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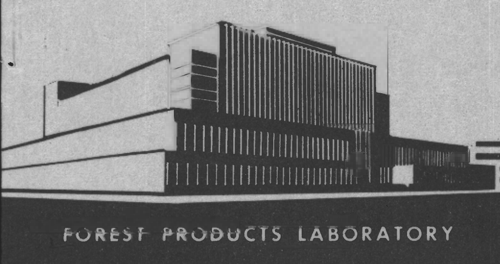

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FOREST SERVICE
In Cooperation with the University of Wisconsin

METHOD OF INDIRECT CALCULATION FOR USE IN LOG GRADE ANALYSES

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Introduction

In the formulation and study of log grades, log value is usually derived from the values per 1,000 board feet or relative values of the various lumber grades admitted for the species in question. Ordinarily, a single value is calculated for each log. This log value may be calculated as either value per 1,000 board feet (V/M) or quality index (Q). In all subsequent analyses which involve either V/M or Q, the particular applied set of lumber values, whether absolute or relative, becomes an integral part of the calculations. Because of the dependency of the final results of any computations involving either V/M or Q on the lumber prices or price relatives, any change in these prices or price relatives will cause corresponding changes in the final results. If it becomes necessary to change the final results of an analysis because of changes in either prices or price relatives, then it is necessary to repeat the entire analysis, beginning with the application of the new prices or price relatives to the lumber grade yields for each log. The purpose of this paper is to show, both algebraically and by numerical examples, an alternate method of calculation, which should be useful in a large variety of applications, in which the prices or price relatives are not introduced until the final step. Having the calculations in this form allows the change from one price structure to another by repetition of only the final step in the analysis.

In order to simplify this presentation, the derivations are given only in terms of quality index, but the general method applies equally well for Value/M.

The prices of the various grades of lumber vary constantly with time. Quality index will change whenever the lumber prices change in such a way that the relative prices are changed. For many purposes, the effect on quality index of fluctuations over short time intervals may not be severe enough to require repeated changes in the analytical results; however, for longer time intervals, changes usually are necessary.

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[1] Maintained at Madison, Wis., in cooperation with the University of Wisconsin.

Various authors[2,3,4] have recommended from 3 to 5 years as a satisfactory base period to be used in establishing the price relatives. A change in the base every 3 to 5 years would usually mean that all data and publications would need to be updated at least that often. A rapid change in economic conditions could, at any time, force the abandonment of the price relatives for the existing base period and present the need for immediate revision of all data, reports, and publications.

The problem of varying price relatives has been studied and discussed by several authors, but there is still considerable disagreement as to how much the price relatives can change without seriously affecting the quality indices and the resulting calculated quantities. One such argument has been that frequently the changes in price relatives tend to be compensating; that is, the increase in price relatives for some lumber grades is offset by decreases for other grades so that the net change in quality index is small. In order to make a direct study of the changes in quality index and quantities calculated from quality index, it would be necessary to work with many different sets of price relatives.

In any of the above-mentioned situations, if the usual procedure of calculating a quality index figure for each log is followed, the entire calculations would have to be repeated for every different set of price relatives. For example, if we want to plot average quality index for a group of logs against time and we have 25 time intervals (25 sets of price relatives), it would be necessary to compute 25 quality indices for each log and subsequently compute 25 averages for the group of logs.

Using the method presented in this paper, it would be necessary to carry out the basic calculations only one time. The final step of the calculations, which is the introduction of the price relatives, would then be repeated 25 times.

As usual, good things are seldom free. The basic calculations as given here will require several times the amount of computing required in the usual procedure. For this reason, there will be situations when the additional computing is not justified. Such a situation would

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[2] Beazley, R. I., and Herrick, A. M. Lumber Price Relatives: Their Application in the Hardwood Quality Index. Purdue University Agr. Exp. Sta. Bul. 610. 1954.

[3] Ellertsen, Birger W., and Lane, Paul. Lumber Price Ratios for Computing Quality Index of Tennessee Valley Hardwoods. TVA Tech. Note No. 15. June 1953.

[4] Purdue University Agricultural Experiment Station. Proceedings of a Symposium on a Standard Hardwood Quality Index. 1952.

perhaps exist when the computations being performed are exploratory or when the results will be used for a short time only. The additional computation should be justified whenever the results are to be used over long periods, and especially when they are to be presented in the form of reports and publications which will have need of periodic revisions as prices change.

