = 0, 1$ are plotted versus Peclet numbers. The magnitudes of the eigenvalues increase with Peclet numbers and asymptotically approach the values for the case of no axial conduction near a Peclet number of 100. This indicates the effect of axial conduction to be negligible for Peclet numbers exceeding 100. This conclusion was also reached by Schneider [55],

Table 1.3. Eigenvalues, eigenfunctions at tube wall, and expansion coefficients for p = 1 and Peclet numbers 5, 10, 20, 30, 50 and 100.

n	λ_{n1}	$R_{n1}(1)$	$\hat{a}_{n1}$ (Eq. 1.116)	$\hat{a}_{n1}$ (Least Squares)	$\int_0^1 r^+ R_{n1}(r^+)dr^+$
Pe = 5					
1	2.3395655	.5034665	-1.4861969	-1.5191623E+00	.2179911
2	4.4798990	-.1308452	1.1298933	8.8401336E-01	.0005677
3	5.9496839	.0647956	-.8617741	-6.3939142E-01	.0038437
4	7.1376609	-.0401927	.6634440	4.9323282E-01	.0006991
5	8.1589039	.0280160	-.5362122	-3.9963237E-01	.0006354
6	9.0676209	-.0209600	.4526074	3.3390404E-01	.0002561
7	9.8940812	.0164431	-.3946338	-2.8342888E-01	.0002107
8	10.6571099	-.0133464	.3523228	2.4148925E-01	.0001153
9	11.3693258	.0111140	-.3200987	-2.0410134E-01	.0000951
10	12.0396638	-.0094421	.2947022	1.6840418E-01	..0000608
11	12.6747203	.0081514	-.2741236	-1.3148986E-01	.0000510
12	13.2795272	-.0071304	.2570740	8.8061658E-02	.0000357
Pe = 10					
1	2.6695425	.4696647	-1.5055578	-1.5523524E+00	.2076048
2	5.5982739	-.1309244	1.3021184	9.8436316E-01	-.0065759
3	7.7140852	.0661968	-1.1911105	-7.7064946E-01	.0050551
4	9.4470416	-.0409682	.9738070	6.0857278E-01	.0000932
5	10.9395496	.0284266	-.7838679	-4.9014303E-01	.0007459
6	12.2654614	-.0211918	.6442110	4.0323442E-01	.0001594
7	13.4684788	.0165833	-.5443459	-3.3671212E-01	.0002257
8	14.5765360	-.0134362	.4717776	2.8277229E-01	.0000897
9	15.6085876	.0111745	-.4175380	-2.3618108E-01	.0000976
10	16.5781218	-.0094844	.3757993	1.9310344E-01	.0000515
11	17.4951109	.0081820	-.3428099	-1.4987862E-01	.0000513
12	18.3671596	-.0071531	.3161269	1.0043446E-01	.0000316
Pe = 20					
1	2.8197826	.4536019	-1.5040813	-1.5422761E+00	.2026325
2	6.4672994	-.1283298	1.2326335	1.0061598E+00	-.0123783
3	9.3694705	.0677853	-1.4076279	-8.6554199E-01	.0071692
4	11.8350853	-.0426121	1.3968095	7.4194318E-01	-.0011056
5	13.9978558	.0295308	-1.2466193	-6.2542009E-01	.0011225
6	15.9351062	-.0218906	1.0633712	5.2351322E-01	-.0000750
7	17.6980099	.0170330	-.8984101	-4.3769187E-01	.0002976
8	19.3224687	-.0137356	.7644877	3.6507794E-01	.0000280
9	20.8344119	.0113810	-.6593968	-3.0216865E-01	.0001146
10	22.2530309	-.0096315	.5773102	2.4497449E-01	.0000300
11	23.5928677	.0082898	-.5126529	-1.8918506E-01	.0000560
12	24.8651860	-.0072340	.4610324	1.2758978E-01	.0000225

Table 1.3. Continued.

n	λ_{n1}	$R_{n1}(1)$	$\hat{a}_{n1}$ (Eq. 1.116)	$\int_0^1 r^+ R_{n1}(r^+)dr^+$
Pe = 30				
1	2.8545778	.4498334	-1.5022573	.2014624
2	6.7747463	-.1263716	1.1175630	-.0142855
3	10.1060500	.0678991	-1.3334928	.0083909
4	13.0364491	-.0435948	1.5110602	-.0020467
5	15.6603235	.0305244	-1.5220368	.0015572
6	18.0423447	-.0226581	1.4107798	-.0003433
7	20.2282494	.0175836	-1.2503466	.0004125
8	22.2524521	-.0141264	1.0876408	-.0000504
9	24.1416343	.0116617	-.9429579	.0001471
10	25.9167186	-.0098370	.8214723	.0000031
11	27.5942514	.0084434	-.7218267	.0000664
12	29.1874493	-.0073511	.6406117	.0000114
Pe = 50				
1	2.8735619	.4477705	-1.5009765	.2008212
2	6.9802511	-.1246824	1.0000705	-.0154775
3	10.6992998	.0672620	-1.1034367	.0094188
4	14.1388678	-.0440567	1.3498381	-.0031129
5	17.3263081	.0315592	-1.5751142	.0022441
6	20.2923130	-.0237911	1.6938552	-.0008407
7	23.0652471	.0185858	-1.6963028	.0006903
8	25.6690302	-.0149384	1.6138519	-.0002397
9	28.1238534	.0122971	-1.4854820	.0002480
10	30.4470629	-.0103296	1.3414231	-.0000694
11	32.6536637	.0088263	-1.2001143	.0001041
12	34.7566170	-.0076514	1.0707484	-.0000187
Pe = 100				
1	2.8818326	.4468702	-1.5003515	.2005413
2	7.0818961	-.1237271	.9272440	-.0160374
3	11.0437856	.0664633	-.8656810	.0099990
4	14.8831962	-.0437110	.9498700	-.0038894
5	18.6004398	.0317845	-1.1274227	.0029304
6	22.1915814	-.0245038	1.3542831	-.0014884
7	25.6562355	.0196010	-1.5852410	.0011865
8	28.9973910	-.0160734	1.7819754	-.0006439
9	32.2200108	.0134205	-1.9201843	.0005328
10	35.3298632	-.0113665	1.9913475	-.0002883
11	38.3328769	.0097447	-1.9996337	.0002519
12	41.2348869	-.0084454	1.9567307	-.0001313

Table 1.4. Eigenvalues, eigenfunctions at tube wall, and expansion coefficients for $p = 2$ and Peclet numbers of 5, 10, 20, 30, 50, and 100.

n	λ_{n2}	$R_{n2}(1)$	$\hat{a}_{n2}$ (Eq. 1.116)	$\int_0^1 r^+ R_{n2}(r^+)dr^+$
Pe = 5				
1.	3.4032501	.2886589	-1.1399639	.1100950
2	5.1919365	-.0510299	1.5343697	.0033174
3	6.5267768	.0193056	-1.7016999	.0014768
4	7.6353599	-.0097093	1.7866822	.0003713
5	8.6030167	.0056906	-1.8533652	.0002033
6	9.4724912	-.0036724	1.9176613	.0000885
7	10.2686159	.0025328	-1.9824711	.0000555
8	11.0072811	-.0018340	2.0477723	.0000307
9	11.6993673	.0013785	-2.1130731	.0000210
10	12.3527068	-.0010672	2.1779397	.0000133
11	12.9731578	.0008462	-2.2420706	.0000097
12	13.5652373	-.0006844	2.3052797	.0000066
Pe = 10				
1	4.2046331	.2367058	-1.2466088	.0952002
2	6.6495793	-.0486234	1.9417994	.0004143
3	8.5690872	.0187870	-2.3281859	.0015878
4	10.1825054	-.0094801	2.4705531	.0002272
5	11.5929365	.0055629	-2.5155926	.0002027
6	12.8585696	-.0035938	2.5342931	.0000686
7	14.0151227	.0024814	-2.5514134	.0000536
8	15.0860360	-.0017988	2.5736529	.0000259
9	16.0875504	.0013535	-2.6017719	.0000201
10	17.0314196	-.0010489	2.6348909	.0000117
11	17.9264526	.0008325	-2.6718690	.0000092
12	18.7794436	-.0006739	2.7117010	.0000060
Pe = 20				
1	4.7440574	.2019163	-1.2952157	.0849643
2	7.9392155	-.0452397	2.1763953	-.0022875
3	10.6041449	.0182943	-3.0511640	.0019445
4	12.9130467	-.0093400	3.6035989	-.0000289
5	14.9615274	.0054807	-3.8445755	.0002371
6	16.8111609	-.0035329	3.9020239	.0000295
7	18.5048086	.0024345	-3.8773293	.0000566
8	20.0731614	-.0017627	3.8260465	.0000166
9	21.5386797	.0013255	-3.7734555	.0000200
10	22.9181788	-.0010269	3.7292457	.0000087
11	24.2245346	.0008151	-3.6960409	.0000089
12	25.4678149	-.0006599	3.6735833	.0000048

Table 1.4. Continued.

n	λ_{n2}	$R_{n2}(1)$	$\hat{a}_{n2}$ (Eq. 1.116)	$\int_0^1 r^+ R_{n2}(r^+)dr^+$
Pe = 30				
1	4.9068008	.1916800	-1.2992654	.0819042
2	8.4693752	-.0431430	2.1380454	-.0032706
3	11.5823963	.0178774	-3.2078639	.0022069
4	14.3511697	-.0092941	4.1805761	-.0002303
5	16.8506410	.0054884	-4.8286529	.0002915
6	19.1330938	-.0035372	5.1511607	-.0000102
7	21.2374062	.0024317	-5.2479027	.0000662
8	23.1937137	-.0017558	5.2185955	.0000069
9	25.0256612	.0013172	-5.1316559	.0000218
10	26.7519063	-.0010187	5.0263589	.0000056
11	28.3872856	.0008074	-4.9225993	.0000092
12	29.9437167	-.0005631	4.8291691	.0000036
Pe = 50				
1	5.0057308	.1855412	-1.2978482	.0800572
2	8.8647177	-.0412323	2.0094935	-.0039035
3	12.4349672	.0172178	-2.9999170	.0024511
4	15.7420018	-.0091433	4.2451264	-.0004715
5	18.8142162	.0055032	-5.5089888	.0003919
6	21.6809832	-.0035837	6.5570951	-.0000825
7	24.3678100	.0024708	-7.2798271	.0000946
8	26.8962058	-.0017814	7.6858694	-.0000145
9	29.2846342	.0013318	-7.8438536	.0000296
10	31.5491292	-.0010258	7.8337849	-.0000014
11	33.7036099	.0008100	-7.7231245	.0000115
12	35.7601053	-.0006529	7.5604440	.0000009
Pe = 100				
1	5.0515937	.1827191	-1.2959977	.0792050
2	9.0777881	-.0400753	1.8908883	-.0042009
3	12.9735566	.0165830	-2.5721933	.0025920
4	16.7502419	-.0088304	3.4810749	-.0006576
5	20.4030340	.0053917	-4.6822397	.0005046
6	23.9294779	-.0035824	6.1446411	-.0001811
7	27.3310077	.0025185	-7.7554763	.0001527
8	30.6117390	-.0018433	9.3629719	-.0000599
9	33.7770876	.0013909	-10.8247955	.0000553
10	36.8328760	-.0010758	12.0411245	-.0000212
11	39.7849263	.0008496	-12.9650754	.0000223
12	42.6389335	-.0006832	13.5956081	-.0000077

Table 1.5. Comparison of the function (-r+) and its 12-term least squares expansion in the range $0 \leq r+ \leq 1$ for $p = 1$ and Peclet number of 5.

r+	(-r+)	$\sum_{n=1}^{12} a_{n1} R_{n1}(r+)$
0.00	0.00000	0.00000
.05	-.05000	-.04908
.10	-.10000	-.10011
.15	-.15000	-.15045
.20	-.20000	-.19959
.25	-.25000	-.24992
.30	-.30000	-.30041
.35	-.35000	-.34973
.40	-.40000	-.39980
.45	-.45000	-.45044
.50	-.50000	-.49984
.55	-.55000	-.54965
.60	-.60000	-.60051
.65	-.65000	-.64999
.70	-.70000	-.69935
.75	-.75000	-.75065
.80	-.80000	-.80035
.85	-.85000	-.84843
.90	-.90000	-.90143
.95	-.95000	-.95183
1.00	-1.00000	-.97340

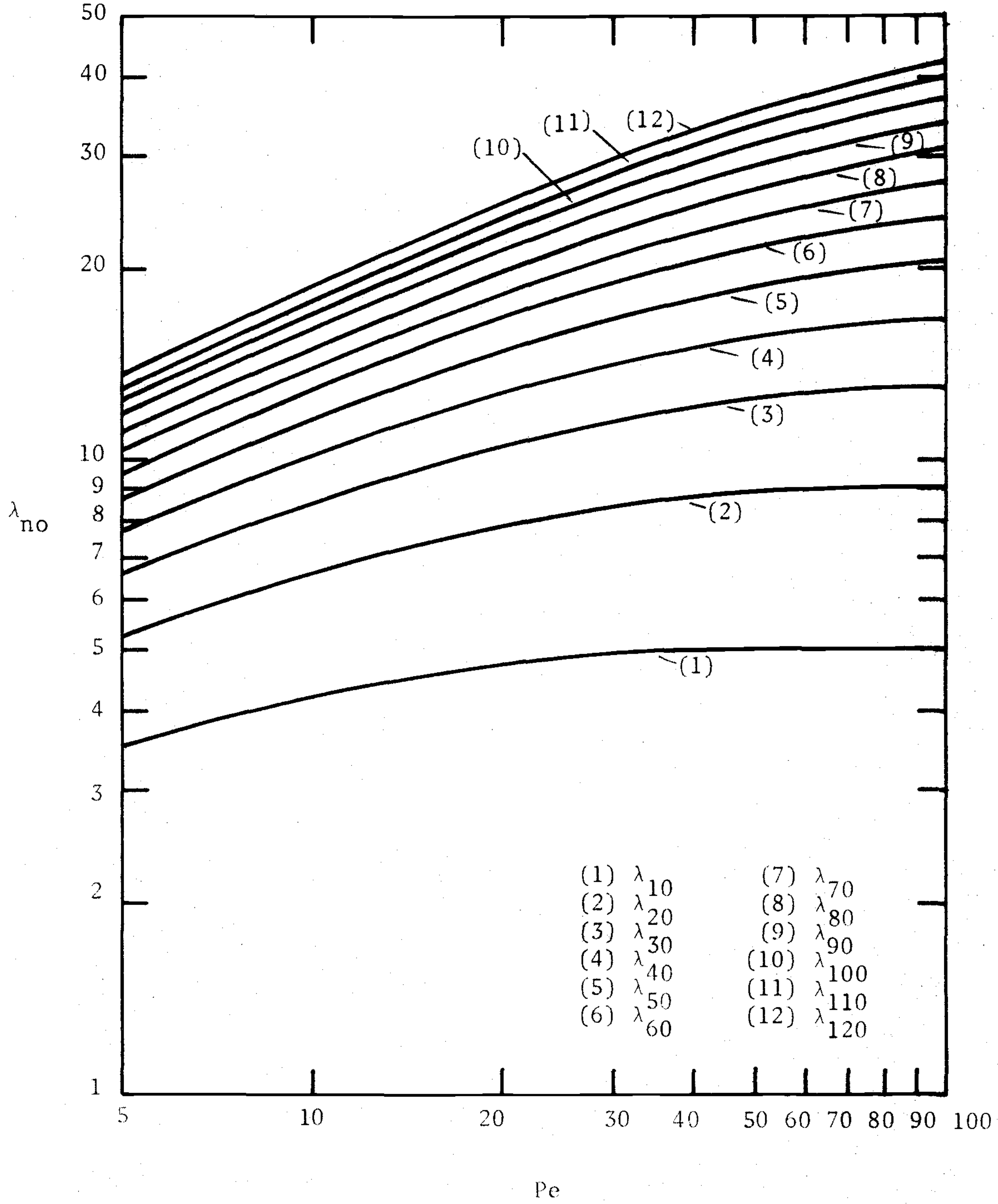

Figure 1.2. Variation of eigenvalues with Peclet number for $p = 0$.

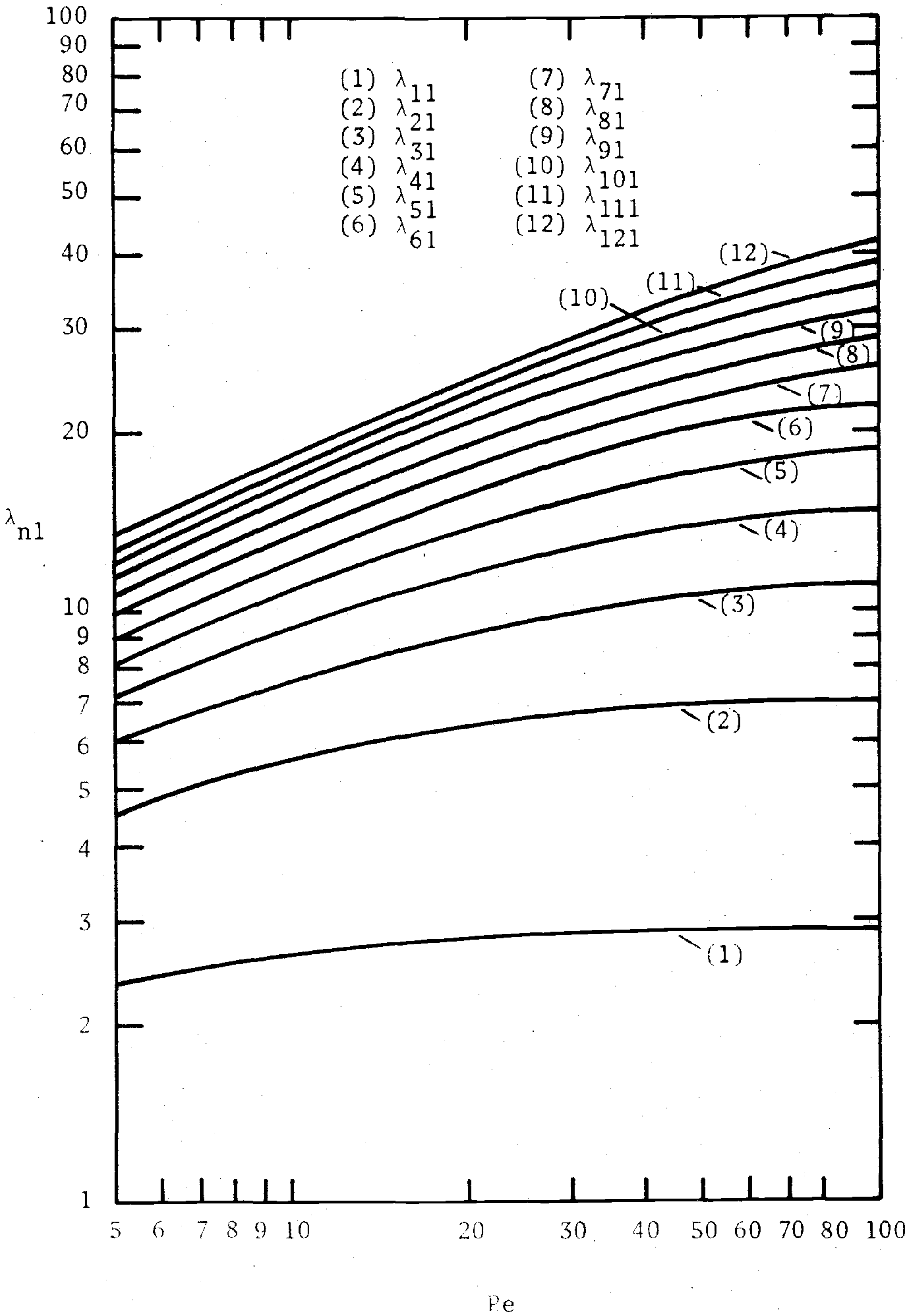

Figure 1.3. Variation of eigenvalues with Peclet number for p = 1.

Singh [61], and Hsu [21] in their analyses of heat transfer including axial conduction for both uniform wall temperature and uniform wall heat flux conditions. In Figures 1.4, 1.5, 1.6, the first two eigenfunctions are shown for $p = 0, 1, 2$ (i.e., R_{00}, R_{10}, R_{01}, R_{11}, R_{02}, R_{12}) and for several values of Peclet number. Figure 1.7 presents similar plots for the third and fourth eigenfunctions for $p = 0$. Finally, the first 12 eigenfunctions for $p = 0, 1, 2$ and Peclet numbers of 5, 10, 20, 30, 50, 100 are included in tabular form in Appendix A. The first five of these eigenfunctions are also represented in graphical form in Appendix B.

For the limiting problem with no axial conduction (i.e., $Pe = \infty$) the eigenfunctions and eigenvalues were obtained from the Sturm Liouville System given by Equation (1.86). The expansion coefficients were determined by (1.104a, b). The first 12 eigenfunctions, eigenvalues, and expansion coefficients are tabulated for $p = 1, 2, 3, 4, 5, 6$ in Table 1.6. These coefficients can be used to determine heat transfer parameters for any arbitrary circumferential wall heat flux that can be expanded in Fourier series up to sixth harmonics. For $p = 0$ the heat flux is uniform and the eigenvalues, eigenfunctions, and expansion coefficients agree with those reported by Siegel, Sparrow, and Hallman [59], and Hsu [18]. The comparison of these results are presented in Table 1.7.

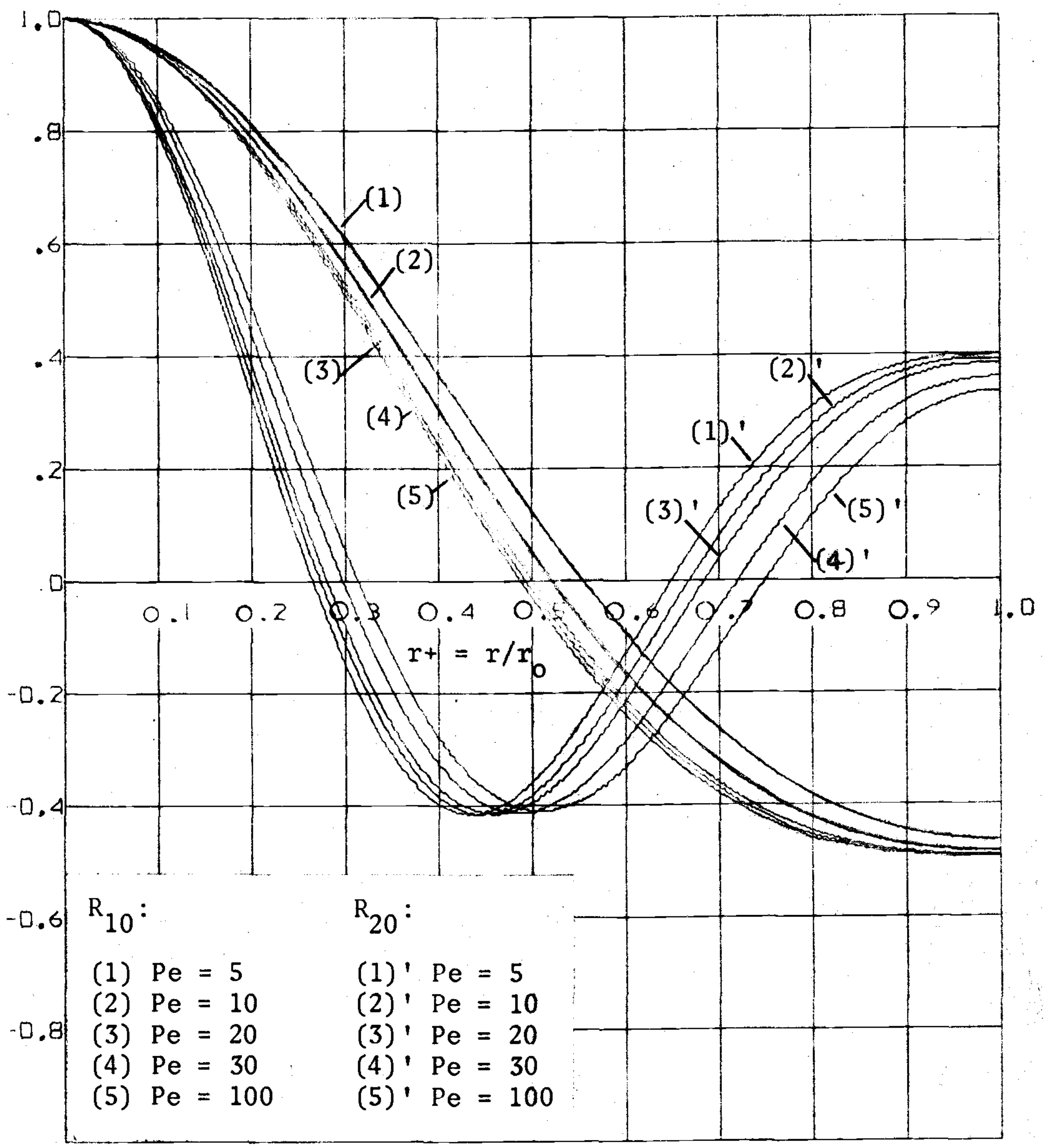

Figure 1.4. The first two eigenfunctions for different Peclet numbers and for $p = 0$ (i.e., uniform wall heat flux).

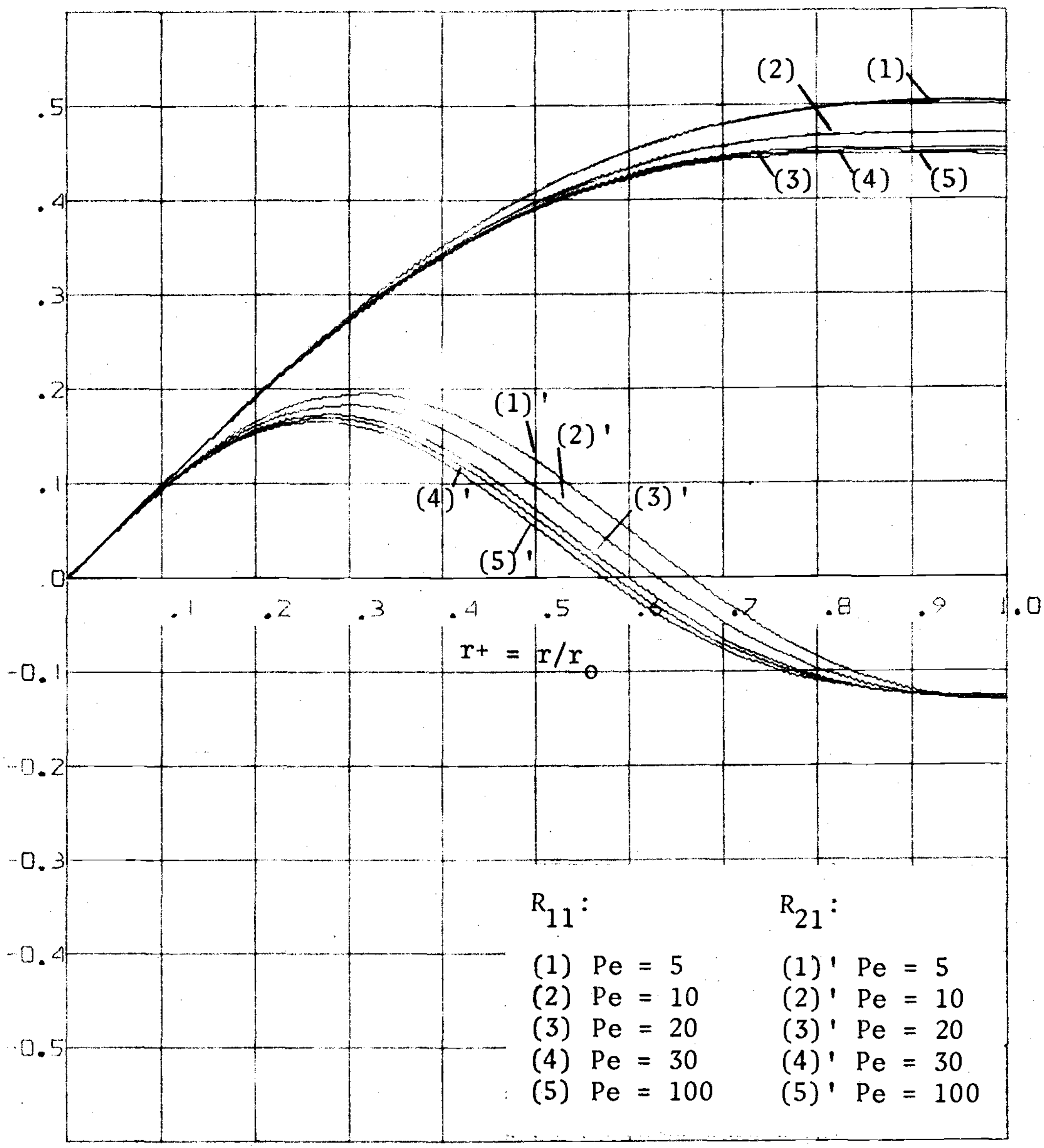

Figure 1.5. The first two eigenfunctions for different Peclet numbers and for p = 1.

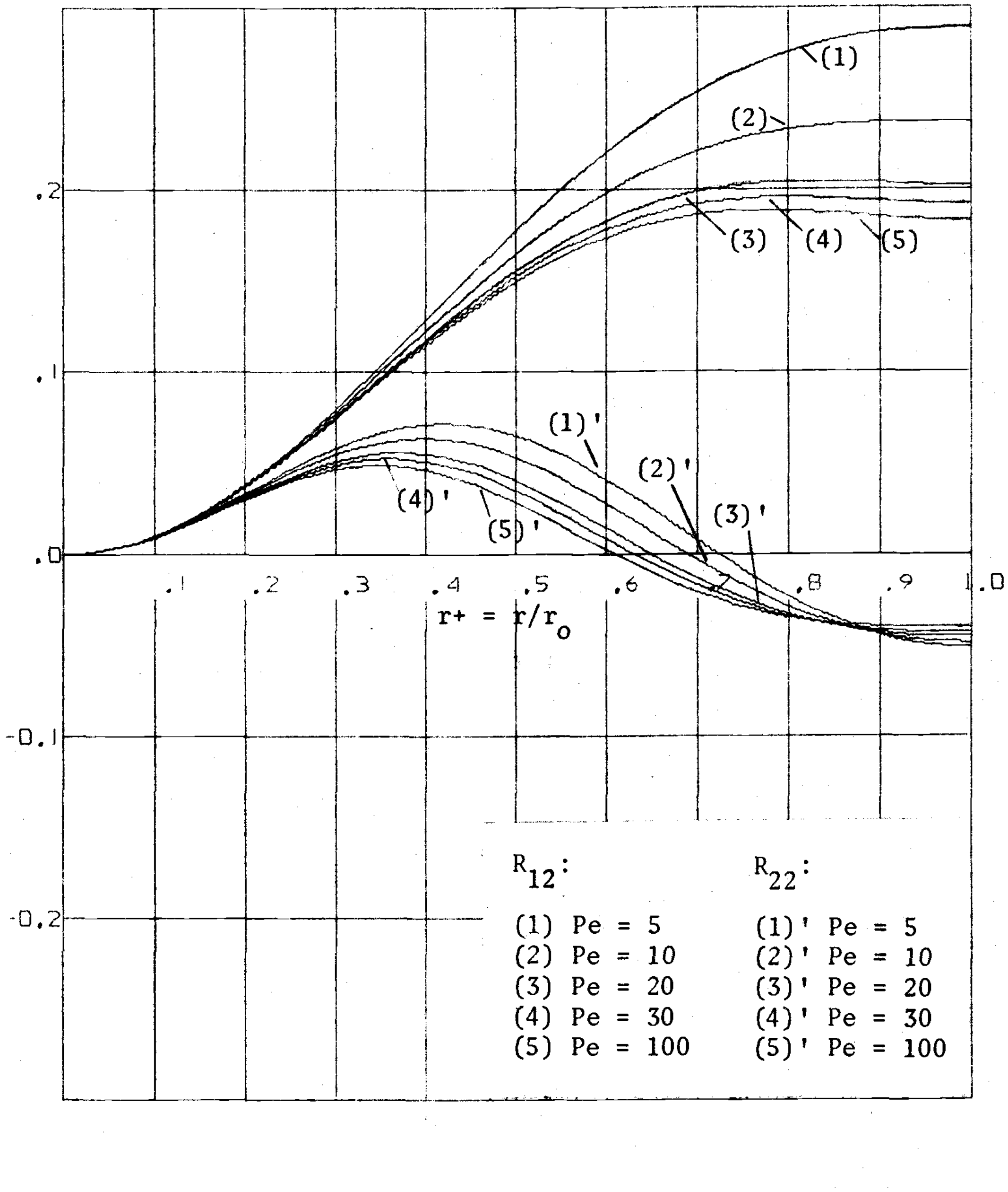

Figure 1.6. The first two eigenfunctions for different Peclet numbers and for p = 2.

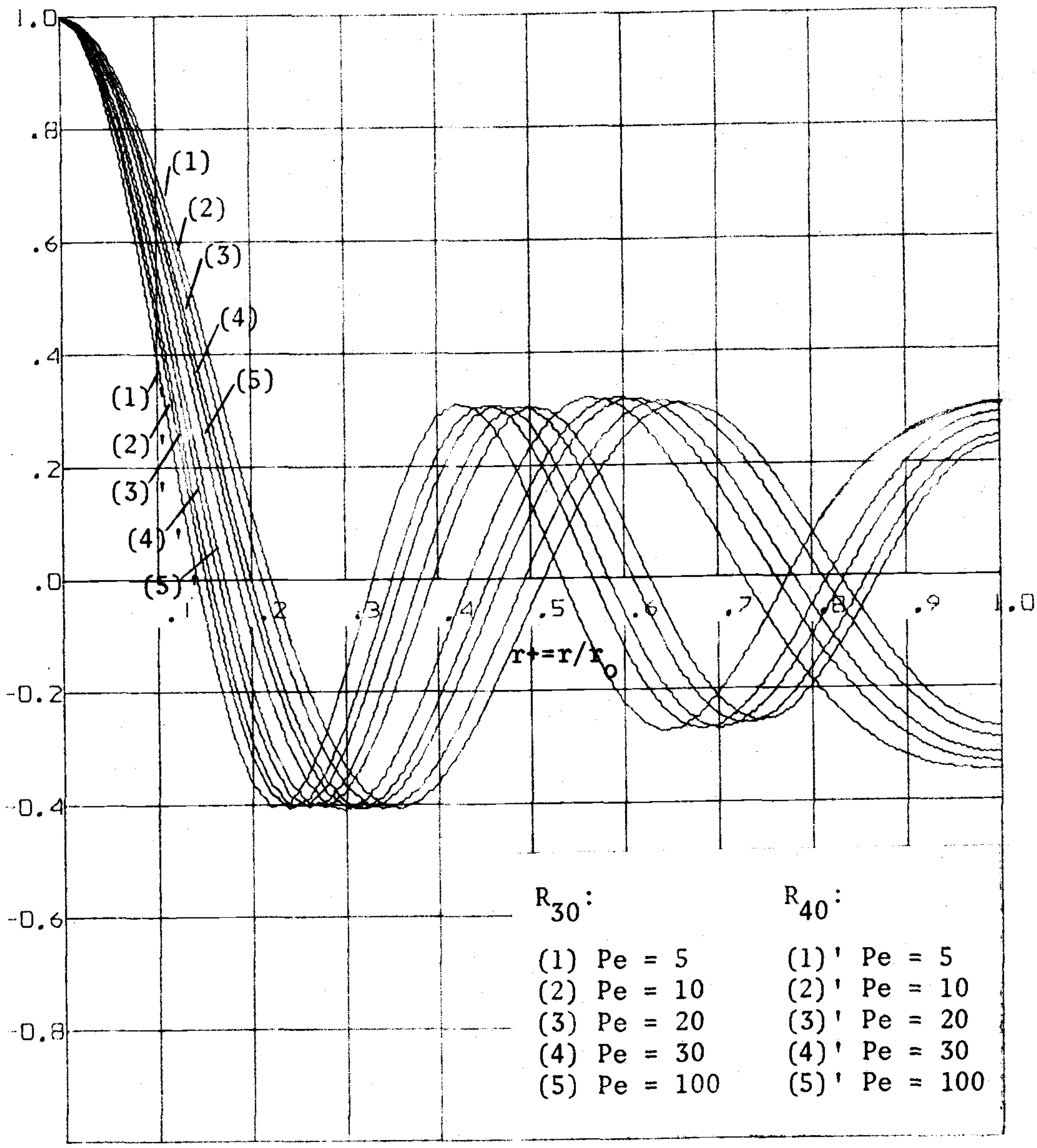

Figure 1.7. The third and fourth eigenfunctions for different Peclet numbers and for p = 0.

Table 1.6. Eigenvalues, eigenfunctions at tube wall, and expansion coefficients for $p = 1, 2, 3, 4, 5, 6,$ and Peclet number of infinity (no axial conduction).

n	λ_{n1}	$R_{n1}(1)$	$\hat{a}_{n1}$
p = 1			
1	2.8846257	.4465660	-1.5001310
2	7.1182769	-.1233654	.8985212
3	11.1789014	.0660488	- .7457699
4	15.2093411	-.0432979	.6623509
5	19.2281902	.0314396	- .6068484
6	23.2412108	-.0242920	.5661359
7	27.2508383	.0195713	- .5344534
8	31.2582968	-.0162501	.5087942
9	35.2642749	.0138029	- .4874053
10	39.2691926	-.0119344	.4691822
11	43.2733218	.0104673	- .4533880
12	47.2768465	-.0092889	.4395086
	λ_{n2}	$R_{n2}(1)$	$\hat{a}_{n2}$
p = 2			
1	5.0675055	.1817437	-1.2951616
2	9.1576064	-.0396193	1.8356290
3	13.1972247	.0162605	-2.2988208
4	17.2202294	-.0085882	2.7163616
5	21.2355173	.0052132	-3.1023198
6	25.2465312	-.0034583	3.4645495
7	29.2549056	.0024404	-3.8080073
8	33.2615237	-.0018022	4.1360765
9	37.2669082	.0013783	-4.4511969
10	41.2713893	-.0010837	4.7552018
11	45.2751869	.0008715	-5.0495135
12	49.2784213	-.0007140	5.3353055

Table 1.6. Continued.

n	λ_{n3}	$R_{n3}(1)$	$\hat{a}_{n3}$
p = 3			
1	7.2301356	.0692356	-1.6041884
2	11.2076358	-.0137637	3.6963680
3	15.2211958	.0047047	-6.2348480
4	19.2343237	-.0021021	9.2193919
5	23.2448353	.0011017	-12.6253758
6	27.2531765	-.0006421	16.4299365
7	31.2599023	.0004038	-20.6137504
8	35.2654301	-.0002688	25.1605122
9	39.2700545	.0001871	-30.0562931
10	43.2739839	-.0001350	35.2890294
11	47.2773684	.0001003	-40.8481299
12	51.2803505	-.0000763	46.7261154
	λ_{n4}	$R_{n4}(1)$	$\hat{a}_{n4}$
p = 4			
1	9.3792135	.0252401	-2.3213776
2	13.2723388	-.0049034	7.4314469
3	17.2549085	.0014777	-15.8125201
4	21.2546254	-.0005792	28.1040147
5	25.2582950	.0002688	-44.8874390
6	29.2627184	-.0001402	66.7027453
7	33.2670067	.0000797	-94.0598892
8	37.2709209	-.0000484	127.4452276
9	41.2744245	.0000309	-167.3256194
10	45.2775446	-.0000206	214.1513339
11	49.2803425	.0000142	-268.3581004
12	53.2833256	-.0000101	330.2625627

Table 1.6. Continued.

n	λ_{n5}	$R_{n5}(1)$	$\hat{a}_{n5}$
$p = 5$			
1	11.5098951	.0089607	-3.6425719
2	15.3489579	-.0017523	14.9425283
3	19.2994476	.0004851	-38.4763055
4	23.2827812	-.0001716	79.9319957
5	27.2774053	.0000720	-145.8859799
6	31.2764169	-.0000341	243.7046747
7	35.2772488	.0000177	-381.4989950
8	39.2788376	-.0000099	568.0910053
9	43.2807099	.0000059	-812.9862309
10	47.2826436	-.0000036	1126.3510676
11	51.2846273	.0000024	-1518.9028058
12	55.2897856	-.0000016	2005.3450201
	λ_{n6}	$R_{n6}(1)$	$\hat{a}_{n6}$
$p = 6$			
1	13.6194985	.0031385	-5.9785255
2	17.4320392	-.0006226	30.0137091
3	21.3537711	.0001630	-90.9529902
4	25.3192435	-.0000533	216.4815922
5	29.3030107	.0000205	-444.5872727
6	33.2951536	-.0000090	824.7249759
7	37.2914389	.0000043	-1418.9510664
8	41.2898936	-.0000022	2302.0232011
9	45.2895293	.0000012	-3567.4619251
10	49.2898175	-.0000007	5318.5921954
11	53.2897917	.0000004	-7680.4664374
12	57.5404175	-.0000002	9271.8561870

Table 1.7. Comparison of eigenvalues, eigenfunctions and expansion coefficients for Peclet number of infinity (no axial conduction) and for uniform wall heat flux with the results of Siegel, Sparrow, and Hallman [59] and Hsu [18].

	Siegel, Sparrow, Hallman			Hsu			Present Work		
n	λ^2_{n0}	$R_{n0}(1)$	$\hat{a}_{n0}$	λ_{n0}	$R_{n0}(1)$	$\hat{a}_{n0}$	λ_{n0}	$R_{n0}(1)$	$\hat{a}_{n0}$
1	25.6796	-.492517	.403483	5.067504	-.492517	.403483	5.0675055	-.4925166	.4034832
2	83.8618	.395508	-.175111	9.157609	.395508	-.175110	9.1576064	.3955085	-.1751100
3	174.167	-.345872	.105594	13.19722	-.345874	.105592	13.1972247	-.3458737	.1055917
4	296.536	.314047	-.0732804	17.22023	.314046	-.0732824	17.2202294	.3140465	-.0732824
5	450.947	-.291252	.0550357	21.23552	-.291251	.0550365	21.2355173	-.2912515	.0550365
6	637.387	.273808	-.043483	25.24653	.273807	-.0434844	25.2465312	.2738070	-.0434844
7	855.850	.259852	.035597	29.25491	-.259853	.0355951	29.2549056	-.2598530	.0355951
8				33.26152	.248332	-.0299085	33.2615237	.2483320	-.0299084
9				37.26691	-.238590	.0256401	37.2669082	-.2385904	.0256401
10				41.27139	.230199	-.0223337	41.2713893	.2301993	-.0223336
11				45.27519	-.222863	.0197069	45.2751871	-.2228631	.0197068
12				49.27846	.216370	-.0175765	49.2784663	.2163696	-.0175763

1.5.2 Discussions of Results for the Special Example $q(\phi) = q_{av}(1+b \cos p\phi)$

With the numerical information obtained in the previous section, we may investigate the simultaneous effects of circumferential wall heat flux and axial conduction on wall temperature and Nusselt number in the entrance region of a tube. Using the obtained eigenvalues, eigenfunctions, and expansion coefficients, the dimensionless wall temperature difference and the local Nusselt numbers have been calculated for various values of the parameters Pe, x+, b, and ϕ from Equations (1.118) and (1.119).

For the well-known limiting case with uniform wall heat flux $(b = 0)$ and no axial conduction, a comparison of these results with Kays' [25] table (8-6) is given by Table 1.8 of the present work.

Table 1.8. Comparison of local Nusselt number for the circular tube; constant wall heat flux; no axial conduction $(Pe \rightarrow \infty)$ with the results of Kays.

x+	Kays' Table (8-6) Nu (x+)	Present Work Nu (x+)
.001	---	15.758
.002	12.00	12.537
.004	9.93	9.986
.010	7.49	7.494
.020	6.14	6.148
.040	5.19	5.198
.10	4.51	4.514
∞	4.36	4.364

For finite Peclet numbers and uniform wall heat flux there are no tabulated Nusselt number values and the Nusselt values obtained in graphical form by Hsu [19] are in error as mentioned before. Table 1.9 presents values of local Nusselt number for various Peclet numbers. Nusselt values for Pe = ∞ are also included in the last column for comparison.

Table 1.9. Local Nusselt numbers for laminar tube flow with uniform wall heat flux, and Peclet numbers of 5, 10, 20, 30, 50, and 100.

x+	Pe = 5 Nu (x+)	Pe = 10 Nu (x+)	Pe = 20 Nu (x+)	Pe = 30 Nu (x+)	Pe = 50 Nu (x+)	Pe = 100 Nu (x+)	Pe = ∞ Nu (x+)
.002	43.306	31.989	23.228	31.075	20.631	15.071	12.537
.004	30.748	21.575	15.645	15.623	12.462	10.711	9.986
.01	17.655	12.399	9.573	8.800	8.019	7.633	7.494
.02	11.455	8.474	7.039	6.586	6.314	6.191	6.148
.04	7.771	6.218	5.552	5.348	5.254	5.212	5.198
.1	5.321	4.780	4.593	4.546	4.526	4.517	4.514
∞	4.364	4.364	4.364	4.364	4.364	4.364	4.364

Graphical representation of the Nusselt numbers tabulated in Table 1.9 is shown in Figure 1.8. If the least squares method for the expansion coefficients had not been employed, the Nusselt number values obtained for low Peclet numbers, would have been larger. This is seen by comparing Figure 1.8 and Figure 1.9. Figure 1.8 corrects Hsu's [21] Figure 4 and illustrates that, as Peclet numbers decrease, the Nusselt values increase in the entrance region and approach 4.364 as $x+ \rightarrow \infty$ for any Peclet number. This was

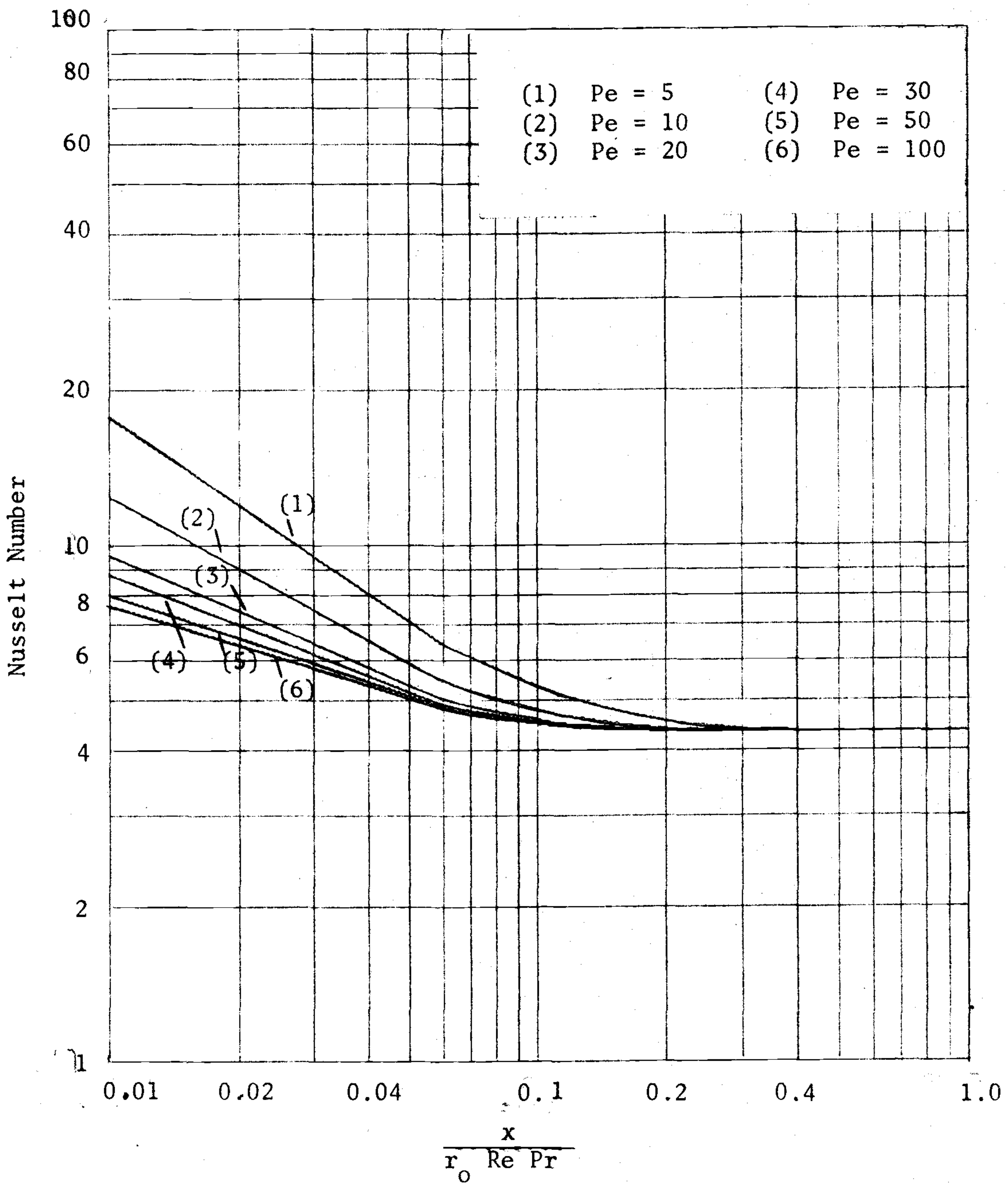

Figure 1.8. Entrance-region local Nusselt numbers for uniform wall heat flux and for different Peclet numbers, employing least squares expansion coefficients.

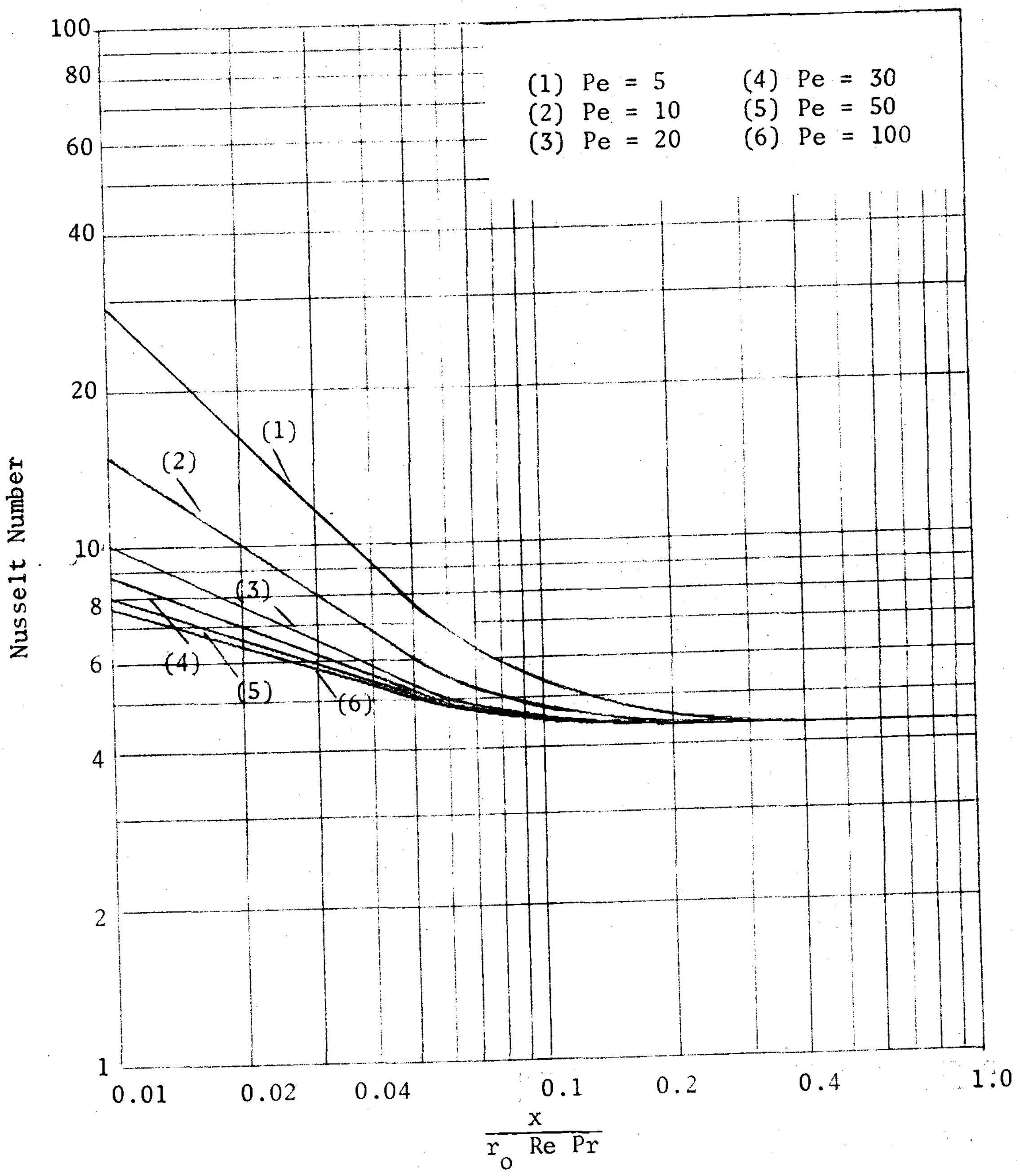

Figure 1.9. Entrance-region local Nusselt numbers for uniform wall heat flux and for different Peclet numbers, employing the approximate expansion coefficients.

verified experimentally by Petukhov and Yushin [45], Eckert and Peterson [10], and Emery and Bailey [11] who independently showed the asymptotic Nusselt number for the case of uniform wall heat flux in liquid metals to be 4.36. Figure 1.8 also illustrates that the entry length increases as Peclet number is decreased. Figure 1.10 shows the relationship between dimensionless wall temperature difference and dimensionless axial distance, for the case of uniform wall heat flux for different Peclet numbers. It is observed that as Peclet number decreases the dimensionless wall temperature decreases in the entrance region and approaches a constant value asymptotically. If the least squares method for the expansion coefficients had not been employed, the wall temperature values for low Peclet number, would have lower values. This is seen by comparing Figures 1.10 and 1.11.

For the case where the heat flux varies around the circumference (i.e., $b = 1$), the dimensionless wall temperature difference (Equation 1.118) has been plotted in Figure 1.12 as a function of angular position ϕ at a section where fully-developed conditions exist (i.e., $x+ = 1$). Since the asymptotic wall temperature is not affected by axial conduction, this plot corresponds to Figure 5 of Reynolds [49] who solved the asymptotic problem with no axial conduction. Figures 1.13, 1.14, 1.15, and 1.16 present corresponding plots for the thermal entrance region (i.e., $x+ = .1, .04, .02, .01$), and Peclet numbers of 5, 10, 20, 30, 50, and 100. It is seen that

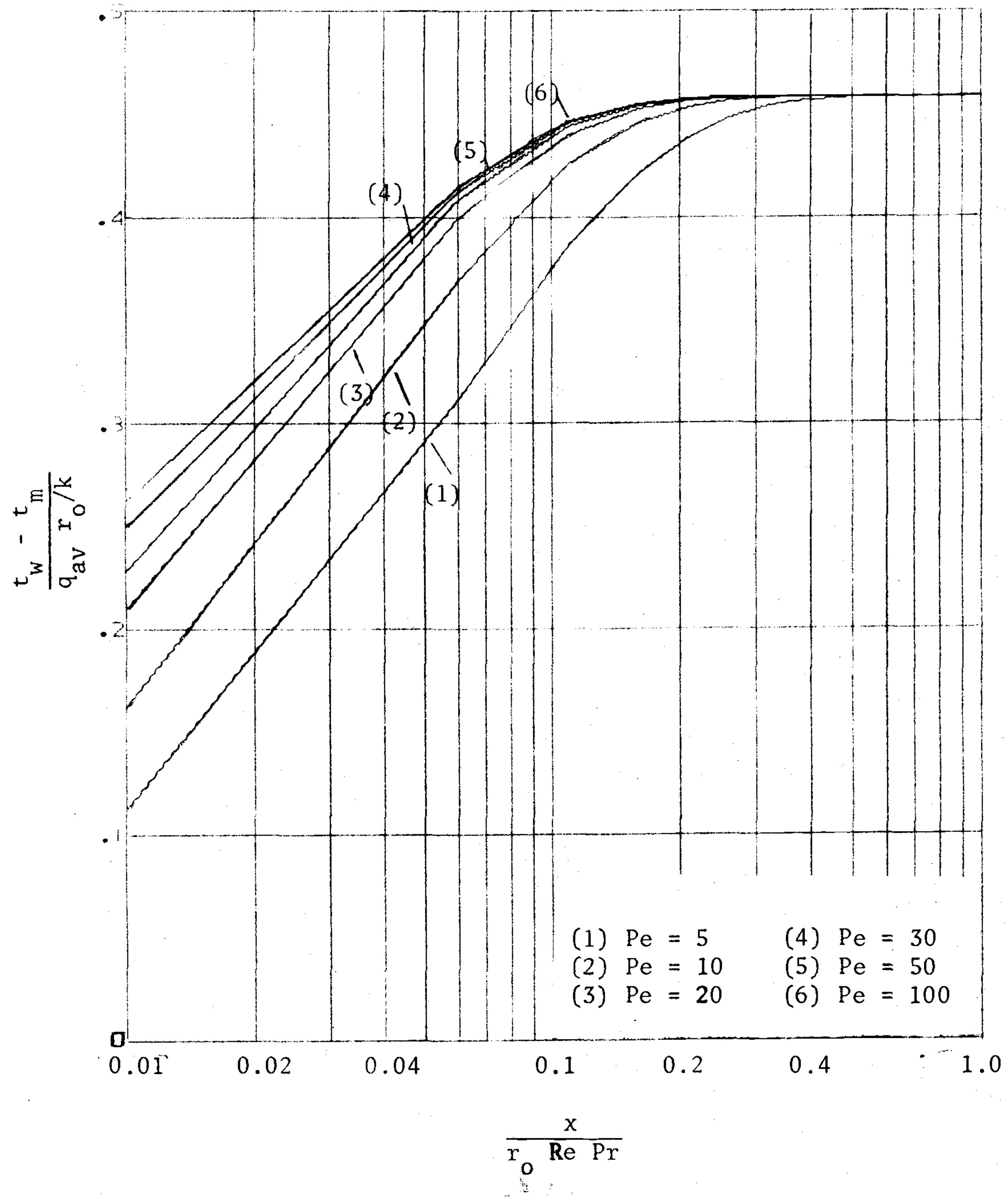

Figure 1.10. Entrance-region local wall-to bulk temperature difference for uniform wall heat flux and for different Peclet numbers, employing least squares expansion coefficients.

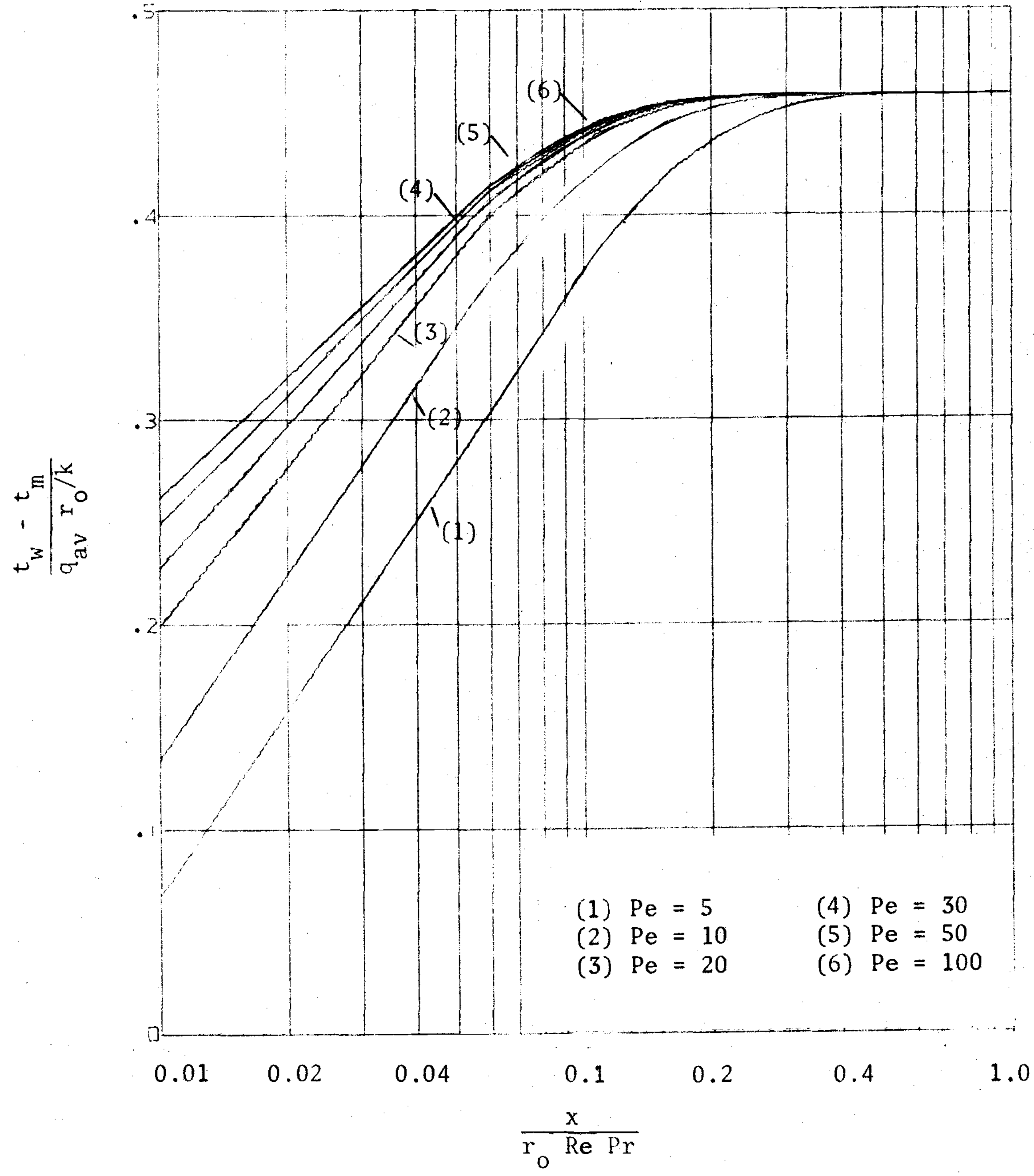

Figure 1.11. Entrance-region local wall-to bulk temperature difference for uniform wall heat flux and for different Peclet numbers, employing the approximate expansion coefficients.

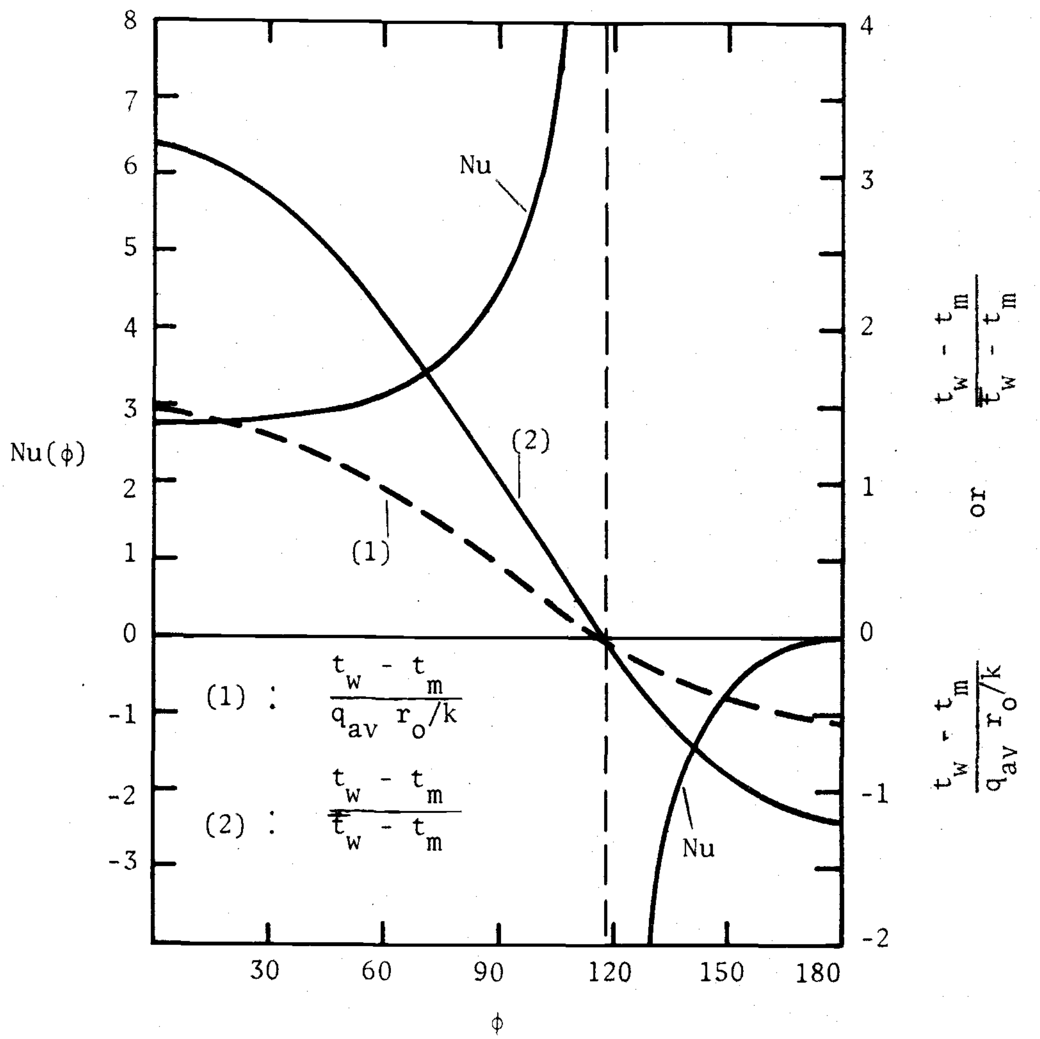

Figure 1. 12. Illustration of effect of prescribed wall heat flux variation $q(\phi) = q_{av}(1 + \cos\phi)$ on wall-to bulk temperature difference and local Nusselt number, at the location far away from the entrance.

