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We give a new characterization of elements in the Veech group of a translation surface. This provides a computational test for Veech group membership. We use this computational test in an algorithm that detects when the Veech group is a lattice (has co-finite area), and in this case computes a fundamental polygon for the action of the Veech group on the hyperbolic plane. A standard result, essentially due to Poincaré, provides that a complete set of generators for the Veech group can then be obtained from the side pairings associated to this fundamental polygon.

Our approach introduces a new computational framework used to formulate a membership criterion for the Veech group of a compact translation surface (X, ω). We represent (X, ω) on a certain non-compact translation surface $\mathbf{O}$ that can be used to represent any translation surface within the $\mathrm{SL}(2, \mathbb{R})$ orbit of the

translation equivalence class of (X, ω). The surface **O** has an easily computed $\mathrm{SL}(2, \mathbb{R})$-action. When this action is restricted to the translation surface representations mentioned above, it corresponds to the usual $\mathrm{SL}(2, \mathbb{R})$-action on the set of equivalence classes of translation surfaces. The Veech group of a compact translation surface is therefore the stabilizer of its representation on **O**.

A New Algorithm for Computing the Veech Group of a Translation Surface

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Major Professor, representing Mathematics

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Dean of the Graduate School

I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

Brandon Edwards, Author

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1 Introduction

1.1 The Problem

A translation surface is an oriented connected topological surface together with a discrete subset, the complement of which supports a 'translation atlas' of chart maps to $\mathbb{C}$ for which the transition functions are translations. Such surfaces have induced Lebesgue measure and flat metric from the Euclidean structure of $\mathbb{C}$. The discrete subset mentioned above consists of cone singularities in this flat metric having cone angles that are integer multiples of 2π.

The translation atlas allows for a well defined notion of whether or not a map between two translation surfaces is affine. Affine maps between translation surfaces have constant Jacobian in local coordinates. For compact translation surfaces, the group of Jacobians for orientation preserving affine self-homeomorphisms is called the Veech group of the surface. The Veech group is a subgroup of $\mathrm{SL}(2,\mathbb{R})$ since compactness forces area conservation. There is a natural action of $\mathrm{SL}(2,\mathbb{R})$ on the set of all translation equivalence classes of translation surfaces, and the Veech group of a compact translation surface is the stabilizer of its equivalence class under this action.

Individual elements of the Veech group can provide information about the geodesic flow along certain directions on the surface. Veech groups are in fact discrete subgroups of $\mathrm{SL}(2, \mathbb{R})$. William Veech [Ve1] proved that when the Veech group of a translation surface is large (specifically when it is a lattice subgroup of $\mathrm{SL}(2, \mathbb{R})$ and so has finite co-volume with respect to Haar measure on $\mathrm{SL}(2, \mathbb{R})$) a dichotomy exists where the geodesic flow along a given direction on the surface is either periodic(orbits are closed curves) or minimal (orbits are dense in the surface). Furthermore, the Lebesgue measure on such surfaces is the unique invariant measure for the flow along a minimal direction. Such surfaces are called lattice surfaces. Along with providing a setting for interesting dynamics, translation surfaces have also shown promise in the study of gas models [DHL].

This thesis addresses the need to identify structures related to a translation surface with which calculations can be performed in order to answer questions about the Veech group. Compact translation surfaces can be defined by identifying sides of polygons. Our algorithms apply to polygons with side lengths in a number field over $\mathbb{Q}$ so that exact arithmetic can be performed.

1.2 Previous Work

In the case of a lattice surface, the Veech group is finitely generated. Recent work ([Sch, Fin, Bo, SW, Mu, BrJu]) has produced algorithms for listing generators of the Veech group in the this case.

It is known that if a translation surface covers the flat torus with ramification over a single point, then the surface is a lattice surface. Gabriela Weitze-Schmithüsen [Sch] produced an algorithm for computing generators of the Veech group for the case of such covers. Her algorithm relates membership in the Veech group to a question about automorphisms of free groups. A generalization of this approach to arbitrary finite coverings of a double n-gon or an n-gon was carried out by her student Myriam Finster [Fin].

For the general case, algorithms have been developed [Bo, SW, Mu] that produce a list of candidate members for the Veech group, then confirm or deny membership for individual candidates using Delaunay triangulations of the surface. The candidates arise as the elements of the group that stabilizes a particular cellular decomposition of the hyperbolic plane. This stabilizer is a (a priori larger) group containing the Veech group. In the case of a lattice surface, the Veech group has finite index in the candidate group and one can complete a list of generators.

Another way to characterize elements of the Veech group itself is provided in [BrJu], but work is required to utilize it for calculation. This characterization is expressed in terms of automorphisms of a polygonal cell complex within the space of all immersions of ellipses into the translation surface.

1.3 Our Approach

In contrast to previous algorithms used to compute Veech groups in the general lattice setting, we introduce a set admitting an easily computable action of $\mathrm{SL}(2,\mathbb{R})$ and an element in that set whose stabilizer is *precisely* the Veech group in question. We produce an algorithm that detects when the Veech group is a lattice, and in this case computes a fundamental polygon for the action of the Veech group on the hyperbolic plane. A standard result, essentially due to Poincaré (see Theorem 3.5.4 in [Kat]), provides that a complete set of generators for the Veech group can then be obtained from the side pairings associated to this fundamental polygon. We thus provide an algorithm for explicitly computing the Veech group of a lattice translation surface.

Our approach introduces a new computational framework used to formulate a membership criterion for the Veech group of a compact translation surface (X,ω). We represent (X,ω) on a certain non-compact translation surface $\mathbf{O}$ that can be used to represent any translation surface within the $\mathrm{SL}(2,\mathbb{R})$ orbit of the translation equivalence class of (X,ω). The surface $\mathbf{O}$ has an easily computed $\mathrm{SL}(2,\mathbb{R})$-action. When this action is restricted to the translation surface representations mentioned above, it corresponds to the $\mathrm{SL}(2,\mathbb{R})$-action on the set of equivalence classes of translation surfaces. The Veech group of a compact translation surface is therefore the stabilizer of its representation on $\mathbf{O}$.

In particular, we utilize the Voronoi decomposition of (X,ω) subordinate to its singular set. A copy of any given open 2-cell of the Voronoi decomposition can be placed on $\mathbf{O}$. This open 2-cell is completely determined by the directed saddle connections lifted from (X,ω) to $\mathbf{O}$. The identifications required to reconstruct (X,ω) from the closures of these copies arise from pairing oppositely directed but otherwise identical directed saddle connections lifted from (X, ω). The pairing is recorded in the form of a $\mathbb{Z}_2$-action on the set of directed saddle connections lifted to $\mathbf{O}$. This $\mathbb{Z}_2$-set constitutes a representative of the equivalence class[1] that we call the *marked periods* of (X, ω). As outlined above, the marked periods of (X, ω) contain a sufficient amount of information to recover the translation equivalence class of (X, ω), and are defined in a space that admits an easily computable action of $\mathrm{SL}(2, \mathbb{R})$. The criterion for Veech group membership is satisfied if a matrix stabilizes the marked periods of (X, ω). In fact we show that a *finite* subset of the marked periods of (X, ω) is sufficient to determine all matrices in the Veech group of (X, ω) whose norm is bounded above by a given value.

As the norm bounds increase, our algorithm constructs 'containment polygons' nesting down to a fundamental polygon for the action of the Veech group on the hyperbolic plane. The fact that we know *all* elements satisfying the given norm bound allows us to simultaneously track increasing metric balls within which we guarantee that the containment polygons coincide exactly with the fundamental polygon. In the lattice case this allows our algorithm completely determine the fundamental polygon in finite time, thus providing us with a finite generating set for the Veech group.

1.4 An Illustrative Example of the Membership Criterion

We now explore a restricted case in order to illustrate the criterion for Veech

[1] There is some freedom in lifting the set of directed saddle connections.

group membership. Consider a compact translation surface, (X, ω), with one singularity P of cone angle 6π. A combinatorial Gauss-Bonet theorem tells us that such a surface has genus 2.

The Voronoi cellular decomposition of such an (X, ω) has only one 2-cell. The closure of any 2-cell in the corresponding decomposition of the universal translation cover $(\tilde{X}, \pi^*\omega)$ is a closed neighborhood of a singularity with piecewise geodesic boundary and a 6π cone singularity in its interior. Choose one such closure and denote it by $\mathrm{Poly}(X)$. We will identify pairs of the geodesic pieces of $\partial\mathrm{Poly}(X)$ in order to reproduce (X, ω) as a translation surface. The shape of $\mathrm{Poly}(X)$ is determined from a finite collection of saddle connections from the singularity within $\mathrm{Poly}(X)$ to other singularities on $(\tilde{X}, \pi^*\omega)$. Gluing data for the reconstruction of (X, ω) from $\mathrm{Poly}(X)$ is obtained by recording when two different outgoing saddle connections from the singularity within $\mathrm{Poly}(X)$ correspond to oppositely oriented but otherwise identical saddle connections on (X, ω). Such observations define a $\mathbb{Z}_2$-action on the set of saddle connections emanating from $\mathrm{Poly}(X)$.

Let $\mathcal{O}_P$ denote the translation surface formed by taking three copies of $\mathbb{C}$, slicing each along the negative imaginary axis, then gluing in a cyclic fashion to produce a triple cover of $\mathbb{C}$ ramified over 0. Note that $\mathcal{O}_P$ is a (non-compact) translation surface with a single 6π cone singularity. We can identify the saddle connections emanating from the singularity in $\mathrm{Poly}(X)$ with line segments emanating from $0 \in \mathcal{O}_P$. We can also carry over the $\mathbb{Z}_2$-action mentioned above as well as identify $\mathrm{Poly}(X)$ with a corresponding polygon within $\mathcal{O}_P$. For the remainder of this subsection we will use these identifications without mention.

We record on $\mathcal{O}_P$ the endpoints of the saddle connections emanating from its singularity. These points along with the induced $\mathbb{Z}_2$-action constitute representatives of the marked periods of (X, ω). We prove that only a finite subset of the marked period representatives is needed to form $\mathrm{Poly}(X)$. The $\mathbb{Z}_2$-action defined on this finite subset contains the information about which sides to identify in $\mathrm{Poly}(X)$ in order to reproduce (X, ω). Let $\rho \in \mathbb{R}$ where $B(0, \rho)$ contains the marked periods needed in order to form $\mathrm{Poly}(X)$ and reconstruct (X, ω).

The $\mathbb{R}$-linear transformations of $\mathbb{C}$ defined through elements of $\mathrm{SL}(2, \mathbb{R})$ can be lifted to affine transformations of $\mathcal{O}_P$. If $M \in \mathrm{SL}(2, \mathbb{R})$ has minimal Frobenius norm $||M|| = \sqrt{2}$ then $M \in \mathrm{SO}(2, \mathbb{R})$ and a lift of its action to $\mathcal{O}$ will preserve $B(0, \rho)$. If the lifted action preserves the marked periods within $B(0, \rho)$ (which includes the $\mathbb{Z}_2$-action), then M is in the Veech group of (X, ω).

If $||M|| > \sqrt{2}$ we consider the singular value decomposition

$$M = O_1 \cdot D \cdot O_2 \text{ where } O_1, O_2 \in \mathrm{SO}(2, \mathbb{R}) \text{ and}$$

$$D = \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} \text{ with } t \in (1, \infty).$$

The image of $B(0, \rho)$ under the lift of the action of M will fail to contain $B(0, \rho)$ as some lines through 0 will be contracted by a factor of t^{-1}. In this case $B(0, R)$ with $R > \rho t$ is sufficiently large to ensure we can compare the image against $B(0, \rho)$. We apply the lift of the action of M to the marked periods within

$B(0, R)$, pushing forward its $\mathbb{Z}_2$-action, then intersect the image with $B(0, \rho)$. If the result is identical to the original marked periods within $B(0, \rho)$ then M is in the Veech group. The value of R grows monotonically as the norm of M. Thus the marked periods within a given $B(0, R)$ can be used to determine *all* elements in the Veech group whose Frobenius norm is bounded above by $||M||$.

1.5 Organization

Section 2 of this dissertation presents background materials. We define translation surfaces as well as a number of related objects. We define some of the objects for the case of a compact translation surface only, including the Veech group, the universal translation cover, the Voronoi cellular decomposition, and the connected components of the auxiliary translation surface, **O**, associated to a compact translation surface. These connected components are used to record the 'marked periods', which contain information regarding the geodesics connecting the singularities of the surface. We also recall the standard action of $\mathrm{SL}(2, \mathbb{R})$ on the set of all equivalence classes of translation surfaces, and show that the Veech group is the stabilizer of the class of the surface under this action.

Section 3 introduces topological embeddings from star-shaped subsets of the auxiliary surface **O** into the universal cover of a compact translation surface. These embeddings are used to define the representatives of the marked periods at each point of a compact translation surface. The union of a choice of representative from each point in the surface is used to define a full representative of the marked periods. Theorem 7 identifies the set of all possible representatives of a fixed surface as an equivalence class, which we define as the 'marked periods' of the surface. The section concludes with a proof of Theorem 8 which states that the marked periods are invariant under translation equivalence.

In section 4 we decompose a compact translation surface by 'unzipping' it along the 1-skeleton of its Voronoi decomposition. We show that the pieces resulting from such a deconstruction can be independently cut from **O** using a finite subset of the marked periods. Additional information encoded in the marked periods for this finite subset can be used to determine the gluing scheme required to reconstruct the surface up to translation equivalence. We thus prove that a finite subset of the marked periods can be used to determine the translation equivalence class of a surface.

Section 5 defines an $\mathrm{SL}(2,\mathbb{R})$-action on the space containing the marked periods. Recall that $\mathrm{SL}(2,\mathbb{R})$ also acts on the space of all translation surfaces. We prove Theorem 16 which establishes that the assignment of marked periods for compact translation surfaces is equivariant with respect to these actions. We combine this with the results of Section 4 to obtain a criterion for Veech group membership. The membership criterion (presented in Theorem 18) can be used to find all elements of the Veech group whose norm is bounded above by some given value using a finite subset of the marked periods determined by this value.

Section 6 provides more background. Veech groups are subgroups of $\mathrm{SL}(2,\mathbb{R})$ that project to 'Fuchsian' subgroups of $\mathrm{SL}(2,\mathbb{R})/\{\pm\mathrm{Id}\}$. We define Fuchsian subgroups of $\mathrm{SL}(2,\mathbb{R})/\{\pm\mathrm{Id}\}$, and introduce the standard isometric action of $\mathrm{SL}(2,\mathbb{R})/\{\pm\mathrm{Id}\}$ on the hyperbolic plane. We also introduce a well-known 'fundamental polygon' associated to the action of a Fuchsian group on the hyperbolic plane. We discuss how this fundamental polygon can be used to obtain a list of generators for the group, and present a generalization of a theorem due to

Poincaré (Theorem 21) that will play a part in determining when the construction of this polygon is complete.

Section 7 introduces an algorithm for computing Veech group elements that produces a list of generators in finite time for the lattice case. Algorithm 7.1 is used for a restricted class of translation surface. Algorithm 7.2 applies to all translation surfaces. Algorithm 7.2 alters the surface to insure that it satisfies the conditions of Algorithm 7.1. The output resulting from applying Algorithm 7.1 on this new surface is then appropriately modified to obtain the elements (generators in the lattice case) of the original surface.

Section 8 provides concluding remarks.

2 Preliminaries

2.1 Translation Surfaces

We provide some basic definitions associated to translation surfaces. See [Zor] for a general reference.

Definition 1. *A* ***translation surface*** *is a pair (Y, ψ) of a connected Riemann surface Y (without punctures) and non-zero holomorphic 1-form ψ defined on Y, such that the zeros of ψ do not have accumulation points in Y. The zeros of ψ are called* ***singularities*** *and a point of (Y, ψ) that is not a singularity is called a* ***regular point****. The* ***order*** *of a singularity P is given as the order of the zero of ψ at P and is denoted by $o(P)$.*

We pick an orientation for the complex plane $\mathbb{C}$. Since holomorphic maps preserve orientation, we can thus induce orientations on all translation surfaces.

If (Y, ψ) is a translation surface with singular set Σ, then ψ induces a Euclidean structure on $Y \setminus \Sigma$ through a collection of local Euclidean chart maps to $\mathbb{C}$.

Definition 2. *Let (Y, ψ) be a translation surface with singular set Σ.*

*If U is a simply connected neighborhood of $Y \setminus \Sigma$ and $y_0 \in \bar{U}$ is a chosen **base point**, we define **regular coordinates** (ζ_{y_0}) on U based at y_0 by:*

$$\zeta_{y_0}(y) := \int_{y_0}^{y} \psi.$$

*If U is a simply connected neighborhood of some $P \in \Sigma$ we define **singular coordinates** (ζ'_P) on U by:*

$$\zeta'_P(y) := \left[(o(P)+1)\int_P^y \psi\right]^{\frac{1}{o(P)+1}}.$$

Note that in regular coordinates the 1-form ψ is represented as $\psi = d\zeta_{y_o}$, regardless of base point y_0. For a singularity P sitting on the boundary of a regular coordinate chart we can base the regular coordinates at P. In this case we have the relationships

$$\zeta_P = \tfrac{1}{o(P)+1}\left[\zeta'_P\right]^{o(P)+1}$$

and

$$\psi = d\zeta_P = (\zeta'_P)^{o(P)} d\zeta'_P.$$

The regular coordinates of a translation surface form a ‘translation atlas’ on $Y \setminus \Sigma$ for which the transition functions are translations. Such an atlas defined on the punctured topological surface underlying $Y \setminus \Sigma$ can be used as an alternate defining object for the translation surface (see [MasTab]).

The translation atlas provides the translation surface with a Lebesgue measure and flat metric induced from the Euclidean structure of $\mathbb{C}$. The singular set consists of cone singularities in this flat metric having cone angles that are integer multiples of 2π.

An oriented Euclidean polygonal subset of the complex plane whose sides are identified in order to define a closed surface may be given the structure of a translation surface provided the identifications occur in a specific way. Each side identification needs to be made with two sides that are parallel, of equal length, and have opposite induced orientations as boundary pieces of the polygon. Such a surface admits an atlas of chart maps to the complex plane whose transition functions are translations, and as such defines a Riemann surface. If z is the complex coordinate from the complex plane containing the polygon, then the one-form given locally by dz defines a non-zero holomorphic one-form on this Riemann surface, which defines a translation surface in the sense of Definition 1. The local flat structure of such a translation surface is identical to the local flat structure of the original polygon. Figure 1 depicts a translation surface given in this way, that has genus two and contains exactly one singularity of cone angle 6π.

Utilizing the real vector space structure of $\mathbb{C}$, the translation atlas allows for a well defined notion of whether or not a map between two translation surfaces is affine (i.e. $\mathbb{R}$-linear plus a translation).

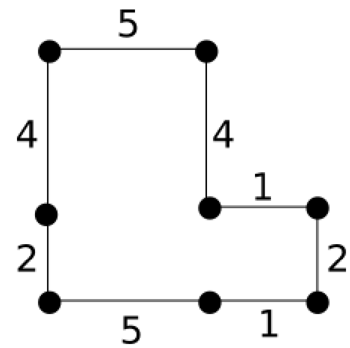

Figure 1: Identifying like numbered edges of the polygon above defines a translation surface of genus two with one singularity of cone angle 6π.

Definition 3. *A map between two translation surfaces that sends the singular set of one into the singular set of another is called* ***affine*** *if it is represented as an affine map in the regular coordinates of the two surfaces.*

Remark 1. *The linear part of an affine map is locally constant and given by the Jacobian of a local representation. Since translation surfaces are connected, this Jacobian matrix is the same regardless of where on the surface the restriction to a regular local neighborhood is performed.*

Definition 4. *If* (Y_1, ψ_1) *and* (Y_2, ψ_2) *are translation surfaces, and* $F : (Y_1, \psi_1) \to (Y_2, \psi_2)$ *is an affine map, we denote the Jacobian of* F *by* dF*.*

Definition 5. *Let (Y, ψ) be a translation surface.*

We define the groups

$$\mathrm{Aff}^+(Y, \psi) := \left\{ F : (Y, \psi) \to (Y, \psi) \;\middle|\; \begin{array}{c} \text{F is an orientation preserving} \\ \text{affine homeomorphism} \end{array} \right\}$$

and

$$\mathrm{Trans}(Y, \psi) := \{F \in \mathrm{Aff}^+(Y, \psi) \,|\, dF = \mathrm{Id}\}.$$

Definition 6. *We define the map* $\mathrm{der} : \mathrm{Aff}^+(Y, \psi) \to \mathrm{GL}(2, \mathbb{R})$ *by* $\mathrm{der}(F) := dF$*.*

Note that the map der is a homomorphism. We can put the definitions above into an exact sequence:

$$\{\mathrm{Id}_Y\} \hookrightarrow \mathrm{Trans}(Y, \psi) \hookrightarrow \mathrm{Aff}^+(Y, \psi) \xrightarrow{\mathrm{der}} \mathrm{GL}(2, \mathbb{R})$$

There is a naturally defined action of $\mathrm{GL}(2, \mathbb{R})$ on the space of all translation surfaces. We will focus on the restriction to $\mathrm{SL}(2, \mathbb{R})$.

Definition 7. *For $M \in \mathrm{SL}(2, \mathbb{R})$ and a translation surface (Y, ψ), we define $M \cdot (Y, \psi)$ to be the translation surface whose translation atlas is the result of post-composing all of the chart maps from (Y, ψ) by M.*

Remark 2. *Note that if $M \in \mathrm{SL}(2, \mathbb{R})$ and (Y, ψ) is a translation surface, then* $\mathrm{Id}_Y : (Y, \psi) \to M \cdot (Y, \psi)$ *is an orientation preserving affine homeomorphism with* $\mathrm{der}(\mathrm{Id}_Y) = M$*.*

We now define what it means for two translation surfaces to be translation equivalent.

Definition 8. *Two translation surfaces (Y_1, ψ_1) and (Y_2, ψ_2) are **translation equivalent** if there is an orientation preserving affine homeomorphism*

$$F : (Y_1, \psi_1) \to (Y_2, \psi_2) \text{ with } dF = \mathrm{Id}.$$

2.2 The Compact Translation Surface (X, ω) and its Veech Group

If (X, ω) is a compact translation surface, then its singular set must be finite since it cannot have any accumulation points in X. Also because area must be preserved, all affine homeomorphisms from (X, ω) to itself must have Jacobian with determinant one. In this subsection we look at the group of linear parts of such affine homeomorphisms.

Definition 9. *If (X, ω) is a compact translation surface then the **Veech group** of (X, ω) is given by:*

$$\Gamma(X, \omega) := \{M \in \mathrm{SL}(2, \mathbb{R}) \,|\, \exists\ F \in \mathrm{Aff}^+(X, \omega) \textit{ such that } \mathrm{der}(F) = M\}$$

Veech groups are in fact discrete subgroups of $\mathrm{SL}(2, \mathbb{R})$. Though this is a standard result (see Proposition 1.3 in [Mol]), it is also a consequence of Remark

9 and Theorem 18. The Veech group of a generic surface is trivial or $\mathbb{Z}_2$ (see Theorem 1.1 of [Mol]).

Remark 3. *There is a left action of group* $\mathrm{SL}(2,\mathbb{R})$ *on itself through left multiplication. Correspondingly there is a unique (up to scaling) left invariant measure defined on* $\mathrm{SL}(2,\mathbb{R})$ *called* ***Haar measure****.*

Definition 10. *A subgroup of* $H \leq \mathrm{SL}(2,\mathbb{R})$ *is called a* ***lattice subgroup*** *if the induced Haar measure on the coset space* $\mathrm{SL}(2,\mathbb{R})/H$ *is finite.*

William Veech proved in [Ve1] that when the Veech group of a translation surface is a lattice subgroup, a dichotomy exists where the geodesic flow along a given direction on the surface is either periodic or minimal (orbits are dense). Furthermore, the Lebesgue measure on such surfaces is the unique invariant measure for the flow along a minimal direction. Such surfaces are called *lattice surfaces.*

Lemma 1. *If* (X,ω) *is a compact translation surface and* $M \in \mathrm{SL}(2,\mathbb{R})$ *then*

$$M \in \Gamma(X,\omega) \textit{ if and only if } (X,\omega) \textit{ and } M \cdot (X,\omega) \textit{ are translation equivalent.}$$

Proof. Let $M \in \mathrm{SL}(2,\mathbb{R})$. By Remark 2 the map $G_M := \mathrm{Id}_X : (X,\omega) \to M\cdot(X,\omega)$ is an affine homeomorphism with $\mathrm{der}(G_M) = M$ and $\mathrm{der}(G_M^{-1}) = M^{-1}$.

Referring to Figure 2, it follows that there exists of an affine homeomorphism

$$F_M : (X,\omega) \to (X,\omega)$$

$$\begin{tikzcd}
(X,\omega) \arrow[r, "G_M"] \arrow[d, "F_M"'] & M\cdot(X,\omega) \arrow[dl, "H_{\mathrm{Id}_{\mathrm{SL}(2,\mathbb{R})}}"] \\
(X,\omega) &
\end{tikzcd}$$

Figure 2: The map F_M exists if and only if $H_{\mathrm{Id}_{\mathrm{SL}(2,\mathbb{R})}}$ exist.

with $\mathrm{der}(F_M) = M$ if and only if there is an affine homeomorphism

$$H_{\mathrm{Id}_{\mathrm{SL}(2,\mathbb{R})}} : M \cdot (X,\omega) \to (X,\omega)$$

with $\mathrm{der}(H_{\mathrm{Id}_{\mathrm{SL}(2,\mathbb{R})}}) = \mathrm{Id}_{\mathrm{SL}(2,\mathbb{R})}$. Such an $H_{\mathrm{Id}_{\mathrm{SL}(2,\mathbb{R})}}$ is a translation equivalence.