General Regression Identities

The lumber grade yield data from a group of logs may be represented in the following array:

Log No.	Proportion of lumber in each grade: Grade 1	Grade 2	. . .	Grade j	. . .	Grade r
1	a_{11}	a_{12}	. . .	a_{1j}	. . .	a_{1r}
2	a_{21}	a_{22}	. . .	a_{2j}	. . .	a_{2r}
3	a_{31}	a_{32}	. . .	a_{3j}	. . .	a_{3r}
⋮	⋮	⋮		⋮		⋮
i	a_{i1}	a_{i2}	. . .	a_{ij}	. . .	a_{ir}
⋮	⋮	⋮	⋮	⋮	⋮	⋮
n	a_{n1}	a_{n2}	. . .	a_{nj}	. . .	a_{nr}

If p_1, p_2, . . ., and p_r are a set of price relatives, then the quality index for a log is

$$Q_i = p_1 a_{i1} + p_2 a_{i2} + \cdots + p_r a_{ir}$$

which may also be written

$$Q_i = \underline{a}'_i \underline{p} \tag{1}$$

where $\underline{p}$ is the $(r \times 1)$ column vector

$$\underline{p} = \begin{bmatrix} p_1 \\ p_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ p_r \end{bmatrix}$$

and $\underline{a}'_i$ is the row vector for the i^{th} row of the array given above. The $(n \times r)$ array will henceforth be denoted as the matrix A.

The data for the independent variables, to be used in a regression with Q as the dependent variable, may be represented by the array:

Log No.	Independent variable: 0	1	. . .	k	. . .	u
1	x_{10}	x_{11}	. . .	x_{1k}	. . .	x_{1u}
2	x_{20}	x_{21}	. . .	x_{2k}	. . .	x_{2u}
3	x_{30}	x_{31}	. . .	x_{3k}	. . .	x_{3u}
⋮	⋮	⋮		⋮		⋮
i	x_{i0}	x_{i1}	. . .	x_{ik}	. . .	x_{iu}
⋮	⋮	⋮		⋮		⋮
n	x_{n0}	x_{n1}	. . .	x_{nk}	. . .	x_{nu}

$(x_{i0} \equiv 1)$

Since a general regression solution is intended, it should be understood that the x_{ik}'s in the array may be any function of the original independent variables such as logarithm, square, etc. This array will be considered as an $(n \times (u + 1))$ matrix and denoted by X.

If we consider the general regression model

$$Q = b_0 + b_1x_1 + . . . + b_ux_u + e$$

the normal equations may be written as the single matrix equation

$$X'X\underline{b} = X'\underline{Q} \tag{2}$$

where

$$\underline{b} = \begin{bmatrix} b_o \\ b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_u \end{bmatrix}_{(u \times 1)} .$$

is the vector of regression coefficients and

$$\underline{Q} = \begin{bmatrix} Q_1 \\ Q_2 \\ \cdot \\ \cdot \\ \cdot \\ Q_n \end{bmatrix}_{(n \times 1)}$$

is the vector of quality indices for the n logs.

The solution of equation (2) for the regression coefficients may be represented simply as:

$$\underline{b} = (X'X)^{-1}X'\underline{Q} \tag{3}$$

Using equation (1), $\underline{Q}$ may be rewritten as:

$$\underline{Q} = \begin{bmatrix} \underline{a}_1'\underline{p} \\ \underline{a}_2'\underline{p} \\ \cdot \\ \cdot \\ \cdot \\ \underline{a}_i'\underline{p} \\ \cdot \\ \cdot \\ \cdot \\ \underline{a}_n'\underline{p} \end{bmatrix} = A\underline{p}$$

$(n \times 1)$

Thus, $\underline{b}$ may be written in terms of the basic data and the price relatives and becomes

$$\underline{b} = (X'X)^{-1}X'A\underline{p} \tag{4}$$

$$= B\underline{p}$$

Note that the matrix product $(X'X)^{-1}X'A = B$ is of order $((u + 1) \times r)$ and can be calculated independently of the price relatives. This means that all calculations except the final step (that is, forming the product $B\underline{p}$) are performed without any measure of value being introduced. Once B has been calculated, it can be multiplied by any vector $\underline{p}$ of price relatives.

In order to show what the matrix B represents, it is necessary to expand $A\underline{p}$.

$$A\underline{p} = \begin{bmatrix} a_{11} & a_{12} \cdots a_{1j} \cdots a_{1r} \\ a_{21} & a_{22} \cdots a_{2j} \cdots a_{2r} \\ \vdots & \vdots \qquad \vdots \qquad \vdots \\ a_{i1} & a_{i2} \cdots a_{ij} \cdots a_{ir} \\ \vdots & \vdots \qquad \vdots \qquad \vdots \\ a_{n1} & a_{n2} \cdots a_{nj} \cdots a_{nr} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_j \\ \vdots \\ p_r \end{bmatrix}$$

