AN ABSTRACT OF THE THESIS OF

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Title: THERMO-ELASTIC-PLASTIC TRANSITION

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The classical theory of elasticity and plasticity does not recognize explicitly the existence of a "transition zone" between elastic and plastic states, which instead, makes extensive use of ad-hoc, semi-empirical laws, such as yield conditions, at the "yield surface" to match both the extreme states. In the present investigation, it is shown that these ad-hoc, semi-empirical laws turn out to be totally unnecessary if one appreciates the existence of a "transition zone" introduced by Seth in recent years, which is quite realistic from the point of view of physical considerations. A transition phenomenon, being an asymptotic one, should be dealt with as a limiting process, and so the transition state should be obtainable from the basic system of equations characterizing the elastic state as a limiting case. The plastic state is similarly to be obtained from the transition state when a certain parameter is made to

approach zero.

In order to appreciate the existence of a transition state, it is basically important at the outset to identify the "transition points" from the differential system which characterizes the physical phenomenon. It is examined in this thesis how and in what manner transition is to be understood in the case of physical phenomena. It is found that there are three ways in which a transition could be identified analytically:

(1) at transition, the differential system characterizing the elastic state should attain some criticality,

(2) the complete breakdown of the macroscopic structure at transition should correspond to the degeneracy of the material or spatial strain ellipsoid,

(3) if we consider the plastic state as an image of the elastic state, then at transition the Jacobian of transformation is bound to behave singularly.

The last condition turns out to be the most general one from which a general yield condition is deduced and it is found that most of the yield conditions present in current literature come out as special cases. Also, it has been seen that our results take into account Bauschinger's effect, while neither Tresca's yield condition nor von-Mises yield condition does. It has also been shown that transition fields naturally being non-linear in character, are sub-harmonic (super-harmonic) fields.

Once one recognizes the "transition zone" as a separate state, the natural question of determining the constitutive equation also arises. In order to answer this question and to illustrate the procedure, four problems of practical interest are discussed in detail. The problems of elastic-plastic transition of shells and tubes subjected to external pressure are solved. Further, by considering the effect of a steady state temperature, the problems of thermo-elastic-plastic transition of shells and tubes are also solved. No yield conditions have been assumed. It is found that if they exist, they come out of the differential system as a consequence of the transition analysis. Some of the results have been compared with those of the classical theory.

Possible scope of future work, where the transition concept may be profitably exploited, has also been discussed.

Thermo-Elastic-Plastic Transition

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CHAPTER I

INTRODUCTION

1.1 Preliminary Remarks.

Transition from one state into another is a common phenomenon in Nature. Boundary Layers, Shocks, elastic-plastic transition, creep, fatigue, shadow-boundary, etc. are some of the familiar examples of transition phenomena.

A state 'A' with some intrinsic properties of its own changes to another state 'B' with different properties than that of 'A'. 'Why' and 'how' this change occurs, are two of the principal questions in continuum mechanics which concerned the early research workers of this subject. Noteworthy mentions are Coulomb, Tresca, Saint Venant, von-Mises, Levy, Hencky, Beltrami, Huber, Prandtl, Reuss and many others. The recent workers in this field are Drucker, Friedrichs, Howard, Green, Prager, Nadai, Sachs, Bland, Hill, Johnson, Seth, Mellor, Thomas among others. While the first question has been dealt with by many authors successfully, the answer to the second question still remains unsatisfactory. To be specific, let us consider the elastic-plastic transition; the elastic state when subjected to internal and external stresses, under suitable circumstances, goes to plastic state whose intrinsic properties are basically different from those of the plastic state. Then a question arises. Is it reasonable to assume that properties of the medium in the elastic state change abruptly to those of the plastic state and that can one divide these two states by non-differentiable, singular or discontinuous surface, so called yield surface?

Most of the authors up to the present time are in common agreement that the situation is not so. A medium can not change from state A into state B without passing through an intermediate state T. Even from the early part of the nineteenth century, Tresca, Stokes, Love and others have studied about this intermediate state, but none of these authors have explained the actual character of this region. Recently Seth [1963] has given a thorough analytical treatment of this intermediate region. He has named this intermediate region as 'Transition Region'. So A passes into B through T. In a large number of cases A and B may be treated as linear fields, but T is essentially a non-linear field, since both A and B dovetail into each other in T. This essentially non-linear transition region T remains untreated by research workers and instead non-differentiable, singular surfaces are introduced to connect the regions A and B. These, in turn have necessitated the increasing use of ad-hoc, semi-empirical laws, such as yield conditions, creep-strain laws, jump conditions across shocks etc. These ad-hoc, semi-empirical laws are based on long experimental results, but very few authors have ever tried to justify these ad-hoc assumptions with analytical basis. Seth's approach towards this end is an attempt at bridging the gap between seemingly unrelated phenomena, the approach which is based on sound mathematical ground.

At a transition, the fundamental structure of the medium undergoes a change. The particles constituting the material re-arrange themselves and give rise to spin, rotation, vorticity and other non-linear effects. Hence a sort of non-linear or non-conservative instability sets in due to the non-conservative nature of the spin forces. That is why T is always non-linear in character. This explains why at boundary layer transition different types of spiral formations or vortex motions are observed. This suggests that at transition, non-linear terms are very important and neglect of which may not represent the real physical phenomenon. Transition fields are, therefore, non-linear, non-conservative and irreversible in nature and should not be treated as superposition of effects.

In the classical theory of elasticity, in particular, the displacements are assumed to be so small that the squares and products of displacement gradients are neglected and the measure of strain thus becomes linear. But non-linear terms are very important at transition state. Linearization of problems has all the advantages of existence, uniqueness and stability. But it also has disadvantages in that it may not be able to explain or represent all the changes and phenomena occuring in a medium. For instance, the well known effects such as the Kelvin and Poynting effects and the occurrence of secondary flows in an elliptic tube cannot be explained by a linear theory. A natural phenomenon is seldom the result of linearized superposed effects. Any event in Nature is the result of a number of others dovetailing into one another and hence any attempt at the exact formulation of a physical problem essentially produces non-linearity in the field equations.

1.2 Present Status of Elastic-Plastic Transition.

Tresca, as early as in 1868, assumed that there exists a 'mid-zone' between the elastic and plastic regions as against Saint Venant's two zone theory. This idea is embodied in the remarks in Todhunter and Pearson's "History of Elasticity and strength of Materials" [1893],

> "Saint Venant distinguishes in his cylinder only two zones, an elastic and plastic one, but Tresca supplies a mid-zone... . Saint Venant's discussion has the theoretical advantage, but it seems not improbable that physically something corresponding to Tresca's mid-zone has an existence".

Although Tresca's 'mid-zone' theory was ignored by later research workers for the sake of analytical convenience, it has long been felt, however, that such a 'mid-zone' actually does exist! As for example in elasticity, the perfect elasticity is one extreme and the ideal plasticity is another. The response of a majority of materials to applied boundary traction and body forces is in between these extremes and it is physically impossible to draw a sharp line between the elastic and plastic states. It has long been realized that the plastic yielding of an elastic material is an asymptotic behavior and as a consequence there arises a necessity of giving a unified treatment which can describe both the behavior patterns under different physical environments. So recent trends in continuum mechanics exhibit an attempt at a global treatment of the changes taking place in a medium. Several authors seem to approach these transition problems in elasticity, plasticity or in fluid mechanics introducing the idea of 'quick transition' zone and 'nonuniformity'. These problems have been treated by using perturbation techniques which are not always satisfactory.

In 1954, Friedrichs [1955] in his address to 'The American Mathematical Society' explains how asymptotic phenomena occur in physical problems. His treatment of these problems employs perturbation techniques to very small regions called 'quick transition regions' and cannot be used for global distribution phenomena.

A few more attempts have been made in this direction notable of which are by Thomas, Green and Seth.

In the Levy-Mises theory the effect of elastic-strain in the plastic range has not been taken into account. This defect was removed in the Prandtl [1924] and Reuss [1930] theory dealing with the general case. Then Thomas [1954, 1955] extended this theory which treats the case of combined constitutive equations of elastic and plastic flow.

Green [1956] has developed a general theory of work-hardening, incompressible materials as a special case of Truesdell's theory of hypo-elasticity and has shown that a yield condition is implied as an asymptotic approach for infinite values of the strain. Based on this result, it has been suggested that a physical condition of plastic yielding can be predicted on the basis of the theory of hypo-elasticity and that plastic yielding itself is a phenomenon associated with very large strains.

But none of the above authors have recognized transition state as a separate state like that of elasticity or plasticity and hence did not consider the existence of constitutive equation in the transition state.

In a series of papers, Seth [1962-1964] has given an entirely different orientation to this interesting problem of transition. He has developed a new 'Transition theory' of elastic-plastic and creep deformation on sound analytical base.

The classical Theory of Elasticity and Plasticity as has already been mentioned, divides the spectrum of deformation of solids into two distinct states, one in which the deformation is recoverable and the other in which it is not. In current literature, both elastic and plastic field equations are solved separately and later joined together by the so called yield condition. Again in the behavior spectrum of materials perfect elasticity is one extreme and ideal plasticity is another and it is physically impossible to draw a sharp line between these two states. It is therefore natural to expect that any physically realistic theory should include mid-zone or transition state. At present, such problems like elastic-plastic deformation, creep, fatigue, boundary layers, shocks are treated by assuming ad-hoc, semi-empirical laws with the result that discontinuities, singular surfaces, non-differentiable regions have to be introduced over which two successive states of a medium are matched together.

Since transition from one state into another is an asymptotic phenomenon as explained by many authors, Seth has argued that at transition, the differential system governing the physical phenomenon should attain some sort of criticality. Once the 'critical points' or 'Transition points' are recognized, the asymptotic solutions at these 'Transition' points give the solutions corresponding to the 'Transition' states. However all the transition points thus obtained from the differential system may or may not correspond to any transition state. Further, in the case of elastic-plastic deformation, the setting in of plasticity is intimately connected with the geometry of deformation rather than the state of stress at a point. For definiteness, if in an axisymmetrical case, r' and r are the distances of an element from the axis of symmetry before and after deformation respectively, then the elastic property of the material at the point breaks down when the differential stretch $\frac{\partial r'}{\partial r}$ becomes zero or infinity. The material is then said to be in the transition state which is supposed to be a 'mid-zone' between elastic and plastic states. The material attains fully plastic state when it tends to become incompressible, that is when Poisson's ratio approaches $\frac{1}{2}$.

1.3 Objective of the Present Study

In order to explain the elastic-plastic transition, it is first necessary to recognize the transition state as an asymptotic one and in this thesis, it is our main aim to eliminate the need for assuming semi-empirical laws, yield condition, creep-strain laws, jump conditions etc. Further, we consider the effects of a steady state temperature in the elastic-plastic transition. We also obtain the constitutive equation corresponding to the transition state. In this investigation, we find that some of the already known solutions of elastic-plastic problems come out as special cases from our analysis.

One of the most interesting results in this work concerns the identification of the transition state. Seth identified the transition state in which the governing differential system shows some criticality. Purushothama [1965] explained that transition state corresponds to the degeneracy of the reciprocal strain ellipsoid. Later Hulsurkar [1967] indicated that this also can be identified as the vanishing of the Jacobian of transformation. Both Purushothama and Hulsurkar did not give all the analytical details. Here these ideas are carried out in detail. Surprisingly, all the three treatments lead to the same result and the most general form of yield condition is obtained from where all the existing yield conditions in the present literature may be obtained as special cases.

1.4 Plan of the Present Investigation

We have divided our work into two parts. Part I is devoted to the theory of elastic-plastic transition. In Chapter 2, the classical theory of elasticity and plasticity is discussed briefly. A discussion on strain measure in classical elasticity is also included in this chapter. The general treatment of transition in cartesian coordinate system is developed in Chapter 3. Here we find that all classical yield conditions come out as special cases from the most general yield condition which is obtained as a result of transition analysis.

In Part II, we apply the concept of transition to some elastic-plastic and thermo-elastic-plastic deformation problems. In Chapter 4, a shell which is made to yield under pressure is discussed. In Chapter 5, we discuss the elastic-plastic deformation of a tube with finite length under pressure. Chapter 6 and 7 are devoted to shells and tubes which are made to yield under pressure and temperature. Chapter 8 contains the summary, general discussion and scope of further work.

CHAPTER 2

BASIC CONCEPTS OF THE THEORY OF ELASTICITY AND PLASTICITY

2.1 Classical Theory of Elasticity and Plasticity

Consider an elastic solid which is subjected to some small external loads for which the deformation is elastic; that is, upon the release of these loads the body resumes its initial unstressed and undeformed state. The range of stresses and strains for which this is true is known as the elastic range. The elastic deformations for a linearly elastic solid which is homogeneous and isotropic are governed by linear Hooke's law:

$$\tau_{k\ell} = \lambda e_{mm}\delta_{k\ell} + 2\mu e_{k\ell} , \qquad (2.1.1)$$

where λ, μ are Lame's elastic constants, $e_{k\ell}$ is the infinitesimal strain tensor and $\tau_{k\ell}$ is the stress tensor.

Experiments indicate that when the external loads are increased gradually, beyond a certain value of loads, the elastic character of the body is destroyed; that is, upon the release of loads the body will not reassume its original state and so retains some permanent deformations. The critical combination of stresses for which the permanent deformation would first set in may be expressed mathematically by a relation of the form

$$f(\tau_{k\ell}) = 0. \tag{2.1.2}$$

This permanent deformation is known as plastic deformation.

The theory of elastic-perfectly-plastic solids has two regions: (1) the elastic region, for which Hooke's law is valid and (2) the flow region, for which the following constitutive equation

$$\tau_{k\ell} = \lambda d_{mm}\delta_{k\ell} + 2\mu d_{k\ell} \tag{2.1.3}$$

may be used, where $d_{k\ell}$ is strain-rate and λ,μ now may be functions of $d_{k\ell}$ or its invariants. Both of these regions are connected by the so-called yield surface.

For an isotropic and homogeneous elastic material, until we reach the yield surface, the material is assumed to remain homogeneous and isotropic. Therefore the yield condition such as (2.1.2) should be expressible only in terms of stress invariants:

$$f(I_1 , I_2 , I_3) = 0, \tag{2.1.4}$$

I's being the three invariants of the stress tensor τ_{ij}. It is argued that during yielding the material becomes incompressible so that $I_1 = 0$. Moreover I_3 is small, and hence (2.1.4) should be of the form:

$$g(I_2) = \text{constant.} \qquad (2.1.5)$$

There are several yield conditions assumed in the classical theory which may be described in the following three categories:

I. Maximum Stress Theory

According to this theory it is the maximum principal stress in the material that determines plasticity and failure regardless of what the other principal stresses may be. This cannot be a true picture of the facts because (a) if this theory were correct, metals would yield under high hydrostatic pressure, (b) fracture of a test speciman would be expected to occur at right angles to the maximum stress.

II. Maximum Strain Theory (Saint Venant's Theory).

According to this theory, the maximum positive elastic extension of the material in a stressed body determines failure by fracture or by plastic flow. This also is not borne out by experiments.

III. Maximum Shear Theory

Following some results on the extension of metals, Tresca, among others, concluded that failure should occur on those planes in the materials, that are subjected to the greatest shear stresses. This is often found to be the case in practice; the most striking example being the form-ation of Lüder's lines in iron and mild steel.

Tresca's yield condition is $\tau_{11} - \tau_{33} = y$, where τ_{11}, τ_{22}, τ_{33} are principal stresses such that $\tau_{11} \geq \tau_{22} \geq \tau_{33}$, which can alternatively be expressed as

$$\tau_{11}{}^{d} = \frac{2}{3}y , \qquad (2.1.6)$$

$\tau_{11}{}^{d}$ being the deviatoric stress provided $\tau_{22} = \tau_{33}$. Note that Tresca's yield condition is not an invariant relation unless two of the stresses are equal. Later modifications of the shear stress theory are as follows:

1. <u>Mohr's Theory</u> or <u>Guest's Law</u>

This states briefly, that the maximum shear stress determines the beginning of plastic flow independently of the other components of shear stresses; that is, when one of these quantities $\tau_{11} - \tau_{22}$, $\tau_{22} - \tau_{33}$, $\tau_{11} - \tau_{33}$ increases to a certain value y. At the same time, according to Mohr, the shear stress S_s in the planes of slip reaches in the limit, a maximum value dependent on the normal stress S_n acting on the same planes and also on the properties of the material.

2. <u>Hencky von-Mises' Theory</u>

According to this theory, the sum of the squares of the principal stress differences should increase to a certain value before failure begins; that is, plastic strain begins when

$$(\tau_{11} - \tau_{22})^2 + (\tau_{22} - \tau_{33})^2 + (\tau_{33} - \tau_{11})^2 = 2y^2, \tag{2.1.7}$$

where y is yield stress in tension.

If two of the principal stresses, say τ_{22} and τ_{33} are equal, then (2.1.7) reduces to the Tresca's yield condition (2.1.6). Again, if, instead of assuming $\tau_{22} = \tau_{33}$, we take $\tau_{33} = \frac{1}{2}(\tau_{11} + \tau_{22})$, which is known as principal line theory, then (2.1.7) reduces to

$$\tau_{11} - \tau_{33} = \frac{y}{\sqrt{3}}, \tag{2.1.8}$$

which is again of the Tresca's form of yield condition. The relation $\tau_{22} = \tau_{33}$ is known as Haar-Kármán hypothesis.

Now, as far as the plastic range is concerned, two of the most important theories are those due to Lévy-Mises and Prandtl-Reuss Theory. The first of these does not take into account the elastic effects in the plastic range, while the second does. Both of these currently used theories divide the analysis into two separate parts, one for the elastic region and the other for the plastic region. The two regions are then joined together with a yield condition.

The Levy-von Mises equation may be expressed in the form

$$de_{ij} = \tau_{ij}^{d}\, d\lambda , \tag{2.1.9}$$

where de_{ij} is the total strain-increment, $d\lambda$ is a scalar factor of proportionality and $\tau_{ij}{}^{d}$ is the stress deviator. Since Lévy and von-Mises used the total strain-increment, and not the plastic strain-increment, these equations are strictly applicable only to a fictitious material in which the elastic effects are absent.

The constitutive equations of plasticity due to Prandtl-Reuss is

$$de^{p}_{ij} = \tau_{ij}{}^{d}\, d\lambda,$$

where de^{p}_{ij} is incremental plastic strain. Hence, the total strain increment is the sum of the elastic-strain increment and the plastic strain increment. Thus,

$$\begin{aligned} de_{ij} &= de^{p}_{ij} + de_{ij}{}^{e} \\ &= \tau_{ij}{}^{d}\, d\lambda + \frac{d\tau_{ij}{}^{d}}{2G} + \frac{1-2\sigma}{3E}\, \delta_{ij} d\tau_{ii}. \end{aligned} \tag{2.1.10}$$

These are applicable to elastic-plastic materials. For further details we refer to the books "Plasticity" by Hill [1950], "Non-linear Theory of Continuous media" by Eringen [1962], "Theory of Perfectly Plastic Solids" by W. Prager and P.G. Hodge [1951].

2.2. Strain Measure in Classical Elasticity Theory

That the strain-measure in elasticity is not uniquely defined is clear from the existence of a number of strain measures in use; e.g., Cauchy, Swainger, Hencky, Almansi and Green measures. The basic principle of defining a strain measure is to consider the difference of the squares of line elements in deformed and undeformed states of the material. An alternative approach is to define the stretch as the ratio of the line elements in the deformed and the initial configuration, and the strain measure as any function of the stretch, which vanishes when the stretch is unity.

Seth [1962a] has defined the generalized strain measure by the relation

$$\lambda_i = (1-n\,e_i^{\frac{1}{m}})^{-\frac{1}{n}} , \qquad (2.2.1)$$

λ_i , e_i being the principal stretches and principal Almansi strains respectively and m and n are two constants known as the measure index and the irreversibility index respectively. This generalized strain measure when $m = 1$, reduces to the known strain measures due to Cauchy, Green, Hencky, Almansi and Swainger for the values of $n = -1,-2,0,2,1$ respectively. Other strain measures can be obtained by choosing different suitable values of m

and n and then forming a linear combination. In viscoelasticity, Narasimhan and Sra [1968] have introduced generalized measures of the deformation-rate involving velocity gradients but also those of a second deformation-rate involving acceleration gradients and have successfully explained viscoelastic behavior of materials.

2.3 Constitutive Theory of Materials

There are some fundamental axioms essential in the design of a theory of continuous media. They are considered to be self-evident as a result of our long experience with the physical world. The basic principles upon which the theory is constructed are:

1. Conservation of mass,
2. Balance of momentum,
3. Balance of moment of momentum,
4. Conservation of energy,
5. Principle of entropy,
6. Conservation of charge,
7. Faraday's Law of induction,
8. Ampere's Law.

These basic principles are valid for all materials irrespective of their constitution.

When different substances of the same mass and geometry are subjected to the same external agents, the response is generally observed to be different. This is caused mainly by the differences in the constitution of the various substances. Therefore in the description of physical phenomena the constitution of bodies plays an essential and important role. It is, therefore, expected that those basic principles generally are not sufficient to predict uniquely the behavior of all substances under prescribed boundary and initial conditions. In order to take account of the nature of different materials, we must, therefore, find additional equations identifying the basic characteristics of the body with respect to the response sought. These equations are known as constitutive equations which characterize the constitution of the medium under consideration. In the theory of continuous media this is done by introducing models appropriate to the particular class of phenomena under scrutiny. There exists certain conditions and invariance requirements [Eringen, 1962] which should be satisfied by all such models.

The present trend of explaining experimental results relating to irreversible phenomena such as elastic-plastic transition, creep, fatigue, etc., consists in assuming complicated constitutive equations, yield conditions, creep strain laws, etc. The resulting constitutive equations are found to involve many unknown response coefficients. The main source of this trouble is the use of a linear or a classical measure of strain, even though the strains occuring in the experiments are non-linear in character. Thus, the order of the strain measure used is not fixed with the result that the constitutive equations involve unknown response coefficients and thus have become very complicated. In order to eliminate the need for assuming unknown response coefficients, yield conditions etc., Seth introduced the generalized measure and transition concepts.

It has already been explained that constitutive equations for different media should be different. From a continuum view point, constitution should be understood in a macroscopic sense. Hence the constitutive equations for the elastic state of a solid should be basically different from that of the plastic state. The fundamental structure of the transition state undergoes a change as a result of two states, elastic and plastic, dovetailing into each other. As a consequence of the above fact, the constitutive equations for the transition state should be different from that of elastic and plastic states.

CHAPTER 3

GENERAL TREATMENT OF TRANSITION IN ELASTIC-PLASTIC DEFORMATION

3.1 Preliminary Remarks

In the previous chapters we have explained how transition occurs in Nature as an asymptotic phenomenon. In this chapter, our objective is to show how the transition state can be identified. Both geometrical and analytical aspects concerning the transition phenomena will be discussed in detail, as identification of the transition state is basically important. Also, in this chapter we shall show that transition fields are sub-harmonic (super-harmonic) fields.