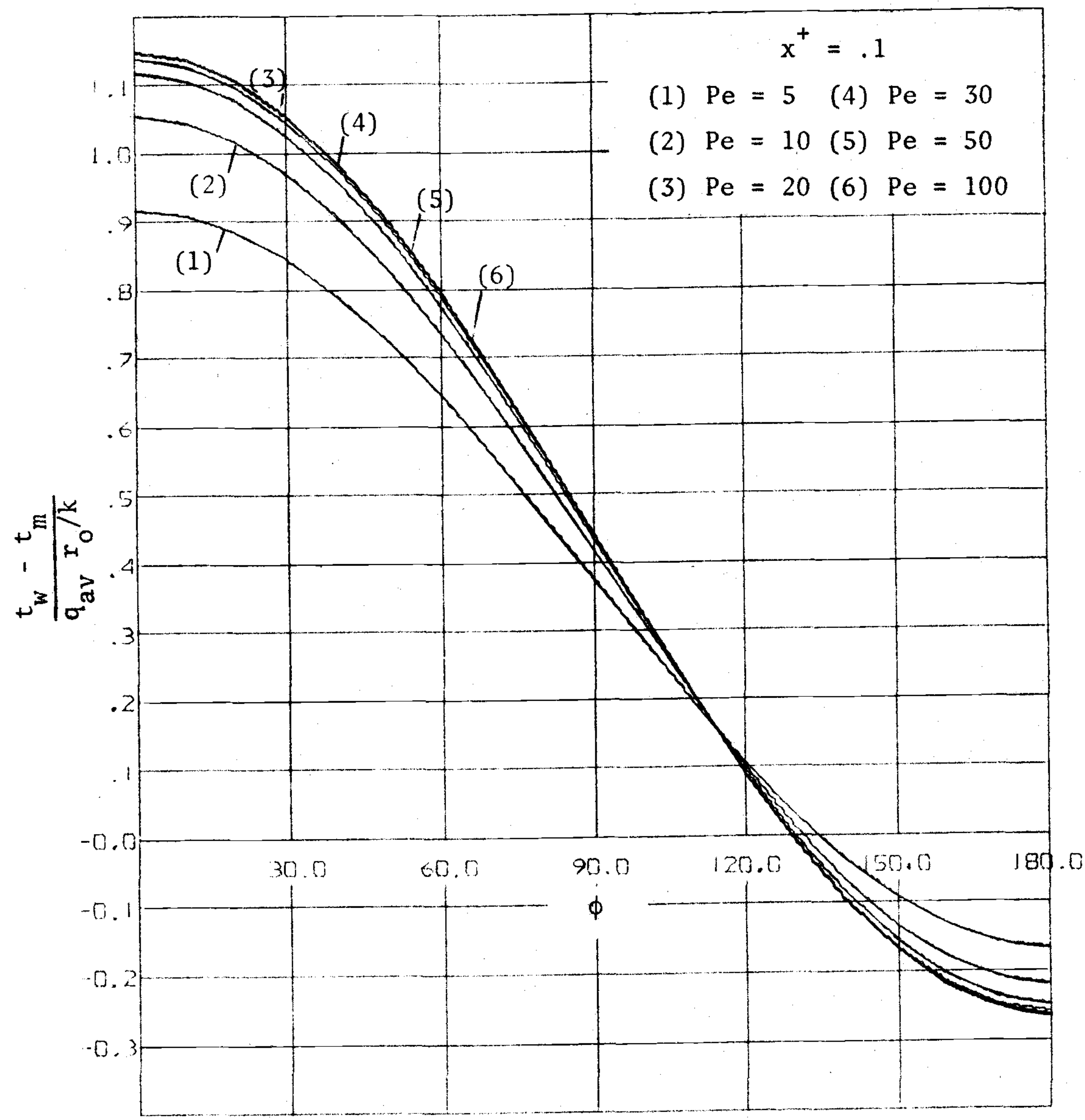

Figure 1.13. Illustration of entrance effect of prescribed wall heat flux variation $q(\phi) = q_{av}(1 + \cos\phi)$ and axial conduction on wall-to bulk temperature difference at the location x+ = .1.

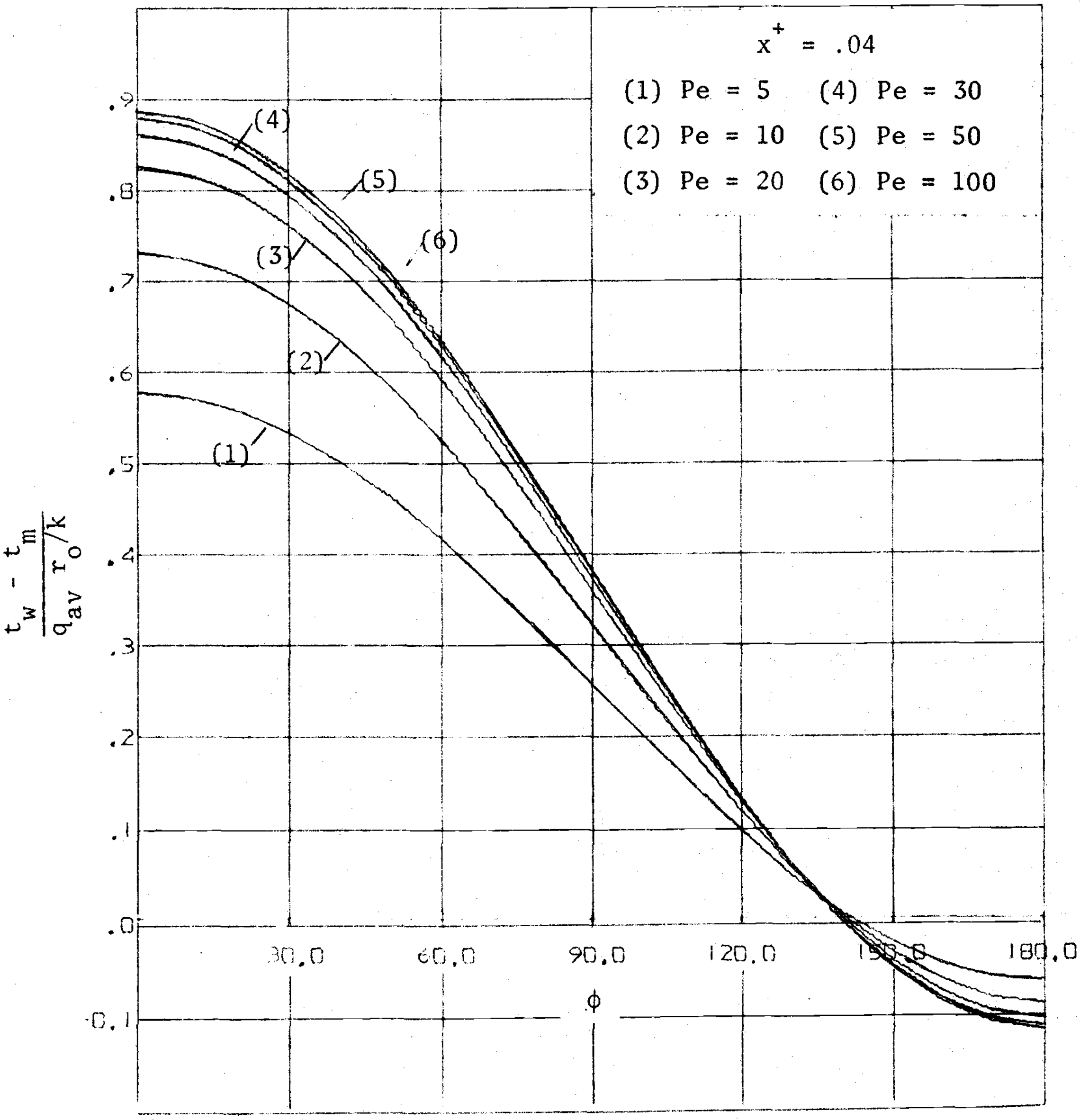

Figure 1. 14. Illustration of entrance effect of prescribed wall heat flux variation $q(\phi) = q_{av}(1 + \cos \phi)$ and axial conduction on wall-to bulk temperature difference at the location x+ = .04.

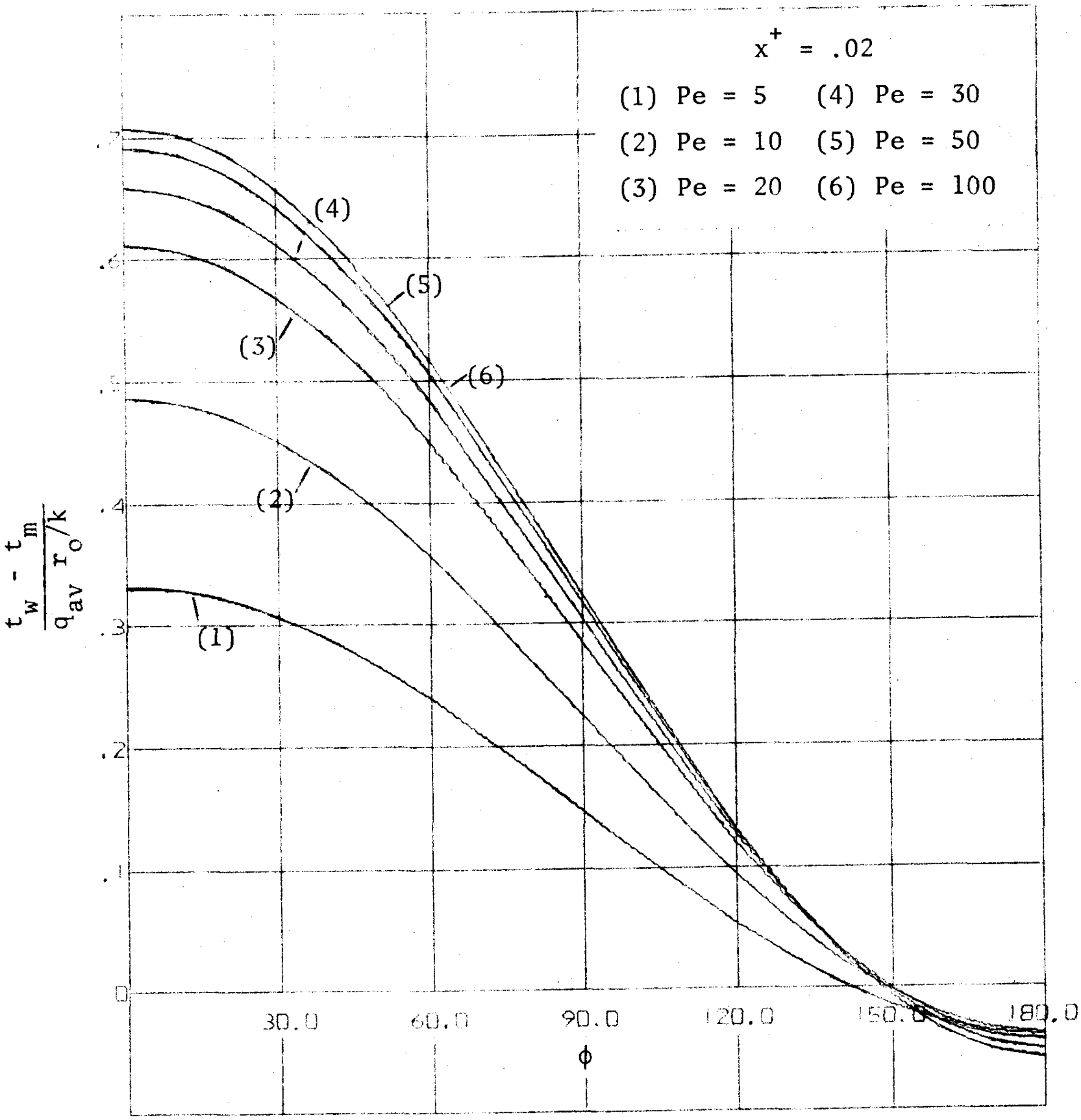

Figure 1.15. Illustration of entrance effect of prescribed wall heat flux variation $q(\phi) = q_{av}(1 + \cos\phi)$ and axial conduction on wall-to bulk temperature difference at the location x+ = .02.

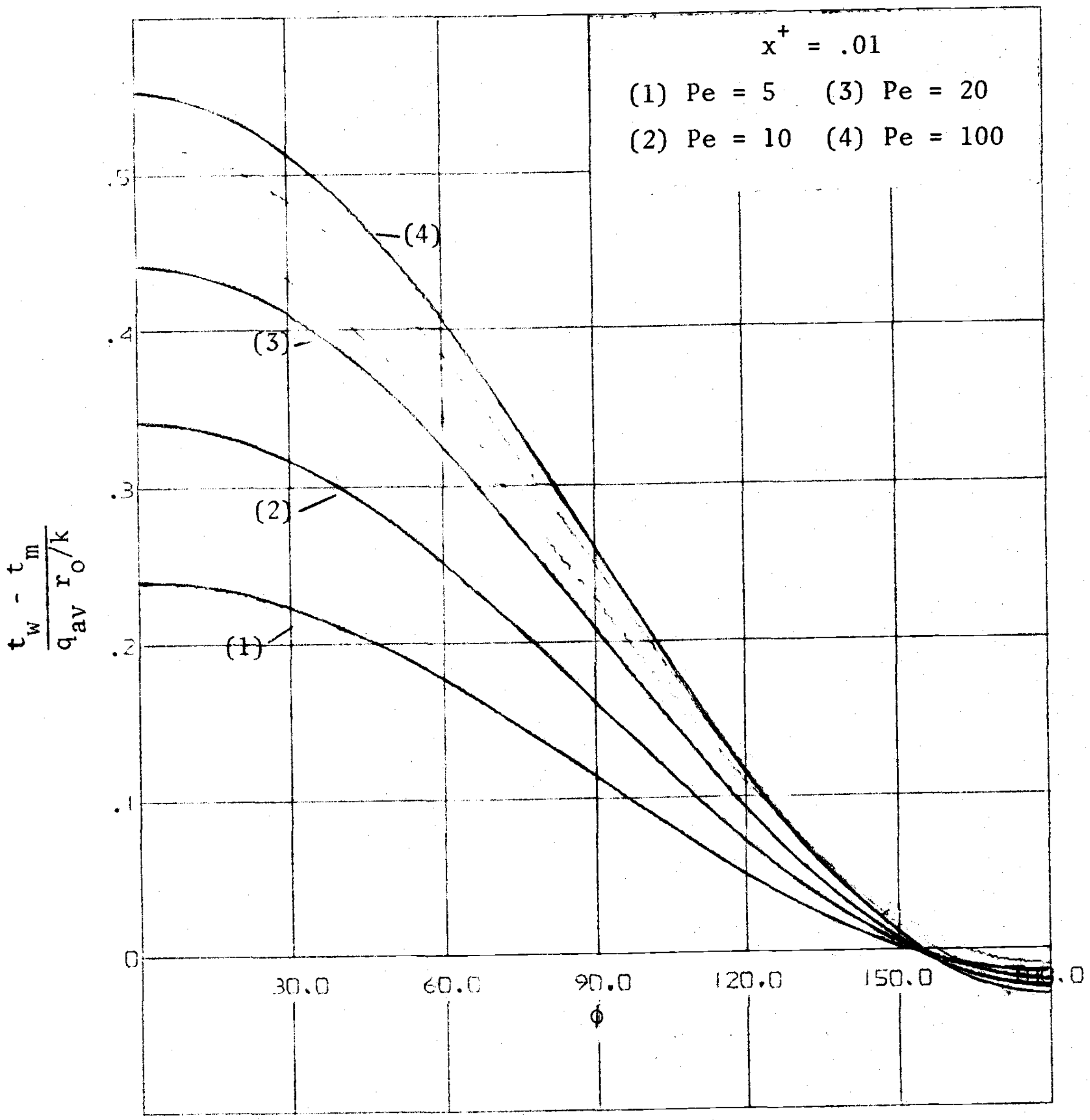

Figure 1. 16. Illustration of entrance effect of prescribed wall heat flux variation $q(\phi) = q_{av}(1 + \cos \phi)$ and axial conduction on wall-to bulk temperature difference at the location x+ = . 01.

there is a significant variation in the dimensionless wall temperature difference $\frac{t_w - t_m}{q_{av} r_0 / k}$ around the tube periphery for the case of a circumferential wall heat flux and axial fluid conduction. By comparing these plots, it is seen that the effect of axial conduction on wall temperature becomes more pronounced in the entrance region. Increased values of the heat flux parameter, b, result in increased temperature variations around the circumference for a given Peclet number as seen by Figures 1.17, 1.18, 1.19, 1.20. The local Nusselt number has been plotted in Figures 1.21, 1.22, 1.23, 1.24 as a function of angular position ϕ. It is found that the local Nusselt number varies over a wide range around the circumference of a tube in the case of a cosine heat flux variation. Furthermore, axial conduction has a pronounced effect on local Nusselt numbers. This effect becomes more significant in the entrance region. Also note that the Nusselt number is infinity at the point where the wall temperature is equal to the mean fluid temperature and becomes negative when the wall temperature is less than the bulk temperature.

Finally, dimensionless wall temperatures and Nusselt numbers are plotted as a function of dimensionless axial position for different Peclet numbers, at the location of maximum heat flux $(\phi = 0)$ in Figures 1.25 and 1.26.

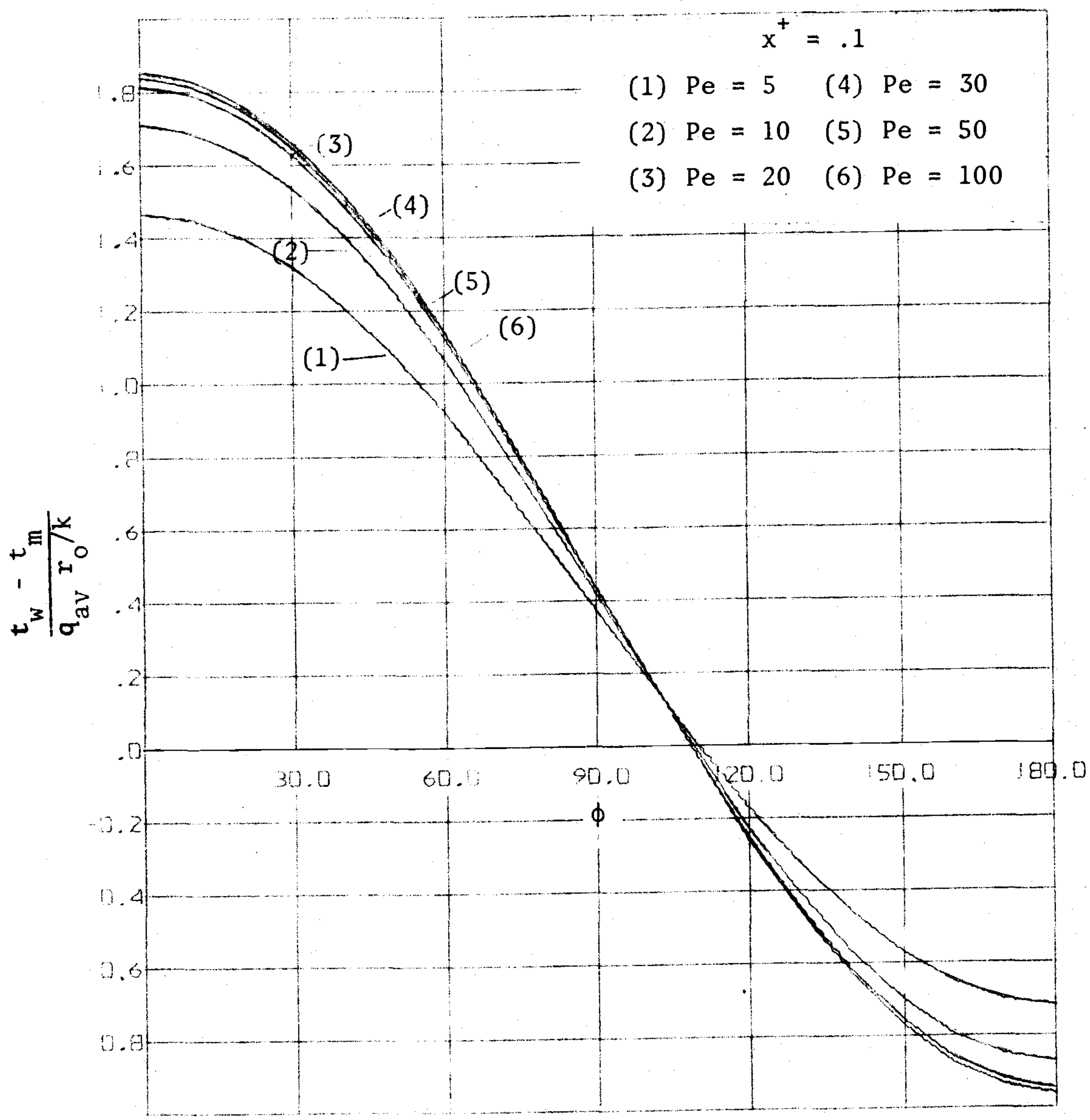

Figure 1. 17. Illustration of entrance effect of prescribed wall heat flux variation $q(\phi) = q_{av}(1 + 2 \cos \phi)$ and axial conduction on wall-to bulk temperature difference at the location x+ = . 1.

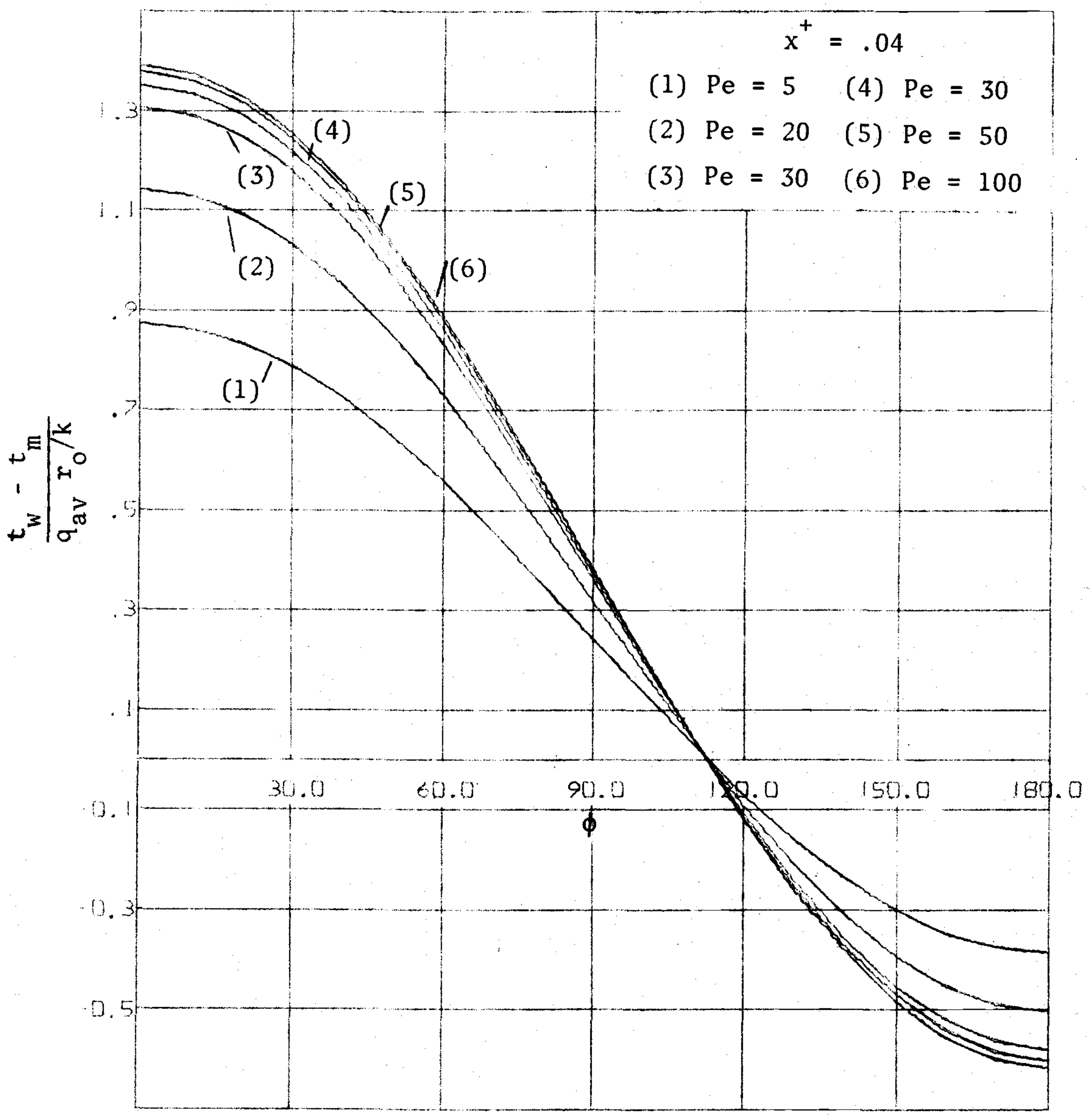

Figure 1. 18. Illustration of entrance effect of prescribed wall heat flux variation $q(\phi) = q_{av}(1 + 2 \cos \phi)$ and axial conduction on wall-to bulk temperature difference at the location x+ = .04.

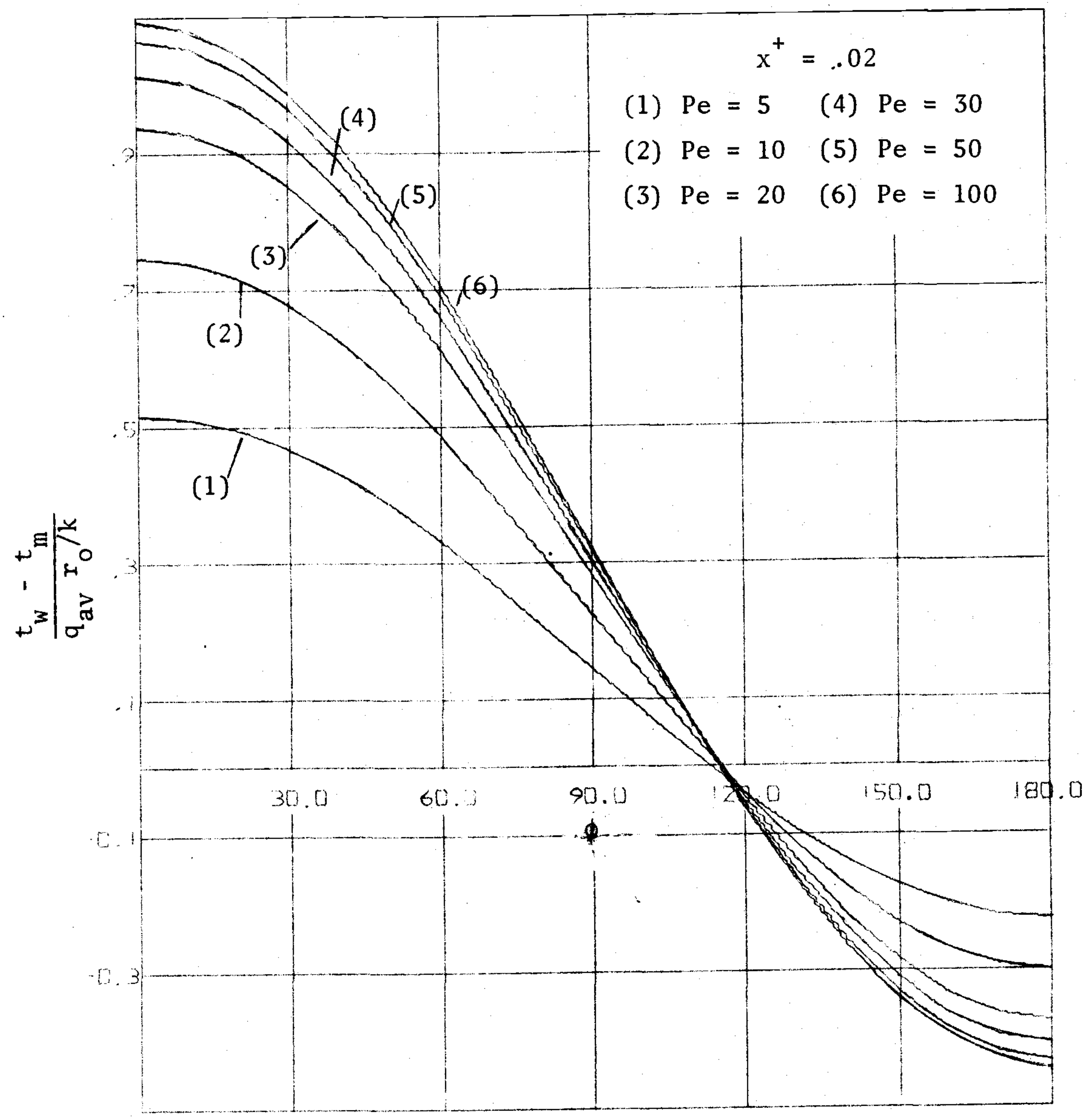

Figure 1.19. Illustration of entrance effect of prescribed wall heat flux variation $q(\phi) = q_{av}(1 + 2 \cos \phi)$ and axial conduction on wall-to bulk temperature difference at the location x+ = .02.

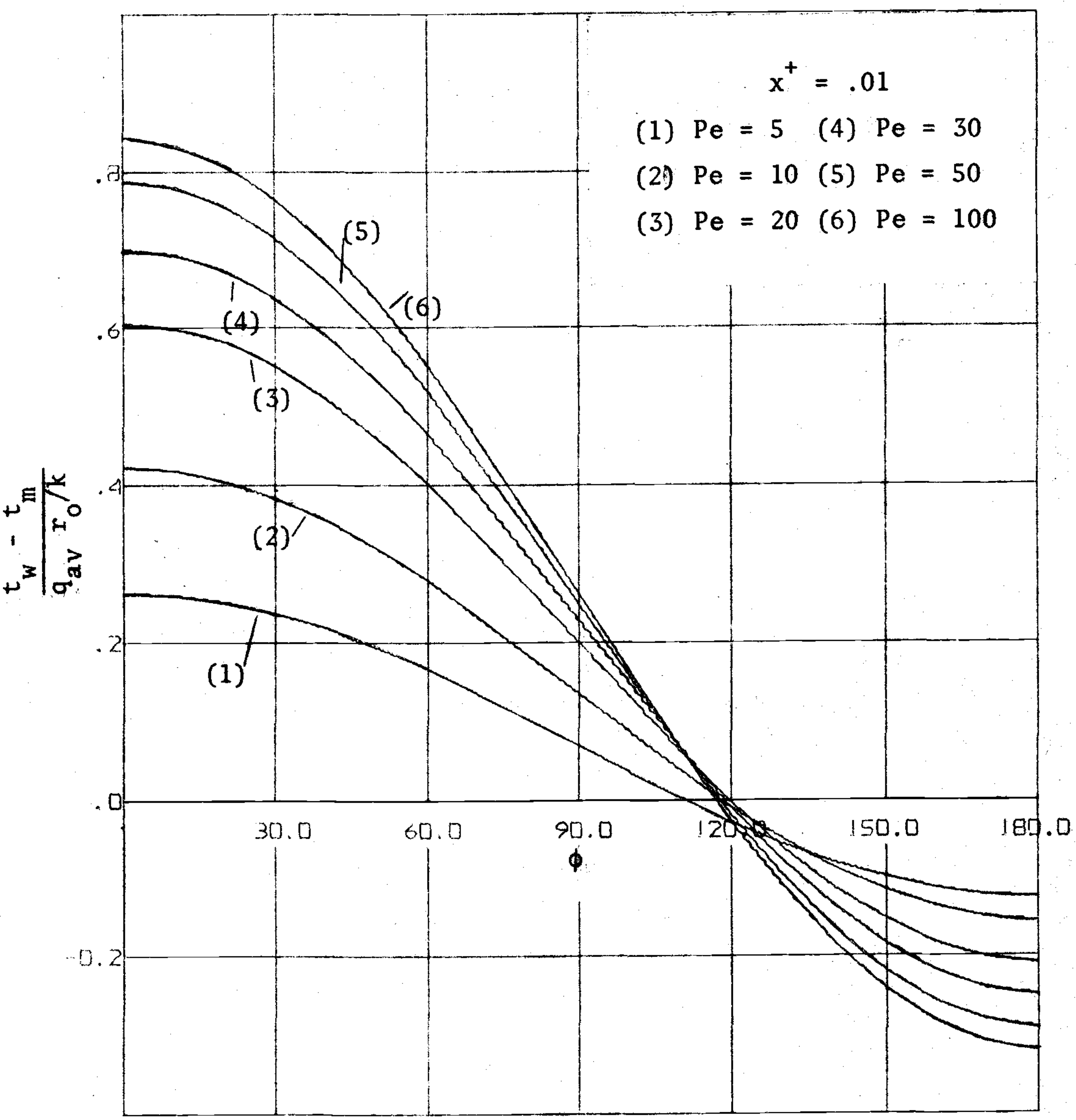

Figure 1.20. Illustration of entrance effect of prescribed wall heat flux variation $q(\phi) = q_{av}(1 + 2\cos\phi)$ and axial conduction on wall-to bulk temperature difference at the location x+ = .01.

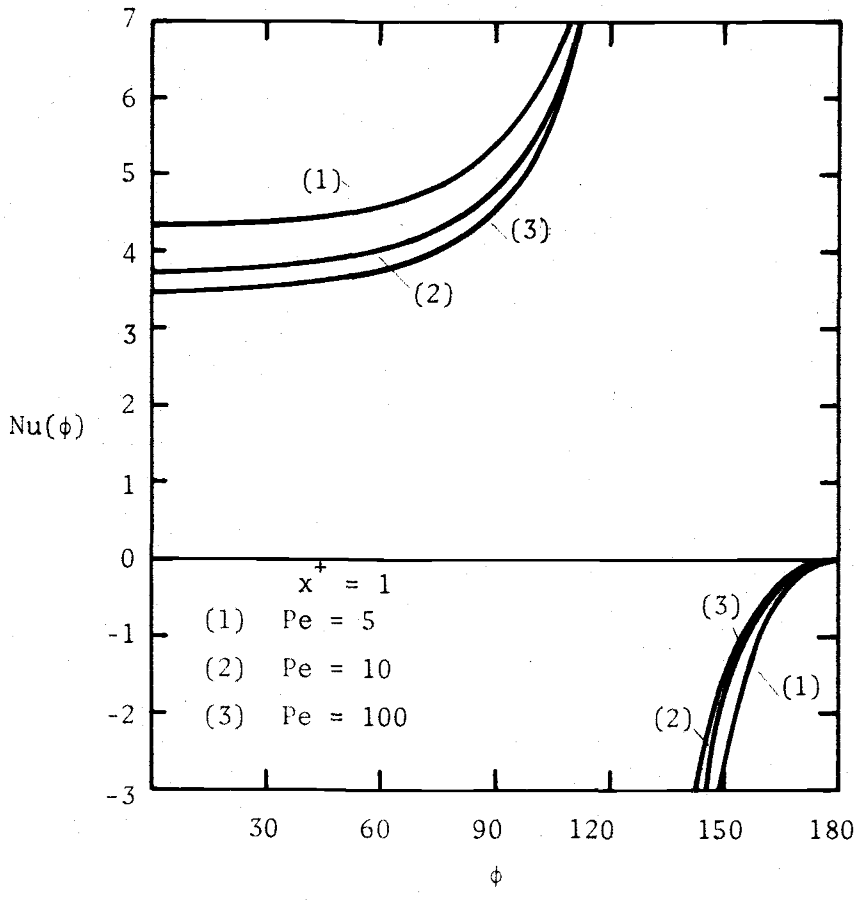

Figure 1.21. Local Nusselt number variation for prescribed wall heat flux $q(\phi) = q_{av}(1 + \cos\phi)$ and for different Peclet numbers at the location x+ = .1.

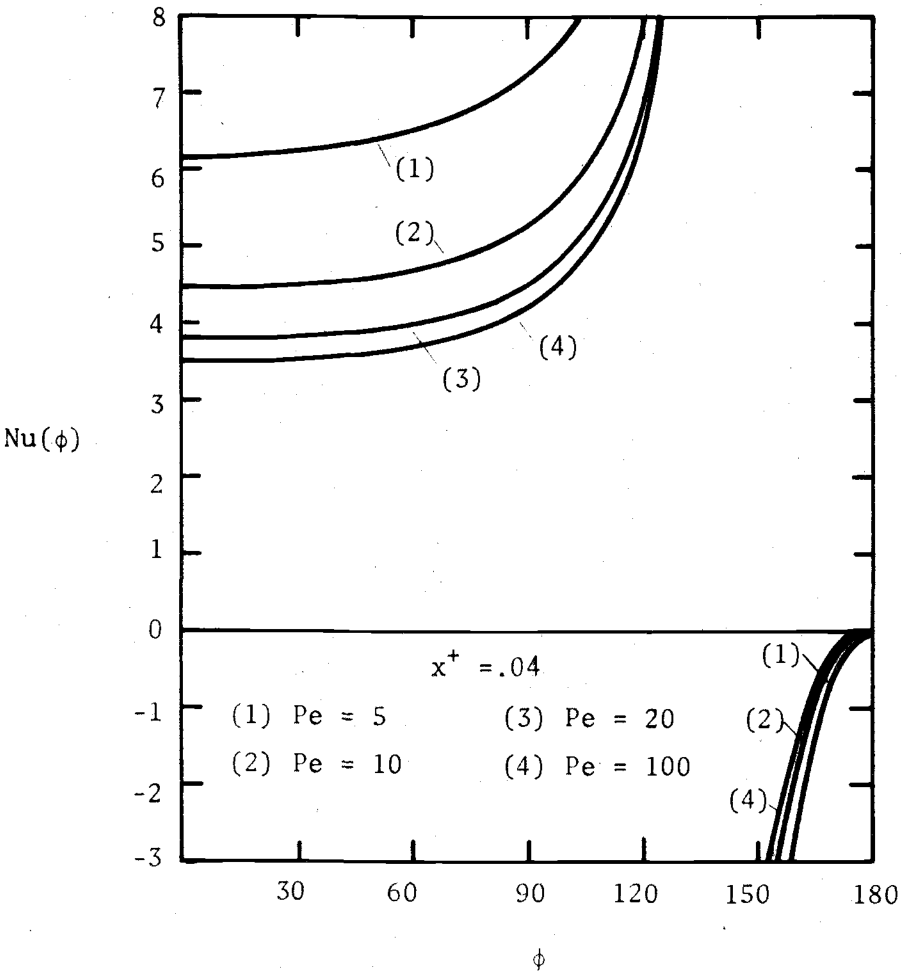

Figure 1.22. Local Nusselt number variation for prescribed wall heat flux $q(\phi) = q_{av}(1 + \cos\phi)$ and for different Peclet numbers at the location $x+ = .04$.

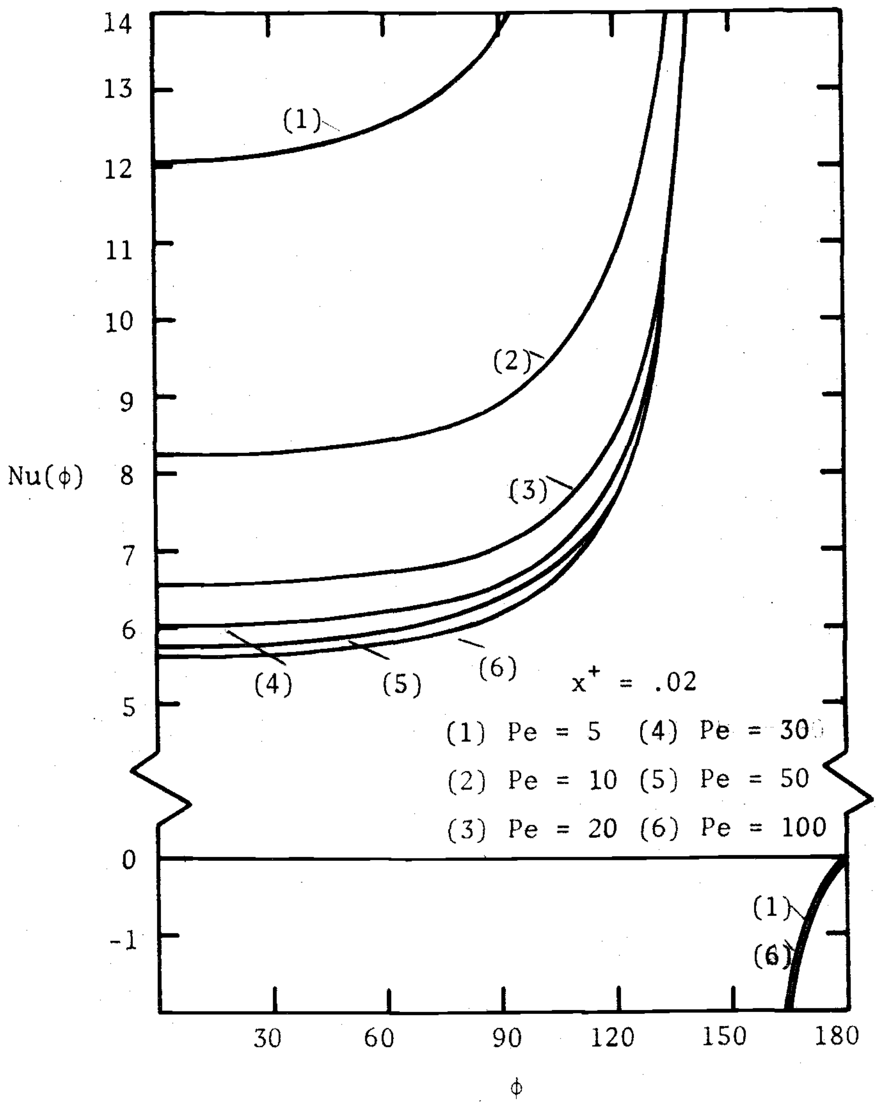

Figure 1.23. Local Nusselt number variation for prescribed wall heat flux $q(\phi) = q_{av}(1 + \cos\phi)$ and for different Peclet numbers at the location x+ = .02.

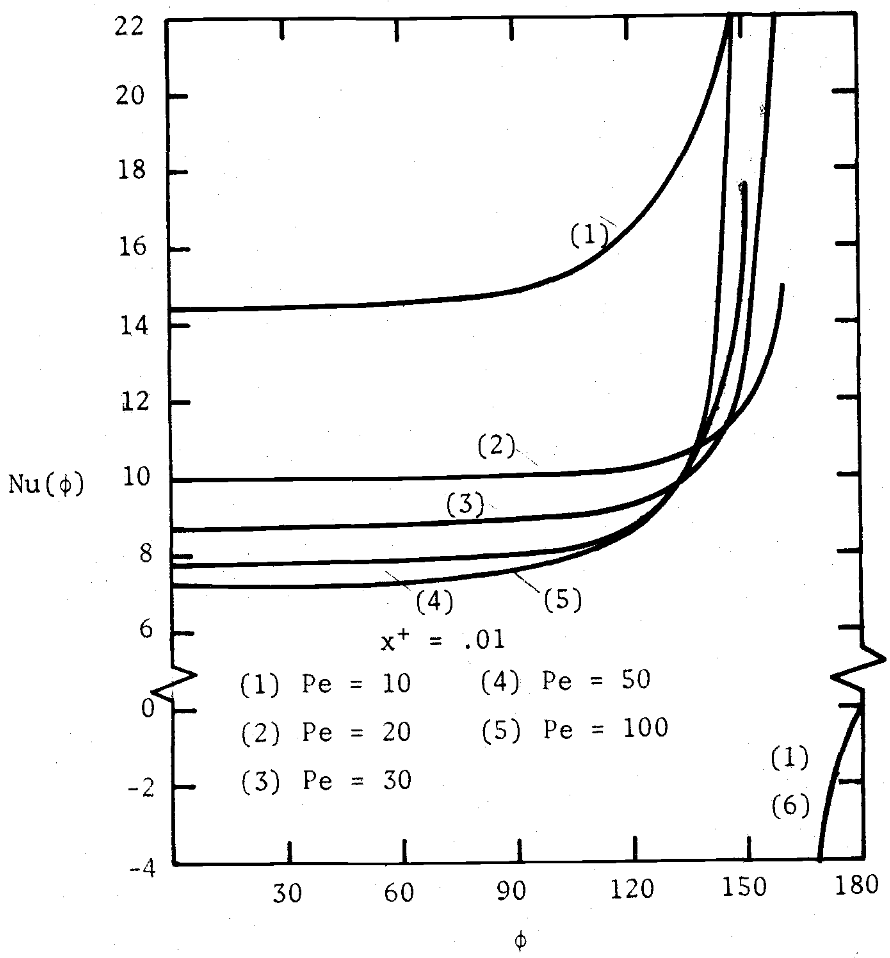

Figure 1.24. Local Nusselt number variation for prescribed wall heat flux $q(\phi) = q_{av}(1 + \cos\phi)$ and for different Peclet numbers at the location $x+ = .01$.

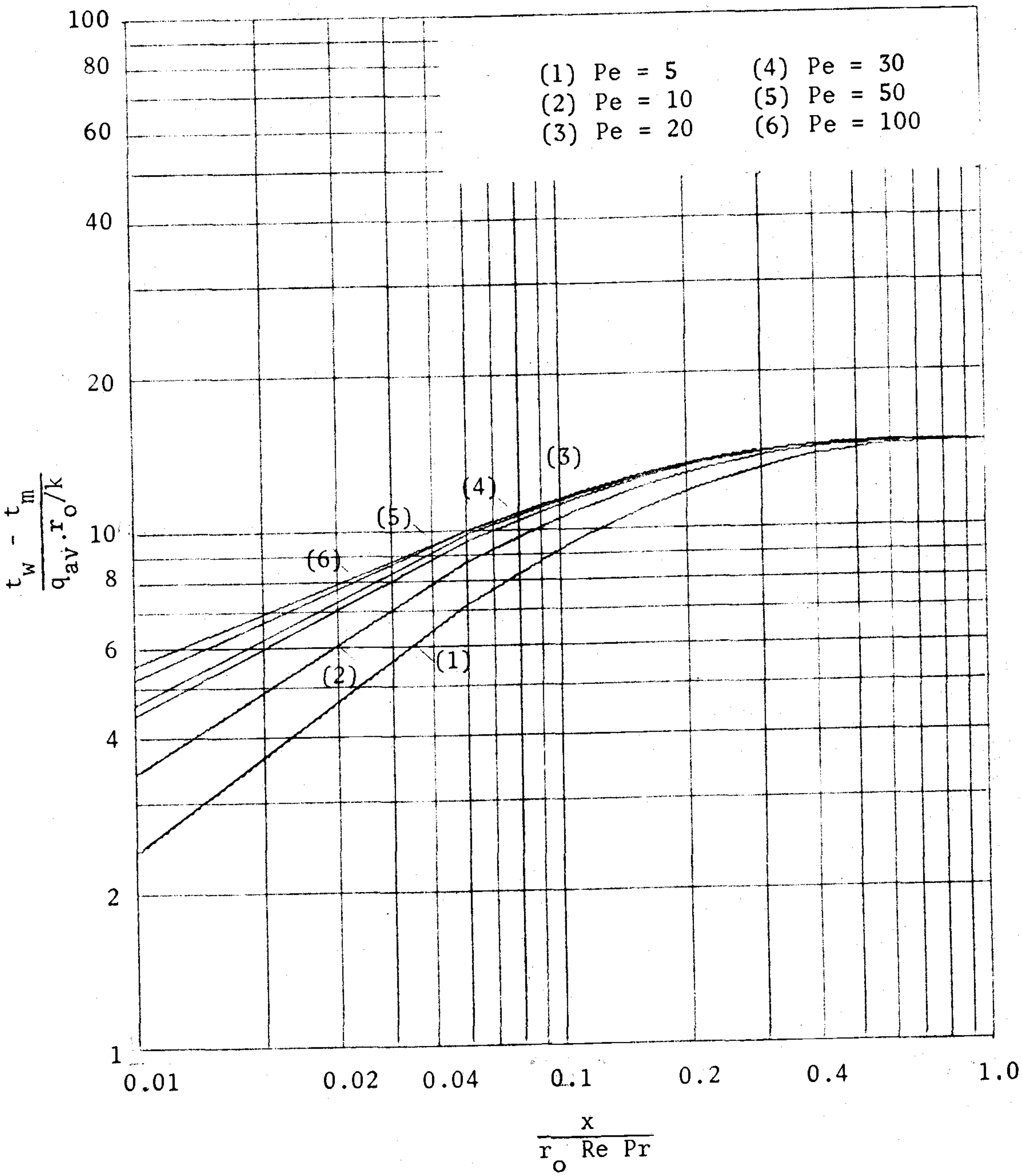

Figure 1.25. Entrance-region local wall-to bulk temperature difference for prescribed wall heat flux variation $q(\phi) = q_{av}(1 + \cos\phi)$ and for different Peclet numbers at the angular position $\phi = 0$ (i.e., maximum wall heat flux).

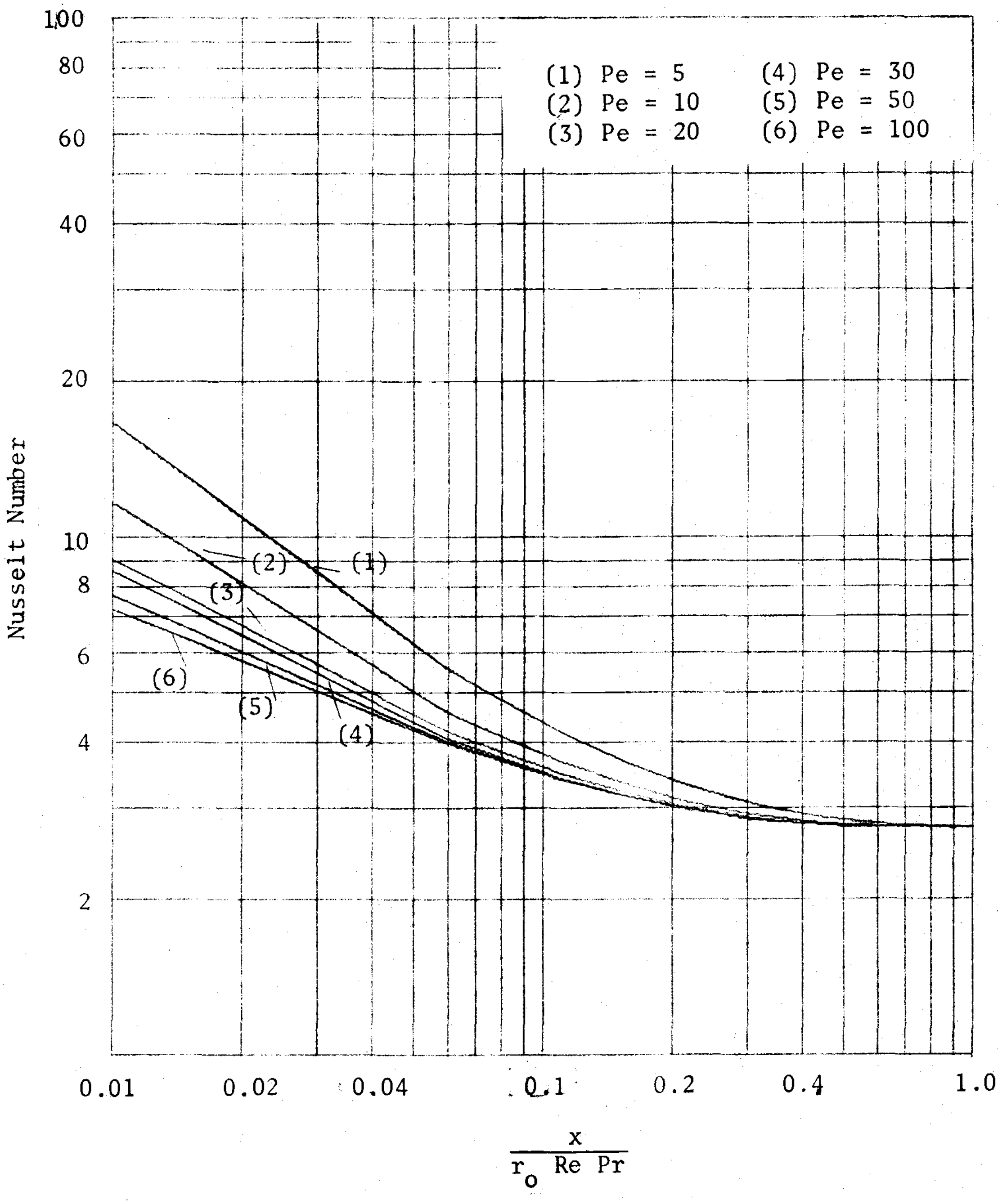

Figure 1.26. Entrance-region local Nusselt numbers for prescribed wall heat flux variation $q(\phi) = q_{av}(1 + \cos \phi)$ and for different Peclet numbers at the angular position $\phi = 0$ (i.e., maximum wall heat flux).

2. NON-NEWTONIAN PROBLEM

2.1 Introduction

2.1.1 Literature Review

In recent years there has been an increasing effort concerned with the theoretical solution of heat transfer for laminar non-Newtonian tube flow with axisymmetric heating, either with uniform wall temperature or uniform wall heat flux.

Grigull [15] obtained an asymptotic, downstream Nusselt number with a uniform wall heat flux for fluids obeying both the power-law and Bingham plastic constitutive equations. Beek and Eggink [3] and Valstar and Beek [70] extended the work of Grigull and considered boundary conditions of both uniform heat flux and uniform temperature in parallel plates and tubes. They showed that the Nusselt number depends primarily on the ratio of maximum velocity to mean velocity. Their results for power-law and Bingham plastic fluids are included in the summary article by Rohsenow [51]. Matsuhisa and Bird [32] obtained similar results for Ellis fluids. Michiyoshi [35] extended the work of Beek and Eggink, and Grigull to Bingham plastic fluids and included the effect of an internal heat source for the condition of uniform wall heat flux.

Sestak and Charles [57] analyzed the influence of arbitrary heat generation terms in the limiting Nusselt number, for non-Newtonian fluids with uniform wall heat flux, by a technique similar to one used by Lyon [30] for liquid metals which does not involve calculating temperature profiles in advance. Expressions were developed by Skelland [62] for asymptotic Nusselt numbers for power-law and Bingham plastic fluids in tubes and parallel plates by assuming a cubic polynomial profile for both cases of uniform wall temperature and uniform wall heat flux. Payvar [44] included the effect of viscous dissipation and developed expressions for the asymptotic Nusselt numbers for three widely used models, namely, the power-law fluids, Bingham plastics, and Ellis fluids.

The case of a fully-developed velocity profile, but thermally developing temperature profile has been the subject of many papers. Lyche and Bird [29] extended the Graetz-Nusselt problem [8, 56] to non-Newtonian fluids using the power-law constitutive equation. They used a separation-of-variables technique to reduce the energy equation into two ordinary differential equations, functionally relating the temperature in the axial and radial directions respectively. The radial ordinary differential equation was solved with the appropriate boundary conditions to give the first three eigenfunctions, eigenvalues, and expansion coefficients. Whiteman and Drake [71] examined the case of uniform tube wall temperature in a study that was similar to that of Lyche and Bird. Whereas the latter authors obtained Nusselt numbers averaged over the heat transfer section, Whiteman and Drake presented only local values of Nusselt number, although over a somewhat larger range of Graetz numbers.

Wissler and Schecter [73] showed how the Graetz-Nusselt problem could be solved to include heat transfer to a slurry behaving as a Bingham plastic, and following other works [4, 7, 14, 67, 68] they considered cases for which heat is generated in the fluid. Foraboschi and Federico [12] extended this problem further to the case where the volumetric heat generation rate varies linearly with local temperatures for power-law non-Newtonian fluids.

For the uniform heat flux boundary condition, the work of Siegel, Sparrow, and Hallman [59] pertaining to Newtonian fluids was extended to non-Newtonian fluids according to the Prandtl-Eyring formula by Shenk and Van Laar [58] and according to power-law and Ellis models by Bird [5]. Schechter and Wissler [53] extended the work of Sparrow and Siegel [53] for Newtonian flow of a heat generating fluid to non-Newtonian flow of a Bingham plastic fluid with constant heat generation and an insulated wall boundary condition. Michiyoshi _et al_. [36] extended the work of the latter authors and included not only the case of a thermally insulated wall but also the cases of uniform wall heat flux and uniform wall temperature. Michiyoshi [37] also considered the power-law heat generating, pseudo-plastic fluids for both a uniform wall heat flux and uniform wall temperature. Their results for no internal heat generation, with uniform wall temperature, agree very well with those of Lyche and Bird [29], and in the case of uniform wall heat flux with a special result of this work, i.e., constant wall heat flux.

Mitsuishi and Miyatake [39] adopted the Ellis model, involving three parameters, for the cases of both uniform wall temperature and uniform wall heat flux. The first three eigenvalues and eigenfunctions and coefficients of the expansion between two limiting values of Newton's law and power-law were obtained. Their eigenvalues, eigenfunctions, and expansion coefficients are in excellent agreement for the case of powerlaw fluids and a uniform wall heat flux with a special result of this work.

The combined hydrodynamic-entry length problem for Newtonian fluids [16, 26, 31, 69] was extended to non-Newtonian power-law fluids by McKillop [33] and later by Yau and Tien [74] using different techniques. Samant and Marner [52] extended the work of McKillop [33] to include Bingham plastic fluids.

Literature in the area of non-uniform or arbitrary variation of heat flux or wall temperature around the periphery of a tube is indeed sparse and, with the exception of the paper by Inman [22] who extended the work of Reynolds [49] to non-Newtonian power-law fluids with variable circumferential wall temperature or heat flux, there exists no other published work.

2.1.2 Present Investigation

The objective of this investigation is to extend the work of Inman [22] to solve the problem of heat transfer in a circular tube with an arbitrary circumferential wall heat flux for the case of a developing temperature profile for power-law pseudo-plastic fluids. The solution is expanded in a power series form that accounts for any arbitrary variation of heat flux around the circumference that can be expressed in terms of a Fourier expansion and the expansion coefficients and the related constants are obtained numerically. This solution is then generalized for any arbitrary variation of wall heat flux in the axial direction.

For the limiting case of power-law pseudo-plastic fluids with uniform wall heat flux, the eigenfunctions at the tube wall and the eigenvalues reduce to Table 1 of Michiyoshi and Matsumoto [36] and Table 4 and 5 of Mitsuishi and Miyatake [39]. Only the first five eigenvalues, eigenfunctions, and related constants were obtained by these authors. This is not sufficient for special problems where the infinite series in the temperature solution converges slowly (i. e., axial variation of wall heat flux is present). In this work the first 12 eigenvalues, eigenfunctions, and the expansion coefficients are obtained.

The problem of Newtonian fluids with an arbitrary circumferential wall heat flux is another limiting case of the present work. The first 12 eigenvalues, eigenfunctions, and expansion coefficients are included for values of the parameter p ranging from $p = 0$ (i.e., the case of uniform wall heat flux) to $p = 5$ (up to fifth harmonic variation in the circumferential wall heat flux). For the case of uniform wall heat flux $(p = 0)$, the related coefficients agree well with the values reported by Siegel, Sparrow and Hallman [59] and Hsu [18]. Finally, a simple result has been obtained for a cosine heat flux variation around the tube periphery which illustrates all the limiting cases and shows how the simultaneous influences of circumferential wall heat flux and non-Newtonian behavior may have an effect on heat transfer results.

2.2 Formulation of Problem

2.2.1 Governing Equations and Boundary Conditions

The problem to be considered is represented schematically in Figure 1.1. A non-Newtonian fluid is flowing in laminar fashion through the tube of constant radius, r_0. The wall heat flux varies circumferentially according to the general function, $q(\phi)$, which can be expressed in terms of Fourier expansion. The applicable form of the governing equations are as follows:

Equation of continuity: $$\frac{\partial \rho}{\partial \hat{\theta}} = -\nabla \cdot \rho\vec{u} \tag{2.1}$$

Equation of motion: $$\rho \frac{D\vec{u}}{D\hat{\theta}} = -\nabla P - \nabla \cdot \overline{\overline{\tau}} + \rho\vec{g} \tag{2.2}$$

Equation of energy: $$\rho c_v \frac{Dt}{D\hat{\theta}} = -\nabla \cdot \vec{q} - t\left(\frac{\partial P}{\partial t}\right)_v \nabla \cdot \vec{u} - \tau : \nabla\vec{u} + Q \tag{2.3}$$

in which $\vec{u}$ is the local fluid velocity, P is the static pressure, t is the temperature, $\vec{q}$ is the heat flux vector, $\overline{\overline{\tau}}$ is the stress tensor, $\tau:\nabla u$ is heat production due to viscous dissipation, $\vec{g}$ is acceleration of gravity and Q is the heat generation rate per unit volume. The following assumptions are made:

(a) The physical properties of the fluid (ρ, c_v, k) are constant.

(b) The viscous dissipation and heat generation are negligible.

(c) The axial conduction term is neglected; this is true in a practical sense for non-Newtonian fluids.

(d) The velocity profiles are fully developed; non-Newtonian fluids are generally very viscous with a short hydrodynamic entry length.

(e) External forces are neglected.

(f) Fourier's law is valid.

Equations (2.2) and (2.3), for steady state flow in cylindrical tubes may be simplified under these assumptions to:

Equation of motion: $$\frac{dP}{dx} + \frac{1}{r}\frac{d}{dr}(r\tau_{rx}) = 0 \tag{2.4}$$

Equation of energy: $$\frac{u(r)}{\alpha}\frac{\partial t}{\partial x} = \frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{1}{r^2}\frac{\partial^2 t}{\partial \phi^2} \tag{2.5}$$

Where $t(x, r, \phi)$ is the local fluid temperature and α is the molecular thermal diffusivity.

For the power-law fluids, the rheological or constitutive equation is

$$\tau_{rx} = -m\left(\frac{du}{dr}\right)^n \tag{2.6}$$

in which m and n are constants which must be determined for the particular fluid in question. Experimental data of the McEachern [75] indicates that the model predicts pressure drop versus flow rate data reasonably well if attention is paid to the range of shear stress over which the parameters are evaluated.