Performing the multiplication gives:

$$A\underline{p} = \begin{bmatrix} a_{11}p_1 + a_{12}p_2 + \cdots + a_{1j}p_j + \cdots + a_{1r}p_r \\ a_{21}p_1 + a_{22}p_2 + \cdots + a_{2j}p_j + \cdots + a_{2r}p_r \\ \vdots \\ a_{i1}p_1 + a_{i2}p_2 + \cdots + a_{ij}p_j + \cdots + a_{ir}p_r \\ \vdots \\ a_{n1}p_1 + a_{n2}p_2 + \cdots + a_{nj}p_j + \cdots + a_{nr}p_r \end{bmatrix}$$

This may be rewritten as:

$$\underline{Ap} = \begin{bmatrix} a_{11} \\ a_{21} \\ \cdot \\ \cdot \\ \cdot \\ a_{i1} \\ \cdot \\ \cdot \\ \cdot \\ a_{n1} \end{bmatrix} p_1 + \begin{bmatrix} a_{12} \\ a_{22} \\ \cdot \\ \cdot \\ \cdot \\ a_{i2} \\ \cdot \\ \cdot \\ \cdot \\ a_{n2} \end{bmatrix} p_2 + \cdot\ \cdot + \begin{bmatrix} a_{1j} \\ a_{2j} \\ \cdot \\ \cdot \\ \cdot \\ a_{ij} \\ \cdot \\ \cdot \\ \cdot \\ a_{nj} \end{bmatrix} p_j + \cdot\ \cdot + \begin{bmatrix} a_{1r} \\ a_{2r} \\ \cdot \\ \cdot \\ \cdot \\ a_{ir} \\ \cdot \\ \cdot \\ \cdot \\ a_{nr} \end{bmatrix} p_r$$

or

$$A\underline{p} = \underline{a}_1 p_1 + \underline{a}_2 p_2 + \cdot \cdot \cdot + \underline{a}_j p_j + \cdot \cdot \cdot + \underline{a}_r p_r \qquad (5)$$

Note that the $\underline{a}_j$'s in equation (5) are column vectors of the matrix A, whereas the $\underline{a}_i$'s in equation (1) were row vectors of A. Substituting equation (5) into equation (4) gives:

$$\underline{b} = (X'X)^{-1}X'\underline{a}_1 p_1 + (X'X)^{-1}X'\underline{a}_2 p_2 + \cdot \cdot \cdot + (X'X)^{-1}X'\underline{a}_j p_j + \cdot \cdot \cdot$$

$$\cdot \cdot \cdot + (X'X)^{-1}X'\underline{a}_r p_r$$

or

$$\underline{b} = \underline{b}_1 p_1 + \underline{b}_2 p_2 + \cdot \cdot \cdot + \underline{b}_j p_j + \cdot \cdot \cdot + \underline{b}_r p_r \qquad (6)$$

Equation (6) states that the coefficients for the regression of Q on the x's are the sums of the like coefficients for the regressions of the

a_j's on the x's weighted by the price relatives. For example, if there are six lumber grades (r = 6) and the regressions are linear in only one independent variable, then

$$b_0 = b_{10}p + b_{20}p_2 + \cdot \cdot \cdot + b_{60}p_6$$

and

$$b_1 = b_{11}p_1 + b_{21}p_2 + \cdot \cdot \cdot + b_{61}p_6$$

where b_0 and b_1 are the coefficients of the regression equation for quality index, Q, that is,

$$\hat{Q} = b_0 + b_1 x$$

and b_{j0} and b_{j1} are the coefficients of the regression equation for proportion of lumber in the j^{th} grade, a_j, that is,

$$\hat{a}_j = b_{j0} + b_{j1}x$$

If we rewrite equation (6) as

$$\underline{b} = [\underline{b}_1 \quad \underline{b}_2 \cdot \cdot \cdot \underline{b}_j \cdot \cdot \cdot \underline{b}_r] \, \underline{p}$$

it becomes apparent that $B = (X'X)^{-1}X'A$ is a matrix in which each of the column vectors gives the coefficients for the regression of one of the lumber grade proportions. For the example above, we have

$$B = \begin{bmatrix} b_{10} & b_{20} & b_{30} & b_{40} & b_{50} & b_{60} \\ b_{11} & b_{21} & b_{31} & b_{41} & b_{51} & b_{61} \end{bmatrix}$$