$\square$

2.3 The Universal Cover $\pi : (\tilde{X}, \pi^*\omega) \to (X,\omega)$

Let (X,ω) be a compact translation surface and let $\pi : \tilde{X} \to X$ be the universal (topological) covering space of the topological space underlying the Riemann surface X. The topological space $\tilde{X}$ can be endowed with a Riemann surface structure by pulling back the local charts of X via π, making π a holomorphism of Riemann surfaces. We will denote this Riemann surface again by $\tilde{X}$. The pull back of ω via π is a non-zero holomorphic 1-form whose zero set $\tilde{\Sigma} := \pi^{-1}(\Sigma)$ cannot have any accumulation point in $\tilde{X}$ due to the local homeomorphic properties of π. Therefore $(\tilde{X}, \pi^*\omega)$ is a translation surface with singular set $\tilde{\Sigma}$ and $\pi : (\tilde{X}, \pi^*(\omega)) \to (X,\omega)$ is a local translation equivalence.

Definition 11. *We call* $\pi : (\tilde{X}, \pi^*(\omega)) \to (X, \omega)$ *the* ***universal translation cover*** *of* (X, ω)*.*

2.4 The Connected Components of the auxiliary surface

In this subsection we introduce the translation surface used to record information about the directed saddle connections starting at a particular singularity of a compact translation surface. The disjoint union taken over such surfaces associated to all the singularities on our compact translation surface will be the auxiliary surface used to hold a representative of our compact translation surface vis-a-vis this information.

We will now introduce terminology to track the number and type of singularities on a compact translation surface. Due to a combinatorial version of the Gauss-Bonnet theorem (see [Sch]), all translation surfaces of genus g must have singularities whose orders sum to $2g-2$.

A compact translation surface (X, ω) of genus g will have singularities whose orders can be arranged into a non-decreasing sequence: $i_1 \leq i_2 \leq ... \leq i_s$ to form the partition of $2g-2$: $i_1 + i_2 + ... + i_s = 2g-2$.

Definition 12. *Let* $\mathcal{H}(i_1, ..., i_s)$ *be the collection of all translation equivalence classes of translation surfaces containing exactly* s *singularities having orders* $i_1 \leq ... \leq i_s$*.*

Remark 4. *There is an action of* $\mathrm{SL}(2,\mathbb{R})$ *on* $\mathcal{H}(i_1,...,i_s)$ *provided for by Definition 7. Lemma 1 tells us that* $\Gamma(X,\omega) = \mathrm{Stab}_{\mathrm{SL}(2,\mathbb{R})}\{(X,\omega)\}$.

Fix $\mathcal{H}(i_1,...,i_s)$ where $i_1 \leq ... \leq i_s$. Let $\mathfrak{s}$ represents the number of distinct values of summands in the partition $i_1+...+i_s = 2g-2$, where for all $k \in \{1,...,\mathfrak{s}\}$ c_k denotes the number of times that the k_{th} value (given by q_k) appears in the partition. Thus $q_1 = i_1 = ... = i_{c_1} \neq q_2 = i_{c_1+1} = ... = i_{c_1+c_2} \neq ... \neq q_{\mathfrak{s}} = i_{s-c_{\mathfrak{s}}+1} = ... = i_s$.

Definition 13. *For all* $k \in \{1,...,\mathfrak{s}\}$ *define the non-compact translation surface* $\mathcal{O}_k := (\mathbb{C}, (q_k+1)z^{q_k}dz)$. *The point* $0 \in \mathcal{O}_k$ *is the only singularity, and is of order* q_k.

We will use 0 as the base point for both the singular and regular coordinates on $\mathcal{O}_k$. For (0-centered) sector neighborhoods having angle less than 2π we use the regular coordinates based at 0 given by

$$\zeta_0(z) = \int_0^z (q_k+1)w^{q_k}dw = z^{q_k+1}.$$

Defined on all of $\mathbb{C}$ are the singular coordinates $\zeta_0'(z) = [(q_k+1)\int_0^z (q_k+1)w^{q_k}dw]^{\frac{1}{1+q_k}}$. There are multiple choices of root for ζ_0', but for simplicity we will take

$$\zeta_0'(z) := (q_k+1)^{\frac{1}{q_k+1}} z$$

where $(q_k+1)^{\frac{1}{q_k+1}}$ is the principal real root.

Remark 5. *The surface $\mathcal{O}_k$ can be viewed as follows. Take $q_k + 1$ copies of the complex plane and slice each along their negative imaginary axes. Denote the resulting edges from the i_{th} copy by l_i and r_i for left and right respectively. Then identify (in the natural way) l_1 with r_2, l_2 with r_3, ... , and finally l_{q_k+1} with r_1. The resulting translation surface is $\mathcal{O}_k$.*

Note in particular that one regular coordinate chart map is defined on any (0 centered) sector of $\mathcal{O}_k$ having angle as measured in regular coordinates of less than 2π. On the underlying surface $\mathbb{C}$ this sector will have angle less than $\frac{2\pi}{o(P)+1}$. It is a consequence of Theorem 3 that such chart maps extend to homeomorphism when the boundary point, 0, is included in its domain.

Definition 14. *Let $proj_k : \mathcal{O}_k \to (\mathbb{C}, dw)$ be given locally by*

$$w = proj_k(z) := \zeta_0(z).$$

Then $proj_k : \mathcal{O}_k \to (\mathbb{C}, dw)$ is a translation cover ramified over $0 \in \mathbb{C}$.

Definition 15. *For $P \in \Sigma$ let $\mathcal{O}_P$ be a copy of $\mathcal{O}_k$ where $k \in \{1, ..., \mathfrak{s}\}$ is such that $q_k = o(P)$. Similarly $proj_P$ will represent the map $proj_k$ in this case.*

2.5 Natural Metrics on (X, ω), $(\tilde{X}, \pi^*\omega)$, and $\mathcal{O}_P$ for $P \in (X, \omega)$

Let (Y, ψ) be an arbitrary translation surface with singular set Σ. Using the regular coordinates on $Y \setminus \Sigma$ we can define a Euclidean Riemannian metric. The associated geodesics will be represented in the regular coordinates as straight lines. A distance can be defined on $Y \setminus \Sigma$ using the infimum of the lengths of all rectifiable (equivalently in this case piecewise linear) curves connecting two given points. The completion of the associated metric space on $Y \setminus \Sigma$ can be achieved by simply re-inserting the isolated singularities of Σ. In such a way we obtain a classical metric on all of Y. The associated distance on Y can (also) be viewed as deriving from the infimum of the lengths of all piecewise linear curves between the given points, provided we understand that these curves may now connect at singularities.

In this subsection we will define geodesics on all of (X, ω), $(\tilde{X}, \pi^*\omega)$, and $\mathcal{O}_P$ for $P \in (X, \omega)$. We will describe the local properties of such curves at singular points, and see that geodesics uniquely realize the distance between points within $(\tilde{X}, \pi^*\omega)$ or $\mathcal{O}_P$ due essentially to the simple connectivity of these spaces. Distances between points in (X, ω) are also realized by geodesics, though uniqueness may not hold.

Definition 16. *If (Y, ψ) is a translation surface with singular set Σ, a **geodesic** in (Y, ψ) is a path that is locally length minimizing over the set of all rectifiable* *curves with respect to the distance metric that ψ induces on Y (identified with the metric completion of $Y \setminus \Sigma$ as described above). We denote the length of a geodesic γ by $l(\gamma)$. A geodesic γ that is purported to '**connect the points**' x_1 and x_2 has domain of definition $[0, 1]$ with $\gamma(0) = x_1$ and $\gamma(1) = x_2$.*

Theorem 1. *(see Theorem 8.1 in [Str]) The geodesics of a translation surface (Y, ψ) with singular set Σ are curves consisting of open ended geodesics of the Riemannian metric on $Y \setminus \Sigma$ connected at the points of Σ in such a way that for each connection point $P \in \Sigma$ the minimum angle (measured between the two legs in the regular coordinates near P) is greater than or equal to π.*

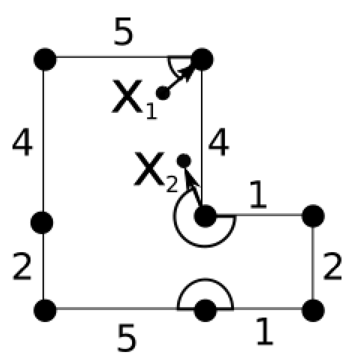

Figure 3: Example of a geodesic from a point X_1 to a point X_2 through a singularity of cone angle 6π. The two directed line segments are geodesics of the Riemannian metric defined on the surface minus the singularity. Also shown is the minimal angle formed at the singularity between the two pieces of the geodesic.

Proof. Suppose α is a geodesic on (Y, ψ). Near regular points of (Y, ψ), α must be a geodesic of the Riemannian metric defined on $Y \setminus \Sigma$. Suppose α contains a singular point P. Then the quadratic differential on Y given by $\phi = \psi^2$ has a zero of order $n = 2 \cdot o(P)$ at P. There is a natural metric associated to the so-called 'half-translation surface' (Y, ϕ) that is precisely the translation surface metric we define above (see Definition 5.3 of [Str]). Therefore α is also a geodesic of the half-translation surface (Y, ϕ). Applying Theorem 8.1 of [Str] to the geodesic α of (Y, ϕ) we have that α is seen in the singular coordinates near P to be two straight lines passing into P connected by an angle of measure greater than $\frac{2\pi}{n+2} = \frac{\pi}{o(P)+1}$. Therefore in regular coordinates this angle will be observed as greater than or equal to π. $\square$

Definition 17. *For $P \in \Sigma$, let the* ***radius*** *of a point $z \in \mathcal{O}_P$ be the distance from z to the singularity $0 \in \mathcal{O}_P$.*

Theorem 2. *Any pair of points in $(\tilde{X}, \pi^*\omega)$ or $\mathcal{O}_P$ for any $P \in \Sigma$ are connected by a unique geodesic.*

Proof. The result holds true regarding $(\tilde{X}, \pi^*\omega)$ by compactness of X and Lemma 18.2 of [Str]. The uniqueness of a geodesic connecting two arbitrary points of $\mathcal{O}_P$ is guaranteed by Theorem 14.2.2 of [Str] and simple connectivity of $\mathcal{O}_P$. Existence follows for $\mathcal{O}_P = (\mathbb{C}, (o(P) + 1)z^{o(P)}dz)$ by Corollary 18.2 of [Str], however we could also simply construct the geodesic connecting two points using Theorem 1. Given two points $z_1, z_2 \in \mathcal{O}_P$ whose arguments using regular coordinates differ by less than π, we can simply connect the two points by a straight line drawn using a single coordinate chart. If their arguments differ by π or more, the geodesic connecting them is given by the concatenation of the geodesic connecting z_1 to 0 and the line connecting 0 to z_2. $\square$

Definition 18. *We will refer to geodesics on a translation surface as **lines**. A line connecting two singularities but containing no singularities otherwise is defined as a **saddle connection**. For all $P \in \Sigma$, we denote the unique geodesic map connecting points z_1 and z_2 in $\mathcal{O}_P$ by ${}_{z_1}\lambda_{z_2}$. We denote the unique geodesic map connecting points y_1 and y_2 in $(\tilde{X}, \pi^*\omega)$ by ${}_{y_1}\gamma_{y_2}$. We denote the images of these lines by $[z_1, z_2]$ and $[y_1, y_2]$ respectively.*

Theorem 2 provides us with a natural concept of convexity in $(\tilde{X}, \pi^*\omega)$ and $\mathcal{O}_P$ for each $P \in \Sigma$.

Definition 19. *If $A \subseteq \mathcal{O}_P$ for some $P \in \Sigma$ or $A \subseteq (\tilde{X}, \pi^*\omega)$, we say A is **star shaped with center** w_0 if $w \in A$ implies $[w_0, w] \subseteq A$. We say A is **convex** if $w_1, w_2 \in A$ implies $[w_1, w_2] \subseteq A$.*

2.6 The Voronoi Decomposition of (X, ω)

We will utilize a cellular decomposition of (X, ω) called the Voronoi decomposition subordinate to the singular set Σ.

Definition 20. *Let*

$$F^{\geq 3} := \{x \in X \,|\, d(x, \Sigma) \text{ is realized by three or more distinct geodesics}\},$$

$$F^2 := \{x \in X \,|\, d(x, \Sigma) \text{ is realized by exactly two distinct geodesics}\}, \text{and}$$

$$F^1 := \{x \in X \,|\, d(x, \Sigma) \text{ is realized by exactly one geodesics}\}.$$

Also, for all $P \in \Sigma$, let:

$$F_P^1 := \{x \in F^1 \mid \text{ the geodesic realizing } d(x, \Sigma) \text{ terminates at } P\} \text{ and}$$

$$F_P := \{x \in X \mid \text{ one of the geodesics realizing } d(x, \Sigma) \text{ terminates at } P\}.$$

Notice that for $P, R \in \Sigma$ with $P \neq R$, we have that $F_P^1 \cap F_R^1 = \emptyset$. We also have that $\bigcup_{P \in \Sigma} F_P = X$.

Remark 6. *The F_P^1 for $P \in \Sigma$ constitute precisely the path components of F^1 and are the open 2-cells of the Voronoi cellular decomposition of (X, ω) subordinate to the set Σ. The open 1-cells and the 0-cells of this decomposition are respectively the path components of F^2 and the elements of $F^{\geq 3}$. See [MasSm] for more details.*

2.7 Extending Local Coordinates to a Developing Map

It is possible to continuously extend a local coordinate chart on a translation surface to a map that is no longer injective while preserving the property that it restricts locally to a coordinate chart at regular interior points of A. In this subsection we discuss the properties of such maps for the surfaces we are considering.

Given a simply connected subset $A \subseteq Y$ of a translation surface (Y, ψ). For $y_0 \in A$, the fact that ψ is a holomorphic 1-form allows for a well defined map

$\text{dev}_{y_0} : A \to \mathbb{C}$ given for all $y \in A$ by

$$\text{dev}_{y_0}(y) := \int_{y_0}^{y} \psi$$

The map dev_{y_0} is the unique continuous extension to all of A of any local chart defined in a neighborhood of a regular point within A subject to the conditions that it restricts to a local coordinate chart in some neighborhood of every regular point of A and that it take y_0 to 0. This map is called the 'developing map' on A based at y_0. Note that developing maps takes saddle connections of A to lines of $\mathbb{C}$.

Theorem 3. *An injective developing map is a homeomorphism onto its image.*

Proof. We show that developing maps are open maps which gives us the result. Specifically we show that every point in the domain of a developing map is contained in a neighborhood on which the map is open. We will freely change the base point in this analysis since translations are open maps.

Suppose that A is a simply connected subset of a translation surface (Y, ψ) and $y_0 \in A$. Let dev_{y_0} be the developing map defined on A and based at y_0. Let $y \in A$.

If y is a regular point, then A can be expanded to include a neighborhood of y. The regular coordinate ζ_y map is a homeomorphism defined near y that will agree exactly with dev_y.

If $y = P$ is a singular point of (Y, ψ), then the agreement of dev_{y_0} with all regular coordinates surrounding P implies that

$$\mathrm{dev}_P(\zeta'_P) = \zeta_P = \tfrac{1}{o(P)+1}(\zeta'_P)^{o(P)+1}$$

where ζ_P and ζ'_P denote the regular and (respectively) singular coordinates based at P (see Definition 2). Since the self map of $\mathbb{C}$ given by $w \mapsto \frac{1}{o(P)+1}w^{o(P)+1}$ is an open map, this completes the proof. $\square$

Remark 7. *Since $\tilde{X}$ is simply connected, there is a developing map defined on all of $(\tilde{X}, \pi^*\omega)$ for any chosen base point $y_0 \in \tilde{X}$. We will exclusively be considering singular base points $y_0 = Q \in \widetilde{\Sigma}$ and will be interested in focusing on the restriction of* dev_Q *to a star shaped subset of $(\tilde{X}, \pi^*\omega)$ with center Q. We will denote this restriction again by* dev_Q*. Further restrictions of this map, which are occasionally considered, will have restrictions reflected in the notation.*

Theorem 4. *Let $P \in \Sigma$ and $A \subset \mathcal{O}_P$ be the union of $\{0\}$ and the open (0 centered) sector in $\mathcal{O}_P$ having angle measured in regular coordinates that is no more than π. Then $proj_P|_A$ and $proj_P|_A^{-1}$ send lines to lines. Furthermore $proj_P|_A$ is a metric space isometric embedding.*

Proof. Using the singular coordinates $\zeta'_0(z) = (o(P)+1)^{\frac{1}{o(P)+1}} z$ on $\mathcal{O}_P = (\mathbb{C}, (o(P)+1)z^{o(P)}dz)$ we see from Section 2 that

$$\mathrm{proj}_P(\zeta'_P) = \tfrac{1}{o(P)+1}(\zeta'_P)^{o(P)+1} = dev_0(\zeta'_P)$$

where dev_0 is the developing map based at $0 \in \mathcal{O}_P$. Since $z \mapsto \frac{1}{o(P)+1}z^{o(P)+1}$ is injective on A, it follows that $\mathrm{proj}_P|_A$ is an injective developing map. Therefore by Theorem 3 $\mathrm{proj}_P|_A$ is an embedding. The fact that proj_P restricts to regular neighborhoods to give Riemannian manifold isometries, together with the fact that the restrictions of the angles at $0 \in A$ preclude the formation of lines containing 0 at anything other than an endpoint, implies that proj_P and proj_P^{-1} take lines to lines. Since A is a convex subset of $\mathcal{O}_P$, it follows that $\mathrm{proj}_P|_A$ is a metric space isometric embedding.

$\square$

Definition 21. *If A contains a directed saddle connection starting at a singularity $P \in A$, the image of this directed saddle connection under dev_P gets an induced direction and is called the* ***development*** *of the directed saddle connection.*

3 The Class of Marked Periods of (X, ω)

The relative periods of ω (relative to the singular set Σ) are an additive subgroup of $\mathbb{C}$ generated by the endpoints to developments of directed saddle connections in $(\tilde{X}, \pi^*\omega)$. Our 'marked periods' are similar to this generating set, but are obtained by replacing the developing map with a more discerning map which distinguishes between saddle connections emanating from distinct singularities and also distinguishes between distinct saddle connections emanating from the same singularity but with identical developments. This latter case occurs when the saddle connections are of equal length and the difference in the angle with which they come off of their singularity Q is an integer multiple of 2π measured using regular coordinates.

Fix a partition of $2g-2$, $i_1 + i_2 + ... + i_s = 2g-2$, with $i_1 \leq i_2 \leq ... \leq i_s$. Let $\mathcal{H}(i_1, ..., i_s)$ be the corresponding stratum of $\mathcal{H}_{2g-2}$. For this section we fix an $(X, \omega) \in \mathcal{H}(i_1, ..., i_s)$ with enumerated singular set $\Sigma = \{P_1, ..., P_k\}$.

3.1 Lifting the Saddle Connections Starting at $P \in (X, \omega)$ to $\mathcal{O}_P$

We wish to investigate the flat structure of (X, ω) by considering the collection of saddle connections on (X, ω) emanating from each singularity $P \in \Sigma$. Fixing a singularity $P \in \Sigma$ and lifting these paths starting at a particular chosen point

$Q \in \pi^{-1}(\{P\})$ produces a star shaped subset of the universal cover.

In this subsection we use the developing map from the universal cover to establish a collection of embeddings of this subset onto star shaped subsets of $\mathcal{O}_P$. Selecting embeddings associated to all singularities on (X, ω) gives us a collection of star shaped subsets of the auxiliary surface $\bigsqcup_{P\in\Sigma} \mathcal{O}_P$ which will be used to reconstruct (X, ω). Due to the dependance on the choice of embedding, we consider this data to be a representative of an equivalence class of such objects.

Theorem 5. *Let $P \in \Sigma$ and choose $Q \in \pi^{-1}(\{P\})$. Let STAR_Q be the union of all directed saddle connections in $(\tilde{X}, \pi^*\omega)$ starting at Q and all separatrices starting at Q that do not otherwise pass through a singularity. Let TIPS_Q denote the endpoints of the directed saddle connections used to form STAR_Q. Then $\mathrm{dev}_Q : \mathrm{STAR}_Q \to \mathbb{C}$ lifts via the ramified cover $proj_P : \mathcal{O}_P \to \mathbb{C}$. The set of all such lifts is an orbit under the left action of $\mathrm{Trans}(\mathcal{O}_P)$ on a particular lift. Each of these lifts is an topological embedding that restricts to the open sub-translation surface $\mathrm{STAR}_Q \setminus \mathrm{TIPS}_Q \subset (\tilde{X}, \pi^*\omega)$ to become a translation equivalence.*

Proof. We know by Subsection 2.7 that dev_Q is a continuous function with $\mathrm{dev}_Q(Q) = 0$ that restricts to a neighborhood of Q, U_Q to have the representation

$$\mathrm{dev}_Q(\zeta'_Q) = \tfrac{1}{o(Q)+1}(\zeta'_Q)^{o(Q)+1} = \tfrac{1}{o(P)+1}(\zeta'_P)^{o(P)+1}$$

which is the same local representation as that of proj_P in the singular coordinates about $0 \in \mathcal{O}_P$. Using the local singular coordinates ζ'_Q near Q, and the singular coordinates ζ_0' on $\mathcal{O}_P$ we define $o(P)+1$ embeddings

$$\{h_k : U_Q \to \mathcal{O}_P \mid k \in \{0,1,...,o(P)\}\ \}$$

by

$$\zeta_0' = h_k(\zeta_Q') := e^{i\frac{2k\pi}{o(P)+1}}\zeta_Q', \quad \text{where } k \in \{0,1,...,o(P)\}.$$

It is clear from the representations of $\mathrm{dev}_Q|_{U_Q}$ and proj_P that h_k is a lift of $\mathrm{dev}_Q|_{U_Q}$ via proj_P for all $k \in \{0,1,...,o(P)\}$. Note that this constitutes all possible lifts, given that the deck transformation group $\mathcal{O}_P \setminus \{0\}$ over $\mathbb{C} \setminus \{0\}$ has order $(o(P)+1)$. In particular any lift of dev_Q must equal one of these h_k when restricted to U_Q. We use this fact in the next paragraph in order to provide for the existence of $o(P)+1$ unique lifts of dev_Q via proj_P.

We start by proving that the restriction $\mathrm{dev}_Q|_{\mathrm{STAR}_Q \setminus \{Q\}}$ maps to the punctured plane $\mathbb{C} \setminus \{0\}$. Suppose that $y \in \mathrm{STAR}_Q$ with $y \neq Q$ and $\mathrm{dev}_Q(y) = 0$. Note that STAR_Q is star shaped with center Q so that $[Q,y] \subseteq \mathrm{STAR}_Q$. Since $\mathrm{dev}_Q|_{\mathrm{STAR}_Q \setminus (\mathrm{TIPS}_Q \cup \{Q\})}$ is a local Riemannian manifold isometry, and

$$[Q,y] \cap (\mathrm{TIPS}_Q \cup \{Q\})$$

contains at most the endpoints y and Q, it follows that $\mathrm{dev}_Q([Q,y])$ is the line in $\mathbb{C}$ from 0 to $\mathrm{dev}_Q(y) = 0$. Since the line from 0 to 0 in $\mathbb{C}$ is the singleton $\{0\}$, it follows that $\mathrm{dev}_Q([Q,y]) = \{0\}$. In particular

$$\mathrm{dev}_Q|_{U_Q}([Q,y] \cap U_Q) = \{0\}$$

which is clearly a contradiction since $y \neq Q$. Therefore if $y \neq Q$ then $\mathrm{dev}_Q(y) \neq 0$ and so indeed $\mathrm{Im}(\mathrm{dev}_Q|_{\mathrm{STAR}_Q \setminus \{Q\}}) \subseteq \mathbb{C} \setminus \{0\}$.

We now use the standard lifting condition for maps from a path connected and locally path connected space into the base of a cover, in order to establish the extensions to all of STAR_Q of the lifts of $\mathrm{dev}_Q|_{U_Q}$ found above. Since STAR_Q is star shaped with center Q and contains an open neighborhood of Q, it follows that $\mathrm{STAR}_Q \setminus \{Q\}$ and $\mathcal{O}_P \setminus \{0\}$ both have fundamental groups isomorphic to $\mathbb{Z}$. Fix base points for the fundamental groups $\pi_1(\mathrm{STAR}_Q \setminus \{Q\})$ and $\pi_1(\mathcal{O}_P \setminus \{0\})$ given as $y_0 \in \mathrm{STAR}_Q \setminus \{Q\}$ and $z_0 \in \mathcal{O}_P \setminus \{0\}$. Using again the local representation of the map dev_Q we see that the generator of $\pi_1[\mathrm{STAR}_Q \setminus \{Q\}]$ maps under $[\mathrm{dev}_Q|_{\mathrm{STAR}_Q \setminus \{Q\}}]_*$ to the image under $[\mathrm{proj}_P|_{\mathcal{O}_P \setminus \{0\}}]_*$ of the generator of $\pi_1[\mathcal{O}_n \setminus \{0\}]$. Thus $\mathrm{dev}_Q|_{\mathrm{STAR}_Q \setminus \{Q\}}$ has $o(P)+1$ lifts via $\mathrm{proj}_P|_{\mathcal{O}_P \setminus \{0\}}$ corresponding to the $o(P)+1$ choices for where y_0 gets sent within the set $\mathrm{proj}_P^{-1}(\{\mathrm{dev}_Q(y_0)\})$. These lifts must coincide with the lifts of $\mathrm{dev}_Q|_{U_Q}$ off of Q. They therefore extend to lifts of dev_Q that send Q to $0 \in \mathcal{O}_P$. If $\widehat{\mathrm{dev}}_Q$ is one of the lifts of dev_Q, then the complete set of lifts is $\{\sigma \circ \widehat{\mathrm{dev}}_Q \,|\, \sigma \in \mathrm{Trans}(\mathcal{O}_P)\}$ (recall that $\mathrm{Trans}(\mathcal{O}_P)$ are the translation equivalences that restrict to $\mathcal{O}_P \setminus \{0\}$ to be the group of deck transformations for the $(o(P)+1)$-fold cover $\mathcal{O}_P \setminus \{0\}$ over $\mathbb{C} \setminus \{0\}$).

