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Title: EFFECTS OF CONJUNCTIVITY ON THE INERTIA OF
COMPLEX MATRICES

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The $n \times n$ complex matrices are studied as to the possibility or impossibility, conjunctively, of increasing, decreasing, or incomparably changing their inertias.

Sample Result 1. Let S be an $n \times n$ matrix and let (1), (2), (3), (4) and (5) denote the following conditions.

- (1) $\text{In } C^*SC \leq \text{In } S$ for every nonsingular $n \times n$ C ;
- (2) $\text{In } C^*SC \leq \text{In } S$ for every $n \times n$ C ;
- (3) $\text{In } C^*SC = \text{In } S$ for every nonsingular $n \times n$ C ;
- (4) the numerical range (= field of values) of S is contained in the imaginary axis or else is disjoint from the nonzero part of the imaginary axis;
- (5) S is a scalar multiple of a hermitian matrix or else is unitarily similar to $S_1 \oplus 0$ for some positive or negative H -stable matrix S_1 .

Then (1) and (2) are equivalent, and so are (3), (4) and (5).

Sample Result 2. If the inertia of an $n \times n$ matrix $S \neq (0,0,n)$, and its column space \neq the column space of S^* , then S is conjunctive with a matrix of smaller inertia and S is also conjunctive with a matrix of incomparable inertia.

Similar results are derived concerning $\pi(S)$ (= the number of eigenvalues of S with positive real part) and $\delta(S)$ (= the number of pure imaginary eigenvalues of S).

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EFFECTS OF CONJUNCTIVITY ON THE INERTIA OF COMPLEX MATRICES

I. INTRODUCTION

In this thesis we consider, unless otherwise specified, only $n \times n$ matrices with entries in the field \mathbf{C} of all complex numbers. Let S be one such. As in [5], we let $\pi(S)$ denote the number of roots of S (i. e., of the polynomial $p(z) = \det(zI - S)$, counted according to multiplicities occurring) with positive real parts. Let $\nu(S) = \pi(-S)$ and $\delta(S) = n - \pi(S) - \nu(S)$. Let $\text{In } S = (\pi(S), \nu(S), \delta(S))$. $\text{In } S$ is called the inertia of S . We sometimes write π, ν, δ and In for $\pi(S), \nu(S), \delta(S)$ and $\text{In}(S)$ when S is understood and no confusion seems likely. In particular, this π is not the ratio of the circumference of a circle to its diameter! By $\text{In } S \leq \text{In } T$ (or $\text{In } T \geq \text{In } S$) we mean $\pi(S) \leq \pi(T)$ and $\nu(S) \leq \nu(T)$. By $\text{In } S < \text{In } T$ (or $\text{In } T > \text{In } S$) we mean $\text{In } S \leq \text{In } T$ and $\text{In } S \neq \text{In } T$. T is said to be semiconjunctive with S if there is an $n \times n$ matrix C such that $T = C*SC$, and conjunctive with S if there is a nonsingular such C . We shall investigate some of the effects conjunctivity and semiconjunctivity have on the inertias of several types (to be later defined) of $n \times n$ complex matrices. In particular we, in some theorems, address ourselves to the following problems, where ϕ stands for any one of π, δ, In .

Problems. Find necessary and sufficient conditions on S that:

(Problem 1) $\phi(C*SC) = \phi(S)$ for all nonsingular $n \times n$ C

(Problem 2) $\phi(C*SC) \geq \phi(S)$ for all nonsingular $n \times n$ C

(Problem 3) $\phi(C*SC) \leq \phi(S)$ for all nonsingular $n \times n$ C

(Problem 4) $\phi(C*SC) \leq \phi(S)$ for all $n \times n$ C

(Problem 5) $\phi(C*SC) \geq \phi(S)$ for all $n \times n$ C

(Problem 6) $\phi(C*SC) = \phi(S)$ for all $n \times n$ C

There are then 18 such problems. We abbreviate them $(1)_{In}$, $(1)_{\pi}$, $(1)_{\delta}$ (and $(1)_{\phi}$ denotes any of these, unless otherwise stipulated) etc.

If there is a matrix T conjunctive with S for which $In T > In S$ we say $In S$ can be conjunctively increased (and use similar terminology regarding π and δ). We sometimes, though rarely, then abbreviate this to

$In \uparrow$

(with obvious analogues concerning decreases and π and δ). If there is a matrix T conjunctive with S for which $In T \not\leq In S$ and $In T \not\geq In S$ we say $In S$ can be conjunctively incomparably changed, and sometimes, though rarely, abbreviate this to

$In i. c.$

We also then say $In T$ is incomparable with $In S$. There are then two ways in which each of π , ν , and δ may be changed, and three

ways (increased, decreased or incomparably changed) in which I_n may be changed. The above 18 problems then concern the impossibility of conjunctively or semiconjunctively making various changes in π , δ , and I_n . One can sometimes more easily adopt the opposite viewpoint and investigate instead the possibility (for a specified type of matrix) of making each type of change, and thus determine nonsolutions of various ones of the 18 $(j)_\phi$.

Let S be an $n \times n$ matrix and let M and N be non-empty (ordered) subsets of $\{1, \dots, n\}$. We denote by $S[M|N]$ the submatrix of S lying in rows whose indices come from M and in columns whose indices come from N . We denote by S_{jk} the entry in the 1×1 submatrix $S[j|k]$. We give meanings to $S[M|N]$, $S(M|N)$, and $S(M|N)$ corresponding to their respective meanings in [1, p. 264-265], e. g., $S[M|N) = S[M|N']$, where N' is the subset complementary to N . We abbreviate further for principal submatrices, putting $S[M] = S[M|M]$ and $S(M) = S(M|M)$. Next we say that a nonsingular matrix C defines the conjunctivity $S \rightarrow C*SC$ and that the order of this conjunctivity is the order of C . Finally, let m be the cardinal of M and let D be an $m \times m$ nonsingular matrix. Then by "the $[M]$ subconjunctivity of order n defined by D " we mean the conjunctivity (of order n) defined by the $n \times n$ matrix C satisfying $C[M] = D$, $C[M|M) = 0$, $C(M|M) = 0$, and $C(M) = I$, where I is the identity matrix of

order $n-m$. (We shall usually not specify the order of a subconjunctivity when it is clear from context.) When $j < k$, the $[j, k]$ subconjunctivity (of order n) defined by

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

will be called "the interchanging $[j, k]$ subconjunctivity (of order n)."

We state now without proof a couple of facts about conjunctivities from [2].

Fact 1.1 ([2, Fact 1.2]). Let A, B, C, D be matrices of respective dimensions $p \times p, p \times q, q \times p, q \times q$, and let A be nonsingular. Then there are uniquely defined matrices C_1 and D_1 such that the conjunctivity of order $p+q$ defined by the matrix (in block form)

$$\begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix}$$

has the following effect:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \rightarrow \begin{bmatrix} A & 0 \\ C_1 & D_1 \end{bmatrix}.$$

Fact 1.2 ([2, Fact 1.3]). Let S be an $n \times n$ matrix and r be the rank of S . Then S is conjunctive with a lower triangular

matrix whose first r diagonal entries are nonzero and whose last $n-r$ columns are zero.

The examination of possible inertia changes in a given matrix due to conjunctivity and semiconjunctivity leads us to study its "cone." Let S be an $n \times n$ complex matrix. By the cone, $\Gamma(S)$, of S we mean the set of all complex numbers X^*SX for which X is an $n \times 1$ complex matrix. From [2] we collect here some of its salient features.

Fact 1.3 ([2, Fact 3.1]). (i) $\Gamma(zS) = (\text{sgn } z)\Gamma(S)$ for every nonzero $z \in \mathbb{C}$;

(ii) Γ is conjunctively invariant, i. e., $\Gamma(C^*SC) = \Gamma(S)$ for every nonsingular C and every S of the same order as C ;

(iii) $\Gamma(S_1) \subset \Gamma(S)$ for every principal submatrix S_1 of S (" \subset " means weak inclusion);

(iv) every diagonal entry of S lies in $\Gamma(S)$;

(v) $\Gamma(S)$ consists of zero, and the numbers T_{11} for which T is conjunctive with S (in particular, $\Gamma(S)$ contains all the eigenvalues of S);

(vi) $\Gamma(S) = \Gamma(S_1) + \Gamma(S_2)$ whenever S is the direct sum of S_1 and S_2 ;

(vii) $\Gamma(S)$ is a convex cone for every S . In fact, $\Gamma(S)$ is the cone generated by the numerical range (= field of values) of S .

Next we introduce ([2] also) some algebraic definitions about S , together with the cone characterization of each defined matrix type. For each $n \times n$ complex matrix S and each real θ (θ will always be real when used in this thesis) we define a (hermitian $n \times n$) matrix $H(\theta;S)$ by

$$H(\theta;S) \equiv \frac{1}{2} (e^{-i\theta} S + e^{i\theta} S^*).$$

(When S is understood, we shall sometimes write $H(\theta)$ instead of $H(\theta;S)$.) The cone of a given matrix is one of six geometric configurations in \mathbb{C} : a point (the origin), a half-line (including its end point 0), a line (through 0), a (plane closed convex) sector (with vertex 0), a half-plane (with 0 on its boundary), or \mathbb{C} itself. When we say "half-plane," "line," or "half-line," we shall mean one of the above special such. In [2] the first of each of the following pairs of "iffs" is taken as the definition of the underlined word. An $n \times n$ complex matrix S is called:

(i) contrahermitian iff $e^{-i\theta} S$ is nonhermitian for all θ
iff $\Gamma(S)$ has nonempty interior;

(ii) cohermitian iff S is not contrahermitian
iff $\Gamma(S)$ is a subcone of some line;

(iii) contradefinite iff $H(\theta)$ is indefinite for all θ ;
iff $\Gamma(S) = \mathbb{C}$ (iff 0 is an interior point of the numerical range of S);

- (iv) codefinite iff S is not contradefinite
iff $\Gamma(S)$ is a subcone of some half-plane;
- (v) unidefinite iff there is a θ for which $e^{-i\theta}S$ is a positive semidefinite nonzero hermitian matrix
iff $\Gamma(S)$ is a half-line;
- (vi) bidefinite iff there is a θ for which $e^{-i\theta}S$ is an indefinite hermitian matrix
iff $\Gamma(S)$ is a line;
- (vii) transdefinite iff there is exactly one value of $e^{i\theta}$ for which $H(\theta)$ is positive semidefinite and nonzero
iff $\Gamma(S)$ is a half-plane;
- (viii) prodefinite iff S is codefinite but not transdefinite nor cohermitian
iff $\Gamma(S)$ is a sector;
- (xi) the zero (matrix) iff $S_{jk} = 0$ for all $j, k = 1, \dots, n$
iff $\Gamma(S)$ is the zero cone.

We wish also to examine the so-called $*$ -regular (or EP or EP_r of [3]) and $*$ -irregular matrices as concerns their inertia changes. An $n \times n$ complex matrix S is called $*$ -regular provided $SX = 0$ implies $S^*X = 0$ for $n \times 1$ complex matrices X . It is called $*$ -irregular if it is not $*$ -regular. The following result is well known but proved here for the sake of completeness.

Lemma 1.1. If H is semidefinite and $X^*HX = 0$ then $HX = 0$.

Proof. Define $\langle X, Y \rangle_H = X^*HY$. Then

$$\begin{aligned} (X^*H^2X)^2 &= \langle X, HX \rangle_H^2 \leq \langle X, X \rangle_H \langle HX, HX \rangle_H \\ &= (X^*HX)(X^*H^3X) \end{aligned}$$

where the inequality follows from the Schwarz Inequality for the semidefinite bilinear form $\langle X, Y \rangle_H$.

Theorem 1.1. Every codefinite matrix is $*$ -regular.

Proof. Let S be codefinite. Then there is a real θ such that $e^{-i\theta}S + e^{i\theta}S^*$ is semidefinite. Let $SX = 0$. Then $X^*SX = 0$ and $X^*S^*X = (X^*SX)^* = 0$ and thus $X^*(e^{-i\theta}S + e^{i\theta}S^*)X = 0$. Then, by Lemma 1.1, $(e^{-i\theta}S + e^{i\theta}S^*)X = 0$ and so $S^*X = 0$. Therefore S is $*$ -regular.

A fundamental characterization of $*$ -regularity is the following theorem, proved in [3, p. 49].

Theorem 1.2. An $n \times n$ matrix S of rank r is $*$ -regular iff S is unitarily similar to the direct sum of a nonsingular $r \times r$ matrix and a zero matrix.

We make use also of the following two well known theorems.

Theorem 1.3 (Sylvester's Law of Inertia). If S is hermitian and T is conjunctive with S , then $\text{In } T = \text{In } S$.

Proof. See, e.g., [4, p. 83].

We denote the diagonal, $(S_{11}, S_{22}, \dots, S_{nn})$, of the $n \times n$ matrix S by diag S .

Theorem 1.4 (Schur's Triangularization Theorem). If S is $n \times n$ and $\lambda_1, \dots, \lambda_n$ is any ordering of its eigenvalues then S is unitarily conjunctive with a lower triangular matrix of diagonal $= (\lambda_1, \lambda_2, \dots, \lambda_n)$.

Proof. See, e.g., [4, p. 67].

A basic theorem, on which nearly all of our results ultimately depend, is the following.

Lemma 1.2 ([2, Lemma 4.1]). (i) Let β be real and γ be nonnegative. Then there is a conjunctivity having the effect

$$e^{i\beta} \begin{bmatrix} 1 & 0 \\ 2\gamma & 1 \end{bmatrix} \rightarrow e^{i\beta} \begin{bmatrix} e^{ia} & 0 \\ 2(\gamma^2 - \sin^2 a)^{1/2} & e^{-ia} \end{bmatrix}$$

for all real a such that

$$|\sin \alpha| \leq \gamma \quad \text{and} \quad (\gamma^2 - \sin^2 \alpha)^{1/2} + \cos \alpha > 0,$$

i. e., for all α when $\gamma > 1$, and for all α such that

$|\sin \alpha| \leq \gamma$ and $\cos \alpha > 0$ when $0 \leq \gamma \leq 1$. (ii) Thus, in par-

ticular, whenever $\frac{1}{2}\pi > |\alpha| \geq |\delta|$, every matrix of the form

$$A_1 = e^{i\beta} \begin{bmatrix} e^{i\alpha} & 0 \\ 2\rho & e^{-i\alpha} \end{bmatrix}$$

is conjunctive with a suitable matrix of the form

$$A_2 = e^{i\beta} \begin{bmatrix} e^{i\delta} & 0 \\ 2\kappa & e^{-i\delta} \end{bmatrix}$$

(where β is the same in A_2 as in A_1), namely, with one for which

$$|\rho|^2 + \sin^2 \alpha = |\kappa|^2 + \sin^2 \delta.$$

(Thus $\kappa \neq 0$ if $\frac{1}{2}\pi > |\alpha| > |\delta|$.) (iii) Furthermore, if

$|\alpha| = \frac{1}{2}\pi$ and $\rho \neq 0$ in A_1 , then A_1 is contradefinite and is

for every δ conjunctive with a suitable matrix of the form A_2

(namely, with one for which $|\rho|^2 + 1 = |\kappa|^2 + \sin^2 \delta$).