Indirect Calculation of Sums of Squares for Q

The total sums of squares for Q is given by $\underline{Q}'\underline{Q}$. Using the previously shown relationship, $\underline{Q} = A\underline{p}$, we may write

$$\underline{Q}'\underline{Q} = \underline{p}'A'A\underline{p}$$

and by defining $S = A'A$, we have

$$\underline{Q}'\underline{Q} = \underline{p}'S\underline{p} \tag{7}$$

Inspection of the matrix A shows that the product $S = A'A$ is the matrix of unadjusted sums of squares and sums of cross products of the lumber grade proportions, that is,

$$S = \begin{bmatrix} \Sigma a_{i1}^2 & \Sigma a_{i1}a_{i2} & \cdots & \Sigma a_{i1}a_{ij} & \cdots & \Sigma a_{i1}a_{ir} \\ \Sigma a_{i2}a_{i1} & \Sigma a_{i2}^2 & \cdots & \Sigma a_{i2}a_{ij} & \cdots & \Sigma a_{i2}a_{ir} \\ & \vdots & \ddots & & & \\ \Sigma a_{ij}a_{i1} & \Sigma a_{ij}a_{i2} & \cdots & \Sigma a_{ij}^2 & \cdots & \Sigma a_{ij}a_{ir} \\ & & & & \ddots & \\ \Sigma a_{ir}a_{i1} & \Sigma a_{ir}a_{i2} & \cdots & \Sigma a_{ir}a_{ij} & \cdots & \Sigma a_{ir}^2 \end{bmatrix} \quad (r \times r). \tag{8}$$

The sum of squares due to regression of Q on the x's is given by $\underline{b}'X'\underline{Q}$. Substituting for $\underline{b} = B\underline{p}$ and for $\underline{Q} = A\underline{p}$ in this expression gives:

$$\text{Regression S.S.} = \underline{p}'B'X'A\underline{p} \tag{9}$$

If we also substitute for $B = (X'X)^{-1}X'A$, then equation (8) becomes

$$\text{Regression S.S.} = \underline{p}'A'X(X'X)^{-1}X'A\underline{p} \tag{10}$$

which shows more clearly that it is a quadratic form.

In order to examine the regression S.S., it is necessary to expand the product X'A.

$$
X'A = \begin{bmatrix} x_{10} & x_{20} & \cdots & x_{i0} & \cdots & x_{n0} \\ x_{11} & x_{21} & \cdots & x_{i1} & \cdots & x_{n1} \\ \vdots & \vdots & & \vdots & & \vdots \\ x_{1k} & x_{2k} & \cdots & x_{ik} & \cdots & x_{nk} \\ \vdots & \vdots & & \vdots & & \vdots \\ x_{1u} & x_{2u} & \cdots & x_{iu} & \cdots & x_{nu} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1r} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{ir} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nr} \end{bmatrix}
$$

or

$$
X'A = \begin{bmatrix} \underline{x}_0' \\ \underline{x}_1' \\ \vdots \\ \underline{x}_k' \\ \vdots \\ \underline{x}_u' \end{bmatrix} \begin{bmatrix} \underline{a}_1 & \underline{a}_2 & \cdots & \underline{a}_j & \cdots & \underline{a}_r \end{bmatrix}
$$

$$
X'A = \begin{bmatrix} \underline{x}_0'\underline{a}_1 & \underline{x}_0'\underline{a}_2 & \cdots & \underline{x}_0'\underline{a}_j & \cdots & \underline{x}_0'\underline{a}_r \\ \underline{x}_1'\underline{a}_1 & \underline{x}_1'\underline{a}_2 & \cdots & \underline{x}_1'\underline{a}_j & \cdots & \underline{x}_1'\underline{a}_r \\ \vdots & \vdots & & \vdots & & \vdots \\ \underline{x}_u'\underline{a}_1 & \underline{x}_u'\underline{a}_2 & \cdots & \underline{x}_u'\underline{a}_j & \cdots & \underline{x}_u'\underline{a}_r \end{bmatrix}
$$

This may be rewritten as:

$$X'A = \begin{bmatrix} \sum a_{i1} & \sum a_{i2} \cdot \cdot \cdot \cdot \cdot & \sum a_{ij} \cdot \cdot \cdot \cdot \cdot & \sum a_{ir} \\ \sum a_{i1}x_{i1} & \sum a_{i2}x_{i1} \cdot \cdot \cdot & \sum a_{ij}x_{i1} \cdot \cdot \cdot & \sum a_{ir}x_{i1} \\ \vdots & \vdots & \vdots & \vdots \\ \sum a_{i1}x_{iu} & \sum a_{i2}x_{iu} \cdot \cdot \cdot & \sum a_{ij}x_{iu} \cdot \cdot \cdot & \sum a_{ir}x_{iu} \end{bmatrix} \qquad (11)$$

Thus, we see that each of the column vectors in the product matrix X'A contains the uncorrected sums of cross products for lumber grade proportion with the independent variables. If we denote each of the vectors as $\underline{g}_j$, then equation (10) may be written as:

$$X'A = \begin{bmatrix} \underline{g}_1 & \underline{g}_2 \cdot \cdot \cdot & \underline{g}_j \cdot \cdot \cdot & \underline{g}_r \end{bmatrix} \qquad (12)$$