Substitution of Equation (2.6) into Equation (2.4) and solution subject to the usual boundary conditions $[u(0)$ is finite and $u(r_0) = 0]$ yields:

$$u = \frac{n}{n+1}\left[-\frac{dP}{dx}/2m\right]^{1/n}\left[r_0\right]^{(n+1)/n}\left[1-\left(\frac{r}{r_0}\right)^{(n+1)/n}\right] \tag{2.7}$$

Using the definition of mean velocity and Equation (2.7) we obtain

$$v = \frac{1}{\pi r_0^2}\int_0^1 2\pi r_0 u(r)dr = \frac{3}{3n+1}\left[-\frac{dP}{dx}/2m\right]^{1/n}\left[r_0\right]^{(n+1)/n} \tag{2.8}$$

The local velocity is expressed in terms of mean velocity by elimination of the term $[-\frac{dP}{dx}/2m]$ between Equations (2.7) and (2.8) to obtain

$$\frac{u}{v} = \frac{3n+1}{n+1}\left[1-\left(\frac{r}{r_0}\right)^{(n+1)/n}\right] \tag{2.9}$$

If $n = 1$ the fluid is Newtonian and Equation (2.9) reduces to the usual parabolic velocity profile; if $n = 0$ plug flow is obtained. Figure 2.1 portrays velocity profiles for several values of n. In this work, we limit our investigation to the values of the index n in the range $0 < n < 1$ i.e., pseudo-plastic fluids. For convenience of the analysis we define

$$s = \frac{n+1}{n} \tag{2.10}$$

and rewrite Equation (2.9) to obtain

$$\frac{u}{v} = \frac{s+2}{s}\left[1-\left(\frac{r}{r_0}\right)^{s}\right] \tag{2.11}$$

Note at $r = 0$, $u = u_{max}$, and Equation (2.11) reduces to

$$\frac{u_{max}}{v} = \frac{s+2}{s} \tag{2.12}$$

We now express Equation (2.11) in terms of maximum velocity using Equation (2.12)

$$\frac{u}{u_{max}} = [1-(\frac{r}{r_0})^s] \qquad (2.13)$$

Note that, if $s = 2$ the fluid is Newtonian; if $s = \infty$ plug flow is obtained, and $s = 1$ is the limiting case of dilatant fluids.

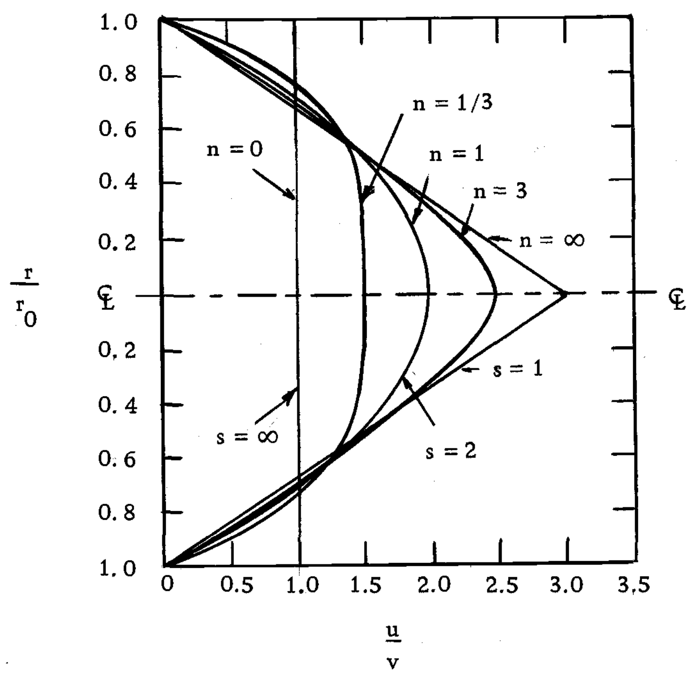

Figure 2.1. Velocity distribution for power-law fluids.

Our consideration is now directed to Equation (2.5), the energy equation, and after the substitution for the velocity function from Equation (2.13) we obtain the following partial differential equation for the temperature profiles:

$$\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{1}{r^2}\frac{\partial^2 t}{\partial \phi^2} = \frac{u_{max}}{\alpha}[1(\frac{r}{r_0})^s]\frac{\partial t}{\partial x} \qquad (2.14)$$

Subject to the boundary conditions:

$$t(0, r, \phi) = t_\epsilon \tag{2.14a}$$

$$k \frac{\partial t}{\partial r} (x, r_0, \phi) = q(\phi) \tag{2.14b}$$

$$t(x, 0, \phi) = \text{finite} \tag{2.14c}$$

$$t(x, r, \phi) = t(x, r, \phi+2\pi) \tag{2.14d}$$

$$\frac{\partial t}{\partial \phi} (x, r, \phi) = \frac{\partial t}{\partial \phi} (x, r, \phi+2\pi) \tag{2.14e}$$

Equation (2.14) may be represented in terms of the following dimensionless variables:

$$\theta = \frac{t - t_\epsilon}{\overline{q}\, 2r_0/\pi k} \tag{2.15a}$$

$$r+ = \frac{r}{r_0} \tag{2.15b}$$

$$x+ = \frac{2s}{s+2} \; \frac{x/r_0}{\text{Re Pr}} = \frac{2v}{u_{max}} \; \frac{x/r_0}{\text{Re Pr}} \tag{2.15c}$$

$$u+ = \frac{u}{v} \tag{2.15d}$$

where

$$\overline{q} = \int_0^{2\pi} q(\phi) d\phi \tag{2.15e}$$

with the requirement that $\overline{q} \neq 0$. Performing the necessary transformations we obtain the dimensionless form of the energy equation as follows:

$$\frac{\partial^2\theta}{\partial r+^2} + \frac{1}{r+}\frac{\partial\theta}{\partial r+} + \frac{1}{r+^2}\frac{\partial^2\theta}{\partial\phi^2} = (1-r+^s)\frac{\partial\theta}{\partial x+} \tag{2.16}$$

Satisfying the boundary conditions:

$$\theta(0, r+, \phi) = 0 \tag{2.16a}$$

$$\frac{\partial\theta}{\partial r+}(x+, 1, \phi) = \frac{q(\phi)}{\overline{q}}\frac{\pi}{2} \tag{2.16b}$$

$$\theta(x+, 0, \phi) = \text{finite} \tag{2.16c}$$

$$\theta(x+, r+, \phi) = \theta(x+, r+, \phi+2\pi) \tag{21.6d}$$

$$\frac{\partial\theta}{\partial\phi}(x+, r+, \phi) = \frac{\partial\theta}{\partial\phi}(x+, r+, \phi+2\pi) \tag{2.16e}$$

Equation (2.16) completes the formulation of the physical problem. Consideration will now be given to its solution.

2.2.2 Fully-Developed and Entry-Length-Equations and Boundary Conditions

Equation (2.16) is a linear differential equation. By experience with heat conduction problems of similar form, a solution can be obtained having the form

$$\theta+(x+, r+, \phi) = \theta(x+, r+, \phi) - \theta_{fd}(x+, r+, \phi) \tag{2.17}$$

in which $\theta_{fd}(x+, r+, \phi)$ is the asymptotic solution obtained far downstream where the temperature profile is fully developed, and $\theta+$ is the entry region solution.

Combining Equations (2.16) and (2.17), we obtain two differential equations and associated boundary conditions for the two regions as follows:

$$\frac{\partial^2 \theta_{fd}}{\partial r+^2} + \frac{1}{r+}\frac{\partial \theta_{fd}}{\partial r+} + \frac{1}{r+^2}\frac{\partial^2 \theta_{fd}}{\partial \phi^2} = (1-r+^s)\frac{\partial \theta_{fd}}{\partial x+} \tag{2.18}$$

$$\frac{\partial \theta_{fd}}{\partial r+}(x+, 1, \phi) = \frac{q(\phi)}{\overline{q}}\frac{\pi}{2} \tag{2.18a}$$

$$\theta_{fd}(x+, 0, \phi) = \text{finite} \tag{2.18b}$$

$$\theta_{fd}(x+, r+, \phi) = \theta_{fd}(x+, r+, \phi+2\pi) \tag{2.18c}$$

$$\frac{\partial \theta_{fd}}{\partial \phi}(x+, r+, \phi) = \frac{\partial \theta_{fd}}{\partial \phi}(x+, r+, \phi+2\pi) \tag{2.18d}$$

$$\frac{\partial^2 \theta+}{\partial r+^2} + \frac{1}{r+}\frac{\partial \theta+}{\partial r+} + \frac{1}{r+^2}\frac{\partial^2 \theta+}{\partial \phi^2} = (1-r+^s)\frac{\partial \theta+}{\partial x+} \tag{2.19}$$

$$\theta+(0, r+, \phi) = -\theta_{fd}(0, r+, \phi) \tag{2.19a}$$

$$\theta+(x+, 0, \phi) = \text{finite} \tag{2.19b}$$

$$\frac{\partial \theta+}{\partial r+}(x+, 1, \phi) = 0 \tag{2.19c}$$

$$\theta+(x+, r+, \phi) = \theta+(x+, r+, \phi+2\pi) \tag{2.19d}$$

$$\frac{\partial \theta+}{\partial \phi}(x+, r+, \phi) = \frac{\partial \theta+}{\partial \phi}(x+, r+, \phi+2\pi) \tag{2.19e}$$

2.3 Discussion of Solution

2.3.1 The Fully-Developed Solution

Equation (2.18) for the fully-developed portion, was solved by Inman [22] utilizing the method of analysis developed by Reynolds [49], namely, by considering an arbitrary variation of heat flux symmetrical about an axis through the center of the pipe. A solution was then obtained for the case of a tube with constant heat flux over a portion of its circumference, insulated over the remainder, and then generalized by superposition to obtain a solution for an arbitrary heat flux, $q(\phi)$. In this work, we utilize a Fourier series approach and assume $q(\phi)$ to be completely arbitrary around the circumference and expressible in a Fourier series.

For the case of a fully-developed temperature profile we have the condition

$$\frac{\partial \theta_{fd}}{\partial x} = \frac{d\theta_m}{dx} = \text{constant} \tag{2.20}$$

An energy balance, for a tube with wall heating as shown in Figure 1.1, yields the expression

$$\pi r_0^2 \rho v c \frac{dt_m}{dx} = \int_0^{2\pi} q(\phi) r_0 d\phi \tag{2.21}$$

Expressing Equation (2.21) in terms of dimensionless variables and combining it with Equation (2.20) we obtain

$$\frac{\partial \theta_{fd}}{\partial x+} = \frac{d\theta_m}{dx+} = \frac{u_{max}}{2v} \tag{2.22}$$

The integration of Equation (2.22) for θ_{fd} yields:

$$\theta_{fd}(x+, r+, \phi) = \frac{u_{max}}{2v} x+ + f+(r+, \phi) \tag{2.23}$$

where $f+(r+, \phi)$ satisfies the following differential equation and boundary conditions. These results are obtained from substituting Equation (2.23) into (2.18).

$$\frac{\partial^2 f+}{\partial r+^2} + \frac{1}{r+}\frac{\partial f+}{\partial r+} + \frac{1}{r+^2}\frac{\partial^2 f+}{\partial \phi^2} = \frac{u_{max}}{2v}(1-r+^s) \tag{2.24}$$

$$\frac{\partial f+}{\partial r+}(r+, \phi) = \frac{q(\phi)}{\bar{q}}\frac{\pi}{2} \tag{2.24a}$$

$$f+(0, \phi) = \text{finite} \tag{2.24b}$$

$$f+(r+, \phi) = f(r+, \phi+2\pi) \tag{2.24c}$$

$$\frac{\partial f+}{\partial \phi}(r+, \phi) = \frac{\partial f+}{\partial \phi}(r+, \phi+2\pi) \tag{2.24d}$$

To eliminate the difficulty arising from the non-homogeneity in Equation (2.24) we express $f+$ as a sum

$$f+(r+, \phi) = F(r+, \phi) + W(r+) \tag{2.25}$$

and include $\frac{u_{max}}{2v}(1-r+^{s})$ in the formulation of the one-dimensional, $W(r+)$, problem. The two problems which result are now

$$\frac{\partial^2 F}{\partial r+^2} + \frac{1}{r+}\frac{\partial F}{\partial r+} + \frac{1}{r+^2}\frac{\partial^2 F}{\partial \phi^2} = 0 \qquad (2.26)$$

$$\frac{\partial F}{\partial r+}(1, \phi) = \frac{q(\phi)}{\bar{q}}\frac{\pi}{2} - \frac{1}{4} \qquad (2.26a)$$

$$F(0, \phi) = \text{finite} \qquad (2.26b)$$

$$F(r+, \phi) = F(r+, \phi+2\pi) \qquad (2.26c)$$

$$\frac{\partial F}{\partial \phi}(r+, \phi) = \frac{\partial F}{\partial \phi}(r+, \phi+2\pi) \qquad (2.26d)$$

and

$$\frac{1}{r+}\frac{d}{dr+}\left(r+\frac{dW}{dr+}\right) = \frac{u_{max}}{2v}(1-r+^{s}) \qquad (2.27)$$

$$W(0) = \text{finite} \qquad (2.27a)$$

Solution to the Equation (2.26) was obtained in Chapter 1 to be

$$F(r+, \phi) = \sum_{n=1}^{\infty} r+^{n}\,[a_n \cos n\phi + b_n \sin n\phi] + C1 \qquad (2.28)$$

where

$$a_n = \frac{1}{2n}\int_0^{2\pi} \frac{q(\phi)}{\bar{q}} \cos n\phi \, d\phi \qquad (2.28a)$$

$$b_n = \frac{1}{2n}\int_0^{2\pi} \frac{q(\phi)}{\bar{q}} \sin n\phi \, d\phi \qquad (2.28b)$$

Equation (2.27) may be solved directly with the boundary condition

(2.27a) incorporated, yielding

$$W = \frac{u_{max}}{2v}\left[\frac{r+^2}{4} - \frac{(r+)^{s+2}}{(s+2)^2}\right] + C2 \tag{2.29}$$

The complete solution for $f+(r+, \phi)$ from Equation (2.24) may now be summarized by combining Equations (2.12), (2.25), (2.28) and (2.29) to yield

$$f+(r+, \phi) = \frac{s+2}{2s}\left[\frac{r+^2}{4} - \frac{(r+)^{s+2}}{(s+2)^2}\right] + \sum_{n=1}^{\infty} r+^n[a_n \cos n\phi + b_n \sin n\phi]$$

$$+ \text{ constant} \tag{2.30}$$

where a_n and b_n are obtained from Equations (2.28a, b) and the constant is still undetermined.

We may now express the fully-developed temperature profile by combining Equations (2.23) and (2.30) to obtain

$$\theta_{fd}(x+, r+, \phi) = \frac{x/r_0}{\text{Re Pe}} + \frac{s+2}{8s} r+^2 - \frac{(r+)^{s+2}}{2(s+2)s}$$

$$+ \sum_{n=1}^{\infty} r+^n[a_n \cos n\phi + b_n \sin n\phi] + \text{constant} \tag{2.31}$$

where a_n and b_n are given by (2.28a, b) and the constant is still undetermined.

2.3.1.1 Calculation of the Average Mean Fluid Temperature.

Integration of Equation (2.22) for θ_{mean} yields

$$\theta_{mean} = \frac{x/r_0}{Re\ Pr} \tag{2.32}$$

By definition, the average mean temperature is evaluated from the following equation

$$\theta_{mean} = \frac{\int_0^{2\pi}\int_0^1 u(r+)\theta(x+, r+, \phi)r+\ dr+\ d\phi}{\int_0^{2\pi}\int_0^1 u(r+)r+\ dr+\ d\phi} \tag{2.33}$$

Equation (2.33) is integrated using Equations (2.13) and (2.31) to obtain

$$\theta_{mean} = \frac{x/r_0}{Re\ Pr} + \frac{(s+2)^2}{16(s+4)s} - \frac{1}{s^2(s+4)} + \frac{1}{2(s+2)s^2} + \text{constant} \tag{2.34}$$

By comparing Equations (2.32) and (2.34), we find the unknown constant to be

$$\text{constant} = \frac{-s^2-6s-12}{16(s+2)(s+4)} \tag{2.35}$$

We may now summarize the complete solution to the fully-developed portion of the problem as follows:

$$\theta_{fd} = \frac{-s^2-6s-12}{16(s+2)(s+4)} + \frac{x/r_0}{\text{Re Pr}} + \frac{s+2}{8s} r+^2 - \frac{r+^{s+2}}{2(s+2)s}$$

$$+ \sum_{n=1}^{\infty} r+^n (a_n \cos n\phi + b_n \sin n\phi) \tag{2.36}$$

where a_n and b_n are given by the Equations (2.28a) and (2.28b).

2.3.2 The Thermal Entry Length Solution

Consideration will now be given to the problem of solving for $\theta+(x+, r+, \phi)$ as posed in Equation (2.19). We assume a series expansion of the following convenient form

$$\theta+(x+, r+, \phi) = \sum_{n=1}^{\infty} \sum_{p=0}^{\infty} e^{-\lambda_{np}^2 x+} R_{np}(r+)[a_{np} \cos p\phi + b_{np} \sin p\phi] \tag{2.37}$$

Substituting Equation (2.37) into (2.19) and simplifying, we see that $\theta+$ satisfies the equation and the boundary conditions provided that $R_{np}(r+)$ is the solution of the following Sturm-Liouville system:

$$\frac{1}{r+} \frac{d}{dr+}\left(r+ \frac{dR_{np}}{dr+}\right) + \left[\lambda_{np}^2 (1-r+^s) - \frac{p^2}{r+^2}\right] R_{np} = 0 \tag{2.38}$$

with the boundary conditions

$$\frac{dR_{np}}{dr+}(1) = 0; \quad R_{np}(0) = \text{finite} \tag{2.38a,b}$$

where p is an integer parameter. For $p = 0$ Equation (2.38) reduces to the characteristic equation for the case with no circumferential wall heat flux variation. For only certain values of the parameter λ_{np}, say $\lambda_{10}, \lambda_{20}, \ldots, \lambda_{11}, \lambda_{21}, \ldots,$; it is possible to obtain a solution to Equation (2.38). For each such λ_{np} a solution to Equation (2.38) is obtained, say $R_{10}, R_{20}, \ldots, R_{11}, R_{21}, \ldots,$; these particular solutions are the eigenfunctions of the problem and the corresponding λ_{np} are the eigenvalues.

<u>2.3.2.1 Analysis of the Eigenvalue Problem.</u> The eigenfunctions of Equation (2.38) cannot be expressed in terms of simple functions. Thus we are forced to employ a power series method to obtain

$$R_{np}(r+) = \sum_{i=0}^{\infty} b_{i;np}(r+)^{i+p} \tag{2.39}$$

It is easily found that the coefficients $b_{i;np}$ satisfy

$$b_{i;np} = \frac{\lambda_{np}^2\left[b_{i-2-s} - b_{i-2}\right]}{i(i+2p)} \tag{2.40}$$

Where

$b_{i;np} = 0$ if $[i-s-2]$ and $[i-2] < 0$

$b_{i;np} = 1$ if $i = 0$

When s is even, every coefficient $b_{i;np}$ is equal to zero whenever i is odd, so Equations (2.39) and (2.40) become:

$$R_{np}(r+) = \sum_{i=0}^{\infty} b_{2i;np}(r+)^{2i+p} \tag{2.41}$$

$$b_{2i;np} = \frac{\lambda_{np}^{2}\left[b_{2i-s-2} - b_{2i-2}\right]}{2i(2i+2p)} \tag{2.42}$$

The eigenvalues are determined by the equation

$$\sum_{i=0}^{\infty} b_{2i;np}(2i+p) = 0 \tag{2.43}$$

following from (2.41) and the boundary condition (2.38a).

2.3.2.2 <u>Determination of Expansion Coefficients.</u> Condition (2.19a) is used to determine the coefficients of the series expansion in Equation (2.37), i.e., a_{np} and b_{np}. Substitution yields

$$\sum_{n=1}^{\infty}\sum_{p=0}^{\infty} R_{np}(r+)[a_{np}\cos p\phi + b_{np}\sin p\phi]$$
$$= \frac{s^2+6s+12}{16(s+2)(s+4)} - \frac{s+2}{8s}r+^2 + \frac{(r+)^{s+2}}{2(s+2)s} - \sum_{n=1}^{\infty} r+^n(a_n\cos n\phi + b_n\sin n\phi) \tag{2.44}$$

We next define the parameters

$$\hat{\theta}_{fd} = \frac{s^2+6s+12}{16(s+2)(s+4)} - \frac{s+2}{8s} r+^2 + \frac{(r+)^{s+2}}{2(s+2)s} - \sum_{n=1}^{\infty} r+^n (a_n \cos n\phi + b_n \sin n\phi) \tag{2.45}$$

$$A_0(r+) = \sum_{n=1}^{\infty} a_{n0} R_{n0}(r+) \tag{2.46a}$$

$$A_p(r+) = \sum_{n=1}^{\infty} a_{np} R_{np}(r+) \tag{2.46b}$$

$$B_p(r+) = \sum_{n=1}^{\infty} b_{np} R_{np}(r+) \tag{2.46c}$$

Now, combining (2.44) with (2.45) and (2.46), we obtain

$$\hat{\theta}_{fd} = A_0(r+) + \sum_{p=1}^{\infty} (A_p(r+) \cos n\phi + B_p(r+) \sin p\phi) \tag{2.47}$$

Equation (2.47) is a complete Fourier series expansion of $\hat{\theta}_{fd}$, therefore

$$A_0(r+) = \frac{1}{2\pi} \int_0^{2\pi} \hat{\theta}_{fd} \, d\phi \tag{2.48a}$$

$$A_p(r+) = \frac{1}{\pi} \int_0^{2\pi} \hat{\theta}_{fd} \cos p\phi \, d\phi \tag{2.48b}$$

$$B_p(r+) = \frac{1}{\pi} \int_0^{2\pi} \hat{\theta}_{fd} \sin p\phi \, d\phi \tag{2.48c}$$

As the eigenfunctions R_{np} of (2.38) form a complete orthogonal set in the interval (0, 1) with respect to the weight function $r+(1-r+^s)$, we have the following orthogonal property.

$$\int_0^1 r+(1-r+^s)R_{np}(r+)R_{mp}(r+)dr+ = 0 \qquad np \neq mp \tag{2.49}$$

Therefore, any arbitrary function defined in this domain may be expanded as an infinite series of these eigenfunctions. This proves the existence of expansions of the form defined in (2.48) to be permissible. Furthermore, a_{n0}, a_{np} and b_{np} are calculated from (2.48) by the following relationships which are obtained after utilizing the orthogonal property of the eigenfunctions, i.e., Equation (2.49)

$$a_{n0} = \frac{\int_0^1 r+(1-r+^s)A_0(r+)R_{n0}(r+)dr+}{\int_0^1 r+(1-r+^s)R_{n0}^2(r+)dr+} \tag{2.50a}$$

$$a_{np} = \frac{\int_0^1 r+(1-r+^s)A_p(r+)R_{np}(r+)dr+}{\int_0^1 r+(1-r+^s)R_{np}^2(r+)dr+} \tag{2.50b}$$

$$b_{np} = \frac{\int_0^1 r+(1-r+^s)B_p(r+)R_{np}(r+)dr+}{\int_0^1 r+(1-r+^s)R_{np}^2(r+)dr+} \tag{2.50c}$$

Combining Equations (2.48) and (2.50), and simplifying, we obtain

$$a_{n0} = \frac{\displaystyle\int_0^1 r+(1-r+^s)\left[\frac{s^2+6s+12}{16(s+2)(s+4)} - \frac{s+2}{8s}\, r+^2 + \frac{(r+)^{s+2}}{2(s+2)s}\right] R_{n0}(r+)dr+}{\displaystyle\int_0^1 r+(1-r+^s)R_{n0}^2(r+)dr+} \tag{2.51a}$$

$$a_{np} = \frac{1}{\pi}\,\frac{\displaystyle\int_0^1\int_0^{2\pi} r+(1-r+^s)\hat{\theta}_{fd}\cos p\phi\, R_{np}(r+)d\phi\, dr+}{\displaystyle\int_0^1 r+(1-r+^s)R_{np}^2(r+)dr+} \tag{2.51b}$$

$$b_{np} = \frac{1}{\pi}\,\frac{\displaystyle\int_0^1\int_0^{2\pi} r+(1-r+^s)\hat{\theta}_{fd}\sin p\phi\, R_{np}(r+)\, d\phi\, dr+}{\displaystyle\int_0^1 r+(1-r+^s)R_{np}^2(r+)dr+} \tag{2.51c}$$

where $\hat{\theta}_{fd}$ is given by (2.45). For any arbitrary variation of circumferential wall heat flux $\hat{\theta}_{fd}$ can be obtained and Equation (2.51) may be integrated to determine a_{n0}, a_{np}, and b_{np}.

2.3.3 Complete Solution

At this point the solution to the thermal entrance region is completed. We may now add this solution to the fully-developed portion, using Equations (2.17), (2.31), and (2.37) to obtain the complete solution as follows:

$$\theta(x+, r+, \phi) = \frac{x/r_0}{\text{Re Pr}} - \left(\frac{s^2+6s+12}{16(s+2)(s+4)}\right) + \frac{s+2}{8s} r+^2 - \frac{1}{2(s+2)s} r+^{s+2}$$

$$+ \sum_{n=1}^{\infty} r+^n(a_n \cos n\phi + b_n \sin n\phi) + \sum_{n=1}^{\infty} a_{n0}R_{n0}e^{-\lambda_{n0}^2 x+}$$

$$+ \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} e^{-\lambda_{np}^2 x+} R_{np}(r+)(a_{np} \cos p\phi + B_{np} \sin p\phi) \quad (2.52)$$

where

$$a_n = \frac{1}{2n} \int_0^{2\pi} \frac{q(\phi)}{\bar{q}} \cos n\phi \, d\phi \quad (2.52a)$$

$$b_n = \frac{1}{2n} \int_0^{2\pi} \frac{q(\phi)}{\bar{q}} \sin n\phi \, d\phi \quad (2.52b)$$

and a_{n0}, a_{np}, and b_{np} are given by Equation (2.51).

2.3.4 Calculation of Nusselt Number

The Nusselt number is defined as

$$\text{Nu}(x, \phi) = \frac{2h(x, r_0, \phi)r_0}{k} = \frac{k \frac{\partial t}{\partial r}(x, r_0, \phi)}{t_w - t_m} \frac{2r_0}{k}$$

or equivalently

$$\text{Nu}(x, \phi) = \frac{q(\phi)2r_0}{(t_w - t_m)k} \quad (2.53)$$

where $t_w(x, \phi) = t(x, r_0, \phi)$, the wall temperature, and $t_m(x)$ is the mean fluid temperature.

It will be convenient to represent the heat flux distribution in the form

$$q(\phi) = \bar{q} f(\phi) \tag{2.54a}$$

where $\bar{q}$ is given by

$$\bar{q} = \int_0^{2\pi} q(\phi) d\phi \tag{2.54b}$$

and $f(\phi)$ is specified angular variation. With this specification on $q(\phi)$, Equation (2.53) reduces to

$$Nu(x+, \phi) = \pi f(\phi) \left[\frac{\bar{q}\, 2r_0/k\pi}{t_w - t_m} \right] \tag{2.54c}$$

Now, expressing Equation (2.52) in terms of mean fluid temperature by combining with Equation (2.32) we obtain,

$$\frac{t-t_m}{\bar{q}\, 2r_0/k\pi} = -\left(\frac{s^2+6s+12}{16(s+2)(s+4)}\right) + \frac{s+2}{8s} r+^2 - \frac{1}{2(s+2)s} r+^{s+2}$$
$$+ \sum_{n=1}^{\infty} r+^n (a_n \cos n\phi + b_n \sin n\phi) + \sum_{n=1}^{\infty} a_{n0} R_{n0}(r+) e^{-\lambda_{n0}^2 x+}$$
$$+ \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} e^{-\lambda_{np}^2 x+} R_{np}(r+)(a_{np} \cos p\phi + b_{np} \sin p\phi) \tag{2.55}$$

When the wall heat flux is specified, the wall temperature is the unknown quantity that is usually of most practical interest. It is found by evaluating Equation (2.55) at $r+ = 1$ to yield

$$\frac{t_w - t_m}{\overline{q}\, 2r_0/k\pi} = \frac{s^2+10s+20}{16(s+2)(s+4)} + \sum_{n=1}^{\infty} (a_n \cos n\phi + b_n \sin n\phi)$$

$$+ \sum_{n=1}^{\infty} a_{n0} R_{n0}(1) e^{-\lambda_{n0}^2 x+}$$

$$+ \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} e^{-\lambda_{np}^2 x+} R_{np}(1)(a_{np} \cos p\phi + b_{np} \sin p\phi) \tag{2.56}$$

Finally, we solve for $Nu(x+, \phi)$ by using Equations (2.54c) and (2.56) to yield

$$Nu(x+, \phi) = \pi f(\phi) \left\{ \frac{s^2+10s+20}{16(s+2)(s+4)} + \sum_{n=1}^{\infty} (a_n \cos n\phi + b_n \sin n\phi) \right.$$

$$+ \sum_{n=1}^{\infty} a_{n0} R_{n0}(1) e^{-\lambda_{n0}^2 x+}$$

$$\left. + \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} e^{-\lambda_{np}^2 x+} R_{np}(1)(a_{np} \cos p\phi + b_{np} \sin p\phi) \right\}^{-1}$$

(2.57)

2.3.5 Axial Non-Uniform Wall Heat Flux

The temperature solution obtained for uniform axial heat input can be used to generate solutions for any arbitrary specified axial variation of wall heat flux, using superposition. This is possible because of the linearity of the energy differential equation. Using the approach used by Siegel, Sparrow, and Hallman [59], for any arbitrary heat flux variation of the form $q(x+, \phi) = \hat{Q}(x+)\,q(\phi)$ Equation (2.52) can be written as follows:

$$\frac{t-t_\epsilon}{r_0/k} = \int_0^{x+} \left\{ \frac{2(s+2)}{s} - 4\sum_{n=1}^{\infty} a_{n0}\lambda_{n0}^2 R_{n0}(r+) e^{-\lambda_{n0}^2 (x+-\zeta)} - 4\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} e^{-\lambda_{np}^2(x+-\zeta)} \lambda_{np}^2 R_{np}(r+)[a_{np}\cos\phi + b_{np}\sin p\phi] \right\} \hat{Q}(\zeta)\,d\zeta$$

2.4 Special Examples

2.4.1 Cosine Heat Flux Variation Around the Tube Periphery

As an illustrative case, a cosine circumferential heat-flux distribution of the form $q(\phi) = q_{av}(1 + b\cos p\phi)$ is considered. A functional relationship of this form is of special interest in nuclear reactor technology. Furthermore, the simultaneous effects of circumferential wall heat flux variation and non-Newtonian velocity distribution on the convection process are investigated. The following cases are considered.

2.4.1.1 Asymptotic Examples. Letting $x+ \rightarrow \infty$, Equation

(2.57) reduces to

$$Nu_{\infty}(\phi) = \pi f(\phi) \left\{ \frac{s^2+10s+20}{16(s+2)(s+4)} + \sum_{n=1}^{\infty} (a_n \cos n\phi + b_n \sin n\phi) \right\}^{-1} \quad (2.58)$$

Solving for a_n, b_n, $f(\phi)$ using Equations (2.28), (2.54) and (2.55) we obtain the following coefficients

$$a_p = \frac{b}{4p}, \quad b_n = 0, \quad f(\phi) = \frac{1+b \cos p\phi}{2\pi} \quad (2.59)$$

Combining (2.58) and (2.59) we have

$$Nu_{\infty}(\phi) = \frac{1+b \cos p\phi}{\frac{s^2+10s+20}{8(s+2)(s+4)} + \frac{b}{2p} \cos p\phi} \quad (2.60)$$

$$p = 1, 2, 3, \ldots$$

For $p = 1$

$$Nu_{\infty}(\phi) = \frac{1+b \cos \phi}{\frac{s^2+10s+20}{8(s+2)(s+4)} + \frac{b}{2} \cos \phi} \quad (2.61)$$

which is the solution obtained by Inman [22] for non-Newtonian power-law fluids, using superposition.

For Newtonian fluids we have $s = 2$ and Equation (2.61) reduces to

$$Nu_{\infty}(\phi) = \frac{1+b \cos \phi}{\frac{11}{48} + \frac{b}{2} \cos \phi} \quad (2.62)$$

which is the solution obtained by Reynolds [49]. Finally, for the constant wall heat flux condition, $b = 0$, and Equation (2.61) reduces to the following asymptotic Nusselt number for non-Newtonian power-law fluids.

$$Nu_\infty = \frac{8(s+2)(s+4)}{s^2+10s+20}$$

Asymptotic solutions for the following cases are:

Slug flow $(s = \infty)$: $Nu_\infty = 8$

Newtonian $(s = 2)$: $Nu_\infty = \frac{48}{11}$

<u>2.4.1.2 Thermal-Entry-Length Examples</u>. Equation (2.45) is reduced to the following form by using Equation (2.59)

$$\hat{\theta}_{fd} = \frac{s^2+6s-12}{16(s+2)(s+4)} - \frac{s+2}{8s}\, r^{+2} + \frac{r^{+s+2}}{2(s+2)s} - \frac{b r^{+p}}{4p} \cos p\phi \qquad (2.63)$$

Now, the coefficients a_{n0}, a_{np}, and b_{np} are simplified using Equations (2.63) and (2.51) to obtain:

$$a_{n0} = \frac{\displaystyle\int_0^1 r^+(1-r^{+s})\left[\frac{s^2+6s+12}{16(s+2)(s+4)} - \frac{s+2}{8s}\, r^{+2} + \frac{r^{+s+2}}{2(s+2)s}\right] R_{n0}(r^+)\,dr^+}{\displaystyle\int_0^1 r^+(1-r^{+s}) R_{n0}^2(r^+)\,dr^+} \qquad (2.64a)$$

$$a_{np} = \frac{-b\int_0^1 (r+)^{p+1}(1-r+^s)R_{np}(r+)dr+}{\int_0^1 r+(1-r+^s)R_{np}^2(r+)dr+} \tag{2.64b}$$

$$b_{np} = 0 \tag{2.64c}$$

The numerators of (2.64a, b) are simplified by substituting for $(r+)^{p+1}(1-r+^s)R_{np}(r+)$ from the characteristic Equation (2.38) and integrating twice by parts to obtain the following general equations for the expansion coefficients.

$$\hat{a}_{n0} = 4\, a_{n0} = \frac{-R_{n0}(1)}{\lambda_{n0}^2 \int_0^1 r+(1-r+^s)R_{n0}^2(r+)dr+} \tag{2.65a}$$

$$\hat{a}_{np} = \frac{4p}{b} a_{np} = \frac{-R_{np}(1)}{\lambda_{np}^2 \int_0^1 r+(1-r+^s)R_{np}^2(r+)dr+} \tag{2.65b}$$

The above equations were used to evaluate the expansion coefficients in this study. The integrals appearing in the denominator were obtained numerically.

For the case where heat flux varies according to $q(\phi) = q_{av}(1+b\cos\phi)$, the only non-zero coefficients are $\hat{a}_{n0}$, and $\hat{a}_{n1}$. The expressions for fluid temperature, wall temperature, and Nusselt number are obtained by simplifying Equations (2.52), (2.56) and (2.57) using (2.59).

$$\frac{t-t_\epsilon}{q_{av}r_0/k} = \frac{4x/r_0}{\text{Re Pr}} - \frac{s^2+6s+12}{4(s+2)(s+4)} + \frac{s+2}{2s} r+^2 - \frac{2}{(s+2)s}(r+)^{s+2}$$

$$+ \sum_{n=1}^{\infty} \hat{a}_{n0} R_{n0}(r+) e^{-\lambda_{n0}^2 x+}$$

$$+ b \cos \phi \left[r+ + \sum_{n=1}^{\infty} \hat{a}_{n1} R_{n1}(r+) e^{-\lambda_{n1}^2 x+} \right] \qquad (2.66)$$

$$\frac{t_w - t_m}{q_{av}r_0/k} = \frac{s^2+10s+20}{4(s+2)(s+4)} + \sum_{n=1}^{\infty} \hat{a}_{n0} R_{n0}(1)\, e^{-\lambda_{n0}^2 x+}$$

$$+ b \cos \phi \left[1 + \sum_{n=1}^{\infty} \hat{a}_{n1} R_{n1}(1) e^{-\lambda_{n1}^2 x+} \right] \qquad (2.67)$$

$$Nu(\phi, x+) = 2(1+b \cos \phi) \left\{ \frac{2+10s+20}{4(s+2)(s+4)} + \sum_{n=1}^{\infty} \hat{a}_{n0} R_{n0}(1) e^{-\lambda_{n0}^2 x+} \right.$$

$$\left. + b \cos \phi \left[1 + \sum_{n=1}^{\infty} \hat{a}_{n1} R_{n1}(1) e^{-\lambda_{n1}^2 x+} \right] \right\}^{-1}$$

$$(2.68)$$

For the limiting case of uniform wall heat flux, $b = 0$, the only non-zero expansion coefficients are $\hat{a}_{n0}$. Equations (2.66), (2.67), and (2.68) reduce to

$$\frac{t-t_\epsilon}{q_{av}r_0/k} = \frac{4x/r_0}{\text{Re Pr}} - \frac{s^2+6s+12}{4(s+2)(s+4)} + \frac{s+2}{2s}\, r+^2 - \frac{2}{(s+2)s}\, r+^{s+2}$$

$$+ \sum_{n=1}^{\infty} \hat{a}_{n0} R_{n0}(r+) e^{-\lambda_{n0}^2 x+} \tag{2.69}$$

$$\frac{t_w - t_m}{q_{av}r_0/k} = \frac{s^2+10s+20}{4(s+2)(s+4)} + \sum_{n=1}^{\infty} \hat{a}_{n0} R_{n0}(1) e^{-\lambda_{n0}^2 x+} \tag{2.70}$$

$$\text{Nu}(x+) = \frac{2}{\dfrac{s^2+10s+20}{4(s+2)(s+4)} + \displaystyle\sum_{n=1}^{\infty} \hat{a}_{n0} R_{n0}(1) e^{-\lambda_{n0}^2 x+}} \tag{2.71}$$

which are the expressions obtained by Bird [5]. Finally for the case of Newtonian fluids, $s = 2$, Equations (2.69), (2.70), and (2.71) reduce to the expressions obtained by Siegel, Sparrow, and Hallman [59] as follows:

$$\frac{t-t_\epsilon}{q_{av}r_0/k} = \frac{4x/r_0}{\text{Re Pr}} - \frac{7}{24} + r+^2 - \frac{r+^4}{4} + \sum_{n=1}^{\infty} \hat{a}_{n0} R_{n0}(r+) e^{-\lambda_{n0}^2 x+} \tag{2.72}$$

$$\frac{t_w - t_m}{q_{av}r_0/k} = \frac{11}{24} + \sum_{n=1}^{\infty} \hat{a}_{n0} R_{n0}(1) e^{-\lambda_{n0}^2 x+} \tag{2.73}$$

$$\text{Nu}(x+) = \frac{2}{\dfrac{11}{24} + \displaystyle\sum_{n=1}^{\infty} \hat{a}_{n0} R_{n0}(1) e^{-\lambda_{n0}^2 x+}} \tag{2.74}$$

2.5 Results

2.5.1 Numerical Determination of the Eigenvalues, Eigenfunctions, and Expansion Coefficients

Using the CDC 6400, the first 12 eigenvalues, eigenfunctions of the characteristic Equation (2.38) for $p = 0, 1, 2, 3, 4, 5$ and several values of the non-Newtonian fluid behavior index, s, have been obtained. The expansion coefficients were evaluated from Equations (2.65a, b). These coefficients are listed in Table C.1. For the limiting case of a Newtonian fluid (i.e., $s = 2$) and the constant wall heat flux condition (i.e., $p = 0$), these coefficients are in excellent agreement with the corresponding values obtained by Siegel, Sparrow, and Hallman [59] and by Hsu [18]. Table 2.1 presents a comparison of these results.

For the case of pseudo-plastic fluids ($s > 2$) and the constant wall heat flux condition, the eigenfunctions at the tube wall, the eigenvalues and expansion coefficients reduce to Table 1 of Michiyoshi and Matsumoto [37] and Tables 4, 5, and 6 of Mitsuishi and Miyatake [39]. These coefficients are summarized and compared in Table 2.2 of the present work.

Michiyoshi and Matsumoto obtained the first five of these coefficients and Mitsuishi and Miyatake obtained the first three of the corresponding coefficients. This is because of the inherent difficulties

Table 2.1. Comparison of eigenvalues, eigenfunctions and expansion coefficients for the Newtonian problem (s = 2) and uniform wall heat flux (p = 0) with the results of Hsu and Siegel, Sparrow, and Hallman.

	Siegel, Sparrow, and Hallman			Hsu			Present Work		
n	λ^2_{n0}	$R_{no}(1)$	$\hat{a}_{n0}$	λ_{n0}	$R_{n0}(1)$	$\hat{a}_{n0}$	λ_{n0}	$R_{n0}(1)$	$\hat{a}_{n0}$
1	25.6796	-.492517	.403483	5.067504	-.492517	.403483	5.0675055	-.4925166	.4034832
2	83.8618	.395508	-.175111	9.157609	.395508	-.175110	9.1576064	.3955085	-.1751100
3	174.167	-.345872	.105594	13.19722	-.345874	.105592	13.1972247	-.3458737	.1055917
4	296.536	.314047	-.0732804	17.22023	.314046	-.0732824	17.2202294	.3140465	-.0732824
5	450.947	-.291252	.0550357	21.23552	-.291251	.0550365	21.2355173	-.2912515	.0550365
6	637.387	.273808	-.043483	25.24653	.273807	-.0434844	25.2465312	.2738070	-.0434844
7	855.850	.259852	.035597	29.25491	-.259853	.0355951	29.2540955	-.2598530	.0355951
8				33.26152	.248332	.0299085	33.2615237	.2483320	-.0299084
9				37.26691	-.238590	.0256401	37.2669082	-.2385904	.0256401
10				41.27139	.230199	-.0223337	41.2713893	.2301993	-.0223336
11				45.27519	-.222863	.0197069	45.2751868	-.2228631	.0197069
12				49.27846	.216370	-.0175765	49.2789682	.2163668	-.0175762

Table 2.2. Comparison of eigenvalues, eigenfunctions, and expansion coefficients for pseudoplastic fluids and uniform wall heat flux with the results of Michiyoshi and Matsumoto [36] and Mitsuishi and Miyatake [39].

n	Michiyoshi and Matsumato			Mitsuishi and Miyatake			Present Work		
	λ^2_{n0}	$R_{n0}(1)$	$\hat{a}_{n0}$	λ^2_{n0}	$R_{n0}(1)$	$\hat{a}_{n0}$	λ_{n0}	$R_{n0}(1)$	$\hat{a}_{n0}$
s = 4									
1	20.7623	-.4594		20.75621	-.459361	.374948	4.5555898	-.4593614	.3749484
2	67.7523	.3684		67.67724	.368174	-.162983	8.2266127	.3681742	-.1629858
3	140.8654	.3219		140.5581	-.321533	.098167	11.8557713	-.3215268	.0981749
4	240.1917	.2923					15.4706322	.2916657	-.0680736
5	366.0957	-.2667					19.0787579	-.2703117	.0510894
6			Not Presented				22.6831188	.2539907	-.0403443
7							26.2851386	-.2409491	.0330107
8							29.8855920	.2301905	-.0277273
9							33.4849398	-.2211002	.0237634
10							37.0834748	.2123750	-.0206940
11							40.6813927	-.2064372	.0182562
12							44.2788297	.2003889	-.0162796
s = 6									
1	18.9927	-.4400		18.98420	-.439976	.362294	4.3570857	-.4399761	.3622953
2	62.2528	.3521		62.183377	.351922	-.156065	7.8856682	.3519216	-.1560682
3	129.5550	-.3072		129.2773	-.306845	.093787	11.3699979	-.3068454	.0937854
4	220.9273	.2782					14.8399911	.2780651	-.0649429
5	338.4745	-.2683					18.3032563	-.2575228	.0486964
6			Not Presented				21.7627592	.2418446	-.0384298
7							25.2199189	-.2293306	.0314287
8							28.6755081	.2190168	-.0263883
9							32.1299868	-.2103090	.0226086
10							35.5836478	.2028178	-.0196830
11							39.0366869	-.1962756	.0173605
12							42.4892407	.1904915	-.0154778

associated with the numerical determination of higher eigenvalues and corresponding eigenfunctions. For special problems where axial variation of wall heat flux is present, the infinite series in the temperature solution converges slowly and the first few eigenvalues, eigenfunctions, and expansion coefficients are not sufficient. For this reason the first 12 of these coefficients were obtained in the present investigation. Another limiting problem is that of a Newtonian fluid flowing in a pipe with an arbitrary circumferential wall heat flux. The related constants for this problem are also presented in Table C.1.

The determination of eigenvalues, eigenfunctions, and expansion coefficients for pseudo-plastic fluids flowing in a tube with an arbitrary circumferential wall heat flux is the main concern of the present work. For several values of non-Newtonian parameter, s (i.e., $s = 4, 6, 8, 10, 12$), and for any arbitrary variation of wall heat flux around the circumference that could be expressed in terms of a Fourier series up to fifth harmonics, the coefficients listed in Table C.1 are useful.

Finally, in Figures 2.2, 2.3, and 2.4 the first two eigenfunctions are shown for $p = 0, 1, 2$ (i.e., $R_{00}, R_{10}, R_{01}, R_{11}, R_{02}, R_{12}$) and for several values of the non-Newtonian parameter, s. Figure 2.5 presents similar plots for the third and the fourth eigenfunctions for $p = 0$.

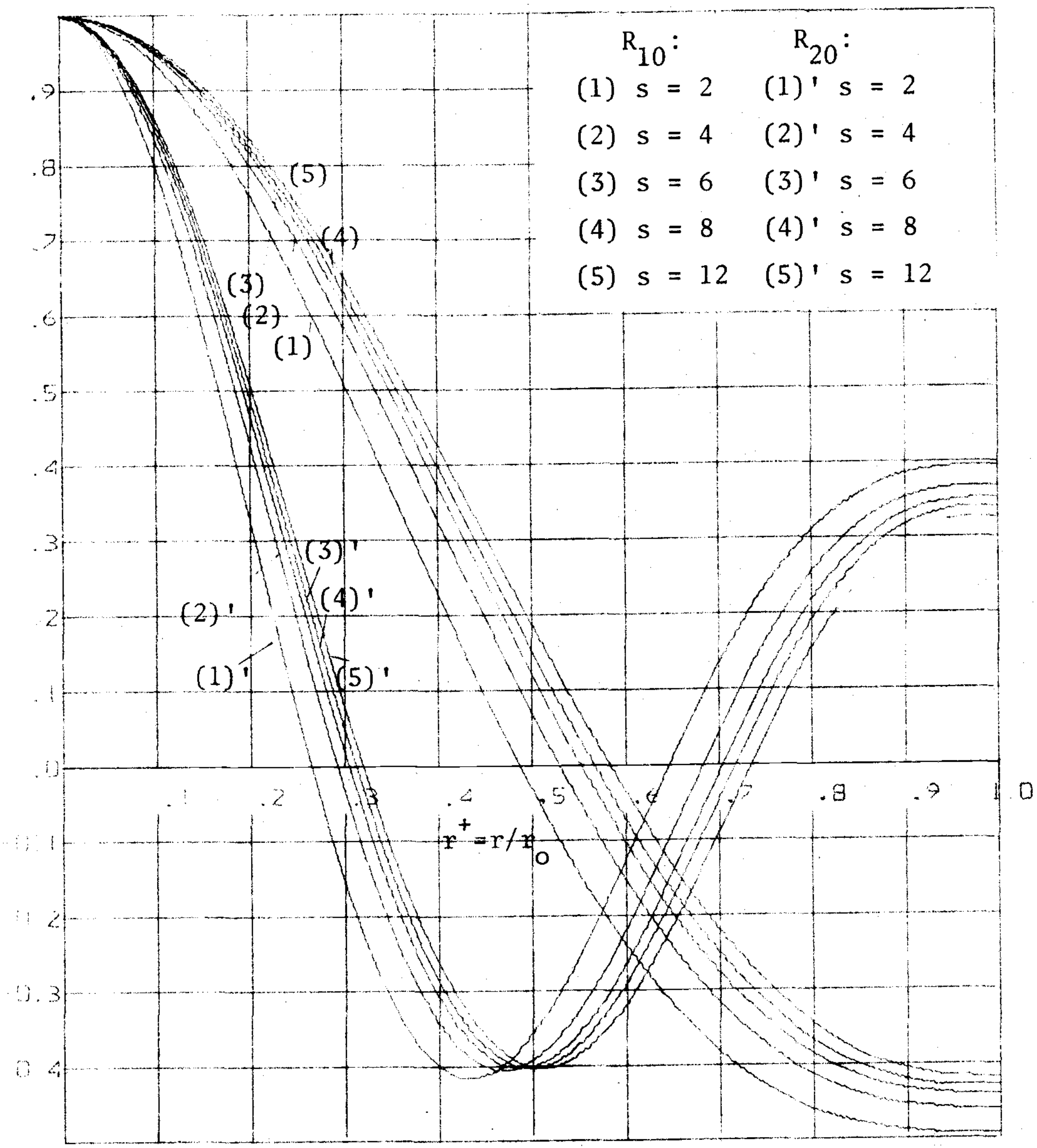

Figure 2.2. The first two eigenfunctions for different non- Newtonian fluid behavior index, s, and for p = 0.

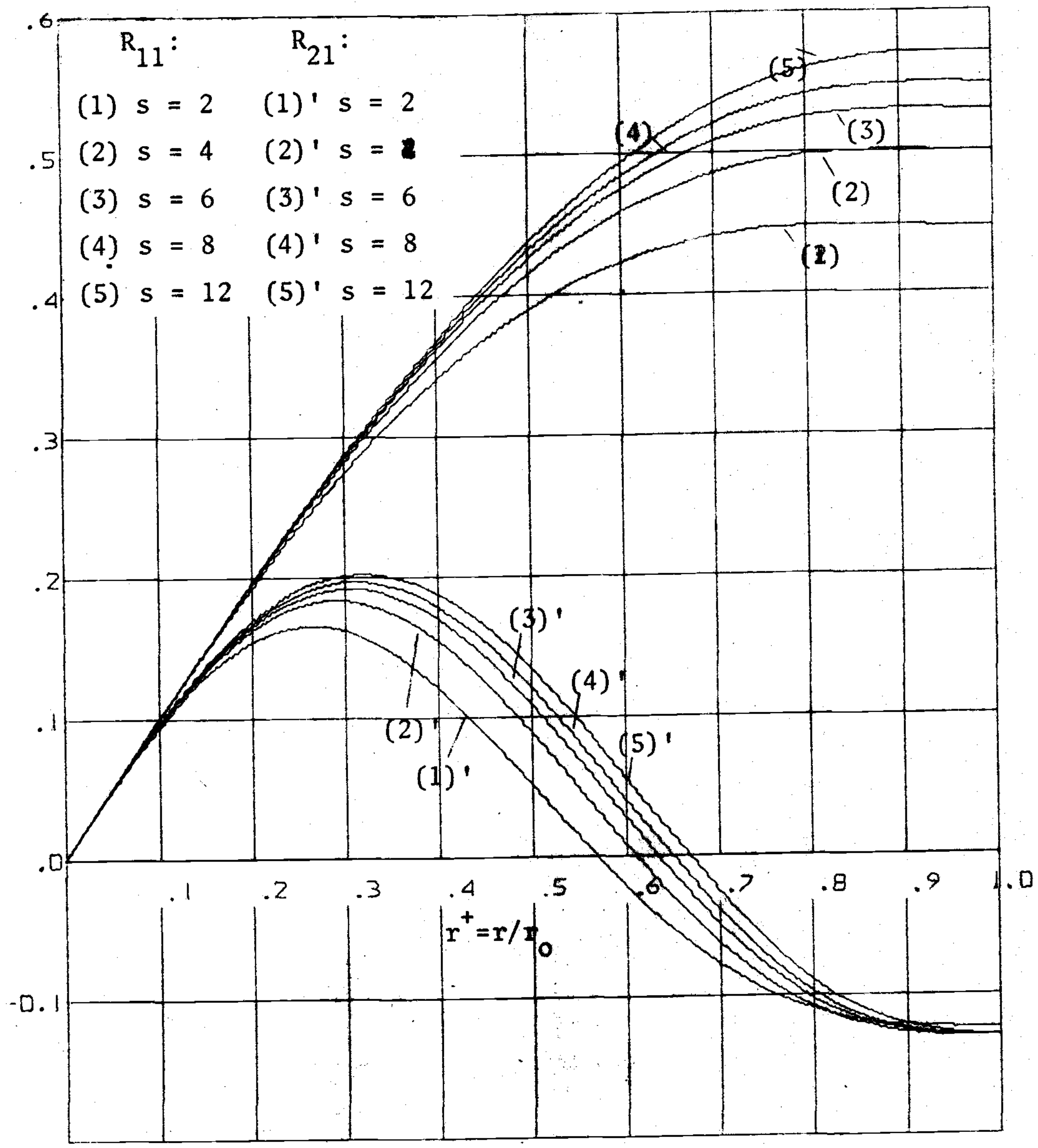

Figure 2.3. The first two eigenfunctions for different non-Newtonian fluid behavior index, s, and for p = 1.

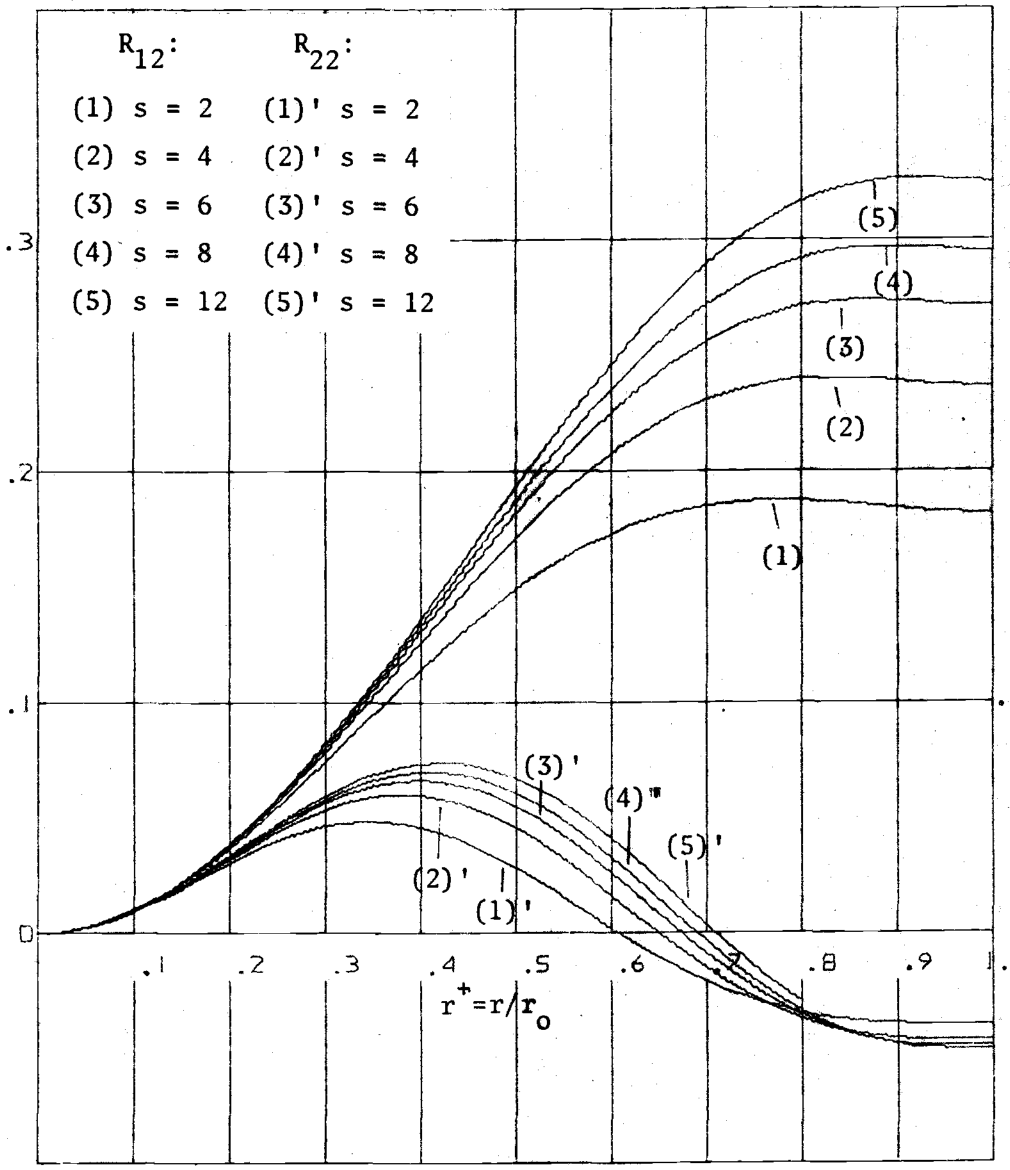

Figure 2.4. The first two eigenfunctions for different non-Newtonian fluid behavior index, s, and for p = 2.

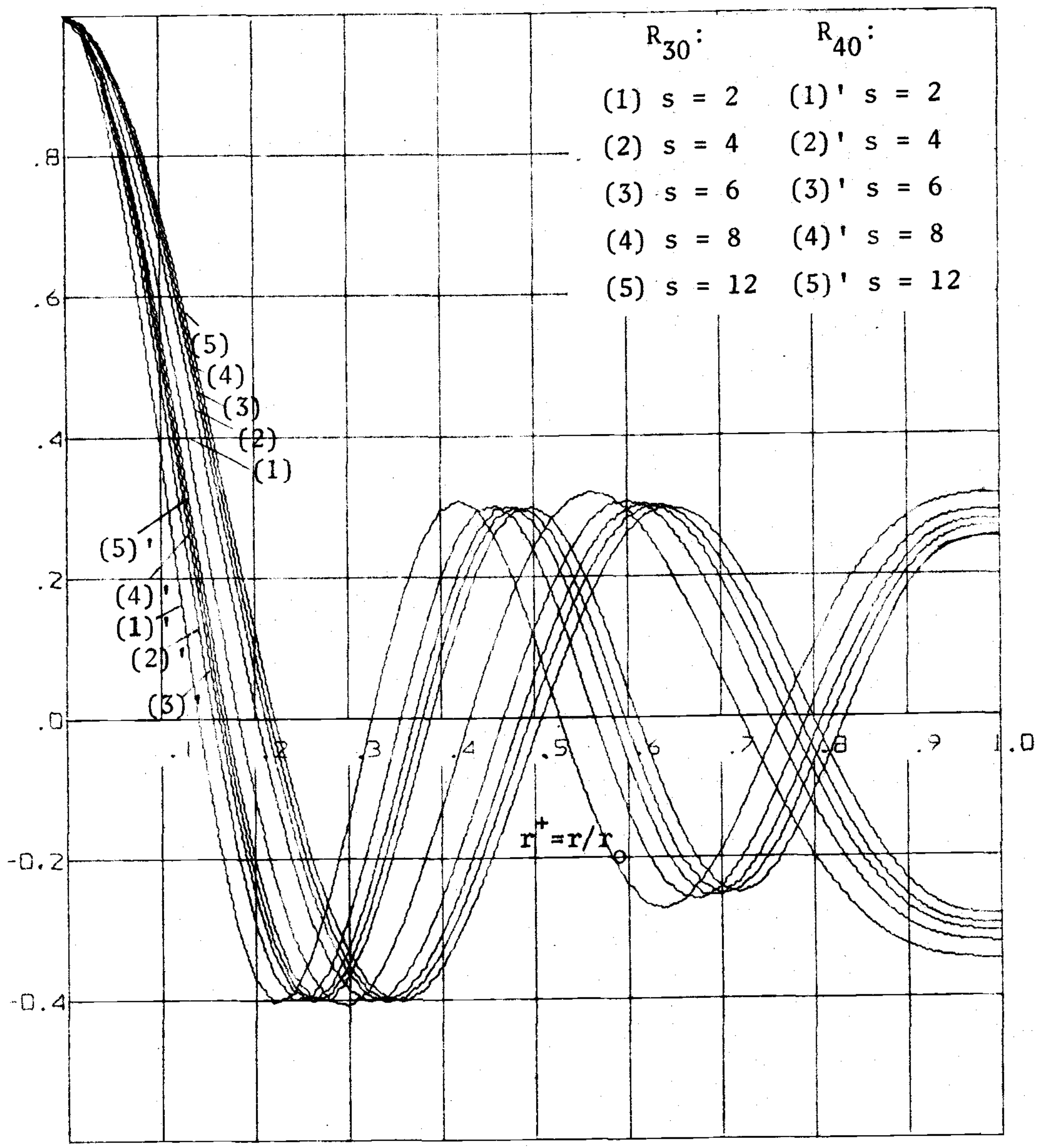

Figure 2.5. The third and fourth eigenfunctions for different non-Newtonian fluid behavior index, s, and for p = 0.

2.5.2 Discussions of Results for the Special Example $q(\phi) = q_{av}(1+b \cos \phi)$

With the numerical information obtained in the previous section, we may investigate the simultaneous effects of circumferential wall heat flux variation and non-Newtonian velocity distributions on wall temperature and Nusselt number in the entrance region of a tube. Using the obtained eigenvalues, eigenfunctions at tube wall, and the expansion coefficients, the dimensionless wall-to-bulk temperature difference and the local Nusselt numbers have been calculated for different values of the parameters s, $x+$, b, and ϕ from Equations (2.67) and (2.68). For the case of a Newtonian fluid $(s = 2)$ and uniform wall heat flux $(b = 0)$ a comparison of these results with Kays' [25] Table 8-6 is given by Table 2.3.

Table 2.3. Comparison of local Nusselt numbers for the circular tube; constant heat rate; thermal entry length with Kays [25].

x+	Kays' Table 8-6 Nu(x+)	Present Work Nu(x+)
.001	Not calculated	15.758
.002	12.00	12.537
.004	9.93	9.986
.010	7.49	7.494
.020	6.14	6.148
.040	5.19	5.198
.10	4.51	4.514
∞	4.36	4.364

For non-Newtonian fluids there are no tabulated Nusselt values for the case of uniform wall heat flux. Table 2.4 presents the local Nusselt number for different non-Newtonian fluid behavior indices.

Table 2.4. Local Nusselt numbers for laminar flow of power-law, non-Newtonian fluids in the thermal entrance region of a circular pipe with uniform wall heat flux where $x+ = ((2s)/(s+2))\ ((x/r_0)/(\mathrm{Re\ Pr}))$.

x+	s = 4 Nu(x+)	s = 6 Nu(x+)	s = 8 Nu(x+)	s = 10 Nu(x+)	s = 12 Nu(x+)
.001	17.927	19.590	20.947	22.095	23.089
.002	14.238	15.534	16.583	17.463	18.220
.004	11.335	12.350	13.163	13.838	14.411
.01	8.507	9.255	9.842	10.320	10.717
.02	6.989	7.559	8.068	8.442	8.748
.04	5.930	6.449	6.838	7.143	7.387
.1	5.195	5.662	6.003	6.264	6.471
∞	5.053	5.517	5.854	6.109	6.310

The graphical representations of the Nusselt numbers tabulated in Tables 2.3 and 2.4 are shown in Figure 2.6. The result is in excellent agreement with the work of Michiyoshi and Matsumoto. Figure 2.6 illustrates the entrance-region local Nusselt numbers for uniform wall heat flux and for different non-Newtonian fluid behavior index, s. It is seen that the local Nusselt values increase as the flow becomes more pseudo-plastic.