In applications of Lemma 1.2 we often use the notation now to be introduced. Let S be an $n \times n$ nonsingular lower triangular matrix and, for $j = 1, 2, \dots, n$, let $\sigma_j = \arg S_{jj}$ (with the usual

ambiguity in the definition of \arg). We then define $\arg \text{diag } S$ by the equation

$$\arg \text{diag } S = (\sigma_1, \sigma_2, \dots, \sigma_n) .$$

We shall write

$$(\sigma_1, \sigma_2, \dots, \sigma_n) \rightarrow (\tau_1, \tau_2, \dots, \tau_n)$$

as an abbreviation for the following conditional: "If S is any (nonsingular lower triangular) matrix whose $\arg \text{diag}$ is $(\sigma_1, \dots, \sigma_n)$, then there is a matrix whose $\arg \text{diag}$ is (τ_1, \dots, τ_n) and which is conjunctive with S ."

II. PRELIMINARIES AND GENERAL LEMMAS

In this chapter we state and prove several lemmas and theorems which apply to both codefinite and contradefinite $n \times n$ matrices.

Let S be an $n \times n$ matrix.

Lemma 2.1. $\Gamma(S) = \{0\} \cup \{T_{11} \mid T \text{ is lower triangular and conjunctive with } S\}$.

Proof. Let $0 \neq \gamma \in \Gamma(S)$. Then, by Fact 1.3(v), $\gamma = V_{11}$ for some matrix V conjunctive with S . Then, by Fact 1.1, V is conjunctive with a matrix $W = \begin{bmatrix} \gamma & 0 \\ C_1 & D_1 \end{bmatrix}$. Then, applying Fact 1.2 to D_1 and the resulting subconjunctivity to W , we see that W is conjunctive with a matrix $T = \begin{bmatrix} \gamma & 0 \\ C_2 & D_2 \end{bmatrix}$ where D_2 (and hence T) is lower triangular. Clearly T is then conjunctive with S .

Lemma 2.2. (a) Any change in π , In , or δ which can be accomplished by a conjunctivity can be accomplished also by a semiconjunctivity. (b) Contrapositively, any change in π , In , or δ which is impossible by semiconjunctivity is impossible also by conjunctivity.

Proof. Every conjunctivity is a semiconjunctivity.

Lemma 2.3 (Extreme Cases). Let S be $n \times n$ of rank r (then $0 \leq \pi \leq r \leq n$ and $0 \leq n-r \leq \delta \leq n$). If $\pi = 0$ (respectively,

$\delta = n-r$) it cannot be semiconjunctively, nor hence conjunctively, decreased. If $\pi = r$ (respectively, $\delta = n$) it cannot be semiconjunctively, nor hence conjunctively, increased. Also, when $\delta = n$, In cannot be incomparably changed nor decreased by semiconjunctivity, nor hence by conjunctivity. Finally, when $\delta = n-r$ then In cannot be semiconjunctively, nor hence conjunctively, increased.

Proof. Mostly obvious. $\delta \geq n-r$ because $\delta \geq$ the number of zero eigenvalues \geq the nullity of $S = n-r$.

Lemma 2.4 (The zero semiconjunctivity). If $\pi > 0$, π can be semiconjunctively decreased. If $\delta < n$, δ can be semiconjunctively increased. If $\text{In} \neq (0, 0, n)$, In can be semiconjunctively decreased.

Proof. The zero semiconjunctivity accomplishes all three.

Lemma 2.5. Let S be an arbitrary nonsingular matrix and $(r_1 e^{i\sigma_1}, \dots, r_n e^{i\sigma_n})$, where $r_j > 0$ for $j = 1, \dots, n$, be any ordering of its n eigenvalues. Let $(r'_1 e^{i\sigma'_1}, \dots, r'_n e^{i\sigma'_n})$ be any permutation of $(r_1 e^{i\sigma_1}, \dots, r_n e^{i\sigma_n})$. Then S is conjunctive with a lower triangular matrix T such that $\text{diag } T = (e^{i\sigma'_1}, \dots, e^{i\sigma'_n})$ (and hence $\text{In } T = \text{In } S$). Thus we may assume without loss of generality, for the purpose of determining what inertias and inertia changes are possible within the conjunctivity class of such S , that

S itself is lower triangular with conveniently ordered diagonal entries of unit modulus.

Proof. By Theorem 1.4, S is unitarily conjunctive with a lower triangular matrix R with $\text{diag } R = (r'_1 e^{i\sigma'_1}, \dots, r'_n e^{i\sigma'_n})$. Then if $C = \text{diag}((r'_1)^{-1/2}, \dots, (r'_n)^{-1/2})$, the matrix $T = C*RC$ satisfies the conditions of the lemma.

Lemma 2.6. Let S be nonsingular contrahermitian and $n \times n$. (a) If $\delta > 0$ then In can be increased, and hence δ decreased, by conjunctivity.

(b) If $\delta > 0$ and $\pi > 0$ then π can be increased by conjunctivity.

Proof. We may assume (by Lemma 2.5) S is lower triangular.

(a) We first assume $n > \delta > 0$. Let $\arg \text{diag } S = (\sigma_1, \dots, \sigma_n)$ with $\cos \sigma_1 = 0 \neq \cos \sigma_2$ and $|\sigma_1 - \sigma_2| < \pi$. This is possible (by Lemma 2.5) since $n > \delta > 0$. By Lemma 1.2,

$$(\sigma_1, \dots, \sigma_n) \rightarrow \left(\frac{1}{2}(\sigma_1 + \sigma_2), \frac{1}{2}(\sigma_1 + \sigma_2), \sigma_3, \dots, \sigma_n \right),$$

(decreasing δ and) increasing In . We next assume $\delta = n$. By contrahermitianness there exists $e^{i\gamma} \in \Gamma(S)$ with $\cos \gamma \neq 0$. By Lemma 2.1, S is conjunctive with a lower triangular matrix

T with $T_{11} = e^{i\gamma}$.

(b) Since $\pi(S) > 0$ and $\delta(S) > 0$ we may choose σ_1 and σ_2 satisfying $\cos \sigma_1 = 0 < \cos \sigma_2$ and $|\sigma_1 - \sigma_2| < \pi$. Then, for some integer m , $\sigma_1 = (2m+1)\frac{\pi}{2}$, and $(2m-1)\frac{\pi}{2} < \sigma_2 < (2m+3)\frac{\pi}{2}$. Because $\cos \sigma_2 > 0$, the stricter inequalities hold:

$$(2m-1)\frac{\pi}{2} < \sigma_2 < (2m+1)\frac{\pi}{2} \quad \text{if } m \text{ is even}$$

or

$$(2m+1)\frac{\pi}{2} < \sigma_2 < (2m+3)\frac{\pi}{2} \quad \text{if } m \text{ is odd.}$$

Then if m is even we have

$$(2m)\frac{\pi}{2} < \frac{\sigma_1 + \sigma_2}{2} < (2m+1)\frac{\pi}{2} \quad (\text{Quadrant I})$$

and if m is odd we have

$$(2m+1)\frac{\pi}{2} < \frac{\sigma_1 + \sigma_2}{2} < (2m+2)\frac{\pi}{2} \quad (\text{Quadrant IV}).$$

Thus in either case $\cos \frac{\sigma_1 + \sigma_2}{2} > 0$, and so the conjunctivity described in the proof of (a) produces a matrix of $\pi = \pi(S) + 1$.

Lemma 2.7. If $\begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix}$ is conjunctive with $\begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}$, where T and Q are $r \times r$ and nonsingular, then T and Q are conjunctive with each other.

Proof. There is a nonsingular $n \times n$ C such that

$C^* \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} C = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}$. We partition C conformably (C_{11} is $r \times r$) and have

$$\begin{bmatrix} C_{11}^* TC_{11} & C_{11}^* TC_{12} \\ C_{12}^* TC_{11} & C_{12}^* TC_{12} \end{bmatrix} = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus $C_{11}^* TC_{11} = Q$, so C_{11} is nonsingular $r \times r$, i.e., of rank r , since otherwise rank Q would be strictly less than r . (The rank of a product \leq the minimum of the ranks of its factors.)

Lemma 2.8. If S is 2×2 and (nonsingular and) $\text{sgn det } S = -1$, then $\text{In } S = (1, 1, 0)$ or $(0, 0, 2)$.

Proof. We may obviously assume without loss of generality that the eigenvalues of S are $e^{i\theta_1}$ and $e^{i\theta_2}$. Then

$$-1 = \text{sgn det } S = \det S = e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}.$$

Therefore $\theta_1 + \theta_2 = \pi \pmod{2\pi}$. Thus $e^{i\theta_1}$ and $e^{i\theta_2}$ have real parts of (equal magnitude and) opposite signs or else both have zero real parts, i.e., $\text{In } S = (1, 1, 0)$ or $(0, 0, 2)$.

Lemma 2.9. *-regularity is a conjunctively invariant property.

Proof. Let S be *-regular, $SX = 0$ and C be nonsingular. Then, if $C^*SC = 0$, we have

$$\begin{aligned} 0 &= SCX = S*CX \text{ (by the *-regularity of } S) = C*S*CX \\ &= (C*SC)*X, \end{aligned}$$

i.e., $C*SC$ is *-regular.

Lemma 2.10. If S is *-regular and T is semiconjunctive with and has the same ranks as S , then T is conjunctive with S .

Proof. We prove first that the statement of the lemma is conjunctively invariant. Namely, let the lemma hold for some $n \times n$ matrix S and every corresponding T , E be nonsingular $n \times n$, $E*SE$ be *-regular, and $\text{rank } C*E*SEC = \text{rank } E*SE$ (hence $= \text{rank } S$). From these we have that S is *-regular (by Lemma 2.9) and $C*E*SEC$ is conjunctive with S and hence with $E*SE$. Therefore to prove the lemma we may without loss of generality, by Theorem 1.2, assume that $S = \begin{bmatrix} N & 0 \\ 0 & 0 \end{bmatrix}$, N nonsingular $r \times r$ where r is the rank of S . Let

$$T = C*SC = \begin{bmatrix} C_{11}^* & C_{21}^* \\ C_{12}^* & C_{22}^* \end{bmatrix} \begin{bmatrix} N & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} C_{11}^*NC_{11} & C_{11}^*NC_{12} \\ C_{12}^*NC_{11} & C_{12}^*NC_{12} \end{bmatrix}$$

for some $n \times n$ matrix C , partitioned conformably with S .

Assume too that $\text{rank } T$ is also r . We observe that the above product is independent of C_{21} and C_{22} , and thus if both were

replaced by zero matrices the resulting equation corresponding to the

above would still hold. Therefore $[C_{11} \ C_{12}]$ has rank r (i.e., full rank), and so the first r rows of C are linearly independent. We complete these to a basis of $C^{1 \times n}$ (= the space of $1 \times n$ complex matrices) and take this basis as the respective rows of an $n \times n$ matrix \widehat{C} , with the first r rows of \widehat{C} the same as those of C . Then \widehat{C} is nonsingular and we have $C*SC = \widehat{C}*S\widehat{C}$, which is conjunctive with S .

Theorem 2.1. If S is $n \times n$ and $*$ -regular of rank 2 with the signum of the product of its two nonzero eigenvalues = -1 and $\delta(S) < n$, then S is a solution of (4) _{ϕ} (and therefore of (3) _{ϕ}) for $\phi = \text{In}$ or π .

Proof. In $S = (1, 1, n-2)$ throughout this theorem and its proof. By Theorem 1.2, S is unitarily conjunctive with a direct sum $N \oplus 0$, where N is nonsingular and 2×2 . Since $\delta(S) < n$, we must hence have $\delta(N) < 2$. Then the eigenvalues of N are the nonzero eigenvalues of S , so $\text{sgn det } N = -1$ and by Lemma 2.8 we have $\text{In } N = (1, 1, 0)$, the possibility $\text{In } N = (0, 0, 2)$ being ruled out by $\delta(N)$ being < 2 . Let $T = C*SC$. Then $\text{rank } T \leq 2$.

First suppose $\text{rank } T = 2$. Then, by Lemma 2.10, T is conjunctive with S and so, by Lemma 2.9, is also $*$ -regular. Thus, by Theorem 1.2, T is unitarily conjunctive with $M \oplus 0$ for some nonsingular 2×2 matrix M . Hence we have $M \oplus 0$ conjunctive

with $N \oplus 0$ and so, by Lemma 2.7, M is conjunctive with N and by Case 1 above $\phi(M) \leq \phi(N)$. Therefore

$\text{In } T = (0, 0, n-2) + \text{In } M \leq (0, 0, n-2) + \text{In } N = \text{In } S$ (and hence also $\pi(T) \leq \pi(S)$) if $\text{rank } T = 2$.

Next suppose $\text{rank } T = 1$. Then T has at most one nonzero eigenvalue. Thus $\text{In } T = (0, 0, n)$ or $(1, 0, n-1)$ or $(0, 1, n-1)$, and $\phi(T) \leq \phi(S)$.

If $\text{rank } T = 0$ then $T = 0$, and the conclusion is obvious.

Theorem 2.2. If S is semiconjunctive with $\begin{bmatrix} T & U_3 \\ U_1 & U_2 \end{bmatrix}$ for some U_1, U_2, U_3 and some nonsingular T , then S is conjunctive with $\begin{bmatrix} T & 0 \\ W & V \end{bmatrix}$ for some W and V (and the same T).