Multiplying by B' gives:

$$B'X'A = \begin{bmatrix} \underline{b}_1'\underline{g}_1 & \underline{b}_1'\underline{g}_2 \cdot \cdot & \cdot \underline{b}_1'\underline{g}_j \cdot \cdot & \cdot \underline{b}_1'\underline{g}_r \\ \underline{b}_2'\underline{g}_1 & \underline{b}_2'\underline{g}_2 \cdot \cdot & \cdot \underline{b}_2'\underline{g}_j \cdot \cdot & \cdot \underline{b}_2'\underline{g}_r \\ \vdots & \vdots & \vdots & \vdots \\ \underline{b}_j'\underline{g}_1 & \underline{b}_j'\underline{g}_2 \cdot \cdot & \cdot \underline{b}_j'\underline{g}_j \cdot \cdot & \cdot \underline{b}_j'\underline{g}_r \\ \vdots & \vdots & \vdots & \vdots \\ \underline{b}_r'\underline{g}_1 & \underline{b}_r'\underline{g}_2 \cdot \cdot & \cdot \underline{b}_r'\underline{g}_j \cdot \cdot & \cdot \underline{b}_r'\underline{g}_r \end{bmatrix} (r \times r) \qquad (13)$$

The elements on the principal diagonal of B'X'A are the regression S.S. for each of the lumber grade proportions. The off diagonal elements are the regression sums of cross products for the pairs of lumber grade proportions.

The residual S.S. (error S.S.) is found by subtracting the regression S.S. from the total S.S., that is,

$$\text{Residual S.S.} = \underline{p}'S\underline{p} - \underline{p}'B'X'A\underline{p} \qquad (14)$$

Order of Calculations

(1) The matrix B is required for the calculation of the regression coefficients of Q and also for the calculation of the regression S.S.; therefore, it should be calculated first. Find B either by direct application of the definition $B = (X'X)^{-1}X'A$ or by solution of regressions for each of the lumber grade proportions.

(2) Form the product X'A, if not already done as a part of (1).

(3) Calculate B'X'A to be used in finding the regression S.S.

(4) Calculate S = A'A to be used in finding total S.S.

The three matrixes B, B'X'A, and S can then be used with any vector $\underline{p}$ to develop the required quantities by using equations (4), (7), (9), and (14) as follows:

(1) The regression coefficients are given by equation (4)

$$\underline{b} = B\underline{p}$$

(2) The regression S.S. is given by applying equation (9)

$$\text{Regression S.S.} = \underline{p}'B'X'A\underline{p}$$

(3) The total S.S. is given by equation (7)

$$\text{Total S.S.} = \underline{p}'S\underline{p}$$

(4) The residual S.S. is given by equation (14)

$$\text{Residual S.S.} = \underline{p}'S\underline{p} - \underline{p}'B'X'A\underline{p}$$

Illustrative Example

In order to illustrate the calculations, we will use the hypothetical data shown in tables 1 and 2. Table 1 gives the lumber tally, by grades, and the diameters inside bark for each of eight logs. Table 2 gives the lumber grade prices in Value/M and also in relative price. The relative prices were computed using the Value/M for #1C as the base.

First, the calculations will be performed in the usual order. For each log, calculate a relative weighted value by multiplying the lumber tally for each grade by its respective price relative and summing for all lumber grades, that is,

$$\begin{pmatrix}\text{FAS}\\ \text{Lbr. Tally}\end{pmatrix}\begin{pmatrix}\text{FAS}\\ \text{Price Rel.}\end{pmatrix} + \begin{pmatrix}\text{Sel.}\\ \text{Lbr. Tally}\end{pmatrix}\begin{pmatrix}\text{Sel.}\\ \text{Price Rel.}\end{pmatrix} +$$

$$\ldots + \begin{pmatrix}\#3\text{C}\\ \text{Lbr. Tally}\end{pmatrix}\begin{pmatrix}\#3\text{C}\\ \text{Price Rel.}\end{pmatrix}$$

For log 1 this is:

$$(10)\ (1.5625) + (12)\ (1.2500) + (30)\ (1.0000) + (20)\ (0.6250)$$

$$+ (8)\ (0.3750) = 76.1250$$

The relative weighted values for all eight logs are shown in column 2 of table 3.

The log quality index, Q, for each log is calculated by dividing the relative weighted value by the total lumber tally, that is,

$$Q = \frac{\text{Relative Weighted Value}}{\text{Total Lumber Tally}}$$

For log 1, Q is:

$$Q = \frac{76.1250}{80} = 0.9516$$

The quality indices for all eight logs are given in column 3, table 3.

