On the Effect of Bottom Friction on Barotropic Motion Over the Continental Shelf

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ABSTRACT

Observations of the velocity fields over the continental shelf and slope off Oregon and off Peru have shown that there is a phase difference in the onshore-offshore direction, with the velocity fluctuations nearshore leading those offshore in time. It is shown here that the effects of bottom Ekman layer friction and cross-shelf depth variation combine to result in such a phase lag in a model for forced or free long barotropic continental shelf waves. The model also shows that bottom friction results in a smaller phase lag between the alongshore components of velocity and wind stress than that predicted by a frictionless model, a feature found in the observations off Oregon.

1. Introduction

In a study of velocity measurements over the continental shelf-slope region off Oregon, Sobey (1977) found that, at frequencies <0.16 cpd, fluctuations near the coast lead those farther offshore in time. Brink *et al.* (1978) have obtained a similar result off Peru. In a study of the modal structure of low-frequency fluctuations off Oregon, Kundu, *et al.* (1975) found that the alongshore component of velocity was more nearly in phase with the local alongshore wind stress than could be accounted for by a frictionless model.

We shall show that these observations can be explained by the effects of bottom friction. Kundu (1977) has shown that a turbulent Ekman-like bottom boundary layer exists off Oregon; Brink *et al.* (1978) have concluded the same off Peru. Gill and Schumann (1974), in their model of long shelf waves, included the effect of bottom friction in a gross sense, but did not consider the problem in detail. We shall present a more explicit model for a barotropic fluid and demonstrate the implications of bottom friction in a continental shelf-slope region.

2. Formulation

We consider a model where a homogeneous fluid is situated on an f-plane which effectively rotates with a uniform angular velocity $\mathbf{\Omega} = \frac{1}{2}f\mathbf{k}$, where $\mathbf{k}$ is a constant unit vector in the z (vertical) direction in a Cartesian coordinate system and f the Coriolis parameter. It is assumed that there is a straight coast along $x = 0$, and that a fluid of depth $H(x)$ exists in $x < 0$. The fluid has a constant depth for $x \leqslant -L$, and $H(0) \neq 0$. We make the usual approximations for long shelf waves, i.e., 1) that there is a rigid lid, 2) that the frequency of the motions ω is small compared to f, 3) that the width of the shelf is small relative to typical alongshore scales, and 4) that only the alongshore component of wind stress is important and that it does not vary appreciably over the scale of the shelf width. With these assumptions, the linear depth-integrated equations of motion are

$$U_x + V_y = 0, \tag{2.1a}$$

$$fV = Hp_x, \tag{2.1b}$$

$$V_t + fU = -Hp_y + \rho^{-1}(\tau_w^y - \tau_B^y), \tag{2.1c}$$

where (U,V) are the depth-integrated (x,y) velocity components, respectively, ρ is the density of the fluid, p the perturbation pressure divided by density, $\tau_w^y(y,t)$ the alongshore component of the wind stress and τ_B^y the alongshore component of the bottom stress. Partial differentiation is denoted by the subscripts (x,y,t).

The bottom stress is evaluated using Ekman-layer dynamics with a constant vertical eddy coefficient A_v. We assume that the depth of the bottom frictional layer is small relative to the depth H of the fluid i.e., that

$$E_0^{1/2} \ll 1, \tag{2.2a}$$

where E_0 is an Ekman number defined by

$$E_0 = A_v(2fH_0^2)^{-1}, \tag{2.2b}$$

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where $H_0 = H(x = 0) \neq 0$. For $\omega \ll f$, for $V \gg U$ [implied by approximations 2) and 3) and (2.1a)], and with the assumption of a small bottom slope ($|H_x| \ll 1$), we obtain

$$\tau_B^y = \frac{\rho}{H(x)} (\tfrac{1}{2}A_v f)^{1/2} V = \rho f \mathrm{E}_0^{1/2} V H_0/H(x). \quad (2.3)$$

For a typical continental shelf-slope region, $|H_x|$ is generally less than 10^{-2} so that the assumption $|H_x| \ll 1$ is justified. Bottom friction is neglected in (2.1b) by assumption (2.2a) but is retained in (2.1c) because $U \ll V$ and $\omega \ll f$. Note that as a consequence of these assumptions, bottom friction enters (2.1c) in the form of an H-dependent drag coefficient multiplied by V.