Proof. Let $C^*SC = \begin{bmatrix} T & U_3 \\ U_1 & U_2 \end{bmatrix}$ where C and S are $n \times n$, U_1 is $(n-k) \times k$, U_2 is $(n-k) \times (n-k)$, U_3 is $k \times (n-k)$, and T is $k \times k$ nonsingular. Partition C as $[C_1 \ C_2]$ with C_1 $n \times k$ and C_2 $n \times (n-k)$. Let \mathcal{U}_1 be the column space of C_1 and let

$$\mathcal{U} = \{X \in \mathbb{C}^{n \times 1} : C_1^*SX = 0\}.$$

Since

$$C^*SC = \begin{bmatrix} C_1^*SC_1 & C_1^*SC_2 \\ C_2^*SC_1 & C_2^*SC_2 \end{bmatrix} = \begin{bmatrix} T & U_3 \\ U_1 & U_2 \end{bmatrix},$$

we have $\text{rank } C_1^*SC_1 = \text{rank } T = k$, so $\text{rank } C_1 = k = \text{rank } C_1^*S$.

Thus $\dim \mathcal{U} = n-k$ and $\dim \mathcal{U}_1 = k$. Also if $X \in \mathcal{U}_1 \cap \mathcal{U}$ then $X = C_1 Y$ for some $k \times 1$ Y and $TY = C_1^* S C_1 Y = C_1^* S X = 0$, which implies $Y = 0$ and hence $X = 0$. Thus $\mathcal{U}_1 \cap \mathcal{U} = 0$, so $\dim(\mathcal{U}_1 \oplus \mathcal{U}) = n$. Let C_3 be an $n \times (n-k)$ matrix whose columns are a basis for \mathcal{U} , and let $D = [C_1 \ C_3]$. Then

$$D^* S D = \begin{bmatrix} C_1^* S C_1 & C_1^* S C_3 \\ C_3^* S C_1 & C_3^* S C_3 \end{bmatrix} = \begin{bmatrix} T & 0 \\ W & V \end{bmatrix}$$

and D is nonsingular (has column rank = $\dim(\mathcal{U}_1 \oplus \mathcal{U}) = n$).

Corollary 2.2.1. $\pi(C^* S C) \leq \pi(S)$ for all nonsingular $n \times n$ C implies $\pi(C^* S C) \leq \pi(S)$ for all $n \times n$ C .

Corollary 2.2.2. $\text{In}(C^* S C) \leq \text{In}(S)$ for all nonsingular $n \times n$ C implies $\text{In}(C^* S C) \leq \text{In}(S)$ for all $n \times n$ C .

Corollary 2.2.3. $\delta(C^* S C) \geq \delta(S)$ for all nonsingular $n \times n$ C implies $\delta(C^* S C) \geq \delta(S)$ for all $n \times n$ C .

Proof of Corollary 2.2.1. (2.2.2 and 2.2.3 are similar.) Let C be any $n \times n$ matrix. Then there is a unitary matrix D such that $D^* C^* S C D = \begin{bmatrix} T & 0 \\ U & N \end{bmatrix}$, where T is nonsingular and N is nilpotent (by Theorem 1.4). Thus, by Theorem 2.2, S is conjunctive with $\begin{bmatrix} T & 0 \\ W & V \end{bmatrix}$ for some W, V . Therefore

$$\pi(C*SC) = \pi(D*C*SCD) = \pi\left(\begin{bmatrix} T & 0 \\ U & N \end{bmatrix}\right) \leq \pi\left(\begin{bmatrix} T & 0 \\ W & V \end{bmatrix}\right) \leq \pi(S) ,$$

the last inequality coming from the hypothesis in Corollary 2.2.1.

III. THE *-IRREGULAR CASE

We begin now our investigation of special classes of matrices, with the *-irregular matrices. This is a subclass of the class of contradefinite matrices, by Theorem 1.1, so some of the results of this chapter are corollaries of results of Chapter IV, but most are not, since in that chapter we concentrate on *-regular contradefinite matrices.

Lemma 3.1 ([1, Fact 1.2]). Let S be an $n \times n$ matrix and let $1 \leq h \leq n$. Suppose that the first $h-1$ rows of S are zero and that $S_{h1} \neq 0$. Let C be the $n \times n$ matrix defined by

$$C_{1j} = -S_{h1}^{-1}S_{hj} \quad \text{for } j = 2, 3, \dots, n$$

and otherwise $C_{ij} = \delta_{ij}$ (the Kronecker δ). Let $T = C*SC$. Then C is nonsingular (hence T is conjunctive with S) and the first $h-1$ rows of T are zero and $T_{h1} = S_{h1}$ and $T_{hj} = 0$ for all $j \geq 2$.

Lemma 3.2 ([1, Fact 1.4]). Let (as above) S be an $n \times n$ matrix, M be a subset of $\{1, 2, \dots, n\}$, m be the cardinal of M , and D be an $m \times m$ nonsingular matrix, and let T be obtained from S by the $[M]$ subconjunctivity defined by D . Then $T[M] = D*S[M]D$, $T[M|M] = D*S[M|M]$, $T(M|M) = S(M|M)D$, and

$T(M) = S(M)$. Hence any zero columns of $S[M|M]$ and $T[M|M]$ correspond, and any zero rows of $S(M|M)$ and $T(M|M)$ correspond.

We call a (square) matrix S trapezoidal provided it is lower triangular and its first r diagonal entries are nonzero (where $r = \text{rank } S$). Thus Fact 1.2 says just that every square matrix is conjunctive with a trapezoidal matrix.

Lemma 3.3 ([1, Fact 1.7]). Let S be trapezoidal, h and ℓ be given such that $r \geq h > \ell$, $S[\ell+1, \ell+2, \dots, h|\ell] = 0$, and $S[h|\ell, \ell+1, \dots, h-1] = 0$. Then the interchanging $[h, \ell]$ subconjunctivity applied to S gives a trapezoidal matrix whose diagonal is the same as that of S except for the interchange of the h th and ℓ th diagonal entries. (The result remains valid if "trapezoidal" is replaced throughout by "lower triangular" and " $r \geq$ " is deleted.)

Lemma 3.4. If Z is strictly lower triangular and $n \times n$, then it is conjunctive with a trapezoidal matrix with inertia $= (0, 0, n)$ ($= \text{In } Z$).

Proof. The result is trivial if $n = 1$ (then $Z = 0$), so suppose $n \geq 2$ and that every $(n-1)$ -square strictly lower triangular matrix is conjunctive with a trapezoidal matrix of inertia $(0, 0, n-1)$. Let Z be $n \times n$, strictly lower triangular and nonzero (there is

nothing to prove if $Z = 0$). Let Z_{hk} be the first nonzero entry in the first nonzero row of Z (there is such since $Z \neq 0$). Then $h > k \geq 1$ because Z is strictly lower triangular.

Case 1. $h > k = 1$ ($Z_{h1} \neq 0$). $Z_{hj} = 0$ for $j \geq h$ by the strict lower triangularity of Z . Therefore, by Lemma 3.1, Z is, by a $[1, 2, \dots, h-1]$ subconjunctivity, conjunctive with a matrix T whose first $h-1$ rows are zero and such that $T_{h1} = Z_{h1} (\neq 0)$ and $T_{hj} = 0$ for $j \geq 2$. Now, by Lemma 3.2, we have $T(1, 2, \dots, h-1) = Z(1, 2, \dots, h-1)$, which is strictly lower triangular. Therefore T is strictly lower triangular. Next, apply to T the $[1, h]$ subconjunctivity defined by the matrix

$$\begin{bmatrix} 1 & 0 \\ \frac{-i}{Z_{h1}} & 1 \end{bmatrix}$$

and get thereby a matrix $T_1 (= P*ZP, \text{ say, with } P \text{ nonsingular})$ which is conjunctive with Z . Then $T_1(1)$ is strictly lower triangular, $T_1[1|1] = i$ and $\text{In } T_1 = \text{In } Z$. Then applying the inductive assumption to $T_1(1)$, we have that there exists a nonsingular $(n-1)$ -square matrix C_1 such that $C_1^* T_1(1) C_1$ is trapezoidal and of inertia $(0, 0, n-1)$. Thus if $C = \begin{bmatrix} 1 & 0 \\ 0 & C_1 \end{bmatrix}$, then $C*T_1C = C*P*ZPC$ is trapezoidal, conjunctive with Z , and of inertia $(0, 0, n)$. For,

$$\begin{bmatrix} i & 0 \\ G_1 & C_1^* T_1(1) C_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & C_1^* \end{bmatrix} \begin{bmatrix} i & 0 \\ G & T_1(1) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & C_1 \end{bmatrix}$$

where

$$G = T_1(1|1] \quad \text{and} \quad G_1 = C_1^* G.$$

Case 2. $h > k > 1$. We apply to Z the interchanging $[1, k]$ subconjunctivity, and, by Lemma 3.3, get thus a strictly lower triangular matrix T . By Lemma 3.2, the first $h-1$ rows of T are zero and $T_{hl} = Z_{hk} \neq 0$, so T is covered by Case 1.

Lemma 3.5. Let $S = \begin{bmatrix} N & 0 \\ G & Z \end{bmatrix}$ be $*$ -irregular where N is nonsingular $r \times r$ and lower triangular and Z is strictly lower triangular. Then S is conjunctive with a trapezoidal matrix of the same inertia.

Proof. $(G, Z) \neq (0, 0)$ because S is $*$ -irregular. If $Z = 0$, S is already trapezoidal. If not, let C be nonsingular and such that $C^* Z C$ is trapezoidal and has inertia = In Z (such C exists by Lemma 3.4). Then $(I_r \oplus C^*) S (I_r \oplus C)$ (where I_r is the $r \times r$ identity matrix) is trapezoidal and has inertia = In S .

Lemma 3.6. If $\mu \neq 0$ then there is a $[1, 2]$ subconjunctivity having the effect

$$\begin{bmatrix} \lambda & 0 & 0 \\ -\lambda & \mu & 0 \\ 1 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \lambda & 0 & 0 \\ * & \mu & 0 \\ * & 1 & 0 \end{bmatrix},$$

where the *'s are numbers of no interest to us.

Proof. Put $d = \lambda\mu^{-1}$ and

$$C = \begin{bmatrix} 1-\bar{d} & 1 & 0 \\ -\bar{d} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$\begin{aligned} & \begin{bmatrix} 1-d & -d & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda & 0 & 0 \\ -\lambda & \mu & 0 \\ 1 & 0 & 0 \end{bmatrix} C \\ &= \begin{bmatrix} \lambda & -\lambda & 0 \\ 0 & \mu & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1-\bar{d} & 1 & 0 \\ -\bar{d} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ -\mu\bar{d} & \mu & 0 \\ 1-\bar{d} & 1 & 0 \end{bmatrix}. \end{aligned}$$

Lemma 3.7 ([1, Fact 1.6]). Let, in Lemma 3.2, S be trapezoidal, $M = \{h+1, h+2, \dots, h+m\}$ be contiguous, and $D*S[M]D = T[M]$ be trapezoidal. Then T is trapezoidal and $T(M) = S(M)$. (The result remains valid if "trapezoidal" is replaced throughout by "lower triangular".)

Lemma 3.8. If S is an $n \times n$ $*$ -irregular trapezoidal matrix of rank r then it is conjunctive with a matrix T such that (i) T is trapezoidal, (ii) $T_{r+1,r} = 1$ and (iii) $T_{jj} = S_{jj}$ for $j = 1, \dots, n$.

Proof (By induction on r). The case $r = 0$ is vacuous (as is the case $n = 1$). If $\text{rank } S = 1$ (which is necessary in the case $n = 2$ since a nonsingular matrix is $*$ -regular), then $n \geq 2$ and we can clearly satisfy (i), (ii) and (iii) by elementary conjunctivity operations. In fact, by $*$ -irregularity and trapezoidality, there is a nonzero element in the first column and below the first row. By elementary conjunctivity operations (involving only rows (and columns) $2, 3, \dots, n$) we can, using it, achieve $T_{21} = 1$ while maintaining the diagonal of S and its trapezoidality, since $S_{ij} = 0$ for $j > 1$ and all i . Let $r \geq 2$ and let the lemma hold for (square) matrices of rank $r-1$, and let S be $n \times n$ $*$ -irregular and trapezoidal of rank r . Then $S(1)$ is trapezoidal and $(n-1)$ -square of rank $r-1$. There are 2 cases.

Case I. $S(1)$ is $*$ -irregular. Then $S(1)$ is, by induction hypothesis, conjunctive with a matrix V such that (i)' V is trapezoidal, (ii)' $V_{r,r-1} = 1$ and (iii)' $V_{jj} = S(1)_{jj}$ for $j = 1, \dots, n-1$, i.e., we have $C_1^* S(1) C_1 = V$ for some nonsingular C_1 . Putting $C = I_1 \oplus C_1$ we have that $C^* S C = T$ satisfies (i), (ii),

and (iii).

Case II. $S(1)$ is $*$ -regular. Let S_{k1} be the last nonzero element in $S(\emptyset|1]$, the first column of S . Such k exists and $k > r$ since S is $*$ -irregular and $S(1)$ is $*$ -regular. We interchange rows $r+1$ and k (and columns $r+1$ and k). Then we multiply row $r+1$ by S_{k1}^{-1} (and column $r+1$ by $\overline{S_{k1}^{-1}}$). Then we add $-(S_{21}+S_{11})$ times the $(r+1)$ st row to the second row (and do the conjugate column operation). Thus we may assume without loss of generality that $S(\emptyset|1] = \text{col}[S_{11}, -S_{11}, *, *, \dots, *, 1, *, *, \dots, *]$, where the 1 is in the $(r+1)$ st position and the numbers designated $*$ are not of interest to us. Then $S_{22} \neq 0$, so by Lemma 3.6, there is a $[1, 2]$ subconjunctivity (say defined by a 2×2 matrix D) taking

$$S[1, 2, r+1] = \begin{bmatrix} S_{11} & 0 & 0 \\ -S_{11} & S_{22} & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ to } \begin{bmatrix} S_{11} & 0 & 0 \\ * & S_{22} & 0 \\ * & 1 & 0 \end{bmatrix}.$$

Let T be $(D* \oplus I_{n-2})S(D \oplus I_{n-2})$. Then, since $\{1, 2\}$ is contiguous and $D*S[1, 2]D$ is trapezoidal, by Lemma 3.7, T is also trapezoidal and $T(1, 2) = S(1, 2)$, so $T_{jj} = S_{jj}$ for $j = 1, \dots, n$. Also, since $(D* \oplus I_1)S[1, 2, r+1](D \oplus I_1) = T[1, 2, r+1]$, we have $T_{r+1, 2} = 1$, so $T(1)$ is $*$ -irregular. Therefore T is covered by Case I and hence is conjunctive with a matrix satisfying (i), (ii), and (iii).