To compute the regression of quality index, Q, on diameter inside bark, x, it was necessary to calculate the following sums of squares and cross products using the last column in table 1 and the last column in table 3:

$$\Sigma Q = 6.9148 \qquad \Sigma x = 102$$

$$\Sigma Q^2 = 6.07887046 \qquad \Sigma x^2 = 1,350$$

$$\Sigma Qx = 90.2403 \qquad (15)$$

The regression coefficients are given by the relationships:

$$b_1 = \frac{\Sigma Qx - (\Sigma Q)(\Sigma x)/N}{\Sigma Q^2 - (\Sigma Q)^2/N}$$

and

$$b_0 = \frac{\Sigma Q - b_1 \Sigma x}{N}$$

Substituting the numerical values into these equations gives

$$b_1 = \frac{90.2403 - (6.9148)(102)/8}{1350 - (102)^2/8} = 0.04195 \tag{16}$$

and

$$b_0 = \frac{6.9148 - (0.04195)(102)}{8} = 0.32944 \tag{17}$$

The estimated regression equation is

$$\hat{Q} = b_0 + b_1 x$$

or

$$\hat{Q} = 0.3294 + 0.04195x \tag{18}$$

The regression S.S. is

$$b_0\sum Q + b_1\sum Qx$$

$$= (0.32944)(6.9148) + (0.04195)(90.2403)$$

$$= 6.064 \tag{19}$$

and the residual S.S. is

$$\sum Q^2 - \text{regression S.S.}$$

$$= 6.079 - 6.064$$

$$= 0.015 \tag{20}$$

We will now calculate the quantities shown in (18), (19), and (20) by using equations (4), (7), (9), and (14). The reader who is unfamiliar with matrix calculations should refer to the appendix before reading the following calculations.

Table 4 shows the grade yield data of table 1 converted from board feet to proportion of total lumber. The first step in the computations is to obtain the column total for each column of table 4. These totals are 0.6692, 0.8772, 2.8999, 2.1602, and 1.3934. Next, compute the sum of cross products of each column of table 4 with the last column of table 1. For example, the sum of cross products of the first column of table 4 with the last column of table 1 is given by:

$$(0.1250)(13) + (0.0500)(14) + (0)(13) + \ldots + (0.1667)(15) = 9.4130$$

The sums of cross products for each of the columns of table 4 with the last column of table 1 are 9.4130, 12.4620, 36.9769, 27.1860, and 15.9608.

In terms of the matrix structure given previously, table 4 is defined as the matrix A, that is

$$A = \begin{bmatrix} 0.1250 & 0.1500 & \cdots & 0.1000 \\ .0500 & .1000 & & .1300 \\ \vdots & \vdots & & \vdots \\ .1667 & .1250 & \cdots & .0833 \end{bmatrix}_{(8 \times 5)} \qquad (21)$$

The X matrix is formed from a column vector of 1's and the last column of table 1.

$$X = \begin{bmatrix} 1 & 13 \\ 1 & 14 \\ 1 & 13 \\ 1 & 10 \\ 1 & 17 \\ 1 & 9 \\ 1 & 11 \\ 1 & 15 \end{bmatrix}_{(8 \times 2)}$$

Premultiplying A by X' yields

$$X'A = \begin{bmatrix} 0.6692 & 0.8772 & 2.8999 & 2.1602 & 1.3934 \\ 9.4130 & 12.4620 & 36.9769 & 27.1860 & 15.9608 \end{bmatrix}_{(2 \times 5)} \qquad (22)$$

Inspection of the matrix X'A shows that its first row is composed of the column sums of table 4, and its second row is composed of the sums of cross products of each of the columns of table 4 with the last column of table 1.

Next, calculate the sums, sums of squares, and sums of cross products of the independent variables. (Since there is only one independent variable, D.I.B., in this example, there are no cross products.) These quantities are the same as previously calculated: $\sum x = 120$, and $\sum x^2 = 1,350$.

Again referring to the matrix definitions, if we premultiply X by its transpose, X', the product is

$$X'X = \begin{bmatrix} N & \sum x \\ \sum x & \sum x^2 \end{bmatrix}$$

which for our example is

$$X'X = \begin{bmatrix} 8 & 120 \\ 120 & 1,350 \end{bmatrix} \tag{23}$$

The inverse of the matrix X'X is

$$(X'X)^{-1} = \begin{bmatrix} 3.40909090 & -0.25757575 \\ -.25757575 & .02020202 \end{bmatrix} \tag{24}$$

The matrix B is calculated by multiplying the matrixes given in (22) and (24).