For any $y \in \mathrm{STAR}_Q$, dev_Q takes the line $[Q,y]$ to a ray from 0 of equal length in $\mathbb{C}$ as argued above. Due to Theorem 4 lifts via proj_P of lines starting at $0 \in \mathbb{C}$ are equal length lines in $\mathcal{O}_P$. Therefore all lifts of dev_Q take lines $[Q, y]$ in STAR_Q to equal length lines starting at P in $\mathcal{O}_P$.

Let $\widehat{\mathrm{dev}}_Q$ be a lift of dev_Q. Suppose $y_1, y_2 \in \mathrm{STAR}_Q$ with $\widehat{\mathrm{dev}}_Q(y_1) = \widehat{\mathrm{dev}}_Q(y_2)$. Recall from above that $\widehat{\mathrm{dev}}_Q|_{U_Q} = h_k$ for some $k \in \{0, ..., o(P)\}$ and so $\widehat{\mathrm{dev}}_Q|_{U_Q}$ is an embedding. Let

$$z \in [\widehat{\mathrm{dev}}_Q(Q), \widehat{\mathrm{dev}}_Q(y_1)] = [\widehat{\mathrm{dev}}_Q(Q), \widehat{\mathrm{dev}}_Q(y_2)]$$

with $z \neq \widehat{\mathrm{dev}}_Q(Q) = 0$ and $z \in \widehat{\mathrm{dev}}_Q(U_Q)$. Since $\widehat{\mathrm{dev}}_Q|_{U_Q}$ is injective it follows that $[Q, y_1] \cap [Q, y_2]$ contains the line $[Q, \widehat{\mathrm{dev}}_Q|_{U_Q}^{-1}(z)]$ which has non-zero length. The line $[Q, \widehat{\mathrm{dev}}_Q|_{U_Q}^{-1}(z)]$ uniquely extends using local translation coordinates in $(\tilde{X}, \pi^*\omega)$ to longer lines until a singularity is encountered. However there are no singularities, other than possibly at the endpoints, in $[Q, y_1]$ and $[Q, y_2]$, and

$$l([Q, y_1]) = l([\widehat{\mathrm{dev}}_Q(Q), \widehat{\mathrm{dev}}_Q(y_1)]) = l([\widehat{\mathrm{dev}}_Q(Q), \widehat{\mathrm{dev}}_Q(y_2))]) = l([Q, y_2]).$$

Therefore in fact $[Q, y_1] = [Q, y_2]$ and so $y_1 = y_2$ and $\widehat{\mathrm{dev}}_Q$ is injective.

We have shown that the restriction $\widehat{\mathrm{dev}}_Q|_{U_Q}$ is a translation equivalence since it is given explicitly as h_k for some $k \in \{0, ..., o(P)\}$. In local neighborhoods about regular points, $\widehat{\mathrm{dev}}_Q|_{(\mathrm{STAR}_Q \setminus \mathrm{TIPS}_Q)}$ also restricts to be a translation equivalence (restriction of dev_Q composed with a local inverse of proj_P). Hence the injective map $\widehat{\mathrm{dev}}_Q|_{(\mathrm{STAR}_Q \setminus \mathrm{TIPS}_Q)}$ is in particular an open map and so a topological embedding. Therefore it is a translation equivalence. $\square$

3.2 Representatives of the Marked Periods at a Point

In this subsection we define a representative of the marked periods of (X, ω) at a singular point of (X, ω) using a particular choice of lift (embedding) from the last subsection. Other representatives are given by choosing different lifts.

Remark 8. *If $P \in \Sigma$ and $Q_1, Q_2 \in \pi^{-1}(\{P\})$ with $\widehat{\mathrm{dev}}_{Q_1}$ being a choice of lift for dev_{Q_1}, then any lift $\widehat{\mathrm{dev}}_{Q_2}$ of dev_{Q_2} satisfies $\widehat{\mathrm{dev}}_{Q_2} = g \circ \widehat{\mathrm{dev}}_{Q_1} \circ \sigma_{Q_2}^{Q_1}$ where $\sigma_{Q_2}^{Q_1} \in \mathrm{Trans}(\widetilde{X}, \pi^*\omega)$ is such that $\sigma_{Q_2}^{Q_1}(Q_2) = Q_1$, and $g \in \mathrm{Trans}(\mathcal{O}_P)$. Since $\mathrm{TIPS}_{Q_2} = \sigma_{Q_1}^{Q_2}(\mathrm{TIPS}_{Q_1})$, it follows that*

$$\widehat{\mathrm{dev}}_{Q_2}(\mathrm{TIPS}_{Q_2}) = g \circ \widehat{\mathrm{dev}}_{Q_1} \circ \sigma_{Q_2}^{Q_1} \circ \sigma_{Q_1}^{Q_2}(\mathrm{TIPS}_{Q_1})) = g \circ \widehat{\mathrm{dev}}_{Q_1}(\mathrm{TIPS}_{Q_1}).$$

Definition 22. *For $P \in \Sigma$, let $Q \in \pi^{-1}(\{P\})$, and let $\widehat{\mathrm{dev}}_Q$ be one of the lifts of $\mathrm{dev}_Q : \mathrm{STAR}_Q \to \mathbb{C}$ via $proj_P : \mathcal{O}_P \to \mathbb{C}$. We define a **representative at P of the marked periods of** (X, ω) (given by $\widehat{\mathrm{dev}}_Q$) to be:*

$$\mathrm{MP}_P(X, \omega) := \widehat{\mathrm{dev}}_Q(\mathrm{TIPS}_Q) \subseteq \mathcal{O}_P.$$

Note by Remark 8 that up to the action of $\mathrm{Trans}(\mathcal{O}_P)$, this definition is independent of the choice of $Q \in \pi^{-1}(\{P\})$.

Remark 9. *The fact that $\tilde{\Sigma}$ has no accumulation points in $\tilde{X}$ implies that $\mathrm{MP}_P(X, \omega)$ has no accumulation points in $\mathcal{O}_P$. In contrast to this sparseness, the directions (from $0 \in \mathcal{O}_P$) for which you encounter an element of $\mathrm{MP}_P(X, \omega)$*

are dense in S^1 *(see [Mas]).*

Definition 23. *For any* $P \in \Sigma$ *and* $r > 0$ *we call*

$$\mathrm{MP}_P^r(X,\omega) := \{\xi \in \mathrm{MP}_P(X,\omega) \mid \textit{ the radius of } \xi \textit{ is less than or equal to } r\},$$

a ***radius*** r ***bounded representative at P of the marked periods of*** (X,ω)*. Also let*

$$MP_P^{>r}(X,\omega) := \{\xi \in \mathrm{MP}_P(X,\omega) \mid \textit{ the radius of } \xi \textit{ is greater than } r\}.$$

Definition 24. *Let* $P \in \Sigma$*. For each representative at* P*,* $\mathrm{MP}_P(X,\omega)$*, given by a particular lift* $\widehat{\mathrm{dev}}_Q$ *for some* $Q \in \pi^{-1}(\{P\})$*, let* $\Lambda_P := \widehat{\mathrm{dev}}_Q(\mathrm{STAR}_Q)$*. We also label* $\widehat{\eta_P}^Q := [\widehat{\mathrm{dev}}_Q]^{-1} : \Lambda_P \to \mathrm{STAR}_Q$ *and* $\eta_P := \pi \circ \widehat{\eta_P}^Q : \Lambda_P \to (X,\omega)$*.*

As the notation suggests, $\widehat{\eta_P}^Q$ is a lift of η_P (via the covering map π). The following diagrams are commutative.

$$\begin{tikzcd} \Lambda_P \arrow[r, "\widehat{\eta_P}^Q", "\text{Homeo.}"'] \arrow[dr, "\eta_P"'] & \mathrm{STAR}_Q \arrow[d, "\pi"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} \Lambda_P \setminus \mathrm{MP}_P(X,\omega) \arrow[r, "\widehat{\eta_P}^Q", "\text{Trans. Eq.}"'] \arrow[dr, "\eta_P"'] & \mathrm{STAR}_Q \setminus \mathrm{TIPS}_Q \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

Figure 4: The set of all saddle connections from $P \in (X,\omega)$ lift via π starting at $Q \in \tilde{X}$ to form STAR_Q. A star shaped subset $\Lambda \subset \mathcal{O}_P$ embeds onto $(\tilde{X}, \pi^*\omega)$ via $\widehat{\eta_P}^Q$.

Remark 10. *We have that* $\widehat{\eta_P}^Q|_{\Lambda_P \setminus \mathrm{MP}_P(X,\omega)}$ *is a translation embedding and* $\eta_P|_{\Lambda_P \setminus \mathrm{MP}_P(X,\omega)}$ *is a local translation embedding. Furthermore,* η_P *takes the out-*

wardly directed radial lines $\{[0,\xi]\,|\,\xi \in \mathrm{MP}_P(X,\omega)\}$ *bijectively onto the set of directed saddle connections starting at* $P \in (X,\omega)$.

3.3 Representatives of the Marked Periods

In this subsection we bring together the representatives of the marked periods at each point in the singular set Σ to form a representative of the marked periods.

Recall from Section 2 that the stratum containing (X,ω), $\mathcal{H}(i_1,...,i_s)$, consists of all translation surfaces with s singularities of orders $i_1,...,i_s$. Recall that $\mathfrak{s}$ represents the number of distinct values of summands in the partition

$$i_1 + ... + i_s = 2g - 2,$$

where for all $k \in \{1,...,\mathfrak{s}\}$ c_k denotes the number of times that the k_{th} value (given by q_k) appears in the partition. Thus

$$\begin{aligned} &q_1 = i_1 = ... = i_{c_1} \neq \\ &q_2 = i_{c_1+1} = ... = i_{c_1+c_2} \neq \cdots \neq \\ &q_{\mathfrak{s}} = i_{s-c_{\mathfrak{s}}+1} = ... = i_s. \end{aligned}$$

Fix an enumeration of the singular set $\Sigma = \{P_1,...,P_s\}$ for the remainder of this section.

Let

$$\mathbf{O} = \bigsqcup_{c_1 \text{ copies}} \mathcal{O}_1 \sqcup \bigsqcup_{c_2 \text{ copies}} \mathcal{O}_2 \sqcup ... \sqcup \bigsqcup_{c_{\mathfrak{s}} \text{ copies}} \mathcal{O}_{\mathfrak{s}}.$$

Given an $(X,\omega) \in \mathcal{H}(i_1,...,i_s)$ with singular set Σ, recall from Definition 15 that $\bigsqcup_{P\in\Sigma} \mathcal{O}_P$ can be identified with $\mathbf{O}$. There are many choices for this identification however, since there is no canonical choice for which of the c_k copies of $\mathcal{O}_k$ we identify with $\mathcal{O}_P$ where $k \in \{1,...,\mathfrak{s}\}$ such that $o(P) = q_k$.

Remark 11. *The collection of all identifications of* $\bigsqcup_{P\in\Sigma} \mathcal{O}_P$ *with* $\boldsymbol{O}$ *is given by* $\mathrm{Trans}(\boldsymbol{O})\iota$ *where* $\iota : \bigsqcup_{P\in\Sigma} \mathcal{O}_P \to \boldsymbol{O}$ *is any given identification.*

We now introduce some notation to track when two distinct marked periods elements represent either end of the same saddle connection on (X,ω).

Definition 25. *Let* $\{\mathrm{MP}_{P_1}(X,\omega), \mathrm{MP}_{P_2}(X,\omega), ..., \mathrm{MP}_{P_s}(X,\omega)\}$ *be a set consisting of representatives (at respectively* $P_1, P_2, ..., P_s$*) of marked periods of* (X,ω)*. For* $P \in \Sigma$ *and* $\xi \in \mathrm{MP}_P(X,\omega)$*. Let* $P' \in \Sigma$ *be such that* $P' = \eta_P(\xi)$*. We define* $\tilde{\xi} \in MP_{P'}(X,\omega)$ *to be the unique element such that* $\eta_{P'} \circ ({}_0\lambda_{\tilde{\xi}})(t) = \eta_P \circ ({}_0\lambda_\xi)(1-t)$ *for all* $t \in [0,1]$*. Hence* $\tilde{\xi} \in MP_{P'}(X,\omega)$ *and* $\xi \in \mathrm{MP}_P(X,\omega)$ *correspond to oppositely directed saddle connections on* (X,ω)*.*

Definition 26. *Let* $\{\mathrm{MP}_{P_1}(X,\omega), \mathrm{MP}_{P_2}(X,\omega), ..., \mathrm{MP}_{P_s}(X,\omega)\}$ *be a set consisting of representatives (at respectively* $P_1, P_2, ..., P_s$*) of marked periods of* (X,ω)*. Furthermore let* $\iota : \bigsqcup_{P\in\Sigma} \mathcal{O}_P \to \boldsymbol{O}$ *be a fixed identification. There is a* $\mathbb{Z}_2$*-action on* $\iota(\bigsqcup_{P\in\Sigma} \mathrm{MP}_P(X,\omega)) \subseteq \boldsymbol{O}$ *generated by the action of the non-trivial element* $\epsilon \in \mathbb{Z}_2 \setminus \{0\}$ *whereby if* $\xi \in \bigsqcup_{P\in\Sigma} \mathrm{MP}_P(X,\omega)$ *then* $\epsilon \cdot \iota(\xi) := \iota(\tilde{\xi})$*. Since this* *action is radius preserving, the subset* $\iota(\bigsqcup_{P\in\Sigma} \mathrm{MP}^r_P(X,\omega)) \subseteq \boldsymbol{O}$ *for any* $r \in \mathbb{R}^+$ *(see Definition 23) also inherits a well defined* $\mathbb{Z}_2$*-action.*

We call the resulting $\mathbb{Z}_2$*-space on* $\iota(\bigsqcup_{P\in\Sigma} \mathrm{MP}_P(X,\omega)) \subseteq \boldsymbol{O}$ *a* ***representative of the marked periods*** *of* (X,ω) *and denote it by* $\mathrm{MP}(X,\omega)$*. We call the* $\mathbb{Z}_2$*-space on* $\iota(\bigsqcup_{P\in\Sigma} \mathrm{MP}^r_P(X,\omega))$ *a* ***radius*** r ***bounded representative*** *of the marked periods of* (X,ω) *and denote it by* $\mathrm{MP}^r(X,\omega)$*.*

The $\mathbb{Z}_2$-action defined on a representative of the marked periods relates the points in each $\mathrm{MP}_P(X,\omega)$ by keeping track of which singularity a particular directed saddle connection terminates at and what other marked period represents traveling in the reverse direction along that same saddle connection. By utilization of the Voronoi cellular decomposition of (X,ω), we will show that a representative of the marked periods completely determines the surface up to translation equivalence . In fact the $\mathbb{Z}_2$-space defined on a finite subset will suffice for determining the surface.

Remark 12. *The selection of a representative of the marked periods for a particular* $(X, \omega) \in \mathcal{H}(i_1, ..., i_s)$ *with singular set* Σ *implicitly assumes a chosen enumeration* $\{P_1, P_2, ..., P_s\}$ *of* Σ*, a chosen set* $\{\mathrm{MP}_{P_1}(X, \omega), \mathrm{MP}_{P_2}(X, \omega), ..., \mathrm{MP}_{P_s}(X, \omega)\}$ *of representatives (at respectively* $P_1, P_2, ..., P_s$*) of marked periods of* (X, ω)*, and a chosen identification* $\iota : \bigsqcup_{P \in \Sigma} \mathcal{O}_P \to \boldsymbol{O}$*. The choice of representative at each* $P \in \{P_1, P_2, ..., P_s\}$ *assumes the selection for all* $k \in \{1, ..., s\}$ *of a point* $Q_k \in \pi^{-1}(\{P_k\})$ *and a lift* $\widehat{\eta_{P_k}}^{Q_k}$ *of* η_{P_k}*.*

3.4 The Class of Marked Periods

A representative of the marked periods of the translation surface
$(X, \omega) \in \mathcal{H}(i_1, ..., i_s)$ is not a canonically defined object. We will show that the collection of all representatives is formed by mapping a particular representative by elements in $\mathrm{Trans}(\mathbf{O})$ and inducing $\mathbb{Z}_2$-actions through the bijections. This indeed defines an equivalence relation on the set of all $\mathbb{Z}_2$-subsets of $\mathbf{O}$.

In this subsection we introduce the canonical object of study, the equivalence class of a particular representative associated to the equivalence relation discussed above.

Definition 27. *For* $P_i \in \Sigma$*, let* $\mathrm{Trans}(\mathcal{O}_{P_i})$ *be identified with a subgroup of* $\mathrm{Trans}(\bigsqcup_{P \in \Sigma} \mathcal{O}_P)$ *whereby for* $F \in \mathrm{Trans}(\mathcal{O}_{P_i})$ *we extend the definition of* F *to include* $F(z) := z$ *for all* $z \in \mathcal{O}_P$ *with* $P \neq P_i$*.*

For $P_i, P_j \in \Sigma$ *with* $o(P_i) = o(P_j) = k$*, recall that* $\mathcal{O}_{P_i} := \mathcal{O}_k =: \mathcal{O}_{P_j}$*. Here* *we let* $\sigma_{i,j} : \mathcal{O}_{P_i} \to \mathcal{O}_{P_j}$ *be given as* $\mathrm{Id}_{\mathcal{O}_k}$.

Theorem 6. *The group of translations* $\mathrm{Trans}(\bigsqcup_{P\in\Sigma} \mathcal{O}_P)$ *is generated by*

$$\bigcup \big\{ \mathrm{Trans}(\mathcal{O}_P) | P \in \Sigma \big\} \bigcup \big\{ \sigma_{i,j} \big| i, j \in \{1, ..., s\}, o(P_i) = o(P_j) \big\}.$$

Proof. Let $F \in \mathrm{Trans}(\bigsqcup_{P\in\Sigma} \mathcal{O}_P)$. By definition F must preserve the singular set of $\bigsqcup_{P\in\Sigma} \mathcal{O}_P$. In fact F at most permutes singularities of the same order. Hence there is a $\sigma \in \langle\ \sigma_{i,j} \mid i, j \in \{1, ..., s\}\ ;\ o(P_i) = o(P_j)\ \rangle$ such that $\sigma \circ F$ fixes all of the singularities of $\bigsqcup_{P\in\Sigma} \mathcal{O}_P$. Thus $\sigma \circ F$ is an affine diffeomorphism that maps each $\mathcal{O}_P$ onto itself having an identity Jacobian in local translation coordinates. Thus $\sigma \circ F = \tau \in \langle\ \bigcup_{P\in\Sigma} \mathrm{Trans}(\mathcal{O}_P)\ \rangle$ so that

$$F = \sigma^{-1} \circ \tau \in \Big\langle \bigcup_{P\in\Sigma} \mathrm{Trans}(\mathcal{O}_P) \bigcup \big\{ \sigma_{i,j} \big| i, j \in \{1, ..., s\} o(P_i) = o(P_j) \big\} \Big\rangle.$$

$\square$

Corollary 1. *Let* $\iota : \bigsqcup_{P\in\Sigma} \mathcal{O}_P \to \boldsymbol{O}$ *be a fixed identification. Let*

$$\mathcal{T} := \iota \circ \Big\langle \bigcup_{P\in\Sigma} \mathrm{Trans}(\mathcal{O}_P) \Big\rangle \circ \iota^{-1},$$

and

$$\mathcal{S} := \iota \circ \Big\langle\ \sigma_{i,j} \mid i, j \in \{1, ..., s\}, o(P_i) = o(P_j)\ \Big\rangle \circ \iota^{-1}.$$

Then the group $\mathrm{Trans}(\boldsymbol{O})$ *is generated by* $\mathcal{T} \cup \mathcal{S}$.

Proof. Since ι is a translation equivalence, this follows from Theorem 6. $\square$

The group $\langle \mathcal{T} \rangle$ consists of translations of $\mathbf{O}$ that rotate individual copies of $\mathcal{O}_1, ..., \mathcal{O}_5$ by integer multiples of 2π. The group $\langle \mathcal{S} \rangle$ permutes the individual surfaces having the same cone angle within $\bigsqcup_{P\in\Sigma} \mathcal{O}_P$, using the identity map on their common base surface.

Definition 28. *Let $P \in \Sigma$ and $\Upsilon := \{A \subseteq \boldsymbol{O} \,|\, A \text{ is equipped with a } \mathbb{Z}_2\text{-action}\}$. If $A_1, A_2 \in \Upsilon$ let $A_1 \sim A_2$ if and only if there is an $F \in \mathrm{Trans}(\boldsymbol{O})$ that takes A_1 bijectively and $\mathbb{Z}_2$-equivariantly onto A_2. Clearly $\sim$ is an equivalence relation on Υ. Let $\Xi := \Upsilon / \sim$.*

Note that any representative of the marked periods of (X, ω), $\mathrm{MP}(X, \omega)$, is an element of Υ.

Theorem 7. *If $\mathrm{MP}(X, \omega) \in \Upsilon$ is a representative of the marked periods of (X, ω), then $\mathrm{MP}(X, \omega) \sim A$ for some $A \subseteq \Upsilon$ if and only if A is another representative of the marked periods of (X, ω).*

If $\mathrm{MP}^r(X, \omega) \in \Upsilon$ is a radius r bounded representative of marked periods of (X, ω), then $\mathrm{MP}^r(X, \omega) \sim A$ for some $A \subseteq \Upsilon$ if and only if A is another radius r bounded representative of marked periods of (X, ω).

Proof. Suppose $\mathrm{MP}(X, \omega) \in \Upsilon$ is a representative of the marked periods of (X, ω), and that $A \subseteq \Upsilon$ is another representative of the marked periods of (X, ω). Then by Remark 8 there are $g_P \in \mathrm{Trans}(\mathcal{O}_P)$ for all $P \in \Sigma$ and an identification $\iota' : \bigsqcup_{P\in\Sigma} \mathcal{O}_P \to \mathbf{O}$ such that $A = \iota'(\bigsqcup_{P\in\Sigma}(g_P(\mathrm{MP}_P(X, \omega)))$ where for each $P \in \Sigma$, $g_P(\mathrm{MP}_P(X, \omega))$ is a representative at P of marked periods of (X, ω) with the map $\eta_P \circ g_P^{-1}$ taking the role of η_P.

Identifying g_P for each $P \in \Sigma$ with an element of $\mathrm{Trans}(\bigsqcup_{P\in\Sigma}\mathcal{O}_P)$ as in Definition 27 we have a bijection

$$[\iota' \circ g_{P_1} \circ ... \circ g_{P_s} \circ \iota^{-1}] : \mathrm{MP}(X,\omega) \to A$$

For each $P \in \Sigma$ and $\xi \in \mathrm{MP}_P(X,\omega)$ we have that ${}_0\lambda_{g_P(\xi)} = g_P \circ_0 \lambda_\xi$.

Let $\epsilon \in \mathbb{Z}_2 \setminus \{0\}$. It follows that for any $P_j, P_k \in \Sigma$, $\xi_1 \in \mathrm{MP}_{P_j}(X,\omega)$, and $\xi_2 \in \mathrm{MP}_{P_k}(X,\omega)$,

$$\iota(\xi_1) = \epsilon \cdot \iota(\xi_2) \quad \Leftrightarrow$$

$$[\eta_{P_j} \circ ({}_0\lambda_{\xi_1})](t) = [\eta_{P_k} \circ ({}_0\lambda_{\xi_2})](1-t) \text{ for all } t \in [0,1] \quad \Leftrightarrow$$

$$[\eta_{P_j} \circ g_{P_j}^{-1} \circ g_{P_j} \circ ({}_0\lambda_{\xi_1})](t) = [\eta_{P_k} \circ g_{P_k}^{-1} \circ g_{P_k} \circ ({}_0\lambda_{\xi_2})](1-t) \text{ for all } t \in [0,1] \quad \Leftrightarrow$$

$$[(\eta_{P_j} \circ g_{P_j}^{-1}) \circ ({}_0\lambda_{g_{P_j}(\xi_1)})](t) = [(\eta_{P_k} \circ g_{P_k}^{-1}) \circ ({}_0\lambda_{g_{P_k}(\xi_2)})](1-t) \text{ for all } t \in [0,1] \quad \Leftrightarrow$$

$$\iota'(g_{P_j}(\xi_1)) = \epsilon \cdot \iota'(g_{P_k}(\xi_2)) \Leftrightarrow$$

$$[\iota' \circ g_{P_1} \circ ... \circ g_{P_s} \circ \iota^{-1}](\iota(\xi_1)) = \epsilon \cdot [\iota' \circ g_{P_1} \circ ... \circ g_{P_s} \circ \iota^{-1}](\iota(\xi_2)).$$

Therefore we have an element $[\iota' \circ g_{P_1} \circ ... \circ g_{P_s} \circ \iota^{-1}] \in \mathrm{Trans}(\mathbf{O})$ that takes $\mathrm{MP}(X,\omega)$ bijectively and $\mathbb{Z}_2$-equivariantly onto A.

Suppose there is an $A \in \Upsilon$ and a $\mathbb{Z}_2$-equivariant $F \in \mathrm{Trans}(\mathbf{O})$ such that $A = F(\mathrm{MP}(X,\omega))$. Since $\mathcal{TS} = \mathcal{ST}$, it follows that there is a $H \in \mathcal{T}$ and a $\sigma \in \mathcal{S}$ such that $F = \sigma \circ H$ and thus $A = \sigma(H(\mathrm{MP}(X,\omega)))$. By Theorem 5 and Remark 8, $(\iota^{-1} \circ H \circ \iota)[\iota^{-1}(\mathrm{MP}(X,\omega))]$ is the disjoint union of representatives at $P_1, ..., P_s \in \Sigma$ of marked periods for (X,ω) . Since $\iota' := \sigma \circ \iota$ is another identification of $\bigsqcup_{P\in\Sigma}\mathcal{O}_P$ and $\mathbf{O}$, it follows that

$$A = \sigma(H(\mathrm{MP}(X,\omega))) = (\sigma \circ \iota) \circ (\iota^{-1} \circ H \circ \iota)[\iota^{-1}(\mathrm{MP}(X,\omega))]$$

is another representative of marked periods of (X,ω).