Lemma 3.9. If S is $*$ -irregular trapezoidal of rank r and $0 \neq t \in \mathbb{C}$, then S is conjunctive with a matrix T such that (i) T is trapezoidal, (ii) $T_{r+1,r} = 1$, (iii) $T_{jj} = S_{jj}$ for $j \neq r$, and (iv) $T_{rr} = t$.

Proof. Lemma 3.8 ensures the existence of a matrix T conjunctive with S and satisfying (i), (ii) and (iii). We then apply the n th order $[r, r+1]$ subconjunctivity defined by

$$\begin{bmatrix} 1 & 0 \\ -\overline{T_{rr}} + t & 1 \end{bmatrix}$$

to get T conjunctive with a matrix satisfying (iv) while still satisfying (i), (ii), and (iii). For, see the following computation of the relevant $([r, r+1])$ part (we abbreviate T_{rr} to z).

$$\begin{bmatrix} 1 & -z+t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\overline{z}+t & 1 \end{bmatrix} = \begin{bmatrix} t & 0 \\ 1 & 0 \end{bmatrix}.$$

Lemma 3.10. Let S be $*$ -irregular $n \times n$ of rank r and let t_1, t_2, \dots, t_r be any nonzero complex numbers. Then S is conjunctive with a matrix T such that (i) T is trapezoidal, (ii) $T_{r+1,r} = 1$, and (iii) $\text{diag } T = (t_1, t_2, \dots, t_r, 0, 0, \dots, 0)$.

Proof (by finite induction). We assume, by Lemma 3.5, that S is trapezoidal $*$ -irregular $n \times n$ of rank r . Then, by Lemma 3.9, S is conjunctive with a matrix of diagonal $= (S_{11}, \dots, S_{r-1, r-1}, t_1, 0, 0, \dots, 0)$ which satisfies (i) and (ii) (and is $*$ -irregular). This matrix is, by Lemma 2.5, conjunctive with a ($*$ -irregular) trapezoidal matrix of diagonal $= (t_1, S_{11}, S_{22}, \dots, S_{r-1, r-1}, 0, 0, \dots, 0)$. This matrix, in turn, is conjunctive, by Lemma 3.8, with a $*$ -irregular trapezoidal matrix of the same diagonal and with 1 in the $(r+1, r)$ position.

Now assume inductively that S is $*$ -irregular trapezoidal of rank r and $S_{r+1, r} = 1$ and $1 \leq k \leq r$ and $\text{diag } S = (t_1, \dots, t_{k-1}, S_{kk}, S_{k+1, k+1}, \dots, S_{rr}, 0, 0, \dots, 0)$. Then, by Lemma 3.9, S is conjunctive with a ($*$ -irregular) trapezoidal matrix with diagonal $= (t_1, \dots, t_{k-1}, S_{kk}, S_{k+1, k+1}, \dots, S_{r-1, r-1}, t_k, 0, 0, \dots, 0)$ where t_k is in the r -th position. This matrix is, by Lemma 2.5, conjunctive with a trapezoidal ($*$ -irregular) matrix of diagonal $= (t_1, \dots, t_k, S_{kk}, S_{k+1, k+1}, \dots, S_{r-1, r-1}, 0, 0, \dots, 0)$ where $S_{r-1, r-1}$ is in the r -th position. This matrix in turn is, by Lemma 3.8, conjunctive with a $*$ -irregular trapezoidal matrix of the same diagonal and with 1 in position $(r+1, r)$. This completes the proof of the induction step, and hence our induction assumption (at the beginning of this paragraph) holds for $k = 1, 2, \dots, r+1$, and for $k = r+1$ it is just the conclusion of Lemma 3.10.

Theorem 3.1. Let S be $n \times n$ *-irregular of rank r and let d and p be any integers satisfying the three inequalities

$$n-r \leq d \leq n$$

$$0 \leq p \leq r$$

and

$$n-r \leq p+d \leq n$$

(or equivalently the two inequalities $0 \leq p \leq n-d$ and $n-r \leq d \leq n$).

Then S is conjunctive with a matrix T such that

In $T = (p, n-p-d, d)$. Hence (since $0 < r < n$):

- (a) if $\delta(S) = n$, then $\text{In } S$ can be conjunctively increased (and hence $\delta(S)$ decreased);
- (b) if $\delta(S) = n-r$, then $\text{In } S$ can be conjunctively decreased (and hence $\delta(S)$ increased) and also $\text{In } S$ can be conjunctively incomparably changed;
- (c) if $n-r < \delta(S) < n$, then $\text{In } S$ can be conjunctively increased and can be conjunctively decreased (hence same for $\delta(S)$) and can be conjunctively incomparably changed;
- (d) if $\pi(S) = 0$, then $\pi(S)$ can be conjunctively increased;
- (e) if $\pi(S) = r$, then $\pi(S)$ can be conjunctively decreased;
- (f) if $0 < \pi(S) < r$, then $\pi(S)$ can be conjunctively increased, and also $\pi(S)$ can be conjunctively decreased.

Proof. Let p and d be given integers which satisfy the

three (or the equivalent two) inequalities. Then we can select complex numbers $t_1, t_2, \dots, t_r, t_{r+1}, \dots, t_n$ with the properties

(i) $t_{r+1} = t_{r+2} = \dots = t_n = 0,$

(ii) exactly p of them have positive real part and

(iii) exactly d of them are pure imaginary. We then apply

Lemma 3.10, with t_1, \dots, t_r , the given r nonzero complex numbers, to obtain S conjunctive with a matrix T of inertia $(p, n-p-d, d)$.

IV. THE *-REGULAR CONTRADEFINITE CASE

In this chapter we study the class of matrices of largest possible cone (= \mathbb{C}) and thus greatest degree of freedom conjunctively to change their inertias. We abbreviate to CRHP the closed right half-plane = $\{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$, to ORHP the open right half-plane = $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and analogously define CLHP and OLHP respectively, for the closed and open left half-planes.

Lemma 4.1. Let $n \geq 3$ and S be $n \times n$ nonsingular contradefinite over \mathbb{C} . Then S is conjunctive with (a) a matrix T such that $T[1, 2, 3]$ is nonsingular contradefinite, and (b) a matrix R such that $R[1, 2]$ is nonsingular contradefinite, and (c) a matrix Q such that $Q[1] = 1$, $Q[1, 1] = 0$, and $Q(1)$ is contradefinite (hence with a lower triangular such Q).

Proof. Follow the proof of [2, Fact 5.1]. If $n = 3$ S is conjunctive with a matrix covered by Case 8, 9, or 10 (hence Subsubcase 10ab) of [2, Fact 5.1], hence S is conjunctive with a matrix R covered by Case 8 (hence R satisfies (b) and (a) of Lemma 4.1) and thus conjunctive with a matrix Q covered by Case 1 [2, Fact 5.1] (hence Q satisfies (c) of Lemma 4.1). For $n > 3$, proceed by induction on n . As in [2, Fact 5.1] S is conjunctive with a Q satisfying (c), and by induction assertion, $Q(1)$ is conjunctive with a

matrix satisfying (b) and (c) and hence so is Q and hence so is S .

Lemma 4.2. Let S be $n \times n$ contradefinite nonsingular and t_1, t_2, \dots, t_n be any complex numbers such that $\text{sgn}(t_1 t_2 \dots t_n) = \text{sgn} \det S$. Then S is conjunctive with a lower triangular matrix T such that $\text{diag } T = (t_1, t_2, \dots, t_n)$ and $T[n-1, n]$ is contradefinite.

Proof. (By routine induction based on Lemma 4.1(c)).

Obviously $t_1 t_2 \dots t_n \neq 0$ because $\det S \neq 0$. Let \mathcal{S}_n be our induction assertion, the statement of Lemma 4.2. \mathcal{S}_1 is vacuously true and \mathcal{S}_2 is an immediate consequence of Lemma 2.1 (just pick T_{11} there = t_1 here). So suppose $n \geq 3$ and \mathcal{S}_{n-1} is true. Let S be nonsingular contradefinite $n \times n$ and $\text{sgn}(t_1 t_2 \dots t_n) = \text{sgn} \det S$. Then $t_1^{-1} S$ is conjunctive (by Lemma 4.1(c)) with a matrix Q such that $Q[1] = 1$, $Q[1|1] = 0$, and $Q(1)$ is contradefinite. Thus S is conjunctive with $t_1 Q$ and hence $\text{sgn} \det S = \text{sgn}(t_1^n \det Q) = \text{sgn}(t_1^n \det Q(1)) = \text{sgn}[t_1 \det(t_1 Q(1))]$. Hence $\text{sgn}(t_2 t_3 \dots t_n) = \text{sgn} \det(t_1 Q(1))$, and $t_1 Q(1)$ is $(n-1) \times (n-1)$ nonsingular contradefinite, so, by \mathcal{S}_{n-1} , $t_1 Q(1)$ is conjunctive with a lower triangular matrix R such that $\text{diag } R = (t_2, t_3, \dots, t_n)$ and $R[n-2, n-1]$ is contradefinite. Applying the corresponding $[2, 3, \dots, n]$ subconjunctivity to $t_1 Q$ we get a lower triangular matrix T such that $\text{diag } T = (t_1, t_2, \dots, t_n)$ and

$T[n-1, n] = R[n-2, n-1]$ is contradefinite. Clearly T is conjunctive with S (because $t_1 Q$ is).

Lemma 4.3. Let S be nonsingular $n \times n$ and contradefinite.

- (a) If $\pi(S) > 0$, then S is conjunctive with a matrix of smaller π .
 (b) If $\delta(S) < n-1$, then S is conjunctive with a matrix of smaller inertia and hence larger δ . (c) If $n \geq 3$ and $\pi(S) < n$, then S is conjunctive with a matrix of larger π .

Proof of (a). Let $re^{i\epsilon}$ be an n th root of $\det S$ lying in the CLHP. By Lemma 4.2, $(\arg \text{diag } S) = (\sigma_1, \sigma_2, \dots, \sigma_n) \rightarrow (\epsilon, \epsilon, \dots, \epsilon)$, decreasing π to zero.

Proof of (b). Let $\theta = \sum_{j=1}^n \sigma_j - (n-1) \frac{\pi}{2}$, where $(\sigma_1, \dots, \sigma_n) = \arg \text{diag } S$. θ is unique mod 2π . Then, using Lemma 4.2, $(\sigma_1, \dots, \sigma_n) \rightarrow (\frac{\pi}{2}, \frac{\pi}{2}, \dots, \frac{\pi}{2}, \alpha, \beta)$ where (α, β) is $(\frac{\pi}{2}, \theta)$ if $\nu(S) = 0$ and $\cos \theta \geq 0$ or if $\pi(S) = 0$ and $\cos \theta \leq 0$; $(\alpha, \beta) = (-\frac{\pi}{2}, \theta + \pi)$ if $\nu(S) = 0$ and $\cos \theta < 0$ or if $\pi(S) = 0$ and $\cos \theta > 0$; and (α, β) may be taken as either of these two otherwise.

Proof of (c). Same as proof of (a), only take $re^{i\epsilon} \in \text{ORHP}$.

This is possible because $n \geq 3$.

Lemma 4.4. Let S be nonsingular $n \times n$ and contradefinite.

If $\delta(S) = n-1$ there is no matrix of larger δ , and hence none of

smaller (i. e., zero) inertia, conjunctive with S .

Proof. $\det iS \notin \mathbb{R}$ (the reals) since $\delta(S) = n-1$ (iS has exactly $n-1$ real roots) and $\operatorname{sgn} \det iS$ is conjunctively invariant. Therefore iS is not conjunctive with a matrix iT with all characteristic roots real (i. e., S is not conjunctive with a matrix T with $\delta(T) = n$), for such a T would have $\det iT \in \mathbb{R}$.

Lemma 4.5. If S is contradefinite and $\pi(S) = 0$, then there is a matrix of larger (i. e., positive) π conjunctive with S .

Proof. Since S is contradefinite, $\Gamma(S) = \mathbb{C}$ and thus contains 1, for example. By Lemma 2.1 (with $T_{11} = 1$) we see that S is conjunctive with a matrix T which has the property that $\pi(T) \geq 1$.

Lemma 4.6. Let S be 2×2 nonsingular contradefinite with $\pi(S) = 1$ and $\operatorname{sgn} \det S = -1$. Then there is no matrix T conjunctive with S for which (a) $\pi(T) = 2$, nor (b) $\nu(T) = 2$, nor (c) $\operatorname{In} T = (0, 1, 1)$. Thus there is no matrix of incomparable inertia nor one of larger inertia conjunctive with S .

Proof. If there were a matrix T of eigenvalues λ_1 and λ_2 conjunctive with S , then by the conjunctive invariance of $\operatorname{sgn} \det S$, we would have $\operatorname{sgn} \lambda_1 \lambda_2 = -1$. It is however well-known that (a) two numbers in ORHP cannot have product -1 , nor can

(b) two numbers in OLHP, nor can (c) two numbers of which one is in OLHP and the other is pure imaginary. Thus the only change conjunctively possible is an inertia decrease to $(0, 0, 2)$.

Lemma 4.7. If S is 2×2 nonsingular and contradefinite and $\pi(S) = 1$ and $\text{sgn det } S \neq -1$, then S is conjunctive with a matrix of larger $\pi (= 2)$.

Proof. Let ϵ be that square root of $\text{sgn det } S$ which $\epsilon \in \text{ORHP}$, and, by Lemma 4.2, S is conjunctive with a lower triangular matrix T of diagonal $= (\epsilon, \epsilon)$. Then $\pi(T) = 2$.

We next prove a fairly weak preliminary result about contradefinite matrices.

Theorem 4.1. If S is $n \times n$ nonsingular and contradefinite then S is conjunctive with a matrix of different inertia.

Proof. The case $n = 1$ is vacuous.

Case 1: $n = 2$ and $\text{sgn det } S = -1$. Here, by Lemma 4.2, S is conjunctive with lower triangular matrices of diagonals $(1, -1)$ and (i, i) , one of which matrices has inertia $\neq \text{In } S$.

Case 2: $n > 2$ or $\text{sgn det } S \neq -1$. Here $\text{sgn det } S$ has n th roots $\omega^k \epsilon$, where $\omega = \exp(\frac{2\pi i}{n})$ and $k = 0, 1, 2, \dots, n-1$; so by Lemma 4.2 S is conjunctive with lower triangular matrices of

diagonals = $\omega^k_\varepsilon(1, 1, \dots, 1)$, at least one of which has inertia $\neq \text{In } S$, because the numbers ω^k_ε do not all lie in ORHP, nor all in OLHP, nor all on the imaginary axis.