Furthermore, the asymptotic Nusselt values are reached at an axial distance, $x+ = .1$, which is unaffected by different values of the parameter, s. Figure 2.7 shows the relationship

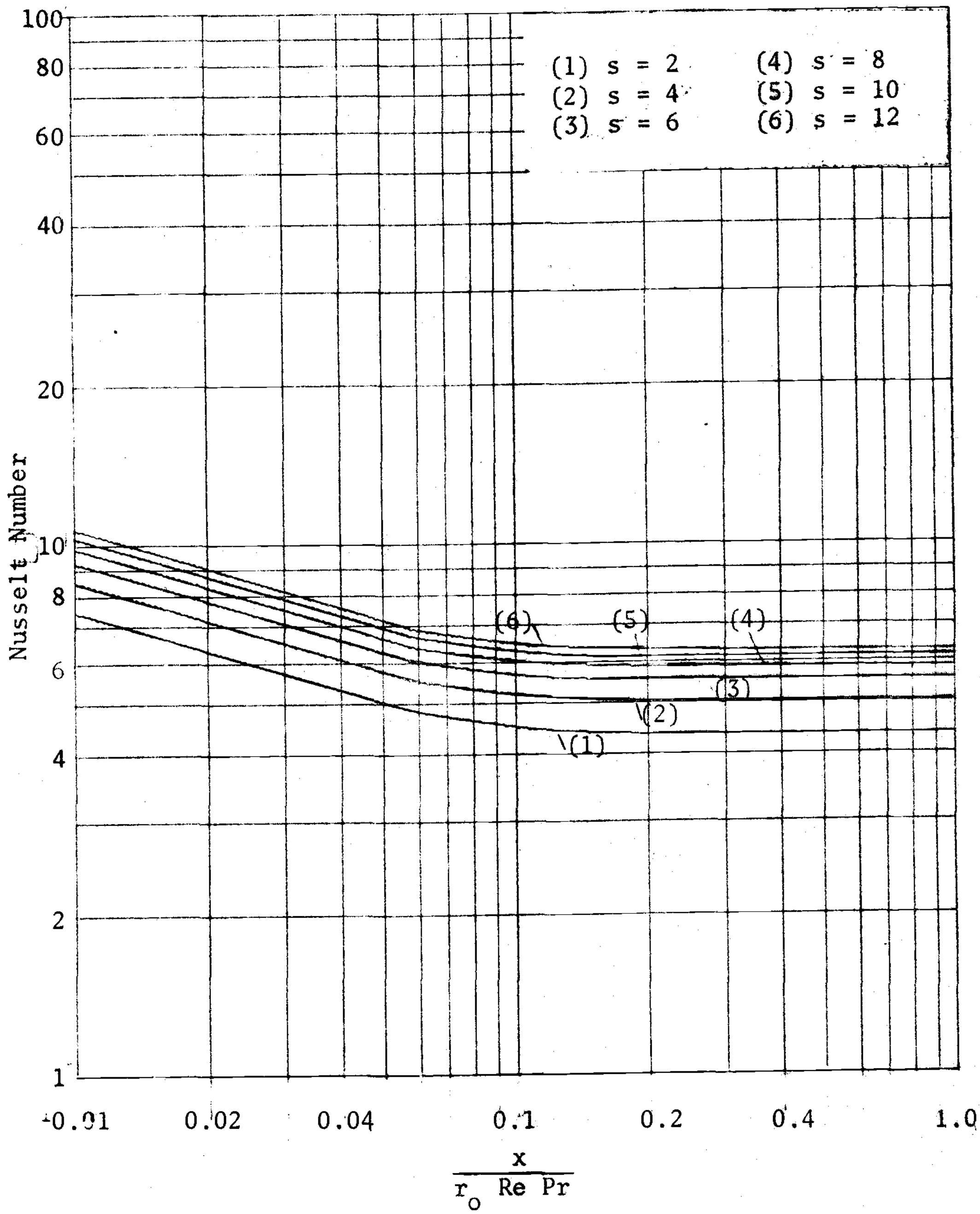

Figure 2.6. Entrance-region local Nusselt numbers for uniform wall heat flux and for different non-Newtonian fluid behavior index, s.

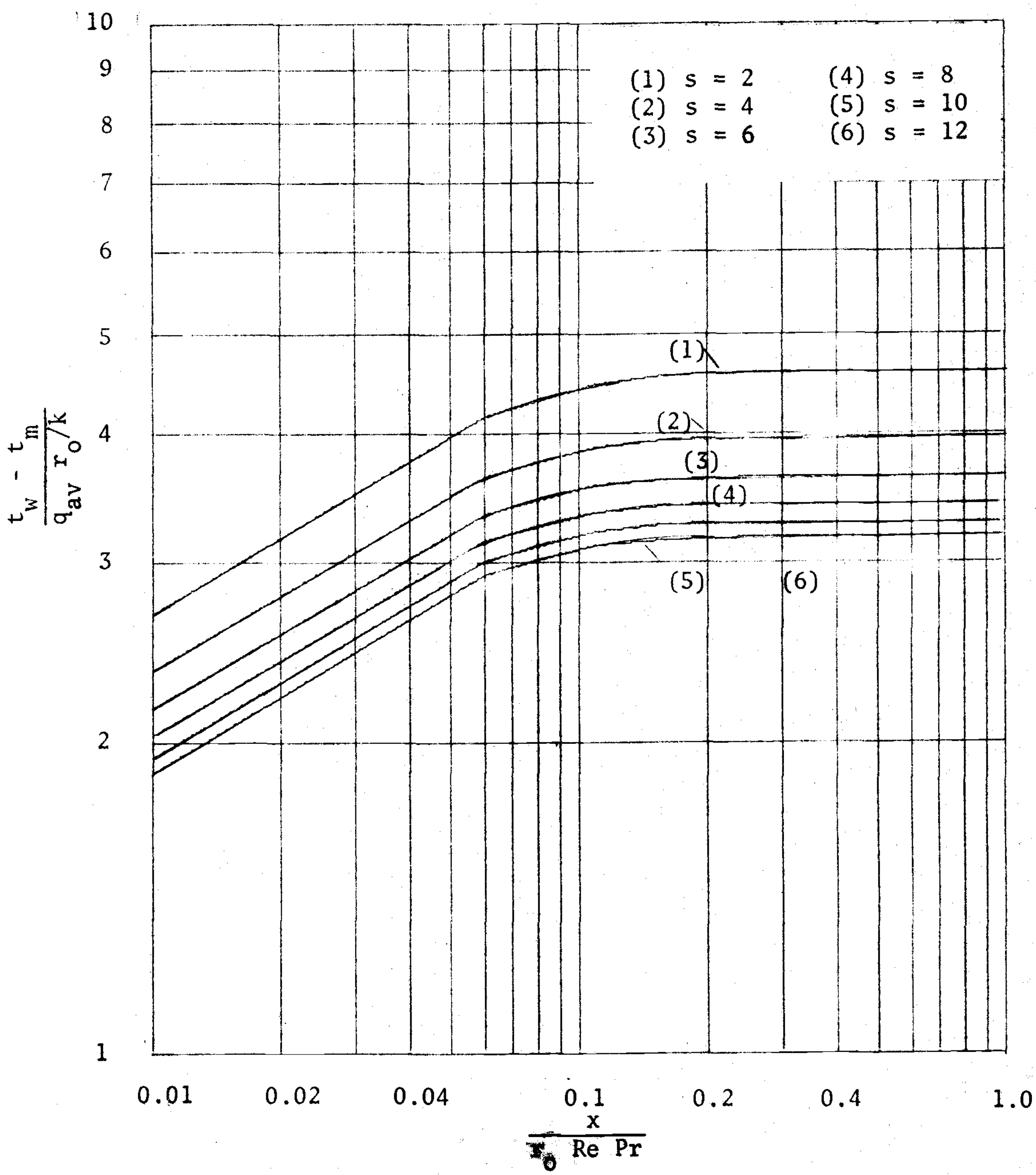

Figure 2.7. Entrance-region local wall-to bulk temperature difference for uniform wall heat flux and for different non-Newtonian fluid behavior index, s.

between dimensionless wall-to bulk temperature difference and dimensionless axial position, for the case of uniform wall heat flux and for different values of the non-Newtonian parameter, s.

For the prescribed wall heat flux $q(\phi) = q_{av}(1+b \cos \phi)$ where $b = 1$, the dimensionless wall-to bulk temperature difference (Equation 2.67) has been plotted in Figure 2.8 as a function of angular position ϕ at a section where fully-developed conditions exist (i.e., $x+ = 1.0$) and for different values of the non-Newtonian behavior parameter, s. It is seen that there is a significant variation in the dimensionless wall temperature difference $\frac{t_w - t_m}{q_{av} r_0 / k}$ around the tube periphery in the presence of a nonuniform peripherial heat flux and non-Newtonian behavior. These results are in excellent agreement with the work of Inman [22]. Figures 2.9, 2.10, 2.11, and 2.12 present the corresponding plots for the thermal entrance region (i.e., $x+ = .1, .04, .02, .01$) respectively. By comparison of these plots, it is seen that the effect of non-Newtonian behavior on wall temperature becomes more pronounced in the entrance region. Increased values of the heat-flux parameter, b, result in increased temperature variations around the circumference, for a given non-Newtonian velocity distribution as seen by Figures 2.13, 2.14, 2.15, and 2.16.

The local Nusselt number has been plotted in Figure 2.17 as a function of angular position ϕ, by using Equation (2.68) for different

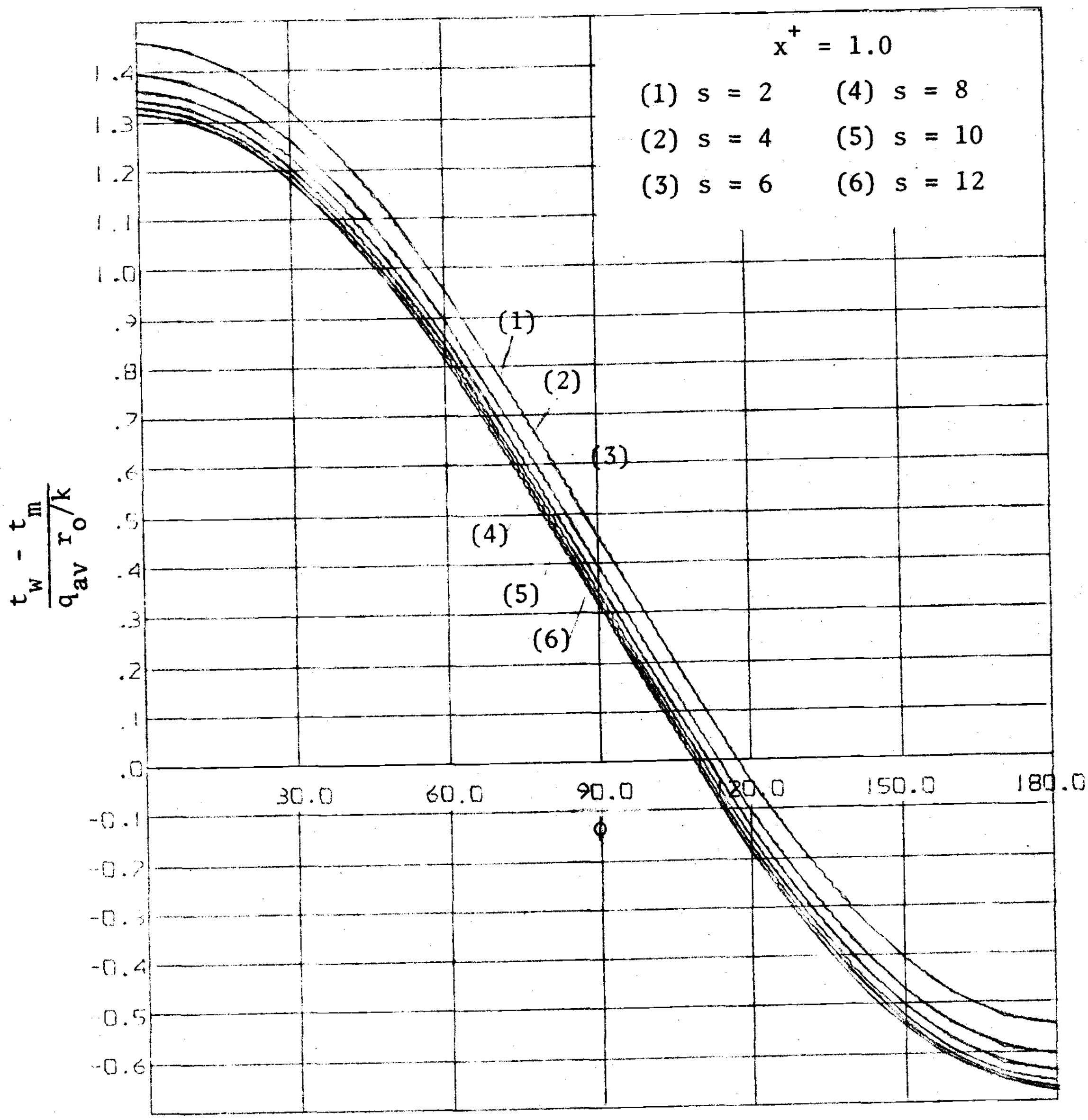

Figure 2.8. Illustration of effect of prescribed wall heat flux variation $q(\phi) = q_{av}(1 + \cos\phi)$ and non-Newtonian influence on wall-to bulk temperature difference at the location far away from the entrance (i.e., x+ = 1).

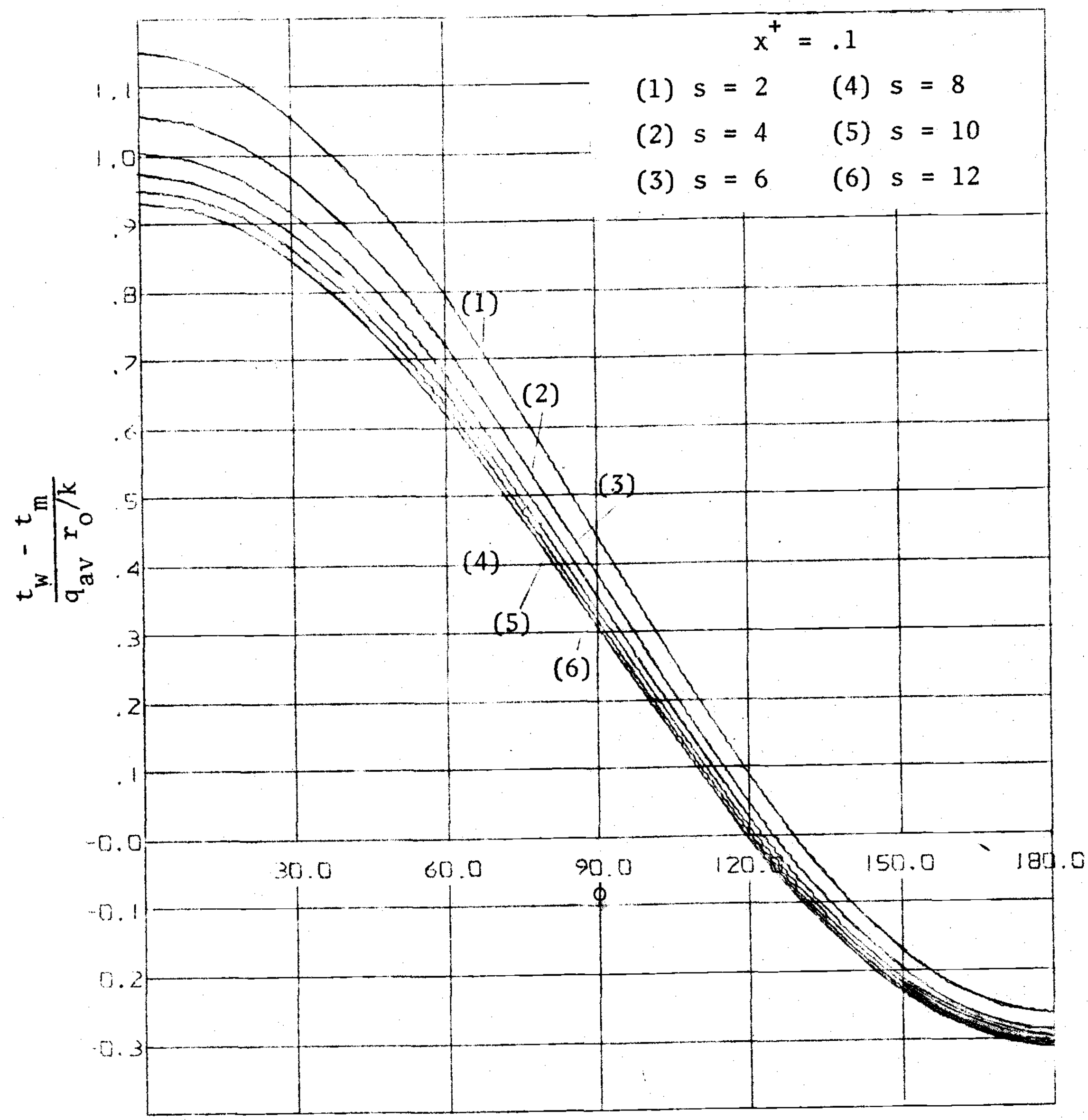

Figure 2.9. Illustration of entrance effect of prescribed wall heat flux variation $q(\phi) = q_{av}(1 + \cos\phi)$ and non-Newtonian influence on wall-to bulk temperature difference at the location x+ = .1.

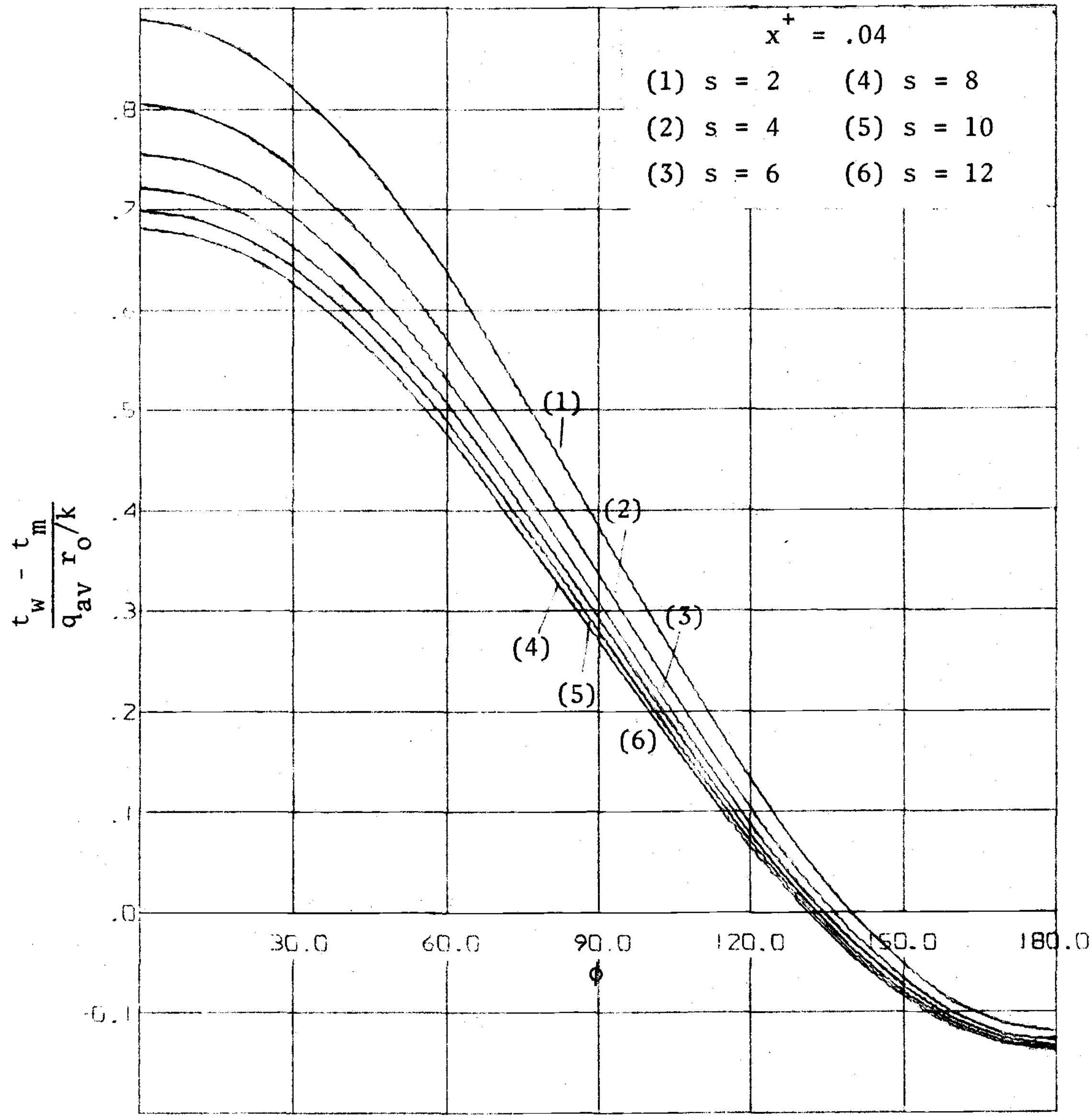

Figure 2.10. Illustration of entrance effect of prescribed wall heat flux variation $q(\phi) = q_{av}(1 + \cos\phi)$ and non-Newtonian influence on wall-to bulk temperature difference at the location x+ = .04.

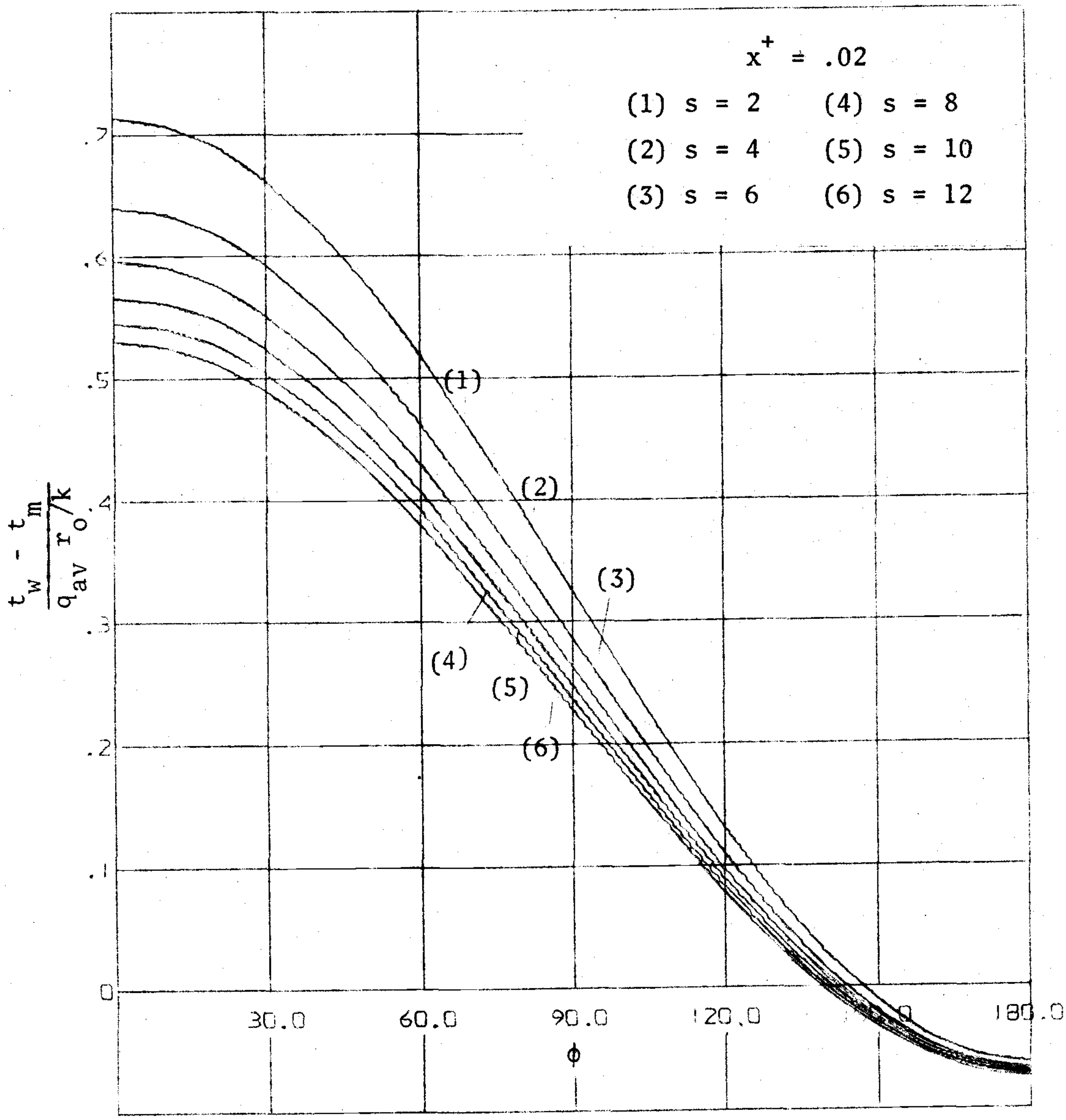

Figure 2.11. Illustration of entrance effect of prescribed wall heat flux variation $q(\phi) = q_{av}(1 + \cos\phi)$ and non-Newtonian influence on wall-to bulk temperature difference at the location x+ = .02.

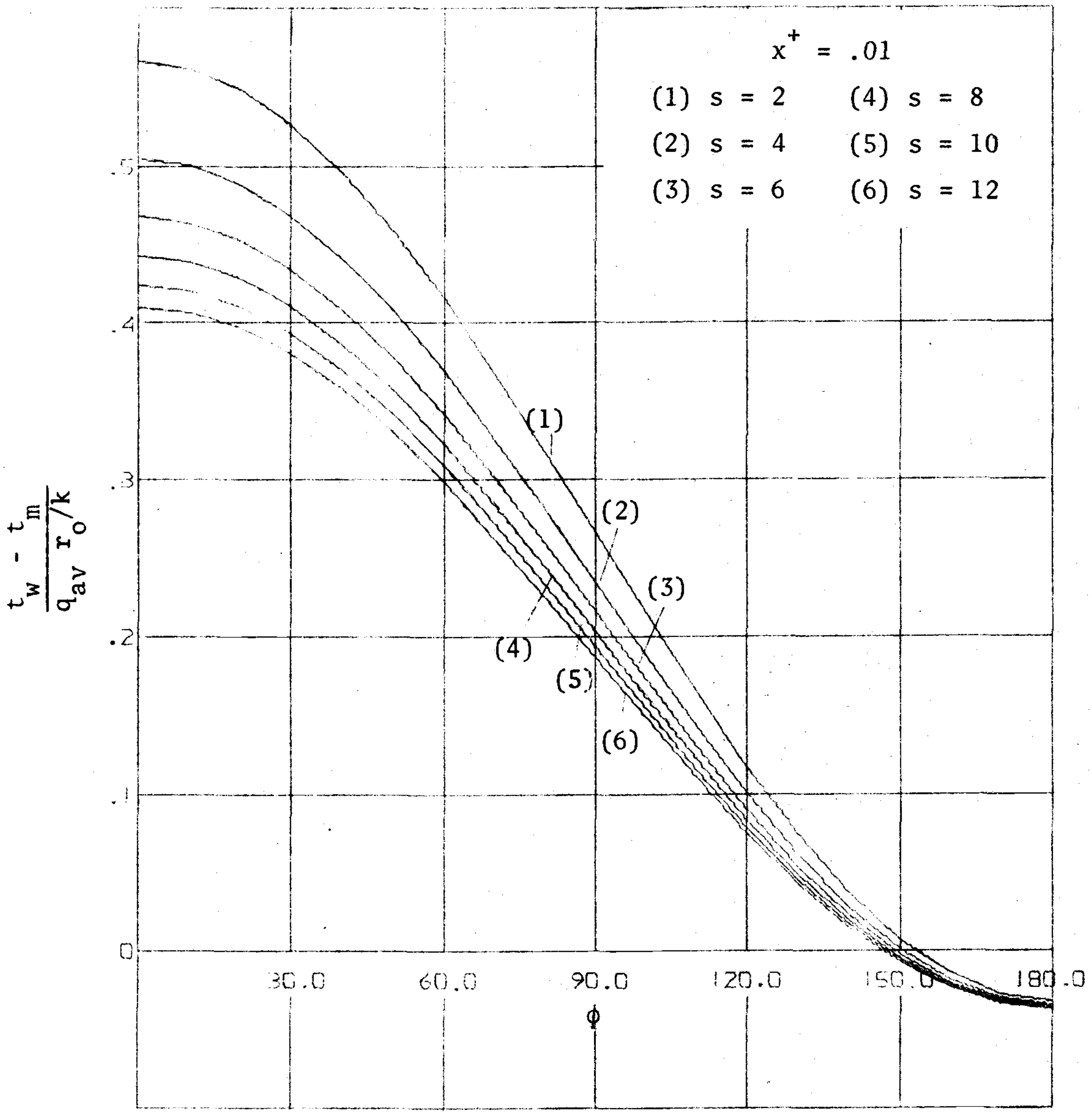

Figure 2. 12. Illustration of entrance effect of prescribed wall heat flux variation $q(\phi) = q_{av}(1 + \cos\phi)$ and non-Newtonian influence on wall-to bulk temperature difference at the location $x+ = .01$.

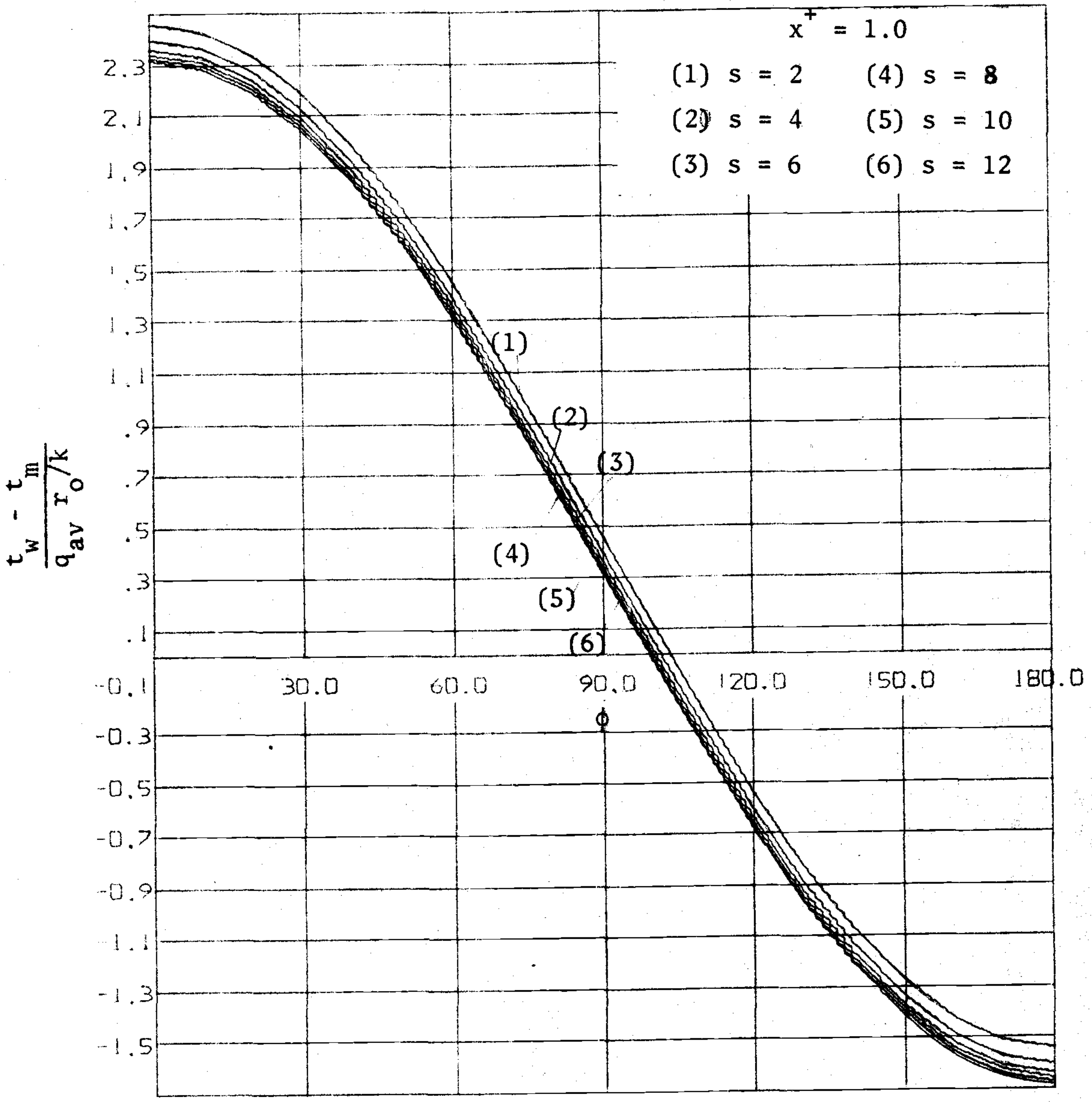

Figure 2. 13. Illustration of effect of prescribed wall heat flux variation $q(\phi) = q_{av}(1 + 2\cos\phi)$ and non-Newtonian influence on wall-to bulk temperature difference at the location far away from the entrance ($x+ = 1.0$).

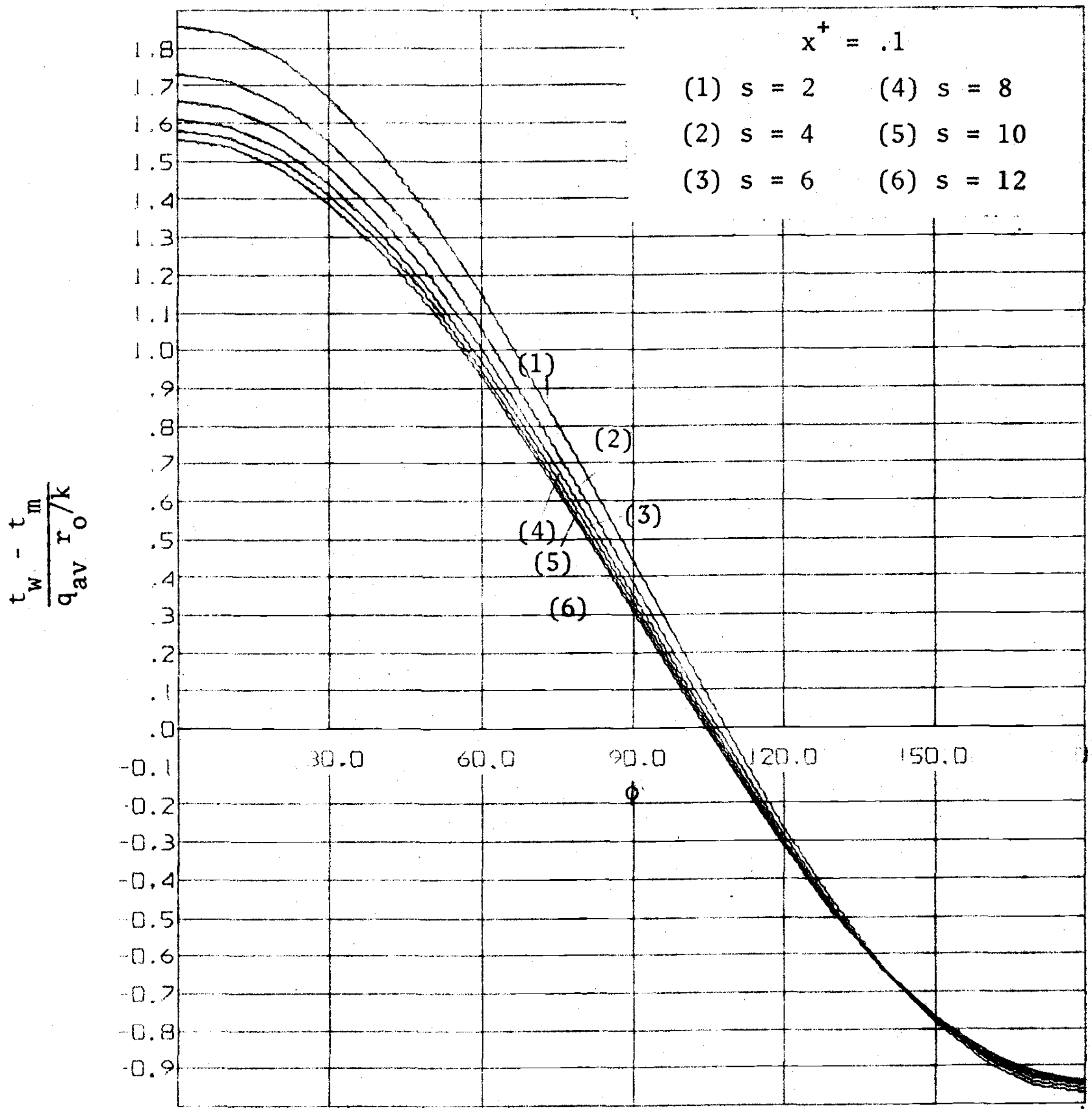

Figure 2.14. Illustration of entrance effect of prescribed wall heat flux variation $q(\phi) = q_{av}(1 + 2 \cos \phi)$ and non-Newtonian influence on wall-to bulk temperature difference at the location x+ = .1.

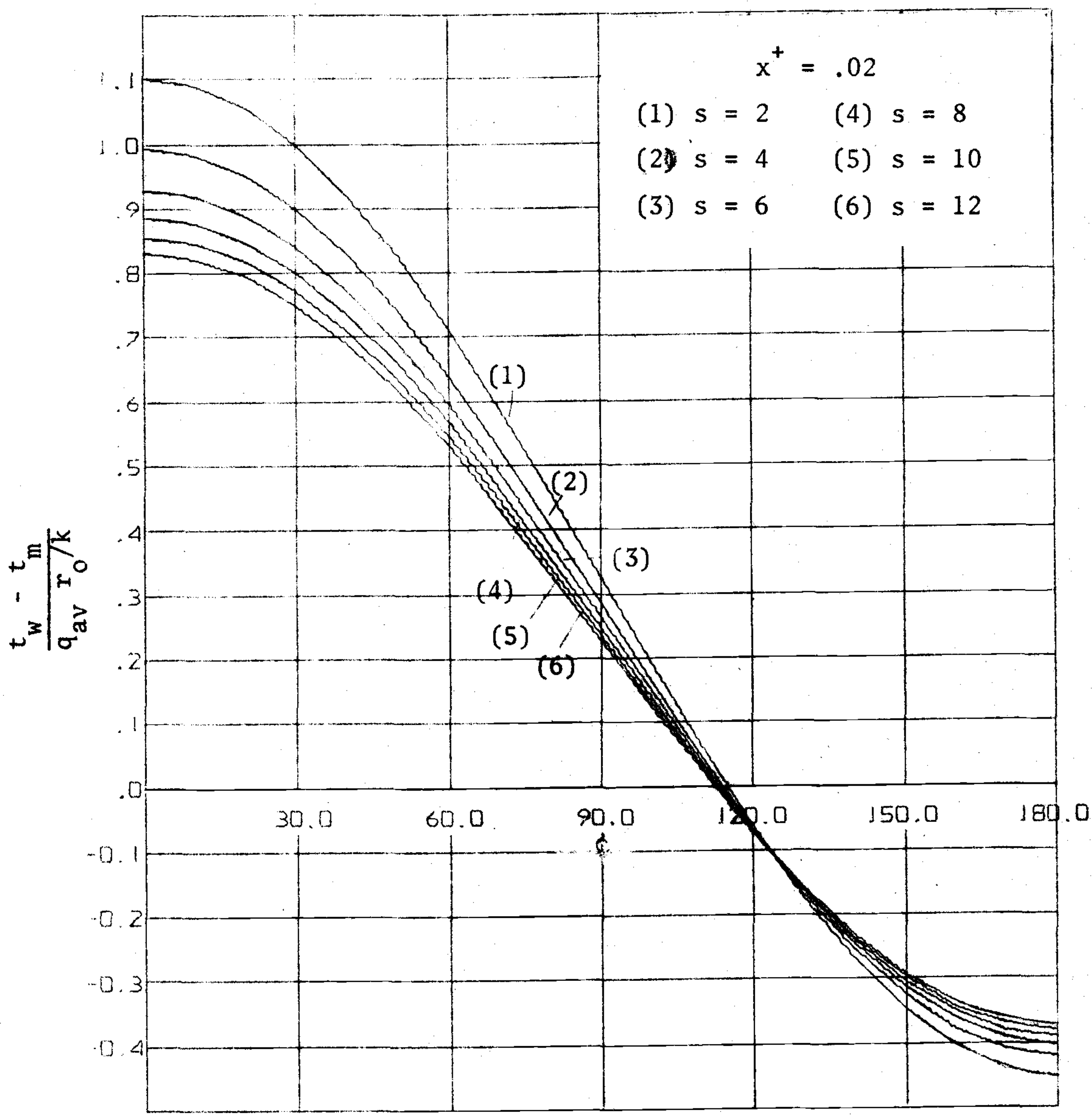

Figure 2. 15. Illustration of entrance effect of prescribed wall heat flux variation $q(\phi) = q_{av}(1 + 2 \cos \phi)$ and non-Newtonian influence on wall-to bulk temperature difference at the location x+ = .02.

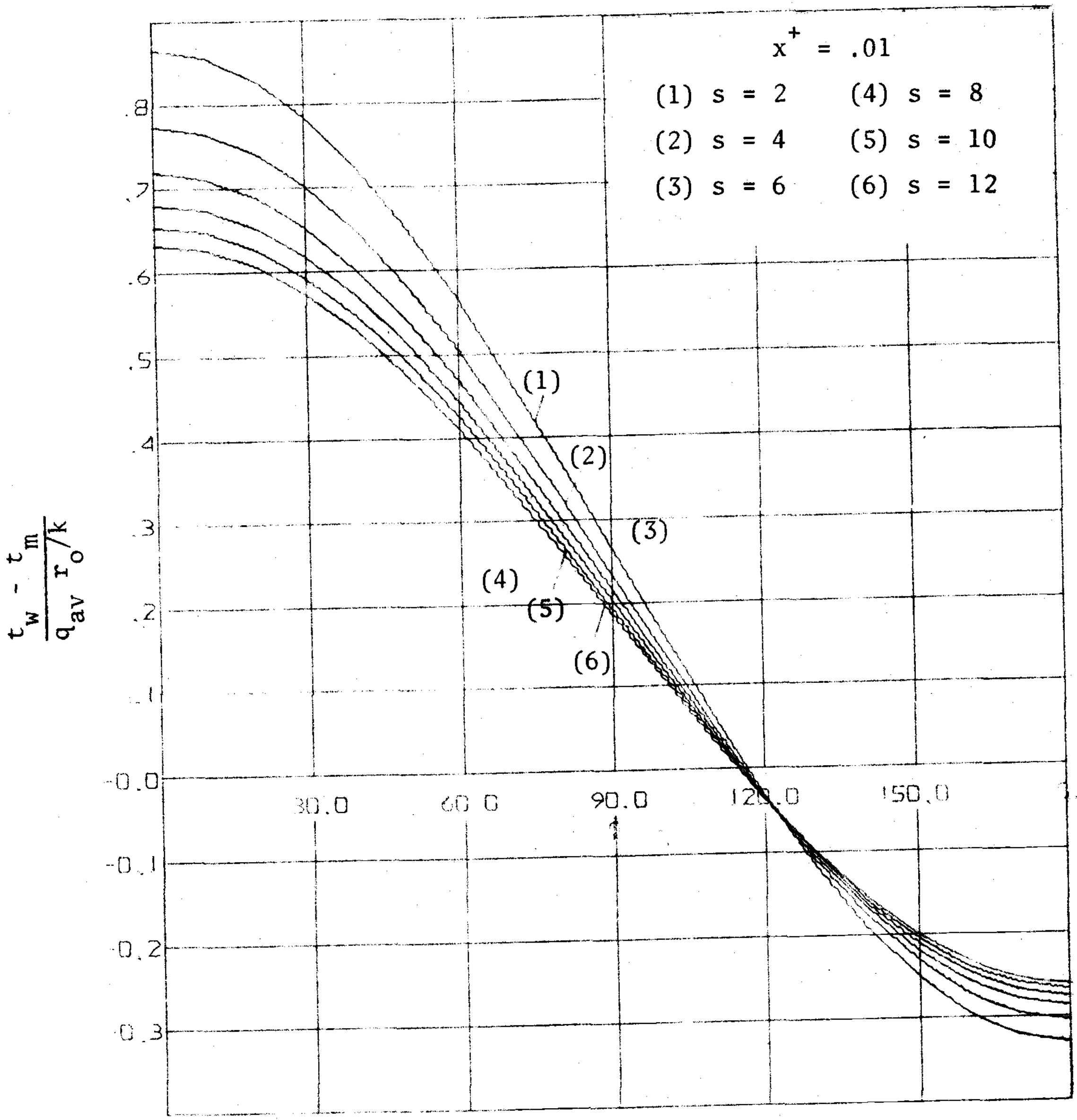

Figure 2. 16. Illustration of entrance effect of prescribed wall heat flux variation $q(\phi) = q_{av}(1 + 2 \cos \phi)$ and non-Newtonian influence on wall-to bulk temperature difference at the location x+ = .01.

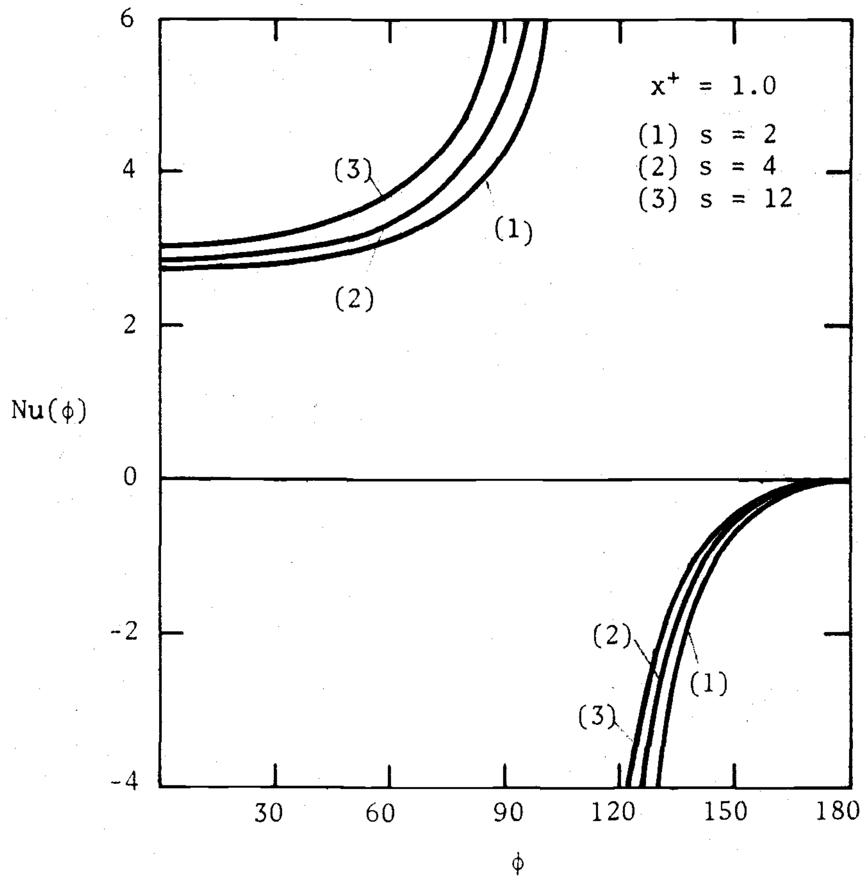

Figure 2.17. Local Nusselt number variation for prescribed wall heat flux $q(\phi) = q_{av}(1 + \cos \phi)$ and for different non-Newtonian behavior index, s, at the location far away from the entrance ($x+ = 1.0$).

values of the non-Newtonian behavior parameter, s, at a location far away from the entrance (i.e., x+ = 1). The corresponding plots for the case of the thermal entrance region (i.e., x+ = .1, .04, .02, .01) are presented in Figures 2.18, 2.19, 2.20, and 2.21. It is found that the local Nusselt values vary over a wide range around the circumference of a tube in the case of a cosine heat flux variation. Furthermore, non-Newtonian behavior has a pronounced effect on local Nusselt numbers. This effect becomes more significant in the entrance region. Also note that the Nusselt number is infinite at the point where the wall temperature is equal to the fluid mean temperature and becomes negative when the wall temperature is less than the bulk temperature.

Finally, dimensionless wall temperatures and Nusselt numbers are plotted as a function of dimensionless axial position for various values of the non-Newtonian behavior index, s, at the location of maximum wall heat flux ($\phi = 0$) in Figures 2.22 and 2.23. In comparing Figures 2.22 and 2.6, it is noted that the fully-developed Nusselt number at the location $\phi = 0$ is less than for the case where the heat flux is uniform around the circumference.

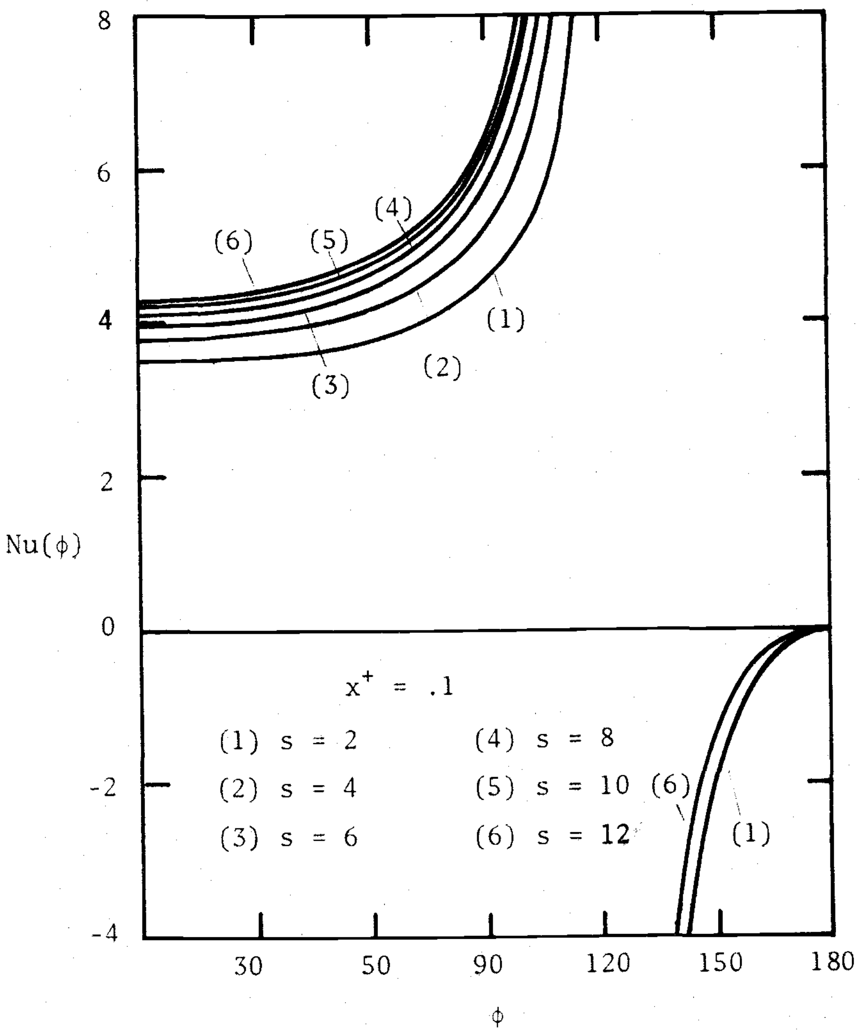

Figure 2.18. Local Nusselt number variation for prescribed wall heat flux $q(\phi) = q_{av}(1 + \cos\phi)$ and for different non-Newtonian behavior index, s, at the location x+ = .1.

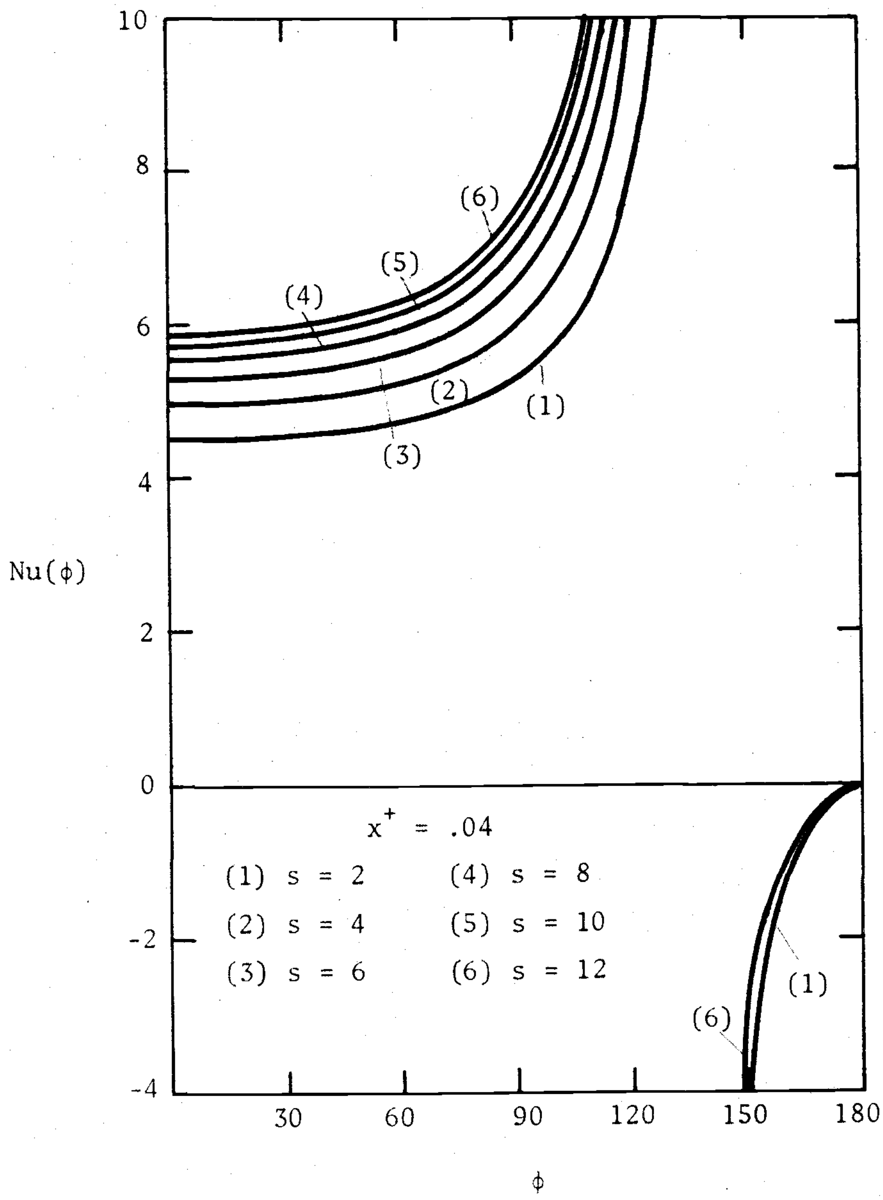

Figure 2.19. Local Nusselt number variation for prescribed wall heat flux of $q(\phi) = q_{av}(1 + \cos \phi)$ and for different non-Newtonian behavior index, s, at the location x+ = .04.

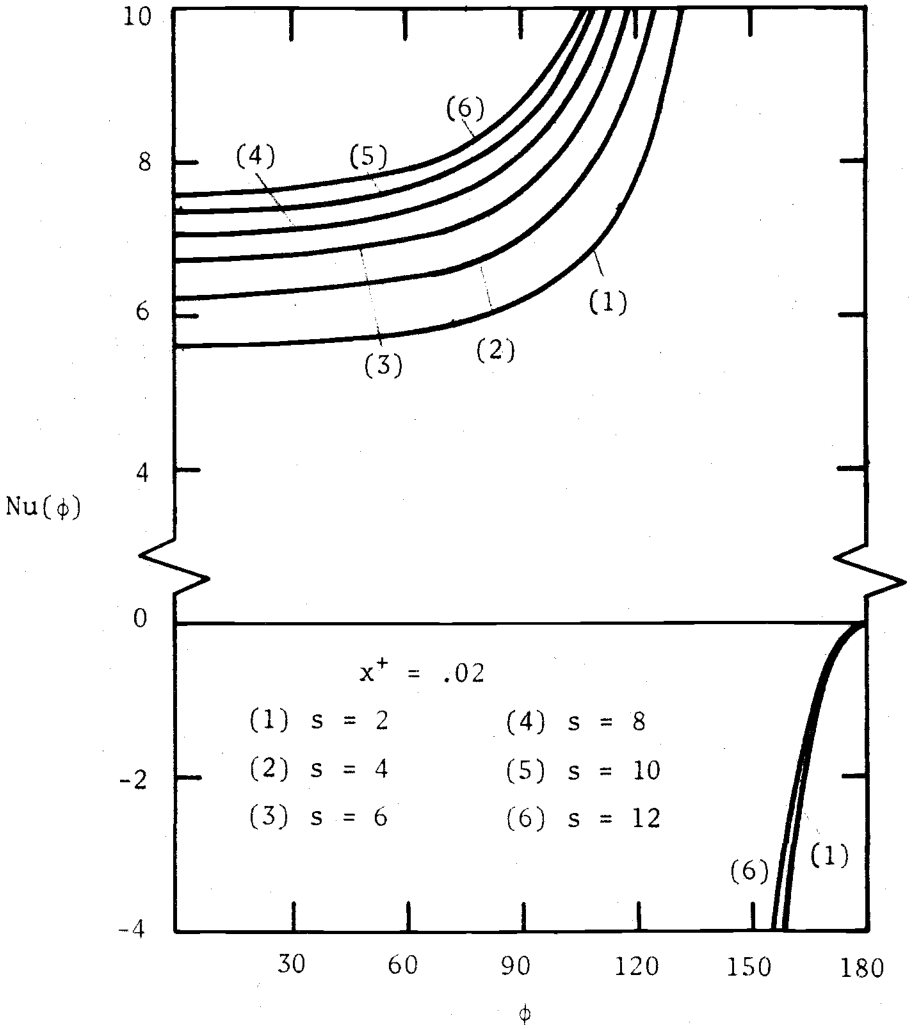

Figure 2.20. Local Nusselt number variation for prescribed wall heat flux $q(\phi) = q_{av}(1 + \cos\phi)$ and for different non-Newtonian behavior index, s, at the location x+ = .02.

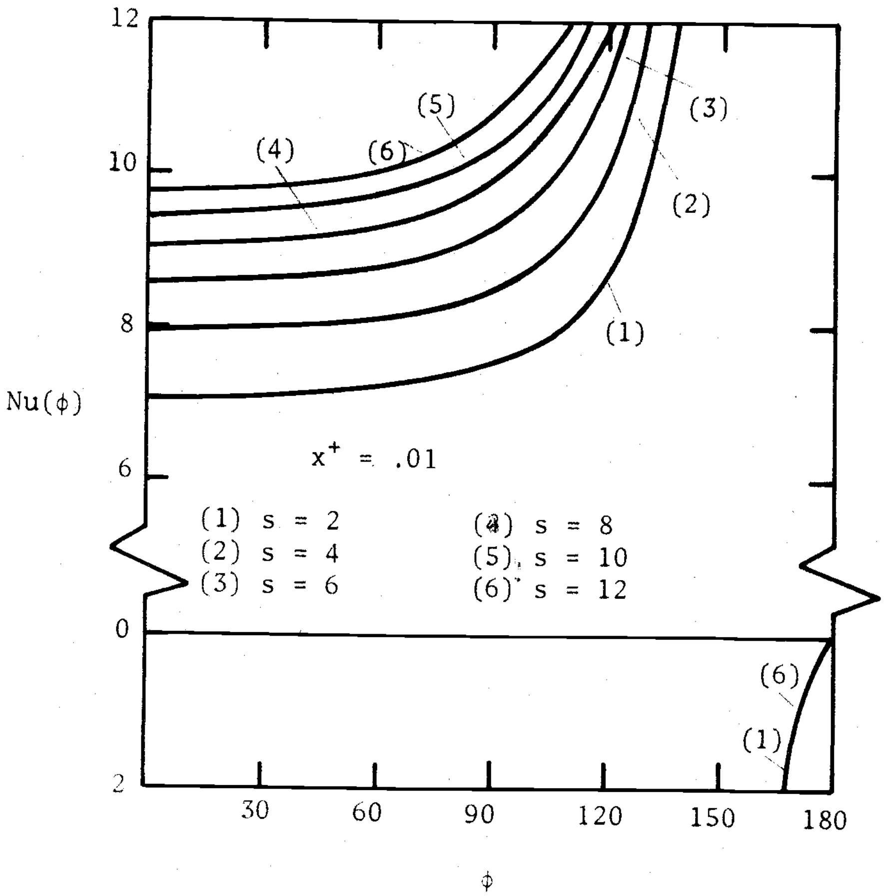

Figure 2.21. Local Nusselt number variation for prescribed wall heat flux $q(\phi) = q_{av}(1 + \cos\phi)$ and for different non-Newtonian behavior index, s, at the location x+ = .01.

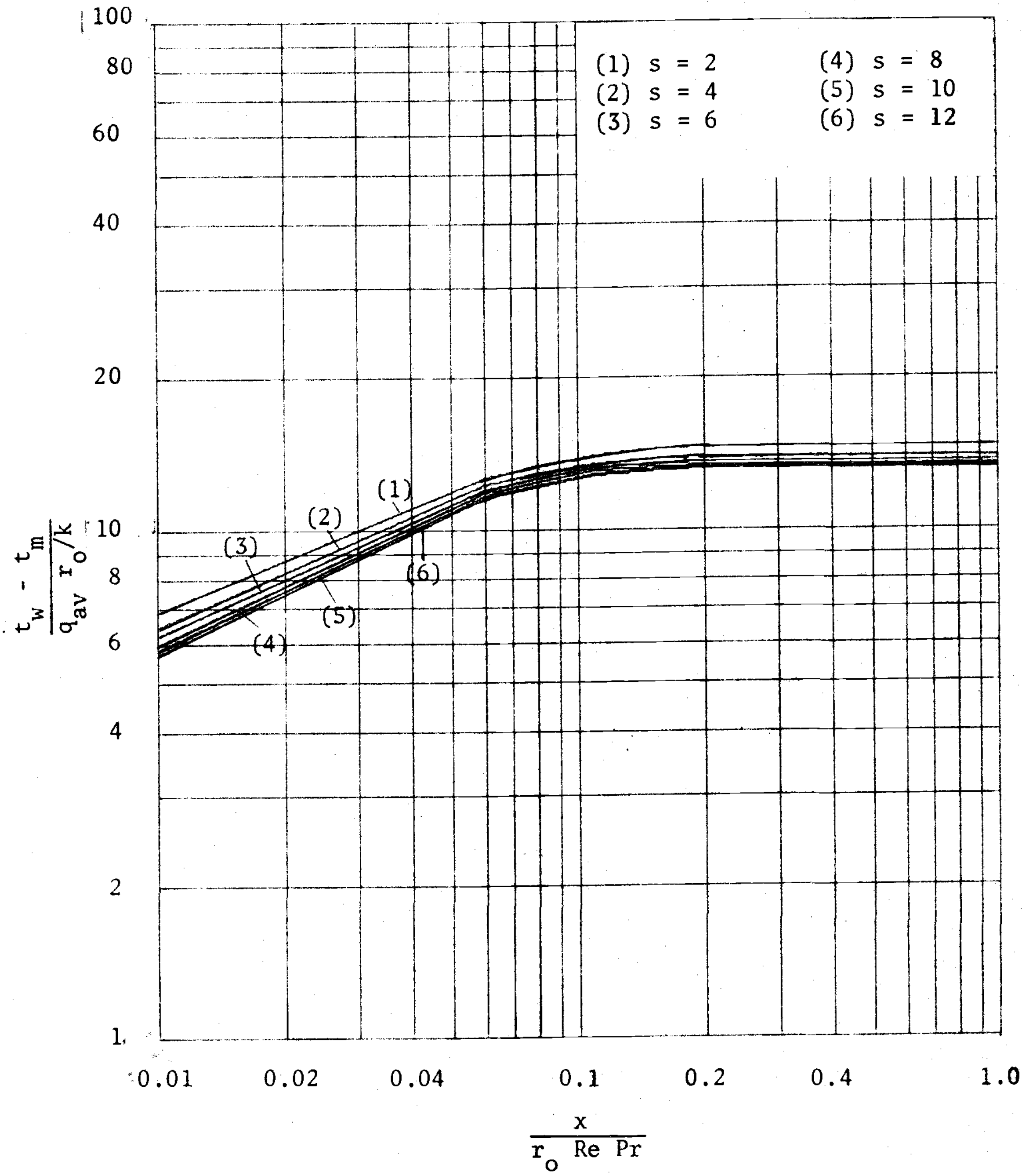

Figure 2.22. Entrance-region local wall-to bulk temperature difference for prescribed wall heat flux variation $q(\phi) = q_{av}(1 + \cos \phi)$ and for different non-Newtonian fluid behavior index, **s**, at the angular position $\phi = 0$ (i.e., maximum wall heat flux).