The continuity equation (2.1a) allows the definition of a stream-function $\psi(x,y,t)$ such that

$$U = \psi_y, \quad V = -\psi_x. \quad (2.4\text{a,b})$$

A mass transport vorticity equation for ψ may be obtained from (2.1b) and (2.1c), i.e.,

$$\psi_{xxt} - \frac{H_x}{H}\psi_{xt} + f\frac{H_x}{H}\psi_y = -f\frac{H_0}{H}\mathrm{E}_0^{1/2}\left(\psi_{xx} - 2\frac{H_x}{H}\psi_x\right) + \frac{H_x}{\rho H}\tau_w^y. \quad (2.5)$$

The boundary conditions for (2.5) (Gill and Schumann, 1974) are

$$\psi_y = 0 \quad \text{at} \quad x = 0, \quad (2.6\text{a})$$

$$\psi_{xt} = 0 \quad \text{at} \quad x = -L. \quad (2.6\text{b})$$

Note that the effect of friction enters (2.5) proportional to the square root of the local Ekman number

$$\mathrm{E} = (H_0/H)^2 \mathrm{E}_0, \quad (2.7)$$

which decreases as H increases.

The problem will be solved by perturbation methods for the limit

$$\mathrm{E}_0^{1/2} \to 0. \quad (2.8)$$

The solution may be written as (Gill and Schumann, 1974)

$$\psi(x,y,t) = \sum_{n=1}^{\infty} Y_n(y,t)\phi_n(x), \quad (2.9)$$

where

$$Y_n(y,t) = Y_{0n}(y,t) + \mathrm{E}_0^{1/2} Y_{1n}(y,t) + \ldots, \quad (2.10)$$

and $\phi_n(x)$ is the solution of the eigenvalue problem

$$\phi_{nxx} - \frac{H_x}{H}\phi_{nx} - \frac{fH_x}{Hc_n}\phi_n = 0, \quad (2.11\text{a})$$

$$\phi_n(0) = 0, \quad \phi_{nx}(-L) = 0, \quad (2.11\text{b,c})$$

where c_n is the phase speed associated with the nth mode. The modes are orthogonal and normalized such that

$$\int_0^{-L} \frac{H_x}{H^2}\phi_n(x)\phi_m(x)dx = \delta_{nm}. \quad (2.12)$$

Substituting (2.9) into (2.5) yields

$$(1/c_n)Y_{0n_t} + Y_{0n_y} + \mathrm{E}_0^{1/2} a_{nn} Y_{0n} = F_n(y,t), \quad (2.13\text{a})$$

$$(1/c_n)Y_{1n_t} + Y_{1n_y} + \mathrm{E}_0^{1/2} a_{nn} Y_{1n} = -\sum_{\substack{m=1\\ m\neq n}}^{\infty} a_{mn} Y_{0m}, \quad (2.13\text{b})$$

where

$$F_n(y,t) = b_n \tau_w^y(y,t)/(\rho f), \quad (2.13\text{c})$$

$$1 = \sum_{n=1}^{\infty} b_n \phi_n(x). \quad (2.13\text{d})$$

The remainder of the right-hand side of (2.5) is represented by

$$-\frac{H_0}{H}\phi_{nx} + \frac{fH_0}{Hc_n}\phi_n = \sum_{m=1}^{\infty} a_{nm}\phi_m, \quad (2.13\text{e})$$

since ϕ_n satisfies (2.11a). The coefficients a_{nm} are expressed simply by using (2.11a,b,c) and (2.12) and by integration by parts:

$$a_{nm} = H_0 \int_{-L}^{0} H^{-2}\phi_{mx}\phi_{nx}dx. \quad (2.13\text{f})$$

The onshore-offshore variation in the importance of friction (2.7) results in the scattering of wave energy among modes, as shown by the right-hand side of (2.13b).

3. An example

Consider an impulsively applied forcing described by

$$\rho^{-1}\tau_w^y(y,t) = B_0 H(t) \exp[il(ct - y)], \quad (3.1)$$

where $H(t)$ is the Heaviside unit function.

The solution, correct to $\mathrm{O}(\mathrm{E}_0^{1/2})$, is

$$\begin{aligned}\psi(x,y,t) &= H(t)\sum_{n=1}^{\infty} A_n\langle \phi_n(x)\{\exp[il(ct - y)] \\ &\quad - \exp[il(c_n t - y) - c_n a_{nn}\mathrm{E}_0^{1/2} t]\} + i\mathrm{E}_0^{1/2} \\ &\quad \times \sum_{\substack{m=1\\ m\neq n}}^{\infty} a_{nm}\phi_m(x)\{c_m l^{-1}(c - c_m)^{-1}\exp[il(ct - y)] \\ &\quad - c_m l^{-1}(c_n - c_m)^{-1}\exp[il(c_n t - y) - c_n a_{nn}\mathrm{E}_0^{1/2} t] \\ &\quad - \alpha_{nm}\exp[il(c_m t - y) - c_m a_{mm}\mathrm{E}_0^{1/2} t]\}\rangle, \quad (3.2\text{a})\end{aligned}$$

where

$$A_n = -ib_n B_0[(cc_n^{-1} - 1)l - ia_{nn}\mathrm{E}_0^{1/2}]^{-1}, \quad (3.2\text{b})$$

$$\alpha_{nm} = (c_n - c)c_m l^{-1} \times [c(c_n - c_m) + c_m(c_m - c_n)]^{-1}. \quad (3.2\text{c})$$