3.2 Identification of the Transition State

When a material at a point has yielded, it is more reasonable to expect that the material at the neighboring points are on their way to yield, rather than assume that they remain in the elastic state as completely opposed to the plastic state of the nearby material. As the plastic yielding of a material is a consequence of collapse of its internal or macroscopic structure, the plastic yielding will be complete or partial depending on the existing physical conditions. This leads us again to the recognition of two material states: a transition state and a plastic state.

There are three different ways to explain how transition may occur from a state A to another state B:

(1) At transition, the differential system defining the elastic state A should attain some criticality.

(2) The complete breakdown of the macroscopic structure at transition should correspond to the degeneracy of the material (spatial) strain ellipsoid. This means that the length of at least one of the axes of the strain ellipsoid should be zero or infinity.

(3) If we consider the plastic state B as an image of the elastic state A, under the transformation

$$x^k = x^k(X^K),$$

then at transition, the Jacobian of the transformation whould be zero or infinity. This means when transition occurs, one to one correspondence between A and B no longer holds.

In the next sections, we discuss the above-mentioned modes of transition starting from the case (3) namely the vanishing of the Jacobian of transformation which is the more general one.

3.3 General Treatment of Transition Theory Corresponding to the Vanishing of the Jacobian of Transformation

Consider the transformation

$$x^k = x^k(X^K),$$

which maps the metric space A into metric space B. If we identify B as the plastic state and A as the elastic state, then the isomorphism of the transformation is presumably destroyed, since this process is irreversible and the material may change from elastic to the plastic, creep or fatigue state. Hence the Jacobian of transformation is bound to behave singularly. The vanishing of the Jacobian will, therefore, correspond to the transition state from A to B.

When a continuum changes from a state A to another state B, as has been explained before, the invariants of the stress and strain tensors undergo some kind of a constraint. This constraint should be obtainable from the condition, namely the vanishing of the Jacobian of transformation, since the latter corresponds to the transition state.

If u, v and w are the displacements along the rectangular cartesian coordinate axes, then

$$X = x - u, \quad Y = y - v, \quad Z = z - w,$$

(X,Y,Z), (x,y,z) being the coordinates of a point in the undeformed and deformed state respectively. Hence the Jacobian

$$J = \begin{vmatrix} 1 - \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} & -\frac{\partial w}{\partial x} \\ -\frac{\partial u}{\partial y} & 1 - \frac{\partial v}{\partial y} & -\frac{\partial w}{\partial y} \\ -\frac{\partial u}{\partial z} & -\frac{\partial v}{\partial z} & 1 - \frac{\partial w}{\partial z} \end{vmatrix}, \qquad (3.3.1)$$

referred to the deformed state and

$$J = \begin{vmatrix} 1 + \frac{\partial u}{\partial X} & \frac{\partial v}{\partial X} & \frac{\partial w}{\partial X} \\ \frac{\partial u}{\partial Y} & 1 + \frac{\partial v}{\partial Y} & \frac{\partial w}{\partial Y} \\ \frac{\partial u}{\partial Z} & \frac{\partial v}{\partial Z} & 1 + \frac{\partial w}{\partial Z} \end{vmatrix}, \qquad (3.3.2)$$

referred to the undeformed state.

From (3.3.1) we have

$$J^2 = \begin{vmatrix} 1 - 2e_{xx} & -2e_{xy} & -2e_{xz} \\ -2e_{yx} & 1 - 2e_{yy} & -2e_{yz} \\ -2e_{zx} & -2e_{zy} & 1 - 2e_{zz} \end{vmatrix}, \qquad (3.3.3)$$

where $1 - 2e_{xx} = -2\frac{\partial u}{\partial x} + (\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial x})^2 + (\frac{\partial w}{\partial x})^2$,

$$-2e_{xy} = -(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}$$

$$+ \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}, \quad \text{etc}, .$$

From (3.3.3), if $J = 0$ which is a transition condition corresponding to asymptotically large extensions, we have

$$1 - 2J_1' + 4J_2' - 8J_3' = 0, \tag{3.3.4}$$

where

$$J_1' = \frac{1}{1!}\,\delta^{k}_{\ell}e^{\ell}_{k},$$

$$J_2' = \frac{1}{2!}\,\delta^{k\,m}_{\ell\,n}e^{\ell}_{k}e^{n}_{m},$$

$$J_3' = \frac{1}{3!}\,\delta^{k\,m\,p}_{\ell\,n\,q}e^{\ell}_{k}e^{n}_{m}e^{q}_{p} = \det e^{k}_{\ell}.$$

The symbols $\delta^{k\,m}_{\ell\,n}$ and $\delta^{k\,m\,p}_{\ell\,n\,q}$ are generalized Kronecker deltas and are defined as

$$\delta^{k\,m\,p\cdots}_{\ell\,n\,q\cdots} = \begin{cases} 1(-1), & \text{when subscripts are distinct numbers taken from } 1,2,3,\ldots \text{ and the superscripts can be brought to the same sequence of the subscripts by an even (odd) permutation;} \\ 0 & \text{otherwise.} \end{cases}$$

Referring to the principal axes of the strain ellipsoid in the deformed state, we have from (3.3.4)

$$8J_3 - 4J_2 + 2J_1 = 1, \qquad (3.3.5)$$

where

$$J_1 = e_{11} + e_{22} + e_{33} ,$$
$$J_2 = e_{11}e_{22} + e_{22}e_{33} + e_{11}e_{33} , \qquad (3.3.5a)$$
$$J_3 = e_{11}e_{22}e_{33} ,$$

e_{11} , e_{22} and e_{33} being the principal strains. Relations similar to (3.3.5) could also be obtained from (3.3.2).

The constraint (3.3.5) should hold at transition irrespective of any type of medium, isotropic, anisotropic, homogeneous or heterogeneous and should come out from the transition condition independent of the constitutive equations and the momentum equations. The strain (stress) invariants are, in general, independent of each other; a functional relation exists between them only at the transition state because of the constraint.

Now from the constitutive equations

$$\tau_{ij} = \lambda\delta_{ij}\, e_{\alpha\alpha} + 2\mu e_{ij}$$

we have

$$e_{11} = \frac{1}{E}[\tau_{11} - \sigma(\tau_{22} + \tau_{33})],$$

$$e_{22} = \frac{1}{E}[\tau_{22} - \sigma(\tau_{11} + \tau_{33})],$$

$$e_{33} = \frac{1}{E}[\tau_{33} - \sigma(\tau_{11} + \tau_{22})],$$

where e_{ii} are principal strains.

Hence

$$J_1 = \frac{1 - 2\sigma}{E} I_1 \ ,$$

$$J_2 = \frac{1}{E^2}[(1 + \sigma)^2 I_2 - \sigma(2 - \sigma)I_1^2],$$

$$J_3 = \frac{1}{E^3}[(1 + \sigma)^3 I_3 - \sigma(1 + \sigma)^2 I_1 I_2 + \sigma^2 I_1^3],$$

where the I_k's are stress invariants and J_k's are strain invariants, E is Young's modulus of elasticity, and σ is Poisson's ratio.

Now from (3.3.5) we have

$$\frac{8}{E^3}[(1 + \sigma)^3 I_3 - \sigma(1 + \sigma)^2 I_1 I_2 + \sigma^2 I_1^3]$$
$$- \frac{4}{E^2}[(1 + \sigma)^2 I_2 - \sigma(2 - \sigma)I_1^2]$$
$$+ \frac{2}{E}(1 - 2\sigma)I_1 = 1. \qquad (3.3.6)$$

The relation (3.3.6) should hold at transition. Now (3.3.6) can be rewritten in a simpler form using the following notations:

$$I_2' = (\tau_{11} - \tau_{22})^2 + (\tau_{22} - \tau_{33})^2 + (\tau_{33} - \tau_{11})^2$$

$$= 2I_1 - 6I_2 \; ,$$

$$T_{ii} = \tau_{ii}/E, \quad I_1/E = K_1 \; , \quad I_2/E^2 = K_2 \; , \quad I_3/E^2 = K_3 \; ,$$

and $\quad K_2' = I_2'/E^2.$

Then (3.3.6) becomes

$$8[(1 + \sigma)^3 K_3 + \frac{1}{6}\sigma(1 + \sigma)^2 K_1 K_2' - \frac{1}{3}\sigma(\sigma^2 - \sigma + 1)K_1^3$$
$$+ \frac{4}{3}[\frac{1}{2}(1 + \sigma)^2 K_2' - (1 - 2\sigma)^2 K_1^2] + 2(1 - 2\sigma)K_1 = 1. \tag{3.3.7}$$

The invariant relation (3.3.7) among the stress invariants should hold good at transition state. This condition involves elastic effects. For the fully plastic state $\sigma \to \frac{1}{2}$ (condition of incompressibility) we have from (3.3.7)

$$3K_2' + 2(27K_3 + \frac{3}{2}K_1K_2' - K_1^3) = 2. \tag{3.3.8}$$

Equation (3.3.8) could be taken as the most general yield condition for all types of media irrespective of their properties.

Rewriting (3.3.8) again we get

$$3[(T_{11}-T_{22})^2+(T_{22}-T_{33})^2+(T_{11}-T_{33})^2]$$
$$+2[(2T_{11}-T_{22}-T_{33})(2T_{22}-T_{11}-T_{33})(2T_{33}-T_{11}-T_{22})]=2. \tag{3.3.9}$$

The general form of the yield condition (3.3.9) given above has been obtained independent of the equations of equilibrium. Equation (3.3.9) can alternately be written as follows:

$$L_1^2+L_2^2+L_3^2+2L_1L_2L_3=2, \tag{3.3.10}$$

$$(T_{11}{}^d)^2+(T_{22}{}^d)^2+(T_{33}{}^d)^2+6T_{11}{}^dT_{22}{}^dT_{33}{}^d=\frac{2}{9}, \tag{3.3.11}$$

where $L_1=(2T_{11}-T_{22}-T_{33})$, etc., and T_{ii}^d are the deviatoric stress tensors in the non-dimensional form.

It is interesting to note that (3.3.9) reduces to Hencky-von Mises yield condition or Tresca's yield condition in some of the following special cases:

I. Principal Line Theory

For the principal line hypothesis one of the L's vanishes and (3.3.10) reduces to Tresca's type:

$$T_{11} - T_{33} = \text{a constant, or } T_{11}{}^{d} = \text{constant,} \tag{3.3.12}$$

$$\tau_{11} - \tau_{33} = \frac{4}{3}y, \qquad E = 2y,$$

where y is yield stress in tension.

II. <u>Haar-Kármán Hypothesis</u>

In the Haar-Kármán hypothesis two of the stresses become equal and again (3.3.10) reduces to the type (3.3.12). Now from (3.3.9) if $T_{11} = T_{22}$, we get

$$3(T_{11} - T_{33})^2 - 2(T_{11} - T_{33})^3 = 1.$$

Solving the above equation we obtain

$$T_{11} - T_{33} = 1, 1, -\frac{1}{2}.$$

Hence we have corresponding to the roots,

(i) $\tau_{33} - \tau_{11} = -2y,$

(ii) $\tau_{33} - \tau_{11} = y, \quad \tau_{11} \leq \tau_{22} \leq \tau_{33}.$

Both (i) and (ii) are yield conditions of Tresca's type. While Tresca's yield condition or von-Mises yield condition does not distinguish yielding in tension and in compression, we have from our analysis (i) representing yield condition in compression and (ii) representing yield condition in tension.

III. Plane Strain

In this case $e_{33} = 0$, so that $J_3 = 0$, and we get from (3.3.5) that

$$4J_2 - 2J_1 + 1 = 0,$$

and in the limiting incompressible case $(\sigma \to \frac{1}{2})$, we get from (3.3.8)

$$K_2' = \frac{2}{3},$$

or

$$(\tau_{11} - \tau_{22})^2 + (\tau_{22} - \tau_{33})^2 + (\tau_{33} - \tau_{11})^2 = \frac{8}{3}y^2, \tag{3.3.13}$$

which is exactly of the same form as von-Mises yield criterion:

$$(\tau_{11} - \tau_{22})^2 + (\tau_{22} - \tau_{33})^2 + (\tau_{33} - \tau_{11})^2 = 2y^2. \tag{3.3.14}$$

We notice here that the constant $2y^2$ is replaced by $\frac{8}{3}y^2$. This is to be expected, as (3.3.14) is only a particular form of (3.3.8).

IV. Longitudinal Strain

If $T_{22} = 0$, and $T_{33} = 0$, then we get from (3.3.9)

$$3T_{11}^{2} + 2T_{11}^{3} = 1.$$

This equation on solving yields the roots for T_{11} as

$$T_{11} = -1, -1, \tfrac{1}{2}.$$

Hence $\tau_{11} = -2y$ and $\tau_{11} = y$, which are yield stresses in compression and tension respectively.

V. Plane Stress

In this case, suppose $T_{33} = 0$ and the yield condition (3.3.8) will reduce to

$$(T_{11} + T_{22} + 1)(2T_{11} - T_{22} - 1)(2T_{22} - T_{11} - 1) = 0. \tag{3.3.15}$$

This yields the following three conditions:

$$T_{11} + T_{22} + 1 = 0, \quad \text{or} \quad T_{\alpha\alpha} = -1, \quad \text{or} \quad \tau_{\alpha\alpha} = -2y,$$

$$2T_{11} - T_{22} - 1 = 0, \quad \text{or} \quad T_{11}^{d} = \tfrac{1}{3}, \quad \text{or} \quad \tau_{11}^{d} = \tfrac{2}{3}y,$$

and

$$2T_{22} - T_{11} - 1 = 0, \quad \text{or} \quad T_{22}^{d} = \tfrac{1}{3}, \quad \text{or} \quad \tau_{22}^{d} = \tfrac{2}{3}y.$$

The last two conditions are exactly Tresca's yield condition (2.2.7). It is interesting to note that in a number of these cases the yield takes place through a deviatoric principal stress taking on a maximum value.

It may be noted that it is only in the case of plane strain (3.3.5) reduces exactly to the Hencky-von-Mises form, the constant now being $\frac{8}{3}y^2$ in place of the classical value $2y^2$. Also Tresca's yield condition is not an invariant relation, unless two of the principal stresses become equal; in that case Tresca's yield condition is a special case of the von-Mises condition.

The yield condition (3.3.9), which is an invariant relation, could be regarded as general yield condition for all types of materials, irrespective of their properties. While von-Mises or Tresca's yield condition does not take into account of the distinction between the yield stress in tension and yield stress in compression, (3.3.9) does, and hence includes Bauschinger's effect.

It may be noted here that all the yield conditions existing in the present literature so far are considered to be self-evident as a result of long experimental consequences. None of the authors in this field thus far has ever tried to establish them on the basis of analytical considerations. This is the first time such an attempt has been made.

3.4 Transition Theory Corresponding to the Degeneracy of the Material or Spatial Strain Ellipsoid

Consider any medium, isotropic or anisotropic, homogeneous or heterogeneous. Suppose X^K and x^k are the coordinates of a point before and after the deformation, given by the transformation

$$x^k = x^k(X^K), \quad k,K = 1,2,3.$$

If G_{ij} , g_{ij} are the fundamental metric tensors in the two coordinate systems, the material strain ellipsoid will be

$$dS^2 = c_{k\ell}\, dx^k dx^\ell, \tag{3.4.1}$$

where Cauchy's deformation tensor $c_{k\ell}$ is given by

$$c_{k\ell} = G_{KL}(X) X^K_{,k}\, X^L_{,\ell}\ , \tag{3.4.2}$$

and

$$c_{k\ell} = 1 - 2e_{k\ell}\quad ,$$

e_{ij} being the finite Almansi measure referred to the current (strained) state. Referring to the principal axes of the strain ellipsoid (3.4.1), we have

$$dS^2 = c_{ii}\, dy^i\, dy^i, \quad i = 1,2,3, \tag{3.4.3}$$

where c_{ii} are the three principal values of the deformation $||c_{k\ell}||$. Also c_{ii} corresponds to the reciprocal of the square of the axes of the material strain ellipsoid. Similarly we can derive the spatial strain ellipsoid (reciprocal strain ellipsoid) which is dual to the material strain ellipsoid.

Now transition will occur when at least one of the c_{ii} tends to zero or infinity. In other words, the ellipsoid tends to become a cylinder, infinite sphere or a pair of planes. This geometrical approach to the transition state does not make use of any of the constitutive equations or dynamical equations of equilibrium.

Let us suppose that $c_{11} \to 0$ at a transition so that if e_{ii} denotes the principal strains and J_1, J_2 and J_3 are the strain invariants, we have from (3.3.5a) after eliminating e_{22} and e_{33}, since $e_{11} = \frac{1}{2}$

$$8J_3 - 4J_2 + 2J_1 = 1. \qquad (3.4.4)$$

This is the same constraint as in (3.3.5), and we may carry out the same analysis as in the previous section. For another type of transition given by $e_{11} = \frac{1}{2}$, $e_{22} = \frac{1}{2}$ we have

$$\tau_{11} = \tau_{22}.$$

Then we get results similar to Haar-Kármán hypothesis. If all the three e_{11}, e_{22} and e_{33} approach $\frac{1}{2}$, we get hydrostatic pressure and (3.4.4) is always satisfied. The case $c_{11} \to \infty$ only amounts to making e_{22} and e_{33} approach $\frac{1}{2}$.

It will be clear later that in the transition treatment of elastic-plastic problems no yield condition needs to be used. The above discussion has been given only to show that the yield condition, when it exists, should come out of the fundamental equations.

Thus we have shown that a transition may occur if at least one of the axes of the strain ellipsoid becomes zero or infinity. It would be interesting to see that the degeneracy of the strain ellipsoid again is intimately connected with the criticality of one or more of the functions X^r, $r = 1,2,3$; where

$$u^r = x^r - X^r, \quad r = 1,2,3;$$

u^r being the displacement vectors parallel to the coordinate axes and x^r, X^r being the coordinates of a point in the deformed and undeformed state respectively.

Suppose X^1 has an extremal value then $X^1_{,i} = 0$. Now from the geometry of the problem we have

$$dS^2 = c_{k\ell}\, dx^x\, dx^\ell,$$

where

$$c_{k\ell} = G_{KL} X^K_{,x} X^L_{,\ell}.$$

Also

$$2e_{ij} = \delta_{ij} - X^{\alpha}_{,i} X^{\alpha}_{,j} ,$$

$$= \delta_{ij} - c_{ij} , \qquad (3.4.5)$$

and

$$c_k = \frac{\partial P}{\partial X^K} \frac{\partial X^K}{\partial x^k} ,$$

p being the spatial position vector. In order to refer the strains to the principal axes of the strain ellipsoid, we must have

$$e_{ij} = 0, \qquad i \neq j.$$

The relation (3.4.5) may also be rewritten as

$$2e_{ij} = \delta_{ij} - (a_{ik} X^r_{,k})(a_{jk} X^r_{,k}), \qquad (3.4.6)$$

where

$$a_{ik} a_{jk} = \delta_{ij} ,$$

and

$$a_{ik} a_{jk} = a_{jk} a_{ik} , \qquad i,j,k = 1,2,3.$$

The a_{ik} 's are the direction cosines of the three principal axes of the strain ellipsoid. We want to find the three principal strains e_{11} , e_{22} and e_{33} such that $e_{ij} = 0$ for $i \neq j$. Now

$$2e_{12} = -(a_{11}X^1_{,x}+a_{12}X^1_{,y}+a_{13}X^1_{,z})(a_{21}X^1_{,x}+a_{22}X^1_{,y}+a_{23}X^1_{,z})$$
$$-(a_{11}X^2_{,x}+a_{12}X^2_{,y}+a_{13}X^2_{,z})(a_{21}X^2_{,x}+a_{22}X^2_{,y}+a_{23}X^2_{,z})$$
$$-(a_{11}X^3_{,x}+a_{12}X^3_{,y}+a_{13}X^3_{,z})(a_{21}X^3_{,x}+a_{22}X^3_{,y}+a_{23}X^3_{,z}).$$

Similarly we can obtain expression for $2e_{13}$ and $2e_{23}$. Also

$$2e_{11} = 1 -(a_{11}X^1_{,x}+a_{12}X^1_{,y}+a_{13}X^1_{,z})^2$$
$$-(a_{11}X^2_{,x}+a_{12}X^2_{,y}+a_{13}X^2_{,z})^2$$
$$-(a_{11}X^3_{,x}+a_{12}X^3_{,y}+a_{13}X^3_{,z})^2.$$

We notice from the above expression that all the shear strains vanish if

$$X^1_{,i} = 0; \quad i = 1,2,3 \quad \text{and} \quad X^1 = X, X^2 = Y, X^3 = Z. \tag{3.4.7}$$

$$a_{21}X^2_{,x} + a_{22}X^2_{,y} + a_{23}X^2_{,z} = 0, \tag{3.4.8a}$$

$$a_{11}X^2_{,x} + a_{12}X^2_{,y} + a_{13}X^2_{,z} = 0, \tag{3.4.8b}$$

$$a_{11}X^3_{,x} + a_{12}X^3_{,y} + a_{13}X^3_{,z} = 0, \tag{3.4.8c}$$

$$a_{31}X^3_{,x} + a_{32}X^3_{,y} + a_{33}X^3_{,z} = 0. \tag{3.4.8d}$$

These equations (3.4.7) and (3.4.8) are necessary and sufficient to make $e_{ij} = 0$ for $i \neq j$ and $e_{11} = \frac{1}{2}$. Thus, we obtain the following principal strains:

$$e_{11} = \frac{1}{2}$$

$$2e_{22} = 1 - (a_{21}X^3_{,x} + a_{22}X^3_{,y} + a_{23}X^3_{,z})^2$$

$$= 1 - X^3_{,i}\, X^3_{,i} .$$

The second line on the right hand side follows from the first on using the sum of the squares of (3.4.8c) and (3.4.8d). A similar treatment gives

$$2e_{33} = 1 - X^2_{,i}\, X^2_{,i} .$$

Hence we have shown that criticality of any one of the functions X^r $(r = 1,2,3)$ is intimately connected with the degeneracy of the strain ellipsoid. A similar discussion can be advanced using the criticality of X^2 and X^3. If all the three functions become critical, then the strain ellipsoid will just reduce to a point.

3.5 Multiple Transition Points

When a deformable solid is subjected to an external loading system it has been observed that the solid first deforms elastically. If the loading is continued plastic flow may set in and if continued further, it gives rise to time dependent continuous deformation known as creep deformation. The elastic-plastic and creep state represent the transition state from the elastic state, as we have seen and can be expected to occur at the critical points of the field equations.

It may be possible that a number of transition states may occur at the same critical point; then the transition function will have different asymptotic values, and the point will be a multiple one, each branch of which will then correspond to a different state.

In general the material from elastic state can go over into (1) plastic state, (2) or to creep state, (3) or first to plastic state and then to creep and vice-versa, depending on the loading.