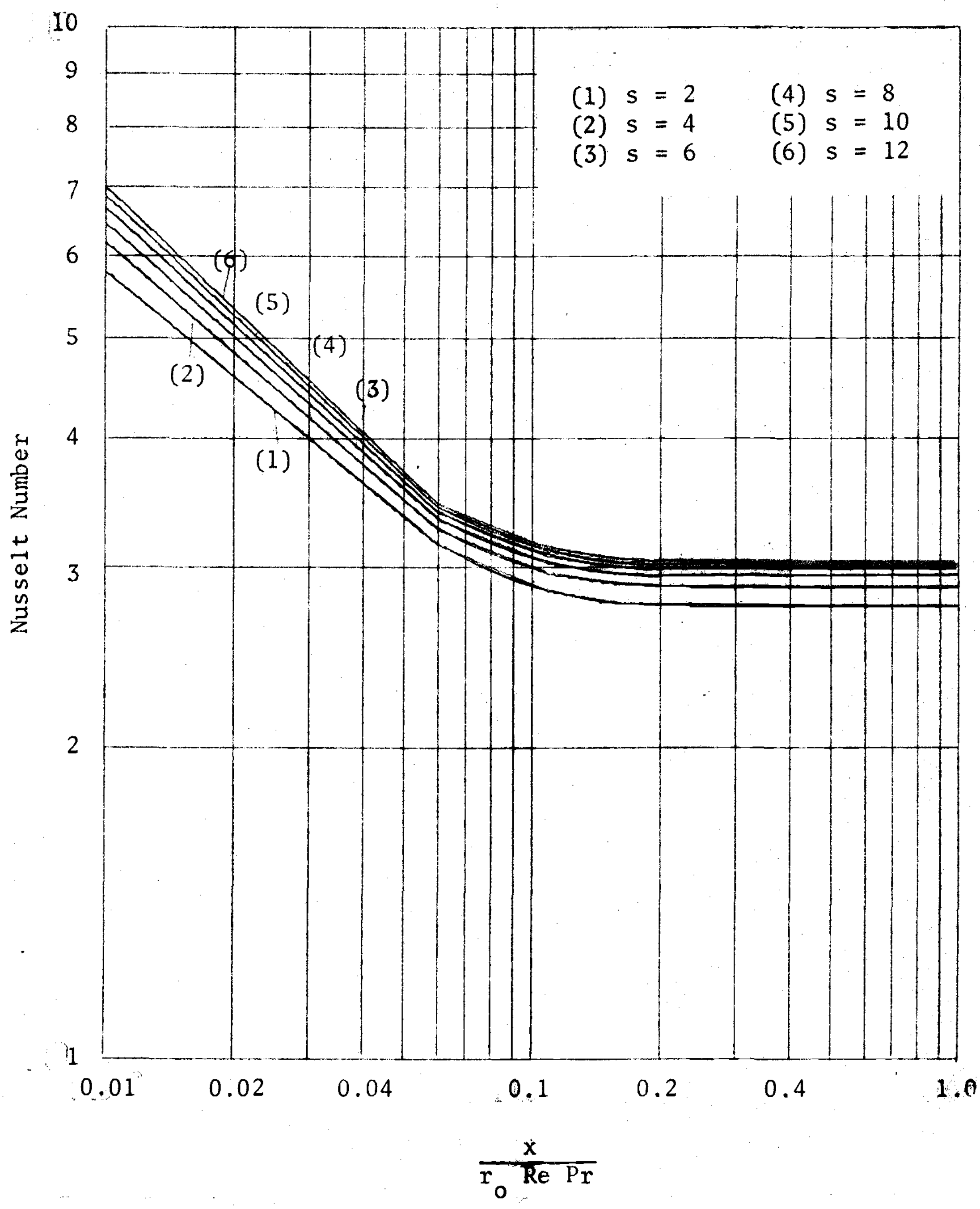

Figure 2.23. Entrance-region local Nusselt numbers for prescribed wall heat flux variation $q(\phi) = q_{av}(1 + \cos\phi)$ and for different non-Newtonian fluid behavior index, s, at the angular position $\phi = 0$ (maximum wall heat flux).

3. CONCLUSIONS AND RECOMMENDATIONS

The work contained in this manuscript is concerned with the analysis of heat transfer in a tube with forced flow under conditions of an arbitrary variation of wall heat flux both axially and circumferentially for cases of Newtonian and non-Newtonian fluids. The following significant results have been achieved:

(1) Analytical results were obtained in such generality and completeness that many of the previously reported work in the heat transfer literature in laminar tube flow are limiting cases of the present work.

(2) An effective new method (i.e., least squares) was presented for obtaining the coefficients of the non-orthogonal power series expansions which arise in the analysis of heat transfer problems when axial fluid conduction is present.

(3) Two considerations were made to correct the errors made in the heat transfer literature for the limiting problem of uniform wall heat flux with the inclusion of axial fluid conduction. The first was the determination of coefficients of the non-orthogonal power series expansion and second, the inclusion of the non-vanishing axial conduction term at the tube entrance which was not included heretofore. Both of these considerations have been included in this work.

(4) The first 12 eigenvalues, eigenfunctions, and expansion coefficients were obtained numerically for any arbitrary variation of circumferential wall heat flux that can be expressed in terms of a Fourier expansion for both Newtonian and non-Newtonian fluids.

(5) By an illustrative example, it was concluded that the circumferential wall heat flux variation has a pronounced effect in both Newtonian and non-Newtonian heat transfer results.

An interesting extension of the work reported here would be to consider the problem presented in Chapter 1 (i.e., laminar flow with an arbitrary variation of wall heat flux both axially and circumferentially with the allowance made for the inclusion of axial heat conduction, viscous heat dissipation, and heat generation) with the following additional features:

(1) include the effect of temperature on fluid properties, i.e., viscosity and density,

(2) include the effect of natural convection, and

(3) solve the coupled energy and momentum equations for the simultaneous development of thermal and velocity profiles.

4. NOMENCLATURE

Variables which are not listed in this nomenclature are defined at the appropriate location within the manuscript. Dimensions are given in mass-length-time-heat-temperature system (M-L-T-Q-t).

a_n	Fourier coefficients
a_{n0}, a_{np}	Expansion coefficients
$\hat{a}_{n0}, \hat{a}_{np}$	Defined as $\hat{a}_{n0} = 4a_{n0}$; $\hat{a}_{np} = \frac{4p}{a_{np}}$
$A_0(r+), A_p(r+)$	Defined by the expansion in Equations (1.63b, c) and (2.46a, b)
b	Heat flux parameter for the special example
b_n	Fourier coefficients
$b_{i;np}$	Coefficients of the power series
b_{np}	Expansion coefficients
$B_p(r+)$	Defined by the expansion in Equations (1.63d) and (1.48c)
C	Coefficient of Equation (1.42)
C1, C2	Constants of integration
c_p	Specific heat at constant pressure, Q/Mt
c_v	Specific heat at constant volume, Q/Mt
D	Coefficient of Equation (1.42)
E	Total error between the function and its power series expansion

$f(\phi)$	Specified angular variation for variable circumferential wall heat flux
$f+(r+, \phi)$	Function of r+ and ϕ satisfying Equations (1.26) and (2.24)
$\hat{f}(r+)$	Function of r+ satisfying Equation (1.10)
$F(r+, \phi)$	Function of r+ and ϕ satisfying the Laplace equation
g	Gravitational constant, L/T^2
g_c	Newton constant relating force and mass, 32.174 lbm ft/lbf sec^2
h	Heat transfer coefficient
J	Mechanical-to-thermal energy conversion factor, 777.66 (ft-lbf)/Btu
k	Thermal conductivity, Q/tLT
K	Defined by Equation (1.7a)
m	Constant in power-law constitutive equation
n	Exponent in the power-law constitutive equation
n	Separation constant in Equation (1.41)
p	Integer parameter in Equations (1.52) and (2.38)
P	Static pressure, M/LT^2
$q(\phi)$	Arbitrary variation of circumferential wall heat flux, Q

$$\frac{1}{2\pi}\int_0^{2\pi}\hat{\Theta}_{fd}d\phi$$

$g(r+)$	Defined as	$\frac{1}{\pi}\int_0^{2\pi}\hat{\Theta}\cos p\phi\, d\phi$
		$\frac{1}{\pi}\int_0^{2\pi}\hat{\Theta}\sin p\phi\, d\phi$
q	Local heat flux, Q/TL^2	
q	Heat flux vector	
$\overline{q}$	Defined by $\overline{q} = \int_0^{2\pi} q(\phi)d\phi$	
Q	Heat generation rate per unit volume, Q/L^2T	
$\hat{Q}(x+)$	Axial variation of wall heat flux	
$R_{np}(r+)$	Eigenfunctions of the characteristic Equations (1.52) or (2.38) for the specified integer parameter p	
r	Radical co-ordinate from the center of the pipe, L	
r_0	Pipe radius, L	
s	Exponent in the power-law constitutive	
$t(x, r, \phi)$	Local fluid temperature, t	
$u(r)$	Local fluid velocity, L/T	
$\vec{u}$	Velocity vector, L/T	
v	Average fluid velocity, L/T	
$W(r+)$	Function of r+ satisfying Equation (132)	
x	Axial co-ordinate from the inlet point, L	

Greek Symbols

α	Thermal diffusivity constant, L^2/T
ζ	Dummy integration variable, L
θ	Local dimensionless fluid temperature
$\theta+$	Dimensionless entrance region temperature
θ_{fd}	Asymptotic dimensionless fluid temperature
$\hat{\theta}$	Defined by Equation (2.45)
Θ	Local dimensionless fluid temperature when heat source and dissipation terms are neglected
$\Theta+$	Dimensionless entrance region temperature when heat source and dissipation terms are neglected
Θ_{fd}	Asymptotic dimensionless fluid temperature when heat source and dissipation terms are neglected
λ	Separation constant defined by Equation (1.34)
λ_{np}	Permissible values of the characteristic Equations (1.52) or (2.38)
μ	Dynamic viscosity coefficient, M/LT
ν	Kinematic viscosity, μ/ρ, L^2/T
π	3.14159
ρ	Fluid density, M/L^3
τ_{rx}	Shear stress in the x-direction
$\bar{\bar{\tau}}$	Stress tensor
ϕ	Angular coordinate, degs

$\Phi(\phi)$	Function of ϕ satisfying Equation (1.35) of the first problem
ω	Weighting function

Standard Dimensionless Parameters

Gz	Graetz number, $\frac{mc_p}{kx} = \frac{\pi}{2}\frac{1}{x+}$
Nu(x+, ϕ)	Local Nusselt number, $\frac{h2r_0}{k}$
Pe	Peclet number, Re Pr
Pr	Prandtl number, $\frac{\mu c_p}{k}$
Re	Reynolds number, $\frac{\rho v 2 r_0}{\mu}$

Dimensionless Parameter Defined in this Manuscript

K	Heat dissipation term, $16\pi(\frac{v^2}{2g_c J c_p})(\frac{k}{\overline{q} r_0})\,Pr$
Q'	Heat source term, $\frac{Qr_0}{\overline{q}}\frac{\pi}{2}$
r+	Radial co-ordinate, $\frac{r}{r_0}$
u+	Fluid velocity, $\frac{u}{v}$
x+	Axial distance, for Newtonian problem: $\frac{x/r_0}{RePr}$, for non-Newtonian problem: $\frac{2s}{s+2}\frac{x/r_0}{RePr}$ or $\frac{2v}{u_{max}}\frac{x/r_0}{Re\,Pr}$
θ	Non-dimensional fluid temperature, $\frac{t-t_\epsilon}{\overline{q}2r_0/k\pi}$

Subscripts

av	Refers to average value
axial	Refers to axial direction

c	Refers to conversion factor
fd	Evaluated far away from the entrance
m	Evaluated at the mixed mean state
max	Refers to maximum value
p	Refers to pressure
t	Refers to solution for constant surface temperature
v	Refers to volume
w	Evaluated at wall condition
x	Refers to x (axial) direction
ϵ	Evaluated at the tube entrance
∞	Evaluated far away from the entrance

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Table A. 1. Tabulation of the first 12 eigenfunctions for p = 0, 1, 2 and different Peclet numbers.

r^+	R_{10}	R_{20}	R_{30}	R_{40}	R_{50}	R_{60}	R_{70}	R_{80}	R_{90}	R_{100}	R_{110}	R_{120}
(1) Pe = 5 , p = 0												
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.05	.9878	.9634	.9273	.8802	.8230	.7568	.6827	.6022	.5165	.4274	.3364	.2450
.10	.9515	.8577	.7253	.5639	.3854	.2023	.0273	-.1281	-.2542	-.3438	-.3930	-.4010
.15	.8928	.6949	.4376	.1625	-.0883	-.2784	-.3841	-.3974	-.3270	-.1957	-.0356	.1186
.20	.8140	.4927	.1252	-.1886	-.3717	-.3921	-.2693	-.0651	.1398	.2730	.2942	.2056
.25	.7192	.2728	-.1490	-.3821	-.3627	-.1499	.1140	.2815	.2737	.1138	-.0953	-.2331
.30	.6093	.0579	-.3348	-.3781	-.1283	.1766	.3009	.1756	-.0729	-.2389	-.2011	-.0072
.35	.4912	-.1314	-.4054	-.2132	.1511	.3003	.1213	-.1597	-.2436	-.0655	.1647	.2028
.40	.3582	-.2785	-.3619	.0210	.2977	.1469	-.1714	-.2260	.0216	.2156	.1067	-.1346
.45	.2445	-.3729	-.2305	.2189	.2391	-.1163	-.2364	.0342	.2194	.0333	-.1894	-.0865
.50	.1240	-.4106	-.0544	.3033	.0360	-.2514	-.0277	.2193	.0230	-.1971	-.0199	.1804
.55	.0101	-.3944	.1188	.2525	-.1705	-.1570	.1927	.0785	-.1945	-.0115	.1799	-.0441
.60	-.0942	-.3322	.2477	.1013	-.2537	.0638	.1787	-.1606	-.0634	.1816	-.0505	-.1360
.65	-.1869	-.2362	.3066	-.0793	-.1750	.2133	-.0330	-.1572	.1649	-.0021	-.1477	.1309
.70	-.2664	-.1207	.2886	-.2150	.0039	.1727	-.1932	.0639	.0996	-.1689	.0998	.0417
.75	-.3318	.0001	.2055	-.2569	.1685	-.0085	-.1318	.1827	-.1281	.0088	.1041	-.1493
.80	-.3832	.1132	.0817	-.1972	.2242	-.1716	.0682	.0454	-.1296	.1587	-.1280	.0536
.85	-.4209	.2083	-.0524	-.0670	.1481	-.1870	.1841	-.1454	.0822	-.0091	-.0589	.1089
.90	-.4460	.2785	-.1681	.0804	-.0069	-.0536	.1004	-.1324	.1491	-.1508	.1387	-.1152
.95	-.4598	.3205	-.2442	.1918	-.1503	.1147	-.0829	.0540	-.0275	.0032	.0187	-.0384
1.00	-.4645	.3340	-.2701	.2321	-.2065	.1878	-.1734	.1618	-.1523	.1443	-.1374	.1314
(2) Pe = 10 , p = 0												
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.05	.9861	.9591	.9204	.8706	.8109	.7423	.6662	.5840	.4971	.4070	.3155	.2240
.10	.9451	.8417	.7007	.5327	.3500	.1658	-.0073	-.1581	-.2773	-.3584	-.3984	-.3974
.15	.8780	.6626	.3933	.1147	-.1303	-.3070	-.3946	-.3890	-.3028	-.1616	.0007	.1494
.20	.7906	.4437	.0690	-.2323	-.3886	-.3782	-.2320	-.0196	.1764	.2880	.2834	.1746
.25	.6843	.2111	-.2017	-.3975	-.3352	-.0981	.1593	.2953	.2506	.0692	-.1347	-.2455
.30	.5646	-.0091	-.3670	-.3525	-.0697	.2193	.2952	.1296	-.1197	-.2493	-.1692	.0390
.35	.4365	-.1940	-.4048	-.1546	.2014	.2902	.0662	-.1983	-.2263	-.0144	.1933	.1800
.40	.3053	-.3273	-.3261	.0869	.3034	.0899	-.2100	-.1962	.0749	.2195	.0591	-.1666
.45	.1757	-.4001	-.1680	.2615	.1943	-.1681	-.2103	.0894	.2116	-.0204	-.1980	-.0388
.50	.0520	-.4120	.0184	.3031	-.0297	-.2501	.0323	.2175	-.0325	-.1950	.0318	.1782
.55	-.0621	-.3697	.1824	.2101	-.2139	-.1049	.2159	.0235	-.1987	.0408	.1676	-.0893
.60	-.1639	-.2852	.2865	.0367	-.2486	.1189	.1406	-.1879	-.0118	.1757	-.0949	-.1037
.65	-.2514	-.1738	.3130	-.1383	-.1284	.2248	-.0853	-.1215	.1814	-.0496	-.1210	.1516
.70	-.3238	-.0510	.2647	-.2461	.0599	.1368	-.2026	.1073	.0588	-.1623	.1301	.0009
.75	-.3811	.0688	.1609	-.2531	.2013	-.0563	-.0961	.1771	-.1531	.0482	.0733	-.1436
.80	-.4239	.1744	.0299	-.1676	.2219	-.1936	.1044	.0088	-.1055	.1546	-.1439	.0823
.85	-.4535	.2581	-.0993	-.0288	.1224	-.1761	.1881	-.1621	.1071	-.0366	-.0344	.0922
.90	-.4719	.3165	-.2034	.1119	-.0346	-.0305	.0825	-.1204	.1433	-.1512	.1448	-.1259
.95	-.4812	.3495	-.2681	.2113	-.1670	.1297	-.0967	.0667	-.0393	.0141	.0088	-.0295
1.00	-.4837	.3595	-.2891	.2457	-.2166	.1956	-.1796	.1670	-.1566	.1480	-.1406	.1343

Table A.1. Continued

(3) Pe = 20 , p = 0

r^*	R_{10}	R_{20}	R_{30}	R_{40}	R_{50}	R_{60}	R_{70}	R_{80}	R_{90}	R_{100}	R_{110}	R_{120}
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.05	.9848	.9541	.9109	.8565	.7921	.7192	.6392	.5536	.4640	.3721	.2793	.1874
.10	.9401	.8231	.6677	.4877	.2971	.1100	-.0608	-.2042	-.3119	-.3787	-.4029	-.3862
.15	.8682	.6254	.3351	.0488	-.1886	-.3445	-.4031	-.3673	-.2563	-.1011	.0618	.1980
.20	.7727	.3883	-.0016	-.2865	-.4027	-.3458	-.1662	.0537	.2290	.3005	.2526	.1136
.25	.6585	.1431	-.2626	-.4058	-.2824	-.0158	.2215	.3009	.1979	-.0090	-.1920	-.2493
.30	.5308	-.0799	-.3954	-.3023	.0186	.2700	.2657	.0464	-.1861	-.2448	-.1000	.1138
.35	.3957	-.2562	-.3872	-.0656	.2606	.2518	-.0256	-.2408	-.1744	.0720	.2191	.1195
.40	.2597	-.3698	-.2639	.1720	.2863	-.0054	-.2479	-.1253	.1528	.1968	-.0291	-.1969
.45	.1254	-.4151	-.0775	.3000	.1096	-.2283	-.1431	.1663	.1681	-.1067	-.1810	.0489
.50	.0062	-.3961	.1115	.2755	-.1239	-.2169	.1229	.1819	-.1186	-.1585	.1136	.1417
.55	-.1131	-.3246	.2525	.1292	-.2541	-.0098	.2210	-.0701	-.1719	.1215	.1154	-.1495
.60	-.2120	-.2171	.3156	-.0607	-.2108	.1892	.0578	-.2021	.0765	.1336	-.1529	-.0273
.65	-.2948	-.0916	.2960	-.2111	-.0410	.2127	-.1581	-.0426	.1785	-.1222	-.0523	.1586
.70	-.3611	.0352	.2093	-.2686	.1412	.0608	-.1900	.1647	-.0206	-.1241	.1610	-.0714
.75	-.4116	.1497	.0835	-.2242	.2336	-.1293	-.0237	.1445	-.1759	.1108	.0075	-.1115
.80	-.4475	.2434	-.0509	-.1065	.1989	-.2133	.1571	-.0566	-.0510	.1297	-.1556	.1246
.85	-.4708	.3120	-.1687	.0377	.0699	-.1451	.1821	-.1803	.1443	-.0844	.0137	.0533
.90	-.4841	.3558	-.2544	.1640	-.0837	.0133	.0461	-.0932	.1265	-.1454	.1497	-.1405
.95	[illegible]	.3740	-.3028	.2442	-.1969	.1570	-.1221	.0906	-.0617	.0350	-.0105	-.0118
1.00	-.4912	.3840	-.3172	.2698	-.2359	.2110	-.1922	.1773	-.1654	.1555	-.1471	.1400

(4) Pe = 30 , p = 0

r^*	R_{10}	R_{20}	R_{30}	R_{40}	R_{50}	R_{60}	R_{70}	R_{80}	R_{90}	R_{100}	R_{110}	R_{120}
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.05	.9844	.9517	.9052	.8470	.7788	.7021	.6185	.5297	.4376	.3436	.2496	.1571
.10	.9386	.8142	.6482	.4584	.2608	.0705	-.0991	-.2370	-.3356	-.3907	-.4019	-.3724
.15	.8648	.6078	.3016	.0079	-.2252	-.3660	-.4028	-.3441	-.2150	-.0511	.1093	.2323
.20	.7671	.3625	-.0406	-.3163	-.4047	-.3152	-.1136	.1072	.2614	.2985	.2168	.0583
.25	.6503	.1121	-.2932	-.4029	-.2395	.0426	.2577	.2906	.1462	-.0706	-.2261	-.2354
.30	.5203	-.1112	-.4046	-.2626	.0772	.2929	.2287	-.0200	-.2243	-.2213	-.0348	.1650
.35	.3830	-.2819	-.3688	-.0071	.2879	.2089	-.0919	-.2530	-.1169	.1341	.2155	.0547
.40	.2444	-.3850	-.2215	.2185	.2575	-.0735	-.2539	-.0575	.1970	.1534	-.0980	-.1936
.45	.1100	-.4164	-.0233	.3093	.0435	-.2520	-.0774	.2063	.1108	-.1615	-.1382	.1147
.50	-.0155	-.3827	.1613	.2413	-.1798	-.1711	.1777	.1295	-.1705	-.1026	.1623	.0842
.55	-.1284	-.2979	.2832	.0675	-.2604	.0618	.1970	-.1344	-.1226	.1675	.0497	-.1715
.60	-.2262	-.1808	.3190	-.1218	-.1645	.2210	-.0144	-.1836	.1367	.0743	-.1731	.0444
.65	-.3073	-.0504	.2720	-.2460	.0266	.1797	-.1953	.0280	.1472	-.1624	.0167	.1333
.70	-.3716	.0763	.1651	-.2658	.1893	-.0049	-.1564	.1894	-.0845	-.0709	.1598	-.1220
.75	-.4199	.1867	.0302	-.1888	.2392	-.1749	.0383	.0994	-.1718	.1494	-.0517	-.0648
.80	-.4536	.2733	-.1022	-.0548	.1668	-.2121	.1870	-.1070	.0020	.0926	-.1471	.1468
.85	-.4749	.3340	-.2103	.0879	.0228	-.1099	.1648	-.1833	.1664	-.1205	.0564	.0128
.90	-.4865	.3704	-.2835	.2014	-.1234	.0519	.0114	-.0644	.1055	-.1333	.1471	-.1471
.95	-.4913	.3874	-.3217	.2676	-.2210	.1805	-.1446	.1121	-.0822	.0546	-.0290	.0054
1.00	-.4922	.3915	-.3320	.2873	-.2524	.2253	-.2043	.1876	-.1741	.1630	-.1537	.1458

Table A.1. Continued

(5) Pe = 50 , p = 0

r^+	R_{10}	R_{20}	R_{30}	R_{40}	R_{50}	R_{60}	R_{70}	R_{80}	R_{90}	R_{100}	R_{110}	R_{120}
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.05	.9842	.9498	.8996	.8364	.7623	.6796	.5901	.4960	.3991	.3014	.2047	.1108
.10	.9376	.8070	.6292	.4258	.2172	.0212	-.1477	-.2783	-.3636	-.4011	-.3927	-.3437
.15	.8627	.5938	.2695	-.0358	-.2652	-.3866	-.3937	-.3036	-.1507	.0224	.1742	.2726
.20	.7636	.3422	-.0767	-.3443	-.3991	-.2684	-.0411	.1743	.2918	.2762	.1485	-.0289
.25	.6454	.0881	-.3193	-.3926	-.1821	.1133	.2903	.2563	.0621	-.1518	-.2508	-.1856
.30	.5138	-.1348	-.4086	-.2134	.1428	.3041	.1628	-.1078	-.2515	-.1581	.0654	.2134
.35	.3752	-.3006	-.3460	.0562	.3046	.1393	-.1693	-.2380	-.0196	.1997	.1687	-.0530
.40	.2356	-.3948	-.1773	.2603	.2063	-.1521	-.2306	.0432	.2231	.0596	-.1754	-.1386
.45	.1006	-.4150	.0288	.3045	-.0388	-.2540	.0205	.2242	.0072	-.2011	-.0382	.1779
.50	-.0251	-.3696	.2048	.1903	-.2310	-.0913	.2215	.0318	-.2034	.0055	.1845	-.0293
.55	-.1376	-.2748	.3046	-.0053	-.2425	.1455	.1301	-.1944	-.0215	.1855	-.0633	-.1430
.60	-.2346	-.1508	.3121	-.1825	-.0902	.2304	-.1098	-.1163	.1870	-.0348	-.1450	.1360
.65	-.3148	-.0175	.2392	-.2689	.1082	.1094	-.2098	.1218	.0628	-.1733	.1151	.0459
.70	-.3778	.1082	.1153	-.2439	.2304	-.0929	-.0809	.1840	-.1578	.0295	.1074	-.1585
.75	-.4247	.2144	-.0249	-.1338	.2222	-.2146	.1212	.0128	-.1267	.1709	-.1299	.0291
.80	-.4570	.2950	-.1521	.0122	.1079	-.1853	.2048	-.1662	.0844	.0141	-.0992	.1462
.85	-.4771	.3489	-.2483	.1472	-.0458	-.0465	.1194	-.1648	.1786	-.1614	.1181	-.0579
.90	-.4877	.3793	-.3080	.2427	-.1766	.1105	-.0472	-.0106	.0603	-.1000	.1280	-.1435
.95	-.4918	.3922	-.3358	.2919	-.2523	.2152	-.1805	.1480	-.1177	.0892	-.0625	.0374
1.00	-.4925	.3947	-.3422	.3045	-.2736	.2472	-.2250	.2063	-.1908	.1777	-.1667	.1573

(6) Pe = 100 , p = 0

r^+	R_{10}	R_{20}	R_{30}	R_{40}	R_{50}	R_{60}	R_{70}	R_{80}	R_{90}	R_{100}	R_{110}	R_{120}
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.05	.9841	.9497	.8958	.8276	.7467	.6557	.5576	.4548	.3499	.2453	.1432	.0458
.10	.9371	.8031	.6162	.3995	.1772	-.0280	-.1982	-.3205	-.3885	-.4017	-.3652	-.2884
.15	.8618	.5860	.2479	-.0696	-.2981	-.3996	-.3722	-.2444	-.0644	.1141	.2451	.3001
.20	.7620	.3309	-.1003	-.3631	-.3863	-.2125	.0395	.2390	.3011	.2148	.0369	-.1425
.25	.6430	.0749	-.3351	-.3791	-.1251	.1784	.3031	.1888	-.0491	-.2272	-.2283	-.0702
.30	.5108	-.1475	-.4086	-.1706	.1963	.2939	.0734	-.1933	-.2350	-.0391	.1770	.2030
.35	.3716	-.3104	-.3280	.1048	.3036	.0560	-.2304	-.1743	.1040	.2193	.0430	-.1731
.40	.2315	-.3995	-.1460	.2852	.1458	-.2148	-.1653	.1504	.1857	-.0821	-.1945	.0102
.45	.0963	-.4132	.0631	.2892	-.1116	-.2216	.1249	.1824	-.1225	-.1604	.1126	.1493
.50	-.0294	-.3614	.2309	.1411	-.2575	.0056	.2192	-.0941	-.1604	.1404	.1014	-.1592
.55	-.1419	-.2612	.3139	-.0640	-.2011	.2082	.0210	-.1983	.1142	.1060	-.1700	.0038
.60	-.2395	-.1337	.3018	-.2226	-.0096	.1978	-.1893	.0086	.1618	-.1564	-.0080	.1512
.65	-.3182	.0009	.2121	-.2776	.1748	.0123	-.1677	.1916	-.0762	-.0856	.1679	-.1103
.70	-.3606	.1256	.0786	-.2115	.2441	-.1723	.0324	.1081	-.1798	.1489	-.0376	-.0871
.75	-.4268	.2292	-.0630	-.0790	.1796	-.2199	.1915	-.1051	-.0090	.1100	-.1613	.1453
.80	-.4585	.3061	-.1846	.0705	.0348	-.1219	.1785	-.1952	.1696	-.1082	.0263	.0560
.85	-.4780	.3562	-.2714	.1940	-.1169	.0399	.0329	-.0956	.1418	-.1669	.1684	-.1469
.90	-.4882	.3832	-.3213	.2718	-.2252	.1776	-.1284	.0781	-.0287	-.0178	.0593	-.0938
.95	-.4919	.3937	-.3419	.3061	-.2771	.2508	-.2254	.2000	-.1745	.1491	-.1241	.0996
1.00	-.4925	.3954	-.3455	.3129	-.2884	.2682	-.2505	.2344	-.2196	.2061	-.1939	.1828

Table A.1. Continued

(1) Pe = 5 , p = 1

r^+	R_{11}	R_{21}	R_{31}	R_{41}	R_{51}	R_{61}	R_{71}	R_{81}	R_{91}	R_{101}	R_{111}	R_{121}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0499	.0494	.0487	.0476	.0463	.0447	.0428	.0408	.0385	.0361	.0335	.0309
.10	.0992	.0956	.0897	.0819	.0725	.0620	.0508	.0395	.0285	.0182	.0091	.0014
.15	.1472	.1353	.1168	.0936	.0683	.0432	.0206	.0023	-.0106	-.0179	-.0198	-.0175
.20	.1935	.1662	.1262	.0808	.0377	.0035	-.0180	-.0262	-.0234	-.0137	-.0020	.0076
.25	.2374	.1864	.1172	.0486	-.0031	-.0294	-.0313	-.0174	.0007	.0133	.0158	.0096
.30	.2786	.1949	.0922	.0078	-.0349	-.0356	-.0125	.0109	.0193	.0116	-.0024	-.0113
.35	.3166	.1918	.0562	-.0291	-.0444	-.0152	.0155	.0214	.0057	-.0108	-.0127	-.0019
.40	.3511	.1777	.0156	-.0518	-.0301	.0135	.0246	.0037	-.0149	-.0105	.0053	.0109
.45	.3819	.1542	-.0229	-.0552	-.0021	.0290	.0085	-.0165	-.0101	.0091	.0099	-.0042
.50	.4090	.1233	-.0534	-.0405	.0240	.0216	-.0143	-.0139	.0098	.0099	-.0072	-.0075
.55	.4322	.0874	-.0719	-.0143	.0354	-.0009	-.0209	.0059	.0127	-.0076	-.0073	.0078
.60	.4516	.0491	-.0767	.0142	.0281	-.0208	-.0065	.0167	-.0042	-.0096	.0083	.0028
.65	.4674	.0107	-.0684	.0360	.0073	-.0242	.0131	.0058	-.0134	.0062	.0050	-.0087
.70	.4798	-.0254	-.0496	.0452	-.0156	-.0101	.0187	-.0115	-.0014	.0096	-.0087	.0017
.75	.4891	-.0576	-.0243	.0401	-.0294	.0101	.0061	-.0136	.0119	-.0047	-.0030	.0073
.80	.4956	-.0846	.0031	.0236	-.0283	.0221	-.0114	.0009	.0065	-.0097	.0087	-.0051
.85	.4998	-.1054	.0284	.0014	-.0142	.0183	-.0174	.0135	-.0083	.0030	.0015	-.0046
.90	.5022	-.1199	.0483	-.0200	.0056	.0025	-.0071	.0094	-.0101	.0097	-.0085	.0068
.95	.5032	-.1282	.0607	-.0350	.0219	-.0141	.0090	-.0055	.0029	-.0010	-.0005	.0015
1.00	.5035	-.1308	.0648	-.0402	.0280	-.0210	.0164	-.0133	.0111	-.0094	.0082	-.0071

(2) Pe = 10 , p = 1

r^+	R_{11}	R_{21}	R_{31}	R_{41}	R_{51}	R_{61}	R_{71}	R_{81}	R_{91}	R_{101}	R_{111}	R_{121}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0499	.0494	.0485	.0474	.0460	.0443	.0424	.0403	.0380	.0355	.0329	.0302
.10	.0991	.0950	.0886	.0804	.0706	.0598	.0485	.0371	.0262	.0161	.0072	-.0002
.15	.1468	.1334	.1135	.0894	.0635	.0385	.0165	-.0009	-.0127	-.0187	-.0196	-.0165
.20	.1926	.1619	.1195	.0731	.0307	-.0017	-.0207	-.0265	-.0218	-.0112	.0004	.0091
.25	.2357	.1788	.1066	.0385	-.0098	-.0315	-.0295	-.0137	.0040	.0147	.0151	.0075
.30	.2756	.1832	.0782	-.0022	-.0377	-.0324	-.0074	.0140	.0189	.0089	-.0050	-.0119
.35	.3121	.1756	.0402	-.0359	-.0414	-.0087	.0188	.0195	.0019	-.0126	-.0112	.0007
.40	.3447	.1571	.0000	-.0529	-.0225	.0185	.0224	-.0009	-.0159	-.0076	.0077	.0100
.45	.3733	.1298	-.0355	-.0500	.0062	.0285	.0029	-.0178	-.0065	.0111	.0077	-.0064
.50	.3978	.0963	-.0608	-.0306	.0286	.0160	-.0175	-.0101	.0122	.0072	-.0091	-.0054
.55	.4182	.0593	-.0729	-.0030	.0341	-.0073	-.0186	.0097	.0101	-.0098	-.0047	.0088
.60	.4346	.0215	-.0713	.0233	.0217	-.0232	-.0012	.0157	-.0075	-.0072	.0095	.0003
.65	.4474	-.0147	-.0579	.0402	-.0006	-.0213	.0160	.0017	-.0123	.0084	.0024	-.0083
.70	.4568	-.0474	-.0362	.0438	-.0210	-.0045	.0168	-.0134	.0019	.0077	-.0093	.0038
.75	.4633	-.0751	-.0105	.0344	-.0302	.0143	.0019	-.0115	.0123	-.0068	-.0008	.0061
.80	.4672	-.0971	.0151	.0161	-.0252	.0224	-.0138	.0039	.0042	-.0085	.0089	-.0062
.85	.4693	-.1132	.0372	-.0054	-.0096	.0158	-.0166	.0140	-.0095	.0046	-.0001	-.0034
.90	.4699	-.1237	.0535	-.0243	.0090	-.0002	-.0051	.0080	-.0092	.0093	-.0085	.0071
.95	.4698	-.1293	.0632	-.0368	.0233	-.0153	.0100	-.0063	.0037	-.0017	.0001	.0010
1.00	.4697	-.1309	.0662	-.0410	.0284	-.0212	.0166	-.0134	.0112	-.0095	.0082	-.0072

Table A.1. Continued

(3) Pe = 20 , p = 1

r^+	R_{11}	R_{21}	R_{31}	R_{41}	R_{51}	R_{61}	R_{71}	R_{81}	R_{91}	R_{101}	R_{111}	R_{121}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0499	.0493	.0483	.0471	.0456	.0438	.0418	.0395	.0371	.0346	.0319	.0291
.10	.0990	.0944	.0872	.0782	.0678	.0564	.0447	.0333	.0224	.0126	.0042	-.0026
.15	.1466	.1315	.1094	.0835	.0566	.0315	.0103	-.0056	-.0155	-.0196	-.0188	-.0145
.20	.1921	.1577	.1112	.0629	.0210	-.0087	-.0239	-.0258	-.0185	-.0069	.0042	.0112
.25	.2348	.1713	.0937	.0256	-.0180	-.0330	-.0255	-.0076	.0088	.0160	.0129	.0037
.30	.2742	.1718	.0615	-.0141	-.0396	-.0263	.0004	.0176	.0167	.0038	-.0086	-.0117
.35	.3099	.1599	.0220	-.0424	-.0351	.0008	.0219	.0150	-.0041	-.0140	-.0076	.0047
.40	.3416	.1375	-.0166	-.0513	-.0110	.0236	.0171	-.0076	-.0155	-.0022	.0104	.0071
.45	.3691	.1070	-.0474	-.0404	.0166	.0249	-.0054	-.0174	-.0001	.0126	.0031	-.0088
.50	.3923	.0715	-.0657	-.0162	.0321	.0063	-.0200	-.0031	.0139	.0017	-.0105	-.0010
.55	.4114	.0339	-.0697	.0114	.0288	-.0154	-.0127	.0141	.0044	-.0115	.0002	.0087
.60	.4264	-.0028	-.0608	.0329	.0107	-.0239	.0067	.0118	-.0113	-.0021	.0096	-.0039
.65	.4378	-.0364	-.0421	.0423	-.0114	-.0144	.0182	-.0050	-.0086	.0105	-.0022	-.0062
.70	.4458	-.0653	-.0180	.0383	-.0265	.0043	.0118	-.0147	.0069	.0035	-.0087	.0065
.75	.4509	-.0885	.0072	.0239	-.0287	.0191	-.0048	-.0068	.0115	-.0093	.0030	.0032
.80	.4537	-.1058	.0298	.0043	-.0187	.0212	-.0164	.0083	-.0002	-.0057	.0082	-.0074
.85	.4547	-.1175	.0476	-.0153	-.0020	.0109	-.0142	.0138	-.0111	.0070	-.0028	-.0010
.90	.4545	-.1244	.0596	-.0307	.0146	-.0048	-.0014	.0053	-.0074	.0083	-.0082	.0074
.95	.4540	-.1276	.0660	-.0398	.0258	-.0174	.0118	-.0079	.0051	-.0029	.0012	.0000
1.00	.4536	-.1283	.0678	-.0426	.0295	-.0219	.0170	-.0137	.0114	-.0096	.0083	-.0072

(4) Pe = 30 , p = 1

r^+	R_{11}	R_{21}	R_{31}	R_{41}	R_{51}	R_{61}	R_{71}	R_{81}	R_{91}	R_{101}	R_{111}	R_{121}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0499	.0493	.0482	.0469	.0453	.0434	.0413	.0389	.0364	.0338	.0310	.0282
.10	.0990	.0941	.0865	.0768	.0658	.0540	.0420	.0304	.0196	.0099	.0019	-.0045
.15	.1466	.1307	.1071	.0798	.0521	.0268	.0061	-.0087	-.0172	-.0198	-.0178	-.0126
.20	.1920	.1560	.1068	.0567	.0149	-.0129	-.0254	-.0247	-.0155	-.0034	.0069	.0123
.25	.2346	.1683	.0869	.0182	-.0225	-.0330	-.0219	-.0030	.0117	.0161	.0106	.0006
.30	.2739	.1673	.0530	-.0203	-.0395	-.0213	.0055	.0190	.0140	-.0002	-.0107	-.0106
.35	.3094	.1538	.0132	-.0449	-.0299	.0068	.0226	.0107	-.0081	-.0137	-.0041	.0074
.40	.3408	.1299	-.0242	-.0487	-.0035	.0255	.0121	-.0116	-.0136	.0021	.0112	.0039
.45	.3681	.0983	-.0521	-.0334	.0220	.0207	-.0106	-.0153	.0046	.0120	-.0009	-.0094
.50	.3910	.0621	-.0664	-.0073	.0321	-.0006	-.0196	.0024	.0134	-.0027	-.0098	.0027
.55	.4098	.0246	-.0663	.0191	.0234	-.0195	-.0071	.0154	-.0005	-.0110	.0041	.0071
.60	.4245	-.0115	-.0538	.0367	.0028	-.0220	.0116	.0074	-.0126	.0022	.0081	-.0065
.65	.4355	-.0440	-.0328	.0412	-.0176	-.0083	.0177	-.0093	-.0045	.0105	-.0055	-.0033
.70	.4432	-.0713	-.0080	.0329	-.0283	.0102	.0069	-.0138	.0098	-.0004	-.0069	.0077
.75	.4480	-.0927	.0163	.0162	-.0258	.0210	-.0093	-.0024	.0094	-.0101	.0058	.0004
.80	.4505	-.1081	.0370	-.0036	-.0130	.0188	-.0171	.0111	-.0037	-.0028	.0067	-.0076
.85	.4512	-.1181	.0523	-.0215	.0038	.0065	-.0115	.0129	-.0116	.0086	-.0048	.0011
.90	.4509	-.1236	.0620	-.0345	.0185	-.0084	.0017	.0028	-.0056	.0071	-.0076	.0073
.95	.4502	-.1259	.0667	-.0416	.0277	-.0191	.0133	-.0093	.0063	-.0040	.0022	-.0008
1.00	.4498	-.1264	.0679	-.0436	.0305	-.0227	.0176	-.0141	.0117	-.0098	.0084	-.0074

Table A.1. Continued

(5) Pe = 50 , p = 1

r^+	R_{11}	R_{21}	R_{31}	R_{41}	R_{51}	R_{61}	R_{71}	R_{81}	R_{91}	R_{101}	R_{111}	R_{121}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0499	.0492	.0482	.0467	.0449	.0429	.0406	.0381	.0354	.0326	.0297	.0268
.10	.0990	.0939	.0858	.0754	.0636	.0510	.0385	.0265	.0157	.0063	-.0013	-.0070
.15	.1466	.1301	.1051	.0760	.0469	.0212	.0011	-.0123	-.0188	-.0195	-.0157	-.0094
.20	.1919	.1548	.1029	.0504	.0084	-.0174	-.0264	-.0222	-.0110	.0013	.0100	.0129
.25	.2345	.1663	.0811	.0110	-.0266	-.0318	-.0166	.0029	.0147	.0147	.0063	-.0038
.30	.2737	.1642	.0458	-.0258	-.0381	-.0146	.0113	.0192	.0092	-.0054	-.0120	-.0074
.35	.3091	.1496	.0059	-.0461	-.0232	.0132	.0215	.0042	-.0122	-.0112	.0015	.0096
.40	.3404	.1247	-.0301	-.0448	.0047	.0256	.0049	-.0152	-.0090	.0074	.0098	-.0014
.45	.3675	.0923	-.0553	-.0256	.0264	.0139	-.0156	-.0103	.0098	.0086	-.0062	-.0076
.50	.3903	.0558	-.0660	.0017	.0299	-.0085	-.0164	.0088	.0099	-.0080	-.0063	.0069
.55	.4089	.0183	-.0623	.0258	.0157	-.0220	.0008	.0141	-.0068	-.0074	.0081	.0024
.60	.4234	-.0173	-.0470	.0388	-.0060	-.0172	.0159	.0003	-.0112	.0075	.0034	-.0080
.65	.4343	-.0489	-.0244	.0381	-.0231	.0000	.0142	-.0131	.0021	.0076	-.0085	.0019
.70	.4417	-.0751	.0006	.0258	-.0279	.0161	-.0005	-.0101	.0116	-.0058	-.0022	.0070
.75	.4464	-.0952	.0238	.0073	-.0203	.0211	-.0140	.0040	.0046	-.0089	.0082	-.0040
.80	.4487	-.1093	.0426	-.0119	-.0052	.0138	-.0159	.0134	-.0081	.0021	.0030	-.0061
.85	.4494	-.1181	.0557	-.0276	.0108	-.0000	-.0067	.0101	-.0109	.0098	-.0073	.0043
.90	.4490	-.1227	.0633	-.0379	.0231	-.0132	.0063	-.0013	-.0022	.0045	-.0059	.0065
.95	.4482	-.1244	.0666	-.0429	.0298	-.0214	.0156	-.0114	.0082	-.0058	.0039	-.0024
1.00	.4478	-.1247	.0673	-.0441	.0316	-.0238	.0186	-.0149	.0123	-.0103	.0088	-.0077

(6) Pe = 100 , p = 1

r^+	R_{11}	R_{21}	R_{31}	R_{41}	R_{51}	R_{61}	R_{71}	R_{81}	R_{91}	R_{101}	R_{111}	R_{121}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0499	.0492	.0481	.0465	.0446	.0424	.0398	.0371	.0341	.0311	.0279	.0247
.10	.0990	.0939	.0854	.0743	.0616	.0480	.0346	.0221	.0111	.0020	-.0050	-.0098
.15	.1465	.1299	.1039	.0732	.0425	.0159	-.0038	-.0156	-.0198	-.0179	-.0121	-.0047
.20	.1919	.1542	.1005	.0459	.0031	-.0209	-.0262	-.0184	-.0051	.0066	.0125	.0117
.25	.2344	.1652	.0775	.0059	-.0294	-.0295	-.0105	.0086	.0161	.0110	.0001	-.0083
.30	.2736	.1626	.0415	-.0292	-.0360	-.0080	.0159	.0170	.0024	-.0103	-.0104	-.0012
.35	.3090	.1475	.0016	-.0463	-.0169	.0181	.0180	-.0032	-.0140	-.0053	.0075	.0084
.40	.3403	.1220	-.0334	-.0413	.0110	.0235	-.0029	-.0158	-.0016	.0110	.0042	-.0072
.45	.3673	.0893	-.0568	-.0196	.0286	.0064	-.0180	-.0026	.0126	.0014	-.0094	-.0012
.50	.3900	.0526	-.0654	.0079	.0263	-.0148	-.0102	.0134	.0025	-.0105	.0013	.0077
.55	.4085	.0152	-.0595	.0298	.0083	-.0215	.0086	.0088	-.0113	.0003	.0083	-.0048
.60	.4230	-.0201	-.0426	.0391	-.0130	-.0102	.0167	-.0076	-.0052	.0099	-.0042	-.0042
.65	.4337	-.0513	-.0192	.0346	-.0258	.0080	.0074	-.0132	.0088	.0003	-.0069	.0070
.70	.4411	-.0768	.0057	.0198	-.0254	.0195	-.0083	-.0026	.0091	-.0095	.0049	.0012
.75	.4457	-.0963	.0281	.0005	-.0139	.0182	-.0162	.0103	-.0030	-.0033	.0070	-.0073
.80	.4480	-.1098	.0456	-.0178	.0022	.0069	-.0116	.0127	-.0111	.0077	-.0034	-.0007
.85	.4485	-.1180	.0573	-.0315	.0168	-.0071	.0004	.0043	-.0071	.0084	-.0083	.0072
.90	.4481	-.1222	.0636	-.0397	.0265	-.0179	.0117	-.0069	.0032	-.0003	-.0019	.0035
.95	.4473	-.1236	.0661	-.0431	.0309	-.0233	.0180	-.0141	.0111	-.0087	.0067	-.0050
1.00	.4469	-.1237	.0665	-.0437	.0318	-.0245	.0196	-.0161	.0134	-.0114	.0097	-.0084

Table A.1. Continued

(1) Pe = 5 , p = 2

r^+	R_{12}	R_{22}	R_{32}	R_{42}	R_{52}	R_{62}	R_{72}	R_{82}	R_{92}	R_{102}	R_{112}	R_{122}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0025	.0025	.0024	.0024	.0024	.0023	.0022	.0022	.0021	.0020	.0019	.0018
.10	.0099	.0095	.0091	.0085	.0078	.0070	.0061	.0053	.0044	.0035	.0027	.0020
.15	.0218	.0202	.0180	.0154	.0124	.0095	.0067	.0041	.0021	.0005	-.0005	-.0011
.20	.0378	.0331	.0263	.0197	.0128	.0068	.0023	-.0007	-.0021	-.0023	-.0017	-.0008
.25	.0572	.0463	.0327	.0192	.0080	.0005	-.0031	-.0035	-.0022	-.0004	.0009	.0013
.30	.0773	.0581	.0346	.0134	.0002	-.0050	-.0044	-.0014	.0011	.0019	.0011	-.0002
.35	.1032	.0669	.0300	.0042	-.0064	-.0058	-.0010	.0022	.0021	.0002	-.0011	-.0010
.40	.1280	.0713	.0212	-.0052	-.0086	-.0019	.0030	.0024	-.0005	-.0016	-.0005	.0008
.45	.1529	.0709	.0094	-.0115	-.0056	.0031	.0033	-.0007	-.0020	-.0001	.0012	.0004
.50	.1770	.0653	-.0031	-.0130	.0003	.0050	-.0000	-.0025	-.0000	.0015	.0000	-.0009
.55	.1996	.0551	-.0137	-.0094	.0055	.0026	-.0030	-.0007	.0018	-.0000	-.0011	.0003
.60	.2202	.0413	-.0205	-.0025	.0070	-.0017	-.0024	.0019	.0005	-.0013	.0004	.0007
.65	.2382	.0252	-.0225	.0047	.0043	-.0043	.0007	.0017	-.0015	.0001	.0009	-.0007
.70	.2535	.0081	-.0197	.0097	-.0007	-.0032	.0029	-.0008	-.0008	.0012	-.0006	-.0002
.75	.2657	-.0085	-.0130	.0107	-.0049	.0004	.0018	-.0021	.0012	-.0001	-.0006	.0008
.80	.2751	-.0233	-.0042	.0078	-.0062	.0035	-.0011	-.0004	.0011	-.0012	.0008	-.0003
.85	.2818	-.0354	.0049	.0024	-.0039	.0036	-.0027	.0017	-.0008	.0001	.0003	-.0006
.90	.2859	-.0442	.0126	-.0036	.0003	.0010	-.0014	.0015	-.0013	.0011	-.0009	.0006
.95	.2881	-.0494	.0176	-.0081	.0042	-.0023	.0012	-.0006	.0003	-.0000	-.0001	.0002
1.00	.2887	-.0510	.0193	-.0097	.0057	-.0037	.0025	-.0018	.0014	-.0011	.0008	-.0007

(2) Pe = 10 , p = 2

r^+	R_{12}	R_{22}	R_{32}	R_{42}	R_{52}	R_{62}	R_{72}	R_{82}	R_{92}	R_{102}	R_{112}	R_{122}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0025	.0025	.0024	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019	.0018
.10	.0098	.0095	.0090	.0084	.0076	.0068	.0060	.0051	.0042	.0033	.0026	.0018
.15	.0216	.0199	.0176	.0148	.0119	.0089	.0061	.0037	.0017	.0002	-.0007	-.0012
.20	.0373	.0322	.0256	.0184	.0116	.0058	.0015	-.0011	-.0022	-.0022	-.0015	-.0006
.25	.0561	.0444	.0304	.0170	.0063	-.0005	-.0034	-.0034	-.0018	-.0001	.0011	.0013
.30	.0771	.0547	.0304	.0106	-.0013	-.0052	-.0039	-.0008	.0014	.0018	.0008	-.0004
.35	.0993	.0614	.0251	.0015	-.0069	-.0050	-.0001	.0025	.0018	-.0001	-.0012	-.0008
.40	.1218	.0636	.0156	-.0069	-.0077	-.0007	.0033	.0019	-.0009	-.0015	-.0002	.0009
.45	.1437	.0609	.0040	-.0117	-.0039	.0037	.0026	-.0013	-.0018	.0003	.0012	.0001
.50	.1641	.0534	-.0072	-.0114	.0019	.0045	-.0008	-.0023	.0005	.0014	-.0003	-.0009
.55	.1824	.0420	-.0158	-.0067	.0060	.0014	-.0031	-.0000	.0017	-.0004	-.0010	.0005
.60	.1982	.0280	-.0203	.0002	.0062	-.0026	-.0017	.0021	-.0000	-.0012	.0006	.0005
.65	.2111	.0125	-.0201	.0064	.0027	-.0041	.0014	.0012	-.0016	.0004	.0007	-.0008
.70	.2212	-.0028	-.0158	.0099	-.0020	-.0023	.0028	-.0012	-.0004	.0011	-.0008	.0000
.75	.2284	-.0169	-.0087	.0096	-.0054	.0012	.0012	-.0019	.0014	-.0004	-.0004	.0007
.80	.2331	-.0287	-.0004	.0060	-.0056	.0036	-.0015	-.0000	.0009	-.0011	.0009	-.0004
.85	.2357	-.0379	.0075	.0007	-.0030	.0032	-.0026	.0018	-.0009	.0003	.0002	-.0005
.90	.2367	-.0441	.0137	-.0046	.0010	.0005	-.0011	.0013	-.0012	.0011	-.0009	.0006
.95	.2368	-.0476	.0176	-.0082	.0043	-.0024	.0014	-.0007	.0003	-.0001	-.0000	.0001
1.00	.2367	-.0486	.0188	-.0095	.0056	-.0036	.0025	-.0018	.0014	-.0010	.0008	-.0007

Table A.1. Continued

(3) Pe = 20 , p = 2

r^+	R_{12}	R_{22}	R_{32}	R_{42}	R_{52}	R_{62}	R_{72}	R_{82}	R_{92}	R_{102}	R_{112}	R_{122}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0025	.0025	.0024	.0024	.0023	.0023	.0022	.0021	.0020	.0019	.0018	.0017
.10	.0098	.0094	.0089	.0082	.0074	.0065	.0057	.0048	.0039	.0030	.0023	.0016
.15	.0215	.0196	.0170	.0141	.0110	.0080	.0052	.0029	.0011	-.0002	-.0010	-.0013
.20	.0372	.0312	.0241	.0167	.0098	.0043	.0005	-.0016	-.0023	-.0020	-.0012	-.0002
.25	.0553	.0423	.0275	.0140	.0040	-.0017	-.0036	-.0029	-.0012	.0004	.0012	.0012
.30	.0754	.0508	.0258	.0070	-.0030	-.0052	-.0029	.0001	.0018	.0016	.0004	-.0007
.35	.0964	.0553	.0192	-.0017	-.0071	-.0036	.0010	.0025	.0012	-.0007	-.0012	-.0005
.40	.1172	.0551	.0092	-.0085	-.0061	.0010	.0033	.0009	-.0014	-.0012	.0003	.0009
.45	.1369	.0501	-.0018	-.0110	-.0014	.0041	.0015	-.0018	-.0012	.0008	.0010	-.0003
.50	.1547	.0409	-.0111	-.0086	.0038	.0033	-.0019	-.0017	.0011	.0010	-.0007	-.0006
.55	.1700	.0285	-.0170	-.0029	.0061	-.0003	-.0027	.0009	.0013	-.0009	-.0006	.0007
.60	.1835	.0145	-.0185	.0035	.0045	-.0034	-.0005	.0020	-.0007	-.0008	.0009	.0001
.65	.1920	.0003	-.0159	.0080	.0004	-.0034	.0021	.0003	-.0014	.0008	.0002	-.0007
.70	.1966	-.0130	-.0102	.0093	-.0036	-.0008	.0023	-.0017	.0002	.0008	-.0009	.0004
.75	.2025	-.0242	-.0029	.0073	-.0055	.0023	.0002	-.0014	.0014	-.0008	-.0000	.0005
.80	.2043	-.0330	.0045	.0031	-.0045	.0036	-.0020	.0006	.0004	-.0008	.0009	-.0006
.85	.2044	-.0392	.0107	-.0017	-.0014	.0023	-.0023	.0018	-.0012	.0006	-.0001	-.0002
.90	.2036	-.0430	.0151	-.0059	.0021	-.0003	-.0005	.0009	-.0010	.0010	-.0008	.0007
.95	.2025	-.0448	.0176	-.0085	.0046	-.0027	.0016	-.0009	.0005	-.0002	.0001	.0001
1.00	.2019	-.0452	.0183	-.0093	.0055	-.0035	.0024	-.0018	.0013	-.0010	.0008	-.0007

(4) Pe = 30 , p = 2

r^+	R_{12}	R_{22}	R_{32}	R_{42}	R_{52}	R_{62}	R_{72}	R_{82}	R_{92}	R_{102}	R_{112}	R_{122}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0025	.0025	.0024	.0024	.0023	.0022	.0022	.0021	.0020	.0019	.0018	.0017
.10	.0098	.0094	.0088	.0081	.0072	.0064	.0054	.0045	.0036	.0028	.0020	.0014
.15	.0215	.0194	.0167	.0136	.0104	.0074	.0046	.0024	.0007	-.0005	-.0011	-.0013
.20	.0368	.0307	.0232	.0155	.0087	.0034	-.0002	-.0019	-.0023	-.0018	-.0009	.0000
.25	.0550	.0412	.0258	.0122	.0026	-.0024	-.0036	-.0025	-.0007	.0008	.0013	.0010
.30	.0749	.0489	.0232	.0049	-.0039	-.0049	-.0022	.0007	.0019	.0013	-.0000	-.0008
.35	.0955	.0525	.0160	-.0034	-.0068	-.0025	.0017	.0024	.0006	-.0010	-.0011	-.0001
.40	.1158	.0512	.0059	-.0090	-.0048	.0019	.0030	.0002	-.0016	-.0008	.0006	.0008
.45	.1344	.0452	-.0045	-.0100	.0002	.0041	.0005	-.0020	-.0007	.0011	.0007	-.0006
.50	.1518	.0353	-.0126	-.0066	.0046	.0023	-.0023	-.0010	.0014	.0006	-.0009	-.0003
.55	.1662	.0227	-.0169	-.0006	.0056	-.0014	-.0022	.0014	.0008	-.0011	-.0002	.0008
.60	.1777	.0089	-.0169	.0051	.0031	-.0036	.0004	.0017	-.0011	-.0004	.0009	-.0002
.65	.1862	-.0047	-.0131	.0084	-.0011	-.0026	.0024	-.0004	-.0011	.0010	-.0001	-.0006
.70	.1917	-.0168	-.0068	.0084	-.0044	.0003	.0018	-.0018	.0007	.0004	-.0008	.0006
.75	.1944	-.0268	.0003	.0056	-.0052	.0029	-.0006	-.0009	.0013	-.0009	.0003	.0003
.80	.1957	-.0342	.0070	.0012	-.0034	.0033	-.0023	.0010	-.0001	-.0006	.0008	-.0007
.85	.1951	-.0391	.0122	-.0033	-.0003	.0016	-.0019	.0017	-.0013	.0008	-.0003	-.0000
.90	.1938	-.0418	.0157	-.0067	.0028	-.0009	-.0001	.0006	-.0008	.0008	-.0008	.0007
.95	.1923	-.0424	.0174	-.0087	.0048	-.0029	.0017	-.0011	.0006	-.0003	.0002	-.0000
1.00	.1917	-.0431	.0179	-.0093	.0055	-.0035	.0024	-.0018	.0013	-.0010	.0008	-.0007

Table A.1. Continued

(5) Pe = 50 , p = 2

r^*	R_{12}	R_{22}	R_{32}	R_{42}	R_{52}	R_{62}	R_{72}	R_{82}	R_{92}	R_{102}	R_{112}	R_{122}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0025	.0025	.0024	.0024	.0023	.0022	.0021	.0020	.0020	.0018	.0017	.0016
.10	.0094	.0093	.0087	.0079	.0070	.0061	.0051	.0042	.0033	.0024	.0017	.0010
.15	.0215	.0193	.0164	.0131	.0097	.0066	.0038	.0017	.0001	-.0008	-.0013	-.0013
.20	.0364	.0304	.0223	.0143	.0073	.0022	-.0009	-.0022	-.0022	-.0014	-.0004	.0004
.25	.0549	.0404	.0242	.0103	.0012	-.0030	-.0035	-.0019	-.0000	.0011	.0012	.0007
.30	.0746	.0475	.0208	.0028	-.0046	-.0044	-.0012	.0013	.0018	.0007	-.0005	-.0009
.35	.0950	.0503	.0130	-.0048	-.0063	-.0012	.0023	.0019	-.0001	-.0012	-.0007	.0003
.40	.1149	.0482	.0030	-.0092	-.0032	.0028	.0023	-.0007	-.0016	-.0002	.0009	.0005
.45	.1335	.0415	-.0067	-.0087	.0018	.0036	-.0006	-.0019	.0001	.0012	.0001	-.0008
.50	.1500	.0311	-.0136	-.0043	.0050	.0009	-.0025	-.0001	.0014	-.0001	-.0009	.0002
.55	.1638	.0184	-.0164	.0016	.0046	-.0024	-.0012	.0017	.0000	-.0011	.0004	.0006
.60	.1747	.0049	-.0149	.0064	.0012	-.0033	.0014	.0009	-.0013	.0003	.0007	-.0006
.65	.1826	-.0081	-.0102	.0082	-.0026	-.0013	.0023	-.0011	-.0004	.0010	-.0006	-.0002
.70	.1876	-.0194	-.0036	.0069	-.0047	.0015	.0008	-.0016	.0011	-.0002	-.0005	.0007
.75	.1901	-.0243	.0032	.0034	-.0043	.0031	-.0014	-.0000	.0008	-.0010	.0006	-.0001
.80	.1905	-.0346	.0091	-.0010	-.0019	.0026	-.0022	.0015	-.0006	-.0001	.0005	-.0006
.85	.1895	-.0386	.0133	-.0048	.0011	.0006	-.0013	.0014	-.0012	.0009	-.0006	.0002
.90	.1873	-.0405	.0159	-.0075	.0037	-.0017	.0006	.0001	-.0004	.0006	-.0006	.0006
.95	.1863	-.0412	.0170	-.0088	.0051	-.0031	.0020	-.0013	.0008	-.0005	.0003	-.0002
1.00	.1855	-.0412	.0172	-.0091	.0055	-.0036	.0025	-.0018	.0013	-.0010	.0008	-.0007

(6) Pe = 100 , p = 2

r^*	R_{12}	R_{22}	R_{32}	R_{42}	R_{52}	R_{62}	R_{72}	R_{82}	R_{92}	R_{102}	R_{112}	R_{122}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0025	.0025	.0024	.0024	.0023	.0022	.0021	.0020	.0019	.0018	.0017	.0016
.10	.0098	.0093	.0087	.0078	.0069	.0058	.0048	.0038	.0028	.0020	.0013	.0007
.15	.0214	.0192	.0161	.0126	.0090	.0058	.0030	.0010	-.0004	-.0011	-.0013	-.0011
.20	.0367	.0301	.0218	.0133	.0062	.0012	-.0015	-.0023	-.0019	-.0009	.0001	.0007
.25	.0548	.0399	.0231	.0089	-.0000	-.0035	-.0030	-.0011	.0007	.0013	.0009	.0001
.30	.0744	.0467	.0192	.0013	-.0050	-.0036	-.0001	.0018	.0014	.0000	-.0009	-.0007
.35	.0947	.0491	.0112	-.0057	-.0055	.0000	.0025	.0011	-.0008	-.0011	-.0001	.0007
.40	.1145	.0466	.0012	-.0096	-.0017	.0032	.0014	-.0014	-.0011	.0005	.0008	-.0001
.45	.1329	.0395	-.0079	-.0074	.0029	.0027	-.0015	-.0013	.0009	.0008	-.0005	-.0005
.50	.1492	.0239	-.0139	-.0025	.0049	-.0005	-.0021	.0009	.0009	-.0008	-.0004	.0006
.55	.1628	.0151	-.0157	.0032	.0033	-.0029	.0000	.0015	-.0008	-.0005	.0008	-.0001
.60	.1734	.0028	-.0134	.0070	-.0004	-.0025	.0020	-.0002	-.0010	.0008	-.0000	-.0006
.65	.1810	-.0099	-.0082	.0076	-.0036	.0001	.0016	-.0015	.0005	.0004	-.0007	.0004
.70	.1857	-.0206	-.0015	.0054	-.0045	.0024	-.0005	-.0008	.0011	-.0008	.0002	.0003
.75	.1879	-.0290	.0050	.0015	-.0031	.0029	-.0020	.0009	.0000	-.0005	.0007	-.0005
.80	.1881	-.0347	.0102	-.0026	-.0003	.0015	-.0017	.0015	-.0011	.0006	-.0001	-.0002
.85	.1875	-.0382	.0138	-.0059	.0024	-.0007	-.0003	.0007	-.0009	.0009	-.0007	.0006
.90	.1852	-.0397	.0158	-.0079	.0043	-.0024	.0013	-.0006	.0002	.0001	-.0002	.0003
.95	.1835	-.0401	.0165	-.0087	.0052	-.0034	.0023	-.0016	.0011	-.0008	.0005	-.0004
1.00	.1827	-.0401	.0166	-.0088	.0054	-.0036	.0025	-.0018	.0014	-.0011	.0008	-.0007

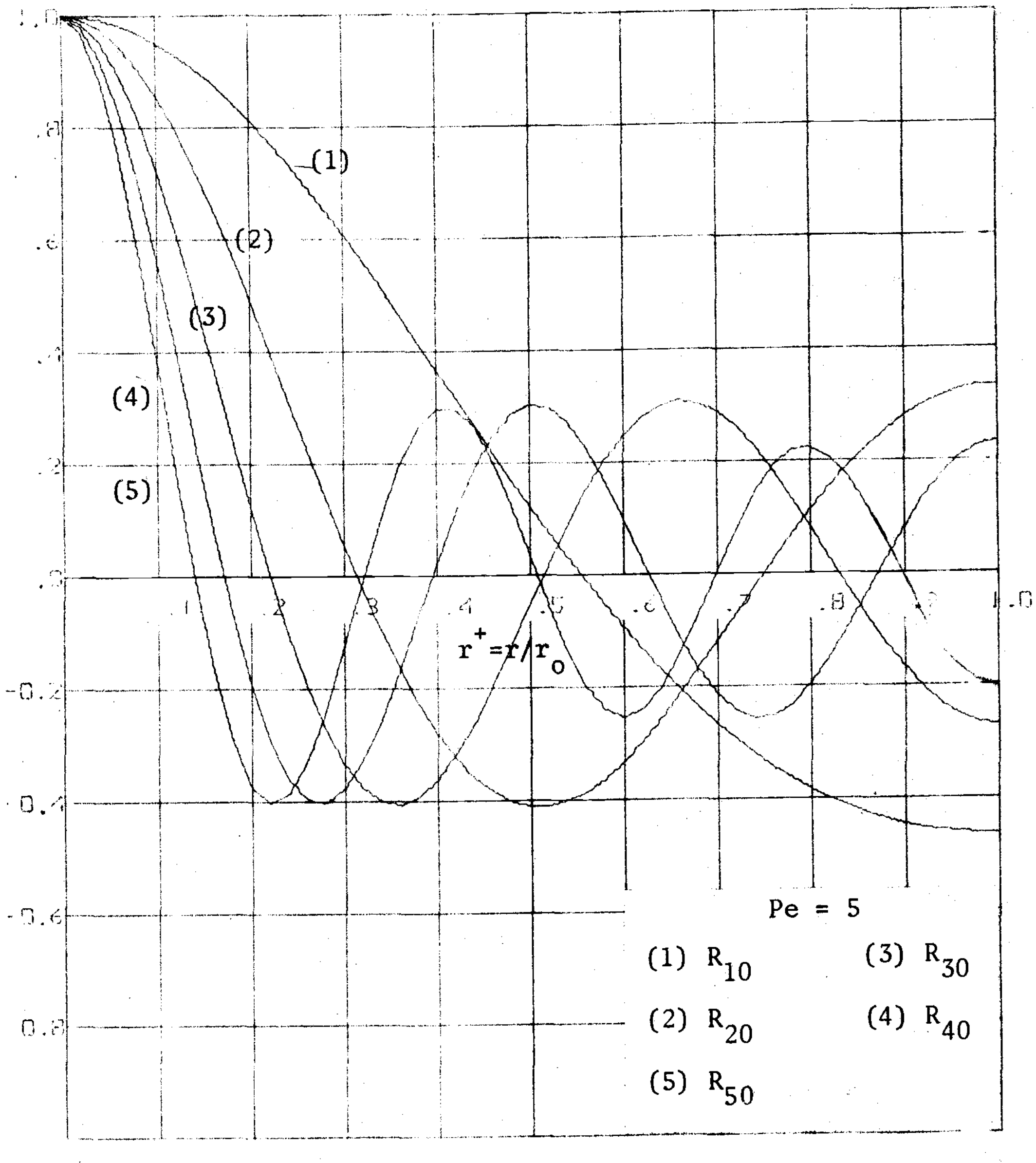

Figure B.1. Graphical representation of the first five eigenfunctions for $p = 0, 1, 2$ and for different Peclet number.