Corollary 4.1.1. If S is $n \times n$ contradefinite and $*$ -regular, then S is conjunctive with a matrix of different inertia.

Proof. If $\text{rank } S = r$, then S is, by Theorem 1.2, unitarily conjunctive with the direct sum T of a nonsingular $r \times r$ matrix N and a zero matrix. Apply the theorem to N and the resulting subconjunctivity to T .

Theorem 4.2. Let S be 2×2 and contradefinite, $\delta(S) < 2$ and $\text{sgn det } S \neq -1$. Then $\text{In } S$ can be incomparably changed by conjunctivity.

Proof (By Lemma 1.2). We assume, by Lemma 2.5, that the eigenvalues λ_1 and λ_2 of S satisfy $|\lambda_1| = |\lambda_2| = 1$.

Case 1. Exactly one of λ_1 and λ_2 is pure imaginary. Then, using Lemma 1.2, we perform the indicated operation:

$$(\lambda_1, \lambda_2) \rightarrow (-\lambda_2, -\lambda_1).$$

Case 2. $(\text{Re } \lambda_1)(\text{Re } \lambda_2) < 0$. Let ε and $-\varepsilon$ be the two square roots of $\text{sgn det } S$. Then, by Lemma 4.2, S is conjunctive with matrices of diagonals $(\varepsilon, \varepsilon)$ and $(-\varepsilon, -\varepsilon)$, each of which has inertia incomparable with $\text{In } S$.

Case 3. $(\operatorname{Re} \lambda_1)(\operatorname{Re} \lambda_2) > 0$. $(\lambda_1, \lambda_2) \rightarrow (-\lambda_2, -\lambda_1)$, e.g., achieves an incomparable inertia change.

Corollary 4.2.1. If S is $n \times n$ contradefinite, $*$ -regular and of rank 2 with $\delta(S) < n$ and the signum of the product of its two nonzero eigenvalues $\neq -1$, then S is conjunctive with a matrix of incomparable inertia.

Proof. See proof of Corollary 4.1.1 with $r = 2$.

Theorem 4.3. If S is 2×2 contradefinite, $\delta(S) < 2$ and $\operatorname{sgn} \det S = -1$, then we can still (compare Theorem 4.2) decrease its inertia.

Proof. Use Lemma 2.1 to do the indicated operation:

$(\lambda_1, \lambda_2) \rightarrow (i, -i\lambda_1\lambda_2) = (i, i)$ (where we have assumed as usual, by use of Lemma 2.5, that $|\lambda_1| = |\lambda_2| = 1$).

Corollary 4.3.1. If S is $n \times n$ contradefinite $*$ -regular and of rank 2 with $\delta(S) < n$ and the signum of the product of its two nonzero eigenvalues $= -1$, then S is conjunctive with a matrix of smaller inertia.

Proof. See proof of Corollary 4.1.1 with $r = 2$.

Theorem 4.4. Let $n \geq 3$, S be $n \times n$ nonsingular and

contradefinite and $\delta(S) < n$. Then S is conjunctive with a matrix of incomparable inertia.

Proof. Suppose $\text{In } S = (n-j-k, j, k)$ where $k < n$. There are two cases.

Case 1. $j > 0$. Since $n \geq 3$ there is an n th root ω of $\det S$ in ORHP. Then, by Lemma 4.2, S is conjunctive with a lower triangular matrix T with diagonal $(\omega, \omega, \dots, \omega)$. Then $\text{In } T$ is incomparable with $\text{In } S$.

Case 2. $n-j-k > 0$. The proof in this case is completely analogous to that of Case 1.

Corollary 4.4.1. Let S be $n \times n$ contradefinite $*$ -regular of rank ≥ 3 with $\delta(S) < n$. Then $\text{In } S$ can be incomparably changed by conjunctivity.

Proof. See proof of Corollary 4.1.1.

The main theorem on contradefinite matrices is next.

Theorem 4.5. Let S be nonsingular $n \times n$ and contradefinite. Let further (A) $n = 2$ or (B) $n \geq 3$. Then the following table tells whether or not the indicated change is possible conjunctively (upper half of the entry) and semiconjunctively (lower half of entry).

Theorem 4.5 Table

	\uparrow In & \downarrow δ ?	\downarrow π ?	\uparrow π ?	\downarrow In & \uparrow δ ?	i. c. In ?	
$\delta > 0$	yes / yes					1
$\delta = 0$	no / no					2
$\pi = 0$		no / no				3
$\pi > 0$		yes / yes				4
$\pi = 0$			yes / yes			5
$\pi = 1$	[sgn det S = -1		no / no			6
		sgn det S \neq -1	yes / yes			7
$\pi = 2$			no / no		(A)	8
$\delta = 0$				yes / yes		9
$\delta = 1$				no / yes		10
$\delta = 2$				no / no		11
$\delta < 2$	[sgn det S = -1			no / no		12
		sgn det S \neq -1		yes / yes		13
$\delta = 2$				no / no		14
$\delta > 0$	yes / yes					15
$\delta = 0$	no / no					16
$\pi = 0$		no / no				17
$\pi > 0$		yes / yes				18
$\pi < n$			yes / yes			19
$\pi = n$			no / no		(B)	20
$\delta < n-1$				yes / yes		21
$\delta = n-1$				no / yes		22
$\delta = n$				no / no		23
$\delta < n$				yes / yes		24
$\delta = n$				no / no		25

Proof. We denote with the subscript u , the upper tabular entry and with the subscript l , the lower tabular entry.

- 1_u : Lemma 2.6(a).
- 1_l : Lemma 2.2(a).
- 2_u and 2_l : Lemma 2.3.
- 3_u and 3_l : Lemma 2.3.
- 4_u : Lemma 4.3(a).
- 4_l : Lemma 2.2(a).
- 5_u : Lemma 4.5.
- 5_l : Lemma 2.2(a).
- 6_u : Lemma 4.6(a).
- 6_l : Corollary 2.2.1.
- 7_u : Lemma 4.7.
- 7_l : Lemma 2.2(a).
- 8_u and 8_l : Lemma 2.3.
- 9_u : Lemma 4.3(b).
- 9_l : Lemma 2.4.
- 10_u : Lemma 4.4.
- 10_l : Lemma 2.4.
- 11_u and 11_l : Lemma 2.3.
- 12_u : Lemma 4.6(c).
- 12_l : Corollary 2.2.2 together with 2_u and 12_u .
- 13_u : Theorem 4.2.

13_1 : Lemma 2.2(a).

14_u and 14_1 : Lemma 2.3.

15_u : Lemma 2.6(a).

15_1 : Lemma 2.2(a).

16_u , 16_1 , 17_u and 17_1 : Lemma 2.3.

18_u : Lemma 4.3(a).

18_1 : Lemma 2.2(a).

19_u : Lemma 4.3(c).

19_1 : Lemma 2.2(a).

20_u and 20_1 : Lemma 2.3.

21_u : Lemma 4.3(b).

21_1 : Lemma 2.2(a).

22_u : Lemma 4.4.

22_1 : Lemma 2.4.

23_u and 23_1 : Lemma 2.3.

24_u : Theorem 4.4.

24_1 : Lemma 2.2(a).

25_u and 25_1 : Lemma 2.3.

Corollary 4.5.1. Let S be $n \times n$ nonsingular and contra-definite and let $\phi \in \{\pi, \delta\}$. Then (a) S is not a solution of $(1)_\phi$ and hence not of $(6)_\phi$, (b) if $\phi > 0$, S is not a solution of $(2)_\phi$ and hence not of $(5)_\phi$, (c) if $\delta < n-1$, S is not a solution of $(3)_\delta$ and

hence not of $(4)_\delta$, (d) if $\pi < n$ and either $n \geq 3$ or (in case $n = 2$) $\text{sgn det } S \neq -1$, then S is not a solution of $(3)_\pi$ and hence not of $(4)_\pi$.

Proof. See Theorem 4.5 Table.

Corollary 4.5.1.1. If S is $k \times k$, $*$ -regular of rank n , and contradefinite and $\phi \in \{\pi, \delta - (k - n)\}$ then the conclusions of Corollary 4.5.1 hold for S , except in (d) we replace "sgn det S " by "sgn product of the nonzero eigenvalues of S ".

Proof. See proof of Corollary 4.1.1.

Corollary 4.5.2. Let S be $k \times k$, $*$ -regular, contradefinite and (A) of rank $n = 2$ or (B) of rank $n \geq 3$. Then the table of Theorem 4.5 holds for S if in it we replace " δ " by " $\delta(S)$ - nullity of S " (the latter = $\delta - (k - n)$) and, in (A), "sgn det S " by "signum of the product of the two nonzero eigenvalues of S ".

Proof. See proof of Corollary 4.1.1.

V. THE COHERMITIAN CASE

We in this chapter consider the matrices with "small" cones (i. e., cones of empty interior), the cohermitian matrices.

Lemma 5.1. If C is $n \times n$ and H is $n \times n$ and hermitian, then $\text{In } C^*HC \leq \text{In } H$.

Proof. Let C be any given $n \times n$ matrix. If $t > 0$ denote by C_t the matrix $C + tI$. Then there is a positive ϵ such that C_t is nonsingular for all t such that $0 < t < \epsilon$. Thus by Theorem 1.3, $\text{In}(C_t^*HC_t) = \text{In } H$ for all t such that $0 < t < \epsilon$. Now let $t \rightarrow 0^+$. We see by the same sort of continuity consideration as in the proof of [5, Corollary 4] that neither π nor ν can increase in the limit.

Theorem 5.1. (a) If S is cohermitian then $\text{In } S$ is conjunctively invariant. (b) If S is skew-hermitian then $\text{In } S$ is semiconjunctively invariant. (c) If S is cohermitian but not skew-hermitian, then the only possible semiconjunctive change in $\text{In } S$ is a decrease, and a decrease is possible.

Proof. (a) Assume S is not skew-hermitian (the skew-hermitian case is considered in part (b)). Then since S is cohermitian there is a number θ in $(-\frac{\pi}{2}, \frac{\pi}{2})$ such that $e^{-i\theta}S$ is

hermitian. Let C be nonsingular. Then since $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we have $\text{In } C*SC = \text{In } C*e^{-i\theta}SC$ and $\text{In } e^{-i\theta}S = \text{In } S$. For, $e^{-i\theta}S$ is hermitian, so has real spectrum, say

$\{r_1, r_2, \dots, r_p, -r_{p+1}, \dots, -r_{p+q}, 0, 0, \dots, 0\}$ with $r_j > 0$ for $1 \leq j \leq p+q$. Then S has spectrum $e^{i\theta}$ times the spectrum of $e^{-i\theta}S$, so S has the same inertia as $e^{-i\theta}S$ because $\cos \theta > 0$. Similarly, $C*SC$ has the same inertia as $C*e^{-i\theta}SC$ because the latter is hermitian and $\cos \theta > 0$. We also have, by Theorem 1.3, that $\text{In } C*e^{-i\theta}SC = \text{In } e^{-i\theta}S$. Thus $\text{In } C*SC = \text{In } S$.

(b) Let S be $n \times n$ and skew-hermitian and C be $n \times n$. Then $(C*SC)* = C*S*C = C*(-S)C = -(C*SC)$, i.e., $C*SC$ is also skew-hermitian. Then $\text{In}(C*SC) = (0, 0, n) = \text{In}(S)$.

(c) Let θ be as in (a) but C be arbitrary $n \times n$. Then $\text{In } C*SC = \text{In } C*e^{-i\theta}SC \leq \text{In } e^{-i\theta}S = \text{In } S$ where here the inequality holds by Lemma 5.1. Also since S is cohermitian and not skew-hermitian, we have that $S \neq 0$ and so has a root in ORHP or in OLHP. Then we can, by Lemma 2.4, decrease In to $(0, 0, n)$.

Corollary 5.1.1. (a) If S is skew-hermitian then it is a solution of $(6)_{\phi}$ and hence of $(1)_{\phi}$, $(2)_{\phi}$, $(3)_{\phi}$, $(4)_{\phi}$, and $(5)_{\phi}$ also.
 (b) If S is cohermitian but not skew-hermitian, then it is a solution of $(1)_{\phi}$ and hence of $(2)_{\phi}$ and $(3)_{\phi}$ also, it is a solution of $(4)_{\text{In}}$ and hence of $(4)_{\pi}$ and $(5)_{\delta}$ also, it is not a solution of

(4) _{δ} nor (6) _{ϕ} nor (5)_{In}, and if further it is not (negative) stable, then it is not a solution of (5) _{π} .

VI. THE CONTRAHERMITIAN CODEFINITE CASE

We turn our attention now to the class of matrices whose cones are subcones, of nonempty interior, of some half-plane. Thus the analysis of these is a bit more difficult and more interesting than the previously considered cases.

We begin by stating without proof three facts from [2]. Let \mathcal{E} denote the set of $n \times n$ complex matrices all of whose real eigenvalues are positive and \mathcal{E}' denote the set of $n \times n$ nonsingular complex matrices S for which $-1 \notin \Gamma(S)$. We shall take Lemma 6.1 as the definition of the function E , though it is otherwise defined in [2].

Lemma 6.1 ([2, Fact 3.2]). Let $S \in \mathcal{E}$ and let s_1, s_2, \dots, s_n be the eigenvalues of S . For $j = 1, 2, \dots, n$ let σ_j be the principal value (i. e., $|\sigma_j| < \pi$) of $\arg s_j$. Then

$$E(S) = \sum_{j=1}^n \sigma_j$$

(thus $E(S)$ is a particular determination of $\arg(\det S)$). Hence

$$e^{iE(S)} = \operatorname{sgn}(\det S).$$

Lemma 6.2 ([2, Fact 3.3]). E is conjunctively invariant on \mathcal{E}' , i. e., $E(C*SC) = E(S)$ for every $n \times n$ nonsingular C and

every $S \in \mathcal{E}'$.

Lemma 6.3 ([2, p. 107]). If $S \in \mathcal{E}'$ and $E(S) = 0$ and $1 \in \Gamma(S)$, then S is conjunctive with a lower triangular matrix of positive diagonal.

Let S be any $n \times n$ nonsingular codefinite and contra-hermitian matrix. Then $\Gamma(S)$ is, by characterizations (i) and (iv) immediately following the definitions (of codefinite and contrahermitian) in Chapter I, a subcone with nonempty interior of some half-plane. Let $\theta = \theta(S)$ denote any determination for the argument of the interior bisector of $\Gamma(S)$. We define a corresponding determination $\sigma = \sigma(S)$ for the "average arg" of the eigenvalues of S by

$$\sigma = \theta + \frac{1}{n} E(e^{-i\theta} S). \quad (*)$$

Lemma 6.4. If S and T are as above and conjunctive with each other, then $\sigma(S) \equiv \sigma(T) \pmod{2\pi}$ and, in fact, if $\theta(S)$ and $\theta(T)$ are chosen equal, then $\sigma(S) = \sigma(T)$.