$$B = (X'X)^{-1}X'A$$

$$B = \begin{bmatrix} -0.14319690 & -0.21945446 & 0.36166995 & 0.36186382 & 0.63911223 \\ .01779192 & .02581213 & .00006416 & -.00720302 & -.03646565 \end{bmatrix} \tag{25}$$

We must now premultiply X'A, equation (22), by the transpose of B to obtain the product

$$B'X'A = \begin{bmatrix} 0.071648 & 0.096111 & 0.242633 & 0.174357 & 0.084443 \\ .096111 & .129165 & .318057 & .227663 & .106194 \\ .242633 & .318057 & 1.051179 & .783024 & .504975 \\ .174357 & .227663 & .783024 & .585877 & .389255 \\ .084443 & .106194 & .504975 & .389255 & .308518 \end{bmatrix} \tag{26}$$

Premultiplying A, equation (21), by its transpose gives

$$A'A = S = \begin{bmatrix} 0.10067014 & 0.08561250 & 0.22283125 & 0.15948125 & 0.10060486 \\ .08561250 & .14295400 & .31870378 & .23345222 & .09647750 \\ .22283125 & .31870378 & 1.10140699 & .78227790 & .47468008 \\ .15948125 & .23345222 & .78227790 & .59984950 & .38513913 \\ .10060486 & .09647750 & .47468008 & .38513913 & .33659843 \end{bmatrix} \tag{27}$$

We now have, in equations (25), (26), and (27), all of the basic calculations. Value in any form has not yet been introduced into the calculations, and it is at this point that we are ready to apply the price relatives shown in table 2. The vector $\underline{p}$ of price relatives is

$$\underline{p} = \begin{bmatrix} 1.5625 \\ 1.2500 \\ 1.0000 \\ .6250 \\ .3750 \end{bmatrix} \quad (5 \times 1) \tag{28}$$

Substituting (25) and (27) into equation (4) and performing the matrix multiplication yields

$$\underline{b} = B\underline{p} = \begin{bmatrix} 0.3294 \\ .04195 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} \tag{29}$$

Substituting the values for b_0, and b_1, from equation (29) into the estimated regression equation gives

$$\hat{Q} = 0.3294 + 0.04195x \tag{30}$$

which is seen to be the same as equation (18).

Substituting (27) and (28) into equation (7) and performing the matrix multiplication yields the total sum of squares for Q.

$$\sum_{i=1}^{n} Q_i^2 = \underline{Q}'\underline{Q} = \underline{p}'S\underline{p} = 6.079 \tag{31}$$

Substituting (26) and (28) into equation (9) gives the regression sum of squares.

$$\text{Regression S.S.} = \underline{p}'B'X'A\underline{p} = 6.064 \tag{32}$$

The residual S.S. is found directly by subtraction.

$$\text{Residual S.S.} = 6.079 - 6.064 = 0.015 \tag{33}$$

The reader can verify that the results obtained in (31), (32), and (33) are the same as those obtained by the first computational method in (15), (19), and (20).

These calculations can quickly be changed from a Q.I. to a V/M basis by using the V/M of the base lumber grade (in this example No. 1 common is the base). Let v be the V/M of the base lumber grade; then equations (30), (31), (32), and (33) may be rewritten as follows:

The estimated regression equation is

$$\widehat{(V/M)} = v\hat{Q} = (80)\hat{Q} = 26.352 + 3.3560x \tag{34}$$

The total S.S. for V/M is

$$\sum_{i=1}^{n}(V/M)_i^2 = v^2\sum_{i=1}^{n}Q_i^2 = (80)^2(6.079) = 38905.6 \tag{35}$$

The regression S.S. is

$$\begin{aligned}\text{Reg. S.S. } (V/M) &= (80)^2(\text{Reg. S.S. } Q) = (80)^2(6.064)\\ &= 38809.6\end{aligned} \tag{36}$$

The residual S.S. is

$$(80)^2(0.015) = 96.0 \tag{37}$$

The quantities in (34), (35), (36), and (37) could have been found directly by using the vector of Values/M in place of the vector of price relatives (28).

APPENDIX

Matrix definition - A matrix is an array of numbers. All matrixes referred to in this paper are rectangular. A matrix with m rows and n columns is said to be of order (m x n). Each number in a matrix is called an element.