3.6 Subharmonicity (Superharmonicity) of Transition Fields

Any transition in harmonic or biharmonic fields, which permeate natural phenomena, may be expected to exhibit itself in terms of the allied subharmonic or superharmonic fields. The latter are non-linear and nonconservative and their studies can be based on a few characteristic properties. Since we are interested in this work only in elastic plastic transition, this section will be devoted to showing that the elastic-plastic transition field is a subharmonic (superharmonic) field.

For simplicity, we are using here a rectangular cartesian frame of reference. The deformation field is given by

$$u^r = x^r - X^r, \quad (r = 1,2,3), \tag{3.6.1}$$

where x^r and X^r being the deformed and undeformed coordinates of a point respectively. If e_{ij} is the strain tensor, then using Almansi strain measure, we have

$$\begin{aligned} 2e_{ij} &= U_{i;j} + u_{j;i} - U_{\alpha;j}\, u^{\alpha}_{;i} \\ &= \delta_{ij} - X^{\alpha}_{,i}\, X^{\alpha}_{,j}\ , \end{aligned}$$

the covariant differentiation reduces to ordinary partial differentiation owing to the choice of rectangular cartesian coordinate system. The stress strain relation

$$\tau_{ij} = \lambda e_{\alpha\alpha}\, \delta_{ij} + 2\mu e_{ij}$$

becomes

$$\tau_{ij} = \lambda e_{\alpha\alpha}\, \delta_{ij} + \mu[\delta_{ij} - X^{\alpha}_{,i}\, X^{\alpha}_{,j}], \tag{3.6.2}$$

where

$$2e_{\alpha\alpha} = 3 - X^{\alpha}_{,i} X^{\alpha}_{,i} \; ; \quad i,j = 1,2,3. \qquad (3.6.2a)$$

Now in the absence of the body forces and if only a steady state deformation is assumed, then the equilibrium equations may be written as

$$\tau_{ij,j} = 0. \qquad (3.6.3)$$

Hence using equations (3.6.2) in (3.6.3) we obtain the following three equations:

$$\begin{aligned}\frac{(\lambda + 2\mu)}{\mu} \frac{\partial}{\partial x}(e_{\alpha\alpha}) &= J_{xy}(X^1; X^1_{,y}) + J_{xz}(X^1; X^1_{,z}) \\ &+ J_{xy}(X^2; X^2_{,y}) + J_{xz}(X^2; X^2_{,z}) \\ &+ J_{xy}(X^3; X^3_{,y}) + J_{xz}(X^3; X^3_{,z})\end{aligned} \qquad (3.6.4)$$

$$\begin{aligned}\frac{\lambda + 2\mu}{\mu} \frac{\partial}{\partial y}(e_{\alpha\alpha}) &= J_{yx}(X^1; X^1_{,x}) + J_{yz}(X^1; X^1_{,z}) \\ &+ J_{yx}(X^2; X^2_{,x}) + J_{yz}(X^2; X^2_{,z}) \\ &+ J_{yx}(X^3; X^3_{,x}) + J_{yz}(X^3; X^3_{,z})\end{aligned} \qquad (3.6.5)$$

and

$$\frac{\lambda + 2\mu}{\mu} \frac{\partial}{\partial z}(e_{\alpha\alpha}) = J_{zx}(X^1;\ X^1_{,x}) + J_{zy}(X^1;\ X^1_{,y})$$

$$+ J_{zx}(X^2;\ X^2_{,x}) + J_{zy}(X^2;\ X^2_{,y})$$

$$+ J_{zx}(X^3;\ X^3_{,x}) + J_{zy}(X^3;\ X^3_{,y}) .$$

(3.6.6)

The above three equations may be rewritten as

$$e_{\alpha\alpha,i} = \tfrac{1}{2}cJ_{ij}(X^r;\ X^r_{,j}), \qquad (3.6.7)$$

$$J_{ii} = 0 \text{ (no sum on } i),$$

where

$$c = \frac{2\mu}{\lambda + 2\mu} = \frac{1 - 2\sigma}{1 - \sigma}, \quad r,i,j = 1,2,3,$$

and

$$J_{xy}(X^1;\ X^1_{,y}) = \begin{vmatrix} \frac{\partial X^1}{\partial x} & \frac{\partial^2 X^1}{\partial x \partial y} \\ \frac{\partial X^1}{\partial y} & \frac{\partial^2 X^1}{\partial y^2} \end{vmatrix}, \text{ etc.}$$

Now differentiating (3.6.4) with respect to y and (3.6.5) with respect to x and then subtracting we get

$$J_{xy}[X^1;\ \nabla^2 X^1] + J_{xy}[X^2;\ \nabla^2 X^2]$$

$$+ J_{xy}[X^3;\ \nabla^2 X^3] = 0. \qquad (3.6.8)$$

A similar treatment will give

$$J_{yz}[X^1;\ \nabla^2 X^1] + J_{yz}[X^2;\ \nabla^2 X^2]$$

$$+ J_{yz}[X^3;\ \nabla^2 X^3] = 0, \qquad (3.6.9)$$

and

$$J_{zx}[X^1;\ \nabla^2 X^1] + J_{zx}[X^2;\ \nabla^2 X^2]$$

$$+ J_{zx}[X^3;\ \nabla^2 X^3] = 0. \qquad (3.6.10)$$

The equations (3.6.8), (3.6.9) and (3.6.10) could further be written as:

$$J_{ij}[X^r;\ \nabla^2 X^r] = 0, \qquad (3.6.11)$$

$$J_{ii} = 0, \quad (\text{no sum on } i),$$

$$r, i, j = 1,2,3.$$

The general solution T of (3.6.11), when it exists, may be written in the form

$$\nabla^2 X^r = \frac{\partial T(X^r)}{\partial X^r} , \qquad (3.6.12)$$

which may be verified to satisfy the equation (3.6.11).

Here T may be any function of X^r; in particular

$$T = F_1(X^1) + F_2(X^2) + F_3(X^3),$$

F_1 , F_2 , F_3 being continuous functions of their arguments.

The equation (3.6.12) shows the subharmonicity (superharmonicity) of the transition field.

The equations (3.6.7) can be integrated with the help of equations (3.6.12) and thus we obtain

$$e_{\alpha\alpha} = \frac{1}{2}c[T - \frac{1}{2}(X^r_{,i} \; X^r_{,i})], \qquad (3.6.13)$$

$$i,r = 1,2,3.$$

From (3.6.2a) and (3.6.13) we get the dilatation as

$$e_{\alpha\alpha} = \frac{c}{2 - c}[T - \frac{3}{2}]. \qquad (3.6.14)$$

For the fully plastic state $c \to 0$ and hence $e_{\alpha\alpha} \to 0$, which is a condition of incompressibility. It is generally assumed that for the fully plastic state, the medium is incompressible which is also borne out by experiments. But here, this result comes out from the transition analysis.

PART II

APPLICATION OF THE TRANSITION THEORY TO CERTAIN ELASTIC-PLASTIC AND THERMO-ELASTIC-PLASTIC DEFORMATION PROBLEMS

CHAPTER 4

ELASTIC-PLASTIC TRANSITION OF SHELLS UNDER UNIFORM PRESSURE

4.1 Preliminary Remarks

This chapter is devoted to the study of elastic-plastic transitions in shells under uniform pressure.

The solutions corresponding to the plastic state for shells under pressure were originally given by Reuss [1930], which were improved by Hill [1949].

Seth [1963] has obtained the stresses and strains in the plastic state using his new theory of transition. Later Hulsurkar [1966] has obtained the solutions for shell and tube under uniform pressure using Seth's generalized measure concept and has extended his solutions to creep deformation using Seth's transition theory. Purushothama [1965] has solved the problem of plastic bending of rectangular metal sheet into circular cylinder by using Seth's [1963] transition concept.

The object of this chapter and the next is to show that not only the stresses and strains may be obtained in the transition and plastic states but also the constitutive equation could be obtained in the transition state. We have shown in the first part of this thesis that with any deformable medium under an external loading system, we associate three states: (1) Elastic, (2) Transition, and

(3) Plastic (or creep). We shall show that there does exist a constitutive equation governing the transition state and it may be obtained from the elastic state with an asymptotic approach. The constitutive equation for the plastic state will follow in a similar manner from the transition state.

The onset of the plastic state is derived without using any of the semi-empirical yield conditions. As the deformation of the medium proceeds, the strain changes from linear to non-linear in character. The classical elastic-plastic model involves a yield condition at the point A, as shown in the Figure 1, joining the elastic and plastic states. It does not take into consideration the non-linear part AB (Fig. 2) through which the transition takes place.

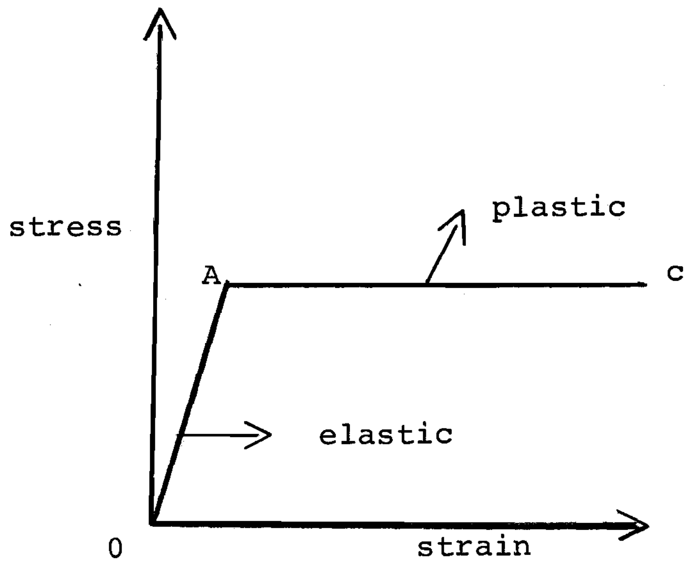

Figure 1

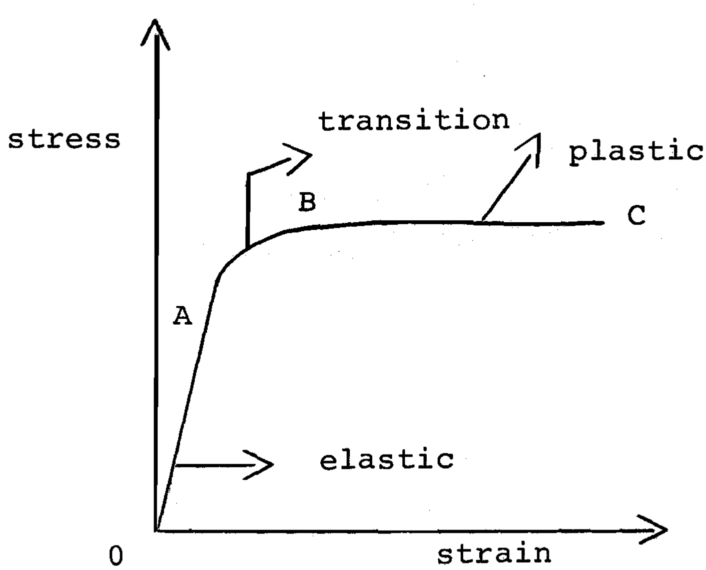

Figure 2

Stress-Strain Diagram.

4.2 Basic Equations

The equations of equilibrium for a continuous medium subjected to an external loading system are

$$\tau^{jk}_{;j} = f^k, \tag{4.2.1}$$

$$m^{ii}_{;j} + \rho \ell^i + \varepsilon^{ijk}\tau_{jk} = 0, \tag{4.2.2}$$

where τ_{ij} are stress tensors, f_k is body force, ρ is density of the medium, m_{ij} are surface couple stresses, ℓ_i is body moment and

$$\varepsilon^{k\ell m} = \frac{e^{k\ell m}}{\sqrt{g}},$$

$$g \equiv \det g_{k\ell},$$

$$e^{k\ell m} = \begin{cases} 1 & \text{when } k\ell m \text{ is an even permutation of } 123, \\ -1 & \text{when } k\ell m \text{ is an odd permutation of } 123, \\ 0 & \text{otherwise.} \end{cases}$$

In the absence of body forces the equation (4.2.1) reduces to

$$\tau^{jk}_{;j} = 0. \tag{4.2.3}$$

When there are no body couples and couple stresses (4.2.2) becomes

$$\tau_{ij} = \tau_{ji} \ . \tag{4.2.4}$$

These equations hold good for any continuum isotropic, anisotropic, homogeneous or heterogeneous.

The constitutive equations given by Hooke's stress-strain relation is

$$\tau_{ij} = \lambda e_{\alpha\alpha}\delta_{ij} + 2\mu e_{ij} \ , \tag{4.2.5}$$

where e_{ij} is the Almansi strain tensor and λ,μ are Lame's constants.

If u_i is the deformation, the six components of e_{ij} are functions of gradient of u_i. Both stress and strain should be referred to the strained framework and not to the unstrained one. For the present discussion the unstrained framework is of no consequence. The Almansi strain measure which will be used here is given by

$$2\varepsilon_{ij} = g_{ij} - G_{ij} + G_{\alpha j}u^{\alpha}_{;i} + G_{i\beta}u^{\beta}_{;j} - G_{\alpha\beta}u^{\alpha}_{;i}u^{\beta}_{;j} \tag{4.2.6}$$

where

$$e_{ij} = \sqrt{g^{ii}g^{jj}}\ \varepsilon_{ij} \ , \quad e_{ij} \ \text{are}$$

physical components, g_{ij} and G_{ij} are metric tensors corresponding to the strained and unstrained states respectively. The equations (4.2.3), (4.2.5) and (4.2.6) are sufficient to determine the fifteen unknowns.

For our future reference the strain components both in cylindrical and spherical coordinate systems are tabulated below. In cylindrical coordinate system:

$$2e_{rr} = 1 -(1 - \frac{\partial u}{\partial r})^2 - r^2(\frac{\partial v}{\partial r})^2(1 - \frac{u}{r})^2 - (\frac{\partial w}{\partial r})^2,$$

$$2e_{\theta\theta} = 1 -(1 - \frac{\partial v}{\partial \theta})^2(1 - \frac{u}{r})^2 - \frac{1}{r^2}[(\frac{\partial u}{\partial \theta})^2 + (\frac{\partial w}{\partial \theta})^2],$$

$$2e_{zz} = 1 -(1 - (\frac{\partial w}{\partial z})^2 - r^2(\frac{\partial v}{\partial z})^2(1 - \frac{u}{r})^2 - (\frac{\partial u}{\partial z})^2,$$

$$2e_{rz} = \frac{\partial u}{\partial z}(1 - \frac{\partial u}{\partial r}) + \frac{\partial w}{\partial r}(1 - \frac{\partial w}{\partial z})^2 - r^2 \frac{\partial v}{\partial r}\frac{\partial v}{\partial z}(1 - \frac{u}{r})^2,$$

$$2e_{\theta z} = \frac{1}{r}\frac{\partial w}{\partial \theta}(1 - \frac{\partial w}{\partial z}) + r\frac{\partial v}{\partial z}(1 - \frac{\partial v}{\partial \theta})(1 - \frac{u}{r})^2$$

$$- \frac{1}{r}\frac{\partial u}{\partial \theta}\frac{\partial u}{\partial z},$$

$$2e_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta}(1 - \frac{\partial u}{\partial r}) + r\frac{\partial v}{\partial r}(1 - \frac{\partial v}{\partial \theta})(1 - \frac{u}{r})^2$$

$$- \frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial \theta}. \qquad (4.2.7)$$

And in spherical coordinate system:

$$2e_{rr} = 2\frac{\partial u}{\partial r} - (\frac{\partial u}{\partial r})^2 - \frac{1}{r^2}[\frac{\partial v}{\partial r} - \frac{v}{r}]^2 - \frac{1}{r^2\sin^2\theta}[\frac{\partial w}{\partial r} - \frac{w}{r}]^2,$$

$$2e_{\theta\theta} = \frac{2}{r^2}(\frac{\partial v}{\partial \theta} + ru) - \frac{1}{r^2}(\frac{\partial u}{\partial \theta} - \frac{v}{r})^2 - \frac{1}{r^4}(\frac{\partial v}{\partial \theta} + ur)^2$$

$$- \frac{1}{r^4\sin^2\theta}(\frac{\partial w}{\partial \theta} - w \cot \theta)^2,$$

$$2e_{\phi\phi} = \frac{1}{r^2\sin^2\theta}[2(\frac{\partial w}{\partial \phi} + r \sin^2\theta.u + \sin\theta \cos\theta.v)$$

$$-(\frac{\partial u}{\partial \phi} - \frac{w}{r})^2 - \frac{1}{r^2}(\frac{\partial v}{\partial \phi} - \cot\theta.w)^2$$

$$- \frac{1}{r^2\sin^2\theta}(\frac{\partial w}{\partial \phi} + r \sin^2\theta.u + \sin\theta.\cos\theta.v)^2],$$

$$2e_{r\theta} = \frac{1}{r}[(\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{2v}{r}) - \frac{\partial u}{\partial r}(\frac{\partial u}{\partial \theta} - \frac{v}{r})$$

$$- \frac{1}{r^2}(\frac{\partial v}{\partial r} - \frac{v}{r})(\frac{\partial v}{\partial \theta} + ur)$$

$$- \frac{1}{r^2\sin^2\theta}(\frac{\partial w}{\partial r} - \frac{w}{r})(\frac{\partial w}{\partial \theta} - w.\cot\theta],$$

$$2e_{\theta\phi} = \frac{1}{r^2\sin^2\theta}[(\frac{\partial v}{\partial \phi} + \frac{\partial w}{\partial \theta} - 2w \cot\theta) - (\frac{\partial u}{\partial r} - \frac{v}{r})(\frac{\partial u}{\partial \phi} - \frac{w}{r})$$

$$- \frac{1}{r^2}(\frac{\partial v}{\partial \theta} + ru)(\frac{\partial v}{\partial \phi} - \cot\theta.w)$$

$$- \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial w}{\partial \theta} - w \cot\theta\right)\left(\frac{\partial w}{\partial \phi} + v \sin\theta \cos\theta + ur \sin^2\theta\right)],$$

$$2e_{r\phi} = \frac{1}{r \sin\theta}[\left(\frac{\partial u}{\partial \phi} + \frac{\partial w}{\partial r} - \frac{2w}{r}\right) - \frac{\partial u}{\partial r}\left(\frac{\partial u}{\partial \phi} - \frac{w}{r}\right)$$

$$- \frac{1}{r^2}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)\left(\frac{\partial v}{\partial \phi} - \cot\theta . w\right)$$

$$- \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial w}{\partial r} - \frac{w}{r}\right)\left(\frac{\partial w}{\partial \phi} + v \sin\theta \cos\theta + ur \sin^2\theta\right)].$$

(4.2.8)

We shall refer to both (4.2.7) and (4.2.8) in the subsequent chapters.

4.3 Formulation of the Problem and Identification of the Transition Points

In this article we consider a spherical shell of radii a and b $(a < b)$, subjected to a uniform external pressure at either of the surfaces. Because of spherical symmetry of the shell we use a spherical coordinate system (r,θ,ϕ) in which the displacements are chosen as:

$$u = r(1-\beta), \quad v = 0, \quad w = 0,$$

where β is a function of $r = (x^2 + y^2 + z^2)^{1/2}$. We assume that τ_{ij} and e_{ij} are single-valued functions of class $c^{(1)}$ and u_i and β are of class $c^{(2)}$ at any point $P(r,\theta,\phi)$ in the shell in the elastic range. Then from (4.2.8) we have

$$\left.\begin{aligned} e_{rr} &= \tfrac{1}{2}[1-(r\beta' + \beta^2], \\ e_{\theta\theta} &= e_{\phi\phi} = \tfrac{1}{2}[1 - \beta^2], \\ e_{r\phi}\ ,\quad & e_{\theta\theta}\ ,\quad e_{r\theta} = 0, \end{aligned}\right\} \tag{4.3.1}$$

and the dilatation $e_{\alpha\alpha} = \frac{3}{2} - \beta^2 - \frac{1}{2}(r\beta' + \beta)^2$. The corresponding stresses are given by (4.2.5)

$$\left.\begin{aligned} \tau_{rr} &= \lambda[\tfrac{3}{2} - \beta^2 - \tfrac{1}{2}(r\beta' + \beta)^2] + \mu[1-(r\beta' + \beta)^2], \\ \tau_{\theta\theta} &= \tau_{\phi\phi} = \lambda[\tfrac{3}{2} - \beta^2 - \tfrac{1}{2}(r\beta' + \beta)^2] + \mu[1 - \beta^2], \\ \tau_{\theta r} &= 0, \quad \tau_{\theta\phi} = 0, \quad \tau_{r\phi} = 0. \end{aligned}\right\} \tag{4.3.2}$$

The only equilibrium equation from (4.2.3) which remains unsatisfied is

$$\frac{\partial\tau_{rr}}{\partial r} + \frac{2(\tau_{rr} - \tau_{\theta\theta})}{r} = 0. \tag{4.3.3}$$

From (4.3.3) and (4.3.2) we get

$$2\beta^2 + (r\beta' + \beta)^2 + 2c\int r\beta'^2 dr = K, \qquad (4.3.4)$$

K being any arbitrary constant and $c = \frac{2\mu}{\lambda+2\mu} = \frac{1-2\sigma}{1-\sigma}$.
The prime indicates differentiation with respect to r.

Differentiating (4.3.4) with respect to r and dividing by β^2 and multiplying by r we get

$$\frac{d\beta}{dP} = -\frac{(P + 1)}{P^2 + (2 + c)P + 3}, \qquad (4.3.5)$$

where $P = \frac{r\beta'}{\beta}$. From (4.3.5) we see that the possible real transition points are

$$P = -1 \qquad \text{and} \qquad P = \pm\infty .$$

Suppose r' and r are the unstrained and strained radii vectors of a point before and after deformation respectively, then $r' = r\beta$ and

$$\frac{\partial r'}{\partial r} = r\beta' + \beta = \beta(1 + P).$$

Hence when $P \to -1$ then $\frac{\partial r'}{\partial r} \to 0$ and so $P = -1$ corresponds to infinite extension. Similarly $P = \pm\infty$ corresponds to infinite contraction. The strain ellipsoid reveals that both above mentioned points are transition points.

4.4 Determination of the Stresses and Strains in the Transition and Plastic States

We first observe that transition from the elastic to plastic state may occur through any one of the following branches:

(a) τ_{rr} ,

(b) $\tau_{\theta\theta}$, and

(c) $\tau_{\theta\theta} - \tau_{rr}$.

In order to determine the stress or the stress-difference through which transition occurs in a material, we examine each of the above three cases separately. Then after obtaining the transition-values of the stresses and their differences for all the three cases (a), (b) and (c) we compare their values. The transition should occur obviously through the largest of these values since the material yields at this largest value.

We shall follow the same technique throughout the rest of this thesis and no further reference will be made.