Suppose $\mathrm{MP}^r(X,\omega) \in \Upsilon$ is a radius r bounded representative of marked periods of (X,ω). Then there is a representative $\mathrm{MP}(X,\omega)$ containing $\mathrm{MP}^r(X,\omega)$.

If $A \subseteq \Upsilon$ is another radius r bounded representative of marked periods of (X,ω), and $B \supset A$ is a corresponding representative, then the $\mathbb{Z}_2$-equivariant bijection in $\mathrm{Trans}(\mathbf{O})$ known to exist from above between $\mathrm{MP}(X,\omega)$ and B will restrict to become a $\mathbb{Z}_2$-equivariant bijection of $\mathrm{MP}^r(X,\omega)$ and A.

Suppose $A \subseteq \Upsilon$ and $F \in \mathrm{Trans}(\mathbf{O})$ that takes $\mathrm{MP}^r(X,\omega)$ bijectively and $\mathbb{Z}_2$-equivariantly onto A. Let $B = F(\mathrm{MP}(X,\omega))$ have $\mathbb{Z}_2$-action induced from $\mathrm{MP}(X,\omega)$ using the fact that F is bijective. From above we know that B is a representative of marked periods of (X,ω). Clearly A is the result of taking all of the elements in B that have distance less than r to the singularity of their connected component, and imposing a $\mathbb{Z}_2$-action by restricting the original $\mathbb{Z}_2$-action of B onto this subset. Thus in fact A is a radius r bounded representative of marked periods of (X,ω). $\square$

Definition 29. *The **marked periods** of (X,ω), denoted $[\mathrm{MP}(X,\omega)]$, is the equivalence class in Ξ consisting of all representative of marked periods $\mathrm{MP}(X,\omega) \in \Upsilon$. Similarly for any $r > 0$, the **radius r bounded marked periods** of (X,ω), denoted $[\mathrm{MP}^r(X,\omega)]$, is the equivalence class in Ξ consisting of* *all possible radius r bounded representatives* $\mathrm{MP}^r(X,\omega)$.

We show next that the class of marked periods depends only on the translation equivalence class of the surface.

Theorem 8. *If the translation surface $(Y,\psi) \in \mathcal{H}(i_1, ..., i_s)$ is translation equivalent to (X,ω), then $[\mathrm{MP}(X,\omega)] = [\mathrm{MP}(Y,\psi)]$.*

Proof. Let Σ_Y correspond to the singular set of (Y,ψ). Let

$$\Sigma_X = \Sigma = \{P_1, P_2, ..., P_s\}$$

and

$$\mathrm{MP}_{P_1}(X,\omega), \mathrm{MP}_{P_2}(X,\omega), ..., \mathrm{MP}_{P_s}(X,\omega)$$

be representatives at these points of marked periods of (X,ω) with corresponding maps $\widehat{\eta_{P_k}}^{Q_k}$ for $k = 1, ..., s$. Let $H : (X,\omega) \to (Y,\psi)$ be a translation equivalence and $\hat{H} : (\tilde{X}, \pi^*\omega) \to (\tilde{Y}, \pi^*\psi)$ be a (translation equivalence) lift of H via the universal translation covers $\pi_X : (\tilde{X}, \pi^*\omega) \to (X,\omega)$ and $\pi_Y : (\tilde{Y}, \pi^*\psi) \to (Y,\psi)$. For all $k \in \{1, ..., s\}$ let $R_k := H(P_k)$, $S_k := \hat{H}(Q_k)$, and $\mathcal{O}_{R_k} := \mathcal{O}_{P_k}$. Following the construction described in Subsections 3 and 3.2, we see that for all $k \in \{1, ..., s\}$, $\mathrm{STAR}_{S_k} = \hat{H}(\mathrm{STAR}_{Q_k})$ and that $\widehat{\eta_{R_k}}^{S_k} := \hat{H} \circ \widehat{\eta_{P_k}}^{Q_k}$ is the result of choosing the lift $(\widehat{\eta_{P_k}}^{Q_k})^{-1} \circ \hat{H}^{-1}$ of the developing map $\mathrm{dev}_{S_k} = \mathrm{dev}_{Q_k} \circ \hat{H}^{-1}$ restricted to STAR_{S_k}. Note that here $\Lambda_{R_k} = \Lambda_{P_k}$ (see Figure 5).

$$
\begin{tikzcd}
(\Lambda_{R_k} = \Lambda_{P_k}) \arrow[r, "\widehat{\eta_{P_k}}^{Q_k}", "\text{Homeo.}"'] \arrow[dr, "\eta_{P_k}"'] \arrow[rr, bend left, "\widehat{\eta_{R_k}}^{S_k}"] \arrow[drr, bend right, "\eta_{R_k}"'] & \mathrm{STAR}_{Q_k} \arrow[r, "\hat{H}", "\text{Trans. Eq.}"'] \arrow[d, "\pi"] & \mathrm{STAR}_{S_k} \arrow[d, "\pi'"] \\
& X \arrow[r, "\mathrm{H}"'] & Y
\end{tikzcd}
$$

Figure 5: Compositions by H and $\hat{H}$ allow us to construct an instance at R_k, $\mathrm{MP}_{R_k}(Y, \psi)$.

Thus for all $k \in \{1, ..., s\}$

$$
\begin{aligned}
\mathrm{MP}_{R_k}(Y, \psi) &:= (\widehat{\eta_{R_k}}^{S_k})^{-1}(\mathrm{TIPS}_{S_k}) \\
&= \left[(\widehat{\eta_{P_k}}^{Q_k})^{-1} \circ \hat{H}^{-1}\right] \left(\hat{H}(\mathrm{TIPS}_{Q_k})\right) \\
&= \mathrm{MP}_{P_k}(X, \omega)
\end{aligned}
$$

are representatives at R_k of marked periods for (Y, ψ).

Given $k \in \{1, ..., s\}$ and $\xi_1, \xi_2 \in \mathrm{MP}_{P_k}(X, \omega)$ we have that $\tilde{\xi_1} = \xi_2$ if and only if the corresponding directed saddle connections in (X, ω) given by the geodesics ${}_0\gamma_{\eta_{P_k}(\xi_1)}$ and ${}_0\gamma_{\eta_{P_k}(\xi_2)}$ are the same saddle connection but oppositely directed. Since the saddle connections corresponding to ξ_1 and ξ_2 considered as elements of $\mathrm{MP}_{R_k}(Y, \psi)$ are given by the geodesics $H \circ_0 \gamma_{\eta_{P_k}(\xi_1)}$ and $H \circ_0 \gamma_{\eta_{P_k}(\xi_2)}$ where H is a homeomorphism, it follows that $\tilde{\xi_1} = \xi_2$ as elements of $\mathrm{MP}_{P_k}(X, \omega)$ if and only if $\tilde{\xi_1} = \xi_2$ as elements of $\mathrm{MP}_{R_k}(Y, \psi)$. Finally, we choose an identification $\iota : \bigsqcup_{R_k \in \Sigma,} \mathcal{O}_{R_k} = \bigsqcup_{P_k \in \Sigma} \mathcal{O}_{P_k} \to \mathbf{O}$, and get that $\iota(\bigsqcup_{k=1,\ldots,s} \mathrm{MP}_{P_k}(X,\omega))$ and $\iota(\bigsqcup_{k=1,\ldots,s} \mathrm{MP}_{R_k}(Y,\psi))$ are representatives of marked periods for (X,ω) and (Y,ψ) respectively. It follows from the work above that $\iota(\bigsqcup_{k=1,\ldots,s} \mathrm{MP}_{P_k}(X,\omega))$ and $\iota(\bigsqcup_{k=1,\ldots,s} \mathrm{MP}_{R_k}(Y,\psi))$ are equal as sets and are endowed with the same $\mathbb{Z}_2$ action. Therefore we have that

$$\begin{aligned}
[\mathrm{MP}(X,\omega)] &= \left[\iota\left(\bigsqcup_{k=1,\ldots,s} \mathrm{MP}_{P_k}(X,\omega)\right)\right] \\
&= \left[\iota\left(\bigsqcup_{k=1,\ldots,s} \mathrm{MP}_{R_k}(Y,\psi)\right)\right] \\
&= [\mathrm{MP}(Y,\psi)].
\end{aligned}$$

$\square$

4 Reconstructing (X,ω) from its Marked Periods

For this section we consider an $(X,\omega) \in \mathcal{H}(i_1,...,i_s)$ with singular set $\Sigma = \{P_1,...,P_k\}$, having a fixed representative of marked periods $\mathrm{MP}(X,\omega)$ corresponding to the set $\bigsqcup_k \mathrm{MP}_{P_k}(X,\omega)$. Recall that a choice of representative $\mathrm{MP}(X,\omega)$ implicitly assumes the choice for each $P_k \in \Sigma$ of Q_k and $\widehat{\eta_{P_k}}^{Q_k}$ (see Remark 12).

4.1 Metric Properties of Embeddings from Subsection 3

For this subsection we fix $k \in \{1,...,s\}$ and let $P = P_k$ and $Q = Q_k$. Recall that $\widehat{\eta_P}^Q : \Lambda_P \to \mathrm{STAR}_Q \subset \tilde{X}$ is the embedding whose image is the union of lifts beginning at Q of all saddle connections on (X,ω) starting at P. The star shaped subset $\Lambda_P \subset \mathcal{O}_P$ will be used to carve out building blocks for reconstructing (X,ω). This process will use the metric on $\mathcal{O}_P$, and will mimic a similar process on $(\tilde{X},\pi^*\omega)$. In this subsection we therefore introduce a few facts regarding how well metric information is carried over by the topological embedding $\widehat{\eta_P}^Q$.

Theorem 9. *The following hold:*

1. *If* $z \in \Lambda_P$ *then* $d(\widehat{\eta_P}^Q(z), Q) = d(z,0)$.

2. *If* $z,w \in \Lambda_P$ *then* $d(\widehat{\eta_P}^Q(z), \widehat{\eta_P}^Q(w)) \geq d(z,w)$ *with equality if and only if*

the interior of $\triangle z0w$ *does not contain a point of* $\mathrm{MP}_P(X,\omega)$.

3. *If* $z \in \Lambda_P$ *and* $d(z,0) \le d(z,\xi)$ *for all* $\xi \in \mathrm{MP}_P(X,\omega)$, *then*

$$d(\widehat{\eta_P}^Q(z), Q) < d(\widehat{\eta_P}^Q(z), y) \text{ for all } y \in \tilde{X} \setminus \mathrm{STAR}_Q.$$

Proof. Let $z \in \Lambda_P$ and $y = \widehat{\eta_P}^Q(z)$. As established in the proof of Theorem 5, $(\widehat{\eta_P}^Q)^{-1} = \widehat{\mathrm{dev}_Q}$ takes the line $[Q,y]$ to a line segment of equal length from $0 \in \Lambda_P$. Since the distance between two points in either Λ_P or $\mathcal{O}_P$ is realized as the length of the unique geodesic connecting them, statement 1 follows.

Suppose $z, w \in \Lambda_P$.

If $z = 0$ or $w = 0$, then statement 2 is true by statement 1 and the fact that $\triangle z0w$ has empty interior.

If neither of z or w is 0 and the legs $[z,0]$ and $[0,w]$ make an angle at 0 that is greater than or equal to π, then by Theorem 1 $[z,0] \cup [0,w] = [z,w]$ and so $d(z,w) = d(z,0) + d(0,w)$. In Theorem 5 we showed that $\widehat{\eta_P}^Q|_{\Lambda_P \setminus \mathrm{MP}_P(X,\omega)}$ is a translation equivalence. Since $0 \in \Lambda_P \setminus \mathrm{MP}_P(X,\omega)$ it follows that the map $\widehat{\eta_P}^Q$ preserves angles at $0 \in \mathcal{O}_P$ between two legs of a geodesic through 0. Therefore $[\widehat{\eta_P}^Q(z), Q] \cup [Q, \widehat{\eta_P}^Q(w)] = [\widehat{\eta_P}^Q(z), \widehat{\eta_P}^Q(w)]$ and so

$$d(\widehat{\eta_P}^Q(z), \widehat{\eta_P}^Q(w)) = d(\widehat{\eta_P}^Q(z), Q) + d(Q, \widehat{\eta_P}^Q(w)).$$

Therefore by statement 1 we have that

$$d\left(\widehat{\eta_P}^Q(z), \widehat{\eta_P}^Q(w)\right) = d\left(\widehat{\eta_P}^Q(z), Q\right) + d\left(Q, \widehat{\eta_P}^Q(w)\right) = d(z,0) + d(0,w) = d(z,w).$$

The equality $[z,0] \cup [0,w] = [z,w]$ also implies that the interior of $\triangle z0w$ is empty. Hence statement 2 holds in this case.

Suppose neither of z or w is 0 and the legs $[z,0]$ and $[0,w]$ make an angle at 0 that is less than π.

Since $\widehat{\eta_P}^Q$ is continuous and $\widehat{\eta_P}^Q(\mathrm{MP}_P(X,\omega)) \subset \tilde{\Sigma}$, the fact that $\tilde{\Sigma}$ has no accumulation points in $\tilde{X}$ implies that $\triangle z0w \cap \mathrm{MP}_P(X,\omega)$ is finite. Let

$$\{\xi_1, ..., \xi_l\} \subset \triangle z0w \cap \mathrm{MP}_P(X,\omega)$$

be such that the polygon determined by the vertex sequence: $z, \xi_1, ..., \xi_l, w$ is equal to the convex hull of $\{z,w\} \cup (\triangle z0w \cap \mathrm{MP}_P(X,\omega))$ (see Figure 6).

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the Euclidean polygons determined by the vertex sequences: $z, \xi_1, ..., \xi_l, w$ and $z, \xi_1, ..., \xi_l, w, 0$ respectively. Then the convexity of $\mathcal{P}_1$ implies that the internal angles at all of the vertices $\xi_1, ..., \xi_l$ within $\mathcal{P}_2$ have measure greater than or equal to π. Since $\mathcal{P}_2 \subseteq \Lambda_P$ and $\widehat{\eta_P}^Q$ takes all of $\xi_1, ..., \xi_l$ to singularities of $(\tilde{X}, \pi^*\omega)$ (which have total angle greater than or equal to 4π), it follows from Remark 10 that the internal angles at all of the vertices $\xi_1, ..., \xi_l$ within $\mathcal{P}_2$ become the (minimal) angles between the legs of the piecewise geodesic $[\widehat{\eta_P}^Q(z), \widehat{\eta_P}^Q(\xi_1)] \cup [\widehat{\eta_P}^Q(\xi_1), \widehat{\eta_P}^Q(\xi_2)] \cup ... \cup [\widehat{\eta_P}^Q(\xi_l), \widehat{\eta_P}^Q(w)]$. Therefore by Theorem 1 the path $[\widehat{\eta_P}^Q(z), \widehat{\eta_P}^Q(\xi_1)] \cup [\widehat{\eta_P}^Q(\xi_1), \widehat{\eta_P}^Q(\xi_2)] \cup ... \cup [\widehat{\eta_P}^Q(\xi_l), \widehat{\eta_P}^Q(w)]$ is

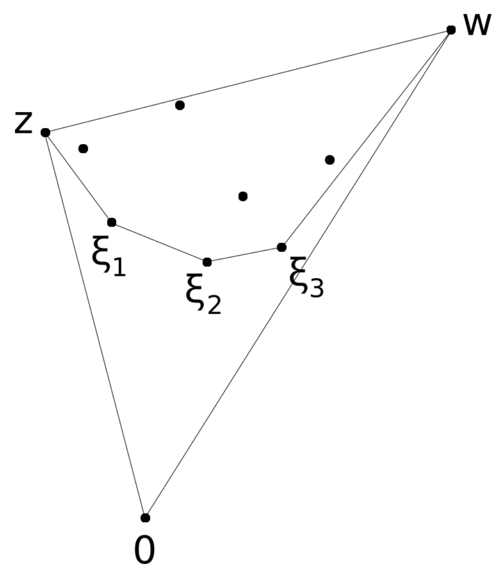

Figure 6: The case $l = 3$. The Euclidean polygon $\mathcal{P}_1$ is the convex hull of $\{z, w\} \cup (\triangle\, z0w \cap \mathrm{MP}_P(X, \omega))$ and is determined by the vertices $z, \xi_1, ..., \xi_l, w$. The Euclidean polygon $\mathcal{P}_2$ is determined by the vertices $z, \xi_1, ..., \xi_l, w, 0$.

a geodesic of $(\tilde{X}, \pi^*\omega)$.

Again using Remark 10 we have that

$$\begin{aligned} d(\widehat{\eta_P}^Q(z), \widehat{\eta_P}^Q(w)) &= d(\widehat{\eta_P}^Q(z), \widehat{\eta_P}^Q(\xi_1)) + d(\widehat{\eta_P}^Q(\xi_1), \widehat{\eta_P}^Q(\xi_2)) + \cdots \\ &\qquad\qquad \cdots + d(\widehat{\eta_P}^Q(\xi_l), \widehat{\eta_P}^Q(w)) \\ &= d(z, \xi_1) + d(\xi_1, \xi_2) + ... + d(\xi_l, w) \\ &\geq d(z, w). \end{aligned}$$

It is clear that equality holds if and only if $\mathrm{MP}_P(X, \omega) \cap \triangle\, z0w \subset [z, w]$. Since $\mathrm{MP}_P(X, \omega) \cap [0, z]$ and $\mathrm{MP}_P(X, \omega) \cap [0, w]$ contain at most z and w respectively, equality holds if and only if the interior of $\triangle z0w$ does not contain a point of $\mathrm{MP}_P(X,\omega)$. This completes the proof of statement 2.

Suppose that there is a $z \in \Lambda_P$ with $d(z,0) \leq d(z,\xi)$ for all $\xi \in \mathrm{MP}_P(X,\omega)$, but there exists a $y \in \tilde{X} \setminus \mathrm{STAR}_Q$ such that $d(\widehat{\eta_P}^Q(z), Q) \geq d(\widehat{\eta_P}^Q(z), y)$.

We first establish the fact that for any point $z_0 \in \mathrm{STAR}_Q$ the geodesic $[\widehat{\eta_P}^Q(z_0), y]$ remains outside of STAR_Q once it exits. In other words

$$[\widehat{\eta_P}^Q(z_0), y] = [\widehat{\eta_P}^Q(z_0), y'] \cup [y', y]$$

where

$$[\widehat{\eta_P}^Q(z_0), y') \subset \mathrm{STAR}_Q$$

and

$$(y', y] \subset \tilde{X} \setminus \mathrm{STAR}_Q$$

with the point y' either being a point of $TIPS_Q$ or lying in $\tilde{X} \setminus \mathrm{STAR}_Q$. Any two points in STAR_Q can be connected by a geodesic lying within STAR_Q either by concatenating their geodesics to 0 (if the angle between these paths is sufficiently large) or by otherwise constructing a path using a convex Euclidean polygon as in the proof of statement 2. Hence there is only one connected component of $[\widehat{\eta_P}^Q(z_0), y] \cap \mathrm{STAR}_Q$ and so the geodesic $[\widehat{\eta_P}^Q(z_0), y]$ indeed remains outside of STAR_Q once it exits.

We now construct a Euclidean triangle in $\mathcal{O}_P$ that can be used to analyze the inequalities that we have assumed.

Let $[\widehat{\eta_P}^Q(z), y] \cap (\mathrm{STAR}_Q \setminus \mathrm{TIPS}_Q) = [\widehat{\eta_P}^Q(z), y')$ where the right hand ')' denotes an open ended geodesic. Let $z' \in \mathcal{O}_P$ denote the right hand limit point of the path $(\widehat{\eta_P}^Q)^{-1}[\widehat{\eta_P}^Q(z), y')$. Then z' lies on a line in $\mathcal{O}_P$ from 0 through a point $\xi \in \mathrm{MP}_P(X, \omega)$.

The geodesic from Q to y can be formed in the following manner. Let $Q' = \widehat{\eta_P}^Q(\xi)$The geodesic $[\widehat{\eta_P}^Q(z), y]$ can be expressed as

$$[\widehat{\eta_P}^Q(z), Q'] \cup [Q', Q''] \cup ... \cup [Q^{(j)}, y],$$

where $Q^{(i)}$ is the unique singularity such that the line $[Q^{(i-1)}, Q^{(i)}]$ extends to contain the point where $[Q^{(i)}, y]$ leaves $\mathrm{STAR}_{Q^{(i)}}$ for all i. In particular this construction shows by uniqueness of geodesics that indeed the line $[Q, y]$ leaves STAR_Q at a point on the extension of $[Q, Q']$ and so $d(Q, y) > d(0, \xi)$.

Suppose that the line $[z, 0]$ makes an angle with $[0, \xi]$ that is greater than or equal to π. Then the path $[\widehat{\eta_P}^Q(z), Q] \cup [Q, y]$ is a geodesic in $(\tilde{X}, \pi^*\omega)$. This contradicts the assumption that $d(\widehat{\eta_P}^Q(z), Q) \geq d(\widehat{\eta_P}^Q(z), y)$. Thus $[z, 0]$ makes an angle with $[0, \xi]$ that is less than π.

Therefore $[z, 0]$,$[0, z']$, and $[z', z]$ form a Euclidean triangle with $\xi \in [0, z']$ (see figure 7). Note the possibility that the interior of this triangle contains points of $\mathrm{MP}_P(X, \omega)$.

Using statements 1 and 2 we get that:

$$d(z, z') \leq d(\widehat{\eta_P}^Q(z), \widehat{\eta_P}^Q(z')) \leq d(\widehat{\eta_P}^Q(z), y) \leq d(\widehat{\eta_P}^Q(z), Q) = d(z, 0)$$

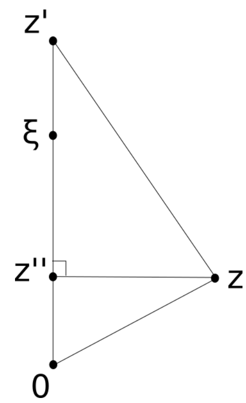

Figure 7: We show that $d(z,z') \leq d(z,0)$. Since $d(z,0) \leq d(z,\xi)$ it follows that in fact $\xi = z'$ and $d(z,0) = d(z,\xi)$.

We also have that $d(z,0) \leq d(z,\xi)$ which together imply that the point z'' in figure 7 indeed lies on the line $[0,z']$.

Furthermore $d(z,z') \leq d(z,0)$ implies that

$$d(z'',z') \leq d(z'',0),$$

and the fact that $d(z,0) \leq d(z,\xi)$ imply the middle inequality of

$$d(z'',z') \leq d(z'',0) \leq d(z'',\xi) \leq d(z'',z'). \tag{1}$$

Therefore $d(z'',\xi) = d(z'',z')$ and so in fact $\xi = z'$. Hence the Geodesic $[\widehat{\eta_P}^Q(z), y]$ exits STAR_Q at the point $\widehat{\eta_P}^Q(\xi)$. Since $y \notin \mathrm{STAR}_Q$ this implies that

$$d(\widehat{\eta_P}^Q(z), y) > d(\widehat{\eta_P}^Q(z), \widehat{\eta_P}^Q(\xi)) \geq d(z,\xi).$$

Inequality 1 also implies that $d(z'',0) = d(z'',\xi)$ so that $d(z,0) = d(z,\xi)$. Therefore we finally have that

$$d(\widehat{\eta_P}^Q(z), Q) = d(z, 0) = d(z, \xi) < d(\widehat{\eta_P}^Q(z), y)$$

which is a contradiction, and so completes the proof of statement 3.

$\square$

4.2 Unzipping (X, ω) along the Voronoi 1-Skeleton, Producing $\bigsqcup_{P \in \Sigma} D_P$

We now establish sets in $\mathcal{O}_P$ for each $P \in \Sigma$ that map translation equivalently to the open 2-cells of the Voronoi decomposition of (X, ω). The closures of these sets serve to represent the building blocks resulting from 'unzipping' (X, ω) along its Voronoi 1-skeleton. Our goal is to establish the fact that these building blocks can be independently constructed within each $\mathcal{O}_P$ using a radius r bounded representative of the marked periods. We thus show that a finite collection of marked periods determines the surface up to translation equivalence.

Definition 30. *For $P \in \Sigma$ let*

$$D_P := \{z \in \mathcal{O}_P | d(z, 0) \leq d(z, \xi) \ \forall\ \xi \in \mathrm{MP}_P(X, \omega)\},$$

and

$$C_P := \{z \in \mathcal{O}_P | d(z,0) < d(z,\xi)\ \forall\ \xi \in \mathrm{MP}_P(X,\omega)\}.$$

Remark 13. *Let $P \in \Sigma$. Clearly we have that $C_P \subseteq D_P$. If $z \in \mathcal{O}_P \setminus \Lambda_P$, then ${}_0\lambda_z(t) = \xi \in \mathrm{MP}_P(X,\omega)$ for some $t \in [0,1)$. Thus $d(z,0) > d(z,\xi)$ and so $z \notin D_P$.Thus $C_P \subseteq D_P \subseteq \Lambda_P$.*

We will next relate D_P and C_P (for each $P \in \Sigma$) to sets within $(\tilde{X}, \pi^*\omega)$ via the embeddings $\widehat{\eta_P}^Q$.

Lemma 2. *Let $P = P_k$ and $Q = Q_k$ for some $k \in \{1, ..., s\}$. Then the following hold.*

1. *We have that*

$$\left\{ y \in (\tilde{X}, \pi^*\omega) \mid d(y,Q) \leq d(y,S)\ \forall\ S \in \tilde{\Sigma} \setminus \{Q\} \right\} \ \subseteq\ \widehat{\eta_P}^Q(\Lambda_P).$$

2. *If $z \in \Lambda_P$ then*

$$z \in D_P \textit{ iff } \quad d(\widehat{\eta_P}^Q(z), Q) \leq d(\widehat{\eta_P}^Q(z), S) \ \textit{ for all } \ S \in \tilde{\Sigma} \setminus \{Q\}.$$

3. *If $z \in \Lambda_P$ then*

$$z \in C_P \textit{ iff } \quad d(\widehat{\eta_P}^Q(z), Q) < d(\widehat{\eta_P}^Q(z), S) \ \textit{ for all } \ S \in \tilde{\Sigma} \setminus \{Q\}.$$

Proof. Let $P = P_k$ and $Q = Q_k$ for some $k \in \{1, ..., s\}$. Suppose $y \notin \mathrm{STAR}_Q$.