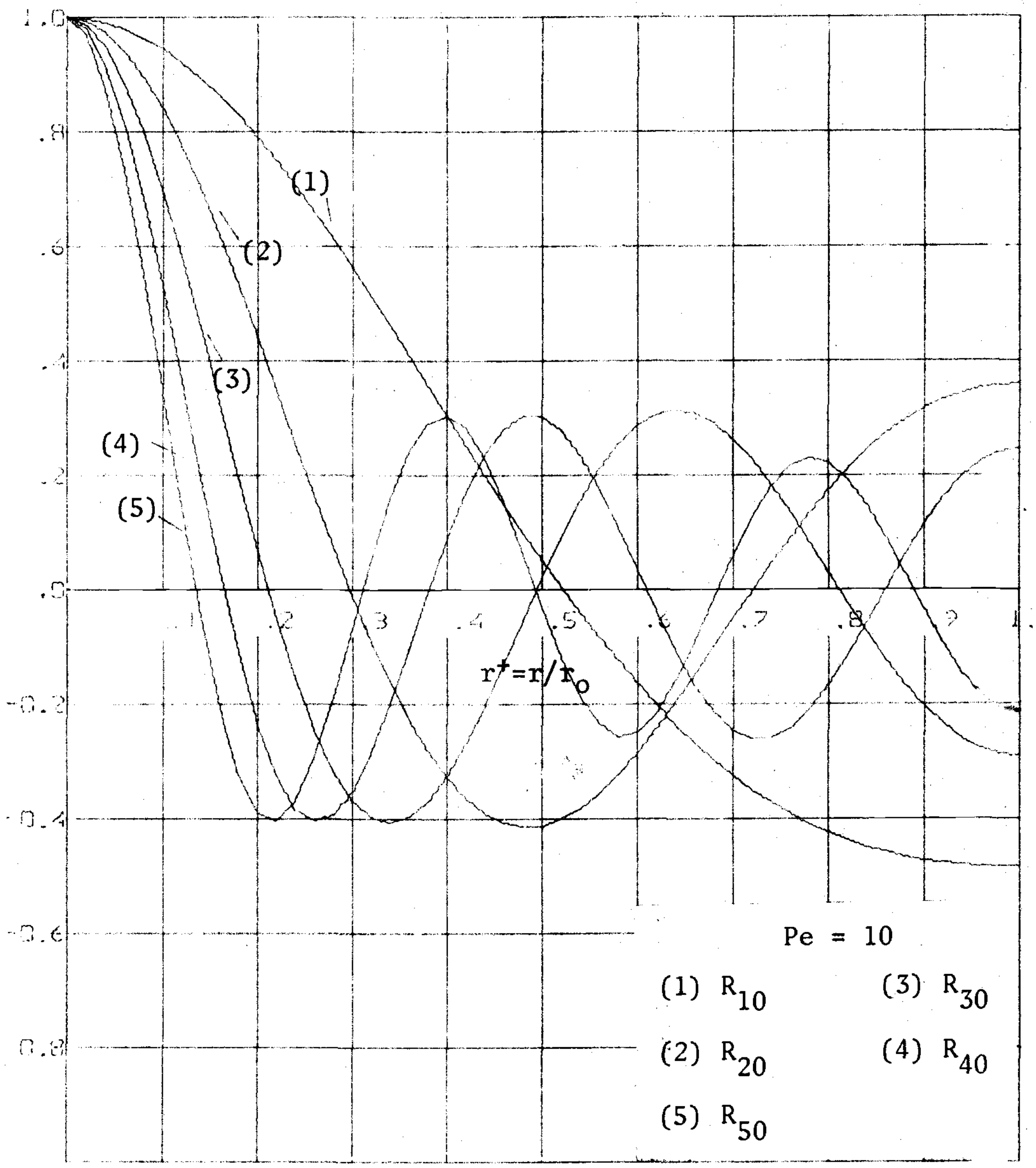

Figure B. 1. Continued.

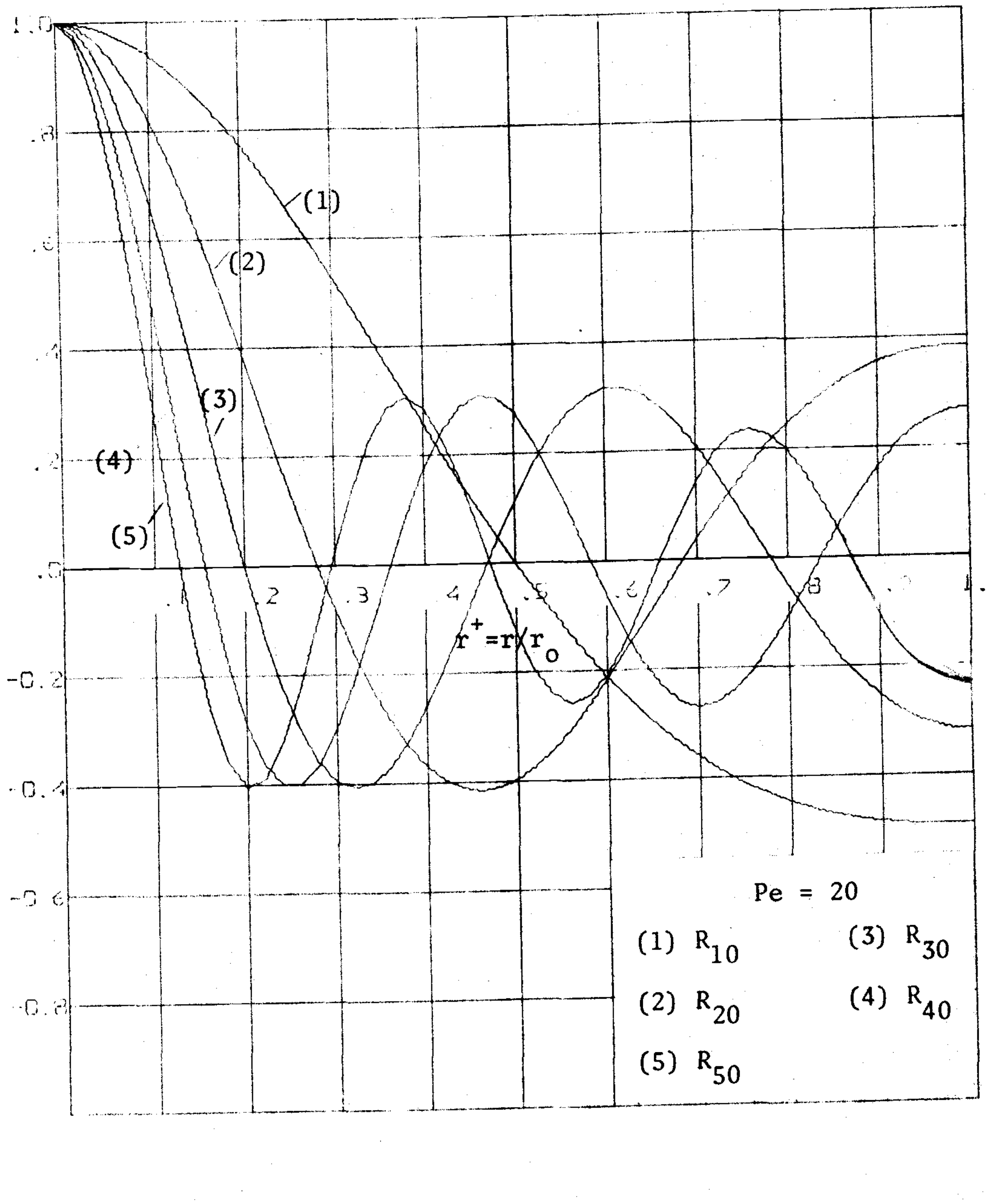

Figure B. 1. Continued.

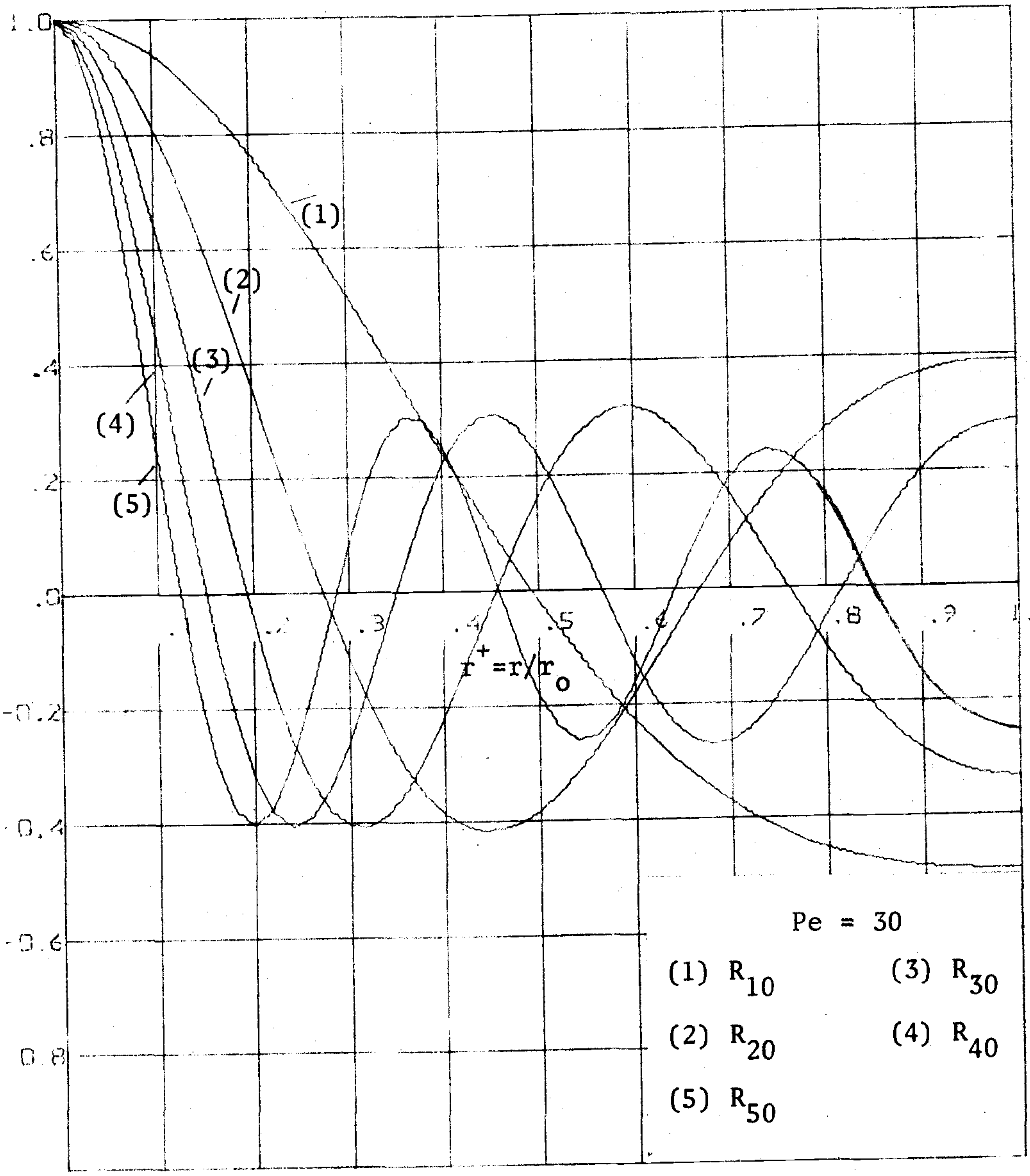

Figure B. 1. Continued.

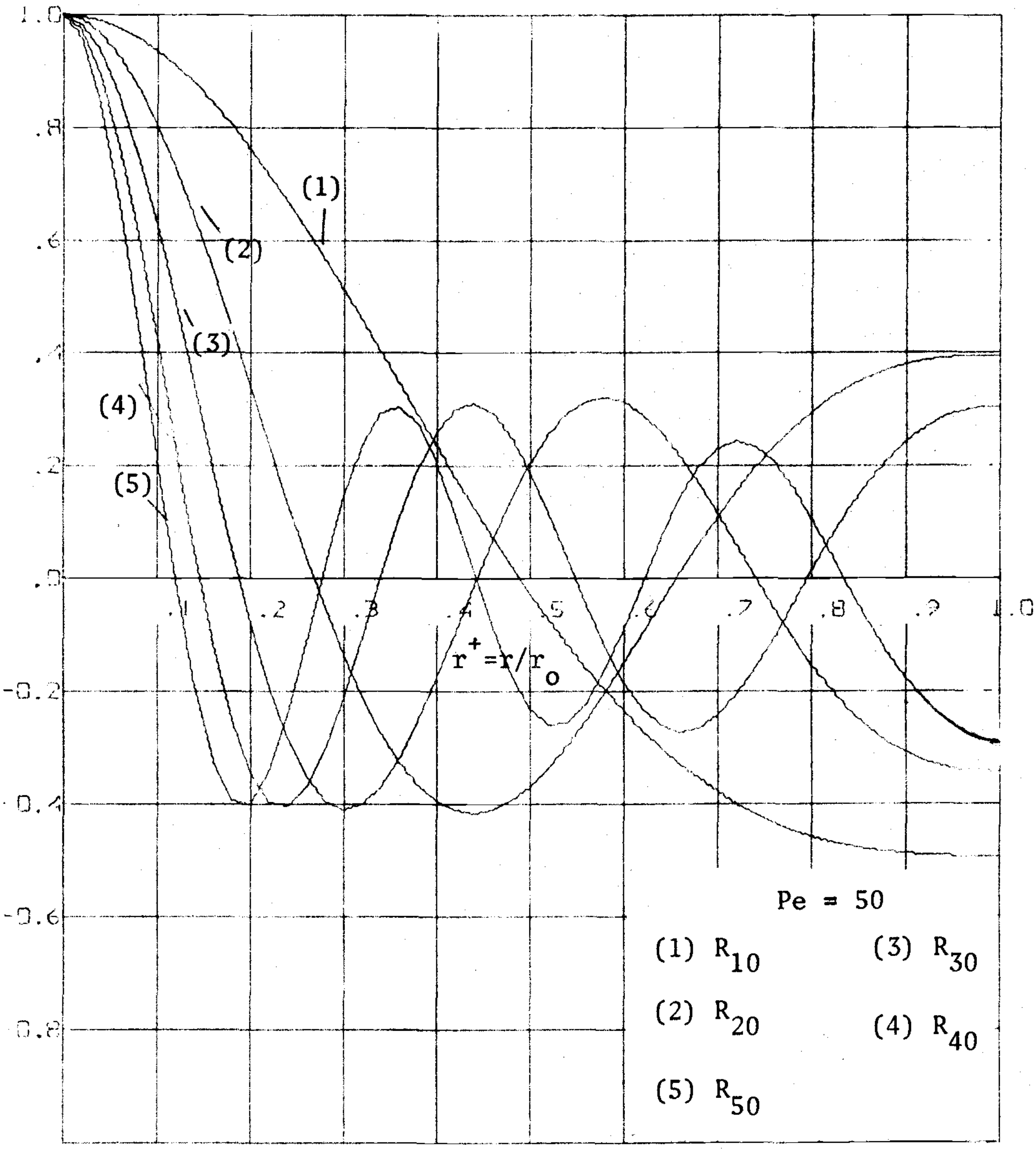

Figure B. 1. Continued.

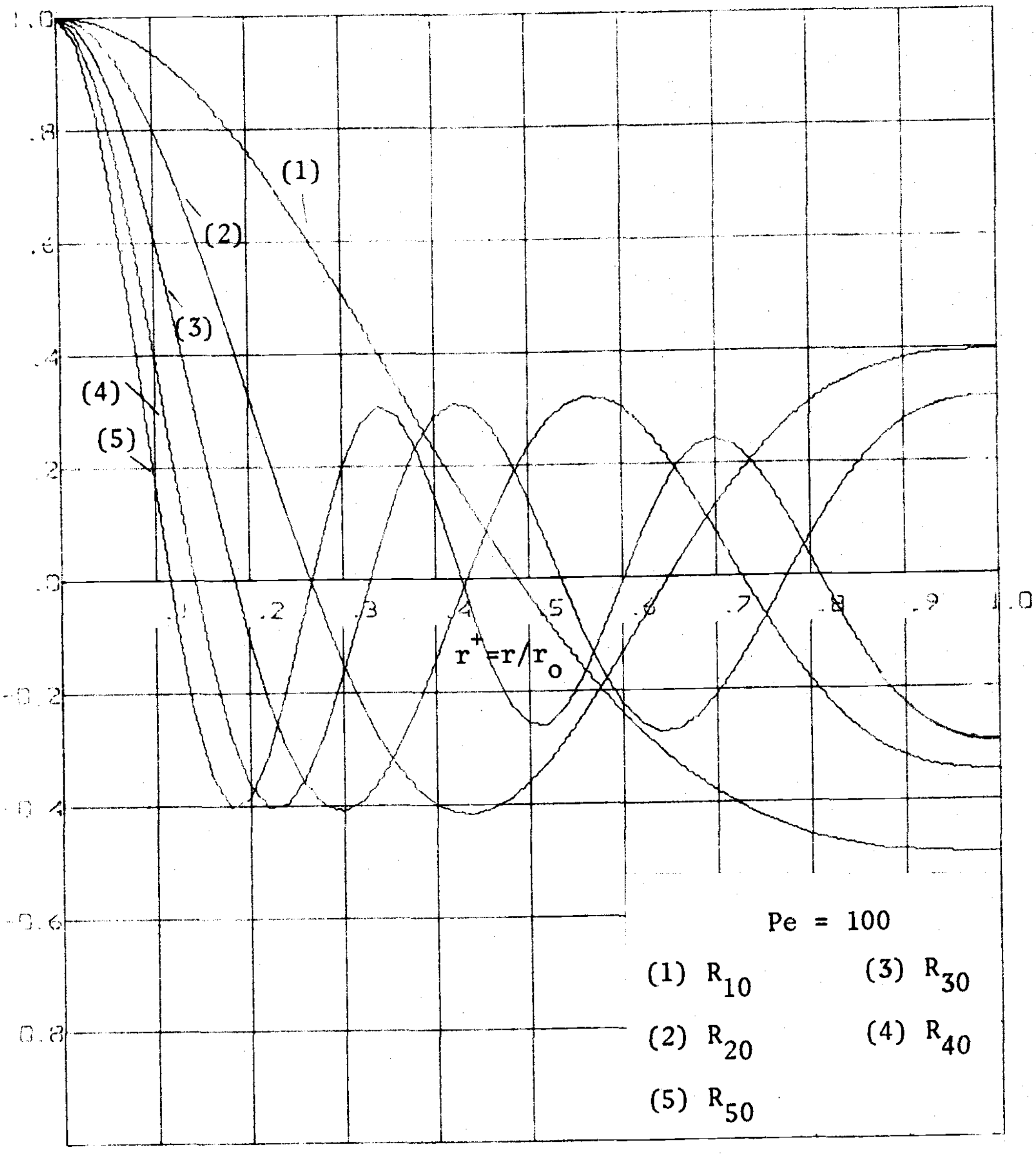

Figure B. 1. Continued.

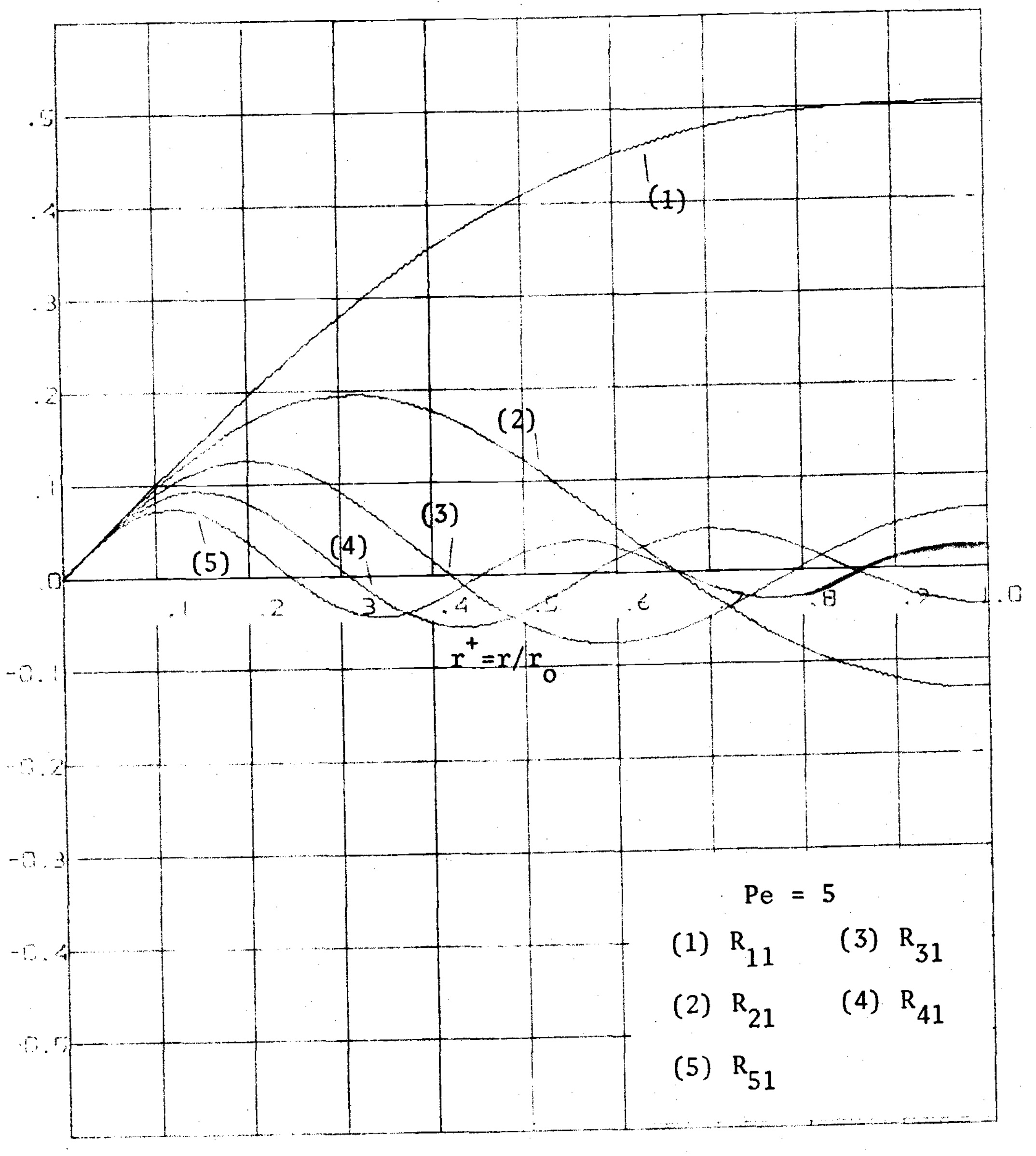

Figure B.1. Continued.

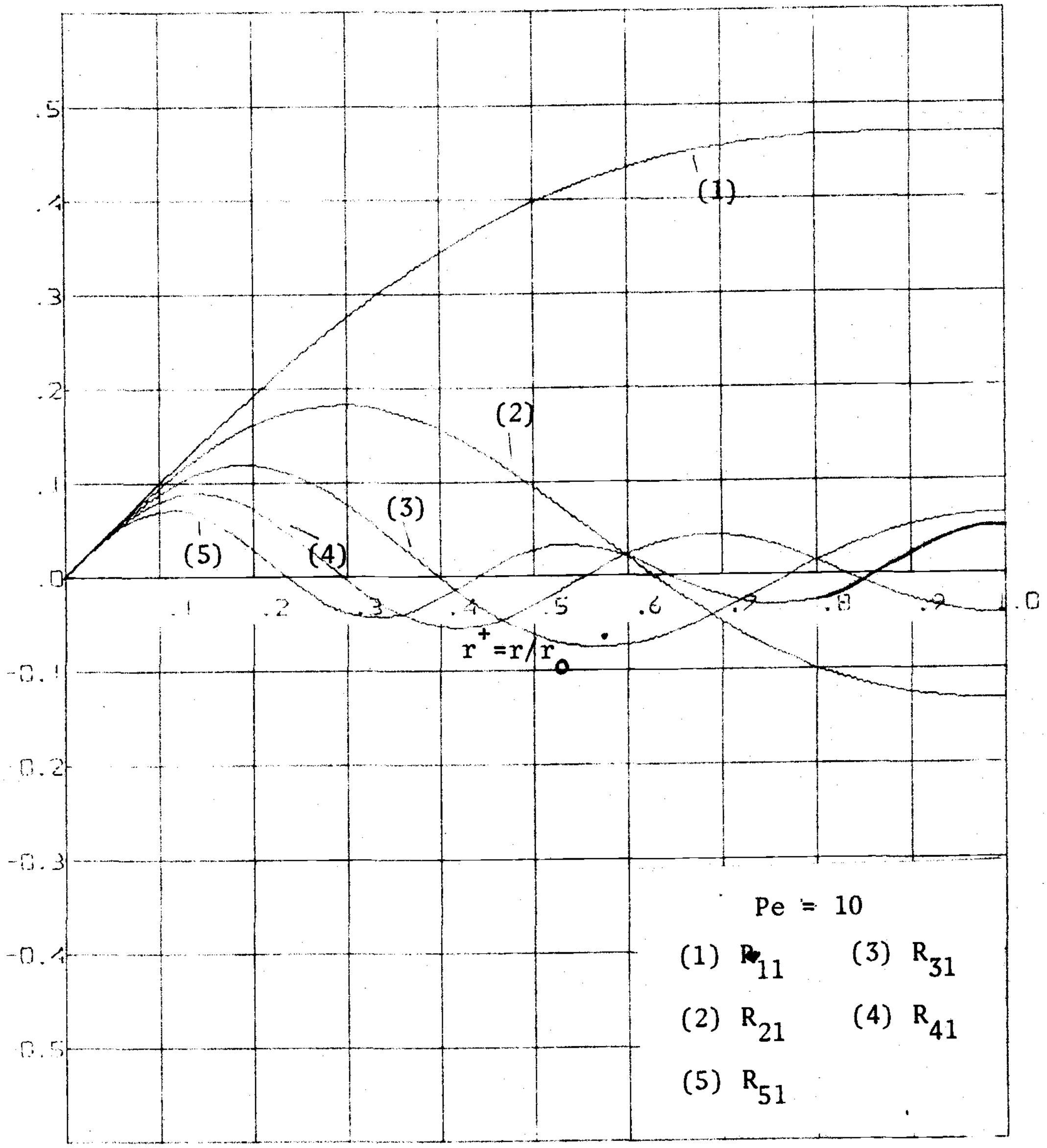

Figure B. 1. Continued.

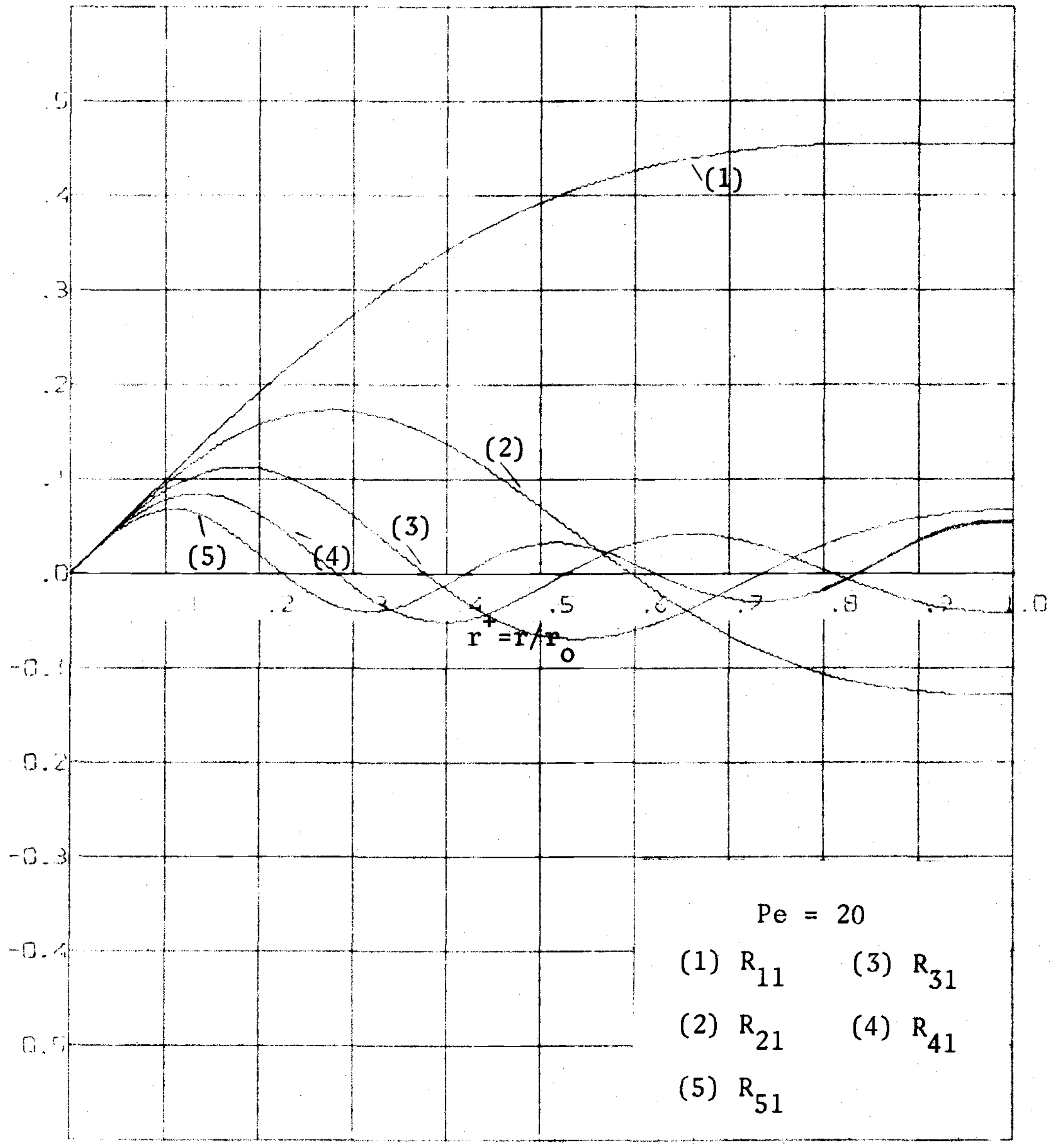

Figure B.1. Continued.

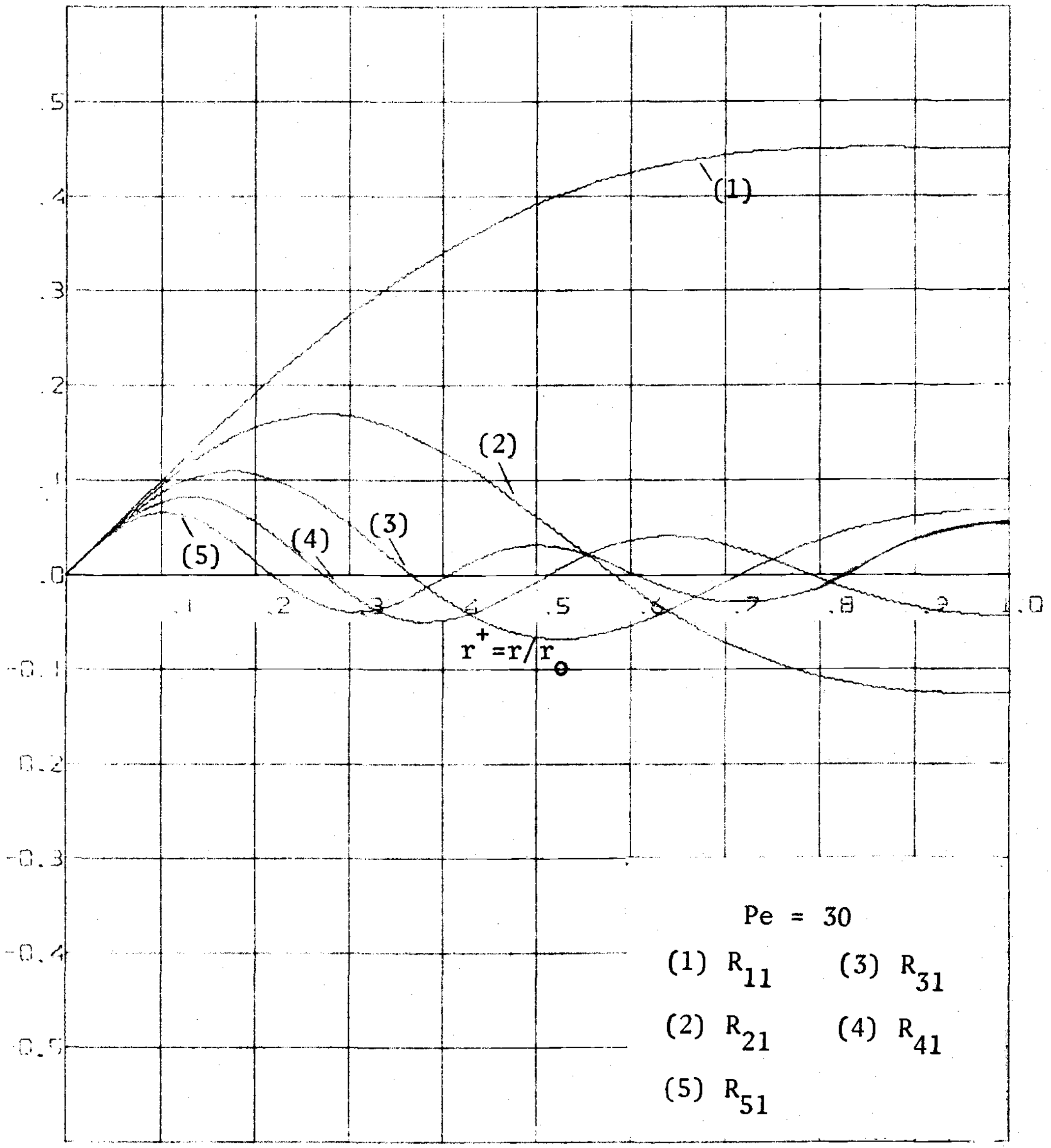

Figure B.1. Continued.

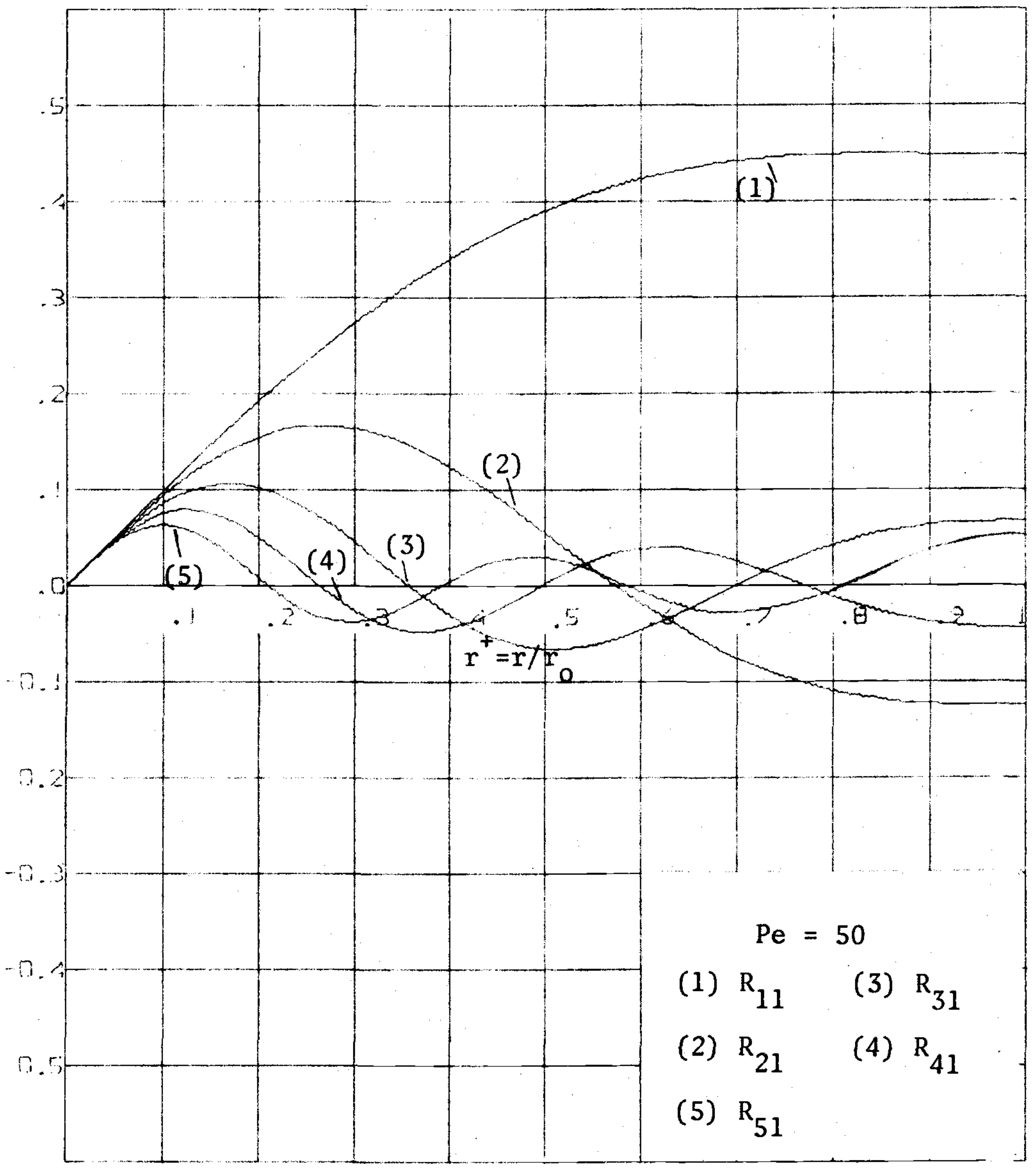

Figure B.1. Continued.

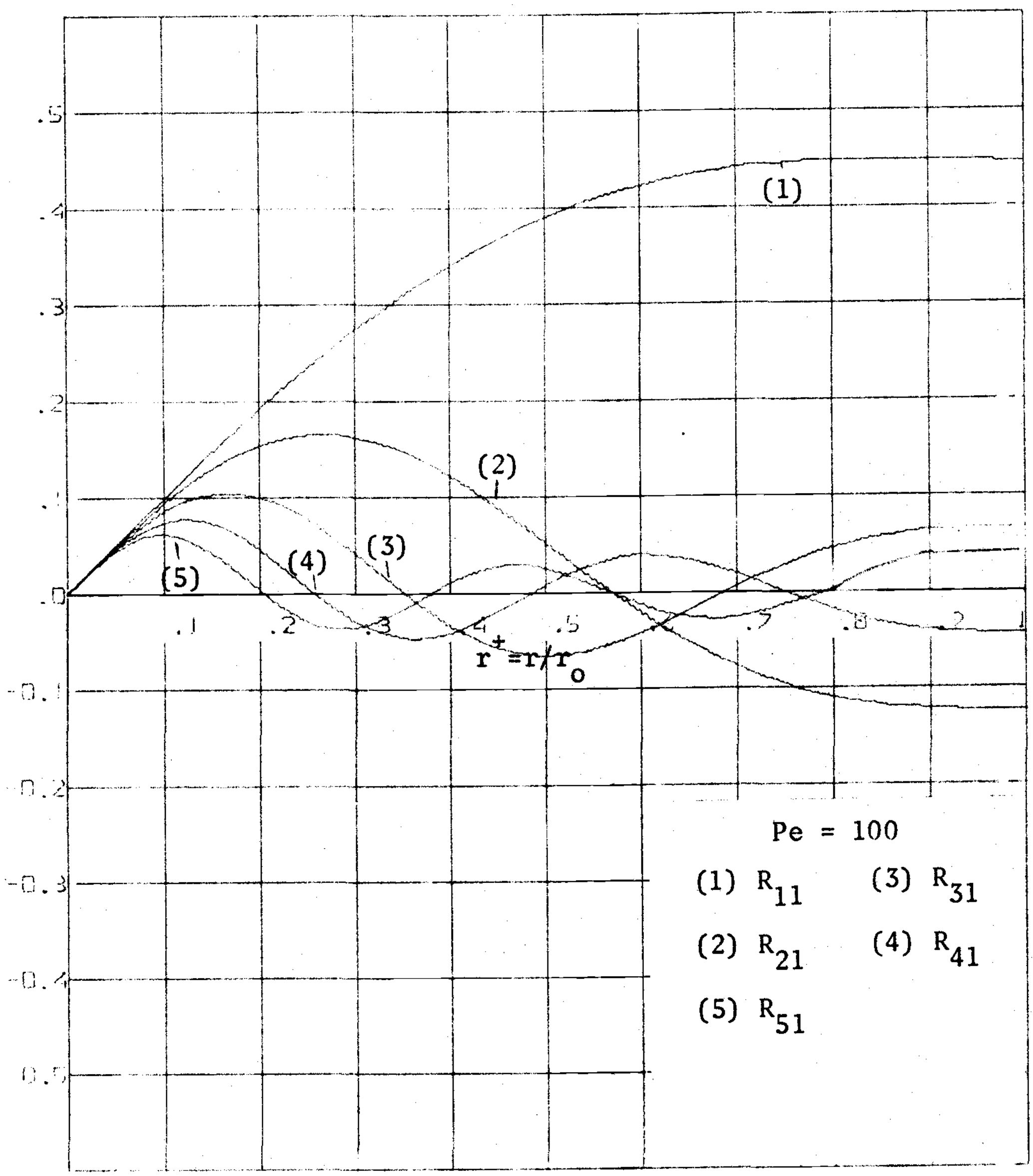

Figure B.1. Continued.

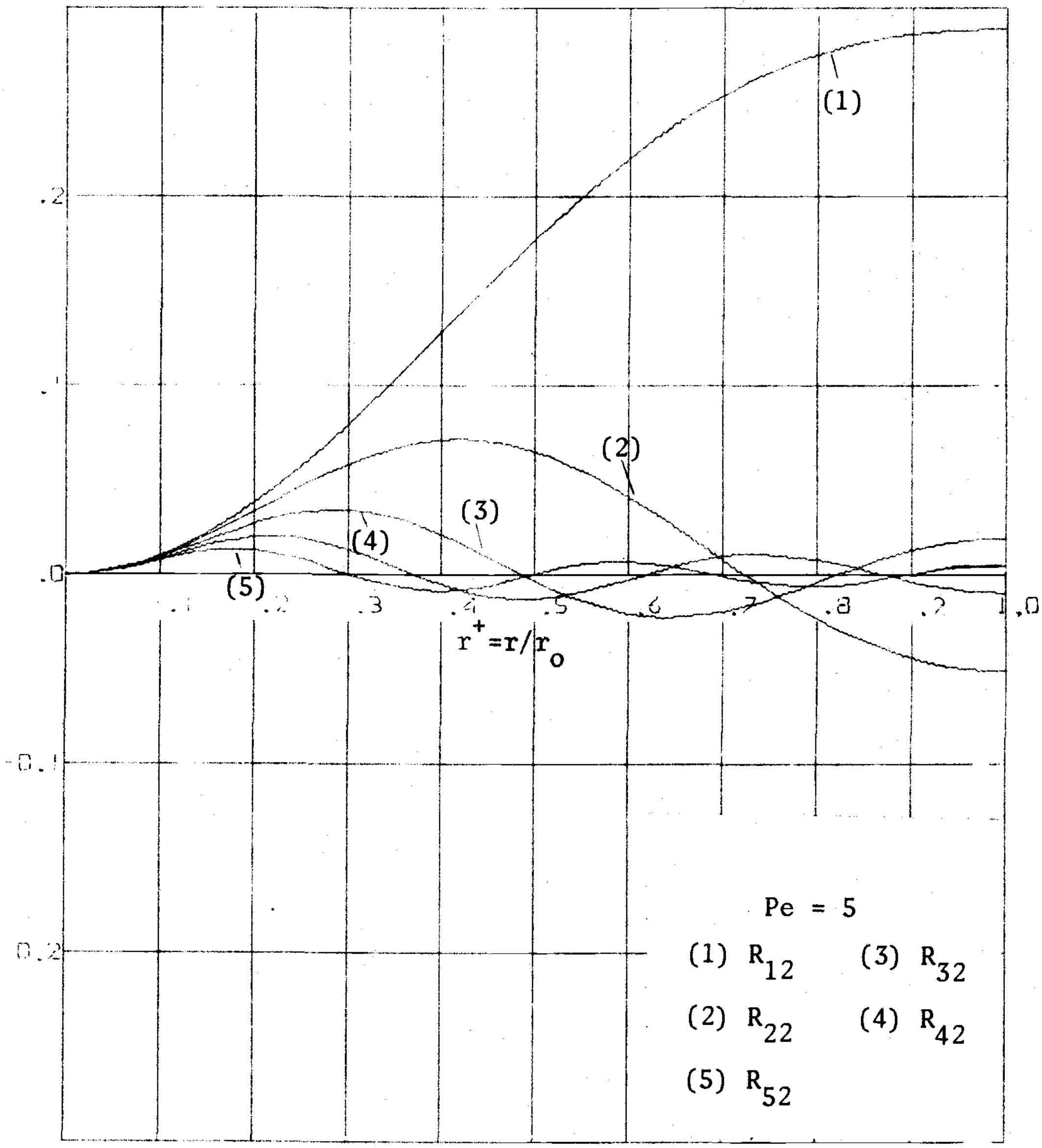

Figure B.1. Continued.

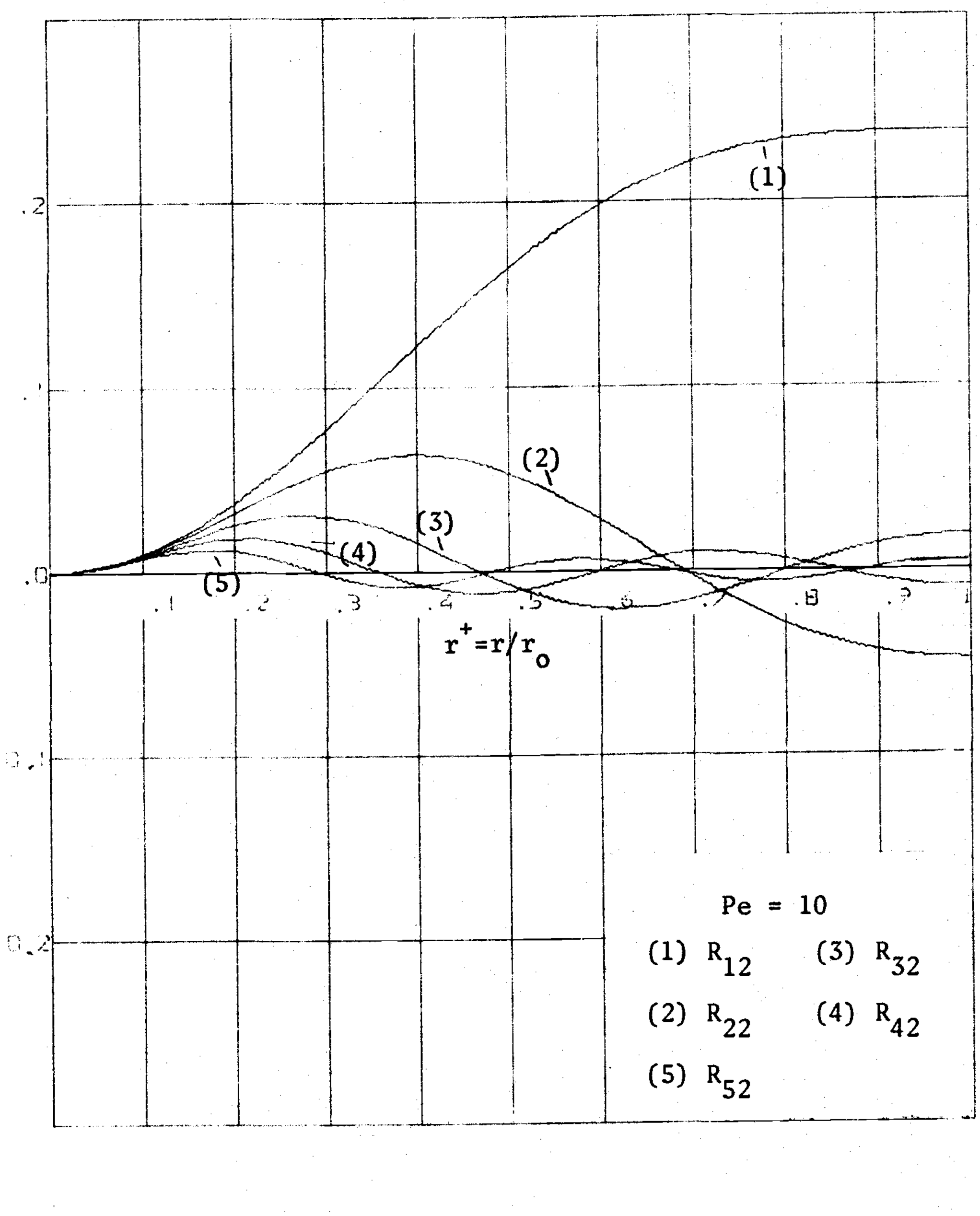

Figure B. 1. Continued.

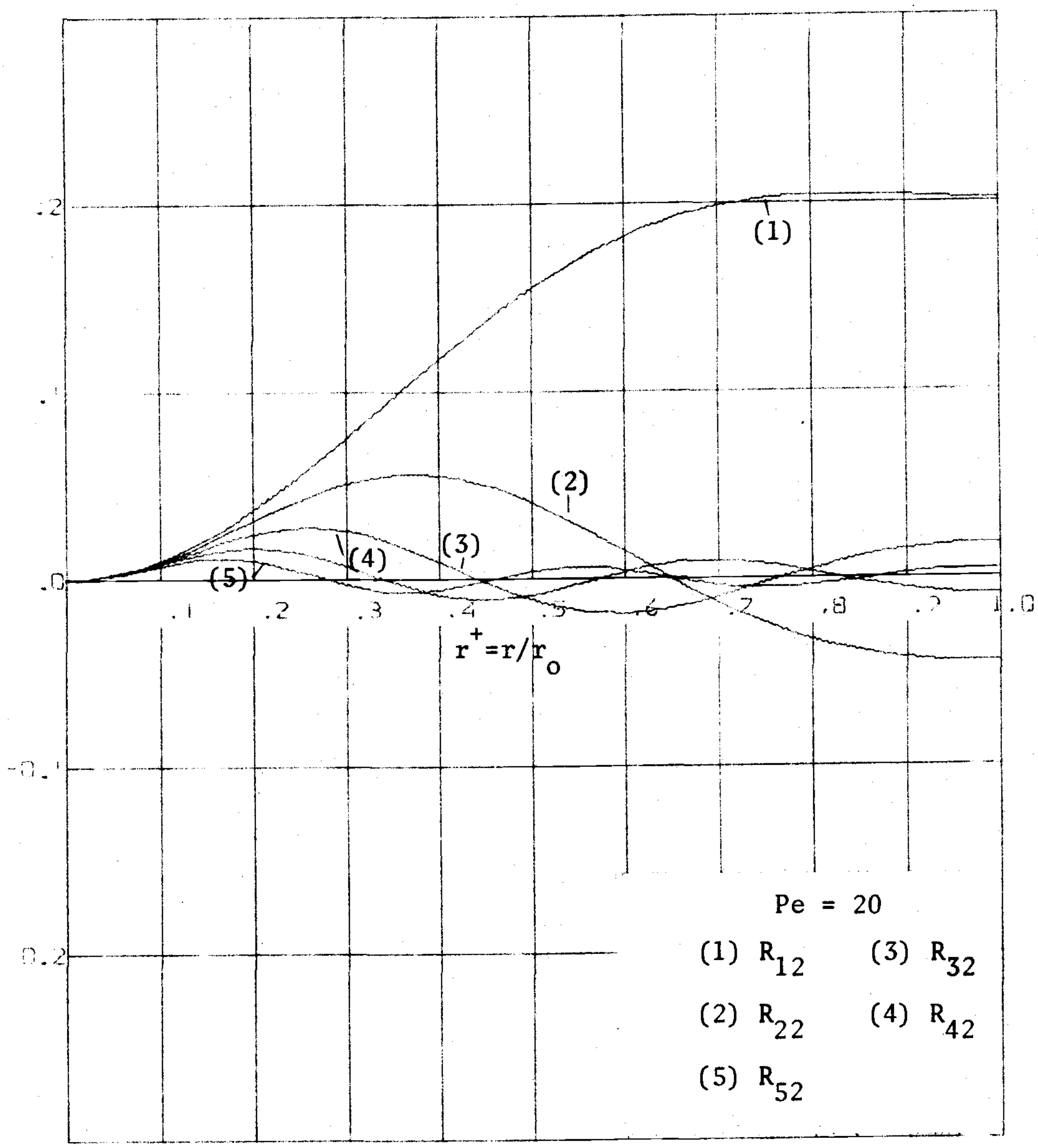

Figure B.1. Continued.

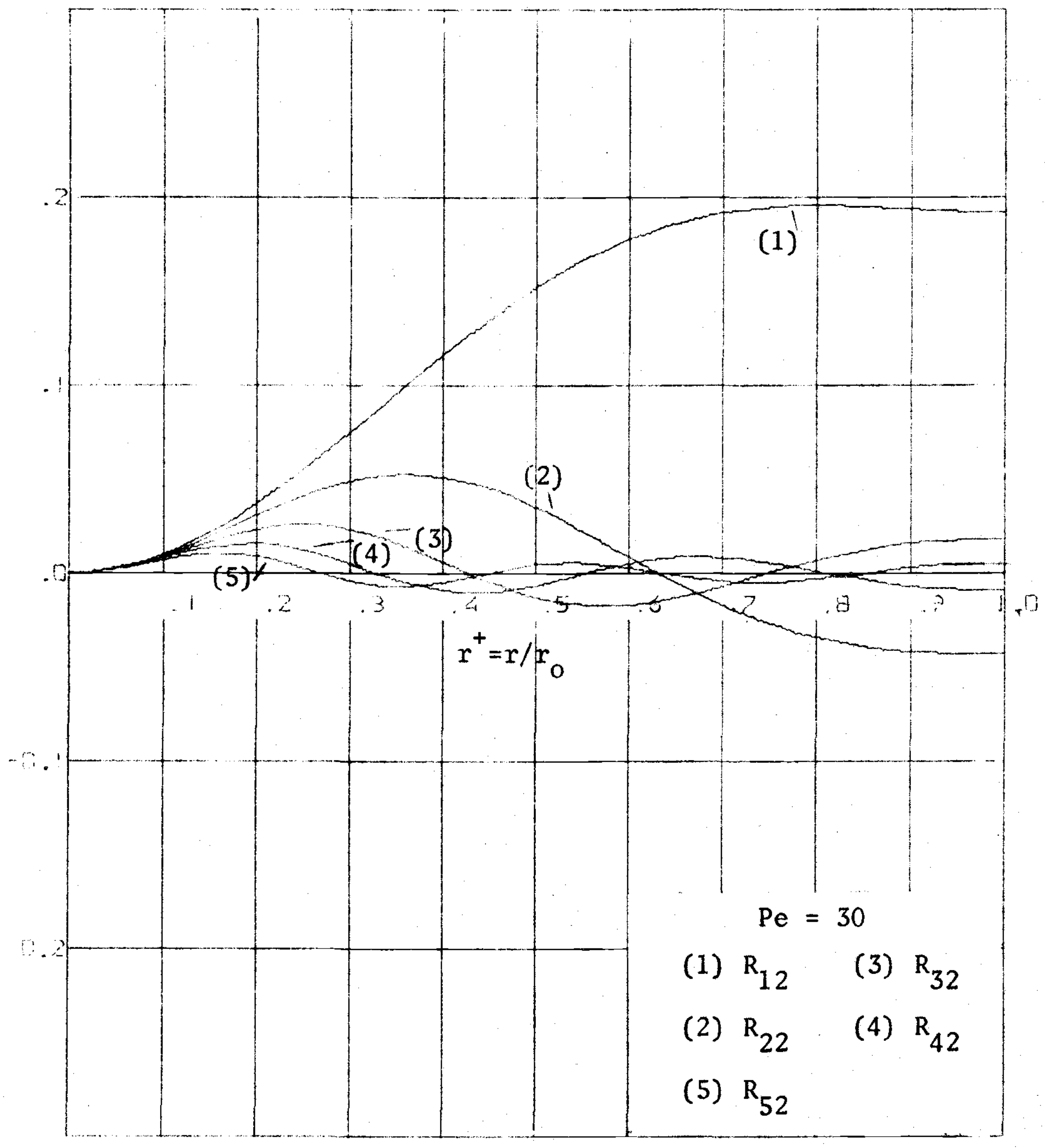

Figure B. 1. Continued.

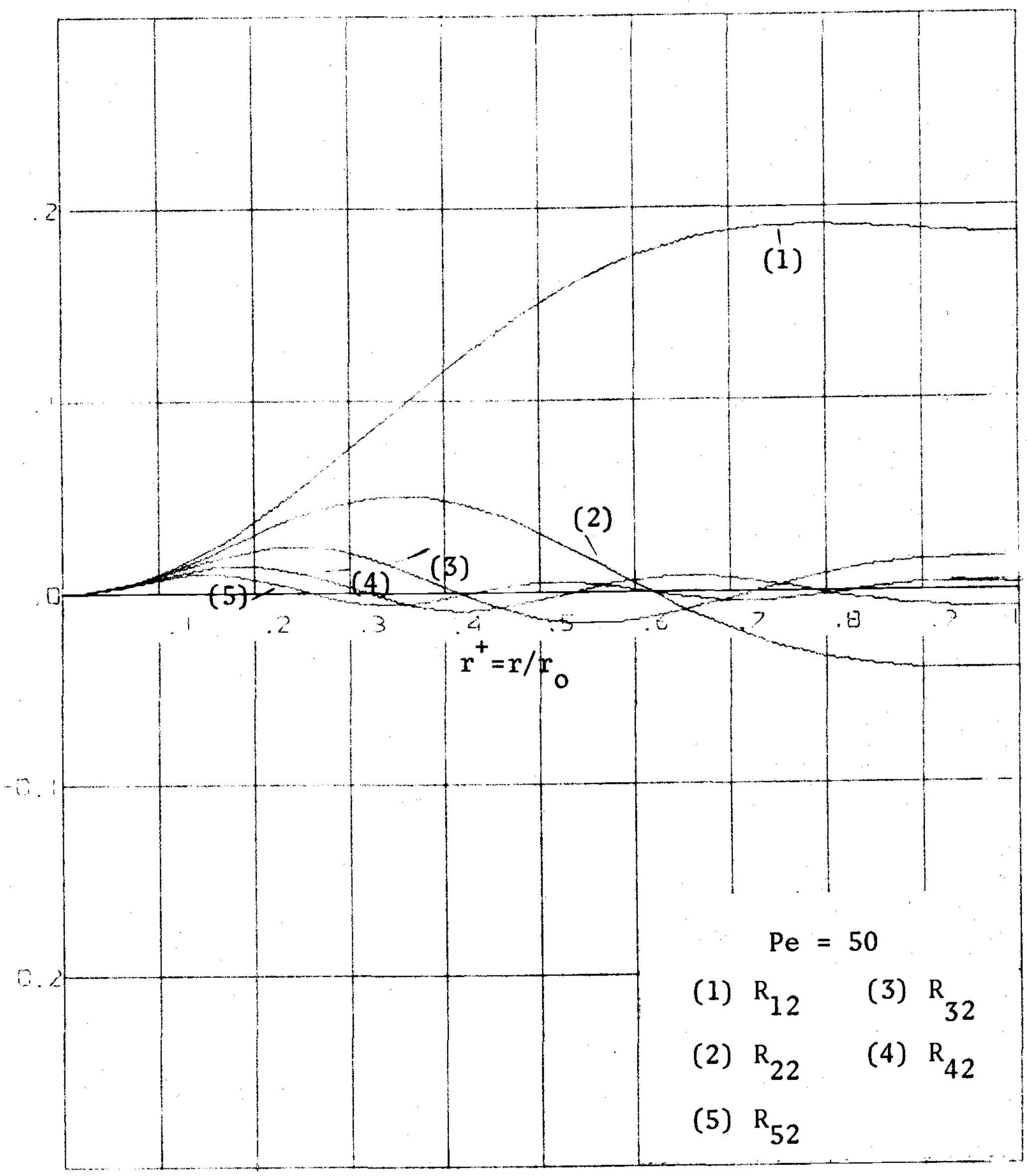

Figure B. 1. Continued.