Case 1: $P \to \pm \infty$, Infinite Contraction

In this case, the shell is subjected to an external pressure on the inner surface and the outer surface is free from external pressure. Now we shall consider the following three cases mentioned above.

(a) Transition Through τ_{rr}:

Here we have from (4.3.2)

$$\tau_{rr} = \lambda[\frac{3}{2} - \beta^2 - \frac{1}{2}(r\beta' + \beta)^2] + [1-(r\beta' + \beta)^2]\mu$$

and thus the setting

$$R \equiv 1 - \frac{2\tau_{rr}}{3\lambda+2\mu} = \frac{\beta^2}{3-2c}[2(1-c)+(1+P)^2]$$

and taking logarithmic differentiation, we have

$$\frac{d(\log R)}{d(\log r)} = \frac{-2cP(2+P)}{2(1-c)+(1+P)^2} .$$

Hence

$$R = A_0 r^{-2c} \qquad \text{as} \qquad P \to \pm\infty,$$

where A_0 is any arbitrary constant, $c = \frac{1-2\sigma}{1-\sigma}$ and σ is the Poisson's ratio of the material. Making use of the boundary condition $\tau_{rr} = 0$ when $r = b$, we get

$$\tau_{rr} = (\frac{2-c}{c})y[1-(\frac{b}{r})^{2c}]. \qquad (4.4.1)$$

The equilibrium equation (4.3.3) gives

$$\tau_{\theta\theta} - \tau_{rr} = (2-c)y(\frac{b}{r})^{2c}. \qquad (4.4.2)$$

The pressure p_i exerted on the inner surface is given by (4.4.1) when $r = a$, as

$$p_i = [(\frac{b}{a})^{2c}-1](\frac{2-c}{c})y. \qquad (4.4.3)$$

When fully plastic state is reached $c \to 0$, and we have

$$\tau_{rr} = 4y\log(\frac{r}{b}),$$

$$\tau_{\theta\theta} - \tau_{rr} = 2y$$

$$p_i = 4y\log(\frac{b}{a}).$$

The corresponding classical values for the same situation are

$$\tau'_{\theta\theta} - \tau'_{rr} = y,$$

$$p'_i = 2y\log(\frac{b}{a}).$$

The reason that our results involve values which are twice the classical values is that the latter do not distinguish between yield stress in compression and yield stress in tension, where as our results take into account the Bauschinger effect.

When $r = a$, the principal stress difference $\tau_{\theta\theta} - \tau_{rr}$ becomes maximum. Hence the yield first occurs at the inner surface. Suppose after a certain time the yield surface moves to a radius d_0, then we have the following zones:

(i)	Transition region,	$a \leq r < b$,
(ii)	Transition-plastic region,	$a \leq r < b$,
(iii)	Fully plastic region,	$a \leq r < b$,

Hence we have the following results corresponding to the three regions:

(i) <u>Transition region</u>: $a \leq r < b$.

For the transition region we have

$$\tau_{rr} = \frac{y(2-c)}{c}\left[1-\left(\frac{b}{r}\right)^{2c}\right],$$

$$\tau_{\theta\theta} - \tau_{rr} = y(2-c)\left(\frac{b}{r}\right)^{2c},$$

both of which show work-hardening at the transition state.

(ii) <u>Transition-plastic region</u>:

For the transition region we have

$$\left.\begin{aligned} \tau_{rr} &= y\left(\frac{2-c}{c}\right)\left[1-\left(\frac{b}{r}\right)^{2c}\right], \\ \tau_{\theta\theta} - \tau_{rr} &= y(2-c)\left(\frac{b}{r}\right)^{2c}, \end{aligned}\right\} \quad d_0 \leq r < b$$

and for the plastic region,

$$\left.\begin{aligned} \tau_{rr} &= 4y\log\left(\frac{r}{d_0}\right) + y\left(\frac{2-c}{c}\right)\left[1-\left(\frac{b}{d_0}\right)^{2c}\right], \\ \tau_{\theta\theta} - \tau_{rr} &= 2y. \end{aligned}\right\} \quad a \leq r < d_0.$$

(iii) Fully plastic state: $(c \to 0)$

In the fully plastic state $c \to 0$, we have

$$\left.\begin{aligned} \tau_{rr} &= 4y\log\frac{r}{b} \\ \tau_{\theta\theta} - \tau_{rr} &= 2y \end{aligned}\right\} \quad a \leq r < b.$$

(b) Transition Through $\tau_{\theta\theta}$.

Let

$$R \equiv 1 - \frac{2\tau_{\theta\theta}}{3\lambda+2\mu} = \frac{\beta^2}{3-2c}[(1-c)(p+1)^2 + (2-c)].$$

Then by applying the same technique as in (a) we obtain

$$\tau_{rr} = \left(\frac{2-c}{c}\right) y \left[1-\left(\frac{b}{r}\right)^{2c}\right], \qquad (4.4.3)$$

$$\tau_{\theta\theta} = \left(\frac{2-c}{c}\right) y \left[1-(1-c)\left(\frac{b}{r}\right)^{2c}\right], \qquad (4.4.4)$$

and

$$\tau_{\theta\theta} - \tau_{rr} = (2-c) y \left(\frac{b}{r}\right)^{2c}. \qquad (4.4.5)$$

Again it is clear that $\tau_{\theta\theta} - \tau_{rr}$ which is the largest of the three quantities τ_{rr} , $\tau_{\theta\theta}$, $(\tau_{\theta\theta} - \tau_{rr})$ attains its maximum at $r = a$. Hence yielding first starts at $r = a$. We have the following results as before:

(i) <u>Transition region</u>:

For the transition region we have

$$\left.\begin{aligned} \tau_{rr} &= \left(\frac{2-c}{c}\right) y \left[1-\left(\frac{b}{r}\right)^{2c}\right] \\ \tau_{\theta\theta} - \tau_{rr} &= (2-c) y \left(\frac{b}{r}\right)^{2c} \end{aligned}\right\} \quad a \leq r < b$$

(ii) <u>Transition-plastic region</u>:

For the transition region we have

$$\left.\begin{aligned} \tau_{rr} &= y \left(\frac{2-c}{c}\right) \left[1-\left(\frac{b}{r}\right)^{2c}\right], \\ \tau_{\theta\theta} - \tau_{rr} &= (2-c) y \left(\frac{b}{r}\right)^{2c}, \end{aligned}\right\} \quad d_0 \leq r < b$$

and

$$\left.\begin{aligned} \tau_{rr} &= 4y\log\frac{r}{d_0} + y\left(\frac{2-c}{c}\right)\left[1-\left(\frac{b}{d_0}\right)^{2c}\right], \\ \tau_{\theta\theta} - \tau_{rr} &= 2y, \end{aligned}\right\} a \leq r < d_0.$$

for the plastic region.

(iii) Fully plastic region:

For the fully plastic state $c \to 0$ and we obtain

$$\left.\begin{aligned} \tau_{rr} &= 4y\log\frac{r}{b}, \\ \tau_{\theta\theta} - \tau_{rr} &= 2y. \end{aligned}\right\} a \leq r < b.$$

(c) Transition Through $(\tau_{\theta\theta} - \tau_{rr})$:

Here let

$$R \equiv \frac{\tau_{\theta\theta} - \tau_{rr}}{\mu} = \beta^2[(P+1)^2-1],$$

and the same procedure as before gives

$$\tau_{rr} = -p_i\left[\frac{1-\left(\frac{b}{r}\right)^{2c}}{1-\left(\frac{b}{a}\right)^{2c}}\right], \tag{4.4.6}$$

$$\tau_{\theta\theta} - \tau_{rr} = -\frac{cp_i}{[1-\left(\frac{b}{a}\right)^{2c}]}\left(\frac{b}{r}\right)^{2c}. \tag{4.4.7}$$

For the fully plastic state, we have

$$\tau_{rr} = p_i \log \frac{r}{b} / \log \frac{b}{a} ,$$

$$\tau_{\theta\theta} - \tau_{rr} = \frac{p_i}{2 \log(\frac{b}{a})} .$$

Thus for the fully plastic state

$$p_i = 4y \log \frac{b}{a} . \qquad (4.4.7a)$$

From the above three cases, careful comparison shows that transition first occurs through $(\tau_{\theta\theta} - \tau_{rr})$ if $(1 - c)(\frac{b}{a})^{2c} < 1$ and the yield condition may be taken as

$$\tau_{\theta\theta} - \tau_{rr} = 2y.$$

The asymptotic values of the strains are similarly determined from

$$e_{rr} = \frac{1}{2}[1-(r\beta' + \beta)^2],$$

$$e_{\theta\theta} = e_{\phi\phi} = \frac{1}{2}[1 - \beta^2], \qquad (4.4.8)$$

and

$e_{\theta\phi}$, $e_{r\theta}$, $e_{r\phi}$ are all zero.

Hence

$$1 - 2e_{rr} = A_0 (\frac{b}{r})^{2c} ,$$

$$e_{rr} - \frac{1}{3}e_{\alpha\alpha} = \frac{1}{6}(B_0 - 3A_0)(\frac{b}{r})^{2c} , \qquad (4.4.8a)$$

$$e_{rr} - e_{\theta\theta} = \frac{1}{4}(B_0 - 3A_0)(\frac{b}{r})^{2c} ,$$

where $e_{\alpha\alpha} = e_{rr} + e_{\theta\theta} + e_{\phi\phi}$, and A_0 , B_0 being arbitrary constants. Also

$$e_{\theta\theta} = e_{\phi\phi} = \frac{1}{2} \quad \text{as} \quad p \to \pm \infty ,$$

hence it can be shown that $A_0 = B_0$.

Case 2: $P \to -1$ Infinite Extension.

In this case, the shell is subjected to external pressure on the outer surface and no pressure is assumed to be present on the inner surface.

The following three types of transition may occur.

(a) Transition Through τ_{rr}:

Let

$$R \equiv 1- \frac{2\tau_{rr}}{3\lambda+2\mu} = \frac{\beta^2}{3-2c}[(P+1)^2 + 2(1-c)],$$

then applying the same method as before we have when $P \to -1$

$$\tau_{rr} = \frac{y(2-c)}{c}\left[1-\left(\frac{r}{a}\right)^{\frac{c}{1-c}}\right], \qquad (4.4.9)$$

$$\tau_{rr} - \tau_{\theta\theta} = \frac{y(2-c)}{2(1-c)}\left(\frac{r}{a}\right)^{\frac{c}{1-c}}, \qquad (4.4.10)$$

here we have used the boundary condition, namely of $r = a$, $\tau_{rr} = 0$.

It is evident from the above results that $|\tau_{\theta\theta}|$ is the largest among all the stresses and that it becomes maximum at $r = b$. Hence the shell begins to yield from the outer surface. Hence we have the following results:

(i) Transition region:

For the transition region, we have

$$\left.\begin{aligned} \tau_{rr} &= \left(\frac{2-c}{c}\right)y\left[1-\left(\frac{r}{a}\right)^{c/1-c}\right], \\ \tau_{rr} - \tau_{\theta\theta} &= \frac{(2-c)}{2(1-c)}y\left(\frac{r}{a}\right)^{\frac{c}{1-c}}. \end{aligned}\right\} \quad a < r \leq b$$

The pressure at $r = b$ which initiates transition is thus:

$$p_0 = \left(\frac{2-c}{c}\right)y\left[\left(\frac{b}{a}\right)^{\frac{c}{1-c}} - 1\right].$$

(ii) Fully plastic region: $(c \to 0)$.

For the fully plastic state $c \to 0$ and we obtain

$$\left.\begin{aligned} \tau_{rr} &= 2y\log\left(\frac{a}{r}\right), \\ \tau_{rr} - \tau_{\theta\theta} &= y \end{aligned}\right\} \quad a < r \leq b.$$

The plastic flow will commence at the outer surface of the shell at pressure p_1 such that

$$p_1 = 2y\log\frac{a}{b},$$

where y is yield stress in tension. The corresponding classical value in a similar situation is also given by

$$p_1' = 2y\log\frac{a}{b}.$$

(iii) Transition-Plastic region:

For the transition region we have,

$$\left.\begin{aligned} \tau_{rr} &= y\left(\frac{2-c}{c}\right)\left[1-\left(\frac{r}{d_0}\right)^{\frac{c}{1-c}}\right] + 2y\log\frac{r}{d_0}, \\ \tau_{rr} - \tau_{\theta\theta} &= \frac{y(2-c)}{2(1-c)}\left(\frac{r}{d_0}\right)^{\frac{c}{1-c}}, \end{aligned}\right\} \quad a < r \leq d_0$$

and

$$\left.\begin{aligned} \tau_{rr} &= 2y\log\frac{a}{r} \\ \tau_{rr} - \tau_{\theta\theta} &= y, \end{aligned}\right\} \quad d_0 < r \leq b$$

for the plastic region.

(b) Transition Through $\tau_{\theta\theta}$:

Let

$$R \equiv 1 - \frac{2\tau_{\theta\theta}}{3\lambda+2\mu} = \frac{\beta^2}{3-2c}[(1-c)(P+1)^2 + (2-c)].$$

A similar treatment as before gives

$$\left.\begin{aligned} \tau_{rr} &= \left(\frac{2-c}{c}\right)y\left[1-\left(\frac{a}{r}\right)^{2c}\right], && (4.4.11) \\ \tau_{\theta\theta} &= \left(\frac{2-c}{c}\right)y\left[1-(1-c)\left(\frac{a}{r}\right)^{2c}\right] && (4.4.12) \end{aligned}\right\} a \leq r < b$$

and

$$\tau_{\theta\theta} - \tau_{rr} = (2-c)y\left(\frac{a}{r}\right)^{2c}. \qquad (4.4.13)$$

Hence for the fully plastic state, we get

$$\left.\begin{aligned} \tau_{rr} &= 4y\log\frac{r}{a}, \\ \tau_{\theta\theta} - \tau_{rr} &= 2y. \end{aligned}\right\} a \leq r < b$$

The above results show that $\tau_{\theta\theta}$ becomes maximum at $r = b$. Hence yield first occurs at the outer surface.

We may again obtain the results for the three regions as in (a).

(c) <u>Transition Through $(\tau_{\theta\theta} - \tau_{rr})$.</u>

By following the same procedure as before we have for transition occuring through $\tau_{\theta\theta} - \tau_{rr}$,

$$\tau_{rr} = \mu K[1-(\frac{a}{r})^{6-2c}], \quad (4.4.14)$$

where K is some parameter such that

$$K = \frac{-p_0}{\mu[1-(\frac{a}{b})^{6-2c}]},$$

p_0 being the pressure applied on the outer surface of the shell. And

$$\tau_{\theta\theta} - \tau_{rr} = \frac{\mu K}{2}(6-2c)(\frac{a}{r})^{6-2c}. \quad (4.4.15)$$

For the fully plastic state,

$$\tau_{rr} = \frac{2}{3}yK_0[1-(\frac{a}{r})^{6}],$$

$$\tau_{\theta\theta} - \tau_{rr} = 2yK_0(\frac{a}{r})^{6},$$

where $K_0 = \lim_{c\to 0} K$ and $\mu = \frac{2-c}{3-2c}y$, so

$$\lim_{c\to 0} \mu = \frac{2}{3}y.$$

We notice from the above results that even in the fully plastic state they show work hardening. Hence transition through $|\tau_{\theta\theta} - \tau_{rr}|$ may lead to the creep state [Hulsurkar, 1966].

Comparing all the results in (a), (b) and (c) in this case, we conclude that $|\tau_{\theta\theta}|$ is the largest one among all. Hence transition has to occur through $\tau_{\theta\theta}$. The yield condition may then be taken as

$$|\tau_{\theta\theta} - \tau_{rr}| = y.$$

The asymptotic solutions for the strain corresponding to $P \to -1$ may similarly be obtained from (4.4.8) and are given below:

$$e_{rr} = \frac{1}{2},$$

$$e_{\alpha\alpha} = \frac{3}{2} - \frac{D_1}{2}\left(\frac{a}{r}\right)^c, \qquad (4.4.16)$$

$$e_{\theta\theta} = \frac{1}{2} - \frac{D_1}{4}\left(\frac{a}{r}\right)^c,$$

where D_1 is some arbitrary parameter, and

$$e_{rr} - e_{\theta\theta} = E\left(\frac{a}{r}\right)^{6-2c},$$

E being another parameter.

4.5 The Constitutive Equations in the Transition and Plastic States

Since we recognize the transition state as a separate state, like that of elastic and plastic ones, asymptotic treatment should lead to the constitutive equations in the transition state as well as in the plastic state. It has been remarked before that transition in a certain material may occur through any of the stresses or stress-differences which ever among them first reaches an extremal value. Further, their transitional paths may coincide or may be distinct. The distinct transitional paths may generally possess distinct constitutive equations. But if all these paths lead to the plastic state, then they should be characterized by a common constitutive equation irrespective of their own transitional courses.

In the transition and plastic states the constitutive equations involve only the deviatoric stresses and strains, since to a reasonable degree of approximation, it has been shown by Bridgman [1923] and verified by Crossland [1954] that the hydrostatic pressure affects neither the initial yield nor the plastic deformation itself. Now we obtain the constitutive equations for the transition and plastic states corresponding to the following cases.

Case 1: $P \to \pm \infty$.

Constitutive equation corresponding to transition through $(\tau_{\theta\theta} - \tau_{rr})$.

In this case transition occurs through $(\tau_{\theta\theta} - \tau_{rr})$, hence the constitutive equation in the transition state should be obtained corresponding to transition through $(\tau_{\theta\theta} - \tau_{rr})$. Hence from (4.4.6) and (4.4.7) we obtain

$$\tau^{d}_{rr} = \frac{2}{3}A\left(\frac{b}{r}\right)^{2c},$$

where

$$A = \frac{cp_i}{[1-(\frac{b}{a})^{2c}]}.$$

Also from (4.4.8a), we get

$$e^{d}_{rr} = \frac{2}{3} \cdot \frac{1}{4}(B_0 - 3A_0)\left(\frac{b}{r}\right)^{2c}.$$

Hence we have

$$e^{d}_{rr} = \frac{B_0 - 3A_0}{4A}\,\tau^{d}_{rr} . \qquad (4.5.1)$$

Similarly we may get

$$e^{d}_{\theta\theta} = \frac{B_0 - 3A_0}{4A}\,\tau^{d}_{\theta\theta} = e^{d}_{\phi\phi} . \qquad (4.5.2)$$

So the constitutive equation in the fully transition state may be written as

$$e^d_{ii} = \eta\tau^d_{ii} , \tag{4.5.3}$$

where $\eta = \dfrac{B_0 - 3A_0}{4A}$, A_0 , B_0 being arbitrary constants and A being a parameter. For the fully plastic state A will be A^1, where

$$A^1 = \frac{-p_i}{2\log \frac{b}{a}} .$$

Hence $A^1 = -2y$ by (4.4.7a). So for the fully plastic state, the constitutive equation will be

$$\delta e^d_{ii} = \eta^1\tau^d_{ii}\delta_\theta , \tag{4.5.4}$$

where θ is a flow parameter and

$$\eta^1 = \frac{A_0}{4y} . \quad (B_0 = A_0).$$

We may rewrite (4.5.4) as

$$\dot{e}^d_{ii} = \eta^1\tau^d_{ii} , \tag{4.5.4a}$$

the dot representing the time derivative, which is the Lévy-von Mises constitutive equation.

If (4.5.4a) should be the constitutive equation for the plastic state, then the same equation must result by considering transitions through τ_{rr} and $\tau_{\theta\theta}$ as well. Note that if the transition through $\tau_{\theta\theta}$ and τ_{rr} lead to some state different from that represented by the state through $(\tau_{\theta\theta} - \tau_{rr})$, one should not expect to get the same constitutive equation in all the cases. But, however, in this problem, transition through τ_{rr} and $\tau_{\theta\theta}$ also lead to the same equation as (4.5.4a).

Case 2: $P \to -1$

Constitutive equation corresponding to transition through $\tau_{\theta\theta}$:

In this case we have already noticed before that the transition occurs through $\tau_{\theta\theta}$. Hence the constitutive equation for the transition state should be obtained through $\tau_{\theta\theta}$. However, a common constitutive equation for the fully plastic state has to be derived through all branches of the transition, if they all go to plastic state.

Now from (4.4.11), (4.4.12), (4.4.13) and (4.4.16) we get

$$e_{\theta\theta}^{d} = \frac{-D_1}{4(2-c)y}\left(\frac{a}{r}\right)^{-c} \tau_{\theta\theta}^{d} \quad ,$$

$$e^d_{rr} = \frac{-D_1}{4(2-c)y}\left(\frac{a}{r}\right)^{-c}\tau^d_{rr} ,$$

and

$$e^d_{\phi\phi} = \frac{-D_1}{4(2-c)y}\left(\frac{a}{r}\right)^{-c}\tau^d_{\phi\phi} .$$

Since

$$I_1 = 2(2-c)y\left(\frac{a}{r}\right)^{2c} + 3\left(\frac{2-c}{c}\right)y\left[1-\left(\frac{a}{r}\right)^{2c}\right],$$

we can express $\left(\frac{a}{r}\right)^{-c}$ by some function of I_1. Hence we may write the above equation as

$$e^d_{ii} = \eta f(I_1)\tau^d_{ii} , \tag{4.5.5}$$

which is the constitutive equation for the transition state. The constitutive equation for the fully plastic state thus becomes

$$\delta e^d_{ii} = \eta' \tau^d_{ii}\delta_\theta , \tag{4.5.6}$$

where

$$\lim_{c \to 0} \eta = \eta' = \frac{-D_1}{8y} ,$$

and

$$\lim_{c \to 0} f(I_1) = 1.$$

The equation (4.5.6) may then be written as

$$\dot{e}^d_{ii} = \eta' \tau^d_{ii}.$$

It can be easily verified that equations similar to (4.5.6) can be obtained by considering the transition through τ_{rr} and $(\tau_{\theta\theta} - \tau_{rr})$ for the fully plastic state.

CHAPTER 5

ELASTIC-PLASTIC TRANSITION OF TUBE UNDER UNIFORM PRESSURE

5.1 Preliminary Remarks

This chapter is devoted to the discussion of elastic-plastic transition in tubes under uniform pressure. Our purpose here is to show, as in the previous chapter, that a constitutive equation does exist in the transition state. The onset of the plastic state is derived without introducing any semi-empirical conditions.

5.2 The Basic Equations

The fundamental equations discussed in §4.2 will remain valid here as well except the equilibrium equations which take different form in cylindrical coordinate system. We have listed already the strain components in the cylindrical coordinate system in equation (4.2.7).

5.3 Formulation of the Problem and Identification of Transition Points

Consider a thick tube of radii a and b $(a < b)$ which is subjected to a uniform external pressure at either of the surfaces. On account of symmetry of the problem, we may assume the following displacements:

$$u = r(1-\beta), \quad v = 0, \quad w = z(1-d_0),$$

where $r = (x^2 + y^2)^{1/2}$ and $d_0 \leq 1$. For the plane strain case, $d_0 = 1$.

Using the Almansi strain measure from (4.2.7) we obtain

$$e_{rr} = \frac{1}{2}[1-(r\beta' + \beta)^2],$$

$$e_{\theta\theta} = \frac{1}{2}[1-\beta^2],$$

$$e_{zz} = \frac{1}{2}[1- d_0^2],$$

$$e_{r\theta}\ ,\ e_{\theta z}\ ,\ e_{rz} = 0,$$

where d_0 is some constant to be determined. The corresponding stresses are

$$\begin{aligned} \tau_{rr} &= \lambda e_{\alpha\alpha} + \mu[1-(r\beta' + \beta)^2], \\ \tau_{\theta\theta} &= \lambda e_{\alpha\alpha} + \mu[1-\beta^2], \\ \tau_{zz} &= \lambda e_{\alpha\alpha} + \mu[1 - d_0{}^2], \\ \tau_{r\theta}\ ,\quad &\tau_{rz}\ ,\quad \tau_{z\theta} = 0, \end{aligned} \qquad (5.3.1)$$

and

$$2e_{\alpha\alpha} = 3 - d_0^2 - (r\beta' + \beta)^2 - \beta^2.$$

The only equilibrium equation which remains to be satisfied is

$$\frac{\partial\tau_{rr}}{\partial r} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = 0. \quad (5.3.2)$$

Putting (5.3.1) in (5.3.2) we get

$$\beta^2 + (r\beta' + \beta)^2 + c\int r\beta'^2 dr = K, \quad (5.3.3)$$

where $c = \frac{2\mu}{\lambda+2\mu}$ and K is some arbitrary constant of integration. The equation (5.3.3) is a non-linear intego-differential equation and no analytical solution has been obtained so far.

Differentiating (5.3.3) with respect to r and putting $p = \frac{r\beta'}{\beta}$ we get

$$\frac{d\beta}{dp} = - \frac{\beta(P + 1)}{P^2 + P(\frac{c}{2} + 2) + 2}. \quad (5.3.4)$$

The possible real transition points are given by

$$P = -1, \quad P = \pm\infty.$$

As before it can be shown that $P \to -1$ corresponds to infinite extension and $P \to \pm\infty$ corresponds to infinite contraction.

5.4 Determination of Stresses and Strains in the Transition and Plastic States

Case 1: $P \to \pm \infty$, Infinite Contraction

In this case the tube is subjected to external pressure on the inner surface of the tube $(a < b)$.

(a) Transition Through τ_{rr}:

For transition through τ_{rr}, we have

$$\tau_{rr} = \lambda[1 - \tfrac{1}{2}(r\beta' + \beta)^2 - \tfrac{1}{2}\beta^2 + \tfrac{1}{2}(1 - d_0{}^2)] + \mu[1-(r\beta' + \beta)^2].$$

Let

$$R \equiv D_0 - \frac{c\tau_{rr}}{\mu} = \beta^2[(1 - c)+(P + 1)^2],$$

where

$$D_0 = (3 - 2c)-(1 - c)d_0{}^2.$$

Hence

$$\frac{d(\log R)}{d(\log r)} = - \frac{cP(P + 2)}{(1 - c)+(P + 1)^2},$$

from which it follows that $R = A_0 r^{-c}$ as $P \to \pm \infty$. The boundary condition, $\tau_{rr} = 0$ at $r = b$, gives

$$\tau_{rr} = \frac{\mu}{c} D_0 [1-(\frac{b}{r})^c]. \tag{5.4.1}$$

The equilibrium equation (5.3.2) gives

$$\tau_{\theta\theta} - \tau_{rr} = \mu D_0 (\frac{b}{r})^c. \tag{5.4.2}$$

In like manner we may obtain

$$\tau_{\theta\theta} - \tau_{zz} = \mu d_0^2 , \tag{5.4.3}$$

and

$$\tau_{zz} - \tau_{rr} = \mu [D_0 (\frac{b}{r})^c - d_0^2]. \tag{5.4.4}$$

In order to obtain d_0 , if F is the force applied at the ends of the cylinder, then

$$F = 2\pi \int_a^b \tau_{zz}\, rdr, \tag{5.4.5}$$

a,b being inner and outer radius of the tube respectively. Substituting the value of τ_{zz} in (5.4.5) we have

$$F = \frac{2\pi\mu(1-c)}{c} D_0 [\frac{1}{2}(b^2 - a^2) - \frac{b^c}{2-c}(b^{2-c} - a^{2-c})]$$

$$+ \pi\mu [D_0 - d_0^2](b^2 - a^2). \tag{5.4.6}$$

When the tube becomes fully plastic the longitudinal force which is to be applied at the ends of the cylinder to keep the length unchanged is given by

$$F' = \frac{2}{3}\pi y[(3 - d_0{}^2)a^2 \log \frac{b}{a} + d_0{}^2(a^2 - b^2)].$$

Thus, in order to maintain the same length for the cylinder the additional normal force required as predicted on the basis of our transition analysis is given by

$$\pi[a^2 p + \frac{2}{3}y(a^2 - b^2)], \qquad (\tau_{rr})_{r=a} = -p,$$

while the corresponding classical value is [Bland, 1956]

$$\pi a^2 p.$$

For the fully plastic state $c \to 0$, and we get the following results:

$$\tau_{rr} = \frac{2}{3}y(3 - d_0{}^2)\log \frac{r}{b},$$

$$\tau_{\theta\theta} - \tau_{rr} = \frac{2}{3}y(3 - d_0{}^2),$$

$$\tau_{\theta\theta} - \tau_{zz} = \frac{2}{3}yd_0{}^2, \tag{5.4.7}$$

$$\tau_{zz} - \tau_{rr} = \frac{2}{3}y(3 - 2d_0{}^2).$$

The plastic flow starts at $r = a$ where the pressure p_i' is given by

$$p_i' = \frac{2}{3}y(3 - d_0^{\,2})\log\frac{b}{a} . \qquad (5.4.7a)$$

For plane strain case, that is, for a tube of infinite length $d_0 = 1$ and we get

$$\begin{aligned} \tau_{rr} &= \frac{4}{3}y\log\frac{r}{b} , \\ \tau_{\theta\theta} - \tau_{rr} &= \frac{4}{3}y , \\ \tau_{\theta\theta} - \tau_{zz} &= \frac{2}{3}y , \qquad (5.4.8) \\ \tau_{zz} - \tau_{rr} &= \frac{2}{3}y . \end{aligned}$$

The maximum principal stress difference in this case is

$$\tau_{\theta\theta} - \tau_{rr} = \frac{4}{3}y ,$$

which is Tresca yield condition, y being yield stress in tension. The von-Mises yield condition becomes

$$(\tau_{rr} - \tau_{\theta\theta})^2 + (\tau_{\theta\theta} - \tau_{zz})^2 + (\tau_{zz} - \tau_{rr})^2 = \frac{8}{3}y^2 .$$

In general, it is seen from equation (5.4.2) that the tube starts yielding at the inner surface $r = a$ since

$|\tau_{\theta\theta} - \tau_{rr}|$ becomes maximum at $r = a$.

When the plastic state has penetrated to a radius $r = d$, we get the following three regions:

(i) Transition region: $a \leq r < b$.

For the transition region, we have

$$\tau_{rr} = \frac{\mu}{c} D_0 [1 - (\frac{b}{r})^c],$$

$$\tau_{\theta\theta} - \tau_{rr} = \mu D_0 (\frac{b}{r})^c,$$

$$\tau_{\theta\theta} - \tau_{zz} = \mu d_0^2,$$

$$\tau_{zz} - \tau_{rr} = \mu [D_0 (\frac{b}{r})^c - d_0^2].$$

(ii) Transition-plastic region:

For the transition region $d \leq r < b$, we have

$$\tau_{rr} = \frac{\mu}{c} D_0 [1 - (\frac{b}{r})^c],$$

$$\tau_{\theta\theta} - \tau_{rr} = \mu D_0 (\frac{b}{r})^c,$$

$$\tau_{\theta\theta} - \tau_{zz} = \mu d_0^2,$$

$$\tau_{zz} - \tau_{rr} = \mu [D_0 (\frac{b}{r})^c - d_0^2].$$

And for the plastic region $a \leq r < d$, we have

$$\tau_{rr} = \frac{2}{3}y(3 - d_0{}^2)\log\frac{r}{d} + \frac{\mu}{c}D_0[1 - (\frac{b}{d})^c],$$

$$\tau_{\theta\theta} - \tau_{rr} = \frac{2}{3}y[3 - d_0{}^2],$$

$$\tau_{\theta\theta} - \tau_{zz} = \frac{2}{3}d_0{}^2,$$

$$\tau_{zz} - \tau_{rr} = \frac{2}{3}y[3 - 2d_0{}^2].$$

(iii) <u>Fully plastic region</u>: $a \leq r < b$.

For the fully plastic region, we get

$$\tau_{rr} = \frac{2}{3}y(3 - d_0{}^2)\log\frac{r}{b},$$

$$\tau_{\theta\theta} - \tau_{rr} = \frac{2}{3}y[3 - d_0{}^2],$$

$$\tau_{\theta\theta} - \tau_{zz} = \frac{2}{3}yd_0{}^2,$$

$$\tau_{zz} - \tau_{rr} = \frac{2}{3}y[3 - 2d_0{}^2].$$

It is clear from the above results that none of the relations is independent of d_0, but we have

$$\tau^d_{\theta\theta} = \tau_{\theta\theta} - \frac{1}{3}(\tau_{rr} + \tau_{\theta\theta} + \tau_{\phi\phi})$$

$$= \frac{2}{3}Y ,$$

a result independent of d_0 and which may, therefore, be considered to be the yield condition. This holds good for all end conditions on the plane ends.

(b) <u>Transition Through $\tau_{\theta\theta}$:</u>

In this case,

$$\tau_{\theta\theta} = \lambda[\frac{3}{2} - \frac{1}{2}\beta^2 - \frac{1}{2}(r\beta' + \beta)^2 - \frac{1}{2}d_0{}^2] + \mu[1 - \beta^2].$$

Putting $R \equiv [\frac{-c\tau_{\theta\theta}}{\mu} + D_0] = \beta^2[1+(1 - c)(P + 1)^2]$ and taking logarithmic differentiation with respect to r we get:

$$\frac{d(\log R)}{d(\log r)} = \frac{-2P[c + (1 - c)\frac{1}{2}cP]}{1 + (1 - c)(P + 1)^2}$$

Hence

$$\tau_{\theta\theta} = \frac{\mu D_0}{c}[1 - A_0 r^{-c}],$$

and the equilibrium equation together with boundary condition gives

$$\tau_{rr} = \frac{\mu D_0}{c}[1 - (\frac{b}{r})^c].$$

And so we have the following:

$$\tau_{\theta\theta} = \frac{\mu D_0}{c}[1-(1 - c)(\frac{b}{r})^c],$$

$$\tau_{\theta\theta} - \tau_{rr} = \mu D_0(\frac{b}{r})^c,$$

$$\tau_{\theta\theta} - \tau_{zz} = \mu d_0^2 ,$$

$$\tau_{zz} - \tau_{rr} = \mu[D_0(\frac{b}{r})^c - d_0^2],$$

which are identical with the results obtained in case (a) and hence all the rest of the results obtained in case (a) will follow automatically.

(c) <u>Transition Through $(\tau_{\theta\theta} - \tau_{rr})$</u>:

In this case let

$$R \equiv \frac{\tau_{\theta\theta} - \tau_{rr}}{\mu} = \beta^2[(P + 1)^2-1],$$

then

$$\frac{d(\log R)}{d(\log r)} = \frac{-2P[P\frac{c}{2} + 2]}{(P + 1)^2-1} .$$

Hence

$$\tau_{\theta\theta} - \tau_{rr} = \mu B_0 (\frac{b}{r})^c , \tag{5.4.9}$$

where B_0 is some arbitrary parameter. From the equilibrium equation we have

$$\tau_{rr} = \frac{\mu B_0}{c}[1-(\frac{b}{r})^c] . \tag{5.4.10}$$

The value of B_0 may be obtained as

$$B_0 = \frac{cp_i}{\mu[(\frac{b}{a})^c - 1]} , \tag{5.4.11}$$

where p_i is the pressure applied at the inner surface $r = a$.

In like manner we get

$$\tau_{\theta\theta} - \tau_{rr} = \mu d_0^2 , \tag{5.4.12}$$

and

$$\tau_{zz} - \tau_{rr} = \mu[B_0 (\frac{b}{r})^c - d_0^2] . \tag{5.4.13}$$

The stresses in the plastic state can be obtained by allowing $c \to 0$. By following a similar procedure as in cases (a) and (b), the three regions of interest namely, (i) transition region, (ii) transition-plastic region and

(iii) the fully plastic region may be discussed.

The transition may occur also through τ_{zz}, $\tau_{zz} - \tau_{rr}$ or $\tau_{zz} - \tau_{rr}$. But, since we are considering only symmetric deformations, their asymptotic solutions can not be determined uniquely. Moreover, it is not the plan of this work to discuss these cases here. From the foregoing results in (a), (b) and (c), it is clear that $|\tau_{\theta\theta} - \tau_{rr}|$ is the largest in value among τ_{rr}, $\tau_{\theta\theta}$, $|\tau_{\theta\theta} - \tau_{zz}|$, etc. in every case. Hence, the transition should occur through $|\tau_{\theta\theta} - \tau_{rr}|$ and the yield condition may therefore be taken as

$$\tau^{d}_{\theta\theta} = \frac{2}{3}Y.$$

Case 2: <u>P → -1 Infinite Extension</u>

As has been explained earlier, this case corresponds to the infinite extension and the tube is subjected to uniform pressure on the outer surface.

(a) <u>Transition Through τ_{rr}.</u>

In this case let

$$R \equiv [-\frac{c}{\mu}\,\tau_{rr} + D_0] = \beta^2[(1 - c)+(P + 1)^2].$$

Hence $R = A_1 r^{\frac{c}{1-c}}$ as $P \to -1$, where A_1 is an arbitrary constant.

Using the boundary condition $\tau_{rr} = 0$ at $r = a$, we get

$$\tau_{rr} = \frac{\mu}{c} D_0 [1-(\frac{r}{a})^{\frac{c}{1-c}}].$$

A similar procedure as before yields

$$\tau_{rr} - \tau_{\theta\theta} = \frac{\mu}{1-c} D_0 (\frac{r}{a})^{\frac{c}{1-c}},$$

$$\tau_{rr} - \tau_{zz} = \mu d_0^{\,2},$$

and

$$\tau_{zz} - \tau_{\theta\theta} = \mu [\frac{D_0}{1-c}(\frac{r}{a})^{\frac{c}{1-c}} - d_0^{\,2}].$$

The maximum principal stress difference is again $\tau_{rr} - \tau_{\theta\theta}$ and depends on both r and d_0. For fully plastic state $c \to 0$, and we get

$$\tau_{rr} = \frac{2}{3}y(3 - d_0^{\,2})\log\frac{a}{r},$$

$$\tau_{rr} - \tau_{\theta\theta} = \frac{2}{3}y(3 - d_0^{\,2}),$$

$$\tau_{rr} - \tau_{zz} = \frac{2}{3}y d_0^{\,2}.$$

In this case, the yield condition may be taken as

$$\tau_{rr}^{d} = \frac{2}{3}y ,$$

since this relation is independent of d_0.

For the case of plane strain $d_0 = 1$, and we get

$$\tau_{rr} = \frac{(2 - c)}{c} [1-(\frac{r}{a})^{\frac{c}{1-c}}],$$

$$\tau_{rr} - \tau_{\theta\theta} = \frac{\mu(2 - c)}{1 - c}(\frac{r}{a})^{\frac{c}{1-c}} ,$$

$$\tau_{rr} - \tau_{zz} = \mu .$$

For the fully plastic state, we have

$$\tau_{rr} = \frac{4}{3}y\log\frac{a}{r} ,$$

$$\tau_{rr} - \tau_{\theta\theta} = \frac{4}{3}y , \quad \tau_{rr} - \tau_{zz} = \frac{2}{3}y ,$$

$$\tau_{zz} - \tau_{\theta\theta} = \frac{2}{3}y.$$

Here $\tau_{zz} = \frac{1}{2}(\tau_{rr} + \tau_{\theta\theta})$ which is again the hypothesis of principal line theory.

(b) Transition Through $\tau_{\theta\theta}$:

Here

$$R \equiv [-\frac{c}{\mu}\tau_{\theta\theta} + D_0] = \beta^2[1+(1 - c)(P + 1)^2],$$

and a similar procedure as before gives the following results:

$$\tau_{rr} = \frac{\mu}{c} D_0[1-(\frac{a}{r})^{c(3-c)}],$$

$$\tau_{\theta\theta} - \tau_{rr} = \mu(3 - c)D_0(\frac{a}{r})^{c(3-c)},$$

$$\tau_{\theta\theta} = \frac{\mu}{c} D_0[1-(1 - c(3-c))(\frac{a}{r})^{c(3-c)}],$$

$$\tau_{\theta\theta} - \tau_{zz} = \mu d_0^2,$$

and

$$\tau_{rr} - \tau_{zz} = \mu d_0^2.$$

The results for the fully plastic state may similarly be obtained when c tends to zero.

(c) Transition Through $(\tau_{\theta\theta} - \tau_{rr})$.

In this case let

$$R \equiv \frac{\tau_{rr} - \tau_{\theta\theta}}{\mu} = \beta^2[1-(P + 1)^2]$$

and we get

$$\tau_{rr} - \tau_{\theta\theta} = \mu B_1 (\frac{a}{r})^{4-c} \quad \text{as} \quad P \to -1. \tag{5.4.14}$$

From the equilibrium equation using the boundary condition $\tau_{rr} = 0$ at $r = a$, we get

$$\tau_{rr} = \frac{\mu B_1}{c-4}[1-(\frac{a}{r})^{4-c}], \tag{5.4.15}$$

where B_1 is a parameter given by

$$B_1 = \frac{-p_0(4-c)}{\mu[1-(\frac{a}{b})^{4-c}]} \quad \text{and} \quad -p_0 = (\tau_{rr})_{r=b}. \tag{5.4.16}$$

Further we have

$$\tau_{\theta\theta} - \tau_{zz} = \mu d_0^{\,2} - \mu B_1 (\frac{a}{r})^{4-c}, \tag{5.4.17}$$

$$\tau_{rr} - \tau_{zz} = \mu d_0^{\,2}. \tag{5.4.18}$$

The asymptotic values of the strains are given below.

<u>For $P \to \pm\infty$</u>:

$$1 - 2e_{rr} = A_1 (\frac{b}{r})^c,$$

$$e_{zz} = \frac{1}{2}(1 - d_0^{\,2}), \tag{5.4.19}$$

(5.4.19) continued

$$3 - d_0^2 - 2e_{\alpha\alpha} = A_1(\frac{b}{r})^c ,$$

$$e_{\theta\theta} = \frac{1}{2} , \quad A_1 \text{ is a parameter.}$$

<u>For $P \to -1$:</u>

$$e_{rr} = \frac{1}{2} ,$$

$$3 - d_0^2 - 2e_{\alpha\alpha} = A_2(\frac{a}{r})^c ,$$

$$e_{\theta\theta} = \frac{1}{2}[1-A_2(\frac{a}{r})^c] , \tag{5.4.20}$$

$$e_{zz} = \frac{1}{2}[1-d_0^2] , \quad A_2 \text{ is a parameter.}$$

Also we have

$$e_{rr} - e_{\theta\theta} = A_3(\frac{a}{r})^{4-c} ,$$

$$e_{rr} - e_{zz} = \frac{1}{2}d_0^2 , \tag{5.4.21}$$

$$e_{\theta\theta} - e_{zz} = \frac{1}{2}d_0^2 - A_3(\frac{a}{r})^{4-c} , \quad A_3 \text{ being}$$

another parameter.

5.5 The Constitutive Equations in the Transition and Plastic States

Now we obtain the constitutive equations for the transition and plastic states corresponding to the following cases.

Case 1: $P \to \pm \infty$

We have already seen before in this case that transition first occurs through $(\tau_{\theta\theta} - \tau_{rr})$ if $(1 - c)(\frac{b}{a})^c < 1$, since then all the cases (a), (b) and (c) $(\tau_{\theta\theta} - \tau_{rr})$ reaches the largest value among the others. Hence we should obtain constitutive equation for the transition state through $(\tau_{\theta\theta} - \tau_{rr})$.

Constitutive Equation Corresponding to Transition through $(\tau_{\theta\theta} - \tau_{rr})$.

From (5.4.9), (5.4.10), (5.4.12), (5.4.13) and (5.4.19) we get the following:

$$e^d_{rr} = \frac{1}{2\mu} \cdot \frac{d_0{}^2 - 2A_1(\frac{b}{r})^c}{d_0{}^2 - 2B_0(\frac{b}{r})^c} \tau^d_{rr} ,$$

$$e^d_{\theta\theta} = \frac{1}{2\mu} \cdot \frac{d_0{}^2 + A_1(\frac{b}{r})^c}{d_0{}^2 + B_0(\frac{b}{r})^c} \tau^d_{\theta\theta} ,$$

and

$$e_{zz}^{d} = \frac{1}{2\mu} \cdot \frac{2d_0^{\;2} - A_1(\frac{b}{r})^c}{2d_0^{\;2} - B_0(\frac{b}{r})^c} \tau_{zz}^{d} .$$

If we choose $A_1 = B_0$, then the above three equations reduce to

$$e_{ii}^{d} = \frac{1}{2\mu} \tau_{ii}^{d} , \qquad (5.5.1)$$

which is the constitutive equation which characterizes the transition state. For the fully plastic state (5.5.1) becomes

$$\dot{e}_{ii}^{d} = \frac{3}{4y} \tau_{ii}^{d} . \qquad (5.5.2)$$

If transition occurs through τ_{rr}, we may get from §5.4 the following equations:

$$e_{rr}^{d} = \frac{1}{2\mu} \cdot \frac{d_0^{\;2} - 2A_1(\frac{b}{r})^c}{d_0^{\;2} - 2D_0(\frac{b}{r})^c} \tau_{rr}^{d} ,$$

$$e_{\theta\theta}^{d} = \frac{1}{2\mu} \cdot \frac{d_0^{\;2} + A_1(\frac{b}{r})^c}{d_0^{\;2} + D_0(\frac{b}{r})^c} \tau_{\theta\theta}^{d} , \qquad (5.5.3)$$

(5.5.3) continued

$$e_{zz}^{d} = \frac{1}{2\mu} \cdot \frac{2d_0^{\;2} - A_1(\frac{b}{r})^c}{2d_0^{\;2} - D_0(\frac{b}{r})^c}\,\tau_{zz}^{d} \; .$$

Equations similar to (5.5.3) may be obtained if transition occurs through $\tau_{\theta\theta}$ from §5.4. The equations (5.5.3) become

$$e_{ii}^{d} = \frac{1}{2\mu}\,\tau_{ii}^{d} \; ,$$

if we take $A_1 = D_0$. Consequently the constitutive equation for the plastic state will also be the same as (5.5.2). Now by the foregoing relations we have

$$A_1 = D_0 = B_0 \; , \tag{5.5.4}$$

which gives

$$A_1 = (3 - 2c)-(1 - c)d_0^{\;2} = \frac{cp_i}{\mu[(\frac{b}{a})^c-1]} \; .$$

Hence

$$p_i = \frac{\mu}{c}[(3 - 2c)-(1 - c)d_0^{\;2}][(\frac{b}{a})^c-1].$$

The pressure on the inside surface $r = a$ necessary to start plastic flow is

$$p_i' = \frac{2}{3}y(3 - d_0^2)\log\frac{b}{a},$$

which is exactly the same pressure obtained in (5.4.7a). Thus (5.5.4) is not an assumption.

Case 2: <u>$P \to -1$</u>

In this case also the transition occurs first through $(\tau_{\theta\theta} - \tau_{rr})$ if

$$(1 - c)\left(\frac{b}{a}\right)^{\frac{c}{1-c}} < 1.$$

Constitutive Equation Corresponding to Transition Through $(\tau_{\theta\theta} - \tau_{rr})$.

As before from (5.4.14), (5.4.15), (5.4.17), (5.4.18) and (5.4.21) we have

$$e^d_{rr} = \frac{1}{2\mu}\cdot\frac{d_0^2 + 2A_3\left(\frac{a}{r}\right)^{4-c}}{d_0^2 + B_1\left(\frac{a}{r}\right)^{4-c}}\tau^d_{rr},$$

$$e^d_{\theta\theta} = \frac{1}{2\mu}\cdot\frac{d_0^2 - 4A_3\left(\frac{a}{r}\right)^{4-c}}{d_0^2 - 2B_1\left(\frac{a}{r}\right)^{4-c}}\tau^d_{\theta\theta}, \qquad (5.5.5)$$

(5.5.5) continued

$$e_{zz}^{d} = \frac{1}{2\mu} \cdot \frac{2d_0^{\ 2} - 2A_3(\frac{a}{r})^{4-c}}{2d_0^{\ 2} - B_1(\frac{a}{r})^{4-c}} \tau_{zz}^{d} \ .$$

Choosing $2A_3 = B_1$, we obtain from (5.5.5)

$$e_{ii}^{d} = \frac{1}{2\mu} \tau_{ii}^{d} \ , \qquad (5.5.6)$$

where

$$2A_3 = \frac{-p_0(4-c)}{\mu[1-(\frac{a}{b})^{4-c}]} \ .$$

For the fully plastic state (5.5.6) becomes

$$\dot{e}_{ii}^{d} = \frac{3}{4y} \tau_{ii}^{d} \ . \qquad (5.5.7)$$

All the other transitional paths also give the same constitutive equation as (5.5.7) in the fully plastic state.

CHAPTER 6

THERMO-ELASTIC-PLASTIC TRANSITION OF SHELLS SUBJECTED TO UNIFORM PRESSURE AND STEADY STATE TEMPERATURE

6.1 Preliminary Remarks

The study of thermo-elasticity has lately received a great impetus and is the object of considerable attention, principally because of relatively recent technological developments in the fields such as high-speed and space flight, nuclear energy for power generation etc. In many cases of machine design of steam turbines and diesel engines, thermal stresses are of great practical importance. The occurrence of very high temperature and pressure, particularly in the chemical plant, has led to new considerations in the design of such containers.

The viscous and elastic properties of metals are highly temperature-dependent. The situation becomes more complicated particularly in the presence of pressure. The material exhibits all sorts of in-elastic behaviors such as plastic, creep, fatigue, buckling, etc. Metals subjected to high pressure and temperature can very easily go into creep state the consideration of which thus becomes important in the design of oil and chemical plants, gas and steam turbines, high speed structures involving areodynamic heating and so on.

The thermo-elastic plastic analysis has been studied by Bland, Derrington, Nadai, Nowacki, Naghdi, Weiner, Whalley among others. Bland [1956], using Tresca's yield condition and its associated flow rule, obtained the solution for the stresses, the elastic and plastic strains, and the displacements when a thick-walled tube of work-hardening material is subjected to internal and external pressure and its surfaces are maintained at different temperatures. In general, a numerical integration is necessary, but the solutions can be expressed explicitly when the work-hardening law is linear. Johnson and Derrington [1958] have obtained an interesting result that the yielding could be caused to occur at any given radius or simultaneously at different positions in the tube (or shell) by making an appropriate choice of temperature, pressure and the inner and outer radii of the tube (or shell).

Wilhoit [1958] has investigated the problem of a ring arriving at the same conclusion as that of Johnson and Derrington.

All the authors mentioned above who have investigated these types of problems have employed the method of superposition and have used ad-hoc semi-empirical laws.

In this chapter and the next, we shall show that without resorting to any ad-hoc, semi-empirical laws, one could arrive at those results existing in current literature as special cases by employing the transition theory.

6.2 Theory of Thermal Stresses

Thermal stresses may arise in a heated body either because of non-uniform temperature distribution or external constraints, or a combination of both. Since the effect of external constraints is easily understood, we shall confine our attention to that of non-uniform temperature.

Imagine a body B as made up of a number of small cubical elements of equal size and shape which fit together to form the continuum under consideration. If the temperature of the body is raised uniformly and the boundary of B is unrestrained, then each element will expand uniformly in all directions. Since they have still to fit together to form the same continuous body, no stresses arise. If, however, the temperature rise is not uniform, each element will expand by a different amount proportinal to its own temperature rise. The resulting cubes of different size cannot, in general, fit together. However, the body must remain continuous, each element must restrain the distortions of its neighbors and as a result, stresses must arise.

The total strains at each point of a heated body are thus made of two parts. The first part is a uniform expansion proportional to the temperature raise θ consisting of only normal strains. If the coefficient of expansion of the material is α, the normal strain in any direction is equal to $\alpha\theta$. The second part comprises of strains required to maintain the continuity of the body as well as those arising from external loads. We now start with a quasi-linear Hooke's law of isothermal elasticity and the response coefficients λ, μ are taken as constant and temperature-independent.

6.3 The Basic Equations

The constitutive equation, which will be used here, is given by the modified Hooke's law:

$$\tau_{ij} = \lambda e_{\alpha\alpha}\delta_{ij} + 2\mu e_{ij} - \omega\theta\delta_{ij} , \qquad (6.3.1)$$

where $\omega = (3\lambda + 2\mu)\alpha$, α being the coefficient of thermal expansion and θ is the rise of temperature. The temperature of any element of a body under elastic stress is, by virtue of the first and second laws of thermodynamics, subjected to the following transient heat conduction equation:

$$K\theta_{;ii} = \rho c_E\dot{\theta} + \omega\theta_0\dot{e}_{\alpha\alpha} , \qquad (6.3.2)$$

where K, ρ, c_E and $e_{\alpha\alpha}$ are thermal conductivity, density, specific heat at constant deformation, and dilatation of the elastic solid under consideration respectively.

The equations (6.3.1), (6.3.2) together with (4.2.1), (4.2.2) and (4.2.6) form the basic equations for coupled thermoelasticity. In the classical theory, it has been shown by Weiner [1957], that the above formulation leads to an unique solution.

Yield functions

For the temperature-dependent case von-Mises yield condition may be written as:

$$f(\tau^d_{ij}, \theta) \equiv \frac{1}{2}\tau^d_{ij}\,\tau^d_{ij} - [K(\theta)]^2 = 0, \qquad (6.3.3)$$

where $K(\theta)$ is the yield stress of the material in simple shear at temperature θ.

The Tresca yield condition may be stated in terms of a set of three yield functions. If τ_{11}, τ_{22} and τ_{33} are the three principal stresses then the three yield functions are defined as follows:

$$f_1(\tau_{11}, \tau_{22}, \tau_{33}, \theta) \equiv |\tau_{11} - \tau_{22}| - 2K(\theta) = 0,$$

$$f_2(\tau_{11}, \tau_{22}, \tau_{33}, \theta) \equiv |\tau_{11} - \tau_{33}| - 2K(\theta) = 0,$$

(6.3.4)

(6.3.4) continued

$$f_3(\tau_{11}, \tau_{22}, \tau_{33}, \theta) \equiv |\tau_{22} - \tau_{33}| - 2K(\theta) = 0.$$

6.4 Formulation of the Problem and Identification of the Transition Points

We consider here a shell of radii a and b $(a < b)$ subjected to uniform pressure on either of the surfaces and a steady state temperature θ applied on the internal surface $r = a$ of the shell. Further, if we assume that there are no body forces, body couples and couple stresses acting on the shell and if only a steady state deformation case is considered, then the basic equations discussed in §6.3 will take the following forms:

$$\tau_{ij} = \lambda e_{\alpha\alpha}\delta_{ij} + 2\mu e_{ij} - \omega\theta\delta_{ij}, \qquad (6.4.1)$$

$$K\theta,_{ii} = 0, \qquad (6.4.2)$$

$$\tau_{ij;j} = 0, \qquad (6.4.3)$$

and

$$\tau_{ij} = \tau_{ji}. \qquad (6.4.4)$$

On account of the spherical symmetry of the problem under consideration, the displacement field may be taken as:

$$u = r(1 - \beta), \quad v = 0, \quad w = 0, \quad \beta = f(r),$$

where $r = (u^2 + y^2 + z^2)^{1/2}$. Then (4.2.8) gives

$$e_{rr} = \frac{1}{2}[1-(\beta + \beta' r)^2],$$

$$e_{\theta\theta} = e_{\phi\phi} = \frac{1}{2}[1 - \beta^2], \qquad (6.4.5)$$

and

$$e_{ij} = 0 \quad \text{for} \quad i \neq j.$$

The dilatation $e_{\alpha\alpha}$ is given by

$$e_{\alpha\alpha} = \frac{3}{2} - \beta^2 - \frac{1}{2}(\beta + r\beta')^2,$$

the prime indicating the differentiation with respect to r. We have the following stresses from (6.4.1):

$$\tau_{rr} = \lambda[\frac{3}{2} - \beta^2 - \frac{1}{2}(r\beta' + \beta)^2] + \mu[1-(r\beta' + \beta)^2]-\omega\theta,$$

$$\tau_{\theta\theta} = \tau_{\phi\phi} = \lambda[\frac{3}{2} - \beta^2 - \frac{1}{2}(r\beta' + \beta)^2] + \mu[1 - \beta^2]-\omega\theta,$$

$$\tau_{r\theta}, \tau_{r\phi}, \tau_{\theta\phi} = 0, \quad \omega = \alpha(3\lambda + 2\mu). \qquad (6.4.6)$$

The equation of equilibrium which remains to be satisfied is

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{2(\tau_{rr} - \tau_{\theta\theta})}{r} = 0. \qquad (6.4.7)$$

The temperature field obtained from (6.4.2) is

$$\theta = \frac{\theta_0 a}{b-a}[\frac{b}{r} - 1],$$

where

$$\theta = \begin{cases} 0 & \text{for} \quad r = b \\ \theta_0 & \text{for} \quad r = a. \end{cases}$$

Now (6.4.7) with the help of (6.4.6) yields the following intego-differential equation

$$2\beta^2 + (r\beta' + \beta)^2 + \frac{2K_0}{r} + 2c\int r\beta'^2 dr = K, \qquad (6.4.9)$$

where

$$2K_0 = \frac{\omega c\theta_0 ab}{\mu(b-a)} \quad \text{and} \quad c = \frac{2\mu}{\lambda + 2\mu} = \frac{1 - 2\sigma}{1 - \sigma}\ .$$

Substituting $z = \frac{1}{r}$ and differentiating with respect to z we get from (6.4.9)

$$2\beta\beta' + 2\beta''(z\beta' - \beta) + K_0 - cz\beta'^2 = 0. \quad (6.4.10)$$

Letting again

$$z = e^t$$

and

$$\beta = p\sqrt{K_0}\, e^{\frac{1}{2}t} ,$$

we obtain from (6.4.10) the following equation:

$$2p(p' + \tfrac{1}{2}p) + [(p'' - \tfrac{1}{4}p)(p' - \tfrac{1}{2}p)]$$

$$- \quad c(p' + \tfrac{1}{2}p)^2 + 1 = 0.$$

$$(6.4.11)$$

The prime in (6.4.11) indicates differentiation with respect to t. The equation (6.4.11) on further substitution

$$p' + \tfrac{1}{2}p = q$$

yields

$$\frac{dp}{dq} = \frac{(\frac{q}{p} - 1)(\frac{q}{p} - \frac{1}{2})}{(\frac{1}{2} + c)(\frac{q}{p})^2 - \frac{5}{2}\frac{q}{p} - \frac{1}{p^2}} . \quad (6.4.12)$$

Here we observe that in the elastic domain $\beta \neq 0$ by assumption, K_0 is always finite and $r \neq 0$, so division by p $(p = \beta\sqrt{r}/\sqrt{K_0})$ is permissible. Finally, putting

$$\frac{q}{p} = F, \quad \frac{1}{p} = Q,$$

we get from (6.4.12),

$$\frac{dQ}{dF} = \frac{Q(F - 1)(F - \frac{1}{2})}{F^3 - (2 + c)F^2 + 3F + Q^2} \cdot \qquad (6.4.13)$$

From (6.4.13) it may be observed that some of the possible transition points of the differential system of the problem under consideration are

$$F = 1, \quad F = \frac{1}{2}, \quad F = \pm\infty .$$

There may be other transition points obtained from the roots of the denominator of (6.4.13). But in this thesis, we do not plan to discuss in detail, instead restrict our analysis to only those transition points given above. Thus the possible transition points may also be written as

$$P = -\frac{1}{2} \quad P = -1, \quad P = \pm\infty ,$$

where $P = \frac{\beta' r}{\beta} = -F$. In order to study further about the nature of these points, we consider the reciprocal deformation ellipsoid:

$$(1 - 2e_{rr})dr^2+(1 - 2e_{\theta\theta})d\theta^2+(1 - 2e_{\phi\phi})d\phi^2 = K^2,$$

which may be written with the help of (6.4.5) as

$$\beta^2(1 - F)dr^2 + \beta^2 d\theta^2 + \beta^2 d\phi^2 = K^2.$$

Hence we now identify these possible transition points given above as follows:

(i) F = 1: Here, one of the axes of the ellipsoid will be infinity, so the ellipsoid becomes a cylinder; consequently $F = 1$ is a transition point.

(ii) $F = \pm \infty$: Here, one of the axes of the ellipsoid becomes zero; so the ellipsoid reduces to a pair of planes. Hence $F = \pm \infty$ are transition points.

(iii) $F = \frac{1}{2}$: In this case, the ellipsoid becomes

$$\frac{\beta^2}{2K^2} dr^2 + \frac{\beta^2}{K^2} d\theta^2 + \frac{\beta^2}{K^2} d\phi^2 = 1,$$

K being any constant. Also $F = \frac{1}{2}$ gives

$$\beta = \frac{A}{\sqrt{r}},$$

where A is an arbitrary constant. Since r cannot be zero or infinity, we conclude that $F = \frac{1}{2}$ is a regular point. Hence the transition points are $F = \pm\infty$ and $F = 1$, the former corresponds to infinite contraction and the latter corresponds to infinite extension.

6.5 Determination of Stresses and Strains in the Transition and Plastic States

We shall determine now the stresses and the strains in the transition and plastic states corresponding to the transition points mentioned in §6.4.

Case 1: $F \to \pm\infty$, Infinite contraction

In this case, the shall is subjected to uniform pressure and steady state temperature on the interior surface of the shell.

(a) Transition Through τ_{rr}

From (6.4.6) we have,

$$2\tau_{rr} = (3\lambda + 2\mu) - 2\lambda\beta^2 - (r\beta' + \beta)^2 - 2\mu(r\beta' + \beta)^2 - 2\beta_0\left(\frac{b}{r} - 1\right),$$

where

$$\beta_0 = \frac{\omega\theta_0 a}{b-a} .$$

Let

$$R \equiv 1- \frac{2\tau_{rr}}{3\lambda+2\mu} - \frac{2\beta_0}{3\lambda+2\mu}(\frac{b}{r} -1) = \frac{2\lambda}{3\lambda+2\mu}\beta^2+ \frac{\lambda+2\mu}{3\lambda+2\mu}(r\beta'+ \beta)^2,$$

then

$$R = \frac{\beta^2}{3 - 2c}[2(1 - c)+(1 - F)^2].$$

Taking logarithmic differentiation with respect to r we get

$$\frac{d(\log R)}{d(\log r)} = \frac{-2cF^2 + 4cF + 2Q^2}{2(1-c)+(1-F)^2} .$$

Hence the solution for τ_{rr} as $F \to \pm \infty$ may be obtained using the boundary condition namely, at $r = b$ $\tau_{rr} = 0$, as

$$\tau_{rr} = \frac{(2-c)}{c} y[(1-(\frac{b}{r})^{2c}]-\beta_0(\frac{b}{r}) - 1). \qquad (6.5.1)$$

The equation of equilibrium (6.4.7) gives

$$\tau_{\theta\theta} - \tau_{rr} = (2-c)y(\frac{b}{r})^{2c}+ \frac{\beta_0}{2}\frac{b}{r} . \qquad (6.5.2)$$

Therefore

$$\tau_{\theta\theta} = \tau_{\phi\phi} = (2-c)y(\frac{b}{r})^{2c}+(\frac{2-c}{c})[1-(\frac{b}{r})^{2c}]$$

$$-\beta_0[\frac{b}{2r} - 1]. \qquad (6.5.3)$$

For the fully plastic state $c \to 0$ and we get the following

$$\tau_{rr} = 4y\log\frac{r}{b} - \beta_0{}^*(\frac{b}{r} - 1), \qquad (6.5.4)$$

$$\tau_{\theta\theta} - \tau_{rr} = 2y + \frac{\beta_0{}^*}{2}(\frac{b}{r}), \qquad (6.5.5)$$

where $\beta_0{}^* = \lim_{c\to 0} \beta_0$.

When a material changes from elastic to plastic state the bulk modulus $K = \frac{1}{3}(3\lambda + 2\mu)$ becomes large. It has been shown by Rosenfield and Averbach [1956] that the coefficient of expansion, α attains its maximum just after the material crosses it's elastic limit and becomes very small when the material reaches the fully plastic state. Hence $\omega[= \alpha(3\lambda + 2\mu]$ reaches a finite fixed value, however small. We denote this number by ω_0. Hence

$$\omega_0 = \lim_{c\to 0} \alpha(3\lambda + 2\mu),$$

and so

$$\beta_0{}^* = \frac{\omega_0\theta_0 a}{b-a}$$

is finite. If the temperature is moderate and the pressure is very high, then

$$\beta_0^* \cong 0,$$

because high pressure reduces the coefficient of expansion [Clark, 1966]. However, we do not consider β_0^* to be zero in our work, which still needs more experimental evidence.

Returning to our original discussion, from (6.5.2), it is clear that the value of $|\tau_{\theta\theta} - \tau_{rr}|$ is the largest among all other stresses, and attains its maximum at $r = a$. Hence the yield will occur at $r = a$ and the yield stress in this case is

$$y^* = 2y + \frac{\beta_0^*}{2}\left(\frac{b}{a}\right). \tag{6.5.6}$$

The interpretation of (6.5.6) shows that presence of pressure and temperature on the same surface of the shell slows down the yield; because in the absence of outcoming heat in §4.4, the yield stress in compression is found to be $2y$. The material in the presence of the outcoming heat needs more stress to yield whence

$$y^* > 2y.$$

Now we can write y^* as

$$y^* = 2y + \frac{\omega_0 \ \theta_0}{2(\frac{b}{a} - 1)} \frac{b}{a} \ ,$$

where $\omega_0 = \lim_{c \to 0} \omega$. The above result shows that y^* depends on both θ_0 , the ratio of radii $\frac{b}{a}$ and yield stress in compression $2y$. Hence a thick-walled shell requires more heat to yield than a thin-walled shell, so long as the pressure remains constant in both the cases. [Wilhoit, 1958].

Derrington and Johnson [1958] have shown that if pressure and temperature are applied separately on the inner surface of a thick-walled spherical shell, yield occurs first at the inside radius in each case. But if both pressure and temperature are applied together on the inner surface, it has been shown that the onset of yield may be caused to occur at any desired radius by making a suitable choice of pressure, temperature and thickness of the shell. An immediate inference from these results is that an outward heat flow induces shear stresses opposite in nature to those caused by the pressure. This leads to the idea of "stress-saving"; that is, by introducing a suitable temperature gradient, the stresses due to pressure may be reduced. This idea is very important in modern applied science.

From (6.5.5) it can be seen that the onset of yield depends on θ_0 , if a,b remain fixed. If θ_0 is small $r(a \leq r < b)$ is small, and vice versa since y^* is fixed. Hence by proper choice of θ_0 the shell may be made to yield at $r = a$, $r = b$ or anywhere in the body of the shell. This is again in agreement with Derrington and Johnson's investigation.

The yield function in this case may be taken as

$$f(\tau_{rr}, \tau_{\theta\theta}, \tau_{\phi\phi}, \theta)$$

$$\equiv |\tau_{\theta\theta} - \tau_{rr}| - \frac{\beta_0{}^*}{2}(\frac{b}{r}) - 2y = 0,$$

which is of Tresca's type of yield function.

(b) <u>Transition Through $\tau_{\theta\theta}$:</u>

From (6.4.6) we have

$$\tau_{\theta\theta} = \lambda[\frac{3}{2} - \beta^2 - \frac{1}{2}(\beta + r\beta')^2] + \mu(1 - \beta^2) - \beta_0(\frac{b}{r} - 1).$$

Here let

$$R \equiv 1 - \frac{2\tau_{\theta\theta}}{3\lambda+2\mu} - \frac{2\beta_0}{3\lambda+2\mu}(\frac{b}{r} - 1) = \beta^2[\frac{2-c}{3-2c} + \frac{1-c}{3-2c}(1-F^2)].$$

Taking logarithmic differentiation with respect to r, we get

$$\frac{d(\log R)}{d(\log r)} = \frac{-2[c(1-c)F^2 + cF - (1-c)Q^2]}{(2-c)+(1-c)F^2}.$$

The solution for $\tau_{\theta\theta}$ may be obtained when $F \to \pm\infty$ as

$$\tau_{\theta\theta} = (\frac{2-c}{c})y[1 - B_0 r^{-2c}] - \beta_0(\frac{b}{r} - 1).$$

The equation of equilibrium (6.4.7) with the help of the boundary condition for τ_{rr} gives

$$\tau_{rr} = (\frac{2-c}{c})y[1-\{1-\frac{c\beta_0}{2(2-c)y}\}(\frac{b}{r})^{2c}]-\frac{\beta_0}{2}(\frac{2b}{r} - 1), \quad (6.5.7)$$

$$\tau_{\theta\theta} = (\frac{2-c}{c})y[1-\{1-\frac{c\beta_0}{2(2-c)y}\}(1-c)(\frac{b}{r})^{2c}-\beta_0(\frac{b}{r} - 1), \quad (6.5.8)$$

$$\tau_{\theta\theta} - \tau_{rr} = (2-c)y[1- \frac{c\beta_0}{2(2-c)y}](\frac{b}{r})^{2c}+ \frac{\beta_0}{2}(\frac{b}{r}). \quad (6.5.9)$$

For the fully plastic state $c \to 0$ and we get the corresponding solutions. The yield function here may be taken as

$$f(\tau_{rr} , \tau_{\theta\theta} , \tau_{\phi\phi} , \theta) \equiv |\tau_{\theta\theta} - \tau_{rr}| - \frac{\beta_0^*}{2}(\frac{b}{r}) - 2y = 0.$$

Thus the yield functions obtained for transitions through τ_{rr} and $\tau_{\theta\theta}$ are the same.

(c) <u>Transition Through $(\tau_{\theta\theta} - \tau_{rr})$.</u>

Here

$$R \equiv \frac{\tau_{\theta\theta} - \tau_{rr}}{\mu} = \beta^2[(1 - F)^2-1],$$

and a similar procedure as before gives,

$$\tau_{\theta\theta} - \tau_{rr} = \mu B_1(\frac{b}{r})^{2c}, \quad (6.5.10)$$

where B_1 is a parameter. The equilibrium equation and the boundary conditions give

$$\tau_{rr} = \frac{p_i}{[(\frac{b}{a})^{2c} - 1]}[1 - (\frac{b}{r})^{2c}], \qquad (6.5.10a)$$

and so

$$\tau_{\theta\theta} - \tau_{rr} = \frac{cp_i}{[(\frac{b}{a})^{2c} - 1]}(\frac{b}{r})^{2c}. \qquad (6.5.11)$$

For the fully plastic state $c \to 0$ and we have

$$\tau_{rr} = p_i \frac{\log(\frac{r}{b})}{\log(\frac{b}{a})}, \qquad (6.5.12)$$

$$\tau_{\theta\theta} - \tau_{rr} = \frac{p_i}{2\log\frac{b}{a}} = y^*, \qquad (6.5.13)$$

where y^* is the yield stress in tension for the thermo-elastic-plastic case. Both the transitions through $\tau_{\theta\theta}$ and τ_{rr} show work-hardening. Only the transition through $(\tau_{\theta\theta} - \tau_{rr})$ lead to a yield condition of the Tresca type. From (6.5.13) we get the pressure p_i when the yield starts as

$$p_i = 2y^*\log(\frac{b}{a}). \qquad (6.5.14)$$

From the foregoing results in this case we have noticed in each case (a), (b) and (c) that $|\tau_{\theta\theta} - \tau_{rr}|$ is the largest in value among all the stresses or their differences. Hence transition should occur through $|\tau_{\theta\theta} - \tau_{rr}|$.

Case 2: <u>$F \to 1$, Infinite extension</u>

In this case the shell is subjected to uniform pressure on the external surface of the shell and a steady state temperature on the internal surface.

(a) <u>Transition Through τ_{rr}:</u>

From (6.4.6) we have

$$2\tau_{rr} = (3\lambda+2\mu)-[2\lambda\beta^2 + (\lambda+2\mu)(r\beta'+\beta)^2-2\beta_0[\frac{b}{r} - 1],$$

and a similar procedure as in Case 1 yields:

$$\tau_{rr} = (\frac{2-c}{c})y[1-A_1 r^{\frac{c}{1-c}} e^{\frac{A_0 r}{1-c}}] - \beta_0(\frac{b}{r} - 1).$$

The equation of equilibrium (6.4.7) gives

$$\tau_{\theta\theta} - \tau_{rr} = - A_1 \frac{(2-c)}{2(1-c)} y\, r^{\frac{c}{1-c}} e^{\frac{A_0 r}{1-c}} [1 + \frac{A_0 r}{c}] + \frac{\beta_0}{2}\frac{b}{r}. \qquad (6.5.15)$$

For the fully plastic state $c \to 0$ and it can be seen from (6.5.15) that $(\tau_{\theta\theta} - \tau_{rr})$ goes to infinity. In order that $(\tau_{\theta\theta} - \tau_{rr})$ may remain finite, we must have $A_0 \equiv 0$. Hence rewriting the above equations for τ_{rr} and $(\tau_{\theta\theta} - \tau_{rr})$ and using the boundary condition for τ_{rr} at $r = a$, we get

$$\tau_{rr} = \frac{y(2-c)}{c}[1-\{1 - \frac{\omega\theta_0 c}{(2-c)y}\}(\frac{r}{a})^{\frac{c}{1-c}}] - \beta_0(\frac{b}{r} - 1),$$

(6.5.16)

$$\tau_{rr} - \tau_{\theta\theta} = \frac{1}{2}[(1 - \frac{c\omega\theta_0}{(2-c)y})(\frac{2-c}{1-c})y(\frac{r}{a})^{\frac{c}{1-c}} - \beta_0 \frac{b}{r}].$$

(6.5.17)

The maximum principal stress difference occurs at $r = b$ which can be seen from (6.5.17).

For fully plastic state,

$$\tau_{rr} = 2y\log\frac{a}{r} - \beta_0{}^*(\frac{b}{r} - 1),$$

$$\tau_{rr} - \tau_{\theta\theta} = y - \frac{\beta_0{}^*}{2}(\frac{b}{r}). \qquad (6.5.17a)$$

If $\tilde{y}$ is yield stress in tension in the thermo-elastic-plastic case, then at $r = b$

$$\tilde{y} = y - \frac{\beta_0{}^*}{2}. \qquad (6.5.18)$$

The equation (6.5.18) shows that in the presence of out flowing heat at the internal surface and tensile stress at the external surface, the shell begins to yield earlier than the case when only tensile force is applied at the external surface. This result is a contrast to that when both temperature and pressure are applied at the internal surface. Rewriting the equation (6.5.18) we have

$$\tilde{y} = y - \frac{\omega_0 \theta_0}{2(\frac{b}{a} - 1)} ,$$

which shows that $\tilde{y}$ depends on θ_0 and the radii of the shell. Again, we notice that the above equation leads to the same conclusion as in Case 1; that is, the larger the thickness of the shell the higher is the temperature necessary to start the yielding.

If both pressure and temperature are applied on the external surface of the shell a similar phenomenon is expected to occur as the case when both of them were applied on the internal surface.

(b) <u>Transition Through $\tau_{\theta\theta}$:</u>

The results corresponding to the transition through $\tau_{\theta\theta}$ may be obtained following a similar procedure as before and may be given below:

$$\tau_{rr} = (\frac{2-c}{c}) y [1 - \{1 - \frac{\beta_0 (2b-a)}{2a} \cdot \frac{c}{(2-c)y}\} (\frac{a}{r})^{2c}] - \frac{\beta_0}{2}(\frac{2b}{r}) - 1) , \qquad (6.5.19)$$

$$\tau_{\theta\theta} - \tau_{rr} = (2-c) y [1 - \frac{\beta_0 (2b-a)}{2a} \cdot \frac{c}{(2-c)y}] (\frac{a}{r})^{2c} + \frac{\beta_0}{2}(\frac{b}{r}) . \qquad (6.5.20)$$

For the fully plastic state $c \to 0$ and we get from the above equations

$$\tau_{rr} = 4y\log\frac{r}{a} + \frac{\beta_0{}^*}{2}[\frac{2b}{a} - \frac{4b}{r} + 1],$$

$$\tau_{\theta\theta} - \tau_{rr} = 2y + \frac{\beta_0{}^*}{2}(\frac{b}{r}). \qquad 6.5.20a)$$

(c) Transition Through $(\tau_{\theta\theta} - \tau_{rr})$.

The stresses in this case may be obtained as before and are given below:

$$\tau_{rr} = \frac{-p_0}{[1-(\frac{b}{a})^{2c-6}]}[1-(\frac{r}{a})^{2c-6}], \qquad (6.5.21)$$

$$\tau_{\theta\theta} = \frac{-p_0}{[1-(\frac{b}{a})^{2c-6}]}[1+(2-c)(\frac{r}{a})^{2c-6}], \qquad (6.5.22)$$

and

$$\tau_{\theta\theta} - \tau_{rr} = \frac{-p_0(3-c)}{[1-(\frac{b}{a})^{2c-6}]}(\frac{r}{a})^{2c-6}, \qquad (6.5.23)$$

where p_0 is the pressure applied at the external surface. For the fully plastic state, we get

$$\tau_{rr} = -p_0\frac{[1-(\frac{a}{r})^6]}{[1-(\frac{a}{b})^6]},$$

and

$$\tau_{\theta\theta} - \tau_{rr} = \frac{-3p_0}{1-(\frac{a}{b})^6} \cdot (\frac{a}{r})^6 . \qquad (6.5.23a)$$

From the above results we again notice that $|\tau_{\theta\theta} - \tau_{rr}|$ is the largest in value in each of the cases (a), (b) and (c) than any other stress or stress-difference. Hence transition should occur through $|\tau_{\theta\theta} - \tau_{rr}|$.

Now, the asymptotic solutions for the strains in the transition state may be obtained:

$F \to \pm \infty$:

From (6.4.5) we have

$$1 - 2e_{rr} = \beta^2[1 - F]^2.$$

Hence

$$\frac{d[\log(1-2e_{rr})]}{d(\log r)} = -2[\frac{cF^2 - 2F - Q^2}{(1 - F)^2}],$$

from which we obtain

$$e_{rr} = \frac{1}{2}[1-D_1(\frac{b}{r})^{2c}] \quad \text{as} \quad F \to \pm \infty, \qquad (6.5.24)$$

where D_1 is a parameter. In like manner we get

$$e_{\alpha\alpha} = \frac{1}{2}[3 - D_1(\frac{b}{r})^{2c}], \qquad (6.5.25)$$

(6.5.25) continued

$$e_{\theta\theta} = \frac{1}{2} = e_{\phi\phi} ,$$

$$e_{\theta\theta} - e_{rr} = \frac{D_1}{2} (\frac{b}{r})^{2c} .$$

F → 1:

Here the values of the strains, obtained as before are:

$$e_{\alpha\alpha} = \frac{1}{2}[3 - D_2 (\frac{a}{r})^c],$$

$$e_{rr} = \frac{1}{2} ,$$

$$e_{\theta\theta} = e_{\phi\phi} = \frac{1}{2}[1 - \frac{D_2}{2}(\frac{a}{r})^c], \qquad (6.5.26)$$

$$e_{\theta\theta} - e_{rr} = K_0 (\frac{a}{r})^{6-2c} .$$

6.6 The Constitutive Equation for the Transition and Plastic States

We shall now obtain the constitutive equation for the transition and plastic states corresponding to the following cases:

Case 1: $F \to \pm \infty$

In this case we have shown earlier that the transition first occurs through $(\tau_{\theta\theta} - \tau_{rr})$, since $(\tau_{\theta\theta} - \tau_{rr})$ is the largest one among all the stresses and it attains its maximum earlier than the other stresses or stress differences. Hence the constitutive equation for the transition state should be obtained through the transition of $(\tau_{\theta\theta} - \tau_{rr})$. Hence from (6.5.10), (6.5.10a), (6.5.24) and (6.5.25) we get the following equations:

$$e^d_{rr} = \frac{D_1}{2\mu B_1} \tau^d_{rr} ,$$

$$e^d_{\theta\theta} = \frac{D_1}{2\mu B_1} \tau^d_{\theta\theta} ,$$

$$e^d_{\phi\phi} = \frac{D_1}{2\mu B_1} \tau^d_{\phi\phi} ,$$

where

$$B_1 = \frac{cp_i}{\mu[(\frac{b}{a})^{2c} - 1]} .$$

Hence the constitutive equation for the transition state is given by

$$e^d_{ii} = \lambda_1 \tau^d_{ii} , \tag{6.6.1}$$

where

$$\lambda_1 = \frac{D_1}{2\mu B_1} .$$

Now allowing, as before, $c \to 0$ in the above equation for the transition state we obtain the constitutive equation

$$\dot{e}^d_{ii} = \lambda_1' \tau^d_{ii} , \tag{6.6.2}$$

where

$$\lambda_1' = \frac{D_1 \log \frac{b}{a}}{\dot{p}_i} = \frac{D_1}{2y^*}$$

by using the equation (6.5.13). Thus the constitutive equation (6.6.2) takes the form

$$\dot{e}^{d}_{ii} = \frac{D_1}{2y^*} \tau^{d}_{ii} , \qquad (6.6.2a)$$

and represents the fully plastic state since the proportionality factor between the stress and strain-rate is obviously a constant $\frac{D_1}{2y^*}$.

Now, if we consider transition through τ_{rr} , the equations (6.5.1), (6.5.2), (6.5.3), (6.5.24) and (6.5.25) yield

$$e^{d}_{ii} = f_1(I_1)\tau^{d}_{ii} , \qquad (6.6.3)$$

where

$$f_1(I_1) = \frac{D_1(\frac{b}{r})^{2c}}{2(2-c)y(\frac{b}{r})^{2c} + \beta_0(\frac{b}{r})} ,$$

I_1 being the first stress invariant. Similarly, the constitutive equation for the transition state through $\tau_{\theta\theta}$ may be obtained from (6.5.7), (6.5.8), (6.5.9), (6.5.24) and (6.5.25) as:

$$e^{d}_{ii} = f_2(I_1)\tau^{d}_{ii} , \qquad (6.6.4)$$

where

$$f_2(I_1) = \frac{D_1(\frac{b}{r})^{2c}}{2(2-c)y\{1 - \frac{c\beta_0}{2(2-c)y}\}(\frac{b}{r})^{2c} + \beta_0(\frac{b}{r})} .$$

We now notice from (6.6.1), (6.6.3) and (6.6.4) that the constitutive equations for the transition states obtained by the transition through $(\tau_{\theta\theta} - \tau_{rr})$, τ_{rr} and $\tau_{\theta\theta}$ respectively, do not lead to the same equation which is not unexpected.

Again allowing $c \to 0$ to go from the transition state to the neighboring state, we have from equations (6.6.3) and (6.6.4) the constitutive equation

$$\dot{e}^{d}_{ii} = \frac{D_1}{2|\tau^{p}_{\theta\theta} - \tau^{p}_{rr}|} \tau^{d}_{ii} , \qquad (6.6.5)$$

where the yield function $|\tau^{p}_{\theta\theta} - \tau^{p}_{rr}| = 2y + \frac{\beta_0^*}{2}(\frac{b}{r})$, being a function of r.

It is important to note here that transition through τ_{rr} or $\tau_{\theta\theta}$ does not lead to the constitutive equation similar to that of (6.6.2) when $c \to 0$. This implies that transition may or may not occur through these branches. In that case, when the transition actually occurs in a material either through τ_{rr} or through $\tau_{\theta\theta}$, the result (6.6.5) implies that the material might go into either creep or fatigue or some other state. Otherwise, there can be no transition at all through τ_{rr} and $\tau_{\theta\theta}$.

Case 2: $\underline{F \to 1}$

In this case, from (6.5.21), (6.5.22), (6.5.23) and (6.5.26) we obtain

$$e^d_{rr} = \frac{K_0}{A} \tau^d_{rr} , \qquad (6.6.8)$$

$$e^d_{\theta\theta} = \frac{K_0}{A} \tau^d_{\theta\theta} , \qquad (6.6.9)$$

and

$$e^d_{\phi\phi} = \frac{K_0}{A} \tau^d_{\phi\phi} , \qquad (6.6.10)$$

where

$$A = \frac{-p_0(3-c)}{[1-(\frac{b}{a})^{2c-6}]} .$$

Hence the constitutive equation in the transition state obtained from the above equations is:

$$e^d_{ii} = \lambda_2 \tau^d_{ii} , \qquad (6.6.11)$$

where

$$\lambda_2 = \frac{K_0}{A} .$$

The constitutive equation for the fully plastic state may be obtained from (6.6.11) as

$$\dot{e}^d_{ii} = \lambda_2' \tau^d_{ii} , \qquad (6.6.12)$$

where

$$\lambda_2' = \frac{K_0[(\frac{a}{b})^6 - 1]}{3p_0} .$$

Now the constitutive equations for transition states, obtained through the transition of τ_{rr} and $\tau_{\theta\theta}$, have entirely different forms. Moreover, they do not become identical with (6.6.12) even when $c \to 0$, implying that after the transition state the material may go into any one neighboring states, creep, fatigue, etc.

CHAPTER 7

THERMO-ELASTIC-PLASTIC TRANSITION OF TUBES SUBJECTED TO UNIFORM PRESSURE AND STEADY STATE TEMPERATURE

7.1 Preliminary Remarks

This chapter is devoted to a discussion of thermo-elastic-plastic transition in tubes under uniform pressure and a steady state temperature.

As has already been mentioned in chapter 6, this problem has been solved by several authors in which the principle of superposition and ad-hoc, semi-empirical laws have been used. In our analysis the principle of superposition does not hold as the strain measure which will be taken for our discussion is not linear and also we do not assume any ad-hoc, semi-empirical laws such as yield conditions.

The basic equations mentioned in §6.2 will remain valid in this chapter as well, except that the equation of equilibrium will be expressed in a cylindrical coordinate system.

7.2 Formulation of the Problem and Identification of the Transition Points

We consider a tube of internal and external radii a and b $(a < b)$ subjected to uniform pressure and a steady state temperature θ applied on the inner surface $r = a$. Further, if we assume that there are no body forces, body couples and couple stresses acting on the tube, and if only a steady deformation problem is considered, the basic equations discussed in §6.3 may be written as in (6.4.1), (6.4.2), (6.4.3) and (6.4.4). On account of axial symmetry of the problem, the displacement field may be chosen as

$$u = r(1-\beta), \quad v = 0, \quad w = z(1-d_0), \quad \beta = f(r),$$

where $r = (x^2 + y^2)^{1/2}$, $d_0(\leq 1)$ being a constant to be determined. By using the Almansi strain measure given by (4.2.7) we have the following expressions for strain:

$$\begin{aligned} e_{rr} &= \tfrac{1}{2}[1-(r\beta' + \beta)^2], \\ e_{\theta\theta} &= \tfrac{1}{2}[1- \beta^2], \\ e_{zz} &= \tfrac{1}{2}[1-d_0{}^2], \\ e_{r\theta}\,,\ e_{\theta z}\,,\ e_{rz} &= 0, \end{aligned} \qquad (7.2.1)$$

and

$$e_{\alpha\alpha} = 1-\tfrac{1}{2}(r\beta' + \beta)^2-\tfrac{1}{2}\beta^2+ \tfrac{1}{2}(1-d_0{}^2),$$

the prime indicates the differentiation with respect to r. The stresses obtained from (6.4.1) and (7.2.1) may be written as:

$$\begin{aligned} \tau_{rr} = \lambda[1-\tfrac{1}{2}(r\beta' + \beta)^2- \tfrac{1}{2}\beta^2 + \tfrac{1}{2}(1-d_0{}^2)] \\ + \mu[1-(r\beta' + \beta)^2]-\omega\theta, \end{aligned}$$

(7.2.2)

(7.2.2) continued

$$\tau_{\theta\theta} = \lambda[1-\tfrac{1}{2}(r\beta' + \beta)^2 - \tfrac{1}{2}\beta^2 + \tfrac{1}{2}(1-d_0^{\,2})] + \mu(1-\beta^2)-\omega\theta,$$

$$\tau_{zz} = \lambda[1-\tfrac{1}{2}(r\beta' + \beta)^2 - \tfrac{1}{2}\beta^2 + \tfrac{1}{2}(1-d_0^{\,2})] + \mu(1-d_0^{\,2})-\omega\theta,$$

and τ_{zr} , $\tau_{r\theta}$, $\tau_{\theta z} = 0$.

The temperature field obtained from (6.4.2) is

$$\theta = \frac{\theta_0 \log \frac{b}{r}}{\log \frac{b}{a}},$$

where

$$\theta = \begin{cases} 0, & \text{for} \quad r = b \\ \theta_0, & \text{for} \quad r = a. \end{cases}$$

the only equilibrium equation which remains to be satisfied is

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = 0, \qquad (7.2.3)$$

the other two equations are satisfied identically. Substituting (7.2.2) in (7.2.3) we have

$$\beta^2 + (r\beta' + \beta)^2 - K_0 \log r + c\int r\beta'^2 dr = K, \qquad (7.2.4)$$

where

$$K_0 = \frac{2\omega\theta_0}{(\lambda+2\mu)\log(\frac{b}{a})} \quad \text{and} \quad c = \frac{2\mu}{\lambda+2\mu} .$$

Setting $\log r = z$ in (6.2.4) and then differentiating with respect to z, we have

$$2\beta\beta' + 2(\beta + \beta')(\beta'' + \beta') - K_0 + c\beta'^2 = 0 \tag{7.2.5}$$

Setting again

$$\beta = \sqrt{K_0}\, p,$$

and

$$p + p' = q,$$

the equation (7.2.5) yields

$$2(q - p)p + 2qq' + c(q - p)^2 = 1,$$

which may also be put in the form

$$\frac{dQ}{dF} = \frac{QF(F - 1)}{F^3 - (1-\frac{c}{2})F^2 - (c-1)F - (\frac{Q^2}{2} - \frac{c}{2} + 1)}, \tag{7.2.6}$$

where $\frac{q}{p} = F$ and $\frac{1}{p} = Q$. We assume here, as in the previous chapters, that β belongs to $c^{(2)}$ and does not vanish throughout the elastic domain. Hence division by p $(p = \frac{\beta}{\sqrt{K_0}})$ in (7.2.6) is permissible. From (7.2.6) it may be observed that some of the possible transition points of the differential system of the problem under consideration are $F = 0$, $F = 1$ and $F = \pm\infty$. There may be other transition points given by the roots of the expression in the denominator of (7.2.6). But in this thesis, we shall restrict, for purposes of illustration, our discussion only to the above-mentioned transition points.

Now it can be shown that

$$F = 1 + P,$$

where $P = \frac{r\beta'}{\beta}$. Hence, $F \to 0$ implies $P \to -1$, $F \to 1$ implies $P \to 0$ and $F \to \pm\infty$ implies $P \to \pm\infty$. In order to study further about the nature of these points, we consider the reciprocal strain ellipsoid

$$(1-2e_{rr})dr^2 + (1-2e_{\theta\theta})d_\theta{}^2 + (1-2e_{zz})d_z{}^2 = K^2$$

(where K is some constant) which may be rewritten with the help of (7.2.1) as

$$\beta^2 F dr^2 + \beta^2 d_\theta{}^2 + d_0{}^2 dz^2 = K^2. \qquad (7.2.7)$$

It may be seen from (7.2.7) that $F = 0$ and $F = \pm\infty$ are transition points; for the first case the ellipsoid becomes a cylinder and for the second case the ellipsoid tends to become a pair of planes. The third point $F = 1$ is a regular point, because, then the ellipsoid (7.2.7) becomes

$$\beta^2 dr^2 + \beta^2 d_\theta{}^2 + d_0{}^2 d_z{}^2 = K^2,$$

where β becomes some constant. Since $F = P + 1$, $F = 0$

corresponds to infinite extension and $F = \pm\infty$ corresponds to infinite contraction.