Recall that STAR_Q contains all of the images of geodesics starting at Q that do not intersect points of $\tilde{\Sigma}$. Thus ${}_y\gamma_Q(t) \in \tilde{\Sigma} \setminus \{Q\}$ for some $t \in [0, 1)$. Hence there is an $S \in \tilde{\Sigma} \setminus \{Q\}$ such that $d(y, Q) > d(y, S)$ which proves statement 1.

Let $z \in D_P$. By Theorem 9 statement 3 and the fact that $\tilde{\Sigma} \setminus \{Q\}$ lies in the closure of $\tilde{X} \setminus \mathrm{STAR}_Q$, it follows that $d(\widehat{\eta_P}^Q(z), Q) \leq d(\widehat{\eta_P}^Q(z), S)$ for all $S \in \tilde{\Sigma} \setminus \{Q\}$.

Suppose $z \notin D_P$ so that there is a $\xi \in \mathrm{MP}_P(X, \omega)$ such that $d(z, \xi) < d(z, 0)$. If $[z, \xi] \subset \Lambda_P$ then $\triangle 0\xi z$ has interior that does not contain any points of $\mathrm{MP}_P(X, \omega)$. Therefore if $S = \widehat{\eta_P}^Q(\xi)$ then $S \in \tilde{\Sigma} \setminus \{Q\}$ and by Theorem 9 it follows that $d(\widehat{\eta_P}^Q(z), S) = d(z, \xi) < d(z, 0) = d(\widehat{\eta_P}^Q(z), Q)$. If $[z, \xi]$ is not contained within Λ_P then following the proof of Theorem 9 the interior of $\triangle 0\xi z$ contains a point $\xi_1 \in \mathrm{MP}_P(X, \omega)$ such that $[z, \xi_1]$ is contained in Λ_P and $d(z, \xi_1) < d(z, 0)$. Following the above argument using ξ_1 instead of ξ completes the proof of statement 2.

To prove statement 3 we now need only establish the fact that if $z \in D_P$ then $d(z, 0) = d(z, \xi)$ for some $\xi \in \mathrm{MP}_P(X, \omega)$ if and only if

$$d(\widehat{\eta_P}^Q(z), S) = d(\widehat{\eta_P}^Q(z), Q)$$

for some $S \in \widetilde{\Sigma} \setminus \{Q\}$.

Suppose that $z \in D_P$ and that $d(z, 0) = d(z, \xi)$ for some $\xi \in \mathrm{MP}_P(X, \omega)$. If the interior of $\triangle 0\xi z$ contains a point of $\mathrm{MP}_P(X, \omega)$ then there is a $\xi_1 \in \mathrm{MP}_P(X, \omega)$ such that $d(z, \xi) < d(z, 0)$ contradicting the fact that $z \in D_P$. Therefore the interior of $\triangle 0\xi z$ does not contain a point of $\mathrm{MP}_P(X,\omega)$ so by Theorem 9, $S = \widehat{\eta_P}^Q(\xi)$ satisfies $d(\widehat{\eta_P}^Q(z), S) = d(\widehat{\eta_P}^Q(z), Q)$.

Suppose $z \in D_P$ and $d(\widehat{\eta_P}^Q(z), S) = d(\widehat{\eta_P}^Q(z), Q)$ for some $S \in \widetilde{\widetilde{\Sigma}} \setminus \{Q\}$. We know that $[\widehat{\eta_P}^Q(z), S] \subset \mathrm{STAR}_Q$, for otherwise we have a contradiction to Theorem 9 statement 3. Therefore $(\widehat{\eta_P}^Q)^{-1}[\widehat{\eta_P}^Q(z), S]$ is a geodesic in Λ_P and in particular letting $\xi = (\widehat{\eta_P}^Q)^{-1}(S) \in \mathrm{MP}_P(X,\omega)$ we have that

$$d(z,\xi) = d(\widehat{\eta_P}^Q(z), S) = d(\widehat{\eta_P}^Q(z), Q) = d(z, 0).$$

$\square$

We would like to finally relate the sets D_P and C_P (for each $P \in \Sigma$) to the Voronoi structure on (X,ω) via the map η_P. In order to make the final step from $(\tilde{X}, \pi^*\omega)$ to (X,ω) we require a lemma that ensures the number of geodesics that realize the minimal distance from a point $y \in \tilde{X}$ to the singular set $\tilde{\Sigma}$ on $\tilde{X}$ is the same as the number of geodesics that realize the minimal distance from the point $\pi(y)$ to the singular set Σ on (X,ω).

Lemma 3. *Let $P = P_k$ and $Q = Q_k$ for some $k \in \{1, ..., s\}$ and let $x \in X$. Let*

$$\mathcal{X} := \{\alpha \text{ a geodesic on } (X,\omega) \,|\, \alpha(0) = x \in X, \alpha(1) = P \in \Sigma, \text{ and } l(\alpha) = d(x,\Sigma)\}$$

and

$$\mathcal{Y} := \{\gamma \text{ a geodesic on } (\tilde{X},\pi^*\omega) \,|\, \gamma(0) \in \pi^{-1}(\{x\}), \gamma(1) = Q, \text{ and } l(\gamma) = d(y,\tilde{\Sigma})\}.$$

If $\pi_* : \mathcal{Y} \to \mathcal{X}$ is given by $\pi_*(\alpha) := \pi \circ \alpha$, then π is a bijection between $\mathcal{Y}$ and $\mathcal{X}$ that sends line segments of $(\tilde{X}, \pi^*\omega)$ ending at Q and realizing the distance to $\tilde{\Sigma}$ to equal length lines of (X, ω) ending at P realizing the distance to Σ.

Proof. Let $\pi_* : \mathcal{Y} \to \mathcal{X}$ be given by $\pi_*(\alpha) := \pi \circ \alpha$. Let $\gamma \in \mathcal{Y}$ with

$$y := \gamma(0) \in \pi^{-1}(\{x\})$$

and $\gamma(1) = Q$. Since π is a local translation equivalence, we have that $\pi \circ \gamma$ is a geodesic segment on (X, ω), having the same length as γ, with

$$(\pi \circ \gamma)(0) = \pi(y) = x$$

and

$$(\pi \circ \gamma)(1) = \pi(Q) = P.$$

Suppose that $l(\pi \circ \gamma) \neq d(x, \Sigma)$. Let α' be a geodesic on (X, ω) with $\alpha'(0) = x$ and $\alpha'(1) = P' \in \Sigma$ such that $l(\alpha') = d(x, \Sigma) < l(\pi \circ \gamma)$. Let γ' be the lift of α' to $(\tilde{X}, \pi^*\omega)$ starting at y . Then γ' is a geodesic connecting some y to some $Q' \in \tilde{\Sigma}$ having the same length as α'. Thus

$$l(\alpha') = l(\gamma') \geq d(y, \tilde{\Sigma}) = l(\gamma) = l(\pi \circ \gamma)$$

which is a contradiction. Therefore $l(\pi \circ \gamma) = d(x, \Sigma)$ and so π_* is well defined.

If $\gamma_1, \gamma_2 \in \mathcal{Y}$ such that $\pi_*(\gamma_1) = \pi_*(\gamma_2)$ then $\gamma_1 = \gamma_2$ by uniqueness of path lifting for a specified endpoint. Thus π_* is injective.

We now prove that π_* is surjective. Suppose that $\alpha \in \mathcal{X}$. Let γ be the lift of α to $(\tilde{X}, \pi^*\omega)$ ending at Q (i.e. γ^{-1} is the lift of α^{-1} starting at Q). Then γ is a geodesic of $(\tilde{X}, \pi^*\omega)$ with $y := \gamma(0) \in \pi^{-1}(\{x\})$ and $\gamma(1) = Q$. If $l(\gamma) > d(y, \tilde{\Sigma})$ then there would be a geodesic connecting y and $\tilde{\Sigma}$ that projects to (X, ω) in order to establish a geodesic connecting x and Σ with length less than α which would be a contradiction. Thus $\gamma \in \mathcal{Y}$ and so π_* is in fact bijective. $\square$

We now relate C_P and D_P (for each $P \in \Sigma$) to cells within the Voronoi decomposition of (X, ω) subordinate to the set Σ.

Theorem 10. *Let $P \in \Sigma$. Then $\eta_P(D_P) = F_P$, $\eta_P(C_P) = F_P^1$, and*

$$\eta_P|_{C_P} : (C_P, proj_P^*(dz)|_{C_P}) \to (F_P^1, \omega|_{F_P^1})$$

is a translation equivalence.

Proof. Let $P = P_k$ and $Q = Q_k$ for some $k \in \{1, ..., s\}$, and define

$$A := \{y \in (\tilde{X}, \pi^*\omega) \,|\, d(y, Q) \leq d(y, S) \text{ for all } S \in \tilde{\Sigma} \setminus \{Q\}\}$$

and

$$B := \{y \in (\tilde{X}, \pi^*\omega) \,|\, d(y, Q) < d(y, S) \text{ for all } S \in \tilde{\Sigma} \setminus \{Q\}\}$$

By Lemma 2 we have that

$$\eta_P(D_P) = \pi \circ \widehat{\eta_P}^Q(D_P) = \pi(A)$$

and

$$\eta_P(C_P) = \pi \circ \widehat{\eta_P}^Q(C_P) = \pi(B).$$

Since the points of A are starting points of lines realizing the distance to $\tilde{\Sigma}$ and ending at Q and points of F_P are starting points of lines realizing the distance to Σ and ending at P, it follows by Lemma 3 that $\eta_P(D_P) = \pi(A) = F_P$.

We will establish that $\pi(B) = F_P^1$ by first showing that any two distinct points of A that lie in the same fiber over X must lie in $A \setminus B$. This will imply that $\pi(B) \subseteq F_P^1$ since any point of $F_P \setminus F_P^1$ has at least two distinct geodesics connecting it to P and realizing the distance to Σ. The starting points of the lifts of these geodesics to $\tilde{X}$ with endpoint Q will be distinct points of A that lie in the same fiber. The fact that $F_P^1 \subseteq \pi(B)$ is again the result of Lemma 3.

Suppose that two distinct points $y_1, y_2 \in A$ satisfy $\pi(y_1) = \pi(y_2)$. Applying the unique deck transformation of the cover $(\tilde{X}, \pi^*\omega) \to (X, \omega)$ that takes y_1 to y_2, to the line segment $[y_1, Q]$, we obtain a new line segment $[y_2, S]$ with $S \neq Q$. Thus we have a $S \in \tilde{\Sigma} \setminus \{Q\}$ with $d(y_1, Q) = d(y_1, S)$ so that $y_1 \in A \setminus B$. By a symmetric argument we have that $y_2 \in A \setminus B$. Therefore $\pi|_B$ is injective, and we have that $\eta_P(C_P) = \pi(B) = F_P^1$.

Note that $\eta_P|_{C_P} = \pi|_B \circ \widehat{\eta_P}^Q|_{C_P}$. However $\widehat{\eta_P}^Q|_{C_P}$ is an embedding, and $\pi|_B$ is an injective local embedding (which is also an embedding). Thus $\eta_P|_{C_P}$ is also an embedding. Since $\eta_P|_{C_P}$ is a composition of local translation equivalences, it follows that $\eta_P|_{C_P} : (C_P, \mathrm{proj}_n^* dz|_{C_P}) \to (F_P^1, \omega|_{F_P^1})$ is a translation equivalence.

$\square$

4.3 Constructing D_P using a Finite Subset of $MP_P(X, \omega)$

We now show that for each $P \in \Sigma$ that the set D_P can be constructed using a finite subset $\mathrm{MP}^r_P(X, \omega) \subset MP_P(X, \omega)$ for a particular $r \in \mathbb{R}^+$ that depends on (X, ω) (recall the definition of $\mathrm{MP}^r_P(X, \omega)$: Definition 23).

Definition 31. *For $P \in \Sigma$, and $\xi \in MP_P(X, \omega)$, let*

$$\Omega^{0,\xi} := \{z \in \mathcal{O}_P \,|\, d(z, 0) \leq d(z, \xi)\}$$

and let

$$[0, \xi]^{\perp} := \text{The perpendicular bisector of the line segment } [0, \xi]$$

Theorem 11. *Let $P \in \Sigma$. For $\xi \in MP_P(X, \omega)$, let*

$$loc(0, \xi) := \{z \in \mathcal{O}_P \,|\, d(z, 0) = d(z, \xi)\}.$$

Then $loc(0, \xi) = \partial\Omega^{0,\xi}$ and $loc(0, \xi) = [0, \xi]^{\perp}$.

Proof. Let $P \in \Sigma$ and $\xi \in MP_P(X, \omega)$.

Suppose $z \in \mathcal{O}_P$ where $[z, 0]$ makes an angle at 0 with $[0, \xi]$ that is greater than or equal to π. By Theorem 1 it follows that $[z, \xi] = [z, 0] \cup [0, \xi]$. Thus $z \notin loc(0, \xi)$ since $d(z, \xi) = d(z, 0) + d(0, \xi)$.

All other $z \in \mathcal{O}_P$ (than the ones considered above) are contained, together with ξ, in a common sector centered at zero with angle equal to π. By Theorem 4, proj_P restricts to such sectors to be an isometric embedding. In such a way it is clear that all of the points of $loc(0,\xi)$ sit in a sector of total angle π containing ξ and spanning an angle of $\pi/2$ on either side of ξ. Therefore Theorem 4 applied to proj_P restricted to this sector tells us that $loc(0,\xi)$ is the isometric image under proj_P^{-1} of the set of points in $\mathbb{C}$ equidistant to 0 and $\mathrm{proj}_P(\xi)$. Since such points constitute a line in $\mathbb{C}$ that is perpendicular to $[0, \mathrm{proj}_P(\xi)]$ and intersects at the midpoint of $[0, \mathrm{proj}_P(\xi)]$, the result follows.

□

Definition 32. *For $P \in \Sigma$ and $K \subseteq \mathrm{MP}_P(X,\omega)$, let $\Theta(K) := \bigcap_{\xi \in K} \Omega^{0,\xi}$.*

Remark 14. *For all $P \in \Sigma$, $D_P = \Theta\left(\mathrm{MP}_P(X,\omega)\right)$.*

Lemma 4. *For any $P \in \Sigma$ and $K \subseteq \mathrm{MP}_P(X,\omega)$, the set $\Theta(K)$ is convex and contains 0.*

Proof. Let $P \in \Sigma$. The fact that $0 \in \Theta(K)$ for any $K \subseteq \mathrm{MP}_P(X,\omega)$ is clear. Since the arbitrary intersection of convex sets in $\mathcal{O}_P$ is convex, It suffices to prove the lemma in the case of $K = \{\xi\}$ for arbitrary $\xi \in \mathrm{MP}_P(X,\omega)$. Let $\xi \in \mathrm{MP}_P(X,\omega)$ and $z_1, z_2 \in \Theta(\{\xi\}) = \Omega^{0,\xi}$ with $z_1 \neq z_2$. Let $h : [0,1] \to \mathbb{R}$ be defined by $h(t) := d\left({}_{z_1}\lambda_{z_2}(t), \xi\right) - d\left({}_{z_1}\lambda_{z_2}(t), 0\right)$. Then h is continuous by continuity of ${}_{z_1}\lambda_{z_2}$ and continuity of the metric, and $h(t) = 0$ if and only if ${}_{z_1}\lambda_{z_2}(t) \in loc(0,\xi)$. Since $z_1, z_2 \in \Theta(\{\xi\}) = \Omega^{0,\xi}$ we have that $h(0) \geq 0$ and $h(1) \geq 0$. If $h(t^*) < 0$ for some $0 < t^* < 1$, then by the intermediate value theorem there would be a $0 \leq t_1 < t^*$ and $t* < t_2 \leq 1$ such that $h(t_1) = h(t_2) = 0$. Thus

${}_{z_1}\lambda_{z_2}$ crosses $loc(0,\xi)$ at two distinct points ${}_{z_1}\lambda_{z_2}(t_1)$ and ${}_{z_1}\lambda_{z_2}(t_2)$. By Theorem 2 it follows that $\mathrm{Im}({}_{z_1}\lambda_{z_2}|_{[t_1,t_2]}) \subseteq loc(0,\xi) \subseteq \Omega^{0,\xi}$ which contradicts the fact that $h(t^*) < 0$. Therefore $[z_1, z_2] \subseteq \Omega^{0,\xi}$.

$\square$

Definition 33. *Using compactness of $\sqcup_{P\in\Sigma} D_P$ for existence, let*

$$\rho(X,\omega) := \max\{d(z,0) | P \in \Sigma, z \in D_P\}.$$

Theorem 12. *Let $\rho \in \mathbb{R}$ with $\rho > \rho(X,\omega)$. For all $P \in \Sigma$ we have that*

$$D_P = \Theta\left(\mathrm{MP}_P^{2\rho}(X,\omega)\right).$$

Proof. We have that

$$D_P = \Theta\left(\mathrm{MP}_P^{2\rho}(X,\omega)\right) \cap \Theta\left(\mathrm{MP}_P^{>2\rho}(X,\omega)\right)$$

If $z \in B(0,\rho)$ then for all $\xi \in \mathrm{MP}_P^{>2\rho}(X,\omega)$ we have that

$$d(z,0) \leq \rho < d(0,\xi) - d(z,0) \leq d(z,\xi).$$

Therefore $B(0,\rho) \subset \Theta\left(\mathrm{MP}_P^{>2\rho}(X,\omega)\right)$. Since $\rho > \max\{d(z,0) \,|\, z \in D_P\}$ we know that $D_P \subseteq B(0,\rho)$ with $D_P \cap \partial B(0,\rho) = \emptyset$. Since

$$\partial B(0,\rho) \subset B(0,\rho) \subseteq \Theta\left(\mathrm{MP}_P^{>2\rho}(X,\omega)\right),$$

if

$$\partial B(0,\rho) \cap \Theta\left(\mathrm{MP}_P^{2\rho}(X,\omega)\right) \neq \emptyset$$

then the fact that

$$D_P = \Theta\left(\mathrm{MP}_P^{2\rho}(X,\omega)\right) \cap \Theta\left(\mathrm{MP}_P^{>2\rho}(X,\omega)\right)$$

implies that $\partial B(0,\rho) \cap D_P \neq \emptyset$ which is a contradiction. Thus since $\Theta\left(\mathrm{MP}_P^{2\rho}(X,\omega)\right)$ is star shaped with center 0 by Lemma 4, it follows that

$$\Theta\left(\mathrm{MP}_P^{2\rho}(X,\omega)\right) \subseteq B(0,\rho).$$

Therefore we have that

$$\begin{aligned}
D_P &= D_P \cap B(0,\rho) \\
&= \Theta\left(\mathrm{MP}_P^{2\rho}(X,\omega)\right) \cap \left[\Theta\left(\mathrm{MP}_P^{>2\rho}(X,\omega)\right) \cap B(0,\rho)\right] \\
&= \Theta\left(\mathrm{MP}_P^{2\rho}(X,\omega)\right) \cap B(0,\rho) \\
&= \Theta\left(\mathrm{MP}_P^{2\rho}(X,\omega)\right)
\end{aligned}$$

□

4.4 Gluing Data for Reconstructing (X, ω) from $\bigsqcup_{P \in \Sigma} D_P$

For each $P \in \Sigma$ we now show that the boundary of D_P consists of a finite collection of geodesic one-manifold segments, and that the $\mathbb{Z}_2$-action on $\bigsqcup_k \mathrm{MP}_{P_k}(X, \omega)$ provides the information needed to glue along pairs of such segments (in $\bigsqcup_{P \in \Sigma} D_P$) to recover the surface (X, ω) up to translation equivalence.

Definition 34. *If $P \in \Sigma$ let*

$$\Pi_P := \{\xi \in \mathrm{MP}_P(X, \omega) \mid D_P \cap loc(0, \xi) \text{ contains more than one point}\}.$$

Remark 15. *If $\rho \in \mathbb{R}$ with $\rho > \rho(X, \omega)$ and $P \in \Sigma$, then $\xi \in \mathrm{MP}_P^{>2\rho}(X, \omega)$ implies that all points $z \in loc(0, \xi)$ satisfy $d(z, 0) > \rho$. Thus $D_P \cap loc(0, \xi) = \emptyset$ for all $\xi \in \mathrm{MP}_P^{>2\rho}(X, \omega)$. Therefore $\Pi_P \subseteq \mathrm{MP}_P^{2\rho}(X, \omega)$.*

Theorem 13. *Let $P \in \Sigma$. For each $\xi \in \Pi_P$, the set $D_P \cap loc(0, \xi)$ is a connected geodesic one-manifold (with boundary) that does not contain the singularity $0 \in \mathcal{O}_P$. Furthermore we have that*

$$\partial D_P = D_P \setminus C_P = \bigcup_{\xi \in \Pi_P} (D_P \cap loc(0, \xi)).$$

Proof. Let $\rho \in \mathbb{R}$ with $\rho > \rho(X, \omega)$ and let $P \in \Sigma$. Since C_P is open and is contained in the closed set D_P, it follows that $\partial D_P \subseteq D_P \setminus C_P$. However by Theorem 12 if $z \in D_P$ then $z \notin C_P$ if and only if $z \in loc(0, \xi_1)$ for some

$\xi_1 \in \mathrm{MP}_P^{2\rho}(X,\omega)$. Thus by Theorem 11 we have

$$\partial D_P \subseteq D_P \setminus C_P = \bigcup_{\xi \in \mathrm{MP}_P^{2\rho}(X,\omega)} (D_P \cap loc(0,\xi)) = \bigcup_{\xi \in \mathrm{MP}_P^{2\rho}(X,\omega)} (D_P \cap \partial\Omega^{0,\xi}).$$

Let $z \in D_P \cap \partial\Omega^{0,\xi_1}$ for some $\xi_1 \in \mathrm{MP}_P^{2\rho}(X,\omega)$. The fact that $z \in \partial\Omega^{0,\xi_1}$ implies that every open neighborhood of z contains points in $\mathcal{O}_P \setminus \Omega^{0,\xi_1}$. Thus every open neighborhood of z contains points in $\mathcal{O}_P \setminus D_P$. Since $z \in D_P$ it follows that $z \in \partial D_P$. Thus in fact $\partial D_P = \bigcup_{\xi \in \mathrm{MP}_P^{2\rho}(X,\omega)} (D_P \cap \partial\Omega^{0,\xi})$, so that

$$\partial D_P = D_P \setminus C_P = \bigcup_{\xi \in \mathrm{MP}_P^{2\rho}(X,\omega)} (D_P \cap loc(0,\xi)).$$

Let $\xi_1 \notin \Pi_P$ and let z be the sole element in the singleton $D_P \cap loc(0,\xi_1)$. Suppose that $z \notin loc(0,\xi) = \partial\Omega^{0,\xi}$ for all $\xi \in \mathrm{MP}_P^{2\rho}(X,\omega) \setminus \{\xi_1\}$, then there is an open neighborhood $U_z \subseteq \mathcal{O}_P$ containing z such that $U_z \subseteq \cap_{\xi \in \mathrm{MP}_P^{2\rho}(X,\omega)\setminus\{\xi_1\}} \Omega^{0,\xi}$. Since $[\Omega^{0,\xi_1} \cap loc(0,\xi_1)] = loc(0,\xi_1)$ and $D_P = \bigcap_{\xi \in \mathrm{MP}_P^{2\rho}(X,\omega)} \Omega^{0,\xi}$ it follows that

$$D_P \cap loc(0,\xi_1) = \bigcap_{\xi \in \mathrm{MP}_P^{2\rho}(X,\omega)} [\Omega^{0,\xi}] \cap loc(0,\xi_1) = \bigcap_{\xi \in \mathrm{MP}_P^{2\rho}(X,\omega)\setminus\{\xi_1\}} [\Omega^{0,\xi}] \cap loc(0,\xi_1)$$

which contains $U_z \cap loc(0,\xi_1)$. However $U_z \cap loc(0,\xi_1)$ clearly contains more than one point since it is the non-empty intersection of a geodesic image with an open set. This contradicts the fact that $D_P \cap loc(0,\xi_1)$ is a singleton. Thus in fact

$D_P \cap loc(0, \xi_1) = \{z\} \subseteq D_P \cap loc(0, \xi_2)$ for some $\xi_2 \in \mathrm{MP}_P^{2\rho}(X, \omega) \setminus \{\xi_1\}$ and so

$$\partial D_P = \bigcup_{\xi \in \mathrm{MP}_P^{2\rho}(X,\omega)} (D_P \cap loc(0, \xi)) = \bigcup_{\xi \in \mathrm{MP}_P^{2\rho}(X,\omega) \setminus \{\xi_1\}} (D_P \cap loc(0, \xi)).$$

Suppose that

$$D_P \neq \bigcap_{\xi \in \mathrm{MP}_P^{2\rho}(X,\omega) \setminus \{\xi_1\}} \Omega^{0,\xi}$$

so that in fact

$$\bigcap_{\xi \in \mathrm{MP}_P^{2\rho}(X,\omega) \setminus \{\xi_1\}} \Omega^{0,\xi} \setminus D_P$$

is non-empty. The sets $\bigcap_{\xi \in \mathrm{MP}_P^{2\rho}(X,\omega) \setminus \{\xi_1\}} \Omega^{0,\xi}$ and D_P are convex by Lemma 4 and are regions (closures of non-empty open sets). This would imply that there is an open ball in $\bigcap_{\xi \in \mathrm{MP}_P^{2\rho}(X,\omega) \setminus \{\xi_1\}} \Omega^{0,\xi} \setminus D_P$. Since 0 is contained in both $\bigcap_{\xi \in \mathrm{MP}_P^{2\rho}(X,\omega) \setminus \{\xi_1\}} \Omega^{0,\xi}$ and D_P, the lines segments from an infinite number of points in this open ball to 0 would then intersect D_P at an infinite number of points of $\partial \Omega^{0,\xi_1}$. This would contradict the assumption that $\xi_1 \notin \Pi_P$. Thus

$$D_P = \bigcap_{\xi \in \mathrm{MP}_P^{2\rho}(X,\omega) \setminus \{\xi_1\}} \Omega^{0,\xi}.$$

Inducting on the elements of $\mathrm{MP}_P^{2\rho}(X, \omega) \setminus \Pi_P$ we can therefore show that

$$\partial D_P = \bigcup_{\xi \in \Pi_P} (D_P \cap loc(0, \xi)).$$

For all $\xi \in \Pi_P$, $loc(0, \xi)$ is a geodesic, and D_P is convex and compact. Thus

$\sharp[D_P \cap loc(0,\xi)] > 1$ implies that $D_P \cap loc(0,\xi)$ is a geodesic segment of $\mathcal{O}_P$. This geodesic segment clearly avoids the singularity $0 \in \mathcal{O}_P$. Thus $D_P \cap loc(0,\xi)$ is a connected geodesic one manifold with boundary that avoids 0.