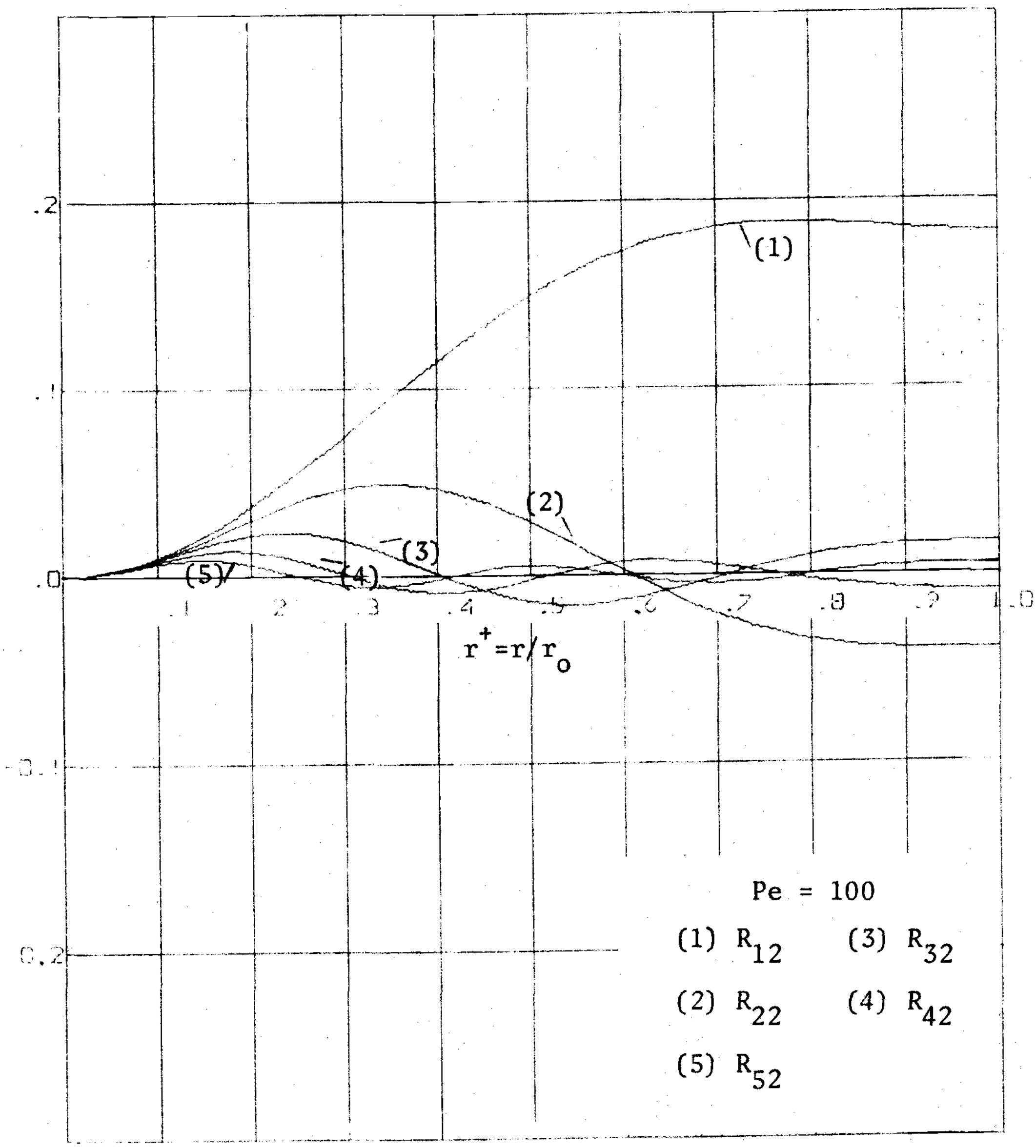

Figure B. 1. Continued.

Table C.1. The first 12 eigenvalues, eigenfunctions and expansion coefficients for p = 1, 2, 3, 4, 5 and for several values of the non-Newtonian behavior index.

(1) s = 2 , p = 0

n	λ_{n0}	$R_{n0}(1)$	$\hat{a}_{n0}$
1	5.0675055	-.4925166	.4034832
2	9.1576064	.3955085	-.1751100
3	13.1572247	-.3458737	.1055917
4	17.2202294	.3140465	-.0732824
5	21.2355173	-.2912515	.0550365
6	25.2465312	.2738070	-.0434844
7	29.2549055	-.2598530	.0355951
8	33.2615237	.2483320	-.0299084
9	37.2659032	-.2385904	.0256401
10	41.2713893	.2301993	-.0223336
11	45.2751868	-.2228631	.0197069
12	49.2784682	.2163688	-.0175762

r^+	R_{10}	R_{20}	R_{30}	R_{40}	R_{50}	R_{60}	R_{70}	R_{80}	R_{90}	R_{100}	R_{110}	R_{120}
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.05	.9840	.9483	.8941	.8232	.7376	.6398	.5328	.4196	.3035	.1877	.0756	-.0300
.10	.9370	.8015	.6106	.3864	.1546	-.0592	-.2328	-.3496	-.4010	-.3871	-.3162	-.2037
.15	.8614	.5831	.2387	-.0859	-.3149	-.4036	-.3489	-.1874	.0171	.1951	.2919	.2833
.20	.7614	.3267	-.1101	-.3711	-.3760	-.1728	.0967	.2771	.2799	.1229	-.0899	-.2330
.25	.6422	.0700	-.3414	-.3708	-.0916	.2151	.2959	.1155	-.1428	-.2522	-.1341	.0901
.30	.5098	-.1523	-.4080	-.1487	.2229	.2755	.0019	-.2387	-.1716	.0913	.2215	.0768
.35	.3703	-.3140	-.3197	.1276	.2959	-.0011	-.2527	-.0915	.1901	.1530	-.1150	-.1835
.40	.2301	-.4011	-.1324	.2944	.1072	-.2419	-.0958	.2089	.0895	-.1855	-.0857	.1677
.45	.0948	-.4124	.0776	.2781	-.1492	-.1829	.1852	.1006	-.1962	-.0277	.1869	-.0347
.50	-.0319	-.3581	.2412	.1148	-.2623	.0697	.1797	-.1751	-.0472	.1876	-.0782	-.1212
.55	-.1433	-.2560	.3167	-.0921	-.1679	.2293	-.0676	-.1374	.1879	-.0526	-.1211	.1624
.60	-.2399	-.1271	.2960	-.2386	.0380	.1537	-.2123	.1165	.0500	-.1643	.1515	-.0318
.65	-.3193	.0078	.1994	-.2696	.2048	-.0563	-.0972	.1845	-.1724	.0771	.0473	-.1379
.70	-.3816	.1322	.0623	-.1910	.2390	-.2077	.1177	-.0025	-.1005	.1612	-.1648	.1150
.75	-.4275	.2347	-.0792	-.0493	.1441	-.1979	.2086	-.1798	.1209	-.0452	-.0321	.0968
.80	-.4590	.3102	-.1980	.0995	-.0124	-.0614	.1191	-.1585	.1782	-.1787	.1616	-.1299
.85	-.4783	.3587	-.2804	.2155	-.1569	.1026	-.0521	.0058	.0357	-.0715	.1012	-.1242
.90	-.4883	.3845	-.3260	.2836	-.2487	.2179	-.1897	.1630	-.1376	.1132	-.0896	.0670
.95	-.4920	.3941	-.3434	.3102	-.2858	.2666	-.2506	.2370	-.2249	.2141	-.2041	.1949
1.00	-.4925	.3955	-.3459	.3140	-.2913	.2738	-.2599	.2483	-.2386	.2302	-.2229	.2164

Table C1. Continued

(2) s = 4 , p = 0

n	λ_{n0}	$R_{n0}(1)$	$\hat{a}_{n0}$
1	4.5555898	-.4593614	.3749484
2	8.2266127	.3681742	-.1629858
3	11.8557713	-.3215268	.0981749
4	15.4706322	.2916657	-.0680736
5	19.0787579	-.2703117	.0510894
6	22.6831188	.2539907	-.0403443
7	26.2851385	-.2409491	.0330107
8	29.8855920	.2301905	-.0277273
9	33.4849398	-.2211002	.0237634
10	37.0834748	.2132750	-.0206940
11	40.6813927	-.2064372	.0182562
12	44.2788297	.2003889	-.0162796

r^+	R_{10}	R_{20}	R_{30}	R_{40}	R_{50}	R_{60}	R_{70}	R_{80}	R_{90}	R_{100}	R_{110}	R_{120}
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.05	.9871	.9581	.9141	.8559	.7851	.7034	.6126	.5150	.4128	.3085	.2043	.1026
.10	.9488	.8378	.6783	.4854	.2772	.0727	-.1101	-.2561	-.3545	-.3997	-.3918	-.3365
.15	.8866	.6541	.3526	.0445	-.2097	-.3647	-.4003	-.3245	-.1702	.0143	.1778	.2790
.20	.8030	.4296	.0177	-.2903	-.4028	-.3121	-.0909	.1441	.2848	.2769	.1388	-.0535
.25	.7012	.1915	-.2475	-.4028	-.2500	.0498	.2702	.2706	.0778	-.1512	-.2499	-.1594
.30	.5851	-.0320	-.3875	-.2866	.0664	.2935	.1991	-.0803	-.2478	-.1498	.0899	.2184
.35	.4592	-.2161	-.3846	-.0386	.2827	.1887	-.1343	-.2413	-.0207	.2046	.1389	-.1053
.40	.3284	-.3424	-.2612	.1967	.2506	-.1033	-.2388	.0308	.2193	.0280	-.1918	-.0751
.45	.1973	-.4014	-.0704	.3015	.0278	-.2517	-.0024	.2196	-.0159	-.1959	.0303	.1769
.50	.0708	-.3934	.1223	.2391	-.1935	-.1193	.2136	.0168	-.1962	.0656	.1510	-.1224
.55	-.0469	-.3274	.2596	.0604	-.2479	.1254	.1221	-.1969	.0337	.1525	-.1413	-.0366
.60	-.1523	-.2196	.3079	-.1333	-.1145	.2230	-.1237	-.0709	.1809	-.1191	-.0434	.1532
.65	-.2427	-.0897	.2643	-.2484	.0908	.0946	-.1947	.1604	-.0287	-.1079	.1628	-.1081
.70	-.3165	.0427	.1521	-.2426	.2209	-.1143	-.0238	.1344	-.1757	.1380	-.0444	-.0604
.75	-.3733	.1609	.0088	-.1344	.2037	-.2119	.1661	-.0838	-.0108	.0931	-.1434	.1515
.80	-.4137	.2540	-.1279	.0192	.0704	-.1362	.1747	-.1847	.1681	-.1297	.0765	-.0169
.85	-.4395	.3173	-.2319	.1586	-.0921	.0315	.0223	-.0682	.1050	-.1318	.1480	-.1537
.90	-.4533	.3525	-.2933	.2486	-.2105	.1760	-.1438	.1132	-.0841	.0563	-.0300	.0053
.95	-.4586	.3662	-.3179	.2860	-.2624	.2435	-.2276	.2137	-.2013	.1899	-.1794	.1694
1.00	-.4594	.3682	-.3215	.2917	-.2703	.2540	-.2409	.2302	-.2211	.2133	-.2064	.2004

Table C1. Continued

(3) s = 6 , p = 0

n	λ_{n0}	$R_{n0}'(1)$	$\hat{a}_{n0}$
1	4.3570857	-.4399761	.3622953
2	7.8856682	.3519216	-.1560682
3	11.3699979	-.3068454	.0937854
4	14.8399911	.2780651	-.0649429
5	18.3032563	-.2575228	.0486964
6	21.7627592	.2418446	-.0384298
7	25.2199189	-.2293306	.0314287
8	28.6755081	.2190168	-.0263883
9	32.1299868	-.2103090	.0226066
10	35.5836478	.2028178	-.0196830
11	39.0366869	-.1962756	.0173605
12	42.4892407	.1904915	-.0154778

r^+	R_{10}	R_{20}	R_{30}	R_{40}	R_{50}	R_{60}	R_{70}	R_{80}	R_{90}	R_{100}	R_{110}	R_{120}
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.05	.9882	.9615	.9208	.8670	.8013	.7252	.6403	.5485	.4517	.3521	.2517	.1527
.10	.9531	.8505	.7020	.5207	.3223	.1236	-.0593	-.2119	-.3236	-.3874	-.4017	-.3694
.15	.8960	.6796	.3948	.0960	-.1622	-.3362	-.4023	-.3608	-.2346	-.0628	.1087	.2384
.20	.8190	.4684	.0696	-.2490	-.3968	-.3515	-.1631	.0712	.2479	.2986	.2139	.0425
.25	.7247	.2404	-.2021	-.3998	-.3023	-.0272	.2248	.2966	.1640	-.0642	-.2283	-.2254
.30	.6164	.0205	-.3663	-.3314	-.0100	.2644	.2570	.0140	-.2142	-.2195	-.0189	.1824
.35	.4979	-.1684	-.3985	-.1125	.2444	.2491	-.0461	-.2474	-.1239	.1389	.2062	.0127
.40	.3733	-.3078	-.3095	.1323	.2868	-.0133	-.2489	-.0752	.1940	.1366	-.1270	-.1709
.45	.2468	-.3866	-.1396	.2830	.1140	-.2315	-.1030	.1989	.0972	-.1755	-.0939	.1575
.50	.1227	-.4012	.0537	.2779	-.1244	-.1934	.1627	.1209	-.1796	-.0558	.1794	-.0017
.55	.0050	-.3567	.2128	.1362	-.2491	.0361	.1908	-.1466	-.0803	.1805	-.0373	-.1424
.60	-.1027	-.2651	.2959	-.0602	-.1818	.2101	-.0316	-.1550	.1674	-.0120	-.1410	.1393
.65	-.1973	-.1436	.2871	-.2122	.0108	.1635	-.1970	.0853	.0755	-.1649	.1251	.0036
.70	-.2767	-.0113	.1983	-.2536	.1825	-.0376	-.1043	.1771	-.1540	.0562	.0609	-.1373
.75	-.3394	.1135	.0622	-.1781	.2209	-.1920	.1096	-.0041	-.0909	.1477	-.1527	.1087
.80	-.3855	.2165	-.0813	-.0318	.1178	-.1711	.1887	-.1725	.1288	-.0672	-.0005	.0624
.85	-.4157	.2898	-.1984	.1191	-.0477	-.0153	.0684	-.1099	.1385	-.1536	.1553	-.1446
.90	-.4324	.3322	-.2716	.2245	-.1836	.1461	-.1108	.0775	-.0460	.0166	.0107	-.0354
.95	-.4390	.3493	-.3021	.2708	-.2473	.2284	-.2122	.1980	-.1851	.1731	-.1619	.1513
1.00	-.4400	.3519	-.3068	.2781	-.2575	.2418	-.2293	.2190	-.2103	.2028	-.1963	.1905

Table C1. Continued

(g) $s = \delta$, $p = 0$

n	λ_{n0}	$R_{n0}(1)$	$\hat{a}_{n0}$
1	4.2462674	-.4286117	.3552787
2	7.7052579	.3407885	-.1514470
3	11.1160782	-.2966706	.0907759
4	14.5120744	.2685743	-.0627721
5	17.9011609	-.2485590	.0470264
6	21.2863862	.2333047	-.0370879
7	24.6692057	-.2211427	.0303166
8	28.0504112	.2111279	-.0254447
9	31.4304745	-.2026789	.0217933
10	34.8096958	.1954148	-.0189683
11	38.1882760	-.1890744	.0167264
12	41.5663555	.1834713	-.0149096

r^+	R_{10}	R_{20}	R_{30}	R_{40}	R_{50}	R_{60}	R_{70}	R_{80}	R_{90}	R_{100}	R_{110}	R_{120}
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.05	.9888	.9632	.9242	.8726	.8095	.7362	.6543	.5655	.4716	.3746	.2763	.1790
.10	.9554	.8570	.7141	.5389	.3457	.1504	-.0318	-.1871	-.3047	-.3774	-.4027	-.3822
.15	.9011	.6929	.4168	.1233	-.1359	-.3183	-.3993	-.3755	-.2652	-.1029	.0690	.2097
.20	.8277	.4888	.0975	-.2252	-.3898	-.3681	-.1990	.0303	.2209	.2997	.2454	.0925
.25	.7375	.2665	-.1764	-.3943	-.3262	-.0683	.1940	.3001	.2035	-.0132	-.2013	-.2438
.30	.6336	.0493	-.3517	-.3513	-.0515	.2407	.2785	.0651	-.1820	-.2408	-.0776	.1418
.35	.5195	-.1411	-.4021	-.1513	.2165	.2728	.0053	-.2339	-.1711	.0873	.2182	.0782
.40	.3989	-.2866	-.3326	.0932	.2970	.0380	-.2380	-.1289	.1584	.1808	-.0689	-.1941
.45	.2756	-.3755	-.1770	.2647	.1586	-.2059	-.1528	.1658	.1506	-.1348	-.1494	.1091
.50	.1536	-.4028	.0126	.2919	-.0777	-.2235	.1171	.1682	-.1419	-.1189	.1557	.0735
.55	.0366	-.3715	.1806	.1769	-.2363	-.0207	.2126	-.0958	-.1369	.1599	.0376	-.1668
.60	-.0717	-.2913	.2824	-.0128	-.2131	.1849	.0295	-.1856	.1285	.0592	-.1663	.0859
.65	-.1683	-.1771	.2958	-.1813	-.0408	.1942	-.1769	.0246	.1297	-.1659	.0669	.0772
.70	-.2507	-.0471	.2251	-.2523	.1478	.0156	-.1469	.1822	-.1140	-.0101	.1172	-.1498
.75	-.3172	.0802	.0975	-.2026	.2229	-.1666	.0621	.0505	-.1329	.1601	-.1279	.0528
.80	-.3670	.1893	-.0479	-.0666	.1471	-.1875	.1869	-.1503	.0884	-.0152	-.0544	.1072
.85	-.4006	.2696	-.1733	.0895	-.0152	-.0483	.0988	-.1345	.1542	-.1580	.1468	-.1229
.90	-.4197	.3177	-.2554	.2061	-.1627	.1227	-.0852	.0500	-.0172	-.0128	.0399	-.0635
.95	-.4274	.3377	-.2910	.2599	-.2363	.2171	-.2007	.1860	-.1726	.1601	-.1482	.1369
1.00	-.4286	.3408	-.2967	.2686	-.2486	.2333	-.2211	.2111	-.2027	.1954	-.1891	.1835

Table C1. Continued

(5) s = 10 , p = 0

n	λ_{n0}	$R_{n0}(1)$	a_{n0}
1	4.1743399	-.4215891	.3509310
2	7.5912699	.3326404	-.1481792
3	10.9583389	-.2890687	.0885513
4	14.3096672	.2614332	-.0611481
5	17.6538079	-.2417851	.0457690
6	20.9939474	.2268320	-.0360736
7	24.3315957	-.2149231	.0294734
8	27.6675726	.2051253	-.0247278
9	31.0023662	-.1968652	.0211729
10	34.3362869	.1897678	-.0184237
11	37.6695427	-.1835761	.0162427
12	41.0022787	.1781068	-.0144758

r^+	R_{10}	R_{20}	R_{30}	R_{40}	R_{50}	R_{60}	R_{70}	R_{80}	R_{90}	R_{100}	R_{110}	R_{120}
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.05	.9891	.9643	.9263	.8761	.8145	.7429	.6628	.5758	.4837	.3883	.2914	.1951
.10	.9569	.8610	.7216	.5500	.3601	.1670	-.0146	-.1712	-.2921	-.3701	-.4019	-.3886
.15	.9044	.7012	.4304	.1403	-.1191	-.3064	-.3960	-.3830	-.2829	-.1272	.0437	.1899
.20	.8332	.5016	.1150	-.2098	-.3842	-.3767	-.2203	.0046	.2019	.2968	.2616	.1221
.25	.7457	.2831	-.1597	-.3895	-.3395	-.0936	.1729	.2986	.2251	.0190	-.1799	-.2490
.30	.6447	.0678	-.3415	-.3622	-.0771	.2234	.2883	.0959	-.1578	-.2477	-.1119	.1109
.35	.5335	-.1232	-.4028	-.1748	.1968	.2839	.0376	-.2199	-.1957	.0516	.2162	.1155
.40	.4154	-.2722	-.3456	.0676	.2998	.0698	-.2255	-.1590	.1297	.2004	-.0280	-.1960
.45	.2943	-.3671	-.1999	.2502	.1845	-.1851	-.1800	.1385	.1774	-.1009	-.1744	.0689
.50	.1739	-.4025	-.0140	.2973	-.0462	-.2370	.0836	.1910	-.1093	-.1513	.1271	.1151
.55	.0578	-.3800	.1580	.2010	-.2229	-.0572	.2181	-.0575	-.1649	.1333	.0834	-.1635
.60	-.0506	-.3078	.2709	.0190	-.2289	.1618	.0689	-.1952	.0930	.1016	-.1651	.0387
.65	-.1483	-.1995	.2991	-.1574	-.0748	.2083	-.1546	-.0183	.1566	-.1505	.0191	.1173
.70	-.2326	-.0723	.2420	-.2475	.1200	.0522	-.1697	.1745	-.0771	-.0557	.1433	-.1382
.75	-.3016	.0559	.1222	-.2175	.2195	-.1431	.0254	.0868	-.1536	.1551	-.0967	.0059
.80	-.3542	.1686	-.0227	-.0916	.1661	-.1949	.1792	-.1273	.0538	.0241	-.0896	.1296
.85	-.3905	.2540	-.1536	.0663	.0097	-.0725	.1196	-.1492	.1605	-.1542	.1325	-.0987
.90	-.4115	.3065	-.2425	.1913	-.1458	.1038	-.0646	.0281	.0053	-.0353	.0615	-.0834
.95	-.4202	.3290	-.2825	.2514	-.2277	.2082	-.1914	.1763	-.1623	.1493	-.1368	.1249
1.00	-.4216	.3326	-.2891	.2614	-.2418	.2268	-.2149	.2051	-.1969	.1898	-.1836	.1781

Table C1. Continued

(6) $s = 12,\ p = 0$

n	λ_{n0}	$R_{n0}(1)$	$\hat{a}_{n0}$
1	4.1236216	-.4170117	.3480447
2	7.5113919	.3265210	-.1457841
3	10.8497602	-.2831251	.0868325
4	14.1714016	.2557998	-.0598742
5	17.4854994	-.2364169	.0447761
6	20.7954319	.2216871	-.0352694
7	24.1027767	-.2099687	.0288031
8	27.4083566	.2003358	-.0241567
9	30.7127682	-.1922203	.0206778
10	34.0162434	.1852513	-.0179885
11	37.3190282	-.1791745	.0158558
12	40.6212730	.1738091	-.0141285

r^+	R_{10}	R_{20}	R_{30}	R_{40}	R_{50}	R_{60}	R_{70}	R_{80}	R_{90}	R_{100}	R_{110}	R_{120}
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.05	.9894	.9650	.9278	.8784	.8178	.7474	.6686	.5828	.4919	.3975	.3016	.2060
.10	.9579	.8638	.7267	.5575	.3699	.1783	-.0028	-.1602	-.2833	-.3646	-.4007	-.3923
.15	.9066	.7069	.4397	.1520	-.1075	-.2978	-.3932	-.3875	-.2944	-.1435	.0263	.1755
.20	.8371	.5106	.1271	-.1989	-.3797	-.3819	-.2343	-.0131	.1880	.2933	.2711	.1415
.25	.7514	.2947	-.1479	-.3856	-.3479	-.1107	.1577	.2961	.2385	.0408	-.1635	-.2497
.30	.6525	.0809	-.3339	-.3690	-.0946	.2106	.2933	.1164	-.1397	-.2496	-.1339	.0880
.35	.5432	-.1103	-.4025	-.1906	.1823	.2899	.0594	-.2080	-.2101	.0264	.2109	.1385
.40	.4271	-.2616	-.3539	.0497	.3000	.0912	-.2145	-.1776	.1079	.2097	.0007	-.1916
.45	.3076	-.3607	-.2153	.2391	.2012	-.1690	-.1964	.1173	.1923	-.0750	-.1864	.0386
.50	.1884	-.4016	-.0326	.2994	-.0240	-.2435	.0591	.2029	-.0840	-.1691	.1025	.1388
.55	.0729	-.3853	.1415	.2166	-.2113	-.0819	.2179	-.0294	-.1795	.1098	.1117	-.1527
.60	-.0355	-.3190	.2616	.0413	-.2375	.1430	.0951	-.1965	.0647	.1272	-.1559	.0030
.65	-.1337	-.2155	.3001	-.1391	-.0983	.2146	-.1351	-.0485	.1698	-.1325	-.0162	.1382
.70	-.2193	-.0908	.2533	-.2420	.0981	.0779	-.1821	.1633	-.0472	-.0862	.1539	-.1201
.75	-.2901	.0373	.1403	-.2269	.2143	-.1231	-.0026	.1112	-.1632	.1440	-.0679	-.0296
.80	-.3448	.1524	-.0032	-.1103	.1790	-.1977	.1697	-.1064	.0255	.0532	-.1121	.1393
.85	-.3831	.2415	-.1376	.0478	.0291	-.0908	.1343	-.1580	.1617	-.1468	.1167	-.0757
.90	-.4058	.2977	-.2320	.1789	-.1317	.0880	-.0475	.0102	.0234	-.0529	.0778	-.0978
.95	-.4154	.3224	-.2758	.2445	-.2206	.2008	-.1836	.1680	-.1536	.1400	-.1271	.1146
1.00	-.4170	.3265	-.2831	.2558	-.2364	.2217	-.2100	.2003	-.1922	.1853	-.1792	.1738

Table C1. Continued

(1) $s = 2$, $p = 1$

n	λ_{n1}	$R_{n1}(1)$	a_{n1}
1	2.8846257	.4465660	-1.5001310
2	7.1182769	-.1233654	.8985212
3	11.1789014	.0660488	-.7457699
4	15.2093411	-.0432979	.6623510
5	19.2281902	.0314396	-.6068484
6	23.2412108	-.0242920	.5661359
7	27.2508383	.0195713	-.5344534
8	31.2582968	-.0162501	.5087942
9	35.2642749	.0138029	-.4874053
10	39.2691926	-.0119344	.4691822
11	43.2733218	.0104673	-.4533880
12	47.2768468	-.0092889	.4395083

r^*	R_{11}	R_{21}	R_{31}	R_{41}	R_{51}	R_{61}	R_{71}	R_{81}	R_{91}	R_{101}	R_{111}	R_{121}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0499	.0492	.0481	.0465	.0444	.0420	.0393	.0362	.0329	.0295	.0259	.0223
.10	.0990	.0938	.0852	.0738	.0605	.0462	.0320	.0188	.0073	-.0018	-.0083	-.0120
.15	.1465	.1298	.1034	.0719	.0402	.0128	-.0069	-.0175	-.0197	-.0154	-.0076	.0006
.20	.1919	.1540	.0996	.0438	.0005	-.0226	-.0255	-.0149	-.0002	.0103	.0127	.0078
.25	.2344	.1648	.0761	.0037	-.0305	-.0277	-.0062	.0122	.0154	.0058	-.0058	-.0100
.30	.2736	.1620	.0399	-.0306	-.0345	-.0039	.0180	.0138	-.0035	-.0120	-.0054	.0055
.35	.3089	.1467	-.0000	-.0462	-.0136	.0202	.0145	-.0082	-.0126	.0015	.0098	.0023
.40	.3402	.1211	-.0346	-.0395	.0140	.0213	-.0077	-.0138	.0049	.0100	-.0033	-.0076
.45	.3672	.0883	-.0573	-.0168	.0291	.0017	-.0176	.0036	.0110	-.0056	-.0065	.0061
.50	.3899	.0515	-.0650	.0106	.0238	-.0175	-.0048	.0139	-.0045	-.0073	.0075	.0011
.55	.4084	.0141	-.0583	.0313	.0043	-.0197	.0127	.0025	-.0107	.0071	.0018	-.0069
.60	.4228	-.0211	-.0407	.0388	-.0161	-.0053	.0148	-.0117	.0021	.0060	-.0081	.0043
.65	.4335	-.0521	-.0171	.0326	-.0264	.0122	.0015	-.0096	.0108	-.0068	.0008	.0042
.70	.4409	-.0775	.0077	.0168	-.0232	.0201	-.0125	.0041	.0028	-.0069	.0079	-.0062
.75	.4454	-.0967	.0298	-.0026	-.0100	.0150	-.0153	.0128	-.0088	.0044	-.0003	-.0028
.80	.4477	-.1099	.0467	-.0203	.0062	.0019	-.0066	.0089	-.0096	.0092	-.0080	.0063
.85	.4483	-.1180	.0578	-.0331	.0197	-.0114	.0058	-.0019	-.0008	.0027	-.0040	.0048
.90	.4478	-.1220	.0637	-.0402	.0278	-.0202	.0150	-.0114	.0086	-.0064	.0047	-.0033
.95	.4470	-.1233	.0658	-.0429	.0310	-.0238	.0190	-.0156	.0131	-.0112	.0097	-.0085
1.00	.4466	-.1234	.0660	-.0433	.0314	-.0243	.0196	-.0163	.0138	-.0119	.0105	-.0093

Table C1. Continued

(2) s = 4 , p = 1

n	λ_{n1}	$R_{n1}(1)$	$\hat{a}_{n1}$
1	2.4180721	.4999121	-1.4253224
2	6.3385399	-.1292929	.7602588
3	10.0074665	.0686698	-.6258631
4	13.6383100	-.0448773	.5542000
5	17.2549635	.0325322	-.5069544
6	20.8645932	-.0251098	.4724688
7	24.4701770	.0202153	-.4457153
8	28.0731954	-.0167757	.4240947
9	31.6744740	.0142433	-.4061010
10	35.2745130	-.0123110	.3907895
11	38.8736344	.0107946	-.3775318
12	42.4720554	-.0095770	.3658909

r^+	R_{11}	R_{21}	R_{31}	R_{41}	R_{51}	R_{61}	R_{71}	R_{81}	R_{91}	R_{101}	R_{111}	R_{121}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0499	.0494	.0485	.0471	.0455	.0435	.0412	.0387	.0359	.0329	.0298	.0266
.10	.0993	.0951	.0880	.0785	.0671	.0546	.0417	.0290	.0173	.0071	-.0011	-.0073
.15	.1475	.1337	.1115	.0841	.0550	.0277	.0054	-.0101	-.0183	-.0196	-.0158	-.0089
.20	.1942	.1624	.1153	.0635	.0183	-.0124	-.0256	-.0237	-.0126	.0006	.0101	.0129
.25	.2388	.1794	.0993	.0258	-.0204	-.0330	-.0206	.0000	.0140	.0149	.0058	-.0049
.30	.2807	.1835	.0678	-.0146	-.0397	-.0211	.0076	.0194	.0103	-.0054	-.0120	-.0059
.35	.3197	.1749	.0276	-.0430	-.0314	.0077	.0224	.0064	-.0119	-.0108	.0029	.0098
.40	.3552	.1548	-.0129	-.0507	-.0046	.0258	.0083	-.0146	-.0090	.0083	.0086	-.0042
.45	.3870	.1251	-.0459	-.0373	.0220	.0184	-.0141	-.0109	.0103	.0070	-.0081	-.0048
.50	.4149	.0887	-.0657	-.0106	.0321	-.0047	-.0174	.0090	.0086	-.0095	-.0028	.0083
.55	.4386	.0487	-.0698	.0178	.0213	-.0215	.0002	.0136	-.0088	-.0040	.0091	-.0030
.60	.4582	.0084	-.0592	.0370	-.0014	-.0184	.0162	-.0022	-.0091	.0097	-.0023	-.0053
.65	.4737	-.0294	-.0377	.0410	-.0216	-.0001	.0129	-.0139	.0066	.0024	-.0076	.0070
.70	.4853	-.0623	-.0106	.0303	-.0284	.0171	-.0040	-.0059	.0104	-.0094	.0050	.0004
.75	.4933	-.0886	.0166	.0104	-.0201	.0206	-.0160	.0093	-.0024	-.0030	.0063	-.0072
.80	.4981	-.1079	.0395	-.0114	-.0028	.0100	-.0129	.0130	-.0114	.0086	-.0054	.0023
.85	.5004	-.1202	.0556	-.0291	.0148	-.0061	.0005	.0032	-.0055	.0067	-.0072	.0072
.90	.5008	-.1267	.0647	-.0399	.0268	-.0187	.0132	-.0092	.0063	-.0040	.0023	-.0009
.95	.5003	-.1290	.0682	-.0443	.0318	-.0243	.0193	-.0157	.0131	-.0111	.0095	-.0082
1.00	.4999	-.1293	.0687	-.0449	.0325	-.0251	.0202	-.0168	.0142	-.0123	.0108	-.0096

Table C1. Continued

(3) $s = 6$, $p = 1$

n	λ_{n1}	$R_{n1}(1)$	$\hat{a}_{n1}$
1	2.2412505	.5296097	-1.3927416
2	6.0574451	-.1294868	.7004967
3	9.5858482	.0684695	-.5744476
4	13.0739614	-.0446537	.5077100
5	16.5470177	.0323305	-.4639007
6	20.0126240	-.0249335	.4320100
7	23.4739405	.0200613	-.4073165
8	26.9325373	-.0166402	.3873889
9	30.3892901	.0141230	-.3708226
10	33.8447290	-.0122034	.3567384
11	37.2991951	.0106976	-.3445525
12	40.7529190	-.0094889	.3338594

r^+	R_{11}	R_{21}	R_{31}	R_{41}	R_{51}	R_{61}	R_{71}	R_{81}	R_{91}	R_{101}	R_{111}	R_{121}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0499	.0494	.0486	.0474	.0458	.0440	.0419	.0395	.0369	.0341	.0312	.0282
.10	.0994	.0955	.0889	.0801	.0695	.0576	.0452	.0329	.0213	.0110	.0022	-.0046
.15	.1479	.1350	.1144	.0886	.0606	.0338	.0110	-.0060	-.0162	-.0198	-.0180	-.0124
.20	.1950	.1655	.1212	.0714	.0262	-.0067	-.0237	-.0257	-.0172	-.0044	.0067	.0124
.25	.2403	.1849	.1087	.0357	-.0141	-.0328	-.0256	-.0063	.0104	.0161	.0105	-.0002
.30	.2834	.1921	.0801	-.0055	-.0391	-.0276	.0007	.0182	.0151	.0003	-.0109	-.0100
.35	.3238	.1869	.0413	-.0383	-.0378	-.0004	.0221	.0131	-.0072	-.0137	-.0032	.0084
.40	.3612	.1702	-.0002	-.0527	-.0145	.0235	.0157	-.0101	-.0139	.0027	.0111	.0017
.45	.3952	.1433	-.0367	-.0457	.0146	.0245	-.0078	-.0160	.0048	.0115	-.0031	-.0088
.50	.4256	.1087	-.0620	-.0222	.0319	.0045	-.0199	.0021	.0130	-.0048	-.0083	.0059
.55	.4522	.0692	-.0724	.0074	.0285	-.0176	-.0081	.0154	-.0027	-.0095	.0072	.0032
.60	.4748	.0277	-.0672	.0317	.0084	-.0227	.0120	.0053	-.0123	.0062	.0040	-.0080
.65	.4932	-.0125	-.0490	.0423	-.0151	-.0084	.0171	-.0117	.0004	.0079	-.0087	.0033
.70	.5076	-.0488	-.0226	.0368	-.0283	.0119	.0030	-.0112	.0117	-.0067	-.0001	.0052
.75	.5181	-.0791	.0063	.0188	-.0250	.0215	-.0134	.0045	.0029	-.0073	.0084	-.0067
.80	.5250	-.1021	.0323	-.0042	-.0091	.0147	-.0157	.0137	-.0102	.0059	-.0019	-.0016
.85	.5288	-.1175	.0517	-.0248	.0103	-.0017	-.0035	.0066	-.0082	.0087	-.0084	.0074
.90	.5301	-.1259	.0632	-.0382	.0248	-.0166	.0110	-.0071	.0041	-.0019	.0002	.0011
.95	.5300	-.1291	.0678	-.0438	.0313	-.0238	.0188	-.0152	.0126	-.0106	.0090	-.0077
1.00	.5296	-.1295	.0685	-.0447	.0323	-.0249	.0201	-.0166	.0141	-.0122	.0107	-.0095

Table C1. Continued

(4) $s = 8$, $p = 1$

n	λ_{n1}	$R_{n1}(1)$	$\hat{a}_{n1}$
1	2.1475599	.5484797	-1.3750784
2	5.9062279	-.1287804	.6670072
3	9.3643434	.0678110	-.5443624
4	12.7801997	-.0441418	.4802841
5	16.1797613	.0319241	-.4383847
6	19.5715278	-.0246013	.4079573
7	22.9588095	.0197829	-.3844363
8	26.3432481	-.0164023	.3654788
9	29.7257587	.0139163	-.3497345
10	33.1068952	-.0120214	.3363600
11	36.4870140	.0105355	-.3247961
12	39.8663560	-.0093432	.3146545

r^+	R_{11}	R_{21}	R_{31}	R_{41}	R_{51}	R_{61}	R_{71}	R_{81}	R_{91}	R_{101}	R_{111}	R_{121}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0499	.0495	.0486	.0475	.0460	.0442	.0422	.0399	.0374	.0347	.0319	.0290
.10	.0994	.0957	.0894	.0809	.0707	.0592	.0471	.0350	.0235	.0131	.0041	-.0031
.15	.1481	.1358	.1159	.0909	.0636	.0371	.0140	-.0036	-.0148	-.0195	-.0188	-.0141
.20	.1954	.1671	.1242	.0755	.0305	-.0034	-.0222	-.0262	-.0193	-.0072	.0045	.0116
.25	.2411	.1878	.1137	.0412	-.0103	-.0321	-.0278	-.0098	.0079	.0160	.0126	.0026
.30	.2847	.1967	.0868	-.0001	-.0380	-.0307	-.0034	.0167	.0171	.0037	-.0092	-.0113
.35	.3259	.1935	.0491	-.0350	-.0406	-.0051	.0209	.0163	-.0038	-.0140	-.0064	.0063
.40	.3642	.1788	.0075	-.0529	-.0200	.0211	.0192	-.0066	-.0154	-.0011	.0109	.0050
.45	.3995	.1538	-.0306	-.0498	.0095	.0269	-.0034	-.0175	.0008	.0125	.0006	-.0094
.50	.4313	.1207	-.0588	-.0289	.0305	.0099	-.0198	-.0026	.0139	-.0009	-.0101	.0028
.55	.4595	.0820	-.0728	.0005	.0317	-.0138	-.0126	.0146	.0017	-.0112	.0042	.0064
.60	.4839	.0405	-.0713	.0272	.0145	-.0239	.0080	.0096	-.0125	.0025	.0073	-.0075
.65	.5043	-.0006	-.0558	.0418	-.0098	-.0134	.0181	-.0086	-.0040	.0100	-.0072	-.0003
.70	.5206	-.0387	-.0306	.0401	-.0267	.0074	.0076	-.0135	.0108	-.0034	-.0038	.0072
.75	.5330	-.0713	-.0013	.0243	-.0275	.0209	-.0105	.0005	.0065	-.0093	.0083	-.0048
.80	.5414	-.0968	.0265	.0012	-.0134	.0175	-.0167	.0132	-.0083	.0032	.0011	-.0042
.85	.5465	-.1143	.0481	-.0210	.0067	.0016	-.0064	.0089	-.0098	.0095	-.0085	.0069
.90	.5486	-.1243	.0614	-.0363	.0230	-.0147	.0091	-.0052	.0023	-.0002	-.0014	.0026
.95	.5488	-.1283	.0670	-.0431	.0307	-.0232	.0182	-.0147	.0121	-.0101	.0085	-.0072
1.00	.5485	-.1288	.0678	-.0441	.0319	-.0246	.0198	-.0164	.0139	-.0120	.0105	-.0093

Table C1. Continued

(5) s = 10 , p = 1

n	λ_{n1}	$R_{n1}(1)$	$\hat{a}_{n1}$
1	2.0894435	.5615120	-1.3642070
2	5.8091194	-.1281300	.6455864
3	9.2268665	.0671110	-.5241574
4	12.5984996	-.0436078	.4616547
5	15.9535869	.0315044	-.4209619
6	19.3005600	-.0242604	.3914787
7	22.6428706	.0194985	-.3687236
8	25.9822265	-.0161598	.3504050
9	29.3195785	.0137061	-.3352056
10	32.6555019	-.0118366	.3223036
11	35.9903670	.0103712	-.3111552
12	39.3244238	-.0091958	.3013835

r^+	R_{11}	R_{21}	R_{31}	R_{41}	R_{51}	R_{61}	R_{71}	R_{81}	R_{91}	R_{101}	R_{111}	R_{121}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0499	.0495	.0487	.0476	.0461	.0444	.0424	.0402	.0377	.0351	.0323	.0294
.10	.0995	.0958	.0897	.0814	.0714	.0601	.0482	.0363	.0248	.0144	.0053	-.0020
.15	.1482	.1362	.1168	.0923	.0654	.0391	.0160	-.0020	-.0138	-.0193	-.0192	-.0150
.20	.1957	.1681	.1261	.0781	.0332	-.0012	-.0211	-.0264	-.0205	-.0089	.0030	.0108
.25	.2416	.1896	.1168	.0447	-.0077	-.0314	-.0290	-.0119	.0062	.0156	.0136	.0043
.30	.2855	.1996	.0911	.0034	-.0371	-.0324	-.0060	.0154	.0180	.0057	-.0079	-.0118
.35	.3271	.1977	.0541	-.0326	-.0421	-.0082	.0197	.0180	-.0015	-.0137	-.0083	.0047
.40	.3661	.1844	.0125	-.0527	-.0234	.0192	.0210	-.0041	-.0159	-.0035	.0103	.0069
.45	.4021	.1607	-.0264	-.0520	.0061	.0280	-.0003	-.0180	-.0019	.0125	.0030	-.0091
.50	.4348	.1287	-.0563	-.0331	.0290	.0133	-.0191	-.0056	.0138	.0017	-.0104	.0004
.55	.4641	.0907	-.0726	-.0042	.0333	-.0108	-.0153	.0134	.0046	-.0115	.0018	.0079
.60	.4896	.0494	-.0736	.0237	.0183	-.0240	.0049	.0121	-.0117	-.0003	.0088	-.0062
.65	.5113	.0079	-.0603	.0408	-.0059	-.0165	.0181	-.0060	-.0067	.0106	-.0053	-.0028
.70	.5290	-.0312	-.0363	.0420	-.0250	.0039	.0106	-.0144	.0093	-.0007	-.0060	.0078
.75	.5426	-.0652	-.0070	.0282	-.0288	.0198	-.0079	-.0025	.0086	-.0100	.0074	-.0028
.80	.5524	-.0924	.0217	.0054	-.0165	.0192	-.0170	.0122	-.0064	.0010	.0032	-.0057
.85	.5584	-.1116	.0449	-.0179	.0037	.0042	-.0086	.0105	-.0107	.0098	-.0081	.0060
.90	.5612	-.1229	.0597	-.0346	.0213	-.0130	.0075	-.0036	.0008	.0012	-.0027	.0037
.95	.5618	-.1275	.0661	-.0424	.0300	-.0226	.0177	-.0142	.0116	-.0096	.0080	-.0067
1.00	.5615	-.1281	.0671	-.0436	.0315	-.0243	.0195	-.0162	.0137	-.0118	.0104	-.0092

Table C1. Continued

(6) s = 12 , p = 1

n	λ_{n1}	$R_{n1}(1)$	$\hat{a}_{n1}$
1	2.0498564	.5710448	-1.3569321
2	5.7405690	-.1276732	.6307846
3	9.1312520	.0664762	-.5095610
4	12.4740348	-.0431136	.4479616
5	15.7994611	.0311152	-.4080749
6	19.1164515	-.0239441	.3792461
7	22.4286088	.0192345	-.3570305
8	25.7377054	-.0159348	.3391668
9	29.0447267	.0135110	-.3243579
10	32.3502683	-.0116651	.3117966
11	35.6547134	.0102187	-.3009491
12	38.9583208	-.0090589	.2914459

r^+	R_{11}	R_{21}	R_{31}	R_{41}	R_{51}	R_{61}	R_{71}	R_{81}	R_{91}	R_{101}	R_{111}	R_{121}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0499	.0495	.0487	.0476	.0462	.0445	.0425	.0403	.0379	.0353	.0326	.0298
.10	.0995	.0959	.0899	.0818	.0719	.0608	.0490	.0371	.0257	.0153	.0062	-.0013
.15	.1482	.1365	.1175	.0933	.0666	.0405	.0173	-.0009	-.0131	-.0190	-.0194	-.0156
.20	.1958	.1688	.1274	.0799	.0351	.0004	-.0203	-.0265	-.0213	-.0100	.0019	.0103
.25	.2419	.1909	.1189	.0471	-.0059	-.0308	-.0298	-.0134	.0049	.0152	.0143	.0055
.30	.2860	.2017	.0940	.0058	-.0364	-.0335	-.0077	.0144	.0185	.0071	-.0068	-.0120
.35	.3280	.2007	.0576	-.0309	-.0430	-.0103	.0188	.0190	.0001	-.0134	-.0095	.0034
.40	.3673	.1883	.0161	-.0523	-.0257	.0178	.0222	-.0023	-.0160	-.0051	.0096	.0080
.45	.4038	.1656	-.0233	-.0534	.0035	.0285	.0018	-.0180	-.0038	.0122	.0046	-.0085
.50	.4372	.1344	-.0543	-.0359	.0277	.0156	-.0183	-.0076	.0134	.0036	-.0104	-.0013
.55	.4671	.0970	-.0723	-.0075	.0342	-.0086	-.0169	.0122	.0065	-.0113	-.0001	.0086
.60	.4935	.0560	-.0750	.0211	.0210	-.0237	.0026	.0135	-.0107	-.0023	.0095	-.0049
.65	.5161	.0143	-.0633	.0399	-.0030	-.0186	.0177	-.0038	-.0085	.0106	-.0037	-.0045
.70	.5347	-.0254	-.0404	.0431	-.0235	.0012	.0126	-.0147	.0079	.0013	-.0074	.0077
.75	.5494	-.0605	-.0114	.0309	-.0295	.0187	-.0057	-.0047	.0100	-.0100	.0063	-.0010
.80	.5601	-.0889	.0179	.0086	-.0188	.0203	-.0169	.0111	-.0047	-.0009	.0048	-.0067
.85	.5669	-.1094	.0423	-.0152	.0013	.0063	-.0101	.0115	-.0112	.0097	-.0075	.0050
.90	.5704	-.1217	.0582	-.0332	.0198	-.0116	.0061	-.0023	-.0004	.0024	-.0037	.0045
.95	.5713	-.1269	.0653	-.0417	.0295	-.0221	.0172	-.0137	.0111	-.0091	.0075	-.0062
1.00	.5710	-.1277	.0665	-.0431	.0311	-.0239	.0192	-.0159	.0135	-.0117	.0102	-.0091

Table C1. Continued

(1) s = 2 , p = 2

n	λ_{n2}	$R_{n2}(1)$	$\hat{a}_{n2}$
1	5.0675055	.1817437	-1.2951616
2	9.1576064	-.0396193	1.8356290
3	13.1972247	.0162605	-2.2988208
4	17.2202294	-.0085882	2.7163616
5	21.2355173	.0052132	-3.1023198
6	25.2465312	-.0034583	3.4645495
7	29.2549055	.0024404	-3.8080073
8	33.2615237	-.0018022	4.1360765
9	37.2663082	.0013783	-4.4511969
10	41.2713893	-.0010837	4.7552018
11	45.2751870	.0008715	-5.0495130
12	49.2784475	-.0007140	5.3352399

r^+	R_{12}	R_{22}	R_{32}	R_{42}	R_{52}	R_{62}	R_{72}	R_{82}	R_{92}	R_{102}	R_{112}	R_{122}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.00[illegible]
.05	.0025	.0025	.0024	.0023	.0023	.0022	.0021	.0020	.0019	.0017	.0016	.0015
.10	.0098	.0093	.0086	.0078	.0067	.0057	.0045	.0035	.0025	.0016	[illegible]	.0002
.15	.0214	.0192	.0160	.0124	.0087	.0053	.0025	.0004	-.0008	-.0013	-.0012	[illegible]
.20	.0367	.0301	.0215	.0128	.0055	.0006	-.0019	-.0023	-.0014	-.0003	.0006	.0008
.25	.0547	.0398	.0226	.0082	-.0006	-.0036	-.0026	-.0004	.0011	.0012	.0004	-.0004
.30	.0744	.0464	.0186	.0006	-.0052	-.0031	.0005	.0019	.0008	-.0006	-.0009	-.0002
.35	.0946	.0486	.0104	-.0061	-.0049	.0008	.0024	.0004	-.0012	-.0006	.0005	.0006
.40	.1143	.0460	.0005	-.0088	-.0009	.0033	.0006	-.0016	-.0004	.0009	.0003	-.0006
.45	.1327	.0388	-.0084	-.0068	.0034	.0020	-.0019	-.0006	.0012	.0000	-.0007	.0002
.50	.1489	.0281	-.0140	-.0016	.0047	-.0013	-.0015	.0014	.0002	-.0009	.0004	.0004
.55	.1624	.0153	-.0154	.0038	.0025	-.0030	.0008	.0009	-.0011	.0003	.0004	-.0005
.60	.1729	.0020	-.0128	.0071	-.0012	-.0017	.0020	-.0009	-.0002	.0008	-.0005	.0001
.65	.1804	-.0105	-.0073	.0072	-.0039	.0010	.0008	-.0013	.0010	-.0004	-.0002	.0004
.70	.1850	-.0211	-.0007	.0046	-.0042	.0027	-.0012	.0001	.0005	-.0007	.0006	-.0004
.75	.1872	-.0292	.0056	.0006	-.0023	.0024	-.0020	.0013	-.0007	.0002	.0001	-.0003
.80	.1873	-.0348	.0107	-.0034	.0005	.0006	-.0010	.0011	-.0010	.0008	-.0006	.0004
.85	.1861	-.0380	.0140	-.0063	.0030	-.0014	.0006	-.0001	-.0002	.0003	-.0004	.0004
.90	.1842	-.0394	.0157	-.0079	.0046	-.0028	.0018	-.0012	.0008	-.0006	.0004	-.0002
.95	.1825	-.0397	.0162	-.0085	.0051	-.0034	.0024	-.0017	.0013	-.0010	.0008	-.0006
1.00	.1817	-.0396	.0163	-.0086	.0052	-.0035	.0024	-.0018	.0014	-.0011	.0009	-.0007

Table C1. Continued

(3) s = 6 , p = 2

n	λ_{n2}	$R_{n2}(1)$	$\hat{a}_{n2}$
1	3.8485504	.2712883	-1.0563109
2	7.6607570	-.0495580	1.2576126
3	11.2200926	.0197894	-1.5392814
4	14.7268591	-.0103438	1.8033920
5	18.2122146	.0062452	-2.0506170
6	21.6865184	-.0041296	2.2839635
7	25.1543076	.0029077	-2.5058940
8	28.6179091	-.0021440	2.7182698
9	32.0786475	.0016378	-2.9225054
10	35.5373358	-.0012866	3.1196969
11	38.9945023	.0010338	-3.3107113
12	42.4505060	-.0008465	3.4962474

r^+	R_{12}	R_{22}	R_{32}	R_{42}	R_{52}	R_{62}	R_{72}	R_{82}	R_{92}	R_{102}	R_{112}	R_{122}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0025	.0025	.0024	.0024	.0023	.0023	.0022	.0021	.0020	.0019	.0018	.0017
.10	.0099	.0095	.0090	.0083	.0075	.0066	.0057	.0047	.0038	.0029	.0020	.0013
.15	.0219	.0201	.0176	.0146	.0114	.0082	.0052	.0028	.0009	-.0004	-.0011	-.0013
.20	.0381	.0327	.0256	.0179	.0106	.0046	.0005	-.0018	-.0023	-.0016	-.0009	.0001
.25	.0578	.0455	.0304	.0159	.0048	-.0016	-.0036	-.0028	-.0008	.0008	.0013	.0009
.30	.0804	.0564	.0301	.0090	-.0026	-.0052	-.0028	.0004	.0019	.0013	-.0001	-.0009
.35	.1050	.0638	.0243	-.0002	-.0072	-.0036	.0013	.0025	.0007	-.0010	-.0010	.0001
.40	.1307	.0663	.0141	-.0081	-.0065	.0011	.0032	.0004	-.0016	-.0007	.0008	.0007
.45	.1564	.0635	.0019	-.0115	-.0015	.0042	.0009	-.0021	-.0006	.0012	.0004	-.0007
.50	.1814	.0551	-.0094	-.0095	.0040	.0029	-.0023	-.0010	.0015	.0003	-.0010	.0001
.55	.2045	.0422	-.0173	-.0034	.0062	-.0011	-.0023	.0016	.0005	-.0012	.0003	.0006
.60	.2251	.0260	-.0201	.0038	.0039	-.0037	.0006	.0014	-.0013	.0001	.0007	-.0006
.65	.2424	.0084	-.0175	.0086	-.0009	-.0026	.0025	-.0009	-.0005	.0010	-.0007	.0000
.70	.2560	-.0089	-.0108	.0093	-.0047	.0009	.0012	-.0017	.0012	-.0004	-.0003	.0006
.75	.2656	-.0240	-.0018	.0060	-.0053	.0034	-.0015	.0001	.0007	-.0009	.0008	-.0005
.80	.2714	-.0358	.0070	.0005	-.0026	.0028	-.0024	.0017	-.0010	.0005	-.0000	-.0003
.85	.2738	-.0438	.0139	-.0049	.0014	.0001	-.0008	.0010	-.0011	.0010	-.0004	.0005
.90	.2737	-.0480	.0180	-.0086	.0046	-.0026	.0015	-.0008	.0004	-.0001	-.0003	.0001
.95	.2722	-.0495	.0196	-.0101	.0060	-.0039	.0027	-.0019	.0015	-.0011	.0009	-.0007
1.00	.2713	-.0496	.0198	-.0103	.0062	-.0041	.0029	-.0021	.0016	-.0013	.0010	-.0008

Table C1. Continued

(2) s = 4 , p = 2

n	λ_{n2}	$R_{n2}(1)$	$\hat{a}_{n2}$
1	4.1882454	.2365451	-1.1280862
2	8.0498635	-.0468768	1.4246992
3	11.7366594	.0189397	-1.7503160
4	15.3803883	-.0099443	2.0535675
5	19.0059996	.0060182	-2.3370427
6	22.6221252	-.0039852	2.6044892
7	26.2326147	.0028088	-2.8588089
8	29.8394623	-.0020726	3.1021619
9	33.4438108	.0015841	-3.3361797
10	37.0463651	-.0012449	3.5621229
11	40.6475846	.0010007	-3.7809870
12	44.2477824	-.0008196	3.9935740

r^+	R_{12}	R_{22}	R_{32}	R_{42}	R_{52}	R_{62}	R_{72}	R_{82}	R_{92}	R_{102}	R_{112}	R_{122}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0025	.0025	.0024	.0024	.0023	.0022	.0022	.0021	.0020	.0019	.0017	.0015
.10	.0099	.0095	.0089	.0082	.0073	.0064	.0054	.0044	.0034	.0025	.0017	[illegible]
.15	.0218	.0199	.0172	.0140	.0106	.0074	.0044	.0020	.0003	-.0008	-.0013	-.0013
.20	.0377	.0320	.0245	.0165	.0091	.0033	-.0004	-.0021	-.0022	-.0014	-.0004	.0005
.25	.0570	.0439	.0282	.0135	.0030	-.0025	-.0036	-.0021	-.0001	.0011	.0012	.0005
.30	.0787	.0536	.0265	.0061	-.0038	-.0049	-.0017	.0012	.0018	.0007	-.0016	-.0009
.35	.1020	.0593	.0197	-.0027	-.0070	-.0022	.0021	.0021	-.0001	-.0013	-.0006	.0005
.40	.1259	.0599	.0091	-.0091	-.0047	.0024	.0026	-.0006	-.0016	-.0000	.0010	.0002
.45	.1492	.0552	-.0024	-.0105	.0006	.0040	-.0004	-.0020	.0003	.0011	-.0002	-.0007
.50	.1711	.0455	-.0121	-.0069	.0050	.0014	-.0026	.0001	.0014	-.0005	-.0007	.0005
.55	.1908	.0320	-.0177	-.0003	.0053	-.0023	-.0012	.0018	-.0004	-.0008	.0007	.0001
.60	.2075	.0161	-.0182	.0058	.0019	-.0035	.0016	.0005	-.0013	.0007	.0002	-.0005
.65	.2208	-.0002	-.0140	.0089	-.0026	-.0012	.0022	-.0015	.0003	.0006	-.0004	.0005
.70	.2306	-.0153	-.0067	.0079	-.0051	.0020	.0001	-.0011	.0013	-.0008	.0002	.0002
.75	.2368	-.0278	.0017	.0039	-.0043	.0033	-.0020	.0008	.0000	-.0005	.0007	-.0006
.80	.2398	-.0371	.0092	-.0014	-.0012	.0020	-.0020	.0017	-.0012	.0008	-.0004	.0001
.85	.2403	-.0430	.0147	-.0059	.0023	-.0006	-.0002	.0006	-.0007	.0008	-.0007	.0006
.90	.2392	-.0460	.0177	-.0087	.0048	-.0028	.0017	-.0011	.0006	-.0003	.0002	-.0000
.95	.2375	-.0469	.0188	-.0098	.0059	-.0038	.0027	-.0019	.0015	-.0011	.0009	-.0007
1.00	.2365	-.0469	.0189	-.0099	.0060	-.0040	.0028	-.0021	.0016	-.0012	.0010	-.0008

Table C1. Continued

(4) s = 8 , p = 2

n	λ_{n2}	$R_{n2}(1)$	$\hat{a}_{n2}$
1	3.6663090	.2950531	-1.0173078
2	7.4580513	-.0506723	1.1669315
3	10.9519576	.0200873	-1.4247026
4	14.3883864	-.0104678	1.6672441
5	17.8016919	.0063096	-1.8944285
6	21.2031205	-.0041677	2.1089010
7	24.5975666	.0029323	-2.3128939
8	27.9875311	-.0021610	2.5081083
9	31.3744346	.0016501	-2.6958407
10	34.7591479	-.0012957	2.8770974
11	38.1422362	.0010409	-3.0526749
12	41.5240830	-.0008521	3.2232157

r^+	R_{12}	R_{22}	R_{32}	R_{42}	R_{52}	R_{62}	R_{72}	R_{82}	R_{92}	R_{102}	R_{112}	R_{122}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	[illegible]
.05	.0025	.0025	.0024	.0024	.0023	.0023	.0022	.0021	.0020	.0019	.0018	[illegible]
.10	.0099	.0095	.0090	.0084	.0076	.0067	.0058	.0049	.0039	.0031	.0022	[illegible]
.15	.0219	.0202	.0178	.0149	.0118	.0086	.0057	.0032	.0012	-.0002	-.0010	-.0013
.20	.0382	.0331	.0262	.0186	.0114	.0053	.0010	-.0015	-.0023	-.0020	-.0011	-.0001
.25	.0582	.0463	.0315	.0172	.0059	-.0010	-.0036	-.0031	-.0012	.0005	.0013	.0011
.30	.0813	.0579	.0319	.0107	-.0017	-.0053	-.0033	-.0001	.0018	.0015	[illegible]	[illegible]
.35	.1065	.0662	.0268	.0013	-.0071	-.0044	.0007	.0026	.0012	-.0008	-.0012	-.0003
.40	.1332	.0699	.0171	-.0073	-.0073	.0003	.0033	.0010	-.0015	-.0011	.0005	.0009
.45	.1603	.0681	.0048	-.0118	-.0028	.0041	.0017	-.0019	-.0011	.0010	.0008	-.0006
.50	.1870	.0609	-.0073	-.0109	.0031	.0037	-.0019	-.0016	.0013	.0007	-.0009	-.0003
.55	.2122	.0487	-.0165	-.0053	.0063	-.0002	-.0028	.0012	.0010	-.0011	-.0001	.0004
.60	.2352	.0327	-.0208	.0022	.0050	-.0035	-.0001	.0018	-.0011	-.0003	.0009	[illegible]
.65	.2552	.0146	-.0195	.0080	.0004	-.0033	.0024	-.0004	-.0010	.0011	-.0004	[illegible]
.70	.2716	-.0037	-.0134	.0099	-.0042	.0000	.0018	-.0019	.0010	.0000	[illegible]	[illegible]
.75	.2839	-.0203	-.0045	.0074	-.0057	.0031	-.0010	-.0004	.0010	-.0011	[illegible]	[illegible]
.80	.2919	-.0338	.0050	.0019	-.0035	.0033	-.0025	.0016	-.0008	.0001	.0003	[illegible]
.85	.2961	-.0432	.0129	-.0040	.0006	.0007	-.0012	.0013	-.0012	.0010	-.0008	.0006
.90	.2970	-.0486	.0178	-.0083	.0043	-.0023	.0012	-.0006	.0002	.0001	-.0002	.0003
.95	.2960	-.0505	.0198	-.0102	.0060	-.0039	.0027	-.0019	.0014	-.0011	.0003	[illegible]
1.00	.2951	-.0507	.0201	-.0105	.0063	-.0042	.0029	-.0022	.0017	-.0013	.0010	[illegible]

Table C1. Continued

(5) s = 10 , p = 2

n	λ_{n2}	$R_{n2}(1)$	$\hat{a}_{n2}$
1	3.5522856	.3122630	-.9931439
2	7.3305881	-.0512269	1.1096445
3	10.7859954	.0201838	-1.3510295
4	14.1798398	-.0104924	1.5793218
5	17.5494519	.0063157	-1.7932951
6	20.9066146	-.0041679	1.9953301
7	24.2564704	.0029307	-2.1875037
8	27.6016402	-.0021587	2.3714100
9	30.9436110	.0016477	-2.5482678
10	34.2832932	-.0012935	2.7190240
11	37.6212774	.0010388	-2.8844288
12	40.9577646	-.0008502	3.0450873

r^+	R_{12}	R_{22}	R_{32}	R_{42}	R_{52}	R_{62}	R_{72}	R_{82}	R_{92}	R_{102}	R_{112}	R_{122}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0025	.0025	.0024	.0024	.0023	.0023	.0022	.0021	.0020	.0019	.0018	.0017
.10	.0099	.0096	.0091	.0084	.0077	.0068	.0059	.0050	.0041	.0032	.0024	.0016
.15	.0220	.0203	.0180	.0151	.0120	.0089	.0060	.0034	.0014	-.0000	-.0009	-.0013
.20	.0383	.0333	.0266	.0191	.0119	.0058	.0013	-.0013	-.0023	-.0021	-.0013	-.0003
.25	.0585	.0467	.0323	.0180	.0066	-.0006	-.0035	-.0032	-.0015	.0003	.0012	.0012
.30	.0818	.0588	.0331	.0117	-.0011	-.0052	-.0037	-.0004	.0017	.0017	.0004	-.0007
.35	.1075	.0677	.0285	.0024	-.0070	-.0048	.0003	.0026	.0014	-.0006	-.0012	-.0004
.40	.1347	.0721	.0190	-.0066	-.0078	-.0003	.0033	.0014	-.0013	-.0013	.0003	.0009
.45	.1627	.0712	.0067	-.0119	-.0036	.0040	.0021	-.0017	-.0014	.0008	.0009	-.0004
.50	.1904	.0647	-.0058	-.0117	.0024	.0042	-.0015	-.0019	.0011	.0010	-.0006	-.0005
.55	.2171	.0530	-.0158	-.0065	.0063	.0005	-.0030	.0008	.0013	-.0010	-.0004	.0003
.60	.2417	.0374	-.0210	.0010	.0057	-.0033	-.0007	.0020	-.0009	-.0006	.0009	-.0002
.65	.2635	.0192	-.0207	.0074	.0013	-.0037	.0022	.0001	-.0013	.0011	-.0002	-.0005
.70	.2819	.0004	-.0153	.0102	-.0036	-.0007	.0022	-.0019	.0007	.0003	-.0004	.0007
.75	.2962	-.0172	-.0065	.0083	-.0058	.0028	-.0005	-.0008	.0013	-.0011	.0006	-.0001
.80	.3061	-.0318	.0034	.0030	-.0041	.0035	-.0025	.0014	-.0005	-.0001	.0005	-.0006
.85	.3118	-.0424	.0119	-.0032	.0000	.0012	-.0016	.0015	-.0013	.0011	-.0008	.0005
.90	.3137	-.0486	.0175	-.0079	.0040	-.0020	.0009	-.0003	-.0000	.0002	-.0003	.0004
.95	.3132	-.0510	.0198	-.0102	.0060	-.0039	.0026	-.0019	.0014	-.0010	.0005	-.0006
1.00	.3123	-.0512	.0202	-.0105	.0063	-.0042	.0029	-.0022	.0016	-.0013	.0010	-.0009

