AN ABSTRACT OF THE THESIS OF

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(Name) (Degree) (Major)

Date thesis is presented Dec 11, '65

Title STABILITY OF ELASTICALLY RESTRAINED FRAMED STRUCTURES BY MATRIX ANALYSIS

Abstract approved [redacted]
(Major Professor)

It has been undertaken here to use the matrix method of structural analysis for the determination of the stability of elastically restrained, simple portal frames. This involves the formulation of the stiffness matrix of the structure in terms of the axial force in the members. The stability criterion, when applied to the stiffness matrix, yields the critical load that may be applied to the structure.

The use of this method, when applied to an example frame, yielded results within 0.1 per cent of a classical approach for the limiting conditions of restraint. These conditions were: (1) no restraint, which produced the side-sway mode of failure and (2) sufficient restraint, which produced the non-sway mode of failure.

STABILITY OF ELASTICALLY RESTRAINED FRAMED STRUCTURES BY MATRIX ANALYSIS

by

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A THESIS

submitted to

OREGON STATE UNIVERSITY

in partial fulfillment of
the requirements for the
degree of

MASTER OF SCIENCE

June 1966

APPROVED:

[redacted]

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Date thesis is presented Jan. 11 '65

Typed by Arlene L. Kampe

ACKNOWLEDGMENTS

The author wishes to express appreciation for the assistance and financial support of the following persons and organizations: The National Science Foundation for financial support through National Science Foundation Graduate Engineering Stipend Number 965 and the staff of the Oregon State University Statistics Computing Laboratory for their suggestions and assistance on the computer program.

Special thanks to Dr. H. I. Laursen for his suggestions on the area of study, for his recommendations throughout the investigation, and for his time and encouragement.

Also special thanks to my wife, Barbara, for her moral and financial support throughout the year.

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STABILITY OF ELASTICALLY RESTRAINED FRAMED STRUCTURES BY MATRIX ANALYSIS

INTRODUCTION

In the design of structures it is necessary to know the load carrying capability of the structure as governed by its stability. Since Euler's development of the column formula, stability has been a point of concern in structures containing columns.

The critical loads as applied to simple portal frames have been found, in general, for both the symmetrical (non-sway) mode of failure and the antisymmetrical (sidesway) mode of failure (2, p. 251-255, 23, p. 66, 149). It is undertaken here to use a method applicable to a large variety of framed structures (12) to determine the critical loads of elastically restrained, simple portal frames. By varying the stiffness of the restraint, the critical loads will vary from those obtained for the sidesway-permitted condition to those obtained for the non-sway condition.

The following limitations or assumptions are made in regard to this investigation:

1. The stresses in the members remain in the elastic range.
2. Only plane frameworks will be considered.
3. There are no out-of-plane deformations.

4. The moment of inertia is constant along the length of each member.
5. Each column is initially perfectly straight.
6. Each column is centrally loaded.
7. Placement of the loads will be such that there will be no primary bending moments.

METHOD OF ANALYSIS

The method of analysis to be used is the displacement method of matrix analysis. The development of this method can be credited mainly to those engaged in aircraft analysis and design (1, 9, 24). To analyze the highly redundant structures used in the aircraft industry by classical methods involves the simultaneous solution of a large number of linear equations. Matrix notation is ideal in such a situation because of its concise notation and its ability to be readily applied to a digital computer.

Notation Used

Figure 1(a) shows the external coordinate system to be used. External displacements are denoted by D and external loads are denoted by Q as shown in Figure 1(b). Quantities as shown are considered to be positive. The circled numbers refer to nodal point designation. External loads will be applied only at the nodal points.

Each member is designated by the number near the center of each member span.

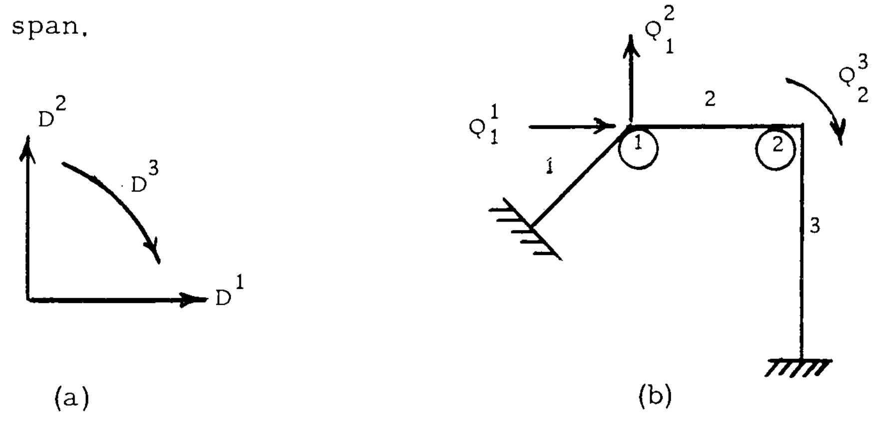

Figure 1. External Load and Displacement Notation

Superscripts used on external loads and displacements denote the particular component as shown in Figure 1(a). Subscripts used on external loads and displacements refer to the nodal point at which they occur. For example in Figure 1(b), Q_2^3 refers to a clockwise moment applied at nodal point number two.

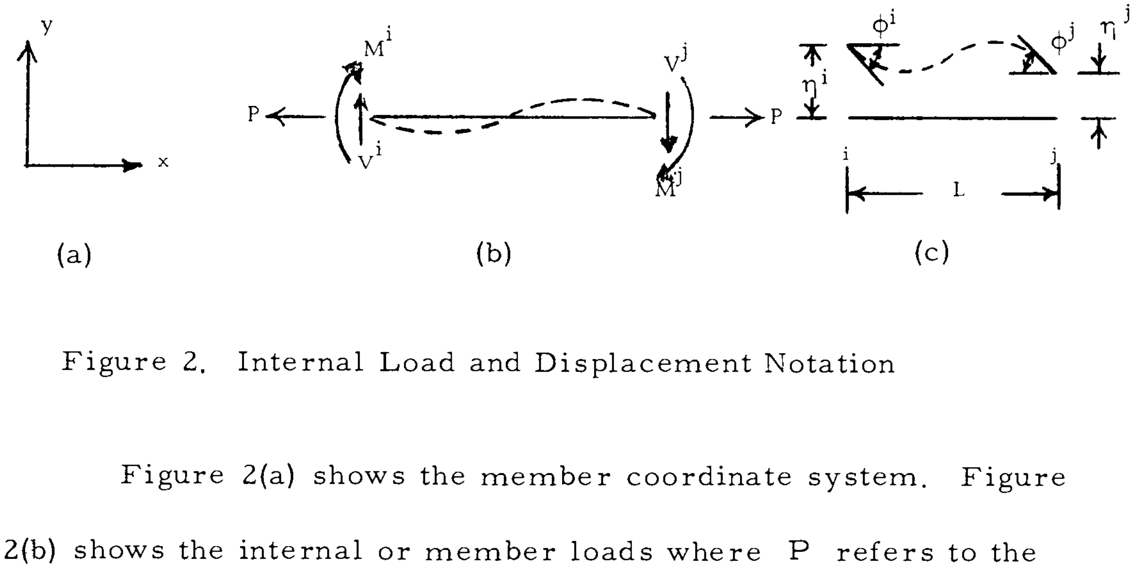

Figure 2. Internal Load and Displacement Notation

Figure 2(a) shows the member coordinate system. Figure 2(b) shows the internal or member loads where P refers to the axial load, M refers to the bending moment, and V refers to the shear. Figure 2(c) shows the internal or member displacements where ϕ refers to rotation and η refers to translation. The quantities as shown in Figures 2(b) and 2(c) are considered to be positive. Not shown in Figure 2(c) is the quantity e which refers to axial deformation. The sign of e is positive when there is extension in the member. The superscripts i and j refer to the member end at which the load or displacement occurs. The letter i refers to the left end and j refers to the right end. Any subscripts used will denote the member number.

Internal loads and displacements in matrix notation are referred to as $\utilde{q}$ and $\utilde{d}$ respectively. Symbols that are underlined with a wavy line denote that the symbol is in matrix form. Equation (1), for example, denotes that the matrix $\utilde{q}$ is composed of the array of elements M^i, V^i, M^j, V^j, and P.

$$\utilde{q} = \begin{bmatrix} M^i \\ V^i \\ M^j \\ V^j \\ P \end{bmatrix} \tag{1}$$

<u>Displacement</u> <u>Transformation</u> <u>Matrix</u>

In using the displacement method of matrix analysis, it is necessary to define a displacement transformation matrix $\utilde{A}$ such that:

$$\utilde{d} = \utilde{A}\,\utilde{D} \tag{2}$$

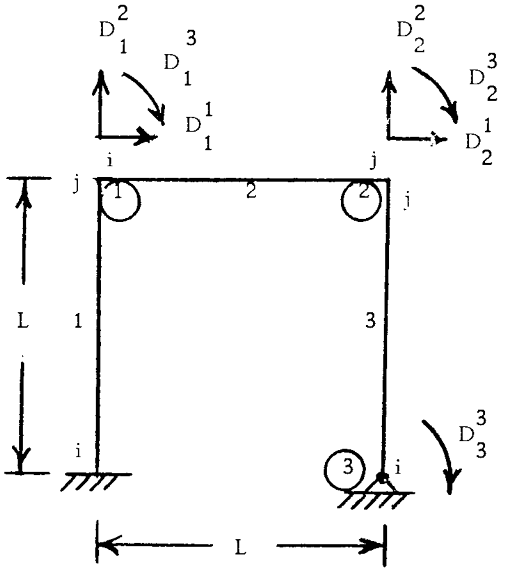

Figure 3. Typical Framed Structure

For example, in Figure 3, neglecting axial deformations and noting that $D_1^2 = D_2^2 = 0.0$ and that $D_1^1 = D_2^1$ we may rewrite equation (2) and get:

$$
\begin{bmatrix} \phi_1^i \\ \eta_1^i \\ \phi_1^j \\ \eta_1^j \\ \phi_2^i \\ \eta_2^i \\ \phi_2^j \\ \eta_2^j \\ \phi_3^i \\ \eta_3^i \\ \phi_3^j \\ \eta_3^j \end{bmatrix}
=
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} D_1^1 = D_2^1 \\ D_1^3 \\ D_2^3 \\ D_3^3 \end{bmatrix}
\qquad (3)
$$

Equation (3) may be rewritten in a more usable form by combining terms and by dividing by L, where needed, to make the matrix dimensionally homogeneous. If this is done we obtain:

$$\begin{bmatrix} \phi_1^i \\ \phi_1^j \\ \frac{\eta_1^i - \eta_1^j}{L} \\ \phi_2^i \\ \phi_2^j \\ \frac{\eta_2^i - \eta_2^j}{L} \\ \phi_3^i \\ \phi_3^j \\ \frac{\eta_3^i - \eta_3^j}{L} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} D_1^1/L = D_2^1/L \\ D_1^3 \\ D_2^3 \\ D_3^3 \end{bmatrix} \qquad (4)$$

If axial deformations only were being considered the displacement transformation matrix would be as shown in equation (5).

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} D_1^1 \\ D_1^2 \\ D_2^1 \\ D_2^2 \end{bmatrix} \qquad (5)$$

Such a transformation is used most often in the truss type of structures.

Member Stiffness Matrix

The stiffness of a member is defined as the amount of force required to deflect the member a unit distance. It is usually expressed by a ratio as in equation (6):

$$k = P/e \tag{6}$$

where k is the stiffness of the member, P is the force applied, and e is the amount of deflection. Equation (6) can be rearranged and expressed in our matrix notation as:

$$\underset{\sim}{q} = \underset{\sim}{k_i}\,\underset{\sim}{d} \tag{7}$$

where $\underset{\sim}{k_i}$ is the individual member stiffness matrix. Thus the expression for the axial force on an elastic restraint or spring becomes:

$$P = k_{sp}\, e \tag{8}$$

where k_{sp} denotes the stiffness of the spring.

Considering axial force P, the differential equation for bending is:

$$EI\frac{d^4y}{dx^4} + P\frac{d^2y}{dx^2} = 0 \tag{9}$$

Using the general solution of this differential equation and the force and deflection boundary conditions, it can be shown that (12)

$$\begin{bmatrix} M^i \\ M^j \\ VL \end{bmatrix} = \frac{EI}{L} \begin{bmatrix} k_1 & k_2 & k_3 \\ k_2 & k_1 & k_3 \\ k_3 & k_3 & k_4 \end{bmatrix} \begin{bmatrix} \phi^i \\ \phi^j \\ \frac{\eta^i - \eta^j}{L} \end{bmatrix} \tag{10}$$

where:

$$k_1 = \frac{\lambda (s - \lambda c)}{2 - 2c - \lambda s} \tag{11a}$$

$$k_2 = \frac{\lambda (\lambda - s)}{2 - 2c - \lambda s} \tag{11b}$$

$$k_3 = \frac{\lambda^2 (c - 1)}{2 - 2c - \lambda s} \tag{11c}$$

$$k_4 = \frac{\lambda^3 s}{2 - 2c - \lambda s} \tag{11d}$$

$$s = \sin \lambda \tag{11e}$$

$$c = \cos \lambda \tag{11f}$$

$$\lambda^2 = \frac{PL^2}{EI} \tag{11g}$$

If P is a tensile force the expressions in equations (11) are replaced by:

$$\bar{k}_1 = \frac{\lambda (\lambda \bar{c} - \bar{s})}{2 - 2\bar{c} + \lambda \bar{s}} \tag{12a}$$

$$\bar{k}_2 = \frac{\lambda (\bar{s} - \lambda)}{2 - 2\bar{c} + \lambda \bar{s}} \tag{12b}$$

$$\bar{k}_3 = \frac{\lambda^2 (\bar{c} - 1)}{2 - 2\bar{c} + \lambda \bar{s}} \tag{12c}$$

$$\bar{k}_4 = \frac{\lambda^3 \bar{s}}{2 - 2\bar{c} + \lambda \bar{s}} \tag{12d}$$

$$\bar{s} = \sinh \lambda \tag{12e}$$

$$\bar{c} = \cosh \lambda \tag{12f}$$

When there is no axial load in the member or when axial force effects are neglected, P equals zero and equation (10) becomes:

$$\begin{bmatrix} M^i \\ M^j \\ VL \end{bmatrix} = \frac{EI}{L} \begin{bmatrix} 4 & 2 & -6 \\ 2 & 4 & -6 \\ -6 & -6 & 12 \end{bmatrix} \begin{bmatrix} \phi^i \\ \phi^j \\ \dfrac{\eta^i - \eta^j}{L} \end{bmatrix} \tag{13}$$

From the individual member stiffness matrixes $\mathbf{k}_i$ a stiffness matrix $\mathbf{k}$ for the structure is formed which appears as follows:

$$\mathbf{k} = \begin{bmatrix} \mathbf{k}_1 & 0 & 0 & \cdots & 0 \\ 0 & \mathbf{k}_2 & 0 & \cdots & 0 \\ 0 & 0 & \mathbf{k}_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \mathbf{k}_n \end{bmatrix} \tag{14}$$

It will be seen in the next section that this matrix is used for determining the stiffness matrix $\mathbf{K}$ for the entire structure.

Structure Stiffness Matrix

Using the principle of virtual work it can be shown that (11):

$$\mathbf{Q} = \mathbf{A}' \mathbf{k} \mathbf{A} \mathbf{D} \tag{15}$$

where $\mathbf{A}'$ denotes $\mathbf{A}$ transpose. Defining the structure stiffness matrix as

$$\underset{\sim}{K} = \underset{\sim}{A'}\,\underset{\sim}{k}\,\underset{\sim}{A} \tag{16}$$

it is seen that the external displacements are related to the external loads by the expression

$$\underset{\sim}{Q} = \underset{\sim}{K}\,\underset{\sim}{D} \tag{17}$$

Stability Criterion

When the load applied to a structure reaches the critical load (that load which will make the structure unstable) its deflection grows infinitely large with virtually no increase in load. From equation (17) the only way $\underset{\sim}{D}$ can change considerably with no change in $\underset{\sim}{Q}$ is for the determinant of $\underset{\sim}{K}$ to go to zero (19, 21). If the determinant of $\underset{\sim}{K}$ equals zero, the structure is said to be in neutral equilibrium and the critical load P_{cr} may be calculated.

APPLICATION TO ELASTICALLY RESTRAINED STRUCTURE

The application of the matrix method to a particular structure requires that the previously mentioned $\underset{\sim}{A}$ and $\underset{\sim}{k}$ matrices be developed for the structure. From these the $\underset{\sim}{K}$ matrix for the structure can be determined.

Development of Required Matrices

Pinned-Base Frame. The notation, geometry, and loading of a pinned-base frame is shown in Figure 4. The lengths and moments of inertia of the respective members are denoted as L_1, L_2, I_1, and I_2.

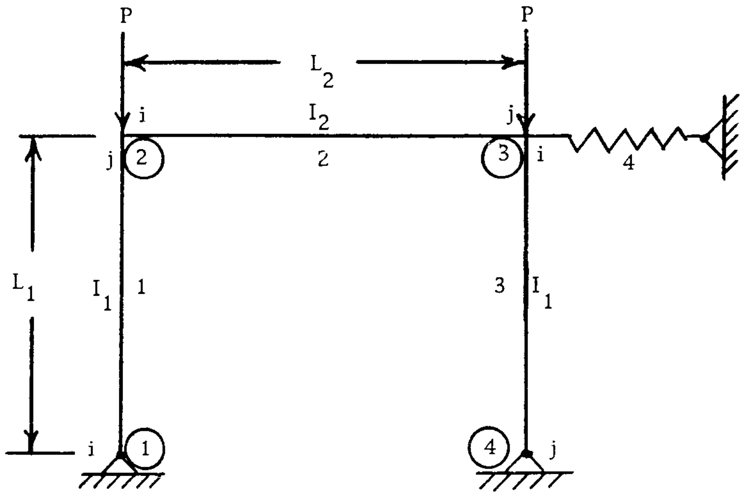

Figure 4. Notation, Geometry, and Loading of Pinned-Base Frame

The elastic restraint, as applied in Figure 4, can represent any of several actual conditions of restraint against sidesway such as shear walls or cross bracing. The spring constant used in the analysis would be equal to the stiffness of the actual restraint.

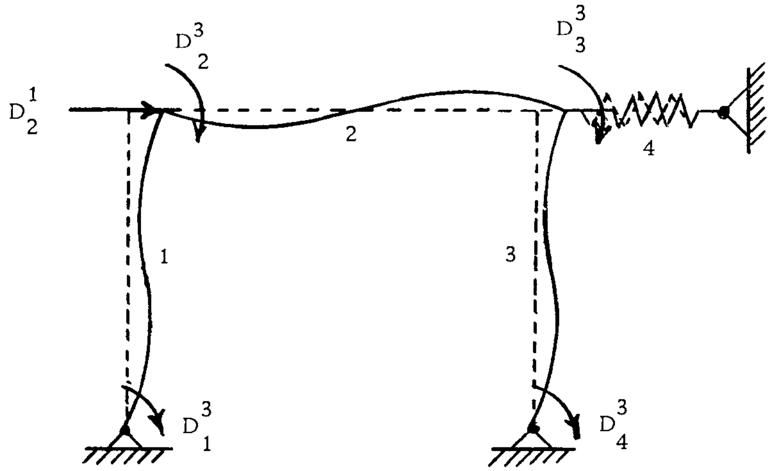

Figure 5. Deflected Structure

Figure 5 shows the deflection that the pinned-base frame is capable of undergoing in the plane of the frame (neglecting axial deformations of all members except member four). The non-deflected shape is shown by dashed lines while the deflected shape is shown in solid lines.

The displacement transformation matrix $\underset{\sim}{A}$ from equation (2) is developed from Figure 5 and is found to be

$$\begin{bmatrix} \phi_1^i \\ \phi_1^j \\ \dfrac{\eta_1^i - \eta_1^j}{L_1} \\ \phi_2^i \\ \phi_2^j \\ \dfrac{\eta_2^i - \eta_2^j}{L_2} \\ \phi_3^i \\ \phi_3^j \\ \dfrac{\eta_3^i - \eta_3^j}{L_1} \\ e_4 / L_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} D_1^3 \\ D_2^1 / L_1 \\ D_2^3 \\ D_3^3 \\ D_4^3 \end{bmatrix} \tag{18}$$

The individual member stiffness matrices for the members in Figure 4 are shown in equations (19).

$$\begin{bmatrix} M_1^i \\ M_1^j \\ V_1 L_1 \end{bmatrix} = \frac{EI_1}{L_1} \begin{bmatrix} k_1 & k_2 & k_3 \\ k_2 & k_1 & k_3 \\ k_3 & k_3 & k_4 \end{bmatrix} \begin{bmatrix} \phi_1^i \\ \phi_1^j \\ \dfrac{\eta_1^i - \eta_1^j}{L_1} \end{bmatrix} \tag{19a}$$

$$\begin{bmatrix} M_2^i \\ M_2^j \\ V_2 L_2 \end{bmatrix} = \frac{EI_2}{L_2} \begin{bmatrix} k_1 & k_2 & k_3 \\ k_2 & k_1 & k_3 \\ k_3 & k_3 & k_4 \end{bmatrix} \begin{bmatrix} \phi_2^i \\ \phi_2^j \\ \dfrac{\eta_2^i - \eta_2^j}{L_2} \end{bmatrix} \tag{19b}$$

$$\begin{bmatrix} M_3^i \\ M_3^j \\ V_3 L_1 \end{bmatrix} = \frac{EI_1}{L_1} \begin{bmatrix} k_1 & k_2 & k_3 \\ k_2 & k_1 & k_3 \\ k_3 & k_3 & k_4 \end{bmatrix} \begin{bmatrix} \phi_3^i \\ \phi_3^j \\ \dfrac{\eta_3^i - \eta_3^j}{L_1} \end{bmatrix} \tag{19c}$$

$$\begin{bmatrix} P_4 L_1 \end{bmatrix} = \begin{bmatrix} L_1^2 \; k_{sp} \end{bmatrix} \begin{bmatrix} e_4 / L_1 \end{bmatrix} \tag{19d}$$

These equations are combined to give the stiffness matrix $\underset{\sim}{k}$ of the structure as shown in equation (20).

The stiffness matrix $\underset{\sim}{K}$ of the entire structure is obtained by substituting the $\underset{\sim}{A}$ matrix from equation (18) and the $\underset{\sim}{k}$ matrix from equation (20) into equation (16). Performing the matrix multiplication one obtains equation (21). The subscript pb indicates this is the stiffness matrix of the pinned-base frame.

$$
\begin{bmatrix} M_1^i \\ M_1^j \\ V_1 L_1 \\ M_2^i \\ M_2^j \\ V_2 L_2 \\ M_3^i \\ M_3^j \\ V_3 L_1 \\ P_4 L_1 \end{bmatrix}
= \frac{EI_1}{L_1}
\begin{bmatrix}
k_1 & k_2 & k_3 & & & & & & & \\
k_2 & k_1 & k_3 & & & & & & & \\
k_3 & k_3 & k_4 & & & & & & & \\
 & & & \multicolumn{3}{c}{\multirow{3}{*}{$\frac{L_1 I_2}{L_2 I_1}\begin{pmatrix} 4 & 2 & -6 \\ 2 & 4 & -6 \\ 6 & -6 & 12 \end{pmatrix}$}} & & & & \\
 & & & & & & & & & \\
 & & & & & & & & & \\
 & & & & & & k_1 & k_2 & k_3 & \\
 & & & & & & k_2 & k_1 & k_3 & \\
 & & & & & & k_3 & k_3 & k_4 & \\
 & & & & & & & & & \frac{L_1^3 k_{sp}}{EI_1}
\end{bmatrix}
\begin{bmatrix} \phi_1^i \\ \phi_1^j \\ \frac{\eta_1^i - \eta_1^j}{L_1} \\ \phi_2^i \\ \phi_2^j \\ \frac{\eta_2^i - \eta_2^j}{L_2} \\ \phi_3^i \\ \phi_3^j \\ \frac{\eta_3^i - \eta_3^j}{L_1} \\ e_4/L_1 \end{bmatrix}
\quad (20)
$$

$$
K_{pb} = \frac{EI_1}{L_1}
\begin{bmatrix}
k_1 & k_3 & k_2 & 0 & 0 \\
k_3 & 2k_4 + \frac{L_1^3 k_{sp}}{EI_1} & k_3 & k_3 & k_3 \\
k_2 & k_3 & k_1 + 4\frac{I_2 L_1}{I_1 L_2} & 2\frac{I_2 L_1}{I_1 L_2} & 0 \\
0 & k_3 & 2\frac{I_2 L_1}{I_1 L_2} & k_1 + 4\frac{I_2 L_1}{I_1 L_2} & k_2 \\
0 & k_3 & 0 & k_2 & k_1
\end{bmatrix}
\quad (21)
$$

<u>Fixed-Base Frame</u>. The fixed-base frame considered has the same form as Figures 4 and 5 except that nodal points one and four are fixed and cannot, therefore, undergo rotation. The displacement

transformation matrix in this case is the same as the three middle columns of the $\underset{\sim}{A}$ matrix for the pinned-base frame.

$$\underset{\sim}{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \tag{22}$$

The stiffness matrix $\underset{\sim}{k}$ of the structure is the same as for the pinned-base frame. This matrix is given by equation (20). Performing the necessary matrix operations the stiffness matrix $\underset{\sim}{K}$ for the entire structure becomes

$$\underset{\sim}{K}_{fb} = \frac{EI_1}{L_1} \begin{bmatrix} 2k_4 + \frac{L_1^3 k_{sp}}{EI_1} & k_3 & k_3 \\ k_3 & k_1 + 4\frac{I_2 L_1}{I_1 L_2} & 2\frac{I_2 L_1}{I_1 L_2} \\ k_3 & 2\frac{I_2 L_1}{I_1 L_2} & k_1 + 4\frac{I_2 L_1}{I_1 L_2} \end{bmatrix} \tag{23}$$

The subscript fb denotes this is the stiffness matrix of the fixed-base frame.

<u>Application of Stability Criterion</u>

As previously mentioned, the criterion for neutral equilibrium is that the determinant of $\underset{\sim}{K}$ is equal to zero. In observing the composition of the $\underset{\sim}{K}$ matrices from equations (21), (23), and (11), it is seen that the condition for the determinant is best obtained by a trial and error solution.

The procedure used is as follows: first, a value for λ is assumed; the sines and cosines of λ are then obtained; these are then substituted into the equations for k_1, k_2, k_3, and k_4, which in turn are substituted into the expression for the determinant of $\underset{\sim}{K}$. If the determinant of $\underset{\sim}{K}$ is zero, the appropriate value of P_{cr} is calculated from the assumed value of λ. If the determinant of $\underset{\sim}{K}$ is positive or negative, a larger or smaller value, respectively, of λ is assumed and the determinant is recalculated.

This procedure may take a large number of trials before a value of λ can be found which will make the determinant of $\underset{\sim}{K}$ equal to zero. A high-speed digital computer can be used very effectively to perform the required calculations.

COMPUTER PROGRAM DESCRIPTION

The program used was written in the FORTRAN language for use on an IBM 1410 Data Processing System. The examples were run at the Oregon State University Statistics Computing Laboratory.

Computer Logic

The program was written in three parts--the main program with two subprograms. Complete listings will be found in the Appendix. The first subprogram (SUBROUTINE MSTFM) simply assembles the member stiffness matrix of the structure from the values given it by the main program. This matrix is then returned to the main program for future calculations.

The second subprogram (FUNCTION DETERM) calculates the value of the determinant of any square matrix given it by the main program.

A brief description of the main program logic follows:

1. All necessary input variables, such as frame geometry, accuracy desired, and the displacement transformation matrix are input to the computer.
2. An initial value of λ is assumed.
3. The SUBROUTINE MSTFM forms the member stiffness matrix from the given geometry and the assumed value of λ.

4. The main program performs the matrix multiplication to get $\underset{\sim}{K} = \underset{\sim}{A}' \underset{\sim}{k} \underset{\sim}{A}$.

5. If the determinant of $\underset{\sim}{K}$ is close enough to zero as specified by the accuracy desired, the value of P_{cr} is calculated and printed out. The computer then returns to the beginning of the program to read a new set of data.

6. If the determinant of $\underset{\sim}{K}$ is positive, λ is increased by the value of DLAMDA and the computer returns to step three.

7. If the determinant of $\underset{\sim}{K}$ is negative, the new value of λ is obtained by using a straight line interpolation between the λ corresponding to the negative value of the determinant of $\underset{\sim}{K}$ and the λ corresponding to the last positive value for the determinant of $\underset{\sim}{K}$. The computer then returns to step three.

Program Generalities

The subprogram FUNCTION DETERM is completely general and will calculate the determinant of any matrix up to the limits of the computer.

The main program would be applicable to many framed structures with only a few minor changes. As the main program is listed in the Appendix it is limited to a symmetric, three-member, framed structure with a single elastic restraint.

The subprogram SUBROUTINE MSTFM was written specifically for the structure configuration used in this work.

RESULTS

Presentation

Input. The examples calculated in this work are based on both a pinned-base frame and a fixed-base frame. The properties of the members selected for the calculations, using the notation of Figure 4, are L_1 = 20 feet, L_2 = 40 feet, I_1 = 1000 in^4, and I_2 = 500 in^4. In each case Young's modulus is equal to 30,000 kips per square inch.

The degree of accuracy to which the determinant of $\underset{\sim}{K}$ was to equal zero was dependent upon the magnitude of the determinant of $\underset{\sim}{K}$ for λ equal to zero and also upon the value input to the computer as data. The accuracy used was equal to the product of the input accuracy, equal to unity in all examples calculated, and 0.01 per cent of the determinant of $\underset{\sim}{K}$ for λ equal to zero.

The value of the increment of λ, DLAMDA, was varied with the stiffness of the restraint to obtain fast convergence of the determinant of $\underset{\sim}{K}$ to zero.

Output. An example of the output using the pinned-base frame is shown in Table 1. This example represents the sidesway condition as can be seen by noting that the spring constant or stiffness of the restraint is equal to zero.

Table 1. Sample Output Data

L_1	L_2	I_1	I_2	SPRING CONSTANT
20.000	40.000	500.000	1000.000	0.000

LAMDA	DETERM(K)
.0000	2880.00
1.0000	1141.46
2.0000	-1913.78
1.3736	-80.48
1.3489	1.85
1.3495	.00

THE CRITICAL LOAD IS 474.294 KIPS

Figure 6 is a graph of λ vs. the spring constant of the restraint for both the pinned-base and the fixed-base frames. Also included are the values from Bleich (2) for the sidesway and non-sway conditions for both frames.

Reliability

The reliability of the results obtained is very good. The maximum per cent error of the values obtained by the computer is 0.1 per cent of those values found in Bleich.

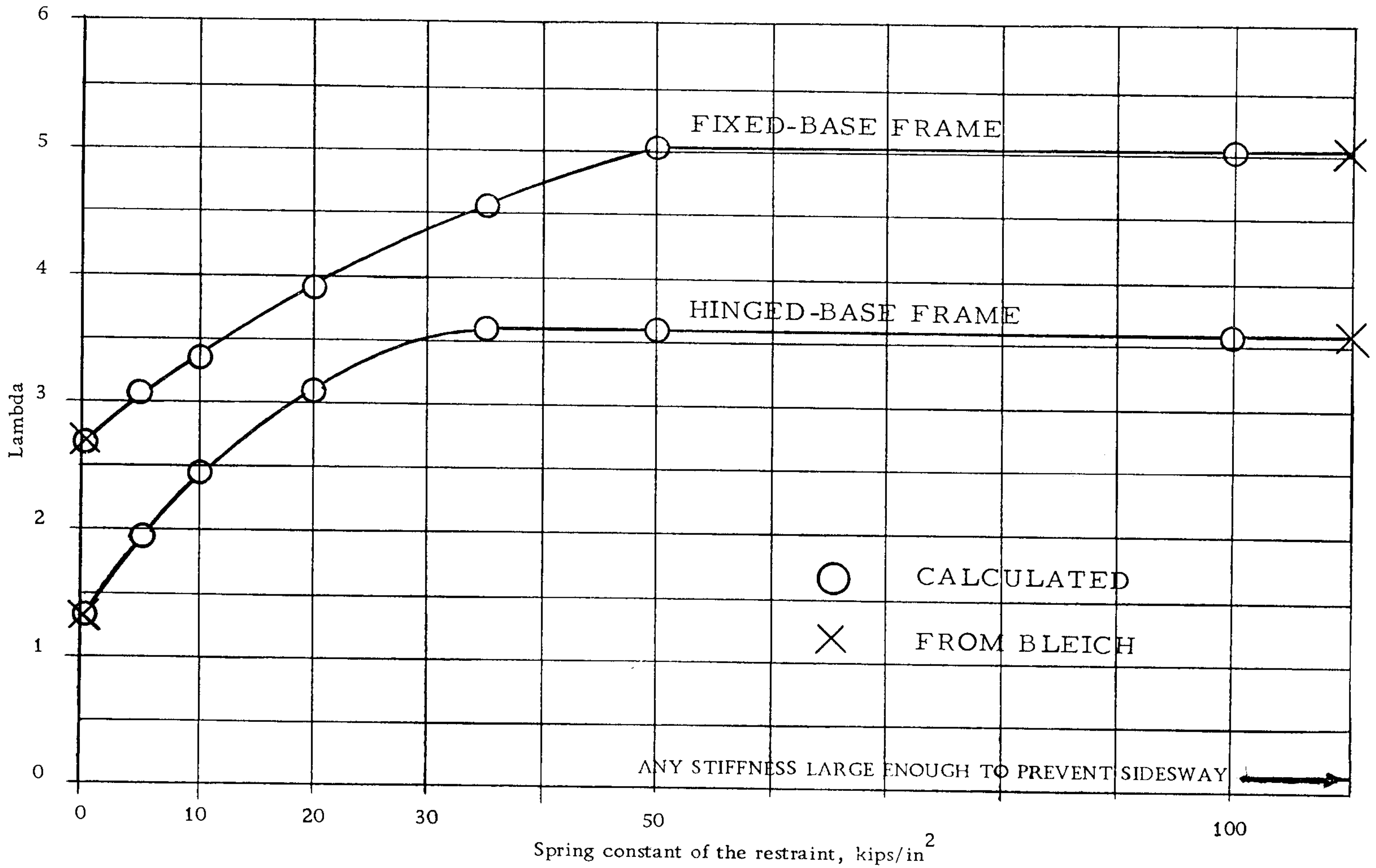

Figure 6. Lambda vs. Spring Constant of the Restraint

Although there are a large number of calculations involved, round-off error does not seem to be a problem within the accuracy desired. This assumption was substantiated by the calculation of one set of data using double precision arithmetic. This calculation showed no change in the sixth significant figure.

CONCLUSIONS

The method presented in this work is a method that shows good accuracy for simple portal frames. The method is versatile and can be applied to many types of framed structures. Framed structures with elastic restraints are especially suited for analysis by the use of this method.

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APPENDIX

NOTATION USED

$\underset{\sim}{A}$	Displacement transformation matrix
$\underset{\sim}{A}'$	$\underset{\sim}{A}$ transpose
c	Cosine
$\underset{\sim}{d}$	Internal displacement matrix
$\underset{\sim}{D}$	External displacement matrix
e	Axial deformation
E	Modulus of elasticity or Young's modulus
i, j	Superscripts referring to member end
I	Moment of inertia
k	Stiffness of a member

$$k_1 = \frac{\lambda(s - \lambda c)}{2 - 2c - \lambda s} \qquad \bar{k}_1 = \frac{\lambda(\lambda\bar{c} - \bar{s})}{2 - 2\bar{c} + \lambda\bar{s}}$$

$$k_2 = \frac{\lambda(\lambda - s)}{2 - 2c - \lambda s} \qquad \bar{k}_2 = \frac{\lambda(\bar{s} - \lambda)}{2 - 2\bar{c} + \lambda\bar{s}}$$

$$k_3 = \frac{\lambda^2(c - 1)}{2 - 2c - \lambda s} \qquad \bar{k}_3 = \frac{\lambda^2(\bar{c} - 1)}{2 - 2\bar{c} + \lambda\bar{s}}$$

$$k_4 = \frac{\lambda^3 s}{2 - 2c - \lambda s} \qquad \bar{k}_4 = \frac{\lambda^3\bar{s}}{2 - 2\bar{c} + \lambda\bar{s}}$$

k_{sp}	Stiffness of an elastic restraint
$\underset{\sim}{k_i}$	Individual member stiffness matrix
$\underset{\sim}{k}$	Member stiffness matrix of entire structure
$\underset{\sim}{K}$	Structure stiffness matrix
L	Length of member
M	Moment

P	Axial force
P_{cr}	Critical axial force
$\underset{\sim}{q}$	Internal load matrix
$\underset{\sim}{Q}$	External load matrix
s	Sine
V	Shear
x, y	Individual member coordinates
η	Translation of end of member

$$\lambda^2 = \frac{PL^2}{EI}$$

ϕ	Rotation of end of member

LIST OF VARIABLES USED IN PROGRAMS

Main Program

A	Displacement transformation matrix
ACC	Accuracy to which determinant of BGK must approach zero. Value input is multiplied by 0.01 per cent of the determinant of BGK for $\lambda = 0$ to get the actual accuracy.
BGK	Structure stiffness matrix
CRDEND	Routine to test for last data card
DETERM	Subprogram to evaluate the determinant of BGK
DLAMDA	Increment of λ
E	Young's modulus in ksi
FI1, FI2	Moment of inertia--corresponds to I_1, I_2 in in^4
FL1, FL2	Length--corresponds to L_1, L_2 in feet
M	Size of SMK matrix
MSTFM	Subroutine to form SMK
N	Degrees of freedom
NTEST, NCOUNT	Temporary values to insure program convergence. If more than 20 increments of λ are required, the program will switch to the next set of data.
PR	Value of the determinant of BGK
SMK	Member stiffness matrix of the entire structure
SPRK	Spring constant or stiffness of restraint in kips/ in
TEMP	Temporary storage
TPR, PPR, TVL, PPL	Temporary values to speed program convergence
VLAMDA	λ

Subroutine MSTFM*

DIV = $2 - 2\cos\lambda - \lambda\sin\lambda$

FACTOR = 1.0 if considering other than member number two

$= \dfrac{L_1 I_2}{L_2 I_1}$ if considering member number two

FK1, FK2, FK3, FK4 = k_1, k_2, k_3, k_4 respectively

L Counting device to determine the member number

Function DETERM*

N Size of BGK matrix

X Dummy variable for BGK

*Variables listed only if not listed in main program or if defined differently from main program.

```
C       MAIN PROGRAM FOR FRAME STABILITY ANALYSIS
        DIMENSION A(10,10),SMK(10,10),BGK(10,10),TEMP(10,10)
   10   FORMAT (2I3,E10.2,5F8.3)
   11   FORMAT (8F10.5)
   12   FORMAT (//6H    L1,7X,2HL2,7X,2HI1,7X,4HI2      ,15HSPRING CONSTANT//,
       15F9.3,//17HVLAMDA  DETERM(K)//)
   13   FORMAT(F8.4,F10.2)
   14   FORMAT (22H  THE CRITICAL LOAD IS  ,F10.3,5H  KIPS)
   15   FORMAT (E14.5,F8.3)
        READ (1,10)M,N,E,FL1,FL2,FI1,FI2,ACC
        DO 1 I=1,M
    1   READ(1,11)(A(I,J),J=1,N)
    2   READ (1,15)SPRK,DLAMDA
        WRITE(3,12)FL1,FL2,FI1,FI2,SPRK
        TPR = 0.0
        PPL = -1.0
        TVL = -1.0
        NTEST=0
        VLAMDA = 0.0
        NCOUNT=0
    3   IF(NCOUNT.GT.20)GO TO 2
        NCOUNT=NCOUNT+1
        CALL MSTFM(SMK,M,SPRK,E,FL1,FL2,FI1,FI2,VLAMDA)
        DO 4 I=1,N
        DO 4 J=1,M
        TEMP(I,J)=0.0
        DO 4 L=1,M
    4   TEMP (I,J)=TEMP(I,J)+A(L,I)*SMK(L,J)
        DO 5 I=1,N
        DO 5 J=1,N
        BGK(I,J)=0.0
        DO5 L=1,M
    5   BGK(I,J)=BGK(I,J)+TEMP(I,L)*A(L,J)
        PR=DETERM(N,BGK)
        WRITE(3,13)VLAMDA,PR
        IF(VLAMDA.EQ.0.)ACC=0.0001*PR*ACC
        IF(ABS  (PR).LE.ACC)GO TO 9
    6   IF(PR.LT.0.)GO TO 8
    7   IF(TPR.NE.0.)GO TO 18
        IF(VLAMDA.EQ.0.) GO TO 18
        IF(NTEST.EQ.2)GO TO 20
        IF(PR.GT.PPR)GO TO 17
   18   PPR=PR
        IF(ABS(VLAMDA-PPL).LE.0.00005)GO TO9
        PPL=VLAMDA
        VLAMDA=VLAMDA+DLAMDA
        IF(TPR.EQ.0.)GO TO 3
        GO TO 16
   17   IF(NTEST.EQ.1)GO TO 19
        DLAMDA=0.25*DLAMDA
        VLAMDA=VLAMDA-3.*DLAMDA
```

```
      NTEST=1
      GO TO 3
   19 VLAMDA=VLAMDA-5.*DLAMDA
      NTEST=2
      GO TO 3
   20 NTEST=0
      GO TO 18
    8 TPR=PR
      IF(ABS(VLAMDA-TVL).LE.0.00005)GO TO 9
      TVL=VLAMDA
   16 VLAMDA=PPL+(PPR*(TVL-PPL))/(ABS (TPR)+PPR)
      GO TO 3
    9 PCR=VLAMDA*VLAMDA*E*FI1/(FL1*FL1*144.)
      WRITE(3,14)PCR
      CALLCRDEND(KK)
      IF(KK.EQ.1)GO TO 2
      STOP
      END
```

```
      SUBROUTINE MSTFM(SMK,M,SPRK,E,FL1,FL2,FI1,FI2,VLAMDA)
      DIMENSION SMK(10,10)
      DO 1 I=1,M
      DO 1 J=1,M
  1   SMK(I,J)=0.0
  7   FK1 = 4.
      FK2 = 2.
      FK3 = -6.
      FK4=12.
      L=3
      FACTOR=(FL1*FI2)/(FL2*FI1)
      GO TO 5
  3   IF(VLAMDA.EQ.0.)GO TO 9
  8   DIV=2.0-2.*COS (VLAMDA)-VLAMDA*SIN (VLAMDA)
      FK1=(VLAMDA*(SIN (VLAMDA)-VLAMDA*COS (VLAMDA)))/DIV
      FK2=(VLAMDA*(VLAMDA-SIN  (VLAMDA)))/DIV
      FK3=(VLAMDA*VLAMDA*(COS  (VLAMDA)-1.0))/DIV
      FK4=(VLAMDA*VLAMDA*VLAMDA*SIN (VLAMDA))/DIV
  9   L=-6
      INCL=6
  4   FACTOR = 1.0
  2   L=L+INC
  5   SMK(L+1,L+1)=FK1*FACTOR
      SMK(L+1,L+2)=FK2*FACTOR
      SMK(L+1,L+3)=FK3*FACTOR
      SMK(L+2,L+1)=FK2*FACTOR
      SMK(L+2,L+2)=FK1*FACTOR
      SMK(L+2,L+3)=FK3*FACTOR
      SMK(L+3,L+1)=FK3*FACTOR
      SMK(L+3,L+2)=FK3*FACTOR
      SMK(L+3,L+3)=FK4*FACTOR
      IF(L.EQ.3)GO TO 3
      IF(L.LT.6)GO TO 2
  6   SMK(M,M)+(FL1*FL1*FL1*SPRK*1728.)/(E*FI1)
      RETURN
      END
```

```
      FUNCTION DETERM(N, X)
      DIMENSION X(10,10)
      M=N-1
      PR=1.
      DO15I=1,N
      XV=X(I,I)
      IF(I.EQ.N)GOTO20
      IF(XV.EQ.0.)GOTO11
    2 PR=PR*XV
      DO7J=I,N
    7 X(I,J)=X(I,J)/XV
   11 DO9L=I,M
      XY=X(L+1,I)
      IF(XY.EQ.0.)GOTO9
   10 DO8K=I,N
    8 X(L+1,K)=X(L+1,K)-XY*X(I,K)
    9 CONTINUE
   15 CONTINUE
   20 PR=PR*XV
      DETERM = PR
      RETURN
      END
```