□

Definition 35. *For each $P \in \Sigma$ and $\xi \in \Pi_P$ we call $s_P^{\xi} := D_P \cap loc(0,\xi)$ a* ***side*** *of D_P.*

Let

$$\eta : \bigsqcup_{P \in \Sigma} D_P \longrightarrow (X, \omega)$$

be defined for $x \in \bigsqcup_{P \in \Sigma} D_P$ by

$$\eta(x) := \eta_P(x), \text{ where } P \in \Sigma \text{ is such that } x \in D_P.$$

By Theorem 10 and Remark 6, $\bigcup_{P \in \Sigma} \mathrm{Im}(\eta_P) = \bigcup_{P \in \Sigma} F_P = X$ and so $\eta(x)$ is surjective.

Let $\sim$ be the equivalence relation on $\bigsqcup_{P \in \Sigma} D_P$ given by $x \sim y$ if and only if $\eta(x) = \eta(y)$. Then η induces a bijective quotient map

$$\bar{\eta} : \left(\bigsqcup_{P \in \Sigma} D_P \ \ / \sim \right) \longrightarrow (X, \omega)$$

Since $\bar{\eta} : [\bigsqcup_{P\Sigma} D_P / \sim] \longrightarrow (X, \omega)$ is a bijective continuous function from a compact space to a Hausdorff space, it is in fact a homeomorphism.

Theorem 14. *Let $P \in \Sigma$, $\xi \in \mathrm{MP}_P(X,\omega)$, and $P' \in \Sigma$ (not necessarily distinct from P) such that $\eta_P(\xi) = P'$. Then $\xi \in \Pi_P$ if and only if $\tilde{\xi} \in \Pi_{P'}$.*

The equivalence relation, $\sim$, given above is characterized completely by the rule that for each $\xi \in \Pi_P$, points in s_P^{ξ} are identified with points in $s_{P'}^{\tilde{\xi}}$ by a translation of open sets containing them. These side pairings preserve the local translation structure coming from $\mathcal{O}_P$ and $\mathcal{O}_{P'}$, and as a result $Z = (\bigsqcup_{P\in\Sigma} D_P/\sim)$ naturally inherits a translation structure giving a translation surface (Z,Ψ). The map $\bar{\eta} : (Z,\Psi) \longrightarrow (X,\omega)$ is a translation equivalence.

Proof. Let $P \in \Sigma$, $\xi \in \mathrm{MP}_P(X,\omega)$, and $P' \in \Sigma$ such that $\eta_P(\xi) = P'$. Let $k_1 \in \{1,...,t\}$ be such that $P = P_{k_1}$ and let $Q = Q_{k_1}$ (with respect to the enumeration of Σ fixed by the representative of the marked periods). Let $k_2 \in \{1,...,t\}$ be such that $P' = P_{k_2}$ and let $Q' = Q_{k_2}$.

Suppose that $\xi \in \Pi_P$ and let $S' = \widehat{\eta_P}^Q(\xi)$. Let $z \in s_P^{\xi} = loc(0,\xi) \cap D_P$ and $y = \widehat{\eta_P}^Q(z)$. Since $z \in D_P$, Lemma 2 tells us that $d(y,Q) \leq d(y,S)$ for all $S \in \tilde{\Sigma} \setminus \{Q\}$. In particular this inequality holds for $S = S'$ due to the fact that $\widehat{\eta_P}^Q$ is injective so that $S' \neq Q$.

Suppose that $d(y,Q) < d(y,S')$. By Theorem 9 we know that

$$d(y,Q) = d(z,0) = d(z,\xi).$$

Thus $d(z,\xi) \neq d(y,S') = d(\widehat{\eta_P}^Q(z), \widehat{\eta_P}^Q(\xi))$ and so by Theorem 9 it follows that the interior of the equilateral triangle $\triangle 0z\xi$ contains a point $\xi_1 \in \mathrm{MP}_P(X,\omega)\setminus\{\xi\}$ However $\triangle 0z\xi$ lies within $B(z, d(z,0))$, touching its boundary only at 0 and ξ.

Therefore we have a $\xi_1 \in \mathrm{MP}_P(X,\omega) \setminus \{\xi\}$ such that $d(z,\xi_1) < d(z,0)$ which contradicts the fact that $z \in D_P$. Therefore in fact $d(y,Q) = d(y,S')$ and so for all $S \in \tilde{\Sigma} \setminus \{S'\}$ we have that $d(y,S') = d(y,Q) \leq d(y,S)$. If $\sigma_{S'}^{Q'}$ denotes the element in $\mathrm{Trans}(\widetilde{X}, \pi^*\omega)$ that sends S' to Q' (note that S' and Q' lie in the same fiber over π), then for all $S \in \tilde{\Sigma} \setminus \{Q'\}$ we have that $d(\sigma_{S'}^{Q'}(y), Q') \leq d(\sigma_{S'}^{Q'}(y), S)$, and in particular

$$d(\sigma_{S'}^{Q'}(y), Q') = d(\sigma_{S'}^{Q'}(y), \sigma_{S'}^{Q'}(Q)) \tag{2}$$

By Lemma 2 and Theorem 13 it follows that $\sigma_{S'}^{Q'}(y) \in \widehat{\eta_{P'}}^{Q'}(\partial D_{P'})$.

By the definition of $\tilde{\xi}$, $\eta_P \circ ({}_0\lambda_\xi)$ and $\eta_{P'} \circ ({}_0\lambda_{\tilde{\xi}})$ are inverse paths in X. Since $[\sigma_{S'}^{Q'} \circ \widehat{\eta_P}^{Q} \circ ({}_0\lambda_\xi)]$ is a lift of $\eta_P \circ ({}_0\lambda_\xi)$ that ends at $\sigma_{S'}^{Q'}(S') = Q'$ and $\widehat{\eta_{P'}}^{Q'} \circ ({}_0\lambda_{\tilde{\xi}})$ is a lift of $\eta_{P'} \circ ({}_0\lambda_{\tilde{\xi}})$ that starts at Q', it follows that $\sigma_{S'}^{Q'} \circ \widehat{\eta_P}^{Q} \circ ({}_0\lambda_\xi)$ and $\widehat{\eta_{P'}}^{Q'} \circ ({}_0\lambda_{\tilde{\xi}})$ are inverse paths in $\tilde{X}$. Thus

$$\sigma_{S'}^{Q'}(Q) = [\sigma_{S'}^{Q'} \circ \widehat{\eta_P}^{Q} \circ ({}_0\lambda_\xi)](0) = [\widehat{\eta_{P'}}^{Q'} \circ ({}_0\lambda_{\tilde{\xi}})](1) = \widehat{\eta_{P'}}^{Q'}(\tilde{\xi}). \tag{3}$$

Therefore by Equations 2 and 3 we have that $d(\sigma_{S'}^{Q'}(y), Q') = d(\sigma_{S'}^{Q'}(y), \widehat{\eta_{P'}}^{Q'}(\tilde{\xi}))$. Thus

$$[(\widehat{\eta_{P'}}^{Q'})^{-1} \circ \sigma_{S'}^{Q'} \circ \widehat{\eta_P}^{Q}](z) = (\widehat{\eta_{P'}}^{Q'})^{-1}\left(\sigma_{S'}^{Q'}(y)\right) \in loc(0,\tilde{\xi}) \cap D_{P'}.$$

We have shown that for all $z \in s_P^\xi$ we have that

$$((\widehat{\eta_{P'}}^{Q'})^{-1} \circ \sigma_{S'}^{Q'} \circ \widehat{\eta_P}^{Q})(z) \in loc(0, \tilde{\xi}) \cap D_{P'}.$$

Since this argument is true for all $z \in s_P^\xi$ (of which there are more than one), the fact that $(\widehat{\eta_{P'}}^{Q'})^{-1} \circ \sigma_{S'}^{Q'} \circ \widehat{\eta_P}^{Q}$ is injective implies that $loc(0, \tilde{\xi}) \cap D_{P'}$ consists of more than one point and so $\tilde{\xi} \in \Pi_{P'}$. Furthermore

$$((\widehat{\eta_{P'}}^{Q'})^{-1} \circ \sigma_{S'}^{Q'} \circ \widehat{\eta_P}^{Q})(s_P^\xi) \subseteq loc(0, \tilde{\xi}) \cap D_{P'} = s_{P'}^{\tilde{\xi}}.$$

Let

$$U_P = [(\widehat{\eta_P}^{Q})^{-1} \circ (\sigma_{S'}^{Q'})^{-1}] \left((\sigma_{S'}^{Q'} \circ \widehat{\eta_P}^{Q})(\Lambda_P) \cap \widehat{\eta_{P'}}^{Q'}(\Lambda_{P'}) \right),$$

$$U_{P'} = (\widehat{\eta_{P'}}^{Q'})^{-1} \left((\sigma_{S'}^{Q'} \circ \widehat{\eta_P}^{Q})(\Lambda_P) \cap \widehat{\eta_{P'}}^{Q'}(\Lambda_{P'}) \right), \text{ and}$$

$$\beta_P^\xi := \left((\widehat{\eta_{P'}}^{Q'})^{-1} \circ \sigma_{S'}^{Q'} \circ \widehat{\eta_P}^{Q} \right) \Big|_{U_P} : U_P \to U_{P'}.$$

Then β_P^ξ is a translation equivalence, and Figure 8 is a commutative diagram.

We have established that for all $P \in \Sigma$ and all $\xi \in \Pi_P$ with $P' \in \tilde{\Sigma}$ such that $P' = \eta_P(\xi)$ we have that $\tilde{\xi} \in \Pi_{P'}$ and that $\beta_P^\xi(s_P^\xi) \subseteq s_{P'}^{\tilde{\xi}}$.

Since $P = \eta_{P'}(\tilde{\xi})$, $\tilde{\tilde{\xi}} = \xi$, and $\widehat{\eta_{P'}}^{Q'}(\tilde{\xi}) = \sigma_{S'}^{Q'}(Q)$, we can run the above argument in reverse to conclude that $\xi \in \Pi_P$ is implied by the fact that $\tilde{\xi} \in \Pi_{P'}$.

$$\begin{tikzcd}
U_P \arrow[rr, "\beta_P^{\xi}"] \arrow[dr, bend left=15, "\sigma_{S'}^{Q'} \circ \widehat{\eta_P}^{Q}"'] \arrow[ddr, bend right=15, "\eta_P"'] & & U_{P'} \arrow[dl, bend right=15, "\widehat{\eta_{P'}}^{Q'}"] \arrow[ddl, bend left=15, "\eta_{P'}"] \\
& {[\sigma_{S'}^{Q'} \circ \widehat{\eta_P}^{Q}](\Lambda_P) \bigcap \widehat{\eta_{P'}}^{Q'}(\Lambda_{P'})} \arrow[d, "\pi"] & \\
& (X, \omega) &
\end{tikzcd}$$

Figure 8: Establishing a translation equivalence, β_P^{ξ}, that will perform gluing along the sides s_P^{ξ} and $s_{P'}^{\tilde{\xi}}$ contained in the neighborhoods U_P and $U_{P'}$ respectively.

We will obtain an analogous map

$$\begin{aligned}
\beta_{P'}^{\tilde{\xi}} &= \left[(\widehat{\eta_P}^{Q})^{-1} \circ (\sigma^{Q}_{\sigma_{S'}^{Q'}(Q)} \circ \widehat{\eta_{P'}}^{Q'}) \right] \Big|_{U_P'} \\
&= \left[(\widehat{\eta_P}^{Q})^{-1} \circ ((\sigma_{S'}^{Q'})^{-1} \circ \widehat{\eta_{P'}}^{Q'}) \right] \Big|_{U_P'} \\
&= (\beta_P^{\xi})^{-1},
\end{aligned}$$

and conclude that $[\beta_P^{\xi}]^{-1}(s_{P'}^{\tilde{\xi}}) \subseteq s_P^{\xi}$. Thus we have open sets U_P containing s_P^{ξ} and $U_{P'}$ containing $s_{P'}^{\tilde{\xi}}$ and a translation equivalence

$$\beta_P^{\xi} : (U_P, (o(P)+1)z^{o(P)}dz) \to (U_{P'}, (o(P')+1)(z)^{o(P')}dz)$$

such that $\beta_P^{\xi}(s_P^{\xi}) = s_{P'}^{\tilde{\xi}}$. In particular s_P^{ξ} and $s_{P'}^{\tilde{\xi}}$ are parallel and of equal length. Also, by commutativity of the diagram above we know that $\sim$ identifies all $z \in s_P^{\xi}$

with $\beta_P^\xi(z) \in s_{P'}^{\tilde{\xi}}$.

Suppose $z_1 \in s_{R_1}^\xi \setminus \mathrm{bdry}(s_{R_1}^\xi)$ where 'bdry$(\cdot)$' denotes manifold boundary. With notation as above $z_2 := \beta_{R_1}^\xi(z_1) \in s_{R_1'}^{\tilde{\xi}} \setminus \mathrm{bdry}(s_{R_1'}^{\tilde{\xi}})$, and open (relative to D_{R_1} and $D_{R_1'}$) neighborhoods N_1 and N_2 of $z_1 \in D_{R_1}$ and $z_2 \in D_{R_1'}$ respectively can be chosen to be Euclidean half-discs that are disjoint and of the same arbitrarily small radius. Since $z_1 \in \partial D_{R_1} \subseteq \Lambda_{R_1} \setminus (\mathrm{MP}_{R_1}(X,\omega) \cup \{0\})$, it follows that $\eta(z_1)$ is a regular point of (X,ω). Note that $N_1 \cap C_{R_1}$ and $N_2 \cap C_{R_1'}$ are the original half-discs without their diameters. Since $\eta|_{\bigsqcup_{P\in\Sigma} C_P}$ is a translation equivalence (and so preserves angles and is injective), the fact that $\eta(z_1) = \eta(z_2)$ therefore implies that $\eta(N_1)$ and $\eta(N_2)$ are two Euclidean half-discs whose union is a full Euclidean disc constituting a metric ball about $\eta(z)$.

Suppose $w \in \partial D_{R_2}$ for some $R_2 \in \Sigma$ with $w \notin \{z_1, z_2\}$. It cannot be the case that $w \sim z_1$ or $w \sim z_2$ since otherwise points of C_{R_2} in proximity of w would be forced to be identified with distinct points in either $N_1 \cap C_{R_1}$ or $N_2 \cap C_{R_1'}$ which contradicts the fact that η is injective on $\bigsqcup_{P\in\Sigma} C_P$. Thus in fact points of $s_R^\xi \setminus bdry(s_R^\xi)$ for any $R \in \Sigma$ and $\xi \in S_R$ are identified by $\sim$ with points of $\bigsqcup_{P\in\Sigma} \partial D_P$ if and only if they are identified by the side pairing of s_R^ξ with $\beta_R^\xi(s_R^\xi)$. Note in particular that the points of $\bigsqcup_{P\in\Sigma,\xi\in\Pi_P} \mathrm{bdry}(s_P^\xi)$ are only identified by $\sim$ with points within the same set.

Recall that

$$\bar{\eta} : \left(\bigsqcup_{P\in\Sigma} D_P \ \ / \sim \right) \longrightarrow (X,\omega)$$

is a homeomorphism. The local holomorphic charts from X on $\eta(N_1) \cup \eta(N_2)$ and the abelian differential $\omega|(\eta(N_1) \cup \eta(N_2))$ can be pulled back, via $\bar{\eta}|_{(N_1 \sqcup N_2)/\sim}$, to give well defined Riemann surface charts and an abelian differential on

$$(N_1 \sqcup N_2/\sim) \subset (\bigsqcup_{P \in \Sigma} D_P/\sim).$$

Notice that this abelian differential will agree with $(\eta|_{\bigsqcup_{P\in\Sigma} C_P})^*(\omega)$ on $N_1 \cap C_{R_1}$ and $N_2 \cap C_{R_1'}$. This process can be carried out for all points in $s_P^\xi \setminus \text{bdry}(s_P^\xi)$ for all $P \in \Sigma$ and $\xi \in \Pi_P$, defining a Riemann surface structure and abelian differential, Ψ, on $(\bigsqcup_{P\in\Sigma} D_P/\sim) \setminus A$ where A is the discrete subset of equivalence classes determined by the set $\bigcup_{P\in\Sigma, \xi\in\Pi_P} \text{bdry}(s_P^\xi)$ (recall that this set is not identified by $\sim$ with any points outside of itself). Let $Z := \bigsqcup_{P\in\Sigma} D_P/\sim$ and $B := \bar{\eta}(A)$. It follows that $\bar{\eta} : Z \setminus A \to X \setminus B$ is a biholomorphism of Riemann surfaces that pulls back $\omega|_{X\setminus B}$ to give $\Psi|_{Z\setminus A}$. By the Riemann removable singularity theorem, there is a unique Riemann surface structure on all of Z such that $\bar{\eta} : Z \to X$ is a biholomorphism. Furthermore Ψ continuously extends to an abelian differential on all of Z which we will also denote Ψ, for which $\Psi = \bar{\eta}^*(\omega)$. Therefore

$$\bar{\eta} : (Z, \Psi) \longrightarrow (X, \omega)$$

is a translation equivalence.

□

Theorem 14 allows us to now prove something stronger than the converse to Theorem 8.

Corollary 2. *Let* $(Y, \psi) \in \mathcal{H}(i_1, ..., i_s)$ *and let* $\rho \in \mathbb{R}$ *such that* $\rho > \rho(X, \omega)$*. If* $[\mathrm{MP}^{2\rho}(X, \omega)] = [MP^{2\rho}(Y, \psi)]$ *then* (X, ω) *and* (Y, ψ) *are translation equivalent.*

Proof. Let $(Y, \psi) \in \mathcal{H}(i_1, ..., i_s)$ have singular set Σ_Y and let $\rho \in \mathbb{R}$ such that $\rho > \rho(X, \omega)$.

Suppose that $[\mathrm{MP}^{2\rho}(X, \omega)] = [MP^{2\rho}(Y, \psi)]$. Then an enumeration $R_1, ..., R_s$ of Σ_Y exists so that for all $k \in \{1, ..., s\}$ the order of the singularity of P_k is the same as for R_k, the surface $\mathcal{O}_{R_k}$ can be identified with $\mathcal{O}_{P_k}$, and the identification ι_Y can be taken as the identification ι used for (X, ω). In this way we can take a radius 2ρ bounded representative of (X, ω) given by

$$\mathrm{MP}^{2\rho}(X, \omega) = \iota(\bigsqcup_{P \in \Sigma} \mathrm{MP}_P^{2\rho}(X, \omega))$$

as a radius 2ρ bounded representative of (Y, ψ) as well.

Thus for all $k \in \{1, ..., s\}$, by Remark 14

$$D_{R_k} \subseteq \Theta(\mathrm{MP}^{2\rho}_{R_k}(Y, \psi)) = \Theta(\mathrm{MP}^{2\rho}_{P_k}(X, \omega)) = D_{P_K}.$$

This implies that $\rho(Y, \psi) \leq \rho(X, \omega)$ so that $\rho > \rho(Y, \psi)$. Therefore by Theorem 12 if $k \in \{1, ..., s\}$ then

$$D_{R_k} = \Theta(\mathrm{MP}^{2\rho}_{R_k}(Y, \psi)) = \Theta(\mathrm{MP}^{2\rho}_{P_k}(X, \omega)) = D_{P_K}.$$

Finally the results of Theorem 14 imply that

$$(X, \omega) \cong \left(\bigsqcup_{P \in \Sigma} D_P / \sim \right) = \left(\bigsqcup_{R \in \Sigma_Y} D_R / \sim \right) \cong (Y, \psi).$$

$\square$

5 The Veech Group from the Marked Periods

5.1 The Action of $\mathrm{SL}(2,\mathbb{R})$ on the Set Containing the Marked Periods

In this subsection we define an action of $\mathrm{SL}(2,\mathbb{R})$ on Ξ (see Definition 28).

Recall for a translation surface (Y,ψ) that $\mathrm{der} : \mathrm{Aff}^+(Y,\psi) \to \mathrm{GL}^+(2,\mathbb{R})$ is a homeomorphism given by $\mathrm{der}(F) = dF$ where dF is the matrix associated with the constant Jacobian of F.

Lemma 5. *We have that* $\mathrm{SL}(2,\mathbb{R}) \subset \mathrm{der}(\mathrm{Aff}^+(\mathbf{O}))$.

Proof. Recall from Subsection 3.3 that

$$\mathbf{O} = \bigsqcup_{c_1 \text{ copies}} \mathcal{O}_1 \sqcup \bigsqcup_{c_2 \text{ copies}} \mathcal{O}_2 \sqcup ... \sqcup \bigsqcup_{c_{\mathfrak{s}} \text{ copies}} \mathcal{O}_{\mathfrak{s}}.$$

where for each $k \in \{1, ..., \mathfrak{s}\}$, $\mathcal{O}_k = (\mathbb{C}, (q_k + 1)z^{q_k} dz)$.

Let $M \in \mathrm{SL}(2,\mathbb{R})$. From Section 2 we have that the map $\mathrm{Id}_{\mathcal{O}_k} : \mathcal{O}_k \to (M \cdot \mathcal{O}_k)$ is an affine homeomorphism with $\mathrm{der}(\mathrm{Id}_{\mathcal{O}_k}) = M$. Furthermore the proof of Theorem 5 carries through if we replace $\tilde{X}$ with $M \cdot \mathcal{O}_k$ and use $Q = 0$, providing a lift $\widehat{\mathrm{dev}}_Q : (M \cdot \mathcal{O}_k) \to \mathcal{O}_k$, of the developing map dev_Q on $M \cdot \mathcal{O}_k$ based at

$Q = 0$ (see Figure 9). Note that $\mathrm{STAR}_Q = M \cdot \mathcal{O}_k$ and that $\widehat{\mathrm{dev}}_Q$ is a translation equivalence since $\mathrm{TIPS}_Q = \emptyset$ in this case.

$$\begin{tikzcd}
\mathcal{O}_k \arrow[d, "\mathrm{proj}_k"'] & M \cdot \mathcal{O}_k \arrow[l, "\widehat{\mathrm{dev}}_Q"'] \arrow[dl, "\mathrm{dev}_Q"] \arrow[dr, phantom, "I"] & \mathcal{O}_k \arrow[l, "\mathrm{Id}_{\mathcal{O}_k}"'] \arrow[ll, bend right, "F_M^k"'] \arrow[d, "\mathrm{proj}_k"] \\
\mathbb{C} & & \mathbb{C} \arrow[ll, "T_M"]
\end{tikzcd}$$

Figure 9: Establishing an affine homeomorphism on an individual component $\mathcal{O}_k$ by composing $\mathrm{Id}_{\mathcal{O}_k}$ (which has Jacobian M) with a lift of the developing map on $M \cdot \mathcal{O}_k$ (which has Jacobian $\mathrm{Id}_{\mathrm{SL}(2,\mathbb{R})}$).

Thus $F_M^k := \widehat{\mathrm{dev}}_Q \circ \mathrm{Id}_{\mathcal{O}_k} : \mathcal{O}_k \to \mathcal{O}_k$ is an affine homeomorphism with $\mathrm{der}(F_M^k) = M$. For each $k \in \{1, ..., \mathfrak{s}\}$ we can apply F_M^k as a self map of each individual copy of $\mathcal{O}_k$ in $\mathbf{O}$.

Define the function $F_M : \mathbf{O} \to \mathbf{O}$ by the rule that for $k \in \{1, ..., \mathfrak{s}\}$ if A is a copy of $\mathcal{O}_k$ in $\mathbf{O}$ then $F_M|_A = F_M^k$ on that copy. Then F_M is an affine self homeomorphism on each connected component of $\mathbf{O}$ with derivative equal to M on that component. Hence $F_M \in \mathrm{Aff}^+(\mathbf{O})$ and $\mathrm{der}(F_M) = M$.

$\square$

Definition 36. *If* $M \in \mathrm{SL}(2,\mathbb{R})$ *and* $[A] \in \Xi$ *let* $M \cdot [A] := [F_M(A)]$ *where* $F_M \in \mathrm{Aff}^+(\mathbf{O})$ *with* $\mathrm{der}(F_M) = M$ *and* $F_M(A)$ *has the* $\mathbb{Z}_2$*-action induced from* A *through the bijection* F_M*.*

Theorem 15. *Definition 36 provides a well defined action of* $\mathrm{SL}(2,\mathbb{R})$ *on* Ξ*.*

Proof. By Lemma 5 if $M \in \mathrm{SL}(2,\mathbb{R})$ there existence an $F_M \in \mathrm{Aff}^+(\mathbf{O})$ with $\mathrm{der}(F_M) = M$. If $F_M, G_M \in \mathrm{Aff}^+(\mathbf{O})$ with $\mathrm{der}(F_M) = \mathrm{der}(G_M) = M$, then $G_M \circ F_M^{-1} : F_M(A) \to G_M(A)$ is a $\mathbb{Z}_2$-equivariant bijection. Since

$$\mathrm{der}(G_M \circ F_M^{-1}) = \mathrm{Id}_{\mathrm{SL}(2,\mathbb{R})}$$

it follows that $G_M \circ F_M^{-1} \in \mathrm{Trans}(\mathbf{O})$ and thus $F_M(A) \sim G_M(A)$. Therefore we have a well defined function from $\mathrm{SL}(2,\mathbb{R})$ to the set of functions from Ξ to itself.