Proof. Since $e^{i\theta(S)} \in \text{int } \Gamma(S)$ (= the interior of $\Gamma(S)$) we have $1 \in \text{int } \Gamma(e^{-i\theta(S)} S)$ and hence, by codefiniteness, $-1 \notin \Gamma(e^{-i\theta(S)} S)$. Thus $E(e^{-i\theta(S)} S)$ is well defined and, by Lemma 6.2, conjunctively invariant. Further, $\Gamma(S) = \Gamma(T)$ and hence $e^{i\theta(S)} = e^{i\theta(T)} = e^{i\theta}$. The conclusion now follows from (*).

To explain our use of the term "average arg" of the eigenvalues of S (and to give a more vivid geometric interpretation of $\sigma(S)$) let us rewrite (*) in terms of the eigenvalues of S , which are, of course, just the diagonal entries of any lower triangular matrix unitarily conjunctive with S . Thus, for this purpose, we may suppose, by Lemma 2.5, that S itself is lower triangular. Let $\sigma_1, \dots, \sigma_n$ be the arguments of the diagonal entries of S , chosen so that the corresponding arguments $\sigma_1 - \theta, \dots, \sigma_n - \theta$ of $e^{-i\theta} S$ are principal values. Then $|\sigma_j - \theta| \leq \frac{\pi}{2}$ for $j = 1, \dots, n$, because $\exp(i\sigma_j) \in \Gamma(S)$ for each $j = 1, \dots, n$ and $\exp i\theta$ is on the interior bisector of $\Gamma(S)$, which is contained in some half-plane. We choose γ such that $0 < \gamma \leq \frac{\pi}{2}$ and such that $\exp i(\theta + \gamma)$ and $\exp i(\theta - \gamma)$ are on the bounding rays of $\Gamma(S)$ (here γ is the opening semi-angle of $\Gamma(S)$). Then $|\sigma_j - \theta| \leq \gamma$ for all j , i.e.,

$$\theta - \gamma \leq \sigma_j \leq \theta + \gamma \quad \text{for } j = 1, \dots, n \quad (**)$$

Lemma 6.5. If σ and σ_j , $j = 1, \dots, n$ are chosen as above, then $\sigma = \frac{1}{n} \sum_{j=1}^n \sigma_j$.

Proof. By Lemma 6.1, (*) becomes

$$\sigma = \theta + \frac{1}{n} \sum_{j=1}^n (\sigma_j - \theta) = \theta + \frac{1}{n} \sum_{j=1}^n \sigma_j - \theta = \frac{1}{n} \sum_{j=1}^n \sigma_j.$$

Lemma 6.6. If S , γ , θ , and σ are as above, then
 $\theta - \gamma < \sigma < \theta + \gamma$.

Proof. By (**) and Lemma 6.5 we clearly have

$$\theta - \gamma \leq \sigma \leq \theta + \gamma .$$

Suppose one of these inequalities were in fact equality, say $\sigma = \theta + \gamma$ ($= \alpha$ for short). If S were diagonal we would thus have $\arg \text{diag } S = (\alpha, \alpha, \dots, \alpha)$. But that would make S unidfinite and contradict the contrahermitianness hypothesis. If S were non-diagonal (recall that S is lower triangular with all diagonal entries of unit modulus) then it would have a 2×2 principal submatrix of the form

$$\begin{bmatrix} e^{i\alpha} & 0 \\ 2ge^{i\alpha} & e^{i\alpha} \end{bmatrix} = e^{i\alpha} \begin{bmatrix} 1 & 0 \\ 2g & 1 \end{bmatrix}, \quad \text{where } g \neq 0.$$

Then $e^{i\alpha}$ would be on the boundary of $\Gamma(e^{i\alpha} \begin{bmatrix} 1 & 0 \\ 2g & 1 \end{bmatrix})$ by Fact 1.3(iii). Then 1 would be on the boundary of $\Gamma(\begin{bmatrix} 1 & 0 \\ 2g & 1 \end{bmatrix})$. But, by Lemma 1.2, we know that that cannot happen with $g \neq 0$. Thus

$$\theta - \gamma < \sigma < \theta + \gamma .$$

Lemma 6.7. If S is as above, then there is a nonsingular matrix C such that C^*SC is lower triangular and $\arg \text{diag } C^*SC = (\sigma, \sigma, \dots, \sigma)$.

Proof. From Lemma 6.6, we have $e^{i\sigma} \in \text{int } \Gamma(S)$, i. e., $1 \in \text{int } \Gamma(e^{-i\sigma}S)$. By codefiniteness, $-1 \notin \Gamma(e^{-i\sigma}S)$. $e^{-i\sigma}S$ is nonsingular (since S is). Also $E(e^{-i\sigma}S) = \sum_{j=1}^n (\sigma_j - \sigma)$ (because $|\sigma_j - \sigma| < \pi$ for $j = 1, \dots, n$) $= n\sigma - n\sigma = 0$. Thus the hypotheses of Lemma 6.3 are met, and so $e^{-i\sigma}S$ is conjunctive with a lower triangular matrix of positive diagonal. Thus S is conjunctive with a lower triangular matrix of $\arg \text{diag } (\sigma, \sigma, \dots, \sigma)$.

Hereafter the letter σ will be understood to mean this average arg.

Lemma 6.8. Let S be lower triangular nonsingular and 3×3 and let $\arg \text{diag } S = (\alpha, \alpha + \pi, \tau)$ with $\alpha < \tau < \alpha + \pi$ and $0 < \epsilon < \min(\tau - \alpha, \alpha + \pi - \tau) (\leq \frac{\pi}{2})$. Then

- (a) $(\alpha, \alpha + \pi, \tau) \rightarrow (\alpha + \epsilon, \alpha + \pi - \epsilon, \tau)$,
- (b) $(\tau, \alpha, \alpha + \pi) \rightarrow (\tau, \alpha + \epsilon, \alpha + \pi - \epsilon)$, and
- (c) $(\alpha, \tau, \alpha + \pi) \rightarrow (\alpha + \epsilon, \tau, \alpha + \pi - \epsilon)$.

Proof of (a). By Lemma 1.2, we have $\arg \text{diag } S = (\alpha, \alpha + \pi, \tau) \rightarrow (\alpha + \epsilon, \alpha + \pi, \tau - \epsilon) \rightarrow (\alpha + \epsilon, \alpha + \pi - \epsilon, \tau)$ if $0 < \epsilon < \min(\tau - \alpha, \alpha + \pi - \tau)$.

Proof of (b) and (c). Lemma 2.5.

Lemma 6.9. If S is lower triangular nonsingular $n \times n$ and

contrahermitian with all diagonal entries on some line through 0 and not all on one side of 0, then S is contradefinite, in fact, S has a 2×2 or 3×3 principal submatrix which is contradefinite.

Proof. We assume without loss of generality that the line is the imaginary axis and that the diagonal entries are of unit modulus, i. e., all diagonal entries are $\pm i$ and they are not all equal. Then diagonality (skew-hermitianness) of S is precluded by the contra-hermitianness hypothesis, and there are two cases to consider.

Case 1. There exists a nonzero off-diagonal entry whose corresponding diagonal entries differ. Then S has a principal 2×2 submatrix of the form $\begin{bmatrix} i & 0 \\ g & -i \end{bmatrix}$ (with $g \neq 0$), which is, by Lemma 1.2(iii), contradefinite, and therefore so is S .

Case 2. All nonzero off-diagonal entries have equal corresponding diagonal entries. Then S has a principal 3×3 submatrix of a form such as

$$T = \begin{bmatrix} i & 0 & 0 \\ g & i & 0 \\ 0 & 0 & -i \end{bmatrix} \quad (\text{with } g \neq 0),$$

which is contradefinite. For, e. g.;, by Lemma 1.2,

$$\begin{bmatrix} i & 0 \\ g & i \end{bmatrix} \rightarrow \begin{bmatrix} ie^{i\varepsilon} & 0 \\ g' & ie^{-i\varepsilon} \end{bmatrix} \quad \text{where } \varepsilon = \arcsin\left[\min\left(\frac{1}{2}, \frac{1}{2}|g|\right)\right] (> 0).$$

Thus T is conjunctive with a matrix of $\arg \text{diag} \left(\frac{\pi}{2} + \epsilon, \frac{\pi}{2} - \epsilon, -\frac{\pi}{2} \right)$ with $0 < \epsilon < \frac{\pi}{2}$ (such an $\arg \text{diag}$ is called "spokelike" in [2]).

Therefore $\Gamma(T) = \mathbb{C}$ and T is contradefinite. Therefore

$\Gamma(S) = \mathbb{C}$ and S is contradefinite.

Lemma 6.10. Let S be nonsingular, codefinite contrahermitian, lower triangular, and $n \times n$ of $\arg \text{diag} = (\sigma_1, \dots, \sigma_n)$ where all the σ_i ($i = 1, \dots, n$) lie in an interval of length π . If $\tau = \frac{1}{n-1} \sum_{i=2}^n \sigma_i$, then S is conjunctive with a lower triangular matrix of $\arg \text{diag} = (\sigma_1, \tau, \tau, \dots, \tau)$.

Proof. Case 1. $S(1)$ not bidefinite. Then we apply if necessary--nothing being necessary if $S(1)$ is cohermitian and hence unidefinite--Lemma 6.7 to $S(1)$ to get $S(1)$ conjunctive with a lower triangular matrix of $\arg \text{diag} (\tau, \tau, \dots, \tau)$, and thus S conjunctive with a lower triangular matrix U with $\arg \text{diag } U = (\sigma_1, \tau, \tau, \dots, \tau)$.

Case 2. $S(1)$ bidefinite (hence diagonal), but $e^{-i\sigma_1} S$ not hermitian (i.e., not all eigenvalues of $S(1)$ on the line $\theta = \sigma_1$). Here without loss of generality (by Lemma 2.5) let $\arg \text{diag } S(1)$ be $(\alpha, \alpha + \pi, \sigma_4, \sigma_5, \dots, \sigma_n)$, where $\alpha < \sigma_1 < \alpha + \pi$. Apply Lemma 6.8(b) to $S[1, 2, 3]$, and the resulting subconjunctivity to S , to get S conjunctive with a matrix covered by Case 1.

Case 3. $S(1)$ bidefinite and $e^{-i\sigma} S(1)$ hermitian (hence diagonal). This case is ruled out by the codefiniteness and contra-hermitianness of S and Lemma 6.9.

Let Γ' be the nonzero part of Γ .

Lemma 6.11. Let S be nonsingular $n \times n$ contrahermitian and codefinite. (a) if $\pi = n$ and $\Gamma' \not\subset \text{ORHP}$, then π can be conjunctively decreased, (b) if $\pi = 0$ and $\Gamma \not\subset \text{CLHP}$, then π can be conjunctively increased, (c) if $\pi\nu = 0 = \delta$ and $\Gamma \not\subset \text{CRHP}$ and $\Gamma \not\subset \text{CLHP}$, then $\text{In } S$ can be incomparably changed by conjunctivity, (d) if $\Gamma \subset \text{CLHP}$, then ($\pi = 0$ and) π can not be increased by semiconjunctivity (and hence not by conjunctivity), (e) if $\Gamma \subset \text{CRHP}$ or $\Gamma \subset \text{CLHP}$ then ($\pi\nu = 0$ and) $\text{In } S$ can not be incomparably changed by semiconjunctivity (and hence not by conjunctivity), (f) if $\Gamma' \subset \text{ORHP}$, then ($\pi = n$ and) π can not be decreased by conjunctivity, (g) if $\Gamma' \subset \text{ORHP}$, then ($0 = \pi\nu = \nu = \delta$ and) δ can not be increased, nor hence In decreased, conjunctively, (h) if $\Gamma' \subset \text{OLHP}$, then ($0 = \pi\nu = \pi = \delta$ and) δ can not be increased, nor hence In decreased, conjunctively.

Proof of (a), (b) and (c). By Lemma 2.1, S is conjunctive with a lower triangular matrix T with T_{11} in (a) $\Gamma' \cap \text{CLHP}$ or

(b) $\Gamma' \cap \text{ORHP}$ or (c) $\Gamma \cap H$, where H is the one of ORHP and OLHP which does not contain an eigenvalue of S .

Proof of (d), (e), (f), (g), and (h). By Lemma 2.1 and the fact that for all $n \times n$ C , $\Gamma(C*SC) \subset \Gamma(S)$, S is not semiconjunctive (hence not conjunctive) with a matrix which has an eigenvalue outside $\Gamma(S)$.

Lemma 6.12. Let S be nonsingular $n \times n$ contrahermitian and codefinite. If $\pi v = 0 < \delta \leq n-1$ and $\Gamma \not\subset \text{CRHP}$ and $\Gamma \not\subset \text{CLHP}$, then $\ln S$ can be incomparably changed by semiconjunctivity.

Proof. Follow a conjunctivity given by Lemma 6.11(c) by the semiconjunctivity defined by $[1] \oplus 0$, which results in $[T_{11}] \oplus 0$.

Lemma 6.13. Let S be nonsingular contrahermitian codefinite and $n \times n$. (a) if $\pi(S) = n$ and $\Gamma' \not\subset \text{ORHP}$, then there is a conjunctivity which decreases \ln and hence increases δ , and (b) the same conclusion holds if $\nu(S) = n$ and $\Gamma' \not\subset \text{OLHP}$.