Matrix addition - Two matrixes which are to be added must be of the same order. Matrixes are added by adding corresponding elements. For example:

$$A + B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$$= \begin{bmatrix} (a_{11} + b_{11}) & (a_{12} + b_{12}) & (a_{13} + b_{13}) \\ (a_{21} + b_{21}) & (a_{22} + b_{22}) & (a_{23} + b_{23}) \end{bmatrix}$$

or a numerical example

$$\begin{bmatrix} 9 & 0 & 2 \\ 6 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 8 & 7 & 5 \\ 3 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 17 & 7 & 7 \\ 9 & 1 & 5 \end{bmatrix}$$

Matrix multiplication - The product AB of two matrixes A and B is defined only if the number of columns in A equals the number of rows in B. The product

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

is found as follows: The element c_{11} of the product matrix C is the sum of products of the elements in the first row of A by the elements in the first column of B, that is,

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$$

Similarly, the other elements of C are found by:

$$c_{12} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}$$

$$c_{21} = a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}$$

$$c_{22} = a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}$$

A numerical example is:

$$\begin{bmatrix} 5 & 3 & 6 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} 35 & 84 \\ 8 & 21 \end{bmatrix}$$

The multiplication of matrixes is not commutative, that is, in general, AB $\neq$ BA. In the numerical example above, if the order of multiplication is reversed, we have

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 5 & 3 & 6 \\ 4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 3 & 9 \\ 30 & 6 & 17 \\ 56 & 12 & 33 \end{bmatrix}$$

Since the order of multiplication must be defined, two terms are introduced: premultiply and postmultiply. In the product AB, we say that B is premultiplied by A or that A is postmultiplied by B.

Transpose of a matrix - The transpose of a matrix A is denoted by A' and is the matrix A with rows and columns interchanged. For example, if

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 5 & 4 & 2 \end{bmatrix}$$

then

$$A' = \begin{bmatrix} 2 & 5 \\ 1 & 4 \\ 3 & 2 \end{bmatrix}$$

Inverse of a matrix - In ordinary algebra, both sides of the equation

$$a = bc$$

may be divided by b and the equation rewritten as

$$\frac{a}{b} = c$$

or

$$b^{-1}a = c$$

The expression b^{-1} is the reciprocal or inverse of b and satisfies the relationship

$$b^{-1}b = 1$$

In matrix algebra, division cannot be performed directly; however, an operation analogous to division is accomplished by using a matrix inverse. The inverse of a square matrix A is denoted by A^{-1} and is a matrix that satisfies the relationship

$$A^{-1}A = I$$

where I is a matrix consisting of 1's on the principle diagonal and 0's elsewhere, that is, for a 3 x 3,

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix I is termed the identity matrix. Just as the product of any number with one is equal to the number itself, that is, $a(1) = (1)a = a$, the product of any matrix with the identity matrix is equal to the matrix, that is,

$$AI = IA = A$$

The matrix equation

$$A = BC$$

may be rewritten as follows: premultiply both sides of the equation by B^{-1}

$$B^{-1}A = B^{-1}BC$$

then using the definition $B^{-1}B = I$, the equation is rewritten as

$$B^{-1}A = IC$$

or

$$B^{-1}A = C$$

Since there is a large volume of literature on matrix inversion, the actual calculation of an inverse matrix is not given here. Good explanation of two methods of inverting matrixes are given in Anderson and Bancroft.[2]

[2] Anderson, R. L., and Bancroft, T. A. Statistical Theory in Research. McGraw-Hill. 1952.

Table 1.--Log lumber tallies, by grade and diameters

Log No.	Lumber Tally (board feet) FAS	Sel	#1C	#2C	#3C	Log total	D.I.B. (=x)
1	10	12	30	20	8	80	13
2	5	10	42	30	13	100	14
3	0	17	30	28	10	85	13
4	7	0	10	13	20	50	10
5	30	35	55	30	10	160	17
6	0	0	20	10	10	40	9
7	0	5	20	20	15	60	11
8	20	15	45	30	10	120	15

Table 2.--Absolute and relative lumber prices by grade

Lumber grade	FAS	Select	No. 1 common	No. 2 common	No. 3 common
Value per thousand	$125	$100	$80	$50	$30
Price relative (p_i)	1.5625	1.2500	1.0000	0.6250	0.3750

Table 3.--Relative weighted value and quality index for each log

Log No.	Relative weighted value	Quality index (=Q)
1	76.1250	0.9516
2	85.9375	.8594
3	72.5000	.8529
4	36.5625	.7313
5	168.1250	1.0508
6	30.0000	.7500
7	44.3750	.7396
8	117.5000	.9792

Table 4.--Lumber grade yields as a proportion of total lumber in the log

Log No.	Proportion of total lumber				
	FAS	Selective	No. 1 common	No. 2 common	No. 3 common
1	0.1250	0.1500	0.3750	0.2500	0.1000
2	.0500	.1000	.4200	.3000	.1300
3	0	.2000	.3529	.3294	.1176
4	.1400	0	.2000	.2600	.4000
5	.1875	.2188	.3437	.1875	.0625
6	0	0	.5000	.2500	.2500
7	0	.0834	.3333	.3333	.2500
8	.1667	.1250	.3750	.2500	.0833

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