Table C1. Continued

(6) s = 12 , p = 2

n	λ_{n2}	$R_{n2}(1)$	$\hat{a}_{n2}$
1	3.4741273	.3252739	-.9768519
2	7.2415619	-.0515673	1.0700972
3	10.6717780	.0201972	-1.2990177
4	14.0375299	-.0104763	1.5168353
5	17.3779367	.0062982	-1.7211900
6	20.7054360	-.0041530	1.9141855
7	24.0253717	.0029185	-2.0977738
8	27.3404601	-.0021488	2.2734683
9	30.6522402	.0016396	-2.4424300
10	33.9616537	-.0012867	2.6055621
11	37.2693113	.0010332	-2.7635810
12	40.5756275	-.0008454	2.9170645

r^+	R_{12}	R_{22}	R_{32}	R_{42}	R_{52}	R_{62}	R_{72}	R_{82}	R_{92}	R_{102}	R_{112}	R_{122}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0025	.0025	.0024	.0024	.0023	.0023	.0022	.0021	.0020	.0020	.0019	.0017
.10	.0099	.0096	.0091	.0085	.0077	.0069	.0060	.0051	.0041	.0033	.0024	.0017
.15	.0220	.0204	.0181	.0153	.0122	.0091	.0062	.0036	.0016	.0001	-.0008	-.0013
.20	.0384	.0335	.0268	.0194	.0122	.0061	.0016	-.0012	-.0023	-.0022	-.0014	-.0004
.25	.0587	.0471	.0328	.0186	.0071	-.0003	-.0034	-.0033	-.0017	.0001	.0012	.0012
.30	.0821	.0594	.0340	.0125	-.0007	-.0052	-.0039	-.0006	.0016	.0017	.0005	-.0006
.35	.1081	.0688	.0296	.0031	-.0068	-.0051	-.0000	.0025	.0016	-.0004	-.0013	-.0006
.40	.1358	.0737	.0204	-.0061	-.0081	-.0007	.0033	.0016	-.0012	-.0014	.0001	.0010
.45	.1643	.0733	.0081	-.0118	-.0042	.0038	.0024	-.0015	-.0015	.0007	.0010	-.0003
.50	.1928	.0674	-.0047	-.0122	.0019	.0044	-.0012	-.0021	.0009	.0011	-.0007	-.0006
.55	.2204	.0562	-.0151	-.0074	.0062	.0010	-.0031	.0005	.0015	-.0008	-.0005	.0008
.60	.2461	.0408	-.0211	.0001	.0061	-.0030	-.0011	.0021	-.0007	-.0008	.0009	-.0001
.65	.2693	.0226	-.0215	.0069	.0020	-.0040	.0020	.0004	-.0014	.0010	.0000	-.0007
.70	.2891	.0035	-.0166	.0103	-.0031	-.0012	.0025	-.0018	.0005	.0006	-.0004	.0005
.75	.3049	-.0146	-.0080	.0090	-.0058	.0025	-.0001	-.0011	.0014	-.0010	.0004	.0001
.80	.3164	-.0301	.0020	.0039	-.0046	.0037	-.0024	.0012	-.0003	-.0003	.0005	-.0007
.85	.3234	-.0415	.0110	-.0025	-.0005	.0016	-.0018	.0017	-.0014	.0010	-.0007	.0004
.90	.3263	-.0485	.0171	-.0076	.0037	-.0017	.0007	-.0001	-.0002	.0004	-.0004	.0005
.95	.3261	-.0512	.0198	-.0101	.0059	-.0038	.0026	-.0018	.0013	-.0010	.0007	-.0006
1.00	.3253	-.0516	.0202	-.0105	.0063	-.0042	.0029	-.0021	.0016	-.0013	.0010	-.0005

Table C1. Continued

(1) s = 2 , p = 3

n	λ_{n3}	$R_{n3}(1)$	$\hat{a}_{n3}$
1	7.2301356	.0692356	-1.6041884
2	11.2076358	-.0137637	3.6963680
3	15.2211958	.0047047	-6.2348480
4	19.2343237	-.0021021	9.2193919
5	23.2448353	.0011017	-12.6253758
6	27.2531765	-.0006421	16.4299365
7	31.2599022	.0004038	-20.6137506
8	35.2654300	-.0002688	25.1605125
9	39.2700545	.0001871	-30.0562938
10	43.2739838	-.0001350	35.2890303
11	47.2773679	.0001003	-40.8481367
12	51.2865172	-.0000763	46.7211641

r^*	R_{13}	R_{23}	R_{33}	R_{43}	R_{53}	R_{63}	R_{73}	R_{83}	R_{93}	R_{103}	R_{113}	R_{123}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
.10	.0010	.0009	.0009	.0008	.0007	.0006	.0005	.0004	.0003	.0003	.0002	.0001
.15	.0031	.0028	.0024	.0020	.0015	.0010	.0007	.0003	.0001	-.0000	-.0001	-.0001
.20	.0070	.0058	.0044	.0029	.0016	.0007	.0001	-.0002	-.0002	-.0002	-.0000	.0000
.25	.0128	.0094	.0059	.0028	.0008	-.0002	-.0004	-.0003	-.0000	.0001	.0001	.0000
.30	.0202	.0129	.0061	.0015	-.0005	-.0007	-.0002	.0002	.0002	.0000	-.0001	-.0001
.35	.0288	.0155	.0049	-.0003	-.0012	-.0003	.0003	.0002	-.0001	-.0001	-.0000	.0001
.40	.0382	.0163	.0023	-.0018	-.0007	.0004	.0003	-.0002	-.0001	.0001	.0001	-.0000
.45	.0476	.0153	-.0006	-.0020	.0003	.0005	-.0002	-.0002	.0001	.0001	-.0001	-.0000
.50	.0562	.0123	-.0031	-.0011	.0009	.0000	-.0003	.0001	.0001	-.0001	.0000	.0001
.55	.0637	.0080	-.0043	.0003	.0008	-.0005	.0000	.0002	-.0001	-.0000	.0001	-.0000
.60	.0694	.0029	-.0041	.0015	.0000	-.0004	.0003	-.0001	-.0001	.0001	-.0001	-.0000
.65	.0734	-.0021	-.0028	.0018	-.0007	.0000	.0002	-.0002	.0001	-.0000	-.0000	.0001
.70	.0754	-.0066	-.0008	.0013	-.0009	.0005	-.0001	-.0000	.0001	-.0001	.0001	-.0000
.75	.0759	-.0100	.0013	.0003	-.0006	.0005	-.0003	.0002	-.0001	.0000	.0000	-.0000
.80	.0750	-.0122	.0030	-.0007	.0000	.0002	-.0002	.0002	-.0001	.0001	-.0001	.0000
.85	.0734	-.0134	.0040	-.0015	.0006	-.0002	.0001	-.0000	-.0000	.0000	-.0000	.0000
.90	.0715	-.0139	.0046	-.0019	.0010	-.0005	.0003	-.0002	.0001	-.0001	.0000	-.0000
.95	.0699	-.0139	.0047	-.0021	.0011	-.0006	.0004	-.0003	.0002	-.0001	.0001	-.0001
1.00	.0692	-.0138	.0047	-.0021	.0011	-.0006	.0004	-.0003	.0002	-.0001	.0001	-.0001

Table C1. Continued

(2) s = 4 , p = 3

n	λ_{n3}	$R_{n3}(1)$	$\hat{a}_{n3}$
1	5.9342531	.1069030	-1.2630710
2	9.7392772	-.0185890	2.6224311
3	13.4395034	.0061753	-4.3084960
4	17.0971823	-.0027291	6.3027699
5	20.7338330	.0014224	-8.5835501
6	24.3585446	-.0008264	11.1336166
7	27.9758117	.0005186	-13.9391509
8	31.5881145	-.0003447	16.9887656
9	35.1969361	.0002397	-20.2728565
10	38.8032188	-.0001728	23.7831699
11	42.4075909	.0001283	-27.5125020
12	46.0104877	-.0000976	31.4544871

r^{+}	R_{13}	R_{23}	R_{33}	R_{43}	R_{53}	R_{63}	R_{73}	R_{83}	R_{93}	R_{103}	R_{113}	R_{123}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
.10	.0010	.0009	.0009	.0008	.0008	.0007	.0006	.0005	.0004	.0003	.0003	.0002
.15	.0032	.0029	.0026	.0022	.0018	.0013	.0010	.0006	.0003	.0001	.0000	-.0001
.20	.0073	.0063	.0050	.0036	.0023	.0013	.0005	.0000	-.0002	-.0002	-.0002	-.0001
.25	.0136	.0106	.0073	.0042	.0018	.0003	-.0004	-.0004	-.0002	-.0000	.0001	.0001
.30	.0221	.0153	.0086	.0033	.0003	-.0008	-.0006	-.0001	.0002	.0002	.0000	-.0001
.35	.0325	.0195	.0081	.0012	-.0012	-.0009	.0000	.0004	.0002	-.0001	-.0001	-.0000
.40	.0445	.0222	.0058	-.0012	-.0015	-.0000	.0005	.0001	-.0002	-.0001	.0001	.0001
.45	.0573	.0227	.0022	-.0027	-.0006	.0007	.0002	-.0003	-.0001	.0001	.0000	-.0001
.50	.0703	.0207	-.0017	-.0026	.0007	.0006	-.0004	-.0002	.0002	.0000	-.0001	.0000
.55	.0824	.0163	-.0047	-.0011	.0013	-.0002	-.0004	.0002	.0000	-.0001	.0001	.0001
.60	.0931	.0102	-.0059	.0008	.0009	-.0007	.0001	.0002	-.0002	.0001	.0001	-.0001
.65	.1017	.0031	-.0053	.0022	-.0002	-.0004	.0004	-.0002	-.0000	.0001	-.0001	.0000
.70	.1078	-.0039	-.0031	.0023	-.0011	.0003	.0001	-.0002	.0002	-.0001	.0000	.0000
.75	.1113	-.0099	-.0002	.0014	-.0011	.0007	-.0003	.0001	.0000	-.0001	.0001	-.0001
.80	.1124	-.0143	.0026	-.0001	-.0004	.0005	-.0004	.0003	-.0002	.0001	-.0000	-.0000
.85	.1117	-.0171	.0046	-.0015	.0005	-.0001	-.0001	.0001	-.0001	.0001	-.0001	.0001
.90	.1099	-.0184	.0058	-.0024	.0011	-.0006	.0003	-.0002	.0001	-.0000	.0000	.0000
.95	.1079	-.0187	.0062	-.0027	.0014	-.0008	.0005	-.0003	.0002	-.0002	.0001	-.0001
1.00	.1069	-.0186	.0062	-.0027	.0014	-.0008	.0005	-.0003	.0002	-.0002	.0001	-.0001

Table C1. Continued

(3) $s = 6$, $p = 3$

n	λ_{n3}	$R_{n3}(1)$	$\hat{a}_{n3}$
1	5.4266823	.1344333	-1.1224946
2	9.2247135	-.0209322	2.2137875
3	12.8158418	.0067999	-3.6236075
4	16.3452410	-.0029785	5.2975473
5	19.8467111	.0015453	-7.2131122
6	23.3329529	-.0008953	9.3548386
7	26.8099116	.0005608	-11.7108026
8	30.2807675	-.0003724	14.2713239
9	33.7473849	.0002587	-17.0282932
10	37.2109328	-.0001863	19.9747640
11	40.6721827	.0001382	-23.1046854
12	44.1316656	-.0001051	26.4127164

r^*	R_{13}	R_{23}	R_{33}	R_{43}	R_{53}	R_{63}	R_{73}	R_{83}	R_{93}	R_{103}	R_{113}	R_{123}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
.10	.0010	.0009	.0009	.0008	.0008	.0007	.0006	.0005	.0005	.0004	.0003	.0002
.15	.0032	.0030	.0027	.0023	.0019	.0015	.0011	.0007	.0004	.0002	.0001	-.0000
.20	.0074	.0064	.0052	.0039	.0026	.0016	.0007	.0002	-.0001	-.0002	-.0002	-.0001
.25	.0139	.0111	.0078	.0048	.0023	.0006	-.0002	-.0005	-.0003	-.0001	.0001	.0001
.30	.0228	.0163	.0096	.0042	.0008	-.0006	-.0007	-.0003	.0001	.0002	.0001	-.0000
.35	.0340	.0213	.0097	.0022	-.0010	-.0011	-.0002	.0003	.0003	-.0000	-.0001	-.0001
.40	.0472	.0250	.0078	-.0005	-.0018	-.0004	.0005	.0003	-.0001	-.0002	.0000	.0001
.45	.0619	.0266	.0042	-.0026	-.0012	.0006	.0005	-.0002	-.0002	.0001	.0001	-.0000
.50	.0771	.0255	-.0001	-.0032	.0002	.0009	-.0002	-.0003	.0001	.0001	-.0001	-.0000
.55	.0922	.0217	-.0040	-.0021	.0013	.0002	-.0005	.0001	.0002	-.0001	-.0000	.0001
.60	.1063	.0156	-.0064	.0000	.0013	-.0007	-.0001	.0003	-.0002	-.0001	.0001	-.0001
.65	.1184	.0078	-.0065	.0020	.0003	-.0007	.0004	-.0001	-.0002	.0002	-.0001	-.0000
.70	.1279	-.0004	-.0047	.0027	-.0009	-.0000	.0003	-.0003	.0002	-.0000	-.0001	.0001
.75	.1345	-.0081	-.0016	.0021	-.0013	.0007	-.0002	-.0001	.0001	-.0001	.0001	-.0000
.80	.1380	-.0142	.0018	.0005	-.0008	.0007	-.0005	.0003	-.0001	.0000	.0000	-.0000
.85	.1388	-.0183	.0045	-.0012	.0002	.0001	-.0002	.0002	-.0002	.0001	-.0001	.0001
.90	.1376	-.0204	.0061	-.0024	.0011	-.0005	.0003	-.0001	.0000	-.0000	-.0000	.0000
.95	.1356	-.0210	.0067	-.0029	.0015	-.0008	.0005	-.0003	.0002	-.0002	.0001	-.0001
1.00	.1344	-.0209	.0068	-.0030	.0015	-.0009	.0006	-.0004	.0003	-.0002	.0001	-.0001

Table C1. Continued

(4) s = 8 , p = 3

n	λ_{n3}	$R_{n3}(1)$	$\hat{a}_{n3}$
1	5.1515661	.1548951	-1.0473732
2	8.9612475	-.0221492	2.0006407
3	12.4951043	.0070969	-3.2719588
4	15.9579257	-.0030918	4.7823553
5	19.3895516	.0015996	-6.5103859
6	22.8043862	-.0009251	8.4419060
7	26.2090563	.0005788	-10.5661211
8	29.6070727	-.0003840	12.8742992
9	33.0004814	.0002666	-15.3591358
10	36.3905596	-.0001919	18.0143788
11	39.7781483	.0001423	-20.8345843
12	43.1638246	-.0001082	23.8149479

r^+	R_{13}	R_{23}	R_{33}	R_{43}	R_{53}	R_{63}	R_{73}	R_{83}	R_{93}	R_{103}	R_{113}	R_{123}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
.10	.0010	.0010	.0009	.0009	.0008	.0007	.0006	.0006	.0005	.0004	.0003	.0003
.15	.0033	.0030	.0027	.0023	.0019	.0015	.0011	.0008	.0005	.0003	.0001	-.0000
.20	.0075	.0065	.0053	.0040	.0028	.0017	.0008	.0003	-.0001	-.0002	-.0002	-.0001
.25	.0141	.0113	.0081	.0051	.0026	.0008	-.0001	-.0005	-.0004	-.0002	.0000	.0001
.30	.0232	.0168	.0102	.0047	.0011	-.0005	-.0008	-.0004	.0000	.0002	.0002	.0000
.35	.0348	.0222	.0106	.0028	-.0008	-.0012	-.0004	.0003	.0003	.0001	-.0001	-.0001
.40	.0487	.0265	.0090	-.0000	-.0019	-.0006	.0004	.0004	-.0001	-.0002	-.0000	.0001
.45	.0643	.0288	.0055	-.0024	-.0015	.0005	.0006	-.0001	-.0003	.0000	.0001	-.0000
.50	.0810	.0284	.0010	-.0034	-.0001	.0010	-.0001	-.0004	.0001	.0002	-.0001	-.0001
.55	.0980	.0252	-.0034	-.0026	.0012	.0004	-.0006	.0000	.0002	-.0001	-.0001	.0001
.60	.1142	.0192	-.0063	-.0006	.0015	-.0005	-.0003	.0004	-.0001	-.0001	.0001	-.0000
.65	.1288	.0113	-.0072	.0016	.0006	-.0008	.0004	.0001	-.0002	.0001	-.0000	-.0001
.70	.1410	.0025	-.0058	.0028	-.0007	-.0002	.0005	-.0003	.0001	.0001	-.0001	.0001
.75	.1502	-.0061	-.0027	.0025	-.0014	.0006	-.0001	-.0002	.0002	-.0002	.0001	-.0000
.80	.1559	-.0133	.0009	.0010	-.0010	.0008	-.0005	.0002	-.0001	-.0000	.0001	-.0001
.85	.1582	-.0184	.0041	-.0009	.0000	.0002	-.0003	.0003	-.0002	.0002	-.0001	.0001
.90	.1579	-.0212	.0062	-.0024	.0010	-.0005	.0002	-.0001	.0000	.0000	-.0000	.0000
.95	.1561	-.0221	.0070	-.0030	.0015	-.0009	.0005	-.0003	.0002	-.0002	.0001	-.0001
1.00	.1549	-.0221	.0071	-.0031	.0016	-.0009	.0006	-.0004	.0003	-.0002	.0001	-.0001

Table C1. Continued

(5) s = 10 , p = 3

n	λ_{n3}	$R_{n3}(1)$	$\hat{a}_{n3}$
1	4.9740773	.1705437	-1.0011779
2	8.7984136	-.0228549	1.8691257
3	12.2977511	.0072539	-3.0541885
4	15.7201383	-.0031481	4.4624984
5	19.1092949	.0016253	-6.0732625
6	22.4806837	-.0009388	7.8732004
7	25.8413548	.0005868	-9.8522380
8	29.1950251	-.0003891	12.0022438
9	32.5438536	.0002700	-14.3164218
10	35.8891857	-.0001943	16.7889580
11	39.2319059	.0001440	-19.4147914
12	42.5726205	-.0001095	22.1894557

r^+	R_{13}	R_{23}	R_{33}	R_{43}	R_{53}	R_{63}	R_{73}	R_{83}	R_{93}	R_{103}	R_{113}	R_{123}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
.10	.0010	.0010	.0009	.0009	.0008	.0007	.0006	.0006	.0005	.0004	.0003	.0003
.15	.0033	.0030	.0027	.0024	.0020	.0016	.0012	.0008	.0005	.0003	.0001	-.0000
.20	.0075	.0066	.0054	.0041	.0029	.0018	.0009	.0003	-.0001	-.0002	-.0002	-.0002
.25	.0142	.0114	.0083	.0053	.0027	.0009	-.0001	-.0004	-.0004	-.0002	.0000	.0001
.30	.0234	.0171	.0105	.0050	.0013	-.0004	-.0008	-.0004	.0000	.0002	.0002	.0000
.35	.0353	.0228	.0112	.0032	-.0006	-.0012	-.0005	.0002	.0003	.0001	-.0001	-.0001
.40	.0496	.0274	.0098	.0003	-.0019	-.0008	.0004	.0004	-.0000	-.0002	-.0001	.0001
.45	.0659	.0302	.0064	-.0023	-.0017	.0004	.0006	-.0001	-.0003	-.0000	.0002	.0000
.50	.0835	.0303	.0018	-.0036	-.0004	.0010	.0001	-.0004	.0000	.0002	-.0000	-.0001
.55	.1016	.0275	-.0028	-.0030	.0011	.0006	-.0005	-.0001	.0003	-.0001	-.0001	.0001
.60	.1194	.0218	-.0062	-.0010	.0016	-.0004	-.0004	.0004	-.0000	-.0002	.0001	.0000
.65	.1358	.0139	-.0075	.0013	.0009	-.0009	.0003	.0002	-.0002	.0001	.0000	-.0001
.70	.1500	.0048	-.0065	.0028	-.0005	-.0004	.0005	-.0003	.0000	.0001	-.0001	.0001
.75	.1612	-.0043	-.0036	.0028	-.0014	.0005	.0000	-.0002	.0002	-.0001	.0001	.0000
.80	.1687	-.0123	.0002	.0013	-.0012	.0008	-.0005	.0002	-.0000	-.0001	.0001	-.0001
.85	.1726	-.0182	.0038	-.0007	-.0002	.0004	-.0004	.0003	-.0002	.0002	-.0001	.0001
.90	.1732	-.0216	.0061	-.0023	.0009	-.0004	.0002	-.0000	-.0000	.0000	-.0001	.0001
.95	.1718	-.0228	.0071	-.0030	.0015	-.0009	.0005	-.0003	.0002	-.0002	.0001	-.0001
1.00	.1705	-.0229	.0073	-.0031	.0016	-.0009	.0006	-.0004	.0003	-.0002	.0001	-.0001

Table C1. Continued

(6) $s = 12$, $p = 3$

n	λ_{n3}	$R_{n3}(1)$	$\hat{a}_{n3}$
1	4.8584264	.1828391	-.9701343
2	8.6860766	-.0233143	1.7794564
3	12.1629538	.0073427	-2.9041035
4	15.5583898	-.0031769	4.2412926
5	18.9191041	.0016374	-5.7703174
6	22.2613455	-.0009448	7.4784604
7	25.5924702	.0005901	-9.3561490
8	28.9163430	-.0003910	11.3956714
9	32.2352044	.0002712	-13.5905881
10	35.5504489	-.0001951	15.9353920
11	38.8629922	.0001446	-18.4252906
12	42.1734619	-.0001099	21.0560552

r^+	R_{13}	R_{23}	R_{33}	R_{43}	R_{53}	R_{63}	R_{73}	R_{83}	R_{93}	R_{103}	R_{113}	R_{123}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
.10	.0010	.0010	.0009	.0009	.0008	.0007	.0007	.0006	.0005	.0004	.0003	.0003
.15	.0033	.0030	.0027	.0024	.0020	.0016	.0012	.0009	.0006	.0003	.0001	.0000
.20	.0075	.0066	.0054	.0042	.0030	.0019	.0010	.0004	-.0000	-.0002	-.0002	-.0002
.25	.0142	.0115	.0084	.0054	.0029	.0010	-.0000	-.0004	-.0004	-.0002	-.0000	.0001
.30	.0236	.0173	.0108	.0052	.0015	-.0004	-.0008	-.0005	-.0000	.0002	.0002	.0001
.35	.0357	.0232	.0116	.0034	-.0005	-.0012	-.0005	.0002	.0003	.0001	-.0001	-.0001
.40	.0502	.0281	.0103	.0006	-.0019	-.0009	.0003	.0005	.0000	-.0002	-.0001	.0001
.45	.0669	.0312	.0070	-.0021	-.0019	.0003	.0007	.0000	-.0003	-.0000	.0001	.0000
.50	.0852	.0317	.0024	-.0036	-.0005	.0010	.0001	-.0004	-.0000	.0002	-.0000	-.0001
.55	.1041	.0292	-.0024	-.0033	.0010	.0007	-.0005	-.0001	.0003	-.0000	-.0001	.0001
.60	.1230	.0237	-.0061	-.0013	.0017	-.0003	-.0004	.0003	.0000	-.0002	.0001	.0000
.65	.1407	.0159	-.0077	.0011	.0011	-.0009	.0002	.0002	-.0003	.0001	.0001	-.0001
.70	.1564	.0067	-.0070	.0028	-.0003	-.0005	.0005	-.0003	-.0000	.0001	-.0001	.0001
.75	.1692	-.0028	-.0042	.0030	-.0014	.0004	.0001	-.0003	.0002	-.0001	.0000	.0000
.80	.1784	-.0113	-.0003	.0016	-.0013	.0008	-.0004	.0002	.0000	-.0001	.0001	-.0001
.85	.1836	-.0178	.0034	-.0004	-.0003	.0005	-.0004	.0003	-.0002	.0001	-.0001	.0000
.90	.1851	-.0217	.0060	-.0022	.0009	-.0003	.0001	-.0000	-.0000	.0001	-.0001	.0001
.95	.1840	-.0232	.0072	-.0030	.0015	-.0009	.0005	-.0003	.0002	-.0001	.0001	-.0001
1.00	.1828	-.0233	.0073	-.0032	.0016	-.0009	.0006	-.0004	.0003	-.0002	.0001	-.0001

Table C1. Continued

(1) s = 2 , p = 4

n	λ_{n4}	$R_{n4}(1)$	$\hat{a}_{n4}$
1	9.3792135	.0252401	-2.3213776
2	13.2723388	-.0049034	7.4314468
3	17.2549085	.0014777	-15.8125201
4	21.2546254	-.0005792	28.1040149
5	25.2582949	.0002688	-44.8874392
6	29.2627184	-.0001402	66.7027461
7	33.2670066	.0000797	-94.0598935
8	37.2709208	-.0000484	127.4452333
9	41.2744242	.0000309	-167.3256393
10	45.2775443	-.0000206	214.1513623
11	49.2803332	.0000142	-268.3575556
12	53.2834587	-.0000101	330.1944123

r^+	R_{14}	R_{24}	R_{34}	R_{44}	R_{54}	R_{64}	R_{74}	R_{84}	R_{94}	R_{104}	R_{114}	R_{124}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.10	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0000	.0000	.0000	.0000	.0000
.15	.0005	.0004	.0004	.0003	.0002	.0002	.0001	.0001	.0000	.0000	.0000	-.0000
.20	.0013	.0011	.0009	.0006	.0004	.0002	.0001	.0000	-.0000	-.0000	-.0000	-.0000
.25	.0030	.0022	.0015	.0008	.0003	.0001	-.0000	-.0001	-.0000	.0000	.0000	.0000
.30	.0055	.0036	.0019	.0007	.0001	-.0001	-.0001	-.0000	.0000	.0000	-.0000	-.0000
.35	.0088	.0048	.0018	.0002	-.0002	-.0001	.0000	.0000	.0000	-.0000	-.0000	.0000
.40	.0127	.0057	.0012	-.0003	-.0003	.0000	.0001	-.0000	-.0000	-.0000	.0000	.0000
.45	.0170	.0058	.0003	-.0006	-.0000	.0001	-.0000	-.0000	.0000	.0000	-.0000	-.0000
.50	.0212	.0050	-.0006	-.0004	.0002	.0001	-.0001	.0000	.0000	-.0000	-.0000	.0000
.55	.0248	.0036	-.0012	-.0001	.0002	-.0001	-.0000	.0000	-.0000	-.0000	.0000	-.0000
.60	.0277	.0017	-.0014	.0003	.0001	-.0001	.0001	-.0000	-.0000	.0000	-.0000	-.0000
.65	.0295	-.0004	-.0010	.0005	-.0001	-.0000	.0001	-.0000	.0000	.0000	-.0000	.0000
.70	.0304	-.0022	-.0004	.0004	-.0002	.0001	-.0000	-.0000	.0000	-.0000	.0000	-.0000
.75	.0302	-.0036	.0003	.0001	-.0002	.0001	-.0001	.0000	-.0000	-.0000	.0000	-.0000
.80	.0294	-.0045	.0009	-.0002	-.0000	.0000	-.0000	.0000	-.0000	.0000	-.0000	.0000
.85	.0281	-.0050	.0013	-.0004	.0001	-.0000	.0000	.0000	-.0000	.0000	-.0000	.0000
.90	.0268	-.0051	.0015	-.0005	.0002	-.0001	.0001	-.0000	.0000	-.0000	.0000	-.0000
.95	.0257	-.0050	.0015	-.0006	.0003	-.0001	.0001	-.0000	.0000	-.0000	.0000	-.0000
1.00	.0252	-.0049	.0015	-.0006	.0003	-.0001	.0001	-.0000	.0000	-.0000	.0000	-.0000

Table C1. Continued

(2) $s = 4,\ p = 4$

n	λ_{n4}	$R_{n4}(1)$	$\hat{a}_{n4}$
1	7.6720294	.0466697	-1.6483456
2	11.4234913	-.0076550	4.8056206
3	15.1291304	.0022024	-9.9102171
4	18.7982257	-.0008472	17.3723162
5	22.4455262	.0003894	-27.5437786
6	26.0792437	-.0002020	40.7499258
7	29.7040013	.0001144	-57.2978837
8	33.3225555	-.0000693	77.4805469
9	36.9366412	.0000442	-101.5790648
10	40.5474027	-.0000294	129.8645775
11	44.1556235	.0000203	-162.5995650
12	47.7618599	-.0000144	200.0387786

r^+	R_{14}	R_{24}	R_{34}	R_{44}	R_{54}	R_{64}	R_{74}	R_{84}	R_{94}	R_{104}	R_{114}	R_{124}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.10	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0000	.0000	.0000	.0000
.15	.0005	.0004	.0004	.0003	.0003	.0002	.0002	.0001	.0001	.0000	.0000	.0000
.20	.0014	.0012	.0010	.0008	.0005	.0003	.0002	.0001	.0000	-.0000	-.0000	-.0000
.25	.0032	.0026	.0018	.0011	.0006	.0002	.0000	-.0001	-.0001	-.0000	-.0000	.0000
.30	.0062	.0044	.0026	.0012	.0003	-.0000	-.0001	-.0001	.0000	.0000	.0000	.0000
.35	.0104	.0064	.0030	.0008	-.0001	-.0002	-.0001	.0000	.0000	.0000	-.0000	-.0000
.40	.0157	.0081	.0026	.0001	-.0004	-.0001	.0001	.0001	-.0000	-.0000	-.0000	.0000
.45	.0220	.0091	.0016	-.0006	-.0003	.0001	.0001	-.0000	-.0000	.0000	.0000	-.0000
.50	.0288	.0090	.0002	-.0009	.0000	.0002	-.0000	-.0001	.0000	.0000	-.0000	-.0000
.55	.0355	.0078	-.0012	-.0006	.0003	.0000	-.0001	.0000	.0000	-.0000	.0000	.0000
.60	.0416	.0054	-.0020	.0000	.0003	-.0002	-.0000	.0001	-.0000	-.0000	.0000	-.0000
.65	.0465	.0023	-.0020	.0006	.0000	-.0001	.0001	-.0000	-.0000	.0000	-.0000	.0000
.70	.0499	-.0009	-.0013	.0007	-.0003	.0000	.0000	-.0001	.0000	-.0000	-.0000	.0000
.75	.0517	-.0038	-.0003	.0005	-.0003	.0002	-.0001	.0000	.0000	-.0000	.0000	-.0000
.80	.0518	-.0059	.0008	.0000	-.0001	.0001	-.0001	.0001	-.0000	.0000	-.0000	-.0000
.85	.0508	-.0072	.0016	-.0004	.0001	-.0000	-.0000	.0000	-.0000	.0000	-.0000	.0000
.90	.0491	-.0077	.0021	-.0007	.0003	-.0001	.0001	-.0000	.0000	-.0000	.0000	.0000
.95	.0474	-.0077	.0022	-.0008	.0004	-.0002	.0001	-.0001	.0000	-.0000	.0000	-.0000
1.00	.0467	-.0077	.0022	-.0008	.0004	-.0002	.0001	-.0001	.0000	-.0000	.0000	-.0000

Table C1. Continued

(3) $s = 6$, $p = 4$

n	λ_{n4}	$R_{n4}(1)$	$\hat{a}_{n4}$
1	6.9967922	.0648243	-1.3852133
2	10.7710244	-.0092586	3.8709821
3	14.3879115	.0025707	-7.9557999
4	17.9395164	-.0009742	13.9517031
5	21.4581720	.0004442	-22.1357527
6	24.9577932	-.0002293	32.7672523
7	28.4453805	.0001295	-46.0920443
8	31.9248549	-.0000782	62.3450037
9	35.3985900	.0000499	-81.7517803
10	38.8681100	-.0000332	104.5301040
11	42.3344399	.0000229	-130.8907809
12	45.7982950	-.0000162	-161.0385159

r^+	R_{14}	R_{24}	P_{34}	R_{44}	R_{54}	R_{64}	R_{74}	R_{84}	R_{94}	R_{104}	R_{114}	R_{124}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.10	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0000	.0000	.0000
.15	.0005	.0004	.0004	.0003	.0003	.0002	.0002	.0001	.0001	.0001	.0000	.0000
.20	.0014	.0013	.0010	.0008	.0006	.0004	.0002	.0001	.0000	-.0000	-.0000	-.0000
.25	.0033	.0027	.0020	.0013	.0007	.0003	.0001	-.0000	-.0001	-.0000	-.0000	.0000
.30	.0065	.0047	.0029	.0015	.0005	.0000	-.0001	-.0001	-.0000	.0000	.0000	.0000
.35	.0110	.0070	.0035	.0012	.0000	-.0002	-.0001	.0000	.0001	.0000	-.0000	-.0000
.40	.0171	.0093	.0034	.0004	-.0004	-.0002	.0000	.0001	.0000	-.0000	-.0000	.0000
.45	.0244	.0109	.0025	-.0004	-.0005	.0000	.0001	.0000	-.0000	-.0000	.0000	.0000
.50	.0328	.0114	.0010	-.0010	-.0001	.0002	.0000	-.0001	.0000	.0000	-.0000	-.0000
.55	.0415	.0106	-.0007	-.0009	.0003	.0001	-.0001	-.0000	.0000	-.0000	-.0000	.0000
.60	.0500	.0083	-.0020	-.0003	.0004	-.0001	-.0001	.0001	-.0000	-.0000	.0000	-.0000
.65	.0576	.0049	-.0025	.0004	.0002	-.0002	.0001	.0000	-.0000	.0000	-.0000	-.0000
.70	.0635	.0010	-.0021	.0009	-.0002	-.0001	.0001	-.0001	.0000	.0000	-.0000	.0000
.75	.0675	-.0028	-.0009	.0008	-.0004	.0001	-.0000	-.0000	.0000	-.0000	.0000	-.0000
.80	.0693	-.0060	.0005	.0003	-.0003	.0002	-.0001	.0001	-.0000	.0000	.0000	-.0000
.85	.0692	-.0081	.0016	-.0004	.0000	.0000	-.0001	.0000	-.0000	.0000	-.0000	.0000
.90	.0677	-.0091	.0023	-.0008	.0003	-.0001	.0001	-.0000	.0000	-.0000	-.0000	.0000
.95	.0658	-.0093	.0026	-.0010	.0004	-.0002	.0001	-.0001	.0000	-.0000	.0000	-.0000
1.00	.0648	-.0093	.0026	-.0010	.0004	-.0002	.0001	-.0001	.0000	-.0000	.0000	-.0000

Table C1. Continued

(4) s = 8 , r = 4

n	λ_{n4}	$R_{n4}(1)$	$\hat{a}_{n4}$
1	6.6277737	.0795666	-1.2480903
2	10.4392255	-.0101993	3.4026294
3	14.0095379	.0027686	-6.9987173
4	17.4995538	-.0010397	12.2852378
5	20.9512792	.0004719	-19.5030504
6	24.3814248	-.0002429	28.8795515
7	27.79[illegible]1085	.0001369	-40.6307553
8	31.20[illegible]7996	-.0000826	54.9631723
9	34.6071395	.0000526	-72.0752502
10	38.0038547	-.0000350	92.1584943
11	41.3970728	.0000241	-115.3983396
12	44.7875839	-.0000171	141.9748492

r^+	R_{14}	R_{24}	R_{34}	R_{44}	R_{54}	R_{64}	R_{74}	R_{84}	R_{94}	R_{104}	R_{114}	R_{124}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.10	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0000	.0000	.0000
.15	.0005	.0004	.0004	.0004	.0003	.0002	.0002	.0001	.0001	.0001	.0000	.0000
.20	.0015	.0013	.0011	.0008	.0006	.0004	.0003	.0001	.0000	-.0000	-.0000	-.0000
.25	.0034	.0028	.0020	.0014	.0008	.0004	.0001	-.0000	-.0001	-.0000	-.0000	.0000
.30	.0066	.0049	.0031	.0016	.0006	.0001	-.0001	-.0001	-.0000	.0000	.0000	.0000
.35	.0114	.0074	.0038	.0014	.0001	-.0002	-.0002	-.0000	.0001	.0000	-.0000	-.0000
.40	.0178	.0099	.0039	.0006	-.0004	-.0003	.0000	.0001	.0000	-.0000	-.0000	.0000
.45	.0258	.0119	.0031	-.0003	-.0005	-.0000	.0001	.0000	-.0001	-.0000	.0000	.0000
.50	.0351	.0129	.0016	-.0010	-.0003	.0002	.0001	-.0001	-.0000	.0000	.0000	-.0000
.55	.0452	.0124	-.0003	-.0011	.0002	.0002	-.0001	-.0000	.0000	-.0000	-.0000	.0000
.60	.0553	.0104	-.0019	-.0005	.0005	-.0001	-.0001	.0001	.0000	-.0000	.0000	.0000
.65	.0649	.0070	-.0027	.0003	.0003	-.0002	.0000	.0000	-.0000	.0000	.0000	-.0000
.70	.0730	.0027	-.0025	.0009	-.0001	-.0001	.0001	-.0001	.0000	.0000	-.0000	.0000
.75	.0790	-.0017	-.0014	.0009	-.0004	.0001	.0000	-.0000	.0000	-.0000	.0000	.0000
.80	.0825	-.0056	.0001	.0004	-.0003	.0002	-.0001	.0000	-.0000	-.0000	.0000	-.0000
.85	.0835	-.0083	.0015	-.0002	-.0000	.0001	-.0001	.0001	-.0000	.0000	-.0000	.0000
.90	.0826	-.0098	.0024	-.0008	.0003	-.0001	.0000	-.0000	-.0000	.0000	-.0000	.0000
.95	.0807	-.0102	.0027	-.0010	.0004	-.0002	.0001	-.0001	.0000	-.0000	.0000	-.0000
1.00	.0796	-.0102	.0028	-.0010	.0005	-.0002	.0001	-.0001	.0001	-.0000	.0000	-.0000

Table C1. Continued

(5) s = 10 , p = 4

n	λ_{n4}	$R_{n4}(1)$	$\hat{a}_{n4}$
1	6.3934867	.0915271	-1.1649294
2	10.2364499	-.0107878	3.1212376
3	13.7780293	.0028849	-6.4254386
4	17.2303579	-.0010769	11.2852831
5	20.6412345	.0004872	-17.9204177
6	24.0290156	-.0002503	26.5390462
7	27.4024792	.0001409	-37.3390999
8	30.7664057	-.0000849	50.5098460
9	34.1236393	.0000540	-66.2331827
10	37.4759847	-.0000359	84.6846521
11	40.8246458	.0000247	-106.0342349
12	44.1704575	-.0000175	130.4469923

r^+	R_{14}	R_{24}	R_{34}	R_{44}	R_{54}	R_{64}	R_{74}	R_{84}	R_{94}	R_{104}	R_{114}	R_{124}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.10	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0000	.0000	.0000
.15	.0005	.0004	.0004	.0004	.0003	.0003	.0002	.0002	.0001	.0001	.0000	.0000
.20	.0015	.0013	.0011	.0009	.0006	.0004	.0003	.0001	.0001	.0000	-.0000	-.0000
.25	.0034	.0028	.0021	.0014	.0008	.0004	.0001	-.0000	-.0001	-.0001	-.0000	-.0000
.30	.0067	.0050	.0032	.0017	.0007	.0001	-.0001	-.0001	-.0001	.0000	.0000	.0000
.35	.0116	.0076	.0040	.0015	.0002	-.0002	-.0002	-.0000	.0000	.0000	.0000	-.0000
.40	.0183	.0103	.0042	.0008	-.0003	-.0003	-.0000	.0001	.0000	-.0000	-.0000	.0000
.45	.0267	.0126	.0035	-.0002	-.0006	-.0001	.0001	.0000	-.0000	-.0000	.0000	.0000
.50	.0366	.0139	.0020	-.0010	-.0003	.0002	.0001	-.0001	-.0000	.0000	.0000	-.0000
.55	.0476	.0137	.0000	-.0012	.0001	.0002	-.0001	-.0001	.0000	.0000	-.0000	.0000
.60	.0589	.0119	-.0018	-.0007	.0005	-.0000	-.0001	.0001	.0000	-.0000	.0000	.0000
.65	.0699	.0085	-.0028	.0001	.0004	-.0002	.0000	.0001	-.0000	.0000	.0000	-.0000
.70	.0796	.0041	-.0028	.0008	-.0000	-.0002	.0001	-.0000	-.0000	.0000	-.0000	.0000
.75	.0874	-.0007	-.0018	.0010	-.0004	.0001	.0000	-.0001	.0000	-.0000	.0000	.0000
.80	.0925	-.0050	-.0002	.0006	-.0004	.0002	-.0001	.0000	.0000	-.0000	.0000	-.0000
.85	.0947	-.0083	.0013	-.0001	-.0001	.0001	-.0001	.0001	-.0000	.0000	-.0000	.0000
.90	.0945	-.0102	.0024	-.0008	.0003	-.0001	.0000	-.0000	-.0000	.0000	-.0000	.0000
.95	.0927	-.0108	.0028	-.0010	.0005	-.0002	.0001	-.0001	.0000	-.0000	.0000	-.0000
1.00	.0915	-.0108	.0029	-.0011	.0005	-.0003	.0001	-.0001	.0001	-.0000	.0000	-.0000

Table C1. Continued

(6) $s = 12$, $p = 4$

n	λ_{n4}	$R_{n4}(1)$	$\hat{a}_{n4}$
1	6.2309998	.1013302	-1.1095087
2	10.0979933	-.0111856	2.9326258
3	13.6206520	.0029565	-6.0393875
4	17.0477484	-.0010994	10.6101226
5	20.4312095	.0004962	-16.8498190
6	23.7905310	-.0002546	24.9535490
7	27.1349446	.0001431	-35.1068657
8	30.4694509	-.0000862	47.4872266
9	33.7970147	.0000548	-62.2651906
10	37.1195133	-.0000364	79.6053627
11	40.4381970	.0000251	-99.6671336
12	43.7539320	-.0000178	122.6052857

r^+	R_{14}	R_{24}	R_{34}	R_{44}	R_{54}	R_{64}	R_{74}	R_{84}	R_{94}	R_{104}	R_{114}	R_{124}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.10	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0000	.0000	.0000
.15	.0005	.0005	.0004	.0004	.0003	.0003	.0002	.0002	.0001	.0001	.0001	.0000
.20	.0015	.0013	.0011	.0009	.0006	.0004	.0003	.0002	.0001	.0000	-.0000	-.0000
.25	.0035	.0028	.0021	.0014	.0009	.0004	.0001	-.0000	-.0001	-.0001	-.0000	-.0000
.30	.0068	.0050	.0033	.0018	.0007	.0001	-.0001	-.0001	-.0001	.0000	.0000	.0000
.35	.0118	.0077	.0042	.0016	.0003	-.0002	-.0002	-.0000	.0000	.0000	.0000	-.0000
.40	.0186	.0106	.0044	.0009	-.0003	-.0003	-.0000	.0001	.0000	-.0000	-.0000	-.0000
.45	.0273	.0131	.0038	-.0001	-.0006	-.0001	.0001	.0001	-.0000	-.0000	.0000	.0000
.50	.0376	.0145	.0023	-.0010	-.0004	.0002	.0001	-.0001	-.0000	.0000	.0000	-.0000
.55	.0492	.0146	.0003	-.0012	.0001	.0003	-.0001	-.0001	.0000	.0000	-.0000	.0000
.60	.0615	.0129	-.0016	-.0008	.0005	.0000	-.0001	.0001	.0000	-.0000	.0000	.0000
.65	.0735	.0097	-.0028	.0000	.0004	-.0002	.0000	.0001	-.0000	.0000	.0000	-.0000
.70	.0846	.0052	-.0030	.0008	.0000	-.0002	.0001	-.0000	-.0000	.0000	-.0000	.0000
.75	.0937	.0002	-.0021	.0010	-.0004	.0000	.0001	-.0001	.0000	-.0000	-.0000	.0000
.80	.1002	-.0045	-.0005	.0007	-.0004	.0002	-.0001	.0000	.0000	-.0000	.0000	-.0000
.85	.1036	-.0082	.0012	-.0000	-.0001	.0001	-.0001	.0001	-.0000	.0000	-.0000	.0000
.90	.1041	-.0104	.0024	-.0007	.0002	-.0001	.0000	.0000	-.0000	.0000	-.0000	.0000
.95	.1026	-.0112	.0029	-.0011	.0005	-.0002	.0001	-.0001	.0000	-.0000	.0000	-.0000
1.00	.1013	-.0112	.0030	-.0011	.0005	-.0003	.0001	-.0001	.0001	-.0000	.0000	-.0000

Table C1. Continued

(1) s = 2 , p = 5

n	λ_{n5}	$R_{n5}(1)$	$\hat{a}_{n5}$
1	11.5093951	.0089607	-3.6425719
2	15.3489579	-.0017523	14.9425284
3	19.2994476	.0004851	-38.4763062
4	23.2827811	-.0001716	79.9319981
5	27.2774051	.0000720	-145.8859915
6	31.2764166	-.0000341	243.7047100
7	35.2772482	.0000177	-381.4990857
8	39.2788365	-.0000099	568.0912561
9	43.2807084	.0000059	-812.9867262
10	47.2826429	-.0000036	1126.3509034
11	51.2844662	.0000024	-1519.0631876
12	55.2883022	-.0000016	1996.5388387

r^+	R_{15}	R_{25}	R_{35}	R_{45}	R_{55}	R_{65}	R_{75}	R_{85}	R_{95}	R_{105}	R_{115}	R_{125}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.10	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.15	.0001	.0001	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.20	.0003	.0002	.0002	.0001	.0001	.0000	.0000	.0000	.0000	-.0000	-.0000	-.0000
.25	.0007	.0005	.0004	.0002	.0001	.0000	.0000	-.0000	-.0000	-.0000	.0000	.0000
.30	.0015	.0010	.0005	.0002	.0001	-.0000	-.0000	-.0000	.0000	.0000	.0000	-.0000
.35	.0027	.0015	.0006	.0001	-.0000	-.0000	-.0000	.0000	.0000	-.0000	-.0000	-.0000
.40	.0043	.0019	.0005	-.0000	-.0001	-.0000	.0000	.0000	-.0000	-.0000	.0000	.0000
.45	.0061	.0021	.0002	-.0001	-.0000	.0000	.0000	-.0000	-.0000	.0000	.0000	-.0000
.50	.0080	.0020	-.0001	-.0002	.0000	.0000	-.0000	-.0000	.0000	-.0000	-.0000	.0000
.55	.0097	.0015	-.0004	-.0001	.0001	-.0000	-.0000	.0000	-.0000	-.0000	.0000	-.0000
.60	.0111	.0008	-.0005	.0001	.0000	-.0000	.0000	.0000	-.0000	.0000	-.0000	-.0000
.65	.0119	-.0000	-.0004	.0001	-.0000	-.0000	.0000	-.0000	.0000	.0000	-.0000	.0000
.70	.0122	-.0008	-.0002	.0001	-.0001	.0000	-.0000	-.0000	.0000	-.0000	.0000	-.0000
.75	.0120	-.0014	.0001	.0001	-.0000	.0000	-.0000	.0000	-.0000	-.0000	.0000	-.0000
.80	.0114	-.0017	.0003	-.0000	-.0000	.0000	-.0000	.0000	-.0000	.0000	-.0000	.0000
.85	.0106	-.0019	.0004	-.0001	.0000	-.0000	.0000	.0000	-.0000	.0000	-.0000	.0000
.90	.0098	-.0019	.0005	-.0002	.0001	-.0000	.0000	-.0000	.0000	-.0000	.0000	-.0000
.95	.0092	-.0018	.0005	-.0002	.0001	-.0000	.0000	-.0000	.0000	-.0000	.0000	-.0000
1.00	.0090	-.0018	.0005	-.0002	.0001	-.0000	.0000	-.0000	.0000	-.0000	.0000	-.0000

Table Cl. Continued

(2) s = 4 , p = 5

n	λ_{n5}	$R_{n5}(1)$	$\hat{a}_{n5}$
1	9.4006426	.0198833	-2.3410735
2	13.1084620	-.0031997	8.7993127
3	16.8127470	.0008280	-21.8642927
4	20.4896963	-.0002845	44.6702776
5	24.1461069	.0001175	-80.7910603
6	27.7883044	-.0000552	134.2198709
7	31.4205312	.0000285	-209.3511215
8	35.0455654	-.0000159	310.9636774
9	38.6652643	.0000094	-444.2061692
10	42.2809062	-.0000058	614.5840041
11	45.8933947	.0000037	-827.9481789
12	49.5033847	-.0000025	1090.4856337

r^+	R_{15}	R_{25}	R_{35}	R_{45}	R_{55}	R_{65}	R_{75}	R_{85}	R_{95}	R_{105}	R_{115}	R_{125}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.10	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.15	.0001	.0001	.0001	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.20	.0003	.0002	.0002	.0002	.0001	.0001	.0000	.0000	.0000	.0000	-.0000	-.0000
.25	.0008	.0006	.0004	.0003	.0002	.0001	.0000	.0000	-.0000	-.0000	-.0000	.0000
.30	.0017	.0012	.0008	.0004	.0002	.0000	-.0000	-.0000	-.0000	.0000	.0000	.0000
.35	.0033	.0021	.0010	.0004	.0000	-.0000	-.0000	-.0000	.0000	.0000	-.0000	-.0000
.40	.0055	.0029	.0011	.0002	-.0001	-.0001	.0000	.0000	.0000	-.0000	-.0000	.0000
.45	.0084	.0036	.0008	-.0001	-.0001	.0000	.0000	.0000	-.0000	-.0000	.0000	.0000
.50	.0118	.0039	.0003	-.0003	-.0000	.0001	.0000	-.0000	.0000	.0000	-.0000	-.0000
.55	.0153	.0035	-.0002	-.0002	.0001	.0000	-.0000	.0000	.0000	-.0000	-.0000	.0000
.60	.0185	.0027	-.0007	-.0001	.0001	-.0000	-.0000	.0000	-.0000	-.0000	.0000	-.0000
.65	.0212	.0013	-.0008	.0002	.0000	-.0000	.0000	-.0000	-.0000	.0000	-.0000	-.0000
.70	.0230	-.0002	-.0006	.0002	-.0001	-.0000	.0000	-.0000	.0000	-.0000	-.0000	.0000
.75	.0238	-.0015	-.0002	.0002	-.0001	.0000	-.0000	-.0000	.0000	-.0000	.0000	-.0000
.80	.0237	-.0025	.0003	.0000	-.0001	.0000	-.0000	.0000	-.0000	.0000	-.0000	-.0000
.85	.0228	-.0031	.0006	-.0001	.0000	.0000	-.0000	.0000	-.0000	.0000	-.0000	.0000
.90	.0215	-.0033	.0008	-.0002	.0001	-.0000	.0000	-.0000	.0000	-.0000	.0000	.0000
.95	.0204	-.0033	.0008	-.0003	.0001	-.0001	.0000	-.0000	.0000	-.0000	.0000	-.0000
1.00	.0199	-.0032	.0008	-.0003	.0001	-.0001	.0000	-.0000	.0000	-.0000	.0000	-.0000

Table C1. Continued

(3) s = 6 , p = 5

n	λ_{n5}	$R_{n5}(1)$	$\hat{a}_{n5}$
1	8.5608291	.0306113	-1.8604233
2	12.3094542	-.0041888	6.7529728
3	15.9451038	.0010304	-16.7287286
4	19.5166508	-.0003461	34.2262721
5	23.0521057	.0001412	-62.0081053
6	26.5654665	-.0000659	103.1599838
7	30.0643363	.0000339	-161.0773711
8	33.5531832	-.0000188	239.4523612
9	37.0348038	.0000111	-342.2622512
10	40.5110381	-.0000068	473.7597640
11	43.9831482	.0000044	-638.4640776
12	47.4520287	-.0000029	841.1532891

r^+	R_{15}	R_{25}	R_{35}	R_{45}	R_{55}	R_{65}	R_{75}	R_{85}	R_{95}	R_{105}	R_{115}	R_{125}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.10	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.15	.0001	.0001	.0001	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.20	.0003	.0002	.0002	.0002	.0001	.0001	.0001	.0000	.0000	.0000	-.0000	-.0000
.25	.0008	.0007	.0005	.0003	.0002	.0001	.0000	.0000	-.0000	-.0000	-.0000	-.0000
.30	.0018	.0013	.0009	.0005	.0002	.0001	-.0000	-.0000	-.0000	-.0000	.0000	.0000
.35	.0036	.0023	.0012	.0005	.0001	-.0000	-.0000	-.0000	.0000	.0000	.0000	-.0000
.40	.0062	.0034	.0014	.0003	-.0001	-.0001	-.0000	.0000	.0000	-.0000	-.0000	-.0000
.45	.0097	.0044	.0012	-.0000	-.0002	-.0000	.0000	.0000	-.0000	-.0000	.0000	.0000
.50	.0139	.0050	.0007	-.0003	-.0001	.0000	.0000	-.0000	-.0000	.0000	.0000	-.0000
.55	.0187	.0050	.0000	-.0004	.0000	.0001	-.0000	-.0000	.0000	-.0000	-.0000	.0000
.60	.0235	.0042	-.0006	-.0002	.0001	-.0000	-.0000	.0000	.0000	-.0000	.0000	.0000
.65	.0279	.0028	-.0010	.0001	.0001	-.0001	.0000	.0000	-.0000	.0000	.0000	-.0000
.70	.0315	.0009	-.0009	.0003	-.0000	-.0000	.0000	-.0000	.0000	.0000	-.0000	.0000
.75	.0337	-.0010	-.0005	.0003	-.0001	.0000	-.0000	-.0000	.0000	-.0000	.0000	-.0000
.80	.0346	-.0026	.0001	.0001	-.0001	.0001	-.0000	.0000	-.0000	-.0000	.0000	-.0000
.85	.0341	-.0037	.0006	-.0001	.0000	.0000	-.0000	.0000	-.0000	.0000	-.0000	.0000
.90	.0328	-.0042	.0009	-.0003	.0001	-.0000	.0000	-.0000	.0000	.0000	-.0000	.0000
.95	.0314	-.0043	.0010	-.0003	.0001	-.0001	.0000	-.0000	.0000	-.0000	.0000	-.0000
1.00	.0306	-.0042	.0010	-.0003	.0001	-.0001	.0000	-.0000	.0000	-.0000	.0000	-.0000

Table C1. Continued

(4) s = 8 , p = 5

n	λ_{n5}	$R_{n5}(1)$	$\hat{a}_{n5}$
1	8.0989038	.0401531	-1.6170777
2	11.9033733	-.0048301	5.7666353
3	15.5044533	.0011494	-14.3194808
4	19.0204428	-.0003808	29.3654395
5	22.4925423	.0001542	-53.2833478
6	25.9388015	-.0000716	88.7330596
7	29.3684688	.0000367	-138.6417775
8	32.7868083	-.0000203	206.1923066
9	36.1970512	.0000120	-294.8130332
10	39.6012954	-.0000074	408.1694765
11	43.0009678	.0000047	-550.1563858
12	46.3970714	-.0000032	724.8915468

r^+	R_{15}	R_{25}	R_{35}	R_{45}	R_{55}	R_{65}	R_{75}	R_{85}	R_{95}	R_{105}	R_{115}	R_{125}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.10	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.15	.0001	.0001	.0001	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.20	.0003	.0003	.0002	.0002	.0001	.0001	.0001	.0000	.0000	.0000	.0000	-.0000
.25	.0008	.0007	.0005	.0004	.0002	.0001	.0001	.0000	-.0000	-.0000	-.0000	-.0000
.30	.0019	.0014	.0009	.0005	.0002	.0001	-.0000	-.0000	-.0000	-.0000	.0000	.0000
.35	.0037	.0024	.0013	.0006	.0001	-.0000	-.0000	-.0000	.0000	.0000	.0000	-.0000
.40	.0065	.0037	.0016	.0004	-.0000	-.0001	-.0000	.0000	.0000	-.0000	-.0000	-.0000
.45	.0104	.0049	.0015	.0001	-.0002	-.0000	.0000	.0000	-.0000	-.0000	.0000	.0000
.50	.0152	.0058	.0010	-.0003	-.0001	.0000	.0000	-.0000	-.0000	.0000	.0000	-.0000
.55	.0208	.0060	.0003	-.0004	.0000	.0001	-.0000	-.0000	.0000	.0000	-.0000	.0000
.60	.0268	.0054	-.0005	-.0003	.0001	.0000	-.0000	.0000	.0000	-.0000	.0000	.0000
.65	.0326	.0040	-.0010	.0000	.0001	-.0001	.0000	.0000	-.0000	.0000	.0000	-.0000
.70	.0377	.0019	-.0011	.0003	.0000	-.0000	.0000	-.0000	-.0000	.0000	-.0000	.0000
.75	.0414	-.0004	-.0007	.0003	-.0001	.0000	.0000	-.0000	.0000	-.0000	.0000	.0000
.80	.0434	-.0025	-.0001	.0002	-.0001	.0001	-.0000	.0000	-.0000	-.0000	.0000	-.0000
.85	.0437	-.0039	.0006	-.0001	-.0000	.0000	-.0000	.0000	-.0000	.0000	-.0000	.0000
.90	.0427	-.0047	.0010	-.0003	.0001	-.0000	.0000	-.0000	-.0000	.0000	-.0000	.0000
.95	.0411	-.0049	.0011	-.0004	.0001	-.0001	.0000	-.0000	.0000	-.0000	.0000	-.0000
1.00	.0402	-.0048	.0011	-.0004	.0002	-.0001	.0000	-.0000	.0000	-.0000	.0000	-.0000

Table C1. Continued

(5) s = 10 , p = 5

n	λ_{n5}	$R_{n5}(1)$	$\hat{a}_{n5}$
1	7.8039530	.0483871	-1.4719884
2	11.6567158	-.0052575	5.1896564
3	15.2361198	.0012239	-12.9196022
4	18.7177989	-.0004019	26.5378478
5	22.1510427	.0001620	-48.1982620
6	25.5562654	-.0000750	80.3103355
7	28.9436676	.0000384	-125.5256458
8	32.3189754	-.0000212	186.7269795
9	35.6856725	.0000125	-267.0192815
10	39.0460087	-.0000077	369.7218605
11	42.4015058	.0000049	-498.3615106
12	45.7532331	-.0000033	656.6663036

r^+	R_{15}	R_{25}	R_{35}	R_{45}	R_{55}	R_{65}	R_{75}	R_{85}	R_{95}	R_{105}	R_{115}	R_{125}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.10	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.15	.0001	.0001	.0001	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.20	.0003	.0003	.0002	.0002	.0001	.0001	.0001	.0000	.0000	.0000	.0000	-.0000
.25	.0008	.0007	.0005	.0004	.0002	.0001	.0001	.0000	-.0000	-.0000	-.0000	-.0000
.30	.0019	.0014	.0010	.0006	.0003	.0001	.0000	-.0000	-.0000	-.0000	.0000	.0000
.35	.0038	.0025	.0014	.0006	.0002	-.0000	-.0000	-.0000	.0000	.0000	.0000	-.0000
.40	.0067	.0039	.0017	.0005	-.0000	-.0001	-.0000	.0000	.0000	.0000	-.0000	-.0000
.45	.0108	.0052	.0017	.0001	-.0002	-.0001	.0000	.0000	-.0000	-.0000	.0000	.0000
.50	.0160	.0063	.0012	-.0002	-.0002	.0000	.0000	-.0000	-.0000	.0000	.0000	-.0000
.55	.0223	.0067	.0005	-.0004	-.0000	.0001	-.0000	-.0000	.0000	.0000	-.0000	-.0000
.60	.0291	.0062	-.0004	-.0004	.0001	.0000	-.0000	.0000	.0000	-.0000	.0000	.0000
.65	.0359	.0049	-.0010	-.0001	.0001	-.0001	-.0000	.0000	-.0000	-.0000	.0000	-.0000
.70	.0422	.0027	-.0012	.0002	.0000	-.0001	.0000	-.0000	-.0000	.0000	-.0000	.0000
.75	.0472	.0002	-.0009	.0004	-.0001	.0000	.0000	-.0000	.0000	-.0000	-.0000	.0000
.80	.0505	-.0022	-.0002	.0003	-.0001	.0001	-.0000	.0000	.0000	-.0000	.0000	-.0000
.85	.0516	-.0040	.0005	-.0000	-.0000	.0000	-.0000	.0000	-.0000	.0000	-.0000	.0000
.90	.0510	-.0050	.0010	-.0003	.0001	-.0000	.0000	.0000	-.0000	.0000	-.0000	.0000
.95	.0494	-.0053	.0012	-.0004	.0002	-.0001	.0000	-.0000	.0000	-.0000	.0000	-.0000
1.00	.0484	-.0053	.0012	-.0004	.0002	-.0001	.0000	-.0000	.0000	-.0000	.0000	-.0000

Table C1. Continued

(6) $s = 12$, $p = 5$

n	λ_{n5}	$R_{n5}(1)$	$\hat{a}_{n5}$
1	7.5983806	.0554438	-1.3763512
2	11.4895393	-.0055569	4.8100590
3	15.0544198	.0012735	-11.9973410
4	18.5130022	-.0004155	24.6702406
5	21.9200992	.0001669	-44.8327789
6	25.2977102	-.0000771	74.7273637
7	28.6566736	.0000394	-116.8214573
8	32.0030248	-.0000218	173.7973922
9	35.3404174	.0000128	-248.5439040
10	38.6712030	-.0000079	344.1488744
11	41.9969684	.0000051	-463.8929663
12	45.3189249	-.0000034	611.2442014

r^+	R_{15}	R_{25}	R_{35}	R_{45}	R_{55}	R_{65}	R_{75}	R_{85}	R_{95}	R_{105}	R_{115}	R_{125}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.10	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.15	.0001	.0001	.0001	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.20	.0003	.0003	.0002	.0002	.0001	.0001	.0001	.0000	.0000	.0000	.0000	-.0000
.25	.0008	.0007	.0005	.0004	.0002	.0001	.0001	.0000	-.0000	-.0000	-.0000	-.0000
.30	.0020	.0015	.0010	.0006	.0003	.0001	.0000	-.0000	-.0000	-.0000	.0000	.0000
.35	.0039	.0026	.0015	.0007	.0002	-.0000	-.0000	-.0000	.0000	.0000	.0000	-.0000
.40	.0069	.0040	.0018	.0005	.0000	-.0001	-.0000	.0000	.0000	.0000	-.0000	-.0000
.45	.0111	.0054	.0018	.0002	-.0002	-.0001	.0000	.0000	-.0000	-.0000	-.0000	.0000
.50	.0166	.0066	.0014	-.0002	-.0002	.0000	.0000	-.0000	-.0000	.0000	.0000	[illegible]
.55	.0233	.0072	.0006	-.0004	-.0000	.0001	-.0000	-.0000	.0000	.0000	-.0000	-.0000
.60	.0307	.0068	-.0003	-.0004	.0001	.0000	-.0000	.0000	.0000	-.0000	-.0000	.0000
.65	.0384	.0055	-.0010	-.0001	.0002	-.0000	-.0000	.0000	-.0000	-.0000	.0000	-.0000
.70	.0457	.0034	-.0013	.0002	.0001	-.0001	.0000	-.0000	-.0000	.0000	-.0000	-.0000
.75	.0518	.0008	-.0010	.0004	-.0001	-.0000	.0000	-.0000	.0000	-.0000	-.0000	.0000
.80	.0561	-.0018	-.0004	.0003	-.0001	.0001	-.0000	.0000	.0000	-.0000	.0000	-.0000
.85	.0581	-.0039	.0004	.0000	-.0001	.0000	-.0000	.0000	-.0000	.0000	-.0000	.0000
.90	.0581	-.0052	.0010	-.0003	.0001	-.0000	.0000	.0000	-.0000	.0000	-.0000	.0000
.95	.0566	-.0056	.0013	-.0004	.0002	-.0001	.0000	-.0000	.0000	-.0000	.0000	-.0000
1.00	.0554	-.0056	.0013	-.0004	.0002	-.0001	.0000	-.0000	.0000	-.0000	.0000	-.0000