Proof of (a). The cone $\Gamma(S)$ has nonempty interior (S is contrahermitian), $\Gamma' \cap \text{ORHP} \neq \emptyset$ ($\pi(S) = n$), and $\Gamma' \cap \text{CLHP} \neq \emptyset$ ($\Gamma' \not\subset \text{ORHP}$). Thus Γ contains i or $-i$; say i (with $\frac{\pi}{2}$ chosen as its argument) $\in \Gamma(S)$. Then, according to

Lemma 2.1, S is conjunctive with a lower triangular matrix T with $\arg \text{diag } T = (\frac{\pi}{2} (= \tau_1), \tau_2, \tau_3, \dots, \tau_n)$ where, since S is codefinite, all the τ_i (including τ_1) can be chosen in some interval of length π . Then we have, for $i = 1, \dots, n$,

$$-\frac{\pi}{2} \leq \tau_i \leq \frac{3\pi}{2}.$$

Let τ denote $\frac{1}{n-1} \sum_{i=2}^n \tau_i$. Then

$$-\frac{\pi}{2} \leq \tau \leq \frac{3\pi}{2}.$$

Now $\pi(S) = n$, i.e., all eigenvalues of S are in ORHP, so their arguments may be taken in $(-\frac{\pi}{2}, \frac{\pi}{2})$, and then we have, by Lemma 6.5, that

$$-\frac{\pi}{2} < \sigma(S) < \frac{\pi}{2}.$$

Further, not all the τ_i (for $i \geq 2$) = $-\frac{\pi}{2}$ (nor all = $\frac{3\pi}{2}$) by Lemma 6.9 (T is codefinite and contrahermitian), so

$$-\frac{\pi}{2} < \tau < \frac{3\pi}{2}.$$

Thus $\sigma(T)$, which = $\frac{1}{n} (\frac{\pi}{2} + \sum_{i=2}^n \tau_i)$, and $\equiv \sigma(S) \pmod{2\pi}$ by Lemma 6.4, must satisfy $-\frac{\pi}{2} < \sigma(T) < \frac{3\pi}{2}$, and hence must equal $\sigma(S)$.

We then have

$$\frac{\pi}{2} + (n-1)\tau = n\sigma(S),$$

from which, together with the fact that $\sigma(S) < \frac{\pi}{2}$, we see that

$$-\frac{\pi}{2} < \tau < \frac{\pi}{2},$$

and so $\cos \tau > 0$. Now, by Lemma 6.10, S is conjunctive with a lower triangular matrix U with $\arg \text{diag } U = (\frac{\pi}{2}, \tau, \tau, \dots, \tau)$, and $\text{In } U = (n-1, 0, 1) < (n, 0, 0) = \text{In } S$.

Proof of (b). Apply part (a) to $-S$.

Lemma 6.14. Let S be nonsingular contrahermitian codefinite and $n \times n$ with $n \geq 3$. If $\pi(S)\nu(S) = 1$ and $\cos \sigma(S) = 0$ then there is a matrix of incomparable inertia semiconjunctive with S .

Proof. Because $n \geq 3$ we have $\delta = n-2 > 0$. We first make our usual Lemma 2.5 assumption about the eigenvalues of S , then use Lemma 1.2 to conjunctively bring closer together the positive real part root and a pure imaginary root. The result is an $n \times n$ matrix T with $\pi(T) \geq 2$, say $\pi(T[1, 2]) = 2$. Follow this conjunctivity by the semiconjunctivity which results in $T[1, 2] \oplus 0$. Its inertia is $(2, 0, n-2)$, incomparable with that of S .

Theorem 6.1. Let S be lower triangular codefinite contra-hermitian nonsingular and $n \times n$ with $\pi(S)\nu(S) > 0$. Then the following hold.

- (a) There is a matrix of smaller inertia, and hence larger δ , conjunctive with S , and
- (b) There is no matrix of incomparable inertia conjunctive with S iff both $\cos \sigma(S) = 0$ and $\pi(S)\nu(S) = 1$.

Proof. Without loss of generality, using Lemma 2.5, codefiniteness and $\pi\nu > 0$, we make the following assumptions.

- (i) the eigenvalues of S are $e^{i\sigma_1}, \dots, e^{i\sigma_n}$,
- (ii) there is a fixed real number α such that, for $i = 1, \dots, n$, $\sigma_i \in [\alpha, \alpha + \pi]$,
- (iii) $\cos \sigma_1 > 0$ and $\cos \sigma_2 < 0$.

Using Lemma 6.9 and Lemma 2.5 again, we further assume (iv) below.

- (iv) $e^{i\sigma_1}, e^{i\sigma_2}$ and $e^{i\sigma_3}$ are not collinear with the origin (if $n \geq 3$).

We may further assume (v) below after applying, if necessary, Lemma 6.8 to $S[1, 2, 3]$.

- (v) $\sin(\sigma_1 - \sigma_2) \neq 0$.

If $n = 2$ then Lemma 6.9 implies (v) directly, without the use of (iv), whose only purpose was (in the case $n \geq 3$) to get (v).

(a) By (i), (iii) and (v) we see that the cone $\Gamma(S[1, 2])$ contains i or $-i$, and so, by Lemma 2.1, $S[1, 2]$ is conjunctive with a lower triangular matrix of diagonal $(\pm i, e^{i\beta})$ for some β . Then S is conjunctive with a lower triangular matrix T of diagonal $(\pm i, e^{i\beta}, e^{i\sigma_3}, e^{i\sigma_4}, \dots, e^{i\sigma_n})$. Since $\delta(T) > \delta(S)$, $\pi(T) \leq \pi(S)$ and $\nu(T) \leq \nu(S)$, we have $\text{In } T < \text{In } S$.

(b) ("if" part). Suppose both $\cos \sigma(S) = 0$ and $\pi(S)\nu(S) = 1$. If S were conjunctive with a matrix T of incomparable inertia, then either $\pi(T) \geq 2$ and $\nu(T) = 0$ or else $\nu(T) \geq 2$ and $\pi(T) = 0$. If the former holds then, by Lemma 6.5, $\cos \sigma(T) > 0$, and if the latter holds then, again by Lemma 6.5, $\cos \sigma(T) < 0$. In either case, Lemma 6.4 has been violated.

(b) ("only if" part). We prove the contrapositive. If $\cos \sigma \neq 0$ then, using Lemma 6.7, $\arg \text{diag } S \rightarrow (\sigma, \sigma, \dots, \sigma)$, i.e., we can conjunctively achieve an incomparable inertia change (to $(n, 0, 0)$ if $\cos \sigma > 0$, or to $(0, n, 0)$ if $\cos \sigma < 0$). If $\cos \sigma = 0$ but $\pi\nu > 1$, then $n > 2$ and we assume for definiteness that $\sigma = \frac{\pi}{2}$ and $\pi(S) > 1$. Then, if $\tau = \frac{1}{n-1} \sum_{i=2}^n \sigma_i$, we have, by (ii) and properties of averages and Lemma 6.5, that

$$-\frac{\pi}{2} \leq \sigma_i \leq \frac{3\pi}{2} \quad \text{for } i = 1, \dots, n,$$

and

$$-\frac{\pi}{2} \leq \tau \leq \frac{3\pi}{2},$$

and, by (iii), that

$$-\frac{\pi}{2} < \sigma_1 < \frac{\pi}{2}.$$

Then by Lemma 6.5

$$n \frac{\pi}{2} - \sigma_1 = n\sigma - \sigma_1 = (n-1)\tau,$$

so

$$(n-1) \frac{\pi}{2} < (n-1)\tau < (n+1) \frac{\pi}{2}.$$

Then

$$\frac{\pi}{2} < \tau < \frac{n+1}{n-1} \frac{\pi}{2} < 3 \frac{\pi}{2},$$

the last inequality holding because $n > 2$. Then $\cos \tau < 0$. By Lemma 6.10 then, $(\sigma_1, \dots, \sigma_n) \rightarrow (\sigma_1, \tau, \tau, \dots, \tau)$, the new inertia being $(1, n-1, 0)$, incomparable with $\text{In } S$.

Corollary 6.1.1. Let (i), (ii), (ii)', (iii), and (iii)' denote the following conditions.

(i) S is lower triangular, codefinite, contrahermitian non-singular $n \times n$ with $\pi(S) > 0$ and $\nu(S) > 0$,

(ii) $n > 2$,

(ii)' $n = 2$,

(iii) $\det S \neq 0$ (i.e., $\text{sgn } \det S \neq -1$),

(iii)' $\det S < 0$.

(a) If (i) and (ii), then (unless $\phi \equiv \delta$ and $\delta = 0$) S is not a

solution of $(j)_\phi$ for $j = 1, \dots, 6$. If (i) and (ii) and $\phi \equiv \delta$ and $\delta = 0$ then S is not a solution of $(j)_\phi$ for $j = 1, 3, 4, 6$ but is a solution of $(j)_\phi$ for $j = 2$ and 5 .

(b) If (i) and (ii)' then S is not a solution of $(j)_\phi$ for $j = 1, 2, 5, 6$ and $\phi \neq \delta$. If further (iii), then S is also not a solution of $(3)_\phi$ and $(4)_\phi$. If on the other hand (iii)', then S is a solution of $(3)_\phi$ and $(4)_\phi$ for $\phi \neq \delta$ and is not a solution of $(3)_\delta$ and $(4)_\delta$.

Proof (partial). The last part of the second assertion of (a) follows from Lemma 2.4. The third assertion of (b) follows from Theorem 2.1. The rest of the proof follows from the theorem.

Corollary 6.1.2. The conclusions of the theorem still hold if the hypotheses of lower triangularity and nonsingularity are deleted. (Here $\sigma(S)$ must be replaced with $\sigma(N)$ where N is nonsingular such that S is unitarily conjunctive with $N \oplus 0$ by Theorem 1.2 and Theorem 1.1.)

Proof. S is, by Theorem 1.1, $*$ -regular. The rest of the proof is thus exactly like that of Corollary 4.1.1.

Theorem 6.2. (a) Let S be $n \times n$ codefinite contrahermitian and have all eigenvalues in CRHP ($\nu(S) = 0$) with at least one nonzero pure imaginary eigenvalue. Then, conjunctively, π and In

can be increased and δ can be decreased.

(b) Same hypothesis as (a) except " $\nu(S) = 0$ " is replaced by " $\pi(S) = 0$." Then π can be conjunctively increased iff $\Gamma(S) \not\subset \text{CLHP}$.

Proof. We first assume that S is lower triangular and non-singular. (a) S is not skew-hermitian, by contrahermitianness. In particular $n > 1$ and there are two cases.

Case 1. S has an eigenvalue in ORHP. Using $\nu = 0$ and Lemma 2.5 we assume without loss of generality that $\text{diag } S = (e^{i\sigma_1}, \dots, e^{i\sigma_n})$ with $|\sigma_j| \leq \frac{\pi}{2}$ for $j = 1, \dots, n$ and $\cos \sigma_1 = 0$ and $\cos \sigma_2 > 0$. Then S is conjunctive with a lower triangular matrix T of $\arg \text{diag} = (\frac{1}{2}(\sigma_1 + \sigma_2), \frac{1}{2}(\sigma_1 + \sigma_2), \sigma_3, \sigma_4, \dots, \sigma_n)$. In $T > \text{In } S$, $\pi(T) > \pi(S)$ and $\delta(T) < \delta(S)$.

Case 2. All eigenvalues of S are pure imaginary. Then, by Lemma 6.9, all eigenvalues are on the positive (or negative) imaginary axis, say $\text{diag } S = (i, i, \dots, i)$. Then since S is not skew-hermitian, S is not diagonal, so we may suppose that $S_{21} \neq 0$. Then S is conjunctive with a matrix T of $\arg \text{diag} = (\frac{\pi}{2} + \epsilon, \frac{\pi}{2} - \epsilon, \frac{\pi}{2}, \frac{\pi}{2}, \dots, \frac{\pi}{2})$ for some small $\epsilon > 0$ (e.g., $\epsilon = \arcsin[\min(\frac{1}{2}, \frac{1}{2}|S_{21}|)]$). Then $\text{In } T > \text{In } S$, $\pi(T) > \pi(S)$ and $\delta(T) < \delta(S)$.

(b) If $\Gamma(S) \not\subset \text{CLHP}$ then there is $e^{ia} \in \Gamma(S)$ such that

$\cos \alpha > 0$. Then, by Lemma 2.1, S is conjunctive with a lower triangular matrix T with $T_{11} = e^{i\alpha}$. Thus $\pi(T) > \pi(S) = 0$. Conversely, assume $\pi(T) > \pi(S) = 0$ where S is conjunctive with T . Then T has an eigenvalue in the ORHP. Therefore $\Gamma(T)$ has an element in ORHP. Then so does $\Gamma(S) (= \Gamma(T))$, i.e., $\Gamma(S) \not\subset \text{CLHP}$. We now show that our assumptions of lower triangularity and nonsingularity did not cause a loss of generality. For, by Theorem 1.1, S is $*$ -regular, and thus, by Theorem 1.2 and Theorem 1.4, is conjunctive with a direct sum of a nonsingular lower triangular matrix N (of order = rank S) and a zero matrix. We have $\text{In } S = (\pi(N), \nu(N), \delta(N) + \text{nullity}(S))$ and $\Gamma(S) = \Gamma(N)$.

Corollary 6.2.1. If S is as in the theorem then it is not a solution of (3) $_{\phi}$ (and hence not of (4) $_{\phi}$, (1) $_{\phi}$ nor (6) $_{\phi}$) for $\phi = \text{In}$ or π and S is not a solution of (2) $_{\delta}$ (and hence not of (5) $_{\delta}$).

Theorem 6.3 (Examples). (a) Let $S_1 = \text{diag}(1, 1, i)$. Then conjunctively π cannot be decreased nor δ increased (nor hence In decreased) nor In incomparably changed.

(b) Let $S_2 = [i] \oplus \exp \frac{\pi}{4} i \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$. Then, conjunctively, π can be decreased, In can be decreased (hence δ increased) and In can be incomparably changed.

(c) Let $S_3 = \text{diag}(1, \exp \frac{3\pi}{4} i, \exp \frac{3\pi}{4} i)$. Then conjunctively (and hence semiconjunctively) π can be increased.

(d) Let $S_4 = \text{diag}(\exp \frac{\pi}{4} i, -1, -1)$. Then π cannot be increased conjunctively nor semiconjunctively.

(e) Let $S_5 = [1] \oplus \begin{bmatrix} i & 0 \\ \sqrt{2} & 1 \end{bmatrix}$. Then, conjunctively, In cannot be incomparably changed.

Proof. (a) $\Gamma(S_1)$ is the closed first quadrant since S_1 is nonsingular diagonal. If π could be conjunctively decreased then S_1 would be conjunctive with some matrix T with $\arg \text{diag } T = (\alpha, \beta, \gamma)$ chosen with $0 \leq \alpha \leq \frac{\pi}{2} = \beta = \gamma$. But then we would have $\sigma(T) = \frac{\alpha + \beta + \gamma}{3} \geq \frac{\pi}{3} > \frac{\pi}{6} = \sigma(S_1)$, contradicting Lemma 6.4. This argument also shows that δ cannot be increased nor In incomparably changed.