This may be rewritten as

$$\psi(x,y,t) = H(t) \sum_{n=1}^{\infty} A_n[|\phi_n(x)| \{\exp[i(clt - ly + \theta_{Dn}(x))] - \exp[i(c_n lt - ly + \theta_{Fn}(x)) - c_n a_{nn} E_0^{1/2} t]\} + i E_0^{1/2} \sum_{\substack{m=1\\m\neq n}}^{\infty} \phi_m(x) a_{nm} \times \exp[il(c_m t - y) - c_m a_{mm} E_0^{1/2} t]], \quad (3.3a)$$

where the x-dependent phase lags are defined by

$$\theta_{Dn} = \tan^{-1}\{E_0^{1/2} \sum_{\substack{m=1\\m\neq n}}^{\infty} a_{nm} c_m \times [l(c - c_m)]^{-1} \phi_m(x)/\phi_n(x)\}, \quad (3.3b)$$

$$\theta_{Fn} = \tan^{-1}\{E_0^{1/2} \sum_{\substack{m=1\\m\neq n}}^{\infty} a_{nm} c_m \times [l(c_n - c_m)]^{-1} \phi_m(x)/\phi_n(x)\}. \quad (3.3c)$$

The response (3.2) consists of five components for each mode number n. The first order solution Y_{0n} consists of a forced wave and a time-decaying free wave with phase velocities c and c_n, respectively. The effect of bottom friction on the first-order response is to damp the free waves through the term $c_n a_{nn} E_0^{1/2}$ in the exponential, and to bring the forced response more nearly into phase with the driving as can be seen from the term $i a_{nn} E_0^{1/2}$ in the denominator of A_n [Eq. (3.2b)]. The second order solution Y_{1n} represents scattered waves in modes $m \neq n$. These consist of two components driven respectively by the first-order forced and free modes, and are summed in (3.3b,c) to yield the x-dependent phase lags $\theta_{Dn}(x)$ and $\theta_{Fn}(x)$. The fifth component represents time-decaying second-order (Y_{1n}) free waves generated in response to the impulsively started second-order response.

Fig. 1 shows the frictionally induced phase shifts $\theta_{Dn}(x)$ and $\theta_{Fn}(x)$ for an example with exponential bottom topography, i.e.,

$$H(x) = H_0 e^{-\lambda x}. \quad (3.4)$$

The values of the parameters are listed in the figure caption. Both $\theta_{D2}(x)$ and $\theta_{F2}(x)$ shift rapidly from near 0° to near 180° at the point where $\phi_2(x)$ changes sign. The main result is that the phase shifts decrease offshore, so that the motions near the coast lead those farther offshore in time.

4. Discussion

The presence of bottom friction over a continental shelf-slope region affects barotropic motion in three ways:

1) It damps free waves with a time scale $T_F = (c_n a_{nn} E_0^{1/2})^{-1}$. For example, with exponential bottom topography and with the parameter values in the caption of Fig. 1, we obtain (for the free, first, $n = 1$ mode) $a_{11} = 4.7 \times 10^{-7}\ \mathrm{cm}^{-1}$ and $T_F = 4.6$ days.

2) It brings the directly forced component of the alongshore flow more nearly into phase with the local driving; i.e., with no friction, Y_{0n} is $\pi/2$ out of phase with $F_n(y,t)$, while in the presence of dissipation, they are more nearly in phase. In (2.13a), the relative importance of friction depends on the ratios $T_F T^{-1}$ and $T_F(c_n l)$, where T is the period of the forcing. In the limit of large dissipation (i.e., $c_n l T_F \ll 1$ and $c l T_F \ll 1$), the balance in (2.13a) is