Note that any choice for $F_{Id_{\mathrm{SL}(2,\mathbb{R})}}$ lies in $\mathrm{Trans}(\mathbf{O})$ so that $Id_{\mathrm{SL}(2,\mathbb{R})} \cdot [A] = [A]$ for all $[A] \in \Xi$. Since der is a homomorphism, if $M_1, M_2 \in \mathrm{SL}(2,\mathbb{R})$ with corresponding $F_{M_1}, F_{M_2} \in \mathrm{Aff}^+(\mathbf{O})$ we have that $F_{M_1M_2} := F_{M_1} \circ F_{M_2} \in \mathrm{Aff}^+(\mathbf{O})$ with $\mathrm{der}(F_{M_1M_2}) = M_1M_2$. Thus for all $[A] \in \Xi$,

$$M_1 \cdot (M_2 \cdot [A]) = [F_{M_1} \circ F_{M_2}(A)] = [F_{M_1M_2}(A)] = (M_1M_2) \cdot [A].$$

□

5.2 The Veech Group as the Stabilizer of the Marked Periods

Let $(X, \omega) \in \mathcal{H}(i_1, ..., i_s)$. Recall from Remark 4 that $\mathrm{SL}(2,\mathbb{R})$ acts on

$\mathcal{H}(i_1, ..., i_s)$ and that $\Gamma(X, \omega) = \mathrm{Stab}_{\mathrm{SL}(2,\mathbb{R})}(X, \omega)$. In this subsection we show that

$$\Gamma(X, \omega) = \mathrm{Stab}_{\mathrm{SL}(2,\mathbb{R})}\left([\mathrm{MP}(X, \omega)]\right).$$

Theorem 16. *If (X, ω) is a translation surface in $\mathcal{H}(i_1, ..., i_s)$ and $M \in \mathrm{SL}(2, \mathbb{R})$ then*

$$[\mathrm{MP}\left(M \cdot (X, \omega)\right)] = M \cdot [\mathrm{MP}(X, \omega)].$$

Proof. Let $(X, \omega) \in \mathcal{H}(i_1, ..., i_s)$ have singular set Σ, and choose a representative $\mathrm{MP}(X, \omega) = \iota(\bigsqcup_{P \in \Sigma} \mathrm{MP}_P(X, \omega))$ of the marked periods of (X, ω).

Let $M \in \mathrm{SL}(2, \mathbb{R})$. By Section 2 the map $\mathrm{Id}_{\tilde{X}} : (M \cdot (\tilde{X}, \pi^*\omega)) \to (\tilde{X}, \pi^*\omega)$ is an affine homeomorphism with $\mathrm{der}(\mathrm{Id}_{\tilde{X}}) = M^{-1}$.

Let $k \in \{1, ..., s\}$, $P = P_k$, and $Q = Q_k$ (with respect to the enumeration of P and Q that come with the choice of a representative of marked periods). Let $M\mathrm{dev}_Q$ be the developing map from the simply connected $M \cdot (\tilde{X}, \pi^*\omega)$ based at $Q \in \tilde{X}$. Recall that this developing map is the unique map that restricts to coordinate neighborhoods of $M \cdot (\tilde{X}, \pi^*\omega)$ to be a translation equivalence and that sends Q to zero. However if $T_M : \mathbb{C} \to \mathbb{C}$ is the $\mathbb{R}$-linear map on $\mathbb{C}$ determined by the matrix M, then $T_M \circ \mathrm{dev}_Q \circ \mathrm{Id}_{\tilde{X}}$ is another map satisfying all of those conditions (see Figure 10).

Therefore $M\mathrm{dev}_Q = T_M \circ \mathrm{dev}_Q \circ \mathrm{Id}_{\tilde{X}}$. Let $k \in \{1, ..., \mathfrak{s}\}$ be such that $\mathcal{O}_P = \mathcal{O}_k$ (and so $\mathrm{proj}_P = \mathrm{proj}_k$). Let $F_M \in \mathrm{Aff}^+(\mathbf{O})$ with $\mathrm{der}(F_M) = M$ shown to exist in the proof of Lemma 5. Note in Figure 9 that the block labeled I indeed commutes since otherwise the developing map proj_P and $T_M \circ \mathrm{proj}_P \circ \left(\iota^{-1} \circ F_M \circ \iota\right)^{-1}$ would

$$
\begin{tikzcd}
\mathcal{O}_P \arrow[d, "\mathrm{proj}_P"'] \arrow[dr, phantom, "I"] & \mathcal{O}_P \arrow[l, "\iota^{-1}\circ F_M\circ\iota"'] \arrow[d, "\mathrm{proj}_P"'] \arrow[dr, phantom, "II"] & (\tilde{X},\pi^*\omega) \arrow[l, "\widehat{\mathrm{dev}}_Q"'] \arrow[dl, "\mathrm{dev}_Q"] & M\cdot(\tilde{X},\pi^*\omega) \arrow[l, "Id_{\tilde{X}}"'] \arrow[lll, bend right, "\widehat{M\mathrm{dev}_Q}"'] \arrow[llld, bend left, "M\mathrm{dev}_Q"] \\
\mathbb{C} & \mathbb{C} \arrow[l, "T_M"'] & &
\end{tikzcd}
$$

Figure 10: The embeddings from $M\cdot(\tilde{X},\pi^*\omega)$ needed to define the marked periods for $M\cdot(X,\omega)$ can be built from the corresponding maps dev_Q on $(\tilde{X},\pi^*\omega)$ along with the map $\mathrm{Id}_{\tilde{X}}$ (having Jacobian M^{-1}) and the lift of the $\mathbb{R}$-linear action of M on $\mathbb{C}$ (having Jacobian M).

be two distinct maps that restrict to coordinate neighborhoods to give translation equivalencies and that take $0\in\mathcal{O}_P$ to $0\in\mathbb{C}$. The block in Figure 10 labeled II is also commutative by Theorem 5, and therefore $\widehat{M\mathrm{dev}_Q} := \iota^{-1}\circ F_M\circ\iota\circ\widehat{\mathrm{dev}}_Q\circ\mathrm{Id}_{\tilde{X}}$ is a lift of the developing map $M\mathrm{dev}_Q$. Since TIPS_Q is the same subset of $\tilde{X}$ whether constructed within the translation surface $M\cdot(\tilde{X},\pi^*\omega)$ or the translation surface $(\tilde{X},\pi^*\omega)$, it follows that a representative at P of the marked periods of $M\cdot(X,\omega)$ is given by

$$
\begin{aligned}
\mathrm{MP}_P\left(M\cdot(X,\omega)\right) &= \widehat{M\mathrm{dev}_Q}(\mathrm{TIPS}_Q)\\
&= \iota^{-1}\circ F_M\circ\iota\circ\widehat{\mathrm{dev}}_Q\circ\mathrm{Id}_{\tilde{X}}(\mathrm{TIPS}_Q)\\
&= (\iota^{-1}\circ F_M\circ\iota)(\mathrm{MP}_P(X,\omega)).
\end{aligned}
$$

Furthermore the path in X corresponding to an element $\xi \in \mathrm{MP}_P(X,\omega)$ is the same as the path in X corresponding to the element

$$(\iota^{-1} \circ F_M \circ \iota)(\xi) \in \mathrm{MP}_P\left(M \cdot (X,\omega)\right).$$

Therefore we have that $\tilde{\xi_1} = \xi_2$ for $\xi_1, \xi_2 \in \mathrm{MP}_P(X,\omega)$ if and only if

$$(\iota^{-1} \circ \widetilde{F_M \circ \iota})(\xi_1) = (\iota^{-1} \circ F_M \circ \iota)(\xi_2).$$

We therefore find that a representative of the marked periods of $M \cdot (X,\omega)$ is given by

$$\begin{aligned} \mathrm{MP}(M \cdot (X,\omega)) &= \iota\left(\bigsqcup_{P\in\Sigma} \mathrm{MP}_P\left(M \cdot (X,\omega)\right)\right) \\ &= F_M\left(\iota(\bigsqcup_{P\in\Sigma} \mathrm{MP}_P(X,\omega))\right) \\ &= F_M\left(\mathrm{MP}(X,\omega)\right). \end{aligned}$$

By Definition 36 we have that

$$[\mathrm{MP}\left(M \cdot (X,\omega)\right)] = M \cdot [\mathrm{MP}(X,\omega)].$$

□

Theorem 17. *If* $(X, \omega) \in \mathcal{H}(i_1, ..., i_s)$ *then* $\Gamma(X, \omega) = \mathrm{Stab}_{\mathrm{SL}(2,\mathbb{R})}([\mathrm{MP}(X, \omega)])$.

Proof. Suppose that $(X, \omega) \in \mathcal{H}(i_1, ..., i_s)$ and $M \in \mathrm{SL}(2, \mathbb{R})$. Then

$$M \in \Gamma(X, \omega)$$

$$\iff \text{(Lemma 1)}$$

$$(X, \omega) \text{ and } M \cdot (X, \omega) \text{ are translation equivalent}$$

$$\iff \text{(Corollary 2, Theorem 8)}$$

$$[\mathrm{MP}(X, \omega)] = [\mathrm{MP}\,(M \cdot (X, \omega))]$$

$$\iff \text{(Theorem 16)}$$

$$[\mathrm{MP}(X, \omega)] = M \cdot [\mathrm{MP}(X, \omega)]$$

$$\iff$$

$$M \in \mathrm{Stab}_{\mathrm{SL}(2,\mathbb{R})}([\mathrm{MP}(X, \omega)])$$

□

5.3 Veech Group Elements from a Finite Subset of the Marked Periods

Given an $(X, \omega) \in \mathcal{H}(i_1, ..., i_s)$ and $M \in \mathrm{SL}(2, \mathbb{R})$ we know from the previous subsection that $M \in \Gamma$ if and only if $M \cdot [\mathrm{MP}(X, \omega)] = [\mathrm{MP}(X, \omega)]$. For calculations we will work with a particular representative $\mathrm{MP}(X, \omega)$ of the marked periods, and replace the action of M by a map $F_M \in \text{Aff}^+(\mathbf{O})$.

In this subsection we show that in fact it is enough to consider only a finite subset of a representative of marked periods, the size of the required subset being dependent on the Frobenius norm of M. In the next section we will see that for a lattice Veech group we can determine when our search for a finite generating set is complete by considering only finite subsets of the group consisting of all elements whose norm is bounded above by some given value.

Definition 37. *Let the* ***radius*** *of a point $z \in \boldsymbol{O}$ be the distance from z to the singularity of the connected component of $\boldsymbol{O}$ containing z.*

The fact that the $\mathbb{Z}_2$-action on a representative of the marked periods is radius preserving allows for a well defined $\mathbb{Z}_2$-action on subsets of a representative defined by restricting values of the radii.

Definition 38. *Suppose $r > 0$ and suppose $A \in \Upsilon$ has a radius preserving $\mathbb{Z}_2$-action. Then the action on A restricts to a well defined $\mathbb{Z}_2$-action on the set*

$$S = \{z \in A | \text{ the radius of } z \text{ is less than or equal to } r\}.$$

Let A^r denote the element of Υ given by the set S with this restricted $\mathbb{Z}_2$-action.

Note that if $A \in \Upsilon$ possesses a radius preserving $\mathbb{Z}_2$-action and $F \in \text{Aff}^+(\mathbf{O})$, then $F(A) \in \Upsilon$ also possesses a radius preserving $\mathbb{Z}_2$-action.

Lemma 6. *Let $A \in \Upsilon$ possess a radius preserving $\mathbb{Z}_2$-action. If $M \in \text{SL}(2,\mathbb{R})$ has minimum singular value ν, then for all $r > 0$ and all $F_M \in \text{Aff}^+(\boldsymbol{O})$ with*

$\operatorname{der}(F_M) = M$,

$$\left(F_M(A^{\frac{r}{\nu}})\right)^r = (F_M(A))^r.$$

Proof. Let $A \in \Upsilon$ possess a radius preserving $\mathbb{Z}_2$-action. Let $M \in \mathrm{SL}(2,\mathbb{R})$ have minimum singular value ν and let $r > 0$. For all $z \in \mathbf{O}$ we have that

$$\operatorname{radius}(F_M(z)) \geq \nu \cdot \operatorname{radius}(z).$$

Thus for all $z \in \mathbf{O}$

$$\operatorname{radius}(F_M(z)) \leq r \quad \Rightarrow \quad \operatorname{radius}(z) \leq \nu^{-1}\operatorname{radius}(F_M(z)) \leq \tfrac{r}{\nu},$$

which implies that

$$(F_M(A))^r \subseteq \left(F_M(A^{\frac{r}{\nu}})\right)^r.$$

Since $A \supseteq A^{\frac{r}{\nu}}$, we also have that $(F_M(A))^r \supseteq \left(F_M(A^{\frac{r}{\nu}})\right)^r$. □

Definition 39. *For $M \in \mathrm{SL}(2,\mathbb{R})$ with*

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we denote the ***Frobenius norm*** *of M by*

$$||M|| := \sqrt{tr(MM^t)} = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

Remark 16. *It is easily shown using the definition of $||M||$ that right and/or* *left multiplication by elements in* $\mathrm{SO}(2,\mathbb{R})$ *leave this norm unchanged. Therefore if* ν *is the minimum singular value of* M *we have that* $||M|| = \sqrt{\nu^2 + \nu^{-2}}$*. Given* $||M||$*, we can solve for the minimum singular value to obtain:*

$$\nu = \sqrt{\tfrac{1}{2}\left(||M||^2 - \sqrt{||M||^4 - 4}\right)}.$$

Definition 40. *Let* $\chi_1 : [\sqrt{2}, \infty) \to (0, 1]$ *be given by*

$$\chi_1(t) := \sqrt{\tfrac{1}{2}\left(t^2 - \sqrt{t^4 - 4}\right)}.$$

Note that χ_1 *is a monotonically decreasing function. Furthermore by Remark 16 if* $M \in \mathrm{SL}(2,\mathbb{R})$ *then* $\chi_1(||M||)$ *is the minimum singular value of* M*.*

We are now ready to prove that a finite subset of a representative of the marked periods is sufficient to test Veech group membership for all matrices in $\mathrm{SL}(2,\mathbb{R})$ up to a given Frobenius norm.

Theorem 18. *Let* $(X,\omega) \in \mathcal{H}(i_1, ..., i_s)$*,* $\rho > \rho(X,\omega)$*, and* $\mathfrak{b} \geq \sqrt{2}$*. Let* $R = \frac{2\rho}{\chi_1(\mathfrak{b})}$ *and let* $\mathrm{MP}^R(X,\omega)$ *be a radius* R *bounded representative of the marked periods of* (X,ω)*. Then for all* $M \in \mathrm{SL}(2,\mathbb{R})$ *with* $||M|| \leq \mathfrak{b}$ *we have that*

$$M \in \Gamma(X,\omega)$$

iff

$$\left(F_M\left(\mathrm{MP}^R(X,\omega)\right)\right)^{2\rho} = \left(\mathrm{MP}^R(X,\omega)\right)^{2\rho} \text{ for some } F_M \in \mathrm{Aff}^+(\boldsymbol{O}) \text{ with } \mathrm{der}(F_M) = M.$$

Proof. Let $(X,\omega) \in \mathcal{H}(i_1,...,i_s)$, $\rho > \rho(X,\omega)$, $\mathfrak{b} \geq \sqrt{2}$, and $R = \frac{2\rho}{\chi_1(\mathfrak{b})}$. Let $\mathrm{MP}^R(X,\omega)$ be a radius R bounded representative of the marked periods of (X,ω), and let $M \in \mathrm{SL}(2,\mathbb{R})$ with $||M|| \leq \mathfrak{b}$. Since $\chi_1(\mathfrak{b}) \leq 1$ it follows that $R \geq 2\rho$ so that

$$\mathrm{MP}^{2\rho}(X,\omega) := \left(\mathrm{MP}^R(X,\omega)\right)^{2\rho} \tag{4}$$

is a radius 2ρ bounded representative of marked periods of (X,ω).

If $M \in \Gamma(X,\omega)$ then $M \cdot (X,\omega)$ is translation equivalent to (X,ω) by Lemma 1. Therefore as a consequence of Theorem 8, $\mathrm{MP}^{2\rho}\left(M \cdot (X,\omega)\right) = \mathrm{MP}^{2\rho}(X,\omega)$ for some bounded representative $\mathrm{MP}^{2\rho}\left(M \cdot (X,\omega)\right)$. Let $F_M \in \mathrm{Aff}^+(\mathbf{O})$ be such that $\mathrm{der}(F_M) = M$ and $\mathrm{MP}^{2\rho}\left(M \cdot (X,\omega)\right) = \left(F_M(\mathrm{MP}(X,\omega))\right)^{2\rho}$ provided for by Theorem 16. Then

$$\left(F_M(\mathrm{MP}(X,\omega))\right)^{2\rho} = \mathrm{MP}^{2\rho}(X,\omega). \tag{5}$$

Since χ_1 is monotonically decreasing it follows that the minimum singular value, ν, of M satisfies $\nu = \chi_1(||M||) \geq \chi_1(\mathfrak{b})$. Thus $R = \frac{2\rho}{\chi_1(\mathfrak{b})} \geq \frac{2\rho}{\nu}$ and so as a consequence to Lemma 6 we obtain

$$\left(F_M(\mathrm{MP}^R(X,\omega))\right)^{2\rho} = \left(F_M(\mathrm{MP}^{\frac{2\rho}{\nu}}(X,\omega))\right)^{2\rho} = \left(F_M(\mathrm{MP}(X,\omega))\right)^{2\rho}. \tag{6}$$

Combining equations 4, 5, and 6 gives us

$$\left(F_M(\mathrm{MP}^R(X,\omega))\right)^{2\rho} = \left(F_M(\mathrm{MP}(X,\omega))\right)^{2\rho} = \mathrm{MP}^{2\rho}(X,\omega) = \left(\mathrm{MP}^R(X,\omega)\right)^{2\rho}.$$

Conversely suppose that $(F_M\left(\mathrm{MP}^R(X,\omega)\right))^{2\rho} = \left(\mathrm{MP}^R(X,\omega)\right)^{2\rho}$ for some $F_M \in \mathrm{Aff}^+(\mathbf{O})$ with $\mathrm{der}(F_M) = M$. By Theorem 16 we then have that

$$\mathrm{MP}^{2\rho}\left(M \cdot (X,\omega)\right) := \left(F_M(\mathrm{MP}(X,\omega))\right)^{2\rho}$$

is a bounded representative of $M \cdot (X,\omega)$. Equation 6 still holds in this case, and again $\mathrm{MP}^{2\rho}(X,\omega) := \left(\mathrm{MP}^R(X,\omega)\right)^{2\rho}$ is a radius 2ρ bounded representative of marked periods of (X,ω). Thus

$$\mathrm{MP}^{2\rho}\left(M \cdot (X,\omega)\right) = \left(F_M(\mathrm{MP}^R(X,\omega))\right)^{2\rho} = \left(\mathrm{MP}^R(X,\omega)\right)^{2\rho} = \mathrm{MP}^{2\rho}(X,\omega).$$

By Corollary 2 it follows that $M \cdot (X,\omega)$ and (X,ω) are translation equivalent. It follows byLemma 1 that $M \in \Gamma(X,\omega)$. □

In the setting of Theorem 18 let $P \in \Sigma$ and fix two points of the set underlying $\mathrm{MP}^{2\rho}_P(X,\omega)$ that project via proj_P to $\mathbb{R}$-linearly independent elements of $\mathbb{C}$. The number of Jacobians of elements $F_M \in \mathrm{Aff}^+(\mathbf{O})$ that satisfy the conditions of Theorem 18 is bounded above by the number of matrices in $\mathrm{SL}(2,\mathbb{R})$ whose inverses take these projections to one of the finite pairs of projections of elements in $\mathrm{MP}^R(X,\omega)$. Since there are a finite number (namely $\#[\mathrm{Trans}(\mathbf{O})]$) of elements in $\mathrm{Aff}^+(\mathbf{O})$ having any given Jacobian, we obtain a finite list of transformations

F_M that could possibly satisfy the right hand condition at the bottom of Theorem 18. As a result, Theorem 18 gives us a finite time algorithm for determining all elements of the Veech group whose Frobenius norm is bounded above by a given value. In the next section we present an extended algorithm that for a lattice Veech group determines a finite generating set in finite time.

6 More Background: The Geometry of Fuchsian Groups

In Section 2 we define Veech groups as subgroups of $\mathrm{SL}(2,\mathbb{R})$. In this section we draw upon a wealth of knowledge regarding the investigation of discrete subgroups of $\mathrm{SL}(2,\mathbb{R})/\{\pm\mathrm{Id}\}$ through their isometric action on the hyperbolic plane. We will use the tools from this section, combined with Theorem 18, to produce an algorithm to compute the Veech group of a translation surface. For more details regarding the material in this section see [Kat] or [Bea].

6.1 Fuchsian Groups and Dirichlet Polygons

We provide some basic definitions.

Definition 41. *We use the standard notation* $\mathrm{PSL}(2,\mathbb{R}) := \mathrm{SL}(2,\mathbb{R})/\{\pm\mathrm{Id}\}$.

The group $\mathrm{PSL}(2,\mathbb{R})$ is a topological group with quotient topology induced from the natural topology of $\mathrm{SL}(2,\mathbb{R})$. The topological group $\mathrm{SL}(2,\mathbb{R})$ admits a unique (up to scaling) left invariant measure, called Haar measure, that also passes through the quotient to give such a measure on $\mathrm{PSL}(2,\mathbb{R})$. The Frobenius norm on $\mathrm{SL}(2,\mathbb{R})$ (see Definition 39) also passes to $\mathrm{PSL}(2,\mathbb{R})$ as it depends only on the squares of the matrix entries. This is the norm we will use for $\mathrm{PSL}(2,\mathbb{R})$.

Definition 42. *A* ***Fuchsian group*** *is a discrete subgroup of* $\mathrm{PSL}(2,\mathbb{R})$.

Definition 43. *For any subset* $\mathcal{A} \subseteq \mathrm{SL}(2,\mathbb{R})$ *we denote the image under projection of this set to* $\mathrm{PSL}(2,\mathbb{R})$ *by* $\bar{\mathcal{A}}$.

Note that the group $\mathrm{PSL}(2,\mathbb{R})$ is obtained by taking the topological group $\mathrm{SL}(2,\mathbb{R})$ and performing the quotient with respect to a finite subgroup. Therefore it easily follows from the fact that $\Gamma(X,\omega)$ is a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ that $\overline{\Gamma(X,\omega)} \leq \mathrm{PSL}(2,\mathbb{R})$ is a Fuchsian group.

Let $\mathbb{H}$ denote the upper half-plane model of the hyperbolic plane. The group $\mathrm{PSL}(2,\mathbb{R})$ acts isometrically on $\mathbb{H}$ through fractional linear transformations defined for $g = \pm \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ by $g \cdot z := \frac{az+b}{cz+d}$.

Definition 44. *A closed region (i.e. the closure of a non-empty open set) is defined to be a* ***fundamental region*** *for the action of a Fuchsian group* $\mathcal{G} \leq \mathrm{PSL}(2,\mathbb{R})$ *on* $\mathbb{H}$ *if every point in* $\mathbb{H}$ *is in the orbit of at least one point in the region, and no two distinct points of the interior of the region lie in the same orbit.*

Given a Fuchsian group $\mathcal{G} \leq \mathrm{PSL}(2,\mathbb{R})$ and a point in $\mathbb{H}$ with trivial stabilizer in $\mathcal{G}$, we can use the orbit of this point to construct a convex polygonal fundamental region in $\mathbb{H}$. We will see that this *fundamental polygon* can be used to obtain generators for the group. In the sequel we will assume that $i \in \mathbb{H}$ has trivial stabilizer in the group. In practice we will perturb the group in order to establish this as fact.

We next establish notation for the hyperbolic half-plane in $\mathbb{H}$ given as all the points closer to $i \in \mathbb{H}$ than to a chosen point in $\mathbb{H} \setminus \{i\}$. A collection of these half-planes will intersect to form our fundamental polygon.

Definition 45. *For $z \in \mathbb{H} \setminus \{i\}$ we let*

$$H_i(z) := \{w \in \mathbb{H} \,|\, d(w,i) \leq d(w,z)\}.$$

We now can construct a fundamental polygon for a Fuchsian group. For more details see Chapter 3 of [Kat].

Definition 46. *If i has trivial stabilizer in a Fuchsian group $\mathcal{G}$ the* ***Dirichlet polygon*** *of $\mathcal{G}$ centered at i is given by*

$$\mathrm{Dir}(\mathcal{G}) := \bigcap_{g \in \mathcal{G}} H_i(g \cdot i).$$

The set $\mathrm{Dir}(\mathcal{G})$ is a convex hyperbolic polygon that may in general contain 'free' sides (contained within the line at infinity). The polygon $\mathrm{Dir}(\mathcal{G})$ is a fundamental region for the action of $\mathcal{G}$ on $\mathbb{H}$.

Given a Fuchsian group $\mathcal{G}$ in which i has a trivial stabilizer, we will refer to the Dirichlet polygon centered at i as *the* Dirichlet polygon for $\mathcal{G}$ regardless of the fact that other choices could be made for the center.