(b) $\Gamma(S_2)$ is, by Lemma 1.2, an upper-right half-plane bounded by $x+y=0$. By that lemma we can conjunctively decrease π , conjunctively decrease In (and hence increase δ), e.g., the $[1, 2]$ subconjunctivity of order 3 having the effect

$$\exp(i \frac{\pi}{4}) \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \rightarrow \exp i \frac{\pi}{4} \begin{bmatrix} \exp(i \frac{\pi}{4}) & 0 \\ \sqrt{2} & \exp(-i \frac{\pi}{4}) \end{bmatrix} = \begin{bmatrix} i & 0 \\ \sqrt{2} \exp(i \frac{\pi}{4}) & 1 \end{bmatrix},$$

sends $\arg \text{diag } S_2 = (\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4})$ to $(\frac{\pi}{2}, \frac{\pi}{2}, 0)$. Using the third order

[1, 2] subconjunctivity having the effect

$$\exp(i \frac{\pi}{4}) \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \rightarrow \exp i \frac{\pi}{4} \begin{bmatrix} \exp(-i \frac{\pi}{3}) & 0 \\ 1 & \exp(i \frac{\pi}{3}) \end{bmatrix},$$

sends $\arg \text{diag } S_2 = (\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4})$ to $(\frac{\pi}{2}, -\frac{\pi}{12}, \frac{7\pi}{12})$, and achieves an incomparable inertia change from $(2, 0, 1)$ to $(1, 1, 1)$.

(c) $\text{Arg diag } S_3 = (0, \frac{3\pi}{4}, \frac{3\pi}{4}) \rightarrow (\frac{3\pi}{8}, \frac{3\pi}{8}, \frac{3\pi}{4})$, using Lemma 1.2 and the resulting third order [1, 2] subconjunctivity.

(d) We show first that $\pi(S_4)$ cannot be conjunctively increased. For otherwise, S_4 would be conjunctive with a matrix T of $\arg \text{diag} = (\alpha, \beta, \gamma)$ where α, β , and γ are chosen such that $\cos \alpha > 0$, $\cos \beta > 0$, $\alpha, \beta \in [\frac{\pi}{4}, \frac{\pi}{2})$ and $\gamma \in [\frac{\pi}{4}, \pi]$. This is so because $\Gamma(T) = \Gamma(S_4) =$ the convex sector with bounding arguments $\frac{\pi}{4}$ and π . Then, by Lemma 6.4, $\sigma(T) = \frac{\alpha+\beta+\gamma}{3} = \sigma(S) = \frac{3\pi}{4}$. But then we would have $\frac{9\pi}{4} = \alpha+\beta+\gamma < \frac{\pi}{2} + \frac{\pi}{2} + \gamma$, or $\gamma > \frac{5\pi}{4}$, a contradiction. $\pi(S_4)$ cannot be semiconjunctively increased either, by Corollary 2.2.1.

(e) By Lemma 2.1, $\Gamma(S_5)$ is an upper-right half-plane bounded by $x+y = 0$. If an incomparable change in $\text{In}(S_5)$ were conjunctively possible, S_5 would be conjunctive with a lower triangular matrix T of $\arg \text{diag} = (\alpha, \beta, \gamma)$ with α, β , and γ chosen, since $\pi(T) \leq 1$ and $\nu(T) \geq 1$, such that

$\alpha, \beta \in \left[\frac{\pi}{2}, \frac{3\pi}{4}\right]$ and $\gamma \in \left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$. But then

$$\sigma(T) = \frac{\alpha + \beta + \gamma}{3} \geq \frac{\pi}{4} > \frac{\pi}{6} = \sigma(S), \quad \text{violating Lemma 6.4.}$$

Our main results on contrahermitian codefinite matrices are collected in the following tabular theorem.

Theorem 6.4. Let S be nonsingular $n \times n$ contrahermitian and codefinite. Then the following table tells whether or not the indicated change is possible conjunctively (upper half of the entry) and (semiconjunctively (lower half of the entry)). (In order to further resolve the questions indicated by $?_1, ?_2, ?_3, ?_4,$ and $?_5,$ more complicated conjunctivity invariants would have to be introduced. We in each case provide examples (from Theorem 6.3) of both outcomes. See the proof.)

Theorem 6.4 Table

	\uparrow In & \downarrow δ ?	\downarrow π ?	\uparrow π ?	\downarrow In & \uparrow δ ?	i. c. In ?	
$\delta > 0$	yes/yes					1
$\delta = 0$	no/no					2
$\pi = 0$		no/no				3
$\pi = 1$	$\left[\begin{array}{l} \cos \sigma \leq 0 \\ \cos \sigma > 0 \end{array} \right.$	yes/yes				4
		no/yes				5
$1 < \pi < n$	$\left[\begin{array}{l} \cos \sigma \leq 0 \\ \cos \sigma > 0 \end{array} \right.$	yes/yes				6
		? ₁ /yes				7
$\pi = n$	$\left[\begin{array}{l} \Gamma' \not\subset \text{ORHP} \\ \Gamma' \subset \text{ORHP} \end{array} \right.$	yes/yes				8
		no/yes				9
$\pi = 0$	$\left[\begin{array}{l} \Gamma \subset \text{CLHP} \\ \Gamma \not\subset \text{CLHP} \end{array} \right.$		no/no			10
			yes/yes			11
$0 < \pi < n-1$	$\left[\begin{array}{l} \delta > 0 \\ \delta = 0 \\ \delta = 1 \end{array} \right.$		yes/yes			12
		$\left[\begin{array}{l} \cos \sigma \leq 0 \\ \cos \sigma > 0 \end{array} \right.$? ₂ */? ₃ *			13
			yes/yes			14
$\pi = n-1$	$\left[\begin{array}{l} \delta = 1 \\ \delta = 0 \end{array} \right.$		yes/yes			15
		$\left[\begin{array}{l} \cos \sigma \leq 0 \\ \cos \sigma > 0 \end{array} \right.$	no/no			16
$\pi = n$	$\left[\begin{array}{l} \delta = 0 \\ \delta = n-1 \\ \delta = n \end{array} \right.$		yes/yes			17
			no/no			18
$\pi \nu = 0$	$\left[\begin{array}{l} \delta = 0 \\ 0 < \delta < n-1 \\ \delta = n-1 \\ \delta = n \end{array} \right.$	$\left[\begin{array}{l} \nu = 0 \\ \pi = 0 \end{array} \right.$	$\left[\begin{array}{l} \Gamma' \subset \text{ORHP} \\ \Gamma' \not\subset \text{ORHP} \end{array} \right.$	no/yes		19
			$\left[\begin{array}{l} \Gamma' \not\subset \text{ORHP} \\ \Gamma' \subset \text{OLHP} \end{array} \right.$	yes/yes		20
			$\left[\begin{array}{l} \Gamma' \subset \text{OLHP} \\ \Gamma' \not\subset \text{OLHP} \end{array} \right.$	no/yes		21
				yes/yes		22
				? ₄ /yes		23
				no/yes		24
$\pi \nu > 0$	$\left[\begin{array}{l} \Gamma \subset \text{CRHP or } \Gamma \subset \text{CLHP} \\ \Gamma \not\subset \text{CRHP and } \Gamma \not\subset \text{CLHP} \end{array} \right.$			no/yes		25
				yes/yes		26
				no/no		27
				yes/yes		28
				? ₅ /yes		29
				no/yes		30
$\pi \nu > 0$	$\left[\begin{array}{l} \pi \nu = 1 \text{ and } \cos \sigma = 0 \\ \pi \nu > 1 \text{ or } \cos \sigma \neq 0 \end{array} \right.$			no/no		31
				no/yes		32
				no/no		33
				yes/yes		34

*According to Lemma 2.2. and Corollary 2.2.1, the cases in which π can (respectively, cannot) be increased conjunctively are precisely those in which π can (respectively, cannot) be increased semiconjunctively. Thus ?₂ and ?₃ "agree" in each case considered.

Proof. As in the proof of Theorem 4.5 we denote with the subscript u , the upper tabular entry and with the subscript l , the lower tabular entry.

1_u : Lemma 2.6(a).

1_l : Lemma 2.2(a).

2_u , 2_l , 3_u , and 3_l : Lemma 2.3.

4_u : Lemma 6.7.

4_l : Lemma 2.2(a).

5_u : Lemma 6.4.

5_l : Lemma 2.4.

6_u : Lemma 6.7.

6_l : Lemma 2.2(a).

7_u : Theorem 6.3(a) and Theorem 6.3(b).

7_l : Lemma 2.4.

8_u : Lemma 6.11(a).

8_l : Lemma 2.2(a) or Lemma 2.4.

9_u : Lemma 6.11(f).

9_l : Lemma 2.4.

10_u : Lemma 2.2(b).

10_l : Lemma 6.11(d).

11_u : Lemma 6.11(b).

11_l : Lemma 2.2(a).

12_u : Lemma 2.6(b).

- 12_1 : Lemma 2.2(a).
- 13_u and 13_1 : Theorem 6.3(c) and Theorem 6.3(d).
- 14_u , 15_u , and 17_u : Lemma 6.7.
- 14_1 , 15_1 , and 17_1 : Lemma 2.2(a).
- 16_u : Lemma 6.4.
- 16_1 : Corollary 2.2.1.
- 18_u and 18_1 : Lemma 2.3.
- 19_u : Lemma 6.11(g).
- 19_1 : Lemma 2.4.
- 20_u : Lemma 6.13(a).
- 20_1 : Lemma 2.4.
- 21_u : Lemma 6.11(h).
- 21_1 : Lemma 2.4.
- 22_u : Lemma 6.13(b).
- 22_1 : Lemma 2.4.
- 23_u : Theorem 6.3(a) and Theorem 6.3(b).
- 23_1 : Lemma 2.4.
- 24_u : Lemma 6.4.
- 24_1 : Lemma 2.4.
- 25_u and 25_1 : Lemma 2.3.
- 26_u : Theorem 6.1(a).
- 26_1 : Lemma 2.4.
- 27_u : Lemma 2.2(b).

- 27_1 : Lemma 6.11(e).
 28_u : Lemma 6.11(c).
 28_1 : Lemma 2.2(a).
 29_u : Theorem 6.3(b) and Theorem 6.3(e).
 29_1 : Lemma 6.12.
 30_u : Lemma 6.4 (since $\cos \sigma \neq 0$).
 30_1 : Lemma 6.12.
 31_u and 31_1 : Lemma 2.3.
 32_u : Theorem 6.1(b).
 32_1 : Lemma 6.14.
 33_u : Theorem 6.1(b).
 33_1 : Corollary 2.2.2 together with 2_u .
 34_u : Theorem 6.1(b).
 34_1 : Lemma 2.2(a).

Corollary 6.4.1. Let S be $k \times k$ contrahermitian codefinite and of rank n . Then if in the table of Theorem 6.4 we replace " δ " by " $\delta(S) - (k-n)$ " and interpret σ as the average arg of the nonzero eigenvalues of S , the table holds for S .

Proof. See proof of Corollary 4.1.1.

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APPENDIX

APPENDIX

The following elementary lemma is an essential observation in the proofs of Corollaries 4.1.1, 4.2.1, 4.3.1, 4.4.1, 4.5.1.1, 4.5.2, 6.1.2 and 6.4.1 and also in the proof of Fact A.1 below.

Lemma A.1. Let N be $n \times n$ nonsingular, let $S = N \oplus 0$ be $k \times k$, and let $\phi \in \{\text{In}, \pi, \delta\}$. Then

(1) if $\phi(D*ND) \leq \phi(N)$ for all nonsingular $n \times n$ D then $\phi(C*SC) \leq \phi(S)$ for all nonsingular $k \times k$ C ;

(2) if $\phi(D*ND) \geq \phi(N)$ for all nonsingular $n \times n$ D then $\phi(C*SC) \geq \phi(S)$ for all nonsingular $k \times k$ C ;

(3) if $\text{In}(D*ND)$ is comparable with $\text{In}(N)$ for all nonsingular $n \times n$ D , then $\text{In}(C*SC)$ is comparable with $\text{In} S$ for all nonsingular $k \times k$ C .

Proof. We prove only (1) for $\phi = \text{In}$, the other proofs being similar. Let C be any given nonsingular $k \times k$ matrix. Then $C*(N \oplus 0)C$ is $*$ -regular, by Lemma 2.9. Therefore, by Theorem 1.2, there is a unitary $k \times k$ matrix V and a nonsingular $n \times n$ matrix M such that $V*C*(N \oplus 0)CV = M \oplus 0$. Then, by Lemma 2.7, M is conjunctive with N . Therefore, by hypothesis, $\text{In} M \leq \text{In} N$. Then

$$\text{In} C*(N \oplus 0)C = \text{In}(M \oplus 0) = (0, 0, k-n) + \text{In} M \leq (0, 0, k-n) + \text{In} N = \text{In}(N \oplus 0).$$

Fact A.1. Let S be $n \times n$ and let (1), (2), (3), (4) and (5) denote the following conditions.

(1) $\text{In } C^*SC \leq \text{In } S$ for all nonsingular $n \times n$ C ;

(2) $\text{In } C^*SC \leq \text{In } S$ for all $n \times n$ C ;

(3) $\text{In } C^*SC = \text{In } S$ for all nonsingular $n \times n$ C ;

(4) $\Gamma(S)$ is a subset of the imaginary axis or else is disjoint from the nonzero part of the imaginary axis;

(5) $\Gamma(S)$ is a subset of some line or else $\Gamma'(S) \subset \text{ORHP}$ or else $\Gamma' \subset \text{OLHP}$.

Then (1) and (2) are equivalent, and so are (3), (4), and (5).

Proof. Trivially, (2) implies (1). (1) implies (2) by Corollary 2.2.2. Using Theorems 3.1 and 1.1 and the cone characterization of contradefiniteness we observe that S cannot be $*$ -irregular and satisfy any of (3), (4), or (5). Then, using Theorem 1.2 and Lemma A.1, we may assume that S is nonsingular.

We show next that (5) implies (3). Let (5) hold. Then if S is cohermitian (i. e., if $\Gamma(S)$ is contained in some line) (3) is satisfied by Theorem 5.1(a). If $\Gamma' \subset \text{ORHP}$ or OLHP then S is codefinite and, if it is also contrahermitian, we see from the Theorem 6.4 Table, 19_u and 21_u that In cannot be decreased, from 2_u that it cannot be increased and from 27_u that it cannot be incomparably changed. Thus (3) holds.

(4) and (5) are obviously equivalent, geometrically.

Finally we show (3) implies (5). Let (3) hold. Then, by the Theorem 4.5 Table, S is codefinite. Then from 2_u , 19_u and 21_u of the Theorem 6.4 Table we see that (5) must hold.