7.3 Determination of Stresses and Strains in the Transition and Plastic States

We shall determine now the stresses and the strains in the transition and plastic states corresponding to the transition points mentioned in §7.2.

Case 1: $F \to \pm\infty$ Infinite Contraction

In this case the tube is subjected to uniform pressure and a steady state temperature on the inner surface of the tube.

(a) Transition Through τ_{rr}:

Equation (7.2.2) may be rewritten

$$R = \beta^2[(1 - c)+F^2], \qquad (7.3.1)$$

where

$$R \equiv -\frac{c}{\mu}\tau_{rr} + D_0 - \beta_0 \log\frac{b}{r},$$

$$D_0 = (3 - 2c)-(1 - c)d_0^{\,2}, \quad \beta_0 = \frac{\omega\theta_0}{\log\frac{b}{a}}$$

and

$$c = \frac{2\mu}{\lambda + 2\mu}.$$

Taking logarithmic differentiation with respect to r we have from (7.3.1)

$$\frac{d(\log R)}{d(\log r)} = \frac{-cF^2 + c + Q^2}{(1-c) + F^2} .$$

Hence

$$R = A_0 r^{-c} \quad \text{as} \quad F \to \pm \infty, \qquad (7.3.1a)$$

where A_0 is some arbitrary constant. The boundary condition, $\tau_{rr} = 0$ when $r = b$, and (7.3.1a) gives

$$\tau_{rr} = \frac{\mu}{c} D_0 [1 - (\frac{b}{r})^c - \beta_0 \log \frac{b}{r} . \qquad (7.3.2)$$

then the equation of equilibrium (7.2.3) with (7.3.2) yields

$$(\tau_{\theta\theta} - \tau_{rr}) = D_0 \mu (\frac{b}{r})^c + \beta_0 . \qquad (7.3.3)$$

Applying the same technique as above we may get

$$\tau_{\theta\theta} - \tau_{zz} = \mu d_0^{\ 2} , \qquad (7.3.4)$$

$$\tau_{rr} - \tau_{zz} = \mu d_0^{\ 2} - \mu D_0 (\frac{b}{r})^c - \beta_0 . \qquad (7.3.5)$$

The yield starts at $r = a$ corresponding to the largest of these stresses and their differences. The pressure for which the material starts to yield may be obtained from (7.3.2) and thus

$$p_i' = \frac{D_0 \mu}{c} [(\frac{b}{a})^c - 1] + \beta_0 \log (\frac{b}{a}) ,$$

while in chapter 5 the cooresponding pressure was p_i^* where

$$p_i^* = \frac{D_0\mu}{c}[(\frac{b}{a})^c - 1].$$

Hence it is clear that the outward flow of heat opposes the yielding. The same phenomenon which has been observed in chapter 5, that yielding may start anywhere in the shell depending on the pressure, temperature and the ratio of the radii, may also be seen to occur in this problem.

In the fully plastic state $c \to 0$ and from (7.3.2) through (7.3.5) we have

$$\begin{aligned}
\tau_{rr} &= [\tfrac{2}{3}y(3-d_0{}^2) + \beta_0^*]\log\frac{r}{b}, \\
\tau_{\theta\theta} - \tau_{rr} &= \tfrac{2}{3}y(3-d_0{}^2) + \beta_0^*, \\
\tau_{\theta\theta} - \tau_{zz} &= \tfrac{2}{3}yd_0{}^2, \\
\tau_{zz} - \tau_{rr} &= 2y + \beta_0{}^* - \tfrac{4}{3}yd_0{}^2.
\end{aligned} \tag{7.3.6}$$

None of the above results in (7.3.6) is independent of d_0, hence no relation may be used as yield condition; however

$$\tau_{\theta\theta}^d = \frac{2}{3}y + \frac{\beta_0^*}{3},$$

which is independent of d_0 and therefore may be used as yield condition.

We may add here that while considering this problem, Bland [1956] has assumed $\tau_{rr} \leq \tau_{zz} \leq \tau_{\theta\theta}$ in both the elastic and plastic state. But in our transition analysis this result comes out automatically.

(b) <u>Transition Through $\tau_{\theta\theta}$:</u>

Here we have

$$\tau_{\theta\theta} = \lambda[1 - \tfrac{1}{2}(r\beta' + \beta)^2 - \tfrac{1}{2}\beta^2 + \tfrac{1}{2}(1 - d_0^2)]$$

$$+ \mu[1 - \beta^2] + \beta_0 \log \frac{b}{r} ,$$

and hence

$$R \equiv [-\frac{cT_{\theta\theta}}{\mu} + D_0] = \beta^2[1 + (1 - c)F^2].$$

A similar technique as before and use of the boundary condition, $\tau_{rr} = 0$ when $r = b$, yield the following results in the transition state:

$$\begin{aligned}
\tau_{rr} &= \frac{\mu}{c}[D_0 - (D_0 - \frac{c}{\mu}\beta_0)(\frac{b}{r})^c] - \beta_0[\log\frac{b}{r} + 1], \\
\tau_{\theta\theta} - \tau_{rr} &= \mu[D_0 - \frac{c}{\mu}\beta_0](\frac{b}{r})^c + \beta_0 , \\
\tau_{\theta\theta} - \tau_{zz} &= \mu d_0^2, \\
\tau_{rr} - \tau_{zz} &= \mu d_0^2 - \mu[D_0 - \frac{c}{\mu}\beta_0](\frac{b}{r})^c - \beta_0.
\end{aligned} \tag{7.3.7}$$

The cooresponding results in the fully plastic state may be obtained from (7.3.7) letting $c \to 0$ and are given below:

$$\tau_{rr} = -\frac{2}{3}y(3-d_0^{\,2})\log\frac{b}{r} - \beta_0^*(\log\frac{b}{r} + 1),$$

$$\tau_{\theta\theta} - \tau_{rr} = \frac{2}{3}y(3-d_0^{\,2}) + \beta_0^* ,$$

$$\tau_{\theta\theta} - \tau_{zz} = \frac{2}{3}yd_0^{\,2} ,$$

$$\tau_{rr} - \tau_{zz} = \frac{4}{3}yd_0^{\,2} - 2y - \beta_0^*.$$

The yield condition again may be taken as

$$\tau_{\theta\theta}^{d} = \frac{2}{3}y + \frac{\beta_0^*}{3} .$$

(c) <u>Transition Through $(\tau_{\theta\theta} - \tau_{rr})$:</u>

We may write here

$$R \equiv \frac{\tau_{rr} - \tau_{\theta\theta}}{\mu} = \beta^2[1 - F^2],$$

and obtain as in (a) and (b)

$$\tau_{rr} - \tau_{\theta\theta} = \mu A_0 r^{-c} \quad \text{as} \quad F \to \pm\infty, \tag{7.3.8}$$

where A_0 is an arbitrary constant. The equation of equilibrium (7.2.3) with (7.3.8) and boundary condition, $\tau_{rr} = 0$ when $r = b$, yield

$$\tau_{rr} = \frac{\mu B_0}{c}[1 - (\frac{b}{r})^c], \tag{7.3.9}$$

$B_0 = -b^{-c}A_0$ being a parameter.

Hence (7.3.8) may be rewritten as

$$\tau_{\theta\theta} - \tau_{rr} = \mu B_0 (\frac{b}{r})^c . \qquad (7.3.10)$$

The parameter B_0 may be obtained from (7.3.9) using the condition

$$(\tau_{rr})_{r=a} = -p_i .$$

Hence

$$B_0 = \frac{cp_i}{\mu[(\frac{b}{a})^c - 1]} .$$

A similar treatment as before gives

$$\tau_{\theta\theta} - \tau_{zz} = \mu d_0^2 , \qquad (7.3.11)$$

and

$$\tau_{rr} - \tau_{zz} = \mu d_0^2 - \mu B_0 (\frac{b}{r})^c . \qquad (7.3.12)$$

Now, we find from (a), (b) and (c) by comparison that $|\tau_{\theta\theta} - \tau_{rr}|$ attains its largest value in each case if $(1 - c)(\frac{b}{a})^c < 1$. Hence transition occurs through $|\tau_{\theta\theta} - \tau_{rr}|$ in the problem under discussion.

Case 2: <u>$F \to 0$, Infinite Extension</u>

In this case the tube is subjected to a uniform pressure on its external surface and a steady state temperature on its interior surface.

In the rest of this section, we shall only state the results obtained in (a), (b) and (c). The same may be verified easily in like manner as in Case 1 (a,b,c). Only the boundary condition will differ from that in Case 1.

(a) Transition Through τ_{rr}:

Here we obtain in the transition state the following results:

$$\tau_{rr} = \frac{\mu}{c}[D_0 - (D_0 - \frac{c\omega\theta_0}{\mu})(\frac{r}{a})^{\frac{c}{1-c}}] - \beta_0 \log \frac{b}{r},$$

$$\tau_{\theta\theta} - \tau_{rr} = [\frac{c\omega\theta_0 - \mu D_0}{1-c}](\frac{r}{a})^{\frac{c}{1-c}} + \beta_0,$$

$$\tau_{rr} - \tau_{zz} = \mu d_0^2, \qquad (7.3.13)$$

$$\tau_{\theta\theta} - \tau_{zz} = \mu d_0^2.$$

In the fully plastic state $c \to 0$ and we have

$$\tau_{rr} = \omega_0\theta_0 - \frac{2}{3}(3-d_0^2)y\log\frac{r}{a} - \beta_0{}^*\log\frac{b}{r},$$

$$\tau_{\theta\theta} - \tau_{rr} = \frac{2}{3}y(3-d_0^2) - \beta_0{}^*,$$

$$\tau_{rr} - \tau_{zz} = \frac{2}{3}yd_0^2, \qquad (7.3.14)$$

and

$$\tau_{\theta\theta} - \tau_{zz} = \frac{2}{3}yd_0^2.$$

None of the above relations in (7.3.14) is independent of d_0, but

$$\tau_{rr}^{d} = \frac{2}{3}Y - \frac{\beta_0{}^*}{3}, \tag{7.3.15}$$

which is independent of d_0. Hence this may be taken as the yield condition.

(b) Transition Through $\tau_{\theta\theta}$:

The solutions for the transition state in this case are:

$$\tau_{rr} = \frac{\mu}{c}[D_0-\{D_0 - \frac{c\beta_0}{\mu}(\log\frac{b}{a} + 1)\}(\frac{a}{r})^{c(3-c)}]$$
$$- \beta_0[\log\frac{b}{r} + 1],$$

$$\tau_{\theta\theta} = \frac{\mu}{c}[D_0-\{D_0 - \frac{c}{\mu}\beta_0(\log\frac{b}{a} + 1)\}(\frac{a}{r})^{c(3-c)}$$
$$- \beta_0[\log\frac{b}{r} + 1], \tag{7.3.16}$$

$$\tau_{\theta\theta} - \tau_{rr} = \mu(3-c)[D_0 - \frac{c}{\mu}\beta_0(\log\frac{b}{a} + 1)](\frac{a}{r})^{c(3-c)} + \beta_0,$$

$$\tau_{rr} - \tau_{zz} = \mu d_0{}^2,$$

$$\tau_{\theta\theta} - \tau_{zz} = \mu d_0{}^2.$$

In the fully plastic state $c \to 0$ and we have

$$\tau_{\theta\theta} - \tau_{rr} = 2y(3-d_0^{\,2}) + \beta_0^* \ ,$$

$$\tau_{rr} - \tau_{zz} = \frac{2}{3}yd_0^{\,2} \ , \qquad (7.3.17)$$

$$\tau_{\theta\theta} - \tau_{zz} = \frac{2}{3}yd_0^{\,2} \ .$$

(c) <u>Transition Through $(\tau_{\theta\theta} - \tau_{rr})$:</u>

In this case, the stresses in the transition state are obtained as

$$\tau_{rr} = \frac{A_0}{3-c}[1 - (\frac{a}{r})^{4-c}] , \qquad (7.3.18)$$

where

$$A_0 = \frac{-(3-c)p_0}{\mu[1 - (\frac{a}{b})^{4-c}]} , \qquad (7.3.19)$$

$$\tau_{\theta\theta} - \tau_{rr} = \mu A_0 (\frac{a}{r})^{4-c} , \qquad (7.3.20)$$

$$\tau_{\theta\theta} - \tau_{zz} = \mu d_0^{\,2} , \qquad (7.3.21)$$

$$\tau_{rr} - \tau_{zz} = \mu d_0^{\,2} . \qquad (7.3.22)$$

In this case when $c \to 0$ we obtain the following results from the above equations:

$$\tau_{rr} = \frac{2}{9}A_0[1 - (\frac{a}{r})^4],$$

$$\tau_{\theta\theta} - \tau_{rr} = \frac{2}{3}A_0[\frac{a}{r})^4,$$

$$\tau_{\theta\theta} - \tau_{zz} = \frac{2}{3}yd_0^2, \qquad (7.3.23)$$

$$\tau_{rr} - \tau_{zz} = \frac{2}{3}yd_0^2.$$

We now compare the above results in (a), (b) and (c) and find that $|\tau_{\theta\theta} - \tau_{rr}|$ attains the largest value in each case. Hence transition should occur through $|\tau_{\theta\theta} - \tau_{rr}|$.

Also we determine the strains in the transition state. The strains in the plastic state may be obtained letting $c \to 0$.

<u>For $F \to \pm \infty$</u> :

The strains in this case are calculated following the same technique as employed in the foregoing cases. Hence from (7.2.1) we have

$$e_{rr} = \frac{1}{2}[1 - E(\frac{b}{r})^c],$$

$$e_{\theta\theta} = \frac{1}{2}, \qquad (7.3.24)$$

$$e_{zz} = \frac{1}{2}[1 - d_0^2],$$

(7.3.24) continued

and

$$e_{\alpha\alpha} = \frac{1}{2}[3-d_0{}^2 - E(\frac{b}{r})^c],$$

where E is some arbitrary parameter. Also we have

$$e_{\theta\theta} - e_{rr} = \frac{1}{2}E(\frac{b}{r})^c,$$

$$e_{\theta\theta} - e_{zz} = \frac{1}{2}d_0{}^2, \qquad (7.3.25)$$

$$e_{rr} - e_{zz} = \frac{1}{2}[d_0{}^2 - E(\frac{b}{r})^c].$$

<u>For $F \to 0$</u> :

The strains in this case are as follows:

$$e_{rr} = \frac{1}{2},$$

$$e_{\alpha\alpha} = \frac{1}{2}[3-d_0{}^2 - G(\frac{a}{r})^c],$$

$$e_{\theta\theta} = \frac{1}{2}[1 - G(\frac{a}{r})^c], \qquad (7.3.26)$$

and

$$e_{zz} = \frac{1}{2}[1 - d_0{}^2],$$

where G is an arbitrary parameter.

Also

$$e_{rr} - e_{\theta\theta} = H(\frac{a}{r})^{4-c},$$

$$e_{rr} - e_{zz} = \frac{1}{2}d_0^{\ 2}, \qquad (7.3.27)$$

and

$$e_{\theta\theta} - e_{zz} = \frac{1}{2}d_0^{\ 2} - H(\frac{a}{r})^{4-c},$$

H being another arbitrary parameter.

7.4 The Constitutive Equations for the Transition and Plastic States

We shall now obtain the constitutive equation for the transition and plastic states corresponding to the following cases.

Case 1: $F \rightarrow \pm \infty$

It has already been remarked before that transition should occur through $|\tau_{\theta\theta} - \tau_{rr}|$, since in all the three possible cases (a), (b) and (c) $|\tau_{\theta\theta} - \tau_{rr}|$ attains the largest value among all the stresses and their differences. Hence, we should obtain the constitutive equation corresponding to the transition through $|\tau_{\theta\theta} - \tau_{rr}|$.

Constitutive Equation Corresponding to the Transition Through $|\tau_{\theta\theta} - \tau_{rr}|$.

The constitutive equations for the transition state may be obtained from the equations (7.3.9), (7.3.10), (7.3.11), (7.3.12) and (7.3.25) and are given below:

$$e^d_{rr} = \frac{1}{2\mu} \cdot \frac{d_0{}^2 - 2E(\frac{b}{r})^c}{d_0{}^2 - 2B_0(\frac{b}{r})^c} \tau^d_{rr} ,$$

$$e^d_{\theta\theta} = \frac{1}{2\mu} \cdot \frac{d_0{}^2 + E(\frac{b}{r})^c}{d_0{}^2 + B_0(\frac{b}{r})^c} \tau^d_{\theta\theta} , \qquad (7.4.1)$$

and

$$e^d_{zz} = \frac{1}{2\mu} \cdot \frac{2d_0{}^2 - E(\frac{b}{r})^c}{2d_0{}^2 - B_0(\frac{b}{r})^c} \tau^d_{zz} .$$

If we choose now $B_0 = E$, then all the equations in (7.4.1) reduce to

$$e^d_{ii} = \frac{1}{2\mu} \tau^d_{ii} , \qquad (7.4.2)$$

which is, therefore, the constitutive equation for the transition state. The parameter E is then given by

$$E = \frac{cp_i}{\mu[(\frac{b}{a})^c - 1]} .$$

For the fully plastic state the constitutive equation may be obtained from (7.4.2) as

$$\dot{e}^d_{ii} = \frac{3}{4y} \tau^d_{ii} . \qquad (7.4.3)$$

It is again important to note that transition through τ_{rr} or $\tau_{\theta\theta}$ does not lead to the constitutive equation similar to that of (7.4.3), since when $c \to 0$, any one of the states creep, fatigue, etc., may be reached the neighboring state after the transition occurs.

Case 2: $F \to 0$

It was observed before in this case that transition should occur through $|\tau_{\theta\theta} - \tau_{rr}|$, since in all the cases (a), (b) and (c), $|\tau_{\theta\theta} - \tau_{rr}|$ attains the largest value among all the stresses or stress-differences. Accordingly in this case also, the constitutive equation should be obtained corresponding to the transition through $|\tau_{\theta\theta} - \tau_{rr}|$.

Constitutive Equation Corresponding to the Transition $|\tau_{\theta\theta} - \tau_{rr}|$

The constitutive equations for the transition state may be obtained from (7.3.18), (7.3.20), (7.3.21), (7.3.22) and (7.3.27) in the following manner:

$$e^{d}_{rr} = \frac{1}{2\mu} \cdot \frac{d_0^{\,2} + 2H\left(\frac{a}{r}\right)^{4-c}}{d_0^{\,2} - A_0\left(\frac{a}{r}\right)^{4-c}} \tau^{d}_{rr} ,$$

$$e^{d}_{\theta\theta} = \frac{1}{2\mu} \cdot \frac{d_0^{\,2} - 4H\left(\frac{a}{r}\right)^{4-c}}{d_0^{\,2} + 2A_0\left(\frac{a}{r}\right)^{4-c}} \tau^{d}_{\theta\theta} , \qquad (7.4.4)$$

(7.4.4) continued

and

$$e_{zz}^{d} = \frac{1}{2\mu} \cdot \frac{2d_0^{\,2} - 2H(\frac{a}{r})^{4-c}}{2d_0^{\,2} + A_0(\frac{a}{r})^{4-c}} \tau_{zz}^{d} .$$

The equations in (7.4.4) reduce to

$$e_{ii}^{d} = \frac{1}{2\mu} \tau_{ii}^{d} , \qquad (7.4.5)$$

if we assume $2H = -A_0$. The parameter H now may be obtained from (7.3.18) as

$$H = \frac{(3-c)p_0}{2\mu[1 - (\frac{a}{b})^{4-c}]} .$$

The constitutive equation for the plastic state may be obtained from (7.4.5) as

$$\dot{e}_{ii}^{d} = \frac{3}{4y} \tau_{rr}^{d} . \qquad (7.4.6)$$

The transition through the other branches τ_{rr} and $\tau_{\theta\theta}$ may not lead to the plastic state, as has already been explained in Case 1, since no constitutive equation like (7.5.6) may be derived.

CHAPTER 8

SUMMARY, DISCUSSION AND SCOPE OF FURTHER WORK

The necessity of increasing use of ad-hoc semi-empirical laws in the classical theory of elastic-plastic transition lies in the fact that the latter does not recognize the existence of the transition state between elastic and plastic ones. We have shown in this thesis that assumptions of yield conditions in such problems become unecessary once we recognize that the transition from elastic to plastic state, as explained by Seth, is an asymptotic process and that transition state is a separate state which can not be replaced by a yield surface as has always been done in the current literature. This treatment in the classical theory amounts to divide two extreme properties of a material by a sharp line which is physically impossible.

It has been clear from our work that identification of the transition state is basically important. There are, at present, three ways to identify the transition state. The most general one among all is the vanishing of the Jacobian of transformation from elastic state to plastic state. An invariant relation among the strain (stress) invariants is obtained from this condition and it is found that most of the yield conditions present in current literature are obtainable from it as special cases. Also our results include the Bauschinger's effect while the classical yield conditions fail to account for it.

The classical theory of elasticity and plasticity makes use of linear strain measure. But we have shown that transition fields are sub-harmonic (super-harmonic) fields and that they are non-linear and non-conservative in character and hence it is very important that a non-linear strain measure such as the Almansi measure should be used in the constitutive equation.

The recognition of "transition state" or "mid-zone" as a separate state necessitates to show the existence of the constitutive equation for that state. In this context, we have used Seth's transition theory to obtain the stresses and strains in the transition state and the same may be obtained for the plastic state when a certain parameter

$$c\left(=\frac{1-2\sigma}{1-\sigma}\right),$$

where σ is the Poisson's ratio of the material, is made to approach zero. From these solutions the constitutive equations for both transition and plastic states are obtained, the latter takes the form of the Lévy-von-Mises equation.

In order to illustrate our concept and procedure we have solved four problems of practical interest. The first two problems are those of shells and tubes subjected to uniform external pressure and the last two are those of shells and tubes subjected to uniform external pressure and also a steady state temperature. It has been shown that transition may occur through any of the stresses or their differences. Hence, it is found necessary to obtain the asymptotic solutions corresponding to all the possible branches of transition. The stress or stress difference which attains the largest value among others at transition should be used to derive the constitutive equation for the transition state and in the limiting form for the plastic state. Some of the results obtained in these problems are compared with those of classical theory to show that the transition analysis provides a satisfactory scientific basis for explaining these irreversible phenomena. The results obtained for the thermo-elastic-plastic problems conform with some of the known experimental results of Johnson, Derrington and Wilhoit. Some of the solutions of the problem of the tube under pressure and steady state temperature given by Bland and others may also be obtained from our results as special cases.

In nature, transitions do occur frequently and the existing classical theory fails to explain them successfully. Thus the transition theory, as it stands, now can be fruitfully exploited to explain a variety of physical phenomena and hence has a very wide application in all applied sciences.

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