6.2 Side Pairings of the Dirichlet Polygon

For this subsection we let $\mathcal{G} \leq \mathrm{PSL}(2,\mathbb{R})$ be a Fuchsian group in which i has a trivial stabilizer.

It is clear from the definition of the Dirichlet polygon of $\mathcal{G}$ that its sides are pieces of the boundaries of $H_i(g \cdot i)$ for various $g \in \mathcal{G}$. We now explore more about what these sides can tell us about the group $\mathcal{G}$.

Definition 47. *The sides of* $\mathrm{Dir}(\mathcal{G})$ *are exactly the sets* $\partial H_i(g \cdot i) \cap \mathrm{Dir}(\mathcal{G})$ *which contain more than one point. We say that a side* $s = \partial H_i(g \cdot i) \cap \mathrm{Dir}(\mathcal{G})$ *is* determined by *the element* g*.*

The next result tells us that if $\mathcal{G}$ has finite co-volume, (i.e. the induce Haar measure of the quotient $\mathrm{PSL}(2,\mathbb{R})/\mathcal{G}$ is finite), then $\mathrm{Dir}(\mathcal{G})$ has a finite number of sides. Again the pass from $\mathrm{SL}(2,\mathbb{R})$ to $\mathrm{PSL}(2,\mathbb{R})$ is achieved by taking the quotient of a finite subgroup. Therefore it is clear that $\Gamma(X,\omega)$ is a lattice (has finite co-volume) if and only if $\overline{\Gamma(X,\omega)}$ has finite co-volume. The following theorem and proof can be found in Chapter 1 Section 5C of [Le1].

Theorem 19. *(see Chapter 1 Section 5C of [Le1]) If* $\mathcal{G}$ *has finite co-volume, then* $\mathrm{Dir}(\mathcal{G})$ *has a finite number of sides and no free sides.*

The set of non-vertex points within a given side of $\mathrm{Dir}(\mathcal{G})$ is the set of all points that are equidistant to i and exactly one other point in the orbit of i. This is the key observation that allows one to prove the following facts (see Chapter 1 Section 4F of [Le1] for more details).

Theorem 20. *The sides of* $\mathrm{Dir}(\mathcal{G})$ *are paired by elements of* $\mathcal{G}$ *called 'side pairing transformations'. A side pairing transformation and its inverse are the same two* *elements that determine (recall Definition 47) the sides being paired. In addition, every side pairing transformation preserves point-wise distance to i.*

Proof. As is seen in Section 4F of [Le1], the sides of $\mathrm{Dir}(\mathcal{G})$ are indeed paired, with a side $s = \partial H_i(g \cdot i) \cap \mathrm{Dir}(\mathcal{G})$ being paired with the side $s' = \partial H_i(g^{-1} \cdot i) \cap \mathrm{Dir}(\mathcal{G})$. Furthermore the element g^{-1} is the side pairing transformation that takes s to s'. If $z \in s$ then it is equidistant to i and $g \cdot i$. Since g^{-1} is an isometry it follows that $d(g^{-1}(z), i) = d(z, g(i)) = d(z, i)$. Therefore the side pairing does in fact preserve point-wise distance to i. □

Suppose $\mathcal{P}$ is a hyperbolic polygon with a finite number of sides, none of which are free sides, that are paired by transformations in $\mathrm{PSL}(2, \mathbb{R})$. Any given vertex of $\mathcal{P}$ can be carried by the side pairing transformations from each adjacent side to other vertices of $\mathcal{P}$. This process partitions the set of vertices into sets called *cycles*. The side pairing transformations used to form a given cycle are called the *cycle transformations* for that cycle. A generalization of a theorem due to Poincaré provides conditions for the cycles and cycle transformations that ensure $\mathcal{P}$ is a fundamental region for the group generated by the side pairings. This theorem, (see Theorem 21) is the key to the finite run time of our algorithm in the lattice case.

Definition 48. *Suppose $\mathcal{P}$ is a hyperbolic polygon with a finite number of sides, none of which are free sides, that are paired by transformations in* $\mathrm{PSL}(2, \mathbb{R})$*. A cycle of finite vertices (i.e. not on the line at infinity) containing l vertices satisfies the* ***elliptic cycle condition*** *if the sum of the internal angles of $\mathcal{P}$ at* *the vertices in the cycle is* $\frac{2\pi}{l}$.

A cycle of vertices at infinity satisfies the ***parabolic cycle condition*** *if every cycle transformation that fixes any given vertex within the cycle is a parabolic transformation (i.e. has a trace-squared value of 4).*

Theorem 21 (Poincaré)**.** *Let* $\mathcal{P}$ *be a hyperbolic polygon with a finite number of sides, none of which are free sides, that are paired by transformations in* $\mathrm{PSL}(2,\mathbb{R})$. *If all of the cycles of vertices within* $\mathbb{H}$ *satisfy the elliptic cycle condition, and all cycles of vertices at infinity satisfy the parabolic cycle condition, then the side pairings of* $\mathcal{P}$ *generate a Fuchsian group, and* $\mathcal{P}$ *is a fundamental region for this group.*

Proof. See p. 227 of [Le2]. $\square$

7 Calculating the Veech Group

7.1 The Algorithm

We now use Theorem 18 along with the tools from Section 6 in order to present an algorithm for computing Veech groups.

We start by defining the set of all elements in $\Gamma(X,\omega)$ that can be determined using Theorem 18 with a radius r bounded instance of the marked periods of (X,ω) for some $r \in \mathbb{R}$.

Definition 49. *Recall Definitions 33 and 40 of ρ and χ_1 respectively. For $r \in \mathbb{R}$ with $r \geq 2\rho$ we define:*

$$A_r := \{M \in \Gamma(X,\omega) | \ ||M|| \leq \mathfrak{b}\} \text{ where } \mathfrak{b} = \chi_1^{-1}(\tfrac{2\rho}{r}).$$

We will need a function that takes the Frobenius norm of a matrix M as input, and produces $\frac{1}{2}d(i, \bar{M} \cdot i)$ as output, representing the minimal distance from i to the boundary of $H_i(\bar{M} \cdot i)$. We need this information, as we increase a subset $\bar{\mathcal{A}} \subseteq \mathrm{PSL}(2,\mathbb{R})$, in order to track how close to i we can possibly observe a change in $\Omega(\bar{\mathcal{A}})$.

Definition 50. *Let $\chi_2 : [\sqrt{2}, \infty) \to [0, \infty)$ be defined for $t \in [\sqrt{2}, \infty)$ by*

$$\chi_2(t) := -ln(\chi_1(t)).$$

Theorem 22. *If $M \in \mathrm{SL}(2, \mathbb{R})$, then the shortest distance in $\mathbb{H}$ from i to $\partial H_i(\bar{M} \cdot i)$ is given by $\chi_2(||M||)$. Therefore $B(i, \chi_2(||M||)) \subset H_i(\bar{M} \cdot i)$.*

Proof. Let $M \in \mathrm{SL}(2, \mathbb{R})$ have singular value decomposition given by

$$M = O_1 \cdot D \cdot O_2 \quad \text{where } D = \begin{bmatrix} \nu & 0 \\ 0 & \nu \end{bmatrix}, \quad O_1, O_2 \in \mathrm{SO}(2, \mathbb{R}),$$

and where we can assume without loss of generality that ν is the minimum singular value of M given be $\nu = \chi_1(||M||)$ (see Definition 40). Note that the shortest distance from i to $\partial H_i(\bar{M} \cdot i)$ is equal to $\frac{1}{2}d(i, \bar{M} \cdot i)$. Since $\bar{M}$ acts by isometry on $\mathbb{H}$, and elements of $\mathrm{SO}(2, \mathbb{R})/\{\pm \mathrm{Id}\}$ fix i, it follows that

$$\tfrac{1}{2}d(i, \bar{M} \cdot i) = \tfrac{1}{2}d(i, \bar{D} \cdot i) = \tfrac{1}{2}d(i, \nu^2 i).$$

We use the hyperbolic line element $ds^2 = \frac{dx^2+dy^2}{y^2}$ to calculate the shortest distance from i to $\partial H_i(\bar{M} \cdot i)$ as

$$\begin{aligned}
\tfrac{1}{2}d(i, \nu^2 i) &= \tfrac{1}{2}\int_{\nu^2 i}^{i} ds \\
&= \tfrac{1}{2}\int_{\nu^2}^{1} \frac{dy}{y} \\
&= -ln(\nu) \\
&= -ln(\chi_1(||M||)) \\
&= \chi_2(||M||).
\end{aligned}$$

□

Note that χ_2 is a monotonically increasing function. Therefore we obtain the following corollary.

Corollary 3. *If* $\mathcal{A} \subseteq \mathrm{SL}(2,\mathbb{R})$, $\mathfrak{b} \geq \sqrt{2}$, *and* $M \in \mathrm{SL}(2,\mathbb{R})$ *with* $||M|| > \mathfrak{b}$, *then*

$$\Omega(\bar{\mathcal{A}}) \cap B(i, \chi_2(\mathfrak{b})) = \Omega(\bar{\mathcal{A}} \cup \{\bar{M}\}) \cap B(i, \chi_2(\mathfrak{b})).$$

As we obtain increasing subsets $A_r \subseteq \Gamma(X,\omega)$ consisting of all elements of $\Gamma(X,\omega)$ whose norms are bounded by $\mathfrak{b} = \chi_1^{-1}(\frac{2\rho}{r})$, the sets $\Omega(\bar{A}_r)$ nest down with a limit of $\Omega(\overline{\Gamma(X,\omega)})$. During this process Corollary 3 provides us with a way to track increasing metric balls centered at $i \in \mathbb{H}$ within which the sets $\Omega(\bar{A}_r)$ and $\Omega(\overline{\Gamma(X,\omega)})$ agree exactly.

Lemma 7. *If $r \in \mathbb{R}$ with $r \geq 2\rho$ and $\mathfrak{b} = \chi_1^{-1}(\frac{2\rho}{r})$, then*

$$\Omega(\bar{A}_r) \cap B(i, \chi_2(\mathfrak{b})) = \Omega(\overline{\Gamma(X,\omega)}) \cap B(i, \chi_2(\mathfrak{b}))$$

Algorithm 7.1. *Calculating the Veech Group of a Translation Surface (Restricted Case).*

Input: *The Voronoi decomposition of a translation surface $(X, \omega) \in \mathcal{H}(i_1, ... i_s)$ for which $\Gamma(X, \omega) \cap \mathrm{SO}(2, \mathbb{R}) \subseteq \{\pm \mathrm{Id}\}$.*

Output:

- *(lattice) A finite generating set for $\Gamma(X, \omega)$.*
- *(non-lattice)The algorithm will not terminate, but will continue to enumerate the elements in the Veech group in order of increasing norm. A stopping condition based on norm or time could be utilized for this case.*

1. *Calculate $\rho = \rho(X, \omega)$ using the Voronoi 2-cells (see Definition 33).*
2. ***Let** $r = 2\rho$ **and let** $\mathfrak{b} = \chi_1^{-1}(\frac{2\rho}{r}) = \sqrt{2}$.*
3. *Calculate $\mathrm{MP}_P^r(X, \omega)$ for all $P \in \Sigma$.*
4. *Calculate $A_r = \{M \in \Gamma(X, \omega) |\ ||M|| \leq \mathfrak{b}\} = \mathrm{SO}(2, \mathbb{R}) \cap \Gamma(X, \omega)$ using Theorem 18.*

5. **If** $-\mathrm{Id} \in A_r$, **then let** $ContainsMinusIdentity = \mathrm{TRUE}$.

6. **else let** $ContainsMinusIdentity = \mathrm{FALSE}$.

7. Let $ContainmentVolume := \infty$.

8. **Do While** $ContainmentVolume = \infty$:

   (a) Double the value of r and **let** $\mathfrak{b} = \chi_1^{-1}(\frac{2\rho}{r})$.

   (b) Calculate $\mathrm{MP}^r_P(X,\omega)$ for all $P \in \Sigma$.

   (c) Use Theorem 18 to complete the set $A_{\frac{r}{2}}$ to

   $A_r = \{M \in \Gamma(X,\omega) | \ ||M|| \leq \mathfrak{b}\}$.

   (d) Construct $\Omega(\bar{A}_r)$.

   (e) **Let** $ContainmentVolume = \nu_{\mathbb{H}}(\Omega(\bar{A}_r))$.

9. **Let** $BoundingRadius = \chi_2(\mathfrak{b})$.

10. **Let** $BoundedPiece = \Omega(\bar{A}_r) \cap B(i, BoundingRadius)$.

11. **Let** $AllSidesRepresented = \mathrm{FALSE}$.

12. **Let** $ParabolicCycles = \mathrm{FALSE}$.

13. **Do While** $AllSidesRepresented = \mathrm{FALSE}$ **or**

    $ParabolicCycles = \mathrm{FALSE}$ **or**

    $\nu_{\mathbb{H}}(BoundedPiece) \leq \frac{1}{2}\nu_{\mathbb{H}}(\Omega(\bar{A}_r))$:

   (a) Double the value of r.

(b) *Recalculate* $\mathrm{MP}_P^r(X,\omega)$ *for each* $P \in \Sigma$, *compute* A_r *and* $\Omega(\bar{A}_r)$.

(c) *Calculate* $BoundingRadius = \chi_2(\mathfrak{b})$.

(d) **Let** $BoundedPiece = \Omega(\bar{A}_r) \cap B(i, BoundingRadius)$.

(e) **If** *the only sides of* $\Omega(\bar{A}_r)$ *not contained in* $BoundedPiece$ *have one endpoint on the line at infinity and the other an interior point of* $B(i, BoundingRadius)$,

then let $AllSidesRepresented = \mathrm{TRUE}$.

i. *Calculate the ideal vertex cycles associated to the side pairing transformations on the sides containing an ideal vertex endpoint.*

ii. **If** *all ideal vertex cycles are parabolic,*

then let $ParabolicCycles = \mathrm{TRUE}$.

iii. **else let** $ParabolicCycles = \mathrm{FALSE}$.

(f) **else let** $AllSidesRepresented = \mathrm{FALSE}$ *and*

let $ParabolicCycles = \mathrm{FALSE}$.

14. **Let** $SidePairingRepresentatives$ *be the set of elements in* A_r *associated to the side pairing transformations of* $\Omega(\bar{A}_r)$.

15. **If** $ContainsMinusIdentity = \mathrm{TRUE}$,

then let $Generators = SidePairingRepresentatives \cup \{-\mathrm{Id}\}$.

else let $Generators = SidePairingRepresentatives$.

16. **`Output`** *Generators.*

Proof of Validity If $\Gamma(X,\omega)$ is not a lattice then the loop at Step 8 will never terminate since it will always be the case that

$$ContainmentVolume \geq \nu_{\mathbb{H}}(\Omega(\overline{\Gamma(X,\omega)})) = \infty.$$

In this case the iterations of the loop at Step 8 establish increasing sets of elements in $\Gamma(X,\omega)$ where each set is an exhaustive list of all elements whose norms are bounded above by some value.

Suppose $\Gamma(X,\omega)$ is a lattice. By Theorem 19 it follows that $\Omega(\overline{\Gamma(X,\omega)})$ is a hyperbolic polygon with a finite number of sides, none of which are free sides. However each (non-free) side of $\Omega(\overline{\Gamma(X,\omega)})$ is determined by the boundary of a half-space $H_i(M \cdot i)$ for some $M \in \Gamma(X,\omega)$. In particular there is a finite set of elements $\mathcal{A} \subseteq \Gamma(X,\omega)$ such that $\Omega(\overline{\Gamma(X,\omega)}) = \Omega(\mathcal{A})$.

Let $\mathfrak{b}$ be the maximum norm of all elements in $\mathcal{A}$, and let $R = \frac{2\rho}{\chi_1(\mathfrak{b})}$. Then $\mathcal{A} \subseteq A_r$ for all $r \geq R$. Thus $\Omega(\bar{A}_r) = \Omega(\bar{\mathcal{A}}) = \Omega(\overline{\Gamma(X,\omega)})$ for all $r \geq R$. Since the sets $\Omega(\bar{A}_r)$ decrease as r increases, there is an $r_1 \leq R$ for which $\nu_{\mathbb{H}}(\Omega(\bar{A}_r)) < \infty$ for all $r > r_1$. Since r increases without bound as a function of the number of iterations in the loop at Step 8, it follows that this loop will terminate at some finite number of iterations.

Suppose that the loop at Step 13 is exited, so that

$$AllSidesRepresented = \text{TRUE}$$

and

$$ParabolicCycles = \text{TRUE}$$

and

$$\nu_{\mathbb{H}}(BoundedPiece) > \frac{1}{2}\nu_{\mathbb{H}}(\Omega(\bar{A}_r))$$

at the end of the loop with a given value of $r = r^*$. The finite volume of $\Omega(\bar{A_{r^*}})$ ensures by Theorem 19 that $\Omega(\bar{A_{r^*}})$ is a hyperbolic polygon with no free sides. Since Theorem 20 ensures that the side pairing transformations of $\Omega(\overline{\Gamma(X,\omega)})$ preserve point-wise distance to i, we know that the side-pieces within

$$\Omega(\bar{A_{r^*}}) \cap B(i, BoundingRadius) = \Omega(\overline{\Gamma(X,\omega)}) \cap B(i, BoundingRadius)$$

possess a complete set of pairing transformations. Theorem 20 also ensures that these side pairing transformations are contained in A_{r^*} since they coincide with the elements that determined the sides. The fact that

$$AllSidesRepresented = \text{TRUE}$$

tells us that all sides of $\Omega(\bar{A_{r^*}})$ which have portions outside of $B(i, BoundingRadius)$ also have non-zero length portions inside the same ball. Therefore the side-piece pairings of $\Omega(\bar{A_{r^*}}) \cap B(i, BoundingRadius)$ induce a complete set of side pairings for $\Omega(\bar{A_{r^*}})$.

These side pairings induce cycle transformations on the finite vertices that satisfy the elliptic cycle condition since these vertices lie within

$B(i, BoundingRadius)$ and this is true for $\Omega(\overline{\Gamma(X,\omega)})$ by Theorem 3.5.3 of [Kat]. The cycle transformations on the vertices at infinity satisfy the parabolic cycle condition since $ParabolicCycles$ = TRUE. Thus by Theorem 21 it follows that $\Omega(\bar{A_{r^*}})$ is a fundamental polygon for the group generated by its side pairings. These side pairings are all elements of $\bar{A_{r^*}} \subseteq \overline{\Gamma(X,\omega)}$. Let $\bar{H} \leq \overline{\Gamma(X,\omega)}$ be the subgroup generated by these side pairings. It follows that

$$[\overline{\Gamma(X,\omega)} : \bar{H}] = \frac{\nu_{\mathbb{H}}(\Omega(\bar{A_{r^*}}))}{\nu_{\mathbb{H}}(\Omega(\overline{\Gamma(X,\omega)}))}.$$

Since the last condition gives us that

$$\nu_{\mathbb{H}}(\Omega(\overline{\Gamma(X,\omega)})) \geq \nu_{\mathbb{H}}(BoundedPiece) > \frac{1}{2}\nu_{\mathbb{H}}(\Omega(\bar{A}_r))$$

it follows that

$$[\overline{\Gamma(X,\omega)} : \bar{H}] = \frac{\nu_{\mathbb{H}}(\Omega(\bar{A_{r^*}}))}{\nu_{\mathbb{H}}(\Omega(\overline{\Gamma(X,\omega)}))} < 2.$$

Thus in fact $\overline{\Gamma(X,\omega)} = \bar{H}$ and so the side pairings for $\Omega(\bar{A_{r^*}})$ generate $\overline{\Gamma(X,\omega)}$.

We now prove that the loop at Step 13 will be exited in a finite number of iterations. It is clear that r increases without bound as a function of the number of iterations of the loop at Step 13. Since $\chi_2(\chi_1^{-1}(\frac{2\rho}{r}))$ increases without bound as a function of r, it follows that $BoundingRadius$ increases without bound as a function of the number of iterations within this loop. For all iterations beyond the finite number of iterations that are required to achieve $r > R$, it follows that $\Omega(\bar{A}_r) = \Omega(\overline{\Gamma(X,\omega)})$ and $BoundingRadius$ continues to increase without bound. Therefore only a finite number of further iterations are required in order for the value of *BoundingRadius* to be large enough to ensure that the only sides of $\Omega(\bar{A}_r) = \Omega(\overline{\Gamma(X,\omega)})$ that are not contained in $B(i, BoundingRadius)$ will have one endpoint on the line at infinity and the other an interior point of $B(i, BoundingRadius)$. This will continue to be true for all subsequent iterations and so the value of *AllSidesRepresented* will continue to be TRUE. The 'induced' side pairings on $\Omega(\bar{A}_r) = \Omega(\overline{\Gamma(X,\omega)})$ are clearly the side pairings of $\Omega(\overline{\Gamma(X,\omega)})$, which by Theorem 9.3.8 of [Bea] induces ideal vertex cycle transformations that are parabolic. This will also remain true for all subsequent iterations so that the value of *ParabolicCycles* will continue to be TRUE. It is also clear that only a finite number of further iterations are required for the inequality

$$\nu_{\mathbb{H}}(\Omega(\overline{\Gamma(X,\omega)}) \cap B(i, BoundingRadius)) > \frac{1}{2}\nu_{\mathbb{H}}(\Omega(\overline{\Gamma(X,\omega)}))$$

to be valid. It then follows that

$$\nu_{\mathbb{H}}(BoundedPiece) = \nu_{\mathbb{H}}(\Omega(\overline{\Gamma(X,\omega)}) \cap B(i, BoundingRadius)) > \frac{1}{2}\nu_{\mathbb{H}}(\Omega(\bar{A}_r)).$$

Hence the loop at Step 13 will indeed terminate in a finite number of iterations at which point the side pairings of $\Omega(\bar{A}_r)$ generate $\overline{\Gamma(X,\omega)}$.

Let $M_1, ..., M_l \in A_r \subseteq \Gamma(X,\omega) \subseteq \mathrm{SL}(2,\mathbb{R})$ be the elements that were collected within the loop at Step 13 that represent the side pairing elements $\bar{M}_1, ..., \bar{M}_l$ of

$\Omega(\bar{A}_r)$. The fact that $\{\bar{M}_1, ..., \bar{M}_l\}$ generates $\overline{\Gamma(X,\omega)}$ implies:

$$\text{Given } M \in \Gamma(X,\omega), \text{ a word } w \text{ in } \{M_1, ..., M_l\} \text{ satisfies } w = \pm M.$$

If $-\mathrm{Id} \in \Gamma(X,\omega)$ it follows that a word in $\{M_1, ..., M_l, -\mathrm{Id}\}$ can be found that equals any element in $\Gamma(X,\omega)$. If $-\mathrm{Id} \notin \Gamma(X,\omega)$ then $w = -M$ can never hold in the statement above, since both w and M are in $\Gamma(X,\omega)$. In this case it is clear that $\{M_1, ..., M_l\}$ generates $\Gamma(X,\omega)$. Therefore Step 15 correctly constructs a generating set for $\Gamma(X,\omega)$. □

When the predicate condition for the previous algorithm is not met, one finds a new translation surface that meets the condition by acting with an element $M_0 \in \mathrm{SL}(2,\mathbb{R})$ close to Id. Once the Veech group of the new surface is computed, the generators of the Veech group of the original surface are obtained by conjugating the generators of the new surface by M_0^{-1}.

Algorithm 7.2. *Calculating the Veech Group of a Translation Surface.*

Input: The Voronoi decomposition of a translation surface $(X,\omega) \in \mathcal{H}(i_1, ...i_s)$.

Output:

- *(lattice) A finite generating set for* $\Gamma(X,\omega)$.
- *(non-lattice) The algorithm will not terminate, but will continue to enumerate the elements in the Veech group in order of increasing norm. A stopping condition based on norm or time could be utilized for this case.*

1. *Calculate* $\rho = \rho(X, \omega)$ *using the Voronoi 2-cells.*

   (see Definition 33).

2. **Let** $r = 2\rho$ *and* **let** $\mathfrak{b} = \chi_1^{-1}(\frac{2\rho}{r}) = \sqrt{2}$.

3. *Calculate* $\mathrm{MP}_P^r(X, \omega)$ *for all* $P \in \Sigma$.

4. *Calculate* $A_r = \{M \in \Gamma(X, \omega) |\ ||M|| \leq \mathfrak{b}\} = \mathrm{SO}(2, \mathbb{R}) \cap \Gamma(X, \omega)$

   using Theorem 18.

5. **If** $M \in A_r$ *with* $M \notin \{\pm Id_{\mathrm{SO}(2,\mathbb{R})}\}$,

   then *find an* $M_0 \in \mathrm{SL}(2, \mathbb{R})$ *such that*

   $\Gamma(M_0 \cdot (X, \omega)) \cap \mathrm{SO}(2, \mathbb{R}) \subseteq \{\pm Id_{\mathrm{SO}(2,\mathbb{R})}\}$.

6. **Let** $(X_0, \omega_0) = M_0 \cdot (X, \omega)$.

7. *Execute Algorithm 7.1 with input:* (X_0, ω_0)

   and rename the output: $AssociatedGenerators$.

8. **Let** $Generators = M_0^{-1} \cdot AssociatedGenerators \cdot M_0$.

9. **Output** $Generators$.

8 Conclusion

We have established a new computational framework and an algorithm for computing elements of the Veech group of translation surfaces defined by polygons with side identifications where the polygonal side lengths lie in a number field so that exact arithmetic can be performed. The algorithm computes subsets of the Veech group defined by elements whose Frobenius norms are bounded above by progressively higher values. The algorithm detects when the Veech group is a lattice, and in this case determines a finite generating set for the group.

A partial implementation of this algorithm has been written using the Sage programming language. Future work could complete this partial implementation in order to make it an available tool to the mathematics community. The characterization of Veech group elements used in our algorithm identifies the elements as stabilizers of the marked periods of the translation surface. This new characterization may be used in future investigations of known Veech groups through the study of their marked periods. In addition, the creation of exotic marked periods objects could be used to define translation surfaces whose Veech groups have not yet been known to exist.

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