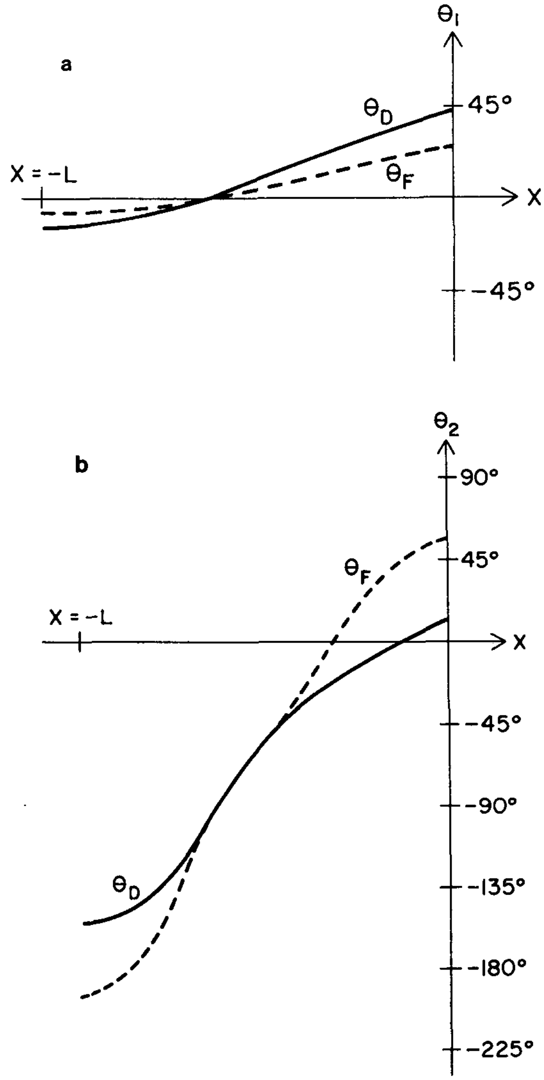

FIG. 1. Values of the functions $\theta_{Dn}(x)$ and $\theta_{Fn}(x)$ for the first two modes (n = 1, 2) in the case of exponential bottom topography. Parameters used are $f = 1.03 \times 10^{-4}\ \mathrm{s}^{-1}$ (latitude 45°N), $l = 2 \times 10^{-8}\ \mathrm{cm}^{-1}$, $\lambda = 3 \times 10^{-7}\ \mathrm{cm}^{-1}$, $L = 10^7$ cm, $H_0 = 10^4$ cm, $c = 278\ \mathrm{cm\ s}^{-1}$, $E_0^{1/2} = 0.12$, $c_1 = 443\ \mathrm{cm\ s}^{-1}$ and $c_2 = 113\ \mathrm{cm\ s}^{-1}$ ($cl = 5.56 \times 10^{-6}\ \mathrm{s}^{-1} = 0.076$ cpd). (a) First mode phase values; (b) second mode phase values.

only between local forcing and bottom friction, so that Y_{0n} is in phase with $F_n(y,t)$.

3) It sets up cross-shelf phase lags such that flow nearshore leads that offshore in time. The greater frictional effect in shallow water (appearing as a larger local Ekman number) retards the flow preferentially near the coast. This causes a perturbation relative vorticity growth (V_{xt}) proportional to minus V. The vorticity generating mechanism for inviscid shelf waves is vortex stretching, where vorticity is developed proportional to $-U$ in the Northern Hemisphere, and where U leads V for free shelf waves. With bottom friction, there is perturbation vorticity development near the coast proportional to $-V$ which leads that caused by U. Thus, vorticity development in shallow water leads that in deeper water where friction plays less of a role.

It is now possible to compare our results with the observations of Sobey (1977). His data consist of current meter records from January through April 1975 (about three months of data) over the shelf off Oregon near 45°N. The moorings were arranged in a line perpendicular to the coast. He computed coherence and phase between several pairs of alongshore velocity records at 50 m depth, but the most relevant pairing is that between Pikake (on the 60 m isobath) and Wisteria (on the 225 m isobath), which are separated by 23 km. For the frequency band centered at 0.06 cpd, he obtained a phase difference between the two of 35° ± 23°, which is equivalent to an observed $\theta'(x) = (1.5 \pm 1)$ deg km^{-1} at the intermediate 140 m isobath. For the data representing the summer of 1973 at a nearby location, Kundu (1977) estimated the depth of the bottom Ekman layer to be 12 m in water 100 m deep. We use this value, along with the parameters listed in the caption for Fig. 1 (chosen to be representative of the region) to obtain $E_0^{1/2} = 0.12$ and $\theta'(x) = 0.7$ deg km^{-1} for a frequency near 0.06 cpd. Although the parameters chosen push the theory to the limit of its validity [from (2.5), $fE_0^{1/2}(cl)^{-1}$ must be small], our calculated value agrees within error of observation, which is encouraging. The agreement, however, depends on the choice of parameters $E_0^{1/2}$, l and c, which are difficult to estimate accurately.

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