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Applying Dynkin's isomorphism: An alternative approach to understand the Markov property of the de Wijs process

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Dynkin's (*Bull. Amer. Math. Soc.* **3** (1980) 975–999) seminal work associates a multidimensional transient symmetric Markov process with a multidimensional Gaussian random field. This association, known as Dynkin's isomorphism, has profoundly influenced the studies of Markov properties of generalized Gaussian random fields. Extending Dykin's isomorphism, we study here a particular generalized Gaussian Markov random field, namely, the de Wijs process that originated in Georges Matheron's pioneering work on mining geostatistics and, following McCullagh (*Ann. Statist.* **30** (2002) 1225–1310), is now receiving renewed attention in spatial statistics. This extension of Dynkin's theory associates the de Wijs process with the (recurrent) Brownian motion on the two dimensional plane, grants us further insight into Matheron's kriging formula for the de Wijs process and highlights previously unexplored relationships of the central Markov models in spatial statistics with Markov processes on the plane.

Keywords: additive functions; Brownian motion; intrinsic autoregressions; kriging; potential kernel; random walk; screening effect; variogram

1. Introduction

After originating in the pioneering work of Georges Matheron, the de Wijs process enjoyed a significant and extensive role in early geostatistical literature [6,19,26,28]. McCullagh's [30] recent work has revived interest in the de Wijs process, both theoretically and in a growing range of applications in spatial statistics; see, for example, [4,5,8–10,13,31,34–36]. In particular, Mondal [34] and Besag and Mondal [5] established a connection between Gaussian Markov random fields on two-dimensional lattices and the de Wijs process on the Euclidean plane, which emerges as a scaling limit of the former. McCullagh and Clifford [31] analyzed agricultural uniformity trials using a spatial formulation that is based on the de Wijs process and a Gaussian white noise random field. See also the related work by Clifford [8,9], Clifford et al. [10]. Mondal [35] considers the exponential functional of the de Wijs process to construct a generalized Cox process to study disease mappings. Mondal [36] indicates a link between the de Wijs process and Tobler's [52] pycnoplylectic interpolation based on the Laplace splines. Dutta and Mondal [13] make explicit use of the connection between intrinsic autoregressions and the de Wijs process and provide approximate matrix free computations for residual maximum likelihood methods for the latter. Furthermore, outside the statistics literature, the de Wijs process appears to originate separately in quantum physics and statistical mechanics as the massless case of the free Gaussian field; see, for example, Chapters 6 and 7 of [17], and in recent probability literature, this massless case has become a subject of intense study; see, for example, [21,48].

Technically, the de Wijs process is a generalized Gaussian random field ([16], Chapter III), whose index set is a certain class of contrasts, that is, non-atomic signed Borel measures on the Euclidean plane with zero total mass. This process corresponds to the logarithmic variogram model and is a generalization of the Brownian motion in two dimensions. It acquires Markov and conformal invariance properties [30,31] and is first-order intrinsic in the sense of [55] and [12]. The Markov property of the de Wijs process was already known to Matheron [27,28], who viewed it from the perspective of kriging predictions. Consider a mean zero Gaussian random field $\{U(x): x \in \mathcal{R}^2\}$. Let B denote a closed contour in $\mathcal{R}^2$. It is natural to call the random field *Markov* if its values along the curve B determine the ordinary kriging predictor for the value $U(x_0)$ at a point x_0 in the interior of B, given its values on and in the exterior of B. This Markovian characterization leads to a kriging predictor for the random variable $U(x_0)$ that takes the form of a contour integral

$$\mathrm{E}\bigl(U(x_0) \mid U(x), x \in B\bigr) = \int_B \upsilon(x, x_0) U(x) \, \mathrm{d}x,$$

where the coefficient function $\upsilon(x, x_0)$, $x \in B$ is such that

$$\int_B \upsilon(x, x_0) \, \mathrm{d}x = 1, \qquad \int_B \upsilon(x, x_0) \operatorname{cov}\bigl(U(x), U(x')\bigr) \, \mathrm{d}x = \operatorname{cov}\bigl(U(x_0), U(x')\bigr) \tag{1.1}$$

for every point x' on B. When $\{U(x): x \in \mathcal{R}^2\}$ is stationary and isotropic, Matheron [27] deduces that it is Markov if and only if

$$\operatorname{cov}\bigl(U(x), U(0)\bigr) \propto K_0\bigl(a\|x\|\bigr), \tag{1.2}$$

where K_0 is the Bessel function of order zero and a is a positive constant. Matheron further notes that his derivations remain intact for the limiting case $a \downarrow 0$ that corresponds to the logarithmic covariance, $\operatorname{cov}(U(x), U(0)) = -\log(\|x\|)$ and thus provides the Markovian characterization of the de Wijs process. However, the respective random fields exist as generalized processes only, and the details of a formal argument go beyond the above kriging formula.

This paper calls attention to the work of Dynkin [14] to present a mathematical formalism to describe the above kriging formula of the de Wijs process, and to connect the field of spatial statistics to the vast, and hitherto unutilized, probabilistic literature on the Markov property of generalized random fields. This body of literature constitutes a fascinating part of probability, much of which emerged in the wake of [24] and [32]. This corpus notably includes [14,15,18,20,22,25,33,38,39,42–45,47,53,54] and the references therein. In many of these works, several notions of Markovianity for generalized Gaussian random fields have emerged and their interrelations and their connections to various related concepts often form a good part of their understanding. For example, a homogeneous and isotropic generalized Gaussian random field whose spectral density is inversely proportional to an even polynomial of the frequencies satisfies a Markov property in the sense of Holley and Stroock but may not be Markov in the sense of Wong. It is interesting to note that Nelson's [38] construction of the free Markov field on the plane actually corresponds to the generalized Gaussian random field with covariance given in (1.2). Using a slightly different notion of Markovianity, Wong [54] arrives, much earlier than Nelson, at the conclusion that the only generalized Gaussian Markov random field again has covariance (1.2). Kallianpur and Mandrekar [20], on the other hand, investigate Markov properties of a generalized Gaussian random field in conjunction with its dual random field. Ekhaguere [15] later provides links between the Markov property due to Nelson [38] and that due to Wong [54]. In contrast, Dynkin's [14] famous work marks an important departure from these earlier studies. In his study, covariances of a generalized Gaussian random field are assumed to arise from the Green function of a symmetric multidimensional Markov process, and the Markov property of this generalized Gaussian random field is then derived from the path properties of the multidimensional Markov process. Thus, for example, the Markov property of (1.2) can be understood in the context of the Markov property of an exponentially killed Brownian motion on the plane. Here our focus is on the limiting case, namely, the de Wijs process. Although its Markov property can be investigated using the work of Nelson or Wong (e.g., by modifying Theorem 1.5 of [18] or by including the case $\alpha = 0$ in Wong's [54] Theorem 2), we take up Dynkin's approach primarily because it provides a precise and computable description of the boundary condition in the kriging formula, and connects closely with Matheron's work. We also piece together many scattered results and extend some known ones to provide this new addition to the body of literature that, respectively, followed Dynkin's and Matheron's works.

The remainder of the paper is structured as follows. Section 2 introduces the de Wijs process as a homogeneous, isotropic and self-similar generalized Gaussian random field. Section 3 explores the association of this process with Brownian motion. Here we show that the covariance formula of the de Wijs process can be written explicitly in terms of an additive function of the Brownian motion. Section 4 studies the Markov property of the de Wijs process by extending the work of Dynkin [14]. Here our main result, namely Theorem 4.1, provides a new interpretation of Matheron's kriging formula in terms of the hitting probabilities of the Brownian motion and as a generalization of the Dirichlet problem. Section 5 focuses on the practical relevance of Matheron's kriging formula. It also considers the relevance of Dynkyn's isomorphisms in lattice approximations of the de Wijs process and concludes with a discussion on the screening effect in kriging.

2. De Wijs process

In this paper, a *generalized random field* on the Euclidean plane $\mathcal{R}^2$ is a stochastic process $\{Z_\sigma : \sigma \in \mathcal{M}\}$ indexed by a vector space $\mathcal{M}$ of non-atomic signed Borel measures on the plane that have total mass zero. We view Z to be a linear functional from the vector space $\mathcal{M}$ to the real numbers such that

$$Z_{b\sigma+\mathrm{d}\nu} = bZ_\sigma + \mathrm{d}Z_\nu \qquad \text{for all } \sigma, \nu \in \mathcal{M}, \text{ and for all } b, d \in \mathcal{R}.$$

We think of the random variable Z_σ as a spatial *contrast*; for instance, if two plots have unit area and σ has a Lebesgue density that is proportional to the difference of the respective indicator functions, then Z_σ might represent the difference of crop yields on these plots.

The generalized random field $\{Z_\sigma : \sigma \in \mathcal{M}\}$ is said to be *homogeneous* if its distribution remains invariant to planar translations, and *isotropic* if its distribution remains invariant to planar rotations. Furthermore, such a homogeneous isotropic generalized random field is *Gaussian* if all finite dimensional marginal distributions are multivariate normal with $\mathrm{E}Z_\sigma = 0$ and

$$\operatorname{cov}(Z_\sigma, Z_\nu) = -\iint \varphi\bigl(\|x-y\|\bigr)\sigma(\mathrm{d}x)\nu(\mathrm{d}y) \tag{2.1}$$

for some real-valued function φ and all non-atomic signed measures $\sigma, \nu \in \mathcal{M}$. Note that in the above covariance formula we can add to $\varphi(\|x-y\|)$ a function $f_1(x) + f_2(y)$ without affecting the integral, and so φ actually belongs to a suitable quotient space (modulo the infinite dimensional subspace of additive functions). In subsequent discussions, we will be implicit about this equivalence relation in the description of φ. The function

$$c(x) = \varphi\bigl(\|x\|\bigr), \qquad x \in \mathcal{R}^2$$

is then called the *generalized variogram* or $-c(x)$ the *generalized covariance function* of the generalized random field. Let $\hat{\sigma}$ and $\hat{\nu}$ denote the Fourier transforms of σ and ν. We can then write (2.1) as

$$\operatorname{cov}(Z_\sigma, Z_\nu) = \int \hat{\sigma}(x)\overline{\hat{\nu}(x)}S(\mathrm{d}x) \tag{2.2}$$

for a certain non-negative tempered measure S which is called the *spectral measure* of the generalized random field ([16], page 264). If the spectral measure is absolutely continuous with respect to the Lebesgue measure on $\mathcal{R}^2$, its Lebesgue density $s(x)$, $x \in \mathcal{R}^2$ is called the *spectral density*. Under slight regularity conditions, the generalized covariance function c and the spectral density s are Fourier transforms of each other.

Specifically, consider $\mathcal{M}$ to be the space of signed Borel measures σ on the Euclidean plane $\mathcal{R}^2$ that satisfy

$$\iint \bigl|\log\bigl(\|x-y\|\bigr)\bigr||\sigma|(\mathrm{d}x)|\sigma|(\mathrm{d}y) < \infty \tag{2.3}$$

and have total mass zero. The *de Wijs process* is then the homogeneous, isotropic and self-similar generalized Gaussian random field (i.e., its distribution also remains invariant to changes of scale) on $\mathcal{R}^2$ with index set $\mathcal{M}$ such that $\mathrm{E}Z_\sigma = 0$ and

$$\operatorname{cov}(Z_\sigma, Z_\nu) = \langle\sigma, \nu\rangle_{\mathcal{M}} = -\iint \log\bigl(\|x-y\|\bigr)\sigma(\mathrm{d}x)\nu(\mathrm{d}y) \tag{2.4}$$

for all signed measures $\sigma, \nu \in \mathcal{M}$. Note that $\mathcal{M}$ has an inner product space structure with inner product $\langle\sigma, \nu\rangle_{\mathcal{M}}$ and norm

$$\|\sigma\|_{\mathcal{M}} = \langle\sigma, \sigma\rangle_{\mathcal{M}}^{1/2}.$$

Indeed, Corollary 2.5 of [29] implies that $\mathcal{M}$ is a vector space, and by Corollary 2.4 and Remark 3.3 in the same reference $\|\sigma\|_{\mathcal{M}} \geq 0$ with equality if and only if σ vanishes identically. The positive definiteness of the covariance matrices associated with the de Wijs structure (2.4) is an immediate consequence of the Gram matrix property. Thus, the de Wijs process has logarithmic variogram; that is, the representation (2.1) holds with the generalized variogram function $c(x) = \varphi(\|x\|) = \log \|x\|$ for $x \in \mathcal{R}^2$, and its spectral density is

$$s(x) = \frac{1}{2\pi \|x\|^2}, \qquad x \in \mathcal{R}^2.$$

3. Association with Brownian motion

We set $T = [0, \infty)$ for consistency in what follows and let $\{W_t, t \in T\}$ be the Brownian motion on the two-dimensional Euclidean plane. Thus, with probability 1, the function $t \to W(t)$ is continuous in t, the components of the increment $W_{t+u} - W_u$ are independent Gaussian random variables each with mean 0 and variance t, and the process $\{W_t, t \in T\}$ has stationary and independent increments. For every x on the plane, let P_x denote the probability law of $\{W_t, t \in T\}$ starting at x and let E_x be its expectation under P_x. For every $t \in T$, let the sub-σ-field $\mathfrak{F}_t$ consist of events observable up to time t, which is the minimum σ-field generated by $\{W_u : 0 \le u \le t\}$. Define $\mathfrak{F}_\infty$ to be the minimum σ-field containing $\bigcup_{t \in T} \mathfrak{F}_t$. The Markov property of the Brownian motion asserts that the conditional law of $\{W_t, t \ge u\}$ given $\{W_t, 0 \le t \le u\}$ depends on W only through W_u. In other words,

$$\mathrm{E}_x(FJ) = \mathrm{E}_x(F\mathrm{E}_{W_u} J)$$

for every x on the plane, every $\mathfrak{F}_u$ measurable positive function F and every measurable function J that depend only on $\{W_t, t \ge u\}$. In particular, the expectation is calculated first with respect to the conditional law of $\{W_t, t \ge u\}$ given $\{W_t, 0 \le t \le u\}$, and then with respect to the marginal law of $\{W_t, 0 \le t \le u\}$. An important generalization of the Markov property is the strong Markov property. When τ is a stopping time, define the stopping field $\mathfrak{F}_\tau$ to be the σ-algebra consisting of all events $A \in \mathfrak{F}_\infty$ such that $A \cap \{\tau \le t\} \in \mathfrak{F}_t$ for every $t \ge 0$. Then the strong Markov property implies that

$$\mathrm{E}_x(FJ) = \mathrm{E}_x(F\mathrm{E}_{W_\tau} J)$$

for every x on the plane, every stopping time τ, every $\mathfrak{F}_\tau$ measurable positive function F and every measurable function J that depends on $\{W_t, t \ge \tau\}$.

Next we define the Green function of the Brownian motion $\{W_t, t \in T\}$. Typically, the Green function is defined for a transient Markov process as the time integral of its transition probability density function, and Dynkin's theory is essentially based on the fact that the Green function of a transient symmetric Markov process can be interpreted as the covariance of a centered Gaussian process. However, the Brownian motion on the plane is recurrent and hence its transition probability density function

$$p_t(x, y) = (2\pi t)^{-1} \exp\left\{-\frac{1}{2t}\|y - x\|^2\right\}$$

is not integrable with respect to t. Thus, we need a modification that will allow us to define the Green function of the Brownian motion and extend Dynkin's result in a straightforward fashion.

To this end, we fix a point x_0 on the unit circle and consider $q_t(x, y) = p_t(0, y - x) - p_t(0, x_0)$. We then apply the definition of Port and Stone ([40], page 70) and obtain the *Green function* or *the potential kernel* of $\{W_t, t \in T\}$ as

$$g(x, y) = \int_0^\infty q_t(x, y)\,\mathrm{d}t = -\pi^{-1} \log \|y - x\|. \tag{3.1}$$

The choice of x_0 will not matter, as we shall see in what follows.

However, note that the covariances of the de Wijs process now satisfy the relationship

$$\operatorname{cov}(Z_\sigma, Z_\nu) = \langle \sigma, \nu \rangle_{\mathcal{M}} = \iint \pi g(x, y)\sigma(\mathrm{d}x)\nu(\mathrm{d}y).$$

Thus, we say that the above relationship *associates* the de Wijs process with the Brownian motion $\{W_t, t \in T\}$, opening an avenue for exploring the properties of the former from those of the latter. For every $\nu \in \mathcal{M}$, we now get

$$\int q_t(x, y)\nu(\mathrm{d}y) = \int p_t(x, y)\nu(\mathrm{d}y) - p_t\big(0, \|x_0\|\big) \int \nu(\mathrm{d}y) = \int p_t(x, y)\nu(\mathrm{d}y)$$

and therefore

$$\int g(x, y)\nu(\mathrm{d}y) = \iint p_t(x, y)\nu(\mathrm{d}y)\,\mathrm{d}t.$$

In other words, the term involving x_0 disappears from the right-hand side of the previous equation. Consequently, when ν is absolutely continuous with the Radon–Nykodyn derivative $\nu(\mathrm{d}y) = \rho(y)\,\mathrm{d}y$, the above equation becomes

$$\int g(x, y)\nu(\mathrm{d}y) = \mathrm{E}_x \int \rho(W_t)\,\mathrm{d}t.$$

Now equation (3.1) can be identified with $\langle \sigma, \nu \rangle_{\mathcal{M}} = \pi \mathrm{E}_\sigma \int \rho(W_t)\,\mathrm{d}t$, where E_σ is the expectation under the probability law of $\{W_t, t \in T\}$ with initial signed measure σ; that is,

$$\mathrm{E}_\sigma \int \rho(W_t)\,\mathrm{d}t = \int \mathrm{E}_x \int \rho(W_t)\,\mathrm{d}t\,\sigma(\mathrm{d}x).$$

We can thus define the *additive function* of the Brownian motion by the measure

$$A_\nu(Q) = \int_Q \rho(W_t)\,\mathrm{d}t \tag{3.2}$$

that satisfies the property that, for every interval $I = (s, u)$ with $s < u$, $A_\nu(I)$ is a functional of $\{W_t, t \in I\}$, and

$$\operatorname{cov}(Z_\sigma, Z_\nu) = \langle \sigma, \nu \rangle_{\mathcal{M}} = \pi \mathrm{E}_\sigma A_\nu(T). \tag{3.3}$$

The collection of all signed measures $\nu \in \mathcal{M}$ that are absolutely continuous forms a dense subspace of $\mathcal{M}$. By passage to limit, it then follows that for every $\nu \in \mathcal{M}$ there exists an additive functional of the Brownian motion such that the above equation holds. In addition, the strong Markov property of the Brownian motion takes the following form

$$\mathrm{E}_\sigma F A_\nu(\tau + Q) = \mathrm{E}_\sigma F \mathrm{E}_{W_\tau} A_\nu(Q) \tag{3.4}$$

for every $\sigma, \nu \in \mathcal{M}$, for every Borel subset Q of T, and for every τ and F as defined earlier. The strengthened relationship that emerges from equation (3.3) in conjunction with equation (3.4) now paves the way to use the Brownian paths to study the properties of the de Wijs process.

4. Markov property of the de Wijs process

As in Section 1, take B to be a simple closed contour on the plane. Then B divides the entire plane into two components, namely, the bounded interior and the unbounded exterior. Let B_I denote the open interior of B with closure $\bar{B}_\mathrm{I}$. Similarly, let B_E be the open exterior of B with closure $\bar{B}_\mathrm{E}$. Our first task is to describe the values of the de Wijs process on the boundary B, and on the inside and the outside of B (e.g., on sets B_I and $\bar{B}_\mathrm{E}$). To this end, there are two approaches. The first approach is due to [39]. Here we describe the values of the de Wijs process on an open set G by the minimum sigma field $\mathcal{A}_G$ generated by all Z_σ such that $\sigma \in \mathcal{M}$ and support of σ is compactly contained in G. Then, for any closed set C the values of the de Wijs process is described by the sigma field

$$\mathcal{A}_C = \bigcap_{G \supset C} \mathcal{A}_G,$$

where the intersection is taken over all open sets G that contain C. Thus, $\mathcal{A}_{B_\mathrm{I}}$, $\mathcal{A}_{\bar{B}_\mathrm{E}}$ and $\mathcal{A}_B$ represent the values of the de Wijs process on the inside, outside and on the boundary, respectively, and the Markov property of the de Wijs process asserts that, for any $\sigma \in \mathcal{M}$ with support of σ compactly contained in B_I,

$$\mathrm{E}(Z_\sigma | \mathcal{A}_{\bar{B}_\mathrm{E}}) = \mathrm{E}(Z_\sigma | \mathcal{A}_B)$$

almost surely in the probability distribution of the de Wijs process. Note that, by construction, the minimum sigma fields $\mathcal{A}_{\bar{B}_\mathrm{E}}$ and $\mathcal{A}_B$ contains neighborhood information, not just the information on the set. The second approach adopted by Dynkin [14] is a simplified version of the above and goes as follows. For a close set C, we define $\mathcal{M}_C$ to be the set of signed Borel measures $\sigma \in \mathcal{M}$ that do not charge on its complement. Then, following [14], the minimum sigma field generated by the collection of random variables $\{Z_\sigma : \sigma \in \mathcal{M}_C\}$ describes the values of the de Wijs process on the C. The Markov property is described in the usual fashion, namely, the values on the inside ($\bar{B}_\mathrm{I}$) and the outside ($\bar{B}_\mathrm{E}$) of B are conditionally independent given the values on B. This leads to the following theorem.

Theorem 4.1. *Let D be any closed set on the plane with a simply connected open interior, and let τ_D be the first hitting time of D by the Brownian motion $\{W_t, t \in T\}$ starting at x on the plane. Denote by V_x the probability measure of W_{τ_D} conditioned on $W_0 = x$. For any initial* *signed measure $\sigma \in \mathcal{M}$ which does not charge on the set D, identify σ_D with the signed measure induced by W_t at the first hitting time of D; that is,*

$$\sigma_D(G) = \int \mathrm{P}_x(W_{\tau_D} \in G)\sigma(\mathrm{d}x) = \int V_x(G)\sigma(\mathrm{d}x)$$

for all Borel subsets G on the plane. Then the conditional expectation of Z_σ, given the values of the de Wijs process on D, is identical to Z_{σ_D}. In other words,

$$\mathrm{E}\big(Z_\sigma | \{Z_\nu, \nu \in \mathcal{M}_D\}\big) = Z_{\sigma_D}. \tag{4.1}$$

Before we turn to the proof, let us first see how Matheron's kriging formula enters into the above theorem. Equation (4.1) implies that for all $\sigma \in \mathcal{M}$ and $\nu \in \mathcal{M}_D$

$$\mathrm{E}Z_\sigma Z_\nu = \mathrm{E}Z_{\sigma_D} Z_\nu.$$

Hence, formula (2.4) applies. This along with the definition of σ_D produces the identity

$$\iint \log\big(\|x-y\|\big)\sigma(\mathrm{d}x)\nu(\mathrm{d}y) = \iint \int \log\big(\|x'-y\|\big) V_x\big(\mathrm{d}x'\big)\sigma(\mathrm{d}x)\nu(\mathrm{d}y)$$

for all $\sigma \in \mathcal{M}$ and $\nu \in \mathcal{M}_D$. Consequently, if x is in the interior of boundary B, we have

$$\int_B V_x\big(\mathrm{d}x'\big) = 1, \qquad \log\big(\|x-y\|\big) = \int_B \log\big(\|x'-y\|\big) V_x\big(\mathrm{d}x'\big)$$

for every point y on B. In short, the coefficient function $v(x', x_0)$ in Matheron's kriging formula (1.1) is the derivative of V_{x_0} at x', and thus corresponds to the probability density function of the Brownian motion at the first hitting time τ_D, a crucial fact that has arguably been missing from the geostatistitical literature. Furthermore, in order for Z to be a linear functional from the vector space $\mathcal{M}$ to the real numbers, we can imagine Z_σ as an integral of the form $\int Z_x \sigma(\mathrm{d}x)$, where the notation Z_x suggests a point-wise intrinsic process with $\mathrm{var}(Z_x - Z_{x'}) = -\log(\|x - x'\|)$. This very imagination of a point-wise Z_x allows us to describe $\hat{Z}_x = \int_B Z_{x'} V_x(\mathrm{d}x')$ as the kriged value of Z_x, for an x in the interior of B. We can take this point further, and even describe the kriging formula from a different angle. First, let ∇ denote the Laplace operator on the plane. If ν is twice differentiable, Theorem 3 of [41], page 525, implies

$$-2\pi\langle\sigma, \nabla\nu\rangle_{\mathcal{M}} = \int \sigma(x)\nu(x)\,\mathrm{d}x, \tag{4.2}$$

which surprisingly asserts that ∇Z_x and $Z_{x'}$ behave as two mean zero uncorrelated Gaussian random variables for $x \neq x'$, and in turn, suggests that the kriged values of ∇Z_x on the interior of B are all zero. An interchange of the Laplace operator and the conditional expectation on Z_x then produce

$$\nabla \hat{Z}_x = 0$$

on the interior of B, implying that the kriging problem is a generalization of the Dirichlet problem in mathematics. Indeed, the probability literature reaffirms that the boundary values of the Brownian motion at first hitting time solve the standard version of the Dirichlet problem and therefore the Matheron's kriging formula (1.1) can be seen as a generalization. For significance of the Dirichlet problem in recent spatial statistics, we refer to the discussion in [30]. We now return to the proof of the theorem.

Proof of Theorem 4.1. First, we verify that $\sigma_D \in \mathcal{M}_D$. As a first step, we argue that σ_D belongs to $\mathcal{M}$. Since $\int \sigma_D(\mathrm{d}x) = \int \sigma(\mathrm{d}x) = 0$, σ_D represents a signed Borel measure with total mass zero. Now, for a non-negative measure μ for which the integral $h_\mu(x) = \int g(x, y)\mu(\mathrm{d}y)$ is finite for every x, the results of [7], pages 193–194, give the identity

$$h_\mu(x) - \mathrm{E}_x h_\mu(W_t) = \int g(x, y)\mu(\mathrm{d}y) - \mathrm{E}_x \int g(W_t, y)\mu(\mathrm{d}y) = \int\int_0^t p_s(x, y)\,\mathrm{d}s\,\mu(\mathrm{d}y) \geq 0.$$

Consequently, h_μ defines an *excessive measure*, and $h_\mu(x) \geq \mathrm{E}_x h_\mu(W_t)$. Hence, the choices $\mu = \nu^+$ and $\mu = \nu^-$ yield

$$h_{\nu^+}(x) \geq \mathrm{E}_x h_{\nu^+}(W_t), \qquad h_{\nu^-}(x) \geq \mathrm{E}_x h_{\nu^-}(W_t).$$

It then follows that

$$\langle \sigma_D^+, \nu^+ \rangle_{\mathcal{M}} = \pi \mathrm{E}_{\sigma^+} h_{\nu^+}(W_{\tau_D}) \leq \pi \int h_{\nu^+}(x)\sigma^+(\mathrm{d}x) = \langle \sigma^+, \nu^+ \rangle_{\mathcal{M}},$$

and, after repeating the same argument, $\langle \sigma_D^-, \nu^- \rangle_{\mathcal{M}} \leq \langle \sigma^-, \nu^- \rangle_{\mathcal{M}}$ and so on. Thus,

$$\langle \sigma_D^+, \nu \rangle_{\mathcal{M}} \leq \left| \langle \sigma^+, \nu^+ \rangle_{\mathcal{M}} \right| + \left| \langle \sigma^+, \nu^- \rangle_{\mathcal{M}} \right|,$$

and a similar upper bound exists for $\langle \sigma_D^-, \nu \rangle_{\mathcal{M}}$. The above bounds imply

$$\langle \sigma_D, \sigma_D \rangle_{\mathcal{M}} \leq \left| \langle \sigma^+, \sigma^+ \rangle_{\mathcal{M}} \right| + 2\left| \langle \sigma^+, \sigma^- \rangle_{\mathcal{M}} \right| + \left| \langle \sigma^-, \sigma^- \rangle_{\mathcal{M}} \right|,$$

which ensures that $\sigma_D \in \mathcal{M}$. Now D is a closed set with a simply connected open interior and so we get

$$|\sigma_D|\left(D^c\right) = \int \mathrm{P}_x(W_{\tau_D} \notin D)|\sigma|(\mathrm{d}x) = 0.$$

Therefore, σ_D belongs to $\mathcal{M}_D$. Next, we establish that

$$\mathrm{E}Z_\sigma Z_\nu = \mathrm{E}Z_{\sigma_D} Z_\nu \qquad \forall \nu \in \mathcal{M}_D.$$

Since σ_D is a measure that satisfies the relation $\sigma_D(G) = \mathrm{E}_\sigma(1_G(W_{\tau_D}))$, for all Borel subsets G of the plane, integrals with respect to σ_D can be defined as appropriate expected values of the functions of W_{τ_D}. In particular, for any element f of an appropriate class of functions, such an integral will satisfy

$$\int f(x)\sigma_D(\mathrm{d}x) = \mathrm{E}_\sigma f(W_{\tau_D}).$$

Now take $f(x) = \mathrm{E}_x A_\nu(T)$. Then, the definition of the additive function asserts that $\mathrm{E} Z_{\sigma_D} Z_\nu = \pi \mathrm{E}_{\sigma_D} A_\nu(T)$, but the above equation also implies

$$\mathrm{E}_{\sigma_D} A_\nu(T) = \int \mathrm{E}_x A_\nu(T) \sigma_D(\mathrm{d}x) = \int f(x)\sigma_D(\mathrm{d}x) = \mathrm{E}_\sigma f(W_{\tau_D}) = \mathrm{E}_\sigma \mathrm{E}_{W_{\tau_D}} A_\nu(T).$$

Consequently, the strong Markov property of the Brownian motion in equation (3.4) applies, and we obtain

$$E_{\sigma_D} A_\nu(T) = \mathrm{E}_\sigma A_\nu(\tau_D + T).$$

Since τ_D is the first hitting time of D, the path of the Brownian motion up to but not including time τ_D lies entirely within the complement of D. However, the signed measure ν concentrates on D making it imminent that $\mathrm{E}_\sigma A_\nu((0, \tau_D)) = 0$. And, therefore

$$\mathrm{cov}(Z_{\sigma_D}, Z_\nu) = \pi \mathrm{E}_{\sigma_D} A_\nu(T) = \pi \mathrm{E}_\sigma A_\nu(\tau_D + T) = \pi \mathrm{E}_\sigma A_\nu(T) = \mathrm{cov}(Z_\sigma, Z_\nu).$$

This completes the proof. □

Interestingly, when values are known along a straight line or on a circle, the analytic formulas for the coefficient function $v(x, x_0)$ are available in closed form, making it possible to apply Theorem 4.1 directly to calculate relevant kriging predictions. For an example, when B is the unit circle, $v(x, x_0)$ becomes the Poisson kernel

$$v(x, x_0) = \frac{1}{2\pi} \frac{1 - \|x_0\|^2}{\|x - x_0\|^2}, \qquad \|x\| = 1, \|x_0\| < 1. \tag{4.3}$$

Furthermore, when the boundary set B is the y-axis, we refer to [27] for a formula for the corresponding coefficient function $v(x, x_0)$.

Finally, we can also discuss predictions for functionals of the values of the de Wijs process inside the boundary B (e.g., $f(Z_\sigma)$ for some suitable function f), given $\{Z_\nu : \nu \in \mathcal{M}_D\}$, but this would require a knowledge of Wick products and Fock spaces and is beyond the scope of this paper.

5. Discussion

In practice, we only select finitely many regular or irregularly distributed sampling locations and observe process values as aggregates or averages over certain non-empty regular or irregular regions around those sampling locations. In both instances there are certain limitations in applying Matheron's kriging formula directly. For example, if we observe only finitely many data values on the unit circle, we won't be able to apply the exact kriging formula in (4.3). Similarly, when the de Wijs process is used as a statistical model for aggregates or averages of spatial variables over non-empty regions in the two-dimensional plane, we simply lose the Markov property because of aggregations or averaging. Examples include agricultural field trials where the variable of interest is the crop yield over plots, or disease mapping where the spatial variable of interest is considered a stochastically degraded version of an underlying unobserved spatial component such as the log relative risk of non-infectious diseases over a geographic region.

However, certain discrete approximations are possible, and, in fact, it is the discrete approximations of the de Wijs process that have played a major role in spatial statistics in the past thirty years; see, for example, [2,3,23], many subsequent papers, and the books by Cressie [11], Banerjee et al. [1] and Rue and Held [46]. These discrete approximations form a subclass of Gaussian Markov random fields on regular and irregular lattices and have lattice graph Laplacians as their precision (i.e., inverse of 'covariance') matrix. The nature of these approximations become clearer when we also note that the log function (i.e., the generalized covariance of the de Wijs process) is the inverse of the Laplacian on the plane. Interestingly, these Gaussian random fields are also associated with the random walks on the lattice graph, as the De Wijs process is with the Browning motion on the plane. As a concrete example, the first order symmetric intrinsic autoregressions on the two dimensional integer lattice $\mathcal{Z}^2$ [5,34] is associated with the simple random walk on $\mathcal{Z}^2$. Thus Dynkin's theory also applies here and we can obtain the coefficient function of a corresponding kriging problem on the discrete lattice $\mathcal{Z}^2$ from probabilities of the simple random walk at the first hitting time. To summarize, the diagram in Figure 1 lists a few important spatial Gaussian Markov models and the associated two-dimensional Markov processes. The top part of this diagram notes the lattice Gaussian Markov random fields along with the spectral densities and, in brackets, the associated lattice Markov processes. The bottom part of the diagram provides limiting continuum Gaussian Markov random fields with corresponding spectral densities and, in brackets, the associated Markov processes. These continuum random fields arise as the scaling limits of corresponding lattice Markov fields from the top part of the diagram; see [37] and [5] for details.

Another interesting point is that aggregates or averages of the de Wijs process retain an approximate Markov property that is known as the screening effect in geostatistics [6,50]. To give a very simple example, we consider a regular lattice in the two-dimensional plane. Let $X_{s,t}$ denote the average value of the spatial variable of interest over the unit square whose center has integer coordinate (s,t) in the Euclidean plane. Given a realization $X^{S,T}$ of $X_{s,t}$ for $s,t=-8,\dots,8$ but

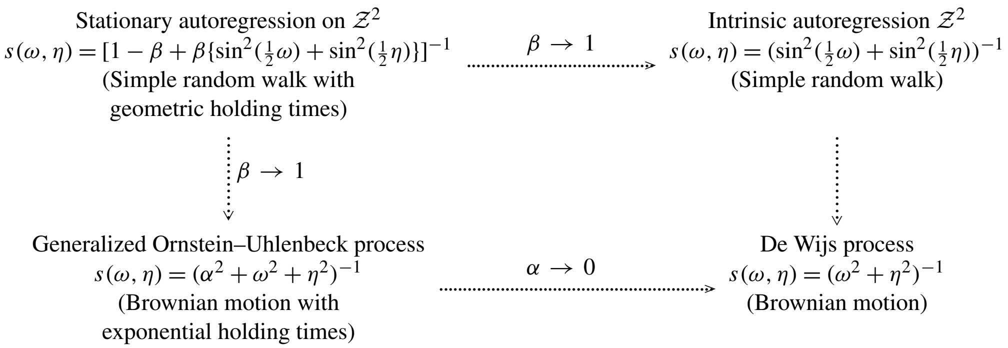

Figure 1. Limit diagram for Gaussian Markov random fields and associated Markov processes. Here $0\le\beta<1$ and $\alpha>0$.

Table 1. Numerical values of the coefficients $\omega_{s,t}$ in the ordinary kriging predictor (5.1) under the regularized de Wijs process

	t							
s	1	2	3	4	5	6	7	8
0	0.342	−0.075	0.017	−0.004	0.001	0.000	0.000	0.000
1	−0.032	−0.001	0.002	−0.001	0.000	0.000	0.000	0.000
2		0.002	−0.001	0.000	0.000	0.000	0.000	0.000
3			0.000	0.000	0.000	0.000	0.000	0.000
4				0.000	0.000	0.000	0.000	0.000
5					0.000	0.000	0.000	0.000
6						0.000	0.000	0.000
7							0.000	0.000
8								0.000

excluding $X_{0,0}$, we can employ exact variogram computations [8,34] to find the coefficients $\omega_{s,t}$ in the conditional expectation or ordinary kriging predictor [49]

$$\mathrm{E}\bigl(X_{0,0}|X^{S,T}\bigr) = \sum \omega_{s,t} X_{s,t} \tag{5.1}$$

under the regularized de Wijs process. The sum on the right-hand extends over the aforementioned index set and the ordinary kriging coefficients $\omega_{s,t}$ add up to 1. Table 1 shows the numerical values of $\omega_{s,t}$ to three decimals; in view of symmetries only 44 of the $17^2 - 1 = 288$ coefficients need to be shown. The screening effect is prominent here in that the immediately neighboring cells dominate, with very few of the remaining cells receiving non-negligible ordinary kriging coefficients. However, it is not known to me if $\omega_{s,t}$ can be interpreted in a meaningful way using certain probability calculations of the Brownian motions, but here one can further try to derive analytic form of $\omega_{s,t}$ for an infinite lattice from the spectral density form of $X_{u,v}$. The same applies for aggregates or averages of the first-order intrinsic autoregression. In general, it would be interesting to know if one can better understand such an approximate Markov property.

Some future directions can be added to this work. For example, the work of [51] provides links between Dynkin's isomorphisms and constructions of statistical designs. Generalizations of Tjur's work in the context of spatial designs would be an interesting matter for future study.

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References

[1] Banerjee, S., Carlin, B.P. and Gelfrand, A.E. (2004). *Hierarchical Modeling and Analysis for Spatial Data*. London: Chapman & Hall.

[2] Besag, J. (1974). Spatial interaction and the statistical analysis of lattice systems. *J. R. Stat. Soc. Ser. B* **36** 192–236. With discussion by D. R. Cox, A. G. Hawkes, P. Clifford, P. Whittle, K. Ord, R. Mead, J. M. Hammersley and M. S. Bartlett and with a reply by the author. MR0373208

[3] Besag, J. (1986). On the statistical analysis of dirty pictures. *J. R. Stat. Soc. Ser. B* **48** 259–302. MR0876840

[4] Besag, J. (2002). Discussion on the paper by McCullagh. *Ann. Statist.* **30** 1267–1277.

[5] Besag, J. and Mondal, D. (2005). First-order intrinsic autoregressions and the de Wijs process. *Biometrika* **92** 909–920. MR2234194

[6] Chilès, J.P. and Delfiner, P. (1999). *Geostatistics: Modeling Spatial Uncertainty. Wiley Series in Probability and Statistics: Applied Probability and Statistics*. New York: Wiley. MR1679557

[7] Chung, K.L. and Walsh, J.B. (2005). *Markov Processes, Brownian Motion, and Time Symmetry*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **249**. New York: Springer. MR2152573

[8] Clifford, D. (2005). Computation of spatial covariance matrices. *J. Comput. Graph. Statist.* **14** 155–167. MR2137895

[9] Clifford, D. (2006). Distribution of increases in residual log likelihood for nested spatial models. *Comm. Statist. Simulation Comput.* **35** 779–788. MR2240044

[10] Clifford, D., McBratney, A.B., Taylor, J. and Whelan, B.M. (2006). Generalized analysis of spatial variation in yield monitor data. *J. Agric. Sci.* **144** 45–51.

[11] Cressie, N.A.C. (1993). *Statistics for Spatial Data. Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics*. New York: Wiley. Revised reprint of the 1991 edition. MR1239641

[12] Dobrushin, R.L. (1979). Gaussian and their subordinated self-similar random generalized fields. *Ann. Probab.* **7** 1–28. MR0515810

[13] Dutta, S. and Mondal, D. (2014). An h-likelihood method for spatial mixed linear models based on intrinsic autoregressions. *J. R. Stat. Soc. Ser. B Stat. Method.* DOI:10.1111/rssb.12084.

[14] Dynkin, E.B. (1980). Markov processes and random fields. *Bull. Amer. Math. Soc.* (*N.S.*) **3** 975–999. MR0585179

[15] Ekhaguere, G.O.S. (1977). On notions of Markov property. *J. Math. Phys.* **18** 2104–2107. MR0456149

[16] Gelfand, I.M. and Vilenkin, N.Y. (1964). *Generalized Functions. Vol.* 4: *Applications of Harmonic Analysis*. New York: Academic Press. Translated from the Russian by Amiel Feinstein. MR0435834

[17] Glimm, J. and Jaffe, A. (1981). *Quantum Physics: A Functional Integral Point of View*. New York: Springer. MR0628000

[18] Holley, R. and Stroock, D. (1980). The D.L.R. conditions for translation invariant Gaussian measures on $\mathcal{S}'(\mathcal{R}^d)$. *Z. Wahrsch. Verw. Gebiete* **53** 293–304. MR0586022

[19] Journel, A.G. and Huijbregts, C.H. (1978). *Mining Geostatistics*. London: Academic Press.

[20] Kallianpur, G. and Mandrekar, V. (1974). The Markov property for generalized Gaussian random fields. *Ann. Inst. Fourier* (*Grenoble*) **24** 143–167. MR0405569

[21] Kenyon, R. (2001). Dominos and the Gaussian free field. *Ann. Probab.* **29** 1128–1137. MR1872739

[22] Künsch, H. (1979). Gaussian Markov random fields. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **26** 53–73. MR0539773

[23] Künsch, H.R. (1987). Intrinsic autoregressions and related models on the two-dimensional lattice. *Biometrika* **74** 517–524. MR0909356

[24] Lévy, P. (1956). A special problem of Brownian motion, and a general theory of Gaussian random functions. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*, 1954–1955, *Vol. II* 133–175. Berkeley: Univ. California Press. MR0090934

[25] Mandrekar, V. and Zhang, S. (1993). Markov property of measure-indexed Gaussian random fields. In *Stochastic Processes* 253–262. New York: Springer. MR1427321

[26] Matheron, G. (1962). *Traité de Géostatistique Appliquée* **1**. Paris: Editions Technip.

[27] Matheron, G. (1963). *Processus Markoviens Normaux Stationaires à n Dimensions. Note Géostatistique* **50**. Paris: Centre de Géostatistique, Ecole des Mines de Paris.

[28] Matheron, G. (1971). *The Theory of Regionalized Variables and Its Applications. Les Cahiers du Centre de Morphologie Mathématique de Fontainebleau* **5**.

[29] Mattner, L. (1997). Strict definiteness of integrals via complete monotonicity of derivatives. *Trans. Amer. Math. Soc.* **349** 3321–3342. MR1422615

[30] McCullagh, P. (2002). What is a statistical model? *Ann. Statist.* **30** 1225–1310. With comments and a rejoinder by the author. MR1936320

[31] McCullagh, P. and Clifford, D. (2006). Evidence for conformal invariance of crop yields. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **462** 2119–2143.

[32] McKean, H.P. Jr. (1963). Brownian motion with a several-dimensional time. *Theory Probab. Appl.* **8** 335–354.

[33] Molchan, G.M. (1971). Characterization of Gaussian fields with Markov property. *Soviet Math. Dokl.* **12** 563–567.

[34] Mondal, D. (2005). Variogram calculations for first-order intrinsic autoregressions and the de Wijs process. Technical Report 479, Department of Statistics, University of Washington.

[35] Mondal, D. (2015). Generalized Gaussian Markov random fields and modeling disease risk. Under revision.

[36] Mondal, D. (2015). On Tobler's pycnophylactic interpolation. Unpublished manuscript, Oregon State Univ.

[37] Moran, P.A.P. (1973). A Gaussian Markovian process on a square lattice. *J. Appl. Probab.* **10** 54–62. MR0353437

[38] Nelson, E. (1973a). Construction of quantum fields from Markoff fields. *J. Funct. Anal.* **12** 97–112. MR0343815

[39] Nelson, E. (1973b). The free Markoff field. *J. Funct. Anal.* **12** 211–227. MR0343816

[40] Port, S.C. and Stone, C.J. (1978). *Brownian Motion and Classical Potential Theory. Probability and Mathematical Statistics*. New York: Academic Press. MR0492329

[41] Rao, M. (1977). *Brownian Motion and Classical Potential Theory. Lecture Notes Series* **47**. Aarhus: Matematisk Institut, Aarhus Univ. MR0440718

[42] Röckner, M. (1983). Markov property of generalized fields and axiomatic potential theory. *Math. Ann.* **264** 153–177. MR0711875

[43] Röckner, M. (1985). Generalized Markov fields and Dirichlet forms. *Acta Appl. Math.* **3** 285–311. MR0790552

[44] Rozanov, Y.A. (1977). Markovian random fields and stochastic partial differential equations. *Mat. USSR–Sb.* **32** 515–534.

[45] Rozanov, Y.A. (1979). Stochastic Markovian fields. In *Developments in Statistics*, *Vol.* 2 203–234. New York: Academic Press. MR0554181

[46] Rue, H. and Held, L. (2005). *Gaussian Markov Random Fields: Theory and Applications. Monographs on Statistics and Applied Probability* **104**. Boca Raton, FL: Chapman & Hall/CRC. MR2130347

[47] Schäfer, J. (1996). Abstract Markov property and local operators. *J. Funct. Anal.* **138** 137–169. MR1391633

[48] Sheffield, S. (2007). Gaussian free fields for mathematicians. *Probab. Theory Related Fields* **139** 521–541. MR2322706

[49] Stein, M.L. (1999). *Interpolation of Spatial Data*: *Some Theory for Kriging*. *Springer Series in Statistics*. New York: Springer. MR1697409

[50] Stein, M.L. (2002). The screening effect in kriging. *Ann. Statist.* **30** 298–323. MR1892665

[51] Tjur, T. (1991). Block designs and electrical networks. *Ann. Statist.* **19** 1010–1027. MR1105858

[52] Tobler, W.R. (1979). Smooth pycnophylactic interpolation for geographical regions. *J. Amer. Statist. Assoc.* **74** 519–536. With a comment by Nira Dyn [Nira Richter-Dyn], Grace Wahba and Wing Hung Wong and a rejoinder by the author. MR0548256

[53] Urbanik, K. (1962). Generalized stationary processes of Markovian character. *Studia Math.* **21** 261–282. MR0150835

[54] Wong, E. (1969). Homogeneous Gauss–Markov random fields. *Ann. Math. Statist.* **40** 1625–1634. MR0263148

[55] Yaglom, A.M. (1957). Certain types of random fields in n-dimensional space similar to stationary stochastic processes. *Theory Probab. Appl.* **2** 